Chapter 4
Lebesgue Points of Higher Dimensional Functions

In Theorem 1.5.4, we have proved the well known theorem of Lebesgue [197], i.e., for one-dimensional Fejér and Cesàro means and for all $f \in L_1(\mathbb{T})$,

$$\lim_{n\to\infty} \sigma_n^\alpha f(x) = f(x)$$

at each Lebesgue point of f. In this chapter, we generalize this result to higher dimensions and to all summability methods considered in Chaps. 2 and 3. We investigate a common generalization of the Cesàro, Riesz and θ-means and define

$$\sigma_n f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n(t)\, dt,$$

where $n \in \mathbb{N}$ or $n \in \mathbb{N}^d$ and $f \in L_1(\mathbb{T}^d)$, $K_n \in L_1(\mathbb{T}^d) \cap L_\infty(\mathbb{T}^d)$. We will give sufficient and/or necessary conditions for K_n such that $\sigma_n f$ is convergent at each Lebesgue point. We will study six versions of Lebesgue points, for different summability methods different Lebesgue points. We consider again the triangular, circular, cubic, the restricted (taken on a cone or cone-like set) and unrestricted rectangular summability as in the previous chapters. The proofs are very different for different summability methods, therefore each case needs new ideas. For each type of Lebesgue points, we introduce different and new type of Hardy-Littlewood maximal functions. We prove that these maximal operators are bounded from $L_p(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ with $1 < p \le \infty$ and we prove also a weak type inequality for $p = 1$. Using this, we obtain that almost every point is a Lebesgue point of an integrable function.

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4.1 ℓ_2-Summability

In this section, we use the notation

$$\sigma_n f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n(t)\, dt \qquad (n \in \mathbb{N}),$$

where $f \in L_1(\mathbb{T}^d)$ and $K_n \in L_1(\mathbb{T}^d) \cap L_\infty(\mathbb{T}^d)$ for all $n \in \mathbb{N}$. If $K_n = K_n^{2,\alpha,\gamma}$, $K_n^{2,\theta}$ or K_n is the one-dimensional Cesàro kernel K_n^α, then we obtain the ℓ_2-Riesz and θ-means $\sigma_n^{2,\alpha,\gamma} f$, $\sigma_n^{2,\theta} f$ or the one-dimensional Cesàro means $\sigma_n^\alpha f$, respectively. The higher dimensional ℓ_2-Riesz kernel $K_n^{2,\alpha,\gamma}$ and the one-dimensional Cesàro kernel K_n^α satisfy all conditions in this subsection. Under some conditions on θ, $K_n^{2,\theta}$ satisfies all conditions, too.

4.1.1 *Hardy-Littlewood Maximal Functions*

We generalize the Hardy-Littlewood maximal function for higher dimensions. As in the one-dimensional case, the Hardy-Littlewood maximal function is bounded on $L_p(\mathbb{T}^d)$ for $1 < p \le \infty$ and it is of weak type $(1, 1)$. We denote by $B_r(c, h)$ $(c \in \mathbb{T}^d, h > 0)$ the r-ball

$$B_r(c, h) := \{x \in \mathbb{T}^d : \|x - c\|_r < h\} \qquad (1 \le r \le \infty)$$

with center c and radius h. For $r = 2$, we omit the index and write simply $B = B_2$. Similarly to the one-dimensional case, the Hardy-Littlewood maximal function can be given by

$$M_p^r f(x) = \sup_{x \in B_r} \left(\frac{1}{|B_r|} \int_{B_r} |f|^p \, d\lambda \right)^{1/p} \qquad (x \in \mathbb{T}^d),$$

where the supremum is taken over all r-balls B_r containing x and $1 \le p, r \le \infty$. In the special case when $r = \infty$, we have to take the supremum over all cubes I with sides parallel to the axes. Note that in the one-dimensional case this definition was given for $p = 1$, only. In this section, we will rather use the next equivalent centered version.

Definition 4.1.1 For $1 \le p < \infty$ and $f \in L_p(\mathbb{T}^d)$, the Hardy-Littlewood maximal function is defined by

$$M_p f(x) := \sup_{h>0} \left(\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x-t)|^p \, dt \right)^{1/p}.$$

If we take the supremum over all $0 < h < \pi$, then we get an equivalent definition. It is easy to see that

$$C_1 M_p f \le M_p^r f \le C_2 M_p f$$

for all $1 \le p < \infty$ and $1 \le r \le \infty$. If $p = 1$, then we omit the notation p and write simply Mf. The next theorem can be proved exactly as Theorem 1.3.3 in the one-dimensional case.

Theorem 4.1.2 *If $1 \le p < \infty$, then the maximal operator M_p is of weak type (p, p), i.e.,*

$$\sup_{\rho>0} \rho\lambda(M_p f > \rho)^{1/p} \le C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Moreover, if $p < r \le \infty$, then

$$\left\|M_p f\right\|_r \le C_r \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

Using the density theorem of Marcinkiewicz and Zygmund (see Theorem 1.3.6), we can formulateLebesgue's differentiation theorem similarly to Corollary 1.3.8.

Corollary 4.1.3 *If $f \in L_1(\mathbb{T}^d)$, then*

$$\lim_{h\to 0} \frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} f(x-t)\, dt = f(x)$$

for almost every $x \in \mathbb{T}^d$.

This implies that the inequality

$$\|f\|_p \le \|Mf\|_p \qquad (1 < p \le \infty)$$

is trivial. Now we introduce the restricted Hardy-Littlewood maximal function by

$$M_{\square,p}^{\infty} f(x) := \left(\sup_{\substack{x\in I, \tau^{-1}\le |I_i|/|I_j|\le\tau \\ i,j=1,\dots,d}} \frac{1}{|I|} \int_I |f|^p \, d\lambda \right)^{1/p} \qquad (x \in \mathbb{T}^d)$$

for some fixed $\tau \ge 1$, where the supremum is taken over all appropriate rectangles

$$I = I_1 \times \cdots \times I_d$$

with sides parallel to the axes. The centered version is given in

Definition 4.1.4 For a fixed $\tau \ge 1$ and $f \in L_p(\mathbb{T}^d)$, the restricted Hardy-Littlewood maximal function is defined by

$$M_{\square,p} f(x) := \sup_{h \in \mathbb{R}^d_\tau} \left(\frac{1}{2^d \prod_{k=1}^d h_k} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t)|^p \, dt \right)^{1/p}.$$

Recall that

$$\mathbb{R}^d_\tau := \{x \in \mathbb{R}^d_+ : \tau^{-1} \le x_i/x_j \le \tau, i, j = 1, \ldots, d\}$$

was defined in (3.3.1). Taking the supremum over all $h \in \mathbb{R}^d_+$, we get a different maximal function, the so called strong Hardy-Littlewood maximal function. We will study this maximal operator in Sect. 4.2.1. Again, it is easy to see that

$$C_1 M_{\square,p} f \le M^{\infty}_{\square,p} f \le C_2 M_{\square,p} f$$

and

$$C_1 M_{\square,p} f \le M_p f \le C_2 M_{\square,p} f \tag{4.1.1}$$

for all $1 \le p \le \infty$. From this follows

Corollary 4.1.5 *If $\tau \ge 1$ is arbitrary and $1 \le p < \infty$, then*

$$\sup_{\rho>0} \rho \lambda(M_{\square,p} f > \rho)^{1/p} \le C \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Moreover, if $p < r \le \infty$, then

$$\left\| M_{\square,p} f \right\|_r \le C_r \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

Corollary 4.1.6 *If $\tau \ge 1$ is arbitrary and $f \in L_1(\mathbb{T}^d)$, then*

$$\lim_{h\to 0, h \in \mathbb{R}^d_\tau} \frac{1}{2^d \prod_{k=1}^d h_k} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} f(x-t) \, dt = f(x)$$

for almost every $x \in \mathbb{T}$.

4.1.2 Lebesgue Points for the ℓ_2-Summability

First of all, we generalize the Herz spaces for higher dimensions.

Definition 4.1.7 For $1 \le q, r \le \infty$, the Herz space $E^r_q(\mathbb{R}^d)$ contains all functions f for which

$$\|f\|_{E^r_q} := \sum_{k=-\infty}^{\infty} 2^{kd(1-1/q)} \left\| f 1_{P^r_k} \right\|_q < \infty,$$

where

$$P_k^r := \{x \in \mathbb{R}^d : 2^{k-1}\pi \le \|x\|_r < 2^k\pi\} = B_r(0, 2^k\pi) \setminus B_r(0, 2^{k-1}\pi).$$

If we modify the definition of P_k^r,

$$P_k^r := \{x \in \mathbb{R}^d : 2^{k-1}\pi \le \|x\|_r < 2^k\pi\} \cap \mathbb{T}^d,$$

then we get the definition of the space $E_q^r(\mathbb{T}^d)$.

These spaces are special cases of the Herz spaces [166] (see also Garcia-Cuerva and Herrero [113]). We immediately obtain the next proposition.

Proposition 4.1.8 *For a fixed* $1 \le q \le \infty$*, the spaces* $E_q^r(\mathbb{X}^d)$ *are equivalent for all* $1 \le r \le \infty$*, where* $\mathbb{X} = \mathbb{R}$ *or* $\mathbb{X} = \mathbb{T}$*.*

For simplicity, we will use usually the sets P_k^∞ and the space $E_q^\infty(\mathbb{X}^d)$ ($\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = \mathbb{T}$). These sets and spaces will be denoted by P_k and $E_q(\mathbb{X}^d)$. This means that we have to take the sum in the $E_q(\mathbb{T}^d)$-norm only for $k \le 0$, i.e.,

$$\|f\|_{E_q(\mathbb{T}^d)} = \sum_{k=-\infty}^{0} 2^{kd(1-1/q)} \left\| f 1_{P_k} \right\|_q < \infty.$$

It is easy to see that

$$L_1(\mathbb{X}^d) = E_1(\mathbb{X}^d) \hookleftarrow E_q(\mathbb{X}^d) \hookleftarrow E_{q'}(\mathbb{X}^d) \hookleftarrow E_\infty(\mathbb{X}^d)$$

for all $1 < q < q' < \infty$, where $\mathbb{X}$ denotes either $\mathbb{R}$ or $\mathbb{T}$. Moreover,

$$E_q(\mathbb{T}^d) \hookleftarrow L_q(\mathbb{T}^d) \qquad (1 \le q \le \infty). \tag{4.1.2}$$

Indeed,

$$\|f\|_{E_q(\mathbb{T}^d)} = \sum_{k=-\infty}^{0} 2^{kd(1-1/q)} \left\| f 1_{P_k} \right\|_q \le \sum_{k=-\infty}^{0} 2^{kd(1-1/q)} \|f\|_q \le \|f\|_q .$$

It is known in the one-dimensional case (see, e.g., Torchinsky [310]) that if there exists an even function η such that η is non-increasing on $\mathbb{R}_+$, $|\widehat{\theta}| \le \eta$, $\eta \in L_1(\mathbb{R})$, then σ_*^θ is of weak type $(1,1)$. Under similar conditions, we will generalize this result to the multi-dimensional setting.

Theorem 4.1.9 *For a measurable function* f*, let the non-increasing majorant be defined by*

$$\eta(x) := \sup_{\|t\|_r \ge \|x\|_r} |f(t)|$$

for some $1 \leq r \leq \infty$. *Then* $f \in E_\infty(\mathbb{X}^d)$ *if and only if* $\eta \in L_1(\mathbb{X}^d)$ *and*

$$C^{-1}\|\eta\|_1 \leq \|f\|_{E_\infty(\mathbb{X}^d)} \leq C\|\eta\|_1,$$

where $\mathbb{X} = \mathbb{R}$ *or* $\mathbb{X} = \mathbb{T}$.

Proof We prove the theorem for $\mathbb{X} = \mathbb{R}$. If $\eta \in L_1(\mathbb{R}^d)$, then

$$\|f\|_{E_\infty} \leq \|\eta\|_{E_\infty} = \sum_{k=-\infty}^{\infty} 2^{kd} \left\|\eta 1_{P_k}\right\|_\infty = \sum_{k=-\infty}^{\infty} 2^{kd}\eta(2^{k-1}\pi) \leq C\|\eta\|_1.$$

For the converse, denote by

$$a_k := \sup_{B_r(0,2^k\pi)\setminus B_r(0,2^{k-1}\pi)} |f| \quad \text{and} \quad \nu' := \sum_{k=-\infty}^{\infty} a_k 1_{B_r(0,2^k\pi)\setminus B_r(0,2^{k-1}\pi)}.$$

Let

$$\nu(x) := \sup_{\|t\|_r \geq \|x\|_r} \nu'(t) \qquad (x \in \mathbb{R}^d).$$

Since $f \in E_\infty(\mathbb{R}^d)$ implies $\lim_{k\to\infty} a_k = 0$, we conclude that there exists an increasing sequence $(n_k)_{k\in\mathbb{Z}}$ of integers such that $(a_{n_k})_{k\in\mathbb{Z}}$ is decreasing and ν can be written in the form

$$\nu = \sum_{k=-\infty}^{\infty} a_{n_k} 1_{B_r(0,2^{n_k}\pi)\setminus B_r(0,2^{n_{k-1}}\pi)}.$$

Thus

$$\begin{aligned}\|\eta\|_1 \leq \|\nu\|_1 &= \sum_{k=-\infty}^{\infty} a_{n_k} \int_{B_r(0,2^{n_k}\pi)\setminus B_r(0,2^{n_{k-1}}\pi)} d\lambda \\ &= C \sum_{k=-\infty}^{\infty} \left(2^{dn_k} - 2^{dn_{k-1}}\right) a_{n_k}.\end{aligned}$$

By Abel rearrangement,

$$\|\eta\|_1 \leq C \sum_{k=-\infty}^{\infty} 2^{dn_{k-1}}(a_{n_{k-1}} - a_{n_k}) \leq C\|f\|_{E_\infty},$$

which proves the theorem. ■

The maximal operator is introduced by

$$\sigma_* f := \sup_{n\in\mathbb{N}} |\sigma_n f|.$$

In the next theorem we show that, under some conditions, the maximal operator can be estimated by the Hardy-Littlewood maximal function pointwise.

Theorem 4.1.10 *If* $1 \le p < \infty$, $1/p + 1/q = 1$ *and*

$$\sup_{n\in\mathbb{N}} \|K_n\|_{E_q(\mathbb{T}^d)} \le C, \tag{4.1.3}$$

then

$$\sigma_* f(x) \le C \left(\sup_{n\in\mathbb{N}} \|K_n\|_{E_q(\mathbb{T}^d)} \right) M_p f(x)$$

for all $f \in L_p(\mathbb{T}^d)$ *and* $x \in \mathbb{T}^d$.

Proof By the definition of $\sigma_n f$,

$$\begin{aligned} |\sigma_n^\theta f(x)| &= \frac{1}{(2\pi)^d} \left| \int_{\mathbb{T}^d} f(x-t) K_n(t)\, dt \right| \\ &\le \frac{1}{(2\pi)^d} \sum_{k=-\infty}^{0} \int_{P_k} |f(x-t)||K_n(t)|\, dt. \end{aligned}$$

Recall that

$$P_k = P_k^\infty = \{x \in \mathbb{R}^d : 2^{k-1}\pi \le \|x\|_\infty < 2^k \pi\}.$$

By Hölder's inequality,

$$|\sigma_n f(x)| \le \frac{1}{(2\pi)^d} \sum_{k=-\infty}^{0} \left(\int_{P_k} |K_n(t)|^q\, dt \right)^{1/q} \left(\int_{P_k} |f(x-t)|^p\, dt \right)^{1/p}.$$

It is easy to see that if

$$G(u) := \left(\int_{-u}^{u} \cdots \int_{-u}^{u} |f(x-t)|^p\, dt \right)^{1/p} \qquad (u > 0),$$

then

$$\frac{G^p(u)}{(2u)^d} \le M_p^p f(x) \qquad (u > 0).$$

Therefore

$$\begin{aligned} |\sigma_n f(x)| &\le C \sum_{k=-\infty}^{0} \left(\int_{P_k} |K_n(t)|^q\, dt \right)^{1/q} G(2^k \pi) \\ &\le C \sum_{k=-\infty}^{0} 2^{kd/p} \left(\int_{P_k} |K_n(t)|^q\, dt \right)^{1/q} M_p f(x) \end{aligned}$$

$$= C \, \|K_n\|_{E_q(\mathbb{T}^d)} \, M_p f(x),$$

which shows the theorem. ■

Note that $K_n \in L_\infty(\mathbb{T}^d) \subset E_\infty(\mathbb{T}^d) \subset E_q(\mathbb{T}^d)$ for all $n \in \mathbb{N}^d$, because of (4.1.2). Theorem 4.1.2 implies immediately

Theorem 4.1.11 *If* $1 \le p < \infty$, $1/p + 1/q = 1$ *and*

$$\sup_{n\in\mathbb{N}} \|K_n\|_{E_q(\mathbb{T}^d)} \le C,$$

then

$$\sup_{\rho>0} \rho \lambda(\sigma_* f > \rho)^{1/p} \le C_p \left(\sup_{n\in\mathbb{N}} \|K_n\|_{E_q(\mathbb{T}^d)} \right) \|f\|_p$$

for all $f \in L_p(\mathbb{T}^d)$. *Moreover, for every* $p < r \le \infty$,

$$\|\sigma_* f\|_r \le C \left(\sup_{n\in\mathbb{N}} \|K_n\|_{E_q(\mathbb{T}^d)} \right) \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

Corollary 4.1.12 *Suppose that* $1 \le p < \infty$, $1/p + 1/q = 1$ *and*

$$\sup_{n\in\mathbb{N}} \|K_n\|_{E_q(\mathbb{T}^d)} \le C.$$

If

$$\lim_{n\to\infty} \widehat{K_n}(k) = 1$$

for all $k \in \mathbb{Z}^d$, *then*

$$\lim_{n\to\infty} \sigma_n f = f \quad a.e.$$

for all $f \in L_p(\mathbb{T}^d)$.

Proof For $f(x) = e^{\imath k\cdot x}$, we have

$$\lim_{n\to\infty} \sigma_n f(x) = \lim_{n\to\infty} \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{\imath k\cdot(x-t)} K_n(t)\,dt = \lim_{n\to\infty} e^{\imath k\cdot x} \widehat{K_n}(k) = e^{\imath k\cdot x}.$$

This means that the convergence holds for all trigonometric polynomials. The corollary follows from Theorem 4.1.11 and from the density theorem. ■

We consider the ℓ_2-θ-means given by

$$\sigma_n^{2,\theta} f(x) := \sum_{k\in\mathbb{Z}^d} \theta\left(\frac{\|k\|_2}{n}\right) \widehat{f}(k) e^{\imath k\cdot x},$$

where $\theta : \mathbb{R} \to \mathbb{R}$. As in Sect. 2.6, we suppose that

$$\sum_{k \in \mathbb{Z}^d} \left| \theta\left(\frac{\|k\|_2}{n}\right) \right| < \infty \tag{4.1.4}$$

and we use the notation

$$\theta_0(x) = \theta(\|x\|_2) \qquad (x \in \mathbb{R}^d).$$

Theorem 4.1.13 *Suppose that* $1 \le p < \infty$ *and* $1/p + 1/q = 1$. *If* (4.1.4) *is satisfied,* $\theta_0 \in L_1(\mathbb{R}^d)$ *and* $\widehat{\theta_0} \in E_q(\mathbb{R}^d)$*, then*

$$\sigma_*^{2,\theta} f(x) \le C \left\|\widehat{\theta_0}\right\|_{E_q(\mathbb{R}^d)} M_p f(x)$$

for all $f \in L_p(\mathbb{T}^d)$ *and* $x \in \mathbb{T}^d$.

Proof Similarly to Lemma 2.2.31,

$$\sigma_n^{2,\theta} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n^{2,\theta}(t)\, dt = n^d \int_{\mathbb{R}^d} f(x-t) \widehat{\theta_0}(nt)\, dt$$

and

$$K_n^{2,\theta}(t) = (2\pi)^d n^d \sum_{j \in \mathbb{Z}^d} \widehat{\theta_0}(n(t_1 + 2j_1\pi), \ldots, n(t_d + 2j_d\pi)). \tag{4.1.5}$$

We will prove that $\widehat{\theta_0} \in E_q(\mathbb{R}^d)$ implies

$$\left\| K_n^{2,\theta} \right\|_{E_q(\mathbb{T}^d)} \le C \left\|\widehat{\theta_0}\right\|_{E_q(\mathbb{R}^d)} \qquad (n \in \mathbb{N}). \tag{4.1.6}$$

First, we investigate the term $j = 0$ of the norm:

$$\begin{aligned} &\left\| n^d \widehat{\theta_0}(nt_1, \ldots, nt_d) \right\|_{E_q(\mathbb{T}^d)} \\ &= \sum_{k=-\infty}^{0} 2^{kd(1-1/q)} n^d \left(\int_{P_k} |\widehat{\theta_0}(nt_1, \ldots, nt_d)|^q\, dt \right)^{1/q} \\ &\le C_q \sum_{k=-\infty}^{0} 2^{kd(1-1/q)} n^{d(1-1/q)} \left(\int_{Q_k} |\widehat{\theta_0}(t_1, \ldots, t_d)|^q\, dt \right)^{1/q}, \end{aligned}$$

where

$$Q_k := \prod_{j=1}^{d} \Big(-n2^k\pi, n2^k\pi \Big) \setminus \prod_{j=1}^{d} \Big(-n2^{k-1}\pi, n2^{k-1}\pi \Big).$$

Suppose that $2^{l-1} < n \le 2^l$ for some $l \in \mathbb{N}$. If

$$Q_{k,l} := \prod_{j=1}^{d} \Big(-2^{k+l}\pi, 2^{k+l}\pi \Big) \setminus \prod_{j=1}^{d} \Big(-2^{k+l-2}\pi, 2^{k+l-2}\pi \Big),$$

then

$$\begin{aligned}
&\left\| n^d \widehat{\theta_0}(nt_1, \ldots, nt_d) \right\|_{E_q(\mathbb{T}^d)} \\
&\quad \le C_q \sum_{k=-\infty}^{0} 2^{kd(1-1/q)} 2^{ld(1-1/q)} \left(\int_{Q_{k,l}} |\widehat{\theta_0}(t_1, \ldots, t_d)|^q \, dt \right)^{1/q} \\
&\quad \le C_q \sum_{k=-\infty}^{0} 2^{(k+l)d(1-1/q)} \left(\sum_{i=k+l-1}^{k+l} \int_{P_i} |\widehat{\theta_0}(t_1, \ldots, t_d)|^q \, dt \right)^{1/q} \\
&\quad \le C_q \sum_{k=-\infty}^{0} \sum_{i=k+l-1}^{k+l} 2^{id(1-1/q)} \left(\int_{P_i} |\widehat{\theta_0}(t_1, \ldots, t_d)|^q \, dt \right)^{1/q} \\
&\quad \le C_q \sum_{i=-\infty}^{l} 2^{id(1-1/q)} \left(\int_{P_i} |\widehat{\theta_0}(t_1, \ldots, t_d)|^q \, dt \right)^{1/q} \\
&\quad \le C_q \left\| \widehat{\theta_0} \right\|_{E_q(\mathbb{R}^d)} .
\end{aligned} \tag{4.1.7}$$

Moreover,

$$\begin{aligned}
&\left\| n^d \sum_{j \in \mathbb{Z}^d, j \ne 0} \widehat{\theta_0}(n_1(t_1 + 2j_1\pi), \ldots, n_d(t_d + 2j_d\pi)) \right\|_{E_q(\mathbb{T}^d)} \\
&\quad = \sum_{k=-\infty}^{0} 2^{kd(1-1/q)} n^d \\
&\qquad \left(\int_{P_k} \left| \sum_{j \in \mathbb{Z}^d, j \ne 0} \widehat{\theta_0}(n(t_1 + 2j_1\pi), \ldots, n(t_d + 2j_d\pi)) \right|^q \, dt \right)^{1/q} \\
&\quad = \sum_{k=-\infty}^{0} 2^{kd(1-1/q)} n^d \\
&\qquad \left(\int_{\mathbb{T}^d} \left| \sum_{j \in \mathbb{Z}^d, j \ne 0} \widehat{\theta_0}(n(t_1 + 2j_1\pi), \ldots, n(t_d + 2j_d\pi)) \right|^q \, dt \right)^{1/q}
\end{aligned}$$

$$\le C_q n^d \left(\int_{\mathbb{T}^d} \left| \sum_{j \in \mathbb{Z}^d, j \neq 0} \widehat{\theta}_0(n(t_1 + 2j_1\pi), \ldots, n(t_d + 2j_d\pi)) \right|^q dt \right)^{1/q}.$$

Let

$$R_i := \left\{ j \in \mathbb{Z}^d : j \neq 0, n(\mathbb{T} + 2j_1\pi) \times \ldots \times n(\mathbb{T} + 2j_d\pi) \cap P_i \neq 0 \right\}.$$

Since $|n(t_m + 2j_m\pi)| \ge 2^{l-1}\pi$ if $j_m \neq 0$, we conclude

$$\begin{aligned}
&\left\| n^d \sum_{j \in \mathbb{Z}^d, j \neq 0} \widehat{\theta}_0(n_1(t_1 + 2j_1\pi), \ldots, n_d(t_d + 2j_d\pi)) \right\|_{E_q(\mathbb{T}^d)} \\
&\quad \le C_q n^d \left(\int_{\mathbb{T}^d} \left| \sum_{i=l}^{\infty} \sum_{j \in R_i} \widehat{\theta}_0(n(t_1 + 2j_1\pi), \ldots, n(t_d + 2j_d\pi)) \right|^q dt \right)^{1/q} \\
&\quad \le C_q \sum_{i=l}^{\infty} n^d \left(\int_{\mathbb{T}^d} \left| \sum_{j \in R_i} \widehat{\theta}_0(n(t_1 + 2j_1\pi), \ldots, n(t_d + 2j_d\pi)) \right|^q dt \right)^{1/q}.
\end{aligned}$$

Since R_i has at most $C2^{id}/n$ members, we get that

$$\begin{aligned}
&\left\| n^d \sum_{j \in \mathbb{Z}^d, j \neq 0} \widehat{\theta}_0(n_1(t_1 + 2j_1\pi), \ldots, n_d(t_d + 2j_d\pi)) \right\|_{E_q(\mathbb{T}^d)} \\
&\le C_q \sum_{i=l}^{\infty} n^d \left(\sum_{j \in R_i} \left(\frac{2^{id}}{n^d} \right)^{q-1} \int_{\mathbb{T}^d} \left| \widehat{\theta}_0(n(t_1 + 2j_1\pi), \ldots, n(t_d + 2j_d\pi)) \right|^q dt \right)^{1/q} \\
&\le C_q \sum_{i=l}^{\infty} 2^{id(1-1/q)} \left(\sum_{j \in R_i} \int_{n_1(\mathbb{T}+2j_1\pi) \times \ldots \times n_d(\mathbb{T}+2j_d\pi)} \left| \widehat{\theta}_0(t_1, \ldots, t_d) \right|^q dt \right)^{1/q} \\
&\le C_q \sum_{i=l}^{\infty} 2^{id(1-1/q)} \left(\int_{P_i} \left| \widehat{\theta}_0(t_1, \ldots, t_d) \right|^q dt \right)^{1/q} \\
&\le C_q \left\| \widehat{\theta}_0 \right\|_{E_q(\mathbb{R}^d)},
\end{aligned} \tag{4.1.8}$$

which proves (4.1.6). The theorem follows from Theorem 4.1.10. ∎

Note that

$$\widehat{\theta}_0 \in E_q(\mathbb{R}^d) \subset E_1(\mathbb{R}^d) \subset L_1(\mathbb{R}^d),$$

thus (2.6.5) is satisfied and θ_0 is continuous.

Theorem 4.1.14 *Suppose that $1 \le p < \infty$ and $1/p + 1/q = 1$. If (4.1.4) is satisfied, $\theta_0 \in L_1(\mathbb{R}^d)$ and $\widehat{\theta_0} \in E_q(\mathbb{R}^d)$, then*

$$\sup_{\rho>0} \rho\lambda(\sigma_*^{2,\theta} f > \rho)^{1/p} \le C_p \left\|\widehat{\theta_0}\right\|_{E_q(\mathbb{R}^d)} \|f\|_p$$

for all $f \in L_p(\mathbb{T}^d)$. Moreover, for every $p < r \le \infty$,

$$\left\|\sigma_*^{2,\theta} f\right\|_r \le C \left\|\widehat{\theta_0}\right\|_{E_q(\mathbb{R}^d)} \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

Corollary 4.1.15 *Suppose that $1 \le p < \infty$ and $1/p + 1/q = 1$. If $\theta(0) = 1$, (4.1.4) is satisfied, $\theta_0 \in L_1(\mathbb{R}^d)$ and $\widehat{\theta_0} \in E_q(\mathbb{R}^d)$, then*

$$\lim_{n\to\infty} \sigma_n^{2,\theta} f = f \quad a.e.$$

for all $f \in L_p(\mathbb{T}^d)$.

Now we prove some converse type results. We know that the weak type inequality of Theorem 4.1.11 implies the almost everywhere convergence

$$\lim_{n\to\infty} \sigma_n f = f \quad \text{a.e.}$$

for all $f \in L_p(\mathbb{T}^d)$ (see Corollary 4.1.12). Conversely, if $1 \le p \le 2$ and the almost everywhere convergence holds for all $f \in L_p(\mathbb{T}^d)$, then σ_* is bounded from $L_p(\mathbb{T}^d)$ to $L_{p,\infty}(\mathbb{T}^d)$, as in Theorem 4.1.11 (see Stein [288]). The converse of Theorem 4.1.10 is given in the next result. More exactly, if $\sigma_* f$ can be estimated pointwise by $M_p f$, then (4.1.3) holds. Before proving this theorem, we need the following definition.

Definition 4.1.16 For $1 \le p < \infty$, we define the space $D_p(\mathbb{T}^d)$ with the norm

$$\|f\|_{D_p(\mathbb{T}^d)} := \sup_{0<r\le\pi} \left(\frac{1}{r^d}\int_{-r}^{r} \cdots \int_{-r}^{r} |f(t)|^p \, dt\right)^{1/p}.$$

Taking the supremum for all $0 < r < \infty$, we obtain the space $D_p(\mathbb{R}^d)$.

Lemma 4.1.17 *For $1 \le p < \infty$, the norm*

$$\|f\|_* = \sup_{k\le 0} 2^{-kd/p} \left\|f 1_{P_k}\right\|_p$$

is an equivalent norm on $D_p(\mathbb{T}^d)$.

Proof Choosing $r = 2^k\pi$ $(k \le 0)$, we conclude

$$2^{-kd/p} \left\|f 1_{P_k}\right\|_p \le C \left(\frac{1}{(2^k\pi)^d}\int_{-2^k\pi}^{2^k\pi} \cdots \int_{-2^k\pi}^{2^k\pi} |f(t)|^p \, dt\right)^{1/p} \le \|f\|_{D_p}.$$

On the other hand, suppose that $2^{N-1}\pi \le r < 2^N\pi$ for some $N \in \mathbb{N}$. Then

$$\begin{aligned}\frac{1}{r^d}\int_{-r}^{r}\cdots\int_{-r}^{r}|f(t)|^p\,dt &\le C2^{-Nd}\int_{-2^N\pi}^{2^N\pi}\cdots\int_{-2^N\pi}^{2^N\pi}|f(t)|^p\,dt\\ &= C2^{-Nd}\sum_{k=-\infty}^{N}\int_{P_k}|f(t)|^p\,dt\\ &\le C2^{-Nd}\sum_{k=-\infty}^{N}2^{kd}\,\|f\|_*^p \le C\,\|f\|_*^p\,,\end{aligned}$$

which shows the lemma. ∎

We can see that $D_p(\mathbb{T}^d) \subset L_p(\mathbb{T}^d)$ and

$$\|f\|_p \le C\|f\|_{D_p(\mathbb{T}^d)} \qquad (f \in D_p(\mathbb{T}^d)).$$

Theorem 4.1.18 *If* $1 \le p < \infty$, $1/p + 1/q = 1$ *and*

$$\sigma_* f(0) \le CM_p f(0) \tag{4.1.9}$$

for all $f \in L_p(\mathbb{T}^d)$, *then*

$$\sup_{n\in\mathbb{N}}\|K_n\|_{E_q(\mathbb{T}^d)} \le C.$$

Proof It is easy to see by Lemma 4.1.17 that

$$\sup_{\|f\|_{D_p(\mathbb{T}^d)}\le 1}\left|\int_{\mathbb{T}^d} f(-t)K_n(t)\,dt\right| = \|K_n\|_{E_q(\mathbb{T}^d)}\,. \tag{4.1.10}$$

There exists a function $f \in D_p(\mathbb{T}^d)$ with $\|f\|_{D_p} \le 1$ such that

$$\frac{\|K_n\|_{E_q(\mathbb{T}^d)}}{2} \le \left|\int_{\mathbb{T}^d} f(-t)K_n(t)\,dt\right|.$$

Since $f \in L_p(\mathbb{R}^d)$, by (4.1.9),

$$\left|\int_{\mathbb{T}^d} f(-t)K_n(t)\,dt\right| = |\sigma_n f(0)| \le CM_p f(0) \qquad (n \in N),$$

which implies

$$\|K_n\|_{E_q(\mathbb{T}^d)} \le CM_p f(0) \le C\|f\|_{D_p} \le C \qquad (n \in N).$$

This proves the result. ■

Note that the norm of $D_p(\mathbb{T}^d)$ is equivalent to

$$\|f\| = \sup_{r\in[0,\pi]^d\cap\mathbb{R}^d_\tau} \left(\frac{1}{\prod_{j=1}^d r_j} \int_{-r_1}^{r_1} \cdots \int_{-r_d}^{r_d} |f(t)|^p \, dt \right)^{1/p}.$$

Now, we introduce the first generalization of Lebesgue points for higher dimensions. Corollary 4.1.3 says that

$$\lim_{h\to 0} \frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} f(x-t)\, dt = f(x)$$

for almost every $x \in \mathbb{T}^d$, where $f \in L_1(\mathbb{T}^d)$. In other words,

$$\lim_{h\to 0} \frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} (f(x-t) - f(x))\, dt = 0,$$

which is equivalent to

$$\lim_{h\to 0} \frac{1}{(2h)^d} \left| \int_{-h}^{h} \cdots \int_{-h}^{h} (f(x-t) - f(x))\, dt \right| = 0.$$

In the next definition, we describe a stronger condition.

Definition 4.1.19 For $1 \le p < \infty$, a point $x \in \mathbb{T}^d$ is called a p-Lebesgue point of $f \in L_p(\mathbb{T}^d)$ if

$$\lim_{h\to 0} \left(\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0.$$

For $p = 1$, the points are said to be Lebesgue points. One can see that using the restricted maximal operator and Corollary 4.1.6, we get an equivalent definition:

$$\lim_{h\to 0, h\in\mathbb{R}^d_\tau} \left(\frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0.$$

If $p < r$ and x is an r-Lebesgue point of f, then it is also a p-Lebesgue point. Indeed, by Hölder's inequality,

$$\left(\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x-t) - f(x)|^p \, dt \right)^{1/p} \le \left(\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x-t) - f(x)|^r \, dt \right)^{1/r}.$$

The following two results can be proved as in the one-dimensional case, see Theorem 1.3.11 and Lemma 1.3.12.

Theorem 4.1.20 *Almost every point $x \in \mathbb{T}^d$ is a p-Lebesgue point of $f \in L_p(\mathbb{T}^d)$* $(1 \le p < \infty)$.

Lemma 4.1.21 *If x is a p-Lebesgue point of $f \in L_p(\mathbb{T}^d)$, then $f(x)$ and $M_p f(x)$ are finite* $(1 \le p < \infty)$.

The next theorem generalizes Theorem 1.5.4.

Theorem 4.1.22 *Suppose that $1 \le p < \infty$, $1/p + 1/q = 1$ and*

$$\sup_{n\in\mathbb{N}} \|K_n\|_{E_q(\mathbb{T}^d)} \le C.$$

If for all $\delta > 0$

$$\lim_{n\to\infty} \|K_n\|_{L_q(\mathbb{T}^d\setminus(-\delta,\delta)^d)} = 0 \tag{4.1.11}$$

and

$$\lim_{n\to\infty} \widehat{K_n}(0) = 1, \tag{4.1.12}$$

then

$$\lim_{n\to\infty} \sigma_n f(x) = f(x)$$

for all p-Lebesgue points of $f \in L_p(\mathbb{T}^d)$.

Proof Now, set

$$G(u) := \left(\int_{-u}^{u} \dots \int_{-u}^{u} |f(x-t) - f(x)|^p \, dt\right)^{1/p} \qquad (u > 0),$$

Since x is a p-Lebesgue point of f, for all $\epsilon > 0$, there exists $m \in \mathbb{Z}$, $m \le 0$ such that

$$\frac{G^p(u)}{(2u)^d} \le \epsilon \qquad \text{if} \qquad 0 < u \le 2^m \pi. \tag{4.1.13}$$

Observe that

$$\sigma_n f(x) - f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} (f(x-t) - f(x)) K_n(t) \, dt + f(x) \left(\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K_n(t) \, dt - 1\right).$$

Thus

$$
\begin{aligned}
|\sigma_n f(x) - f(x)| &\le C \int_{\mathbb{T}^d} |f(x-t) - f(x)||K_n(t)|\,dt + \left| f(x) \left(\widehat{K_n}(0) - 1\right)\right| \\
&= C \int_{-2^m\pi}^{2^m\pi} \dots \int_{-2^m\pi}^{2^m\pi} |f(x-t) - f(x)||K_n(t)|\,dt \\
&\quad + C \int_{\mathbb{T}^d \setminus (-2^m\pi, 2^m\pi)^d} |f(x-t) - f(x)||K_n(t)|\,dt \\
&\quad + \left| f(x) \left(\widehat{K_n}(0) - 1\right)\right| \\
&=: A_1(x) + A_2(x) + A_3(x).
\end{aligned}
$$

We estimate $A_1(x)$ by

$$
\begin{aligned}
A_1(x) &= C \sum_{k=-\infty}^{m} \int_{P_k} |f(x-t) - f(x)||K_n(t)|\,dt \\
&\le C \sum_{k=-\infty}^{m} \left(\int_{P_k} |K_n(t)|^q\,dt \right)^{1/q} \left(\int_{P_k} |f(x-t) - f(x)|^p\,dt \right)^{1/p} \\
&\le C \sum_{k=-\infty}^{m} \left(\int_{P_k} |K_n(t)|^q\,dt \right)^{1/q} G(2^k\pi).
\end{aligned}
$$

Then, by (4.1.13),

$$
A_1(x) \le C_p \epsilon \sum_{k=-\infty}^{m} 2^{kd/p} \left(\int_{P_k} |K_n(t)|^q\,dt \right)^{1/q} \le C_p \epsilon \|K_n\|_{E_q(\mathbb{T}^d)}.
$$

For $0 < \delta < 2^m\pi$, we have

$$
\begin{aligned}
A_2(x) &\le C \int_{\mathbb{T}^d \setminus (-\delta,\delta)^d} |f(x-t) - f(x)||K_n(t)|\,dt \\
&\le C \left(\int_{\mathbb{T}^d \setminus (-\delta,\delta)^d} |K_n(t)|^q\,dt \right)^{1/q} \left(\|f\|_p + |f(x)| \right),
\end{aligned}
$$

which tends to 0 as $n \to \infty$. Moreover, $A_3(x) \to 0$ as $n \to \infty$, too. This completes the proof of the theorem. ■

Observe that (4.1.2) and $\delta' < 2^k\pi < \delta$ imply

$$
\begin{aligned}
\|K_n\|_{E_q(\mathbb{T}^d \setminus (-\delta,\delta)^d)} &\le \|K_n\|_{L_q(\mathbb{T}^d \setminus (-\delta,\delta)^d)} \\
&\le \|K_n\|_{L_q(\mathbb{T}^d \setminus (-2^k\pi, 2^k\pi)^d)} \\
&\le \left(\sum_{l=k+1}^{0} \int_{P_l} |K_n(t)|^q\,dt \right)^{1/q}
\end{aligned}
$$

$$\leq C_\delta \sum_{l=k+1}^{0} 2^{kd(1-1/q)} \left(\int_{P_l} |K_n(t)|^q \, dt \right)^{1/q}$$
$$\leq C_\delta \|K_n\|_{E_q(\mathbb{T}^d \setminus (-2^k\pi, 2^k\pi)^d)}$$
$$\leq C_\delta \|K_n\|_{E_q(\mathbb{T}^d \setminus (-\delta', \delta')^d)}. \tag{4.1.14}$$

Then condition (4.1.11) is equivalent to

$$\lim_{n\to\infty} \|K_n\|_{E_q(\mathbb{T}^d \setminus (-\delta,\delta)^d)} = 0.$$

In the case $\widehat{\theta_0} \in E_q(\mathbb{R}^d)$, we can formulate a somewhat simpler version of the preceding theorem.

Theorem 4.1.23 *Suppose that* $1 \leq p < \infty$ *and* $1/p + 1/q = 1$. *If* $\theta(0) = 1$, (4.1.4) *is satisfied,* $\theta_0 \in L_1(\mathbb{R}^d)$ *and* $\widehat{\theta_0} \in E_q(\mathbb{R}^d)$, *then*

$$\lim_{n\to\infty} \sigma_n^{2,\theta} f(x) = f(x)$$

for all p-Lebesgue points of $f \in L_p(\mathbb{T}^d)$.

Proof We have seen in Theorem 4.1.13 that $\widehat{\theta_0} \in E_q(\mathbb{R}^d)$ implies

$$\left\| K_n^{2,\theta} \right\|_{E_q(\mathbb{T}^d)} \leq C \left\| \widehat{\theta_0} \right\|_{E_q(\mathbb{R}^d)} \qquad (n \in \mathbb{N}),$$

so the first condition of Theorem 4.1.22 is satisfied.

On the other hand, let $2^{k_0}\pi < \delta$ and $2^{l-1} \leq n < 2^l$ as in the proof of Theorem 4.1.13. We get similarly to (4.1.7) and (4.1.8) that

$$\left\| K_n^{2,\theta} \right\|_{E_q(\mathbb{T}^d \setminus (-\delta,\delta)^d)} \leq C_q \sum_{i=k_0+l-1}^{\infty} 2^{id(1-1/q)} \left(\int_{P_i} |\widehat{\theta_0}(t_1, \ldots, t_d)|^q \, dt \right)^{1/q}$$
$$+ C_q \sum_{i=l}^{\infty} 2^{id(1-1/q)} \left(\int_{P_i} |\widehat{\theta_0}(t_1, \ldots, t_d)|^q \, dt \right)^{1/q},$$

which tends to 0 as $n \to \infty$, since $\widehat{\theta_0} \in E_q(\mathbb{R}^d)$. Then (4.1.11) follows from (4.1.14). Finally, by (4.1.5),

$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K_n(t) \, dt = n^d \sum_{j \in \mathbb{Z}^d} \int_{\mathbb{T}^d} \widehat{\theta_0}(n(t_1 + 2j_1\pi), \ldots, n(t_d + 2j_d\pi)) \, dt$$
$$= n^d \int_{\mathbb{R}^d} \widehat{\theta_0}(nt) \, dt = \theta_0(1) = 1,$$

which finishes the proof of our theorem. ∎

Since each point of continuity is a Lebesgue point, we have

Corollary 4.1.24 *If the conditions of Theorem 4.1.22 or Theorem 4.1.23 are satisfied and if $f \in L_p(\mathbb{T}^d)$ is continuous at a point x, then*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x).$$

The converse of Theorem 4.1.22 holds also.

Theorem 4.1.25 *Suppose that $1 \le p < \infty$ and $1/p + 1/q = 1$. If*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x)$$

for all p-Lebesgue points of $f \in L_p(\mathbb{T}^d)$ then

$$\sup_{n\in\mathbb{N}} \|K_n\|_{E_q(\mathbb{T}^d)} \le C.$$

Proof The space $D_p^0(\mathbb{T}^d)$ consists of all functions $f \in D_p(\mathbb{T}^d)$ for which $f(0) = 0$ and 0 is a p-Lebesgue point of f, in other words

$$\lim_{h\to 0} \left(\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(t)|^p \, dt \right)^{1/p} = 0.$$

We will show that $D_p^0(\mathbb{T}^d)$ is a Banach space. Let (f_n) be a Cauchy sequence in $D_p^0(\mathbb{T}^d)$, i.e.,

$$\|f_n - f_m\|_{D_p^0(\mathbb{T}^d)} \to 0 \qquad \text{as} \qquad n, m \to \infty.$$

Then there exists a subsequence (f_{ν_n}) such that

$$\left\| f_{\nu_{n+1}} - f_{\nu_n} \right\|_{D_p^0(\mathbb{T}^d)} \le 2^{-n}.$$

Then

$$\left\| \sum_{n=0}^{\infty} \left| f_{\nu_{n+1}} - f_{\nu_n} \right| \right\|_{L_p(\mathbb{T}^d)} \le \left\| \sum_{n=0}^{\infty} \left| f_{\nu_{n+1}} - f_{\nu_n} \right| \right\|_{\mathbb{D}_p^0(\mathbb{T}^d)} \le 2,$$

thus the series

$$\sum_{n=0}^{\infty} \left| f_{\nu_{n+1}} - f_{\nu_n} \right|$$

is almost everywhere finite. That is to say the sequence (f_{ν_n}) is almost everywhere convergent. Let

$$f := \lim_{n\to\infty} f_{\nu_n} \qquad \text{and} \qquad f(0) = 0.$$

For all $\epsilon > 0$, there exists N such that

$$\| f - f_{\nu_N} \|_{D_p^0(\mathbb{T}^d)} \le \sum_{n=N}^{\infty} \| f_{\nu_{n+1}} - f_{\nu_n} \|_{D_p^0(\mathbb{T}^d)} \le \sum_{n=N}^{\infty} 2^{-n} < \epsilon.$$

If $h > 0$ is small enough, then

$$\left(\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f_{\nu_N}(t)|^p \, dt \right)^{1/p} < \epsilon.$$

Hence

$$\left(\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(t)|^p \, dt \right)^{1/p} \le C \| f - f_{\nu_N} \|_{D_p^0(\mathbb{T}^d)} + \left(\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f_{\nu_N}(t)|^p \, dt \right)^{1/p} < 2\epsilon,$$

whenever h is small enough. From this it follows that $f \in D_p^0(\mathbb{T}^d)$ and 0 is a Lebesgue point of f. Thus $D_p^0(\mathbb{T}^d)$ is a Banach space, indeed.

We get from the conditions of the theorem that

$$\lim_{n\to\infty} \sigma_n f(0) = 0 \qquad \text{for all} \qquad f \in D_p^0(\mathbb{T}^d).$$

Thus the operators

$$U_n : D_p^0(\mathbb{T}^d) \to \mathbb{R}, \qquad U_n f := \sigma_n f(0) \qquad (n \in \mathbb{N})$$

are uniformly bounded by the Banach-Steinhaus theorem. Observe that in (4.1.10), we may suppose that f is 0 in a neighborhood of 0. Then

$$\begin{aligned} C &\ge \|U_n\| \\ &= \sup_{\|f\|_{D_p^0(\mathbb{T}^d)} \le 1} \left| \int_{\mathbb{T}^d} f(-t) K_n(t) \, dt \right| \\ &= \sup_{\|f\|_{D_p(\mathbb{T}^d)} \le 1} \left| \int_{\mathbb{T}^d} f(-t) K_n(t) \, dt \right| \\ &= \|K_n\|_{E_q(\mathbb{T}^d)} \end{aligned}$$

for all $n \in \mathbb{N}$. ■

Corollary 4.1.26 *Suppose that* $1 \le p < \infty$, $1/p + 1/q = 1$, (4.1.11) *and* (4.1.12) *hold. Then*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x)$$

for all p-Lebesgue points of $f \in L_p(\mathbb{T}^d)$ *if and only if*

$$\sup_{n\in\mathbb{N}} \|K_n\|_{E_q(\mathbb{T}^d)} \le C.$$

Note that these results can be found in Feichtinger and Weisz [104]. We know that our results can be applied to the one-dimensional Cesàro summability (see Sect. 1.5). Moreover, the Riesz, Weierstrass, Picard and Bessel summations given in Sect. 3.7.4 (Examples 3.7.28, 3.7.29, 3.7.30) satisfy all conditions of this section, too.

Corollary 4.1.27 *Suppose that* θ *is one of the Examples 3.7.28, 3.7.29 or 3.7.30. Then*

$$\lim_{n\to\infty} \sigma_n^{2,\theta} f(x) = f(x)$$

for all Lebesgue points of $f \in L_1(\mathbb{T}^d)$*. Moreover,*

$$\sup_{\rho>0} \rho\lambda(\sigma_*^{2,\theta} > \rho) \le C \left\|\widehat{\theta}_0\right\|_{E_\infty(\mathbb{R}^d)} \|f\|_1 \qquad (f \in L_1(\mathbb{T}^d))$$

and, for every $1 < p \le \infty$*,*

$$\left\|\sigma_*^{2,\theta} f\right\|_p \le C_p \left\|\widehat{\theta}_0\right\|_{E_\infty(\mathbb{R}^d)} \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

4.2 Unrestricted Rectangular Summability

Here we study the operators

$$\sigma_n f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n(t)\, dt \qquad (n \in \mathbb{N}^d),$$

where $f \in L_1(\mathbb{T}^d)$ and $K_n \in L_1(\mathbb{T}^d) \cap L_\infty(\mathbb{T}^d)$ for all $n \in \mathbb{N}^d$. The higher dimensional rectangular Cesàro and Riesz kernels, K_n^α and $K_n^{\alpha,\gamma}$ satisfy the conditions of this section. The kernel K_n^θ is also investigated.

4.2.1 Strong Hardy-Littlewood Maximal Functions

A second generalization of the one-dimensional maximal function is the so-called strong Hardy-Littlewood maximal function given by

$$M'_{s,p} f(x) := \sup_{x\in I} \left(\frac{1}{|I|} \int_I |f|^p \, d\lambda \right)^{1/p} \qquad (x \in \mathbb{T}^d),$$

where $f \in L_p(\mathbb{T}^d)$ and the supremum is taken over all rectangles

$$I = I_1 \times \cdots \times I_d \subset \mathbb{T}^d$$

with sides parallel to the axes and containing x. This maximal function is different from M_p and from $M_{\Box,p}$ defined in Sect. 4.1.1, it remains bounded on $L_p(\mathbb{T}^d)$ with $1 < p \le \infty$, but it is not of weak type $(1, 1)$. The reason for this is that in the definition the ratio of the sides of the rectangles can be large. We will use again the next centered version of the strong maximal function.

Definition 4.2.1 For $1 \le p < \infty$ and $f \in L_p(\mathbb{T}^d)$ the strong Hardy-Littlewood maximal function is defined by

$$M_{s,p} f(x) := \sup_{h \in \mathbb{R}_+^d} \left(\frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t)|^p \, dt \right)^{1/p}.$$

Taking the supremum over all $h \in (0, \pi)^d$, we get an equivalent definition. It is easy to see that

$$C_1 M_{s,p} f \le M'_{s,p} f \le C_2 M_{s,p} f$$

for all $1 \le p < \infty$. If $p = 1$, then we omit the notation p and write simply $M_s f$. In the one-dimensional case M_s is the usual Hardy-Littlewood maximal function and so, it is of weak type $(1, 1)$. For higher dimensions it is known that there is a function $f \in L_1(\mathbb{T}^d)$ such that $M_s f = \infty$ almost everywhere (see Jessen, Marcinkiewicz and Zygmund [177] and Saks [268]). Thus M_s cannot be of weak type $(1, 1)$, however, with the help of the $L_p(\log L)^k(\mathbb{T}^d)$ spaces, we can show a weak type inequality. Set $\log^+ u := \max(0, \log u)$.

Definition 4.2.2 For $k \in \mathbb{N}$ and $1 \le p < \infty$, a measurable function f is in the set $L_p(\log L)^k(\mathbb{T}^d)$ if

$$\|f\|_{L_p(\log L)^k} := \left(\int_{\mathbb{T}^d} |f|^p (\log^+ |f|)^k \, d\lambda \right)^{1/p} < \infty.$$

If $p = \infty$, then set $L_\infty(\log L)^k(\mathbb{T}^d) = L_\infty(\mathbb{T}^d)$.

For $k = 0$, we get back the $L_p(\mathbb{T}^d)$ spaces. We have for all $k \in \mathbb{P}$ and $1 \le p < r \le \infty$ that

$$L_p(\mathbb{T}^d) \supset L_p(\log L)^{k-1}(\mathbb{T}^d) \supset L_p(\log L)^k(\mathbb{T}^d) \supset L_r(\mathbb{T}^d).$$

Theorem 4.2.3 *If $f \in L(\log L)^{d-1}(\mathbb{T}^d)$, then*

$$\sup_{\rho>0} \rho \lambda(M_s f > \rho) \le C + C \left\| |f| \left(\log^+ |f|\right)^{d-1} \right\|_1.$$

Moreover, for $1 < p \leq \infty$*, we have*

$$\|M_s f\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Proof Let us denote the one-dimensional Hardy-Littlewood maximal function in the ith dimension by $M^{(i)}$. Then

$$M_s f \leq M^{(1)} \circ M^{(2)} \circ \cdots \circ M^{(d)} f.$$

By Theorems 1.3.3 and 1.3.5,

$$\begin{aligned}
\sup_{\rho>0} \rho\lambda(M_s f > \rho) &= \sup_{\rho>0} \rho\lambda(M^{(1)} \circ M^{(2)} \circ \cdots \circ M^{(d)} f > \rho) \\
&\leq \left\| M^{(2)} \circ \cdots \circ M^{(d)} f \right\|_1 \\
&\leq C + C \left\| M^{(3)} \circ \cdots \circ M^{(d)} f \right\|_{L_1(\log L)(\mathbb{T}^d)} \\
&\leq \ldots \leq C + C \|f\|_{L_1(\log L)^{d-1}(\mathbb{T}^d)} .
\end{aligned}$$

The second inequality of Theorem 4.2.3 follows similarly. ■

Similarly to Corollary 4.1.3, we obtain

Corollary 4.2.4 *If* $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$*, then*

$$\lim_{h\to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} f(x-t)\, dt = f(x)$$

for almost every $x \in \mathbb{T}^d$*.*

Note that this convergence result does not hold for al $f \in L_1(\mathbb{T}^d)$ (see Jessen, Marcinkiewicz and Zygmund [177] and Saks [268]). Since $M_{s,p}^p f = M_s(|f|^p)$ for $1 \leq p < \infty$, we have

Corollary 4.2.5 *If* $1 \leq p < \infty$ *and* $f \in L_p(\log L)^{d-1}(\mathbb{T}^d)$*, then*

$$\sup_{\rho>0} \rho\lambda(M_{s,p} f > \rho)^{1/p} \leq C_p + C_p \|f\|_{L_p(\log L)^{d-1}} .$$

For $p < r \leq \infty$*,*

$$\left\| M_{s,p} f \right\|_r \leq C_r \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

4.2.2 Lebesgue Points for the Unrestricted Rectangular Summability

To formulate the generalization of Lebesgue's theorem for the unrestricted rectangular summability, we have to modify slightly the definition of the space $E_q(\mathbb{R}^d)$.

Definition 4.2.6 For $1 \le q \le \infty$, the Herz space $E_q'(\mathbb{R}^d)$ resp. $E_q'(\mathbb{T}^d)$ contains all functions f for which

$$\|f\|_{E_q'(\mathbb{R}^d)} := \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \left(\prod_{j=1}^{d} 2^{k_j(1-1/q)} \right) \left\| f 1_{P_k} \right\|_q < \infty$$

resp.

$$\|f\|_{E_q'(\mathbb{T}^d)} := \sum_{k_1=-\infty}^{0} \cdots \sum_{k_d=-\infty}^{0} \left(\prod_{j=1}^{d} 2^{k_j(1-1/q)} \right) \left\| f 1_{P_k} \right\|_q < \infty,$$

where

$$P_k := P_{k_1} \times \cdots \times P_{k_d} \qquad (k \in \mathbb{Z}^d)$$

and

$$P_i = \{x \in \mathbb{R} : 2^{i-1}\pi \le |x| < 2^i \pi\} \qquad (i \in \mathbb{Z}).$$

Again,

$$L_1(\mathbb{X}^d) = E_1'(\mathbb{X}^d) \hookleftarrow E_q'(\mathbb{X}^d) \hookleftarrow E_{q'}'(\mathbb{X}^d) \hookleftarrow E_\infty'(\mathbb{X}^d), \qquad 1 < q < q' < \infty,$$

where $\mathbb{X} = \mathbb{R}$ or $\mathbb{T}$ and

$$E_q'(\mathbb{T}^d) \hookleftarrow L_q(\mathbb{T}^d) \qquad (1 \le q \le \infty).$$

It is easy to see that $E_q'(\mathbb{X}^d) \supset E_q(\mathbb{X}^d)$ and

$$\|f\|_{E_q'} \le C \|f\|_{E_q} \qquad (1 \le q \le \infty).$$

Here we will estimate pointwise the maximal operator

$$\sigma_* f := \sup_{n \in \mathbb{N}^d} |\sigma_n f|$$

by the strong Hardy-Littlewood maximal function. Since the condition (4.1.11) is not true for rectangular summability kernels (e.g., for the Cesàro or Riesz kernels, K_n^α, $K_n^{\alpha,\gamma}$), we use here other conditions and other ideas. We introduce the functions

$$\widetilde{K}_n(t) := \left(\prod_{j=1}^d n_j\right)^{-1} \left(1_{(-\pi,\pi)^d} K_n\right)\left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right)$$
$$= \begin{cases} \left(\prod_{j=1}^d n_j\right)^{-1} K_n\left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right), & \text{if } |t_1| \le \pi n_1, \ldots, |t_d| \le \pi n_d; \\ 0, & \text{else.} \end{cases}$$

Theorem 4.2.7 *For all $n \in \mathbb{N}^d$,*

$$c \left\| K_n \right\|_{E_q'(\mathbb{T}^d)} \le \left\| \widetilde{K}_n \right\|_{E_q'(\mathbb{R}^d)} \le C \left\| K_n \right\|_{E_q'(\mathbb{T}^d)}.$$

Proof We have

$$\left\| \widetilde{K}_n \right\|_{E_q'(\mathbb{R}^d)} = \left(\prod_{j=1}^d n_j\right)^{-1} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \left(\prod_{j=1}^d 2^{k_j(1-1/q)}\right)$$
$$\left(\int_{P_{k_1}} \cdots \int_{P_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n\right)\left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right)\right|^q \, dt\right)^{1/q}$$
$$= \left(\prod_{j=1}^d n_j\right)^{-1+1/q} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \left(\prod_{j=1}^d 2^{k_j(1-1/q)}\right)$$
$$\left(\int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} \left| \left(1_{(-\pi,\pi)^d} K_n\right)(t)\right|^q \, dt\right)^{1/q},$$

where

$$P_{k_j}(n_j) := \{x \in \mathbb{R} : 2^{k_j-1}\pi/n_j \le |x| < 2^{k_j}\pi/n_j\} \qquad (j = 1, \ldots, d).$$

Choosing $l_j \in \mathbb{N}$ such that $2^{l_j-1} < n_j \le 2^{l_j}$, we conclude that

$$P_{k_j}(n_j) \subset \{x \in \mathbb{R} : 2^{k_j-l_j-1}\pi \le |x| < 2^{k_j-l_j+1}\pi\} =: R_{k_j,l_j} \qquad (j = 1, \ldots, d)$$

and

$$\left\| \widetilde{K}_n \right\|_{E_q'(\mathbb{R}^d)} \le C \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \left(\prod_{j=1}^d 2^{(k_j-l_j)(1-1/q)}\right)$$
$$\left(\int_{R_{k_1,l_1}} \cdots \int_{R_{k_d,l_d}} \left| \left(1_{(-\pi,\pi)^d} K_n\right)(t)\right|^q \, dt\right)^{1/q}$$

$$\leq C \sum_{i_1=-\infty}^{\infty} \cdots \sum_{i_d=-\infty}^{\infty} \left(\prod_{j=1}^{d} 2^{i_j(1-1/q)} \right)$$
$$\left(\int_{P_{i_1}} \cdots \int_{P_{i_d}} \left| \left(1_{(-\pi,\pi)^d} K_n \right)(t) \right|^q \, dt \right)^{1/q}$$
$$\leq C \left\| 1_{(-\pi,\pi)^d} K_n \right\|_{E'_q(\mathbb{R}^d)}$$
$$= C \, \|K_n\|_{E'_q(\mathbb{T}^d)} .$$

The other inequality can be shown in the same way. ■

Now we formulate the analogue of Theorem 4.1.10.

Theorem 4.2.8 *If* $1 \leq p < \infty$, $1/p + 1/q = 1$ *and*

$$\sup_{n \in \mathbb{N}^d} \|K_n\|_{E'_q(\mathbb{T}^d)} \leq C, \tag{4.2.1}$$

then

$$\sigma_* f(x) \leq C \left(\sup_{n \in \mathbb{N}^d} \|K_n\|_{E'_q(\mathbb{T}^d)} \right) M_{s,p} f(x)$$

for all $f \in L_p(\mathbb{T}^d)$ *and* $x \in \mathbb{T}^d$.

Proof Observe that

$$|\sigma_n f(x)|$$
$$= \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} f(x-t) \left(1_{(-\pi,\pi)^d} K_n \right)(t) \, dt \right|$$
$$= \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t)| \left| \left(1_{(-\pi,\pi)^d} K_n \right)(t) \right| \, dt.$$

By Hölder's inequality,

$$|\sigma_n f(x)|$$
$$\leq \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \left(\int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t)|^p \, dt \right)^{1/p}$$
$$\left(\int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} \left| \left(1_{(-\pi,\pi)^d} K_n \right)(t) \right|^q \, dt \right)^{1/q}$$
$$= \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \left(\int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t)|^p \, dt \right)^{1/p}$$

$$\left(\prod_{j=1}^{d} n_j\right)^{-1/q} \left(\int_{P_{k_1}} \cdots \int_{P_{k_d}} \left|\left(1_{(-\pi,\pi)^d} K_n\right)\left(\frac{t_1}{n_1},\ldots,\frac{t_d}{n_d}\right)\right|^q dt\right)^{1/q}.$$

If we define

$$G(u) := \left(\int_{-u_1}^{u_1} \cdots \int_{-u_d}^{u_d} |f(x-t)|^p \, dt\right)^{1/p} \qquad (u \in \mathbb{R}_+^d),$$

then

$$\frac{G^p(u)}{\prod_{j=1}^d (2u_j)} \le M_{s,p}^p f(x) \qquad (u \in \mathbb{R}_+^d).$$

Thus

$$\begin{aligned} |\sigma_n f(x)| \le {} & \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} G\left(\frac{2^{k_1}\pi}{n_1},\ldots,\frac{2^{k_d}\pi}{n_d}\right)\left(\prod_{j=1}^{d} n_j\right)^{-1/q} \\ & \left(\int_{P_{k_1}} \cdots \int_{P_{k_d}} \left|\left(1_{(-\pi,\pi)^d} K_n\right)\left(\frac{t_1}{n_1},\ldots,\frac{t_d}{n_d}\right)\right|^q dt\right)^{1/q} \qquad (4.2.2) \\ \le {} & \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \left(\prod_{j=1}^{d} 2^{k_j/p}\right)\left(\prod_{j=1}^{d} n_j\right)^{-1} M_{s,p} f(x) \\ & \left(\int_{P_{k_1}} \cdots \int_{P_{k_d}} \left|\left(1_{(-\pi,\pi)^d} K_n\right)\left(\frac{t_1}{n_1},\ldots,\frac{t_d}{n_d}\right)\right|^q dt\right)^{1/q} \\ = {} & C \left\|\widetilde{K}_n\right\|_{E_q'(\mathbb{R}^d)} M_{s,p} f(x). \end{aligned}$$

The result follows from Theorem 4.2.7. ■

The following result comes from Corollary 4.2.5.

Corollary 4.2.9 *If* $1 \le p < \infty$, $1/p + 1/q = 1$ *and*

$$\sup_{n\in\mathbb{N}^d} \|K_n\|_{E_q'(\mathbb{T}^d)} \le C,$$

then

$$\sup_{\rho>0} \rho\lambda(\sigma_* f > \rho)^{1/p} \le C_p \left(\sup_{n\in\mathbb{N}^d} \|K_n\|_{E_q'(\mathbb{T}^d)}\right)\left(1 + \|f\|_{L_p(\log L)^{d-1}}\right)$$

for all $f \in L_p(\log L)^{d-1}(\mathbb{T}^d)$. *Moreover, for every* $p < r \le \infty$,

$$\|\sigma_* f\|_r \le C \left(\sup_{n\in\mathbb{N}^d} \|K_n\|_{E'_q(\mathbb{T}^d)} \right) \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

Corollary 4.2.10 *Suppose that* $1 \le p < \infty$*,* $1/p + 1/q = 1$ *and*

$$\sup_{n\in\mathbb{N}^d} \|K_n\|_{E'_q(\mathbb{T}^d)} \le C,$$

If

$$\lim_{n\to\infty} \widehat{K_n}(k) = 1$$

for all $k \in \mathbb{Z}^d$*, then*

$$\lim_{n\to\infty} \sigma_n f = f \quad a.e.$$

for all $f \in L_p(\log L)^{d-1}(\mathbb{T}^d)$.

Recall that $L_p(\log L)^k(\mathbb{T}^d) \supset L_r(\mathbb{T}^d)$ with $1 \le p < r \le \infty$. In this section, we study the rectangular θ-means,

$$\sigma_n^\theta f(x) := \sum_{k_1\in\mathbb{Z}} \cdots \sum_{k_d\in\mathbb{Z}} \theta\left(\frac{-k_1}{n_1}, \ldots, \frac{-k_d}{n_d}\right) \widehat{f}(k) e^{\imath k\cdot x},$$

where $\theta \in W(C, \ell_1)(\mathbb{R}^d)$.

Theorem 4.2.11 *Suppose that* $1 \le p < \infty$ *and* $1/p + 1/q = 1$*. If* $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ *and* $\widehat{\theta} \in E'_q(\mathbb{R}^d)$*, then*

$$\sigma_*^\theta f(x) \le C \left\|\widehat{\theta}\right\|_{E'_q(\mathbb{R}^d)} M_{s,p} f(x)$$

for all $f \in L_p(\mathbb{T}^d)$ *and* $x \in \mathbb{T}^d$.

Proof Since $\widehat{\theta} \in L_1(\mathbb{R}^d)$ and, by Theorem 3.7.6,

$$\sigma_n^\theta f(x) = \left(\prod_{j=1}^d n_j\right) \int_{\mathbb{R}^d} f(x-t)\widehat{\theta}(n_1 t_1, \ldots, n_d t_d)\, dt,$$

we can repeat the proof of Theorem 4.2.8 step by step. ■

Corollary 4.2.12 *Suppose that* $1 \le p < \infty$ *and* $1/p + 1/q = 1$*. If* $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ *and* $\widehat{\theta} \in E'_q(\mathbb{R}^d)$*, then*

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\theta f > \rho)^{1/p} \le C_p \left\|\widehat{\theta}\right\|_{E'_q(\mathbb{R}^d)} \left(1 + \|f\|_{L_p(\log L)^{d-1}}\right)$$

for all $f \in L_p(\log L)^{d-1}(\mathbb{T}^d)$*. Moreover, for every* $p < r \le \infty$*,*

$$\left\| \sigma_*^\theta f \right\|_r \leq C \left\| \widehat{\theta} \right\|_{E_q'(\mathbb{R}^d)} \| f \|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

Corollary 4.2.13 *Suppose that* $1 \leq p < \infty$ *and* $1/p + 1/q = 1$. *If* $\theta \in W(C, \ell_1)(\mathbb{R}^d)$, $\theta(0) = 1$ *and* $\widehat{\theta} \in E_q'(\mathbb{R}^d)$, *then*

$$\lim_{n \to \infty} \sigma_n^\theta f = f \quad a.e.$$

for all $f \in L_p(\log L)^{d-1}(\mathbb{T}^d)$.

For the converse theorems, we need

Definition 4.2.14 For $1 \leq p < \infty$, we define the space $D_p'(\mathbb{T}^d)$ with the norm

$$\| f \|_{D_p'(\mathbb{T}^d)} := \sup_{r \in (0,\pi)^d} \left(\frac{1}{\prod_{j=1}^d r_j} \int_{-r_1}^{r_1} \cdots \int_{-r_d}^{r_d} |f(t)|^p \, dt \right)^{1/p}.$$

The next two results can be proved as Lemma 4.1.17 and Theorem 4.1.18.

Lemma 4.2.15 *For* $1 \leq p < \infty$, *the norm*

$$\| f \|_* = \sup_{k_1 \leq 0, \ldots, k_d \leq 0} \left(\prod_{j=1}^d 2^{k_j/p} \right) \left\| f 1_{P_k} \right\|_p$$

is an equivalent norm on $D_p'(\mathbb{T}^d)$.

The converse of Theorem 4.2.8 reads as follows.

Theorem 4.2.16 *If* $1 \leq p < \infty$, $1/p + 1/q = 1$ *and*

$$\sigma_* f(0) \leq C M_{s,p} f(0)$$

for all $f \in L_p(\mathbb{T}^d)$, *then*

$$\sup_{n \in \mathbb{N}^d} \| K_n \|_{E_q'(\mathbb{T}^d)} \leq C.$$

We are going to study the second generalization of Lebesgue points for higher dimensions. By Corollary 4.2.4,

$$\lim_{h \to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} f(x - t) \, dt = f(x)$$

for almost every $x \in \mathbb{T}^d$, where $f \in L(\log L)^{d-1}(\mathbb{T}^d)$. This is equivalent to

$$\lim_{h \to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \left| \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} (f(x - t) - f(x)) \, dt \right| = 0.$$

Definition 4.2.17 For $1 \le p < \infty$, a point $x \in \mathbb{T}^d$ is called a strong p-Lebesgue point of $f \in L_p(\mathbb{T}^d)$ if

$$\lim_{h \to 0} \left(\frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0.$$

For $p = 1$, the points are called strong Lebesgue points. If $p < r$, then all strong r-Lebesgue points are strong p-Lebesgue points. The next result can be proved as Theorem 4.1.20

Theorem 4.2.18 *Almost every point $x \in \mathbb{T}^d$ is a strong p-Lebesgue point of $f \in L_p(\log L)^{d-1}(\mathbb{T}^d)$ $(1 \le p < \infty)$.*

This is not true for $f \in L_p(\mathbb{T}^d)$. The reason for this is again that in the definition of the strong Lebesgue points the ratio of the sides of the rectangles can be large. To be able to obtain convergence at strong Lebesgue points, we have to modify slightly condition (4.2.1).

Theorem 4.2.19 *Suppose that $1 \le p < \infty$, $1/p + 1/q = 1$ and*

$$\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \left(\prod_{j=1}^{d} 2^{k_j(1-1/q)} \right) \sup_{n \in \mathbb{N}^d} \left\| \widetilde{K}_n 1_{Q_k} \right\|_q \le C. \tag{4.2.3}$$

If

$$\lim_{n \to \infty} \widehat{K_n}(0) = 1,$$

$M_{s,p} f(x)$ is finite and x is a strong p-Lebesgue point of $f \in L_p(\log L)^{d-1}(\mathbb{T}^d)$, then

$$\lim_{n \to \infty} \sigma_n f(x) = f(x).$$

Proof Similarly to Theorem 4.2.8, let

$$G(u) := \left(\int_{-u_1}^{u_1} \cdots \int_{-u_d}^{u_d} |f(x-t) - f(x)|^p \, dt \right)^{1/p} \qquad (u \in \mathbb{R}_+^d).$$

Since x is a strong p-Lebesgue point of f, for all $\epsilon > 0$, we can find an integer $m \le 0$ such that

$$\frac{G^p(u)}{\prod_{j=1}^d (2u_j)} \le \epsilon \qquad \text{if} \qquad 0 < u_j \le 2^m \pi,\ j = 1, \ldots, d. \tag{4.2.4}$$

Let $\{\pi_1, \ldots, \pi_d\}$ be a permutation of $\{1, \ldots, d\}$ and $1 \le j \le d$. Then

$$\begin{aligned}|\sigma_n f(x) - f(x)| &\le \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |f(x-t) - f(x)| \left| \left(1_{(-\pi,\pi)^d} K_n\right)(t) \right| \, dt \\ &\quad + \left| f(x) \left(\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K_n(t)\, dt - 1 \right) \right| \\ &= A_1(x) + A_2(x) + A_3(x),\end{aligned}$$

where

$$A_1(x) := \frac{1}{(2\pi)^d} \sum_{k_1=-\infty}^{m+\lfloor \log_2 n_1 \rfloor} \cdots \sum_{k_d=-\infty}^{m+\lfloor \log_2 n_d \rfloor} \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t) - f(x)| \left| \left(1_{(-\pi,\pi)^d} K_n\right)(t) \right| \, dt$$

and

$$A_2(x) := \frac{1}{(2\pi)^d} \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=m+\lfloor \log_2 n_{\pi_1} \rfloor+1}^{\infty} \cdots \sum_{k_{\pi_j}=m+\lfloor \log_2 n_{\pi_j} \rfloor+1}^{\infty} \sum_{k_{\pi_{j+1}}=-\infty}^{\infty} \cdots \sum_{k_{\pi_d}=-\infty}^{\infty} \int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t) - f(x)| \left| \left(1_{(-\pi,\pi)^d} K_n\right)(t) \right| \, dt,$$

and

$$A_3(x) := \left| f(x) \left(\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K_n(t)\, dt - 1 \right) \right| = \left| f(x) \left(\widehat{K_n}(0) - 1 \right) \right|.$$

It is clear that

$$\lim_{n\to\infty} A_3(x) = 0.$$

As in (4.2.2),

$$\begin{aligned}&A_1(x) \\ &\le C \sum_{k_1=-\infty}^{m+\lfloor \log_2 n_1 \rfloor} \cdots \sum_{k_d=-\infty}^{m+\lfloor \log_2 n_d \rfloor} \left(\int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t) - f(x)|^p \, dt \right)^{1/p} \\ &\quad \left(\prod_{j=1}^d n_j \right)^{-1/q} \left(\int_{P_{k_1}} \cdots \int_{P_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n\right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right|^q dt \right)^{1/q} \\ &\le C \sum_{k_1=-\infty}^{m+\lfloor \log_2 n_1 \rfloor} \cdots \sum_{k_d=-\infty}^{m+\lfloor \log_2 n_d \rfloor} G\left(\frac{2^{k_1}\pi}{n_1}, \ldots, \frac{2^{k_d}\pi}{n_d} \right) \left(\prod_{j=1}^d n_j \right)^{-1/q}\end{aligned}$$

$$\left(\int_{P_{k_1}} \cdots \int_{P_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n \right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right|^q dt \right)^{1/q}.$$

Inequality (4.2.4) and $2^{k_j}/n_j \le 2^m n_j/n_j = 2^m$ imply

$$\begin{aligned} A_1(x) &\le C_p \epsilon \sum_{k_1=-\infty}^{m+\lfloor \log_2 n_1 \rfloor} \cdots \sum_{k_d=-\infty}^{m+\lfloor \log_2 n_d \rfloor} \left(\prod_{j=1}^d 2^{k_j/p} \right) \left(\prod_{j=1}^d n_j \right)^{-1} \\ &\qquad \left(\int_{P_{k_1}} \cdots \int_{P_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n \right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right|^q dt \right)^{1/q} \\ &\le C_p \epsilon \left\| \widetilde{K}_n \right\|_{E_q'(\mathbb{R}^d)}. \end{aligned}$$

Similarly,

$$\begin{aligned} &A_2(x) \\ &\le C \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=m+\lfloor \log_2 n_{\pi_1} \rfloor+1}^{\infty} \cdots \sum_{k_{\pi_j}=m+\lfloor \log_2 n_{\pi_j} \rfloor+1}^{\infty} \sum_{k_{\pi_{j+1}}=-\infty}^{\infty} \cdots \sum_{k_{\pi_d}=-\infty}^{\infty} \\ &\quad \left(\int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t) - f(x)|^p \, dt \right)^{1/p} \\ &\quad \left(\prod_{j=1}^d n_j \right)^{-1/q} \left(\int_{P_{k_1}} \cdots \int_{P_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n \right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right|^q dt \right)^{1/q}. \end{aligned}$$

We supposed that $M_{s,p} f(x)$ is finite and x is a strong p-Lebesgue point of f, so we have

$$\begin{aligned} &\left(\int_{P_{k_1}(n_1)} \cdots \int_{P_{k_d}(n_d)} |f(x-t) - f(x)|^p \, dt \right)^{1/p} \\ &\qquad \le C_p \left(\prod_{j=1}^d \frac{2^{k_j}}{n_j} \right)^{1/p} \left(M_{s,p} f(x) + |f(x)| \right). \end{aligned}$$

Consequently,

$$\begin{aligned} &A_2(x) \\ &\le C_p \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=m+\lfloor \log_2 n_{\pi_1} \rfloor+1}^{\infty} \cdots \sum_{k_{\pi_j}=m+\lfloor \log_2 n_{\pi_j} \rfloor+1}^{\infty} \sum_{k_{\pi_{j+1}}=-\infty}^{\infty} \cdots \sum_{k_{\pi_d}=-\infty}^{\infty} \end{aligned}$$

$$
\left(\prod_{j=1}^{d} 2^{k_j/p}\right)\left(M_{s,p} f(x) + |f(x)|\right)
$$

$$
\left(\int_{P_{k_1}} \cdots \int_{P_{k_d}} \left| \left(\prod_{j=1}^{d} n_j\right)^{-1} \left(1_{(-\pi,\pi)^d} K_n\right)\left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right)\right|^q dt\right)^{1/q}
$$

$$
\leq C_p \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=m+\lfloor \log_2 n_{\pi_1}\rfloor+1}^{\infty} \cdots \sum_{k_{\pi_j}=m+\lfloor \log_2 n_{\pi_j}\rfloor+1}^{\infty} \sum_{k_{\pi_{j+1}}=-\infty}^{\infty} \cdots \sum_{k_{\pi_d}=-\infty}^{\infty}
$$

$$
\left(\prod_{j=1}^{d} 2^{k_j/p}\right) \sup_{n\in\mathbb{N}^d} \left(\int_{P_{k_1}} \cdots \int_{P_{k_d}} \left|\widetilde{K}_n(t)\right|^q dt\right)^{1/q} \left(M_{s,p} f(x) + |f(x)|\right).
$$

Therefore (4.2.3) and the fact $\lfloor \log_2 n_{\pi_j} \rfloor \to \infty$ as $T \to \infty$ imply that $A_2(x) \to 0$ as $n \to \infty$. ■

Obviously, (4.2.3) implies

$$
\sup_{n\in\mathbb{N}^d} \|\widetilde{K}_n\|_{E_q'(\mathbb{R}^d)} \leq C,
$$

which is equivalent to

$$
\sup_{n\in\mathbb{N}^d} \|K_n\|_{E_q'(\mathbb{T}^d)} \leq C,
$$

by Theorem 4.2.7. If $\widetilde{K}_n$ can be estimated by a function $g \in E_q'(\mathbb{R}^d)$ which is independent of n, then (4.2.3) holds clearly. This is true for the Cesàro kernel $K_n^\alpha = K_{n_1}^\alpha \otimes \cdots \otimes K_{n_d}^\alpha$ and for the Riesz kernel $K_n^{\alpha,\gamma} = K_{n_1}^{\alpha,\gamma} \otimes \cdots \otimes K_{n_d}^{\alpha,\gamma}$. Indeed, for the one-dimensional Cesàro kernel functions

$$
\begin{aligned}
\frac{1}{n_j}\left|K_{n_j}^\alpha\left(\frac{t}{n_j}\right)\right| &\leq \frac{C}{n_j} \min\left\{n_j, \frac{n_j}{|t|^{\alpha+1}}\right\} \\
&= C \min\left\{1, |t|^{-\alpha-1}\right\} \in E_\infty(\mathbb{R})
\end{aligned}
\tag{4.2.5}
$$

by Theorem 1.4.16. Similarly, by (3.3.12),

$$
\begin{aligned}
\frac{1}{n_j}\left|K_{n_j}^{\alpha,\gamma}\left(\frac{t}{n_j}\right)\right| &\leq \frac{C}{n_j} \min\left\{n_j, \frac{n_j}{|t|^{\min(\alpha,1)+1}}\right\} \\
&= C \min\left\{1, |t|^{-\min(\alpha,1)-1}\right\} \in E_\infty(\mathbb{R}).
\end{aligned}
\tag{4.2.6}
$$

Hence, by (4.2.7), $K_n^\alpha, K_n^{\alpha,\gamma} \in E_\infty'(\mathbb{R}^d)$.

Corollary 4.2.20 *If $0 < \alpha \leq 1$, $M_{s,p} f(x)$ is finite and x is a strong Lebesgue point of $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty} \sigma_n^\alpha f(x) = f(x).$$

Moreover,

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\alpha > \rho) \le C \left(1 + \|f\|_{L_1(\log L)^{d-1}}\right)$$

for all $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$ *and, for every* $1 < p \le \infty$,

$$\left\|\sigma_*^\alpha f\right\|_p \le C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

The same hold for the Riesz summation $\sigma_n^{\alpha,\gamma}$ *if* $0 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$.

Considering different parameters α_j in the jth coordinate, we obtain the same results. The next result can be proved in the same way as Theorem 4.2.19.

Theorem 4.2.21 *Suppose that* $1 \le p < \infty$, $1/p + 1/q = 1$, $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ *and* $\widehat{\theta} \in E_q'(\mathbb{R}^d)$. *If* $\theta(0) = 1$, $M_{s,p} f(x)$ *is finite and* x *is a strong* p*-Lebesgue point of* $f \in L_p(\log L)^{d-1}(\mathbb{T}^d)$, *then*

$$\lim_{n\to\infty} \sigma_n^\theta f(x) = f(x).$$

Corollary 4.2.22 *If the conditions of Theorem 4.2.19 or Theorem 4.2.21 are satisfied and if* $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$ *is continuous at a point* x, *then*

$$\lim_{n\to\infty} \sigma_n^\theta f(x) = f(x).$$

Now we show the partial converse of Theorem 4.2.19.

Theorem 4.2.23 *Suppose that* $1 \le p < \infty$ *and* $1/p + 1/q = 1$. *If*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x)$$

for all strong p*-Lebesgue points of* $f \in L_p(\mathbb{T}^d)$ *then*

$$\sup_{n\in\mathbb{N}^d} \|K_n\|_{E_q'(\mathbb{T}^d)} \le C.$$

Proof We define $D_p'^0(\mathbb{T}^d)$ as the set of all functions $f \in D_p'(\mathbb{T}^d)$ for which $f(0) = 0$ and 0 is a strong p-Lebesgue point of f, i.e.,

$$\lim_{h\to 0} \left(\frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \dots \int_{-h_d}^{h_d} |f(t)|^p \, dt \right)^{1/p} = 0.$$

Then we can show that $D_p'^0(\mathbb{T}^d)$ is a Banach space and the proof can be finished as in Theorems 4.1.25 and 4.2.16. ■

In the next subsection, we give some further examples for the θ-summation satisfying the above conditions.

4.2.3 Some Applications

Now we suppose that

$$K_n = K_{n_1}^{(1)} \otimes \cdots \otimes K_{n_d}^{(d)} \qquad (n \in \mathbb{N}^d)$$

and

$$\theta = \theta_1 \otimes \cdots \otimes \theta_d.$$

For these functions, we have

$$\|K_n\|_{E_q'(\mathbb{T}^d)} = \prod_{j=1}^{d} \left\| K_{n_j}^{(j)} \right\|_{E_q(\mathbb{T})} \qquad (n \in \mathbb{N}^d) \tag{4.2.7}$$

and a similar formula holds for $\|\theta\|_{E_q'(\mathbb{R}^d)}$. Hence for these functions, it is enough to consider the one-dimensional Herz spaces $E_q(\mathbb{X})$ ($\mathbb{X} = \mathbb{T}, \mathbb{R}$). As we have seen in Corollary 4.2.20, the rectangular Cesàro and Riesz summation satisfy the conditions of the preceding subsection.

Now, we present some sufficient condition on θ such that $\widehat{\theta} \in E_\infty(\mathbb{R})$. The next theorem was proved in Herz [166], Peetre [254] and Girardi and Weis [130].

Lemma 4.2.24 *If $\theta \in B_{1,1}^1(\mathbb{R})$, then $\widehat{\theta} \in E_\infty(\mathbb{R})$ and*

$$\left\|\widehat{\theta}\right\|_{E_\infty} \le C_p \|\theta\|_{B_{1,1}^1}.$$

A function f belongs to the weighted Wiener amalgam space $W(L_\infty, \ell_1^{v_s})(\mathbb{R})$ if

$$\|f\|_{W(L_\infty, \ell_1^{v_s})} := \sum_{k=-\infty}^{\infty} \sup_{x \in [0,1)} |f(x+k)| v_s(k) < \infty,$$

where $v_s(x) := (1 + |x|)^s$ $(x \in \mathbb{R})$.

Lemma 4.2.25 *If $\theta \in W(L_\infty, \ell_1^{v_1})(\mathbb{R})$, then $\theta \in E_\infty(\mathbb{R})$ and*

$$\|\theta\|_{E_\infty} \le C \|\theta\|_{W(L_\infty, \ell_1^{v_1})}.$$

Proof The inequalities

$$\begin{aligned}\|\theta\|_{E_\infty} &= \sum_{k=-\infty}^{\infty} 2^k \sup_{P_k} |\theta| \\ &\le 2 \sup_{(-\pi,\pi)} |\theta| + C \sum_{k=0}^{\infty} 2^k \sum_{j:(-\pi,\pi)+2j\pi \cap P_k \neq \emptyset} \sup_{(-\pi,\pi)+2j\pi} |\theta| \\ &\le C \sum_{j=-\infty}^{\infty} (1+|j|) \sup_{(-\pi,\pi)^d+2j\pi} |\theta| \\ &= C\|\theta\|_{W(L_\infty,\ell_1^{v_1})}\end{aligned}$$

prove the result. ■

We generalize Feichtinger's algebra and introduce its weighted version.

Definition 4.2.26 Let $g_0(x) := e^{-\pi\|x\|_2^2}$ be the Gauss function. We define the weighted Feichtinger's algebra or modulation space $M_1^{v_s}(\mathbb{R}^d)$ ($s \ge 0$) by

$$M_1^{v_s}(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) : \|f\|_{M_1^{v_s}} := \left\| S_{g_0} f \cdot v_s \right\|_{L_1(\mathbb{R}^{2d})} < \infty \right\},$$

where $v_s(x,\omega) := v_s(\omega) = (1+|\omega|)^s$ $(x,\omega \in \mathbb{R}^d)$.

Any other non-zero Schwartz function defines the same space and an equivalent norm (see, e.g., Feichtinger [100] and Gröchenig [152]).

Lemma 4.2.27 *If $\theta \in M_1^{v_1}(\mathbb{R})$, then $\widehat{\theta} \in E_\infty(\mathbb{R})$ and*

$$\left\|\widehat{\theta}\right\|_{E_\infty} \le \|\theta\|_{M_1^{v_1}}.$$

Proof By Lemma 4.2.25,

$$\left\|\widehat{\theta}\right\|_{E_\infty} \le C \left\|\widehat{\theta}\right\|_{W(L_\infty,\ell_1^{v_1})} \le C \|f\|_{M_1^{v_1}},$$

where the second inequality can be found in Gröchenig [152, p. 249]. ■

Corollary 4.2.28 *Suppose that $\theta = \theta_1 \otimes \cdots \otimes \theta_d \in W(C,\ell_1)(\mathbb{R}^d)$ and $\theta(0) = 1$. If $\theta_j \in M_1^{v_1}(\mathbb{R})$ for all $j = 1,\ldots,d$, $M_s f(x)$ is finite and x is a strong Lebesgue point of $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty} \sigma_n^\theta f(x) = f(x).$$

Moreover,

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\theta > \rho) \le C \left(\prod_{j=1}^{d} \|\theta_j\|_{M_1^{v_1}} \right) \left(1 + \|f\|_{L_1(\log L)^{d-1}}\right)$$

for all $f \in L_1(\log L)^{d-1}(\mathbb{T}^d)$. *Also, for every* $1 < p \le \infty$,

$$\left\|\sigma_*^\theta f\right\|_p \le C_p \left(\prod_{j=1}^d \left\|\theta_j\right\|_{M_1^{v_1}}\right) \|f\|_p \qquad (f \in L_p(\mathbb{R}^d)).$$

In the next theorem, we give a sufficient result for θ to be in $M_1^{v_s}(\mathbb{R})$.

Theorem 4.2.29 *If* $\theta \in V_1^k(\mathbb{R})$ *for some* $k \ge 2$, *then* $\theta \in M_1^{v_s}(\mathbb{R})$ *for all* $0 \le s < k-1$ *and*

$$\|\theta\|_{M_1^{v_s}} \le C_s \|\theta\|_{V_1^k}.$$

This theorem can be proved as was Theorem 3.7.14. Note that $V_1^k(\mathbb{R}^d)$ was defined in Definition 3.7.13.

The space $V_1^2(\mathbb{R})$ is not contained in $M_1^{v_1}(\mathbb{R})$. However, the same results hold as in Corollary 4.2.28.

Theorem 4.2.30 *If* $\theta \in V_1^2(\mathbb{R})$, *then* $\widehat{\theta} \in E_\infty(\mathbb{R})$.

Proof The inequality

$$\left|\widehat{\theta}(x)\right| \le \frac{C}{x^2} \qquad (x \ne 0) \tag{4.2.8}$$

can be shown similarly to Theorem 3.7.14. $\widehat{\theta} \in E_\infty(\mathbb{R})$ follows from Theorem 4.1.9. ■

Corollary 4.2.31 *Suppose that* $\theta = \theta_1 \otimes \cdots \otimes \theta_d \in W(C, \ell_1)(\mathbb{R}^d)$ *and* $\theta(0) = 1$. *If* $\theta_j \in V_1^2(\mathbb{R})$ *for all* $j = 1, \ldots, d$, *then the results of Corollary 4.2.28 holds.*

Note that for all examples of Sect. 2.6.3, we have $\theta \in V_1^2(\mathbb{R})$ or (4.2.8). This means that all results of Sect. 4.2.2 hold if each θ_j denotes either the Cesàro summation or one of the examples of Sect. 2.6.3.

4.3 Restricted Rectangular Summability over a Cone

Let again

$$\sigma_n f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n(t)\, dt \qquad (n \in \mathbb{N}^d),$$

where $f \in L_1(\mathbb{T}^d)$ and $K_n \in L_1(\mathbb{T}^d) \cap L_\infty(\mathbb{T}^d)$. Here we suppose that $n \in \mathbb{N}^d$ is in the cone $\mathbb{R}_\tau^d$. Recall that $\mathbb{R}_\tau^d$ is defined by

$$\mathbb{R}_\tau^d := \{x \in \mathbb{R}_+^d : \tau^{-1} \le x_i/x_j \le \tau, i, j = 1, \ldots, d\},$$

where $\tau \ge 1$ is fixed. The higher dimensional rectangular Cesàro and Riesz kernels, K_n^α and $K_n^{\alpha,\gamma}$ satisfy the conditions of this section.

4.3.1 Hardy-Littlewood Maximal Functions

It would be a straightforward idea that for the restricted rectangular summability, we use the restricted Hardy-Littlewood maximal function $M_{\square,p}$ defined in Theorem 4.1.4. However, this would be not useful because the restricted maximal function is equivalent to the usual maximal function $M_p f$ (see (4.1.1)). So we have to introduce a third generalization of the maximal function.

Definition 4.3.1 For $\omega > 0$, $1 \le p < \infty$ and $f \in L_p(\mathbb{T}^d)$, the Hardy-Littlewood maximal function $\mathcal{M}_p^{\omega,1} f$ is given by

$$\mathcal{M}_p^{\omega,1} f(x) := \sup_{i\in\mathbb{N}^d, h>0} 2^{-\omega\|i\|_1}\left(\frac{1}{(2h)^d 2^{\|i\|_1}} \int_{-2^{i_1}h}^{2^{i_1}h} \cdots \int_{-2^{i_d}h}^{2^{i_d}h} |f(x-t)|^p \, dt\right)^{1/p}.$$

For $p = 1$, we write simply $\mathcal{M}^{\omega,1} f$. If $\omega = 0$, we get back the definition of the strong Hardy-Littlewood maximal function $M_{s,p} f$. In contrary to the strong maximal function, due to the weight $2^{-\omega\|i\|_1}$, the weak type (p, p) inequality will be true for $\mathcal{M}_p^{\omega}$. It is clear that

$$\mathcal{M}_p^{\omega_1,1} f \le \mathcal{M}_p^{\omega_2,1} f \qquad \text{for } \omega_1 > \omega_2 > 0 \text{ and } 1 \le p < \infty.$$

Let us point out the definition in the two-dimensional case. We have

$$\mathcal{M}_p^{\omega,1} f(x_1, x_2) = \sup_{i_1,i_2\in\mathbb{N}, h>0} 2^{-\omega(i_1+i_2)}\left(\frac{1}{4\cdot 2^{i_1+i_2}h^2} \int_{-2^{i_1}h}^{2^{i_1}h} \int_{-2^{i_2}h}^{2^{i_2}h} |f(x_1-t_1, x_2-t_2)|^p \, dt\right)^{1/p}.$$

To prove inequalities for $\mathcal{M}_p^{\omega,1} f$, we need another generalization of the maximal function $M_p f$. Let $\mu(h)$ and $\nu(h)$ be two continuous functions of $h \ge 0$, strictly increasing to ∞ and 0 at $h = 0$. Let

$$M_p^{1,\mu,\nu} f(x_1, x_2) := \sup_{h>0} \left(\frac{1}{4\mu(h)\nu(h)} \int_{-\mu(h)}^{\mu(h)} \int_{-\nu(h)}^{\nu(h)} |f(x_1-t_1, x_2-t_2)|^p \, dt\right)^{1/p},$$

where $f \in L_p(\mathbb{T}^2)$. If $\mu(h) = \nu(h) = h$, then we get back the usual Hardy-Littlewood maximal function $M_p f$ investigated in Sect. 4.1.1. The next result can be proved in the same way as Theorem 4.1.2.

Theorem 4.3.2 *If* $1 \le p < \infty$, *then*

$$\sup_{\rho>0} \rho\lambda(M_p^{1,\mu,\nu} f > \rho)^{1/p} \le C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^2)).$$

Moreover, if $p < r \le \infty$, then

$$\left\| M_p^{1,\mu,\nu} f \right\|_r \le C_r \|f\|_r \qquad (f \in L_r(\mathbb{T}^2)),$$

where the constants C_p and C_r are independent of μ and ν.

Using this theorem, we can prove the inequalities for $\mathcal{M}_p^{\omega,1}$.

Theorem 4.3.3 *If $\omega > 0$ and $1 \le p < \infty$, then*

$$\sup_{\rho>0} \rho \lambda(\mathcal{M}_p^{\omega,1} f > \rho)^{1/p} \le C \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Moreover, if $p < r \le \infty$, then

$$\left\| \mathcal{M}_p^{\omega,1} f \right\|_r \le C_r \|f\|_r \qquad (f \in L_r(\mathbb{T}^d).$$

Proof Applying Theorem 4.3.2 to $\mu(h) = 2^{i_1} h$ and $\nu(h) = 2^{i_2} h$, we obtain

$$\begin{aligned}
\rho^p \lambda(\mathcal{M}_p^{\omega,1} f > \rho) &\le \rho \lambda \Bigg(\bigcup_{i_1,i_2=0}^{\infty} 2^{-\omega(i_1+i_2)} \sup_{h>0} \Bigg(\frac{1}{4 \cdot 2^{i_1+i_2} h^2} \\
&\qquad \int_{-2^{i_1} h}^{2^{i_1} h} \int_{-2^{i_2} h}^{2^{i_2} h} |f(x_1 - t_1, x_2 - t_2)|^p \, dt \Bigg)^{1/p} > \rho \Bigg) \\
&\le \rho^p \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \lambda \left(M_p^{1,\mu,\nu} f > 2^{\omega(i_1+i_2)} \rho \right) \\
&\le C_p \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} 2^{-\omega p(i_1+i_2)} \|f\|_p^p \\
&\le C_p \|f\|_p^p
\end{aligned}$$

for all $f \in L_1(\mathbb{T}^2)$ and $\rho > 0$. The inequality

$$\left\| \mathcal{M}^{\omega,1} f \right\|_p \le C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^2), 1 < p \le \infty)$$

can be shown similarly. ∎

4.3.2 Lebesgue Points for the Summability over a Cone

We briefly write $L_p^\omega(\mathbb{R}^d)$ $(\omega \ge 0)$ instead of the weighted Lebesgue space $L_p^\omega(\mathbb{R}^d, \lambda)$ equipped with the norm

$$\|f\|_{L_p^\omega} := \left(\int_{\mathbb{R}^d} |f(x)(1+|x|)^\omega|^p \, dx \right)^{1/p} \qquad (1 \le p < \infty),$$

with the usual modification for $p = \infty$. If $\omega = 0$, then we get back the $L_p(\mathbb{R}^d)$ spaces. Clearly, $L_p(\mathbb{R}^d) \supset L_p^\omega(\mathbb{R}^d)$.

In this subsection, we introduce a new type of Herz spaces, the so-called weighted inhomogeneous Herz spaces.

Definition 4.3.4 For $\omega \ge 0$ and $1 \le q \le \infty$, the weighted Herz space $E_q^\omega(\mathbb{R}^d)$ contains all functions f for which

$$\|f\|_{E_q^\omega} := \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \left(\prod_{j=1}^{d} 2^{k_j(\omega+1-1/q)} \right) \|f 1_{Q_k}\|_q < \infty,$$

where

$$Q_k := Q_{k_1} \times \cdots \times Q_{k_d} \qquad (k \in \mathbb{N}^d)$$

and

$$Q_i = \{x \in \mathbb{R} : 2^{i-1}\pi \le |x| < 2^i \pi\} \qquad (i \in \mathbb{N}_+), \qquad Q_0 := (-\pi, \pi).$$

It is clear that

$$E_q^\omega(\mathbb{R}^d) \supset E_q^{\omega'}(\mathbb{R}^d) \qquad 0 \le \omega < \omega' < \infty$$

and

$$L_1(\mathbb{R}^d) \supset L_1^\omega(\mathbb{R}^d) = E_1^\omega(\mathbb{R}^d) \supset E_q^\omega(\mathbb{R}^d) \supset E_{q'}^\omega(\mathbb{R}^d) \supset E_\infty^\omega(\mathbb{R}^d)$$

for any $1 < q < q' < \infty$ with continuous embeddings. Moreover,

$$E_q^\omega(\mathbb{R}^d) \subset L_q^\omega(\mathbb{R}^d) \qquad \text{and} \qquad \|f\|_{L_q^\omega(\mathbb{R}^d)} \le C_q \|f\|_{E_q^\omega(\mathbb{R}^d)}.$$

Indeed,

$$\int_{\mathbb{R}^d} |f(t)|^q \, dt \le \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \left(\prod_{j=1}^{d} 2^{k_j(\omega+1-1/q)q} \right) \int_{Q_k} |f(t)|^q \, dt,$$

which implies the inequality. The connection between $E_q^0(\mathbb{R}^d)$ and $E_q'(\mathbb{R}^d)$ is the following. First of all,

$$E_q^0(\mathbb{R}^d) \subset E_q'(\mathbb{R}^d) \qquad \text{and} \qquad \|f\|_{E_q'(\mathbb{R}^d)} \le C_q \|f\|_{E_q^0(\mathbb{R}^d)}.$$

We prove this for one dimension, only:

$$\sum_{k=-\infty}^{0} 2^{k(1-1/q)} \left\| f 1_{P_k} \right\|_q \le \sum_{k=-\infty}^{0} 2^{k(1-1/q)} \left\| f 1_{(-\pi,\pi)} \right\|_q \le C_q \left\| f \right\|_{E_q^0(\mathbb{R})} .$$

We get the inequality for higher dimensions similarly. Summarizing these results, we can see that

$$E_q^0(\mathbb{R}^d) = L_q(\mathbb{R}^d) \cap E_q'(\mathbb{R}^d) \quad \text{and} \quad \| f \|_{E_q^0(\mathbb{R}^d)} \sim \| f \|_q + \| f \|_{E_q'(\mathbb{R}^d)}$$

with equivalent norms. Usually, $\widehat{\theta} \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$, thus $\widehat{\theta} \in E_q'(\mathbb{R}^d)$ if and only if $\widehat{\theta} \in E_q^0(\mathbb{R}^d)$, but $\|\widehat{\theta}\|_{E_q'(\mathbb{R}^d)} \le C_q \|\widehat{\theta}\|_{E_q^0(\mathbb{R}^d)}$, $(1 \le q \le \infty)$.

We show that $f \in E_\infty^\omega(\mathbb{R})$ if and only if f has a decreasing majorant function belonging to $L_1^\omega(\mathbb{R})$.

Theorem 4.3.5 *Let $\omega \ge 0$ and $\eta(x) := \sup_{|t|\ge|x|} |f(t)|$. Then $f \in E_\infty^\omega(\mathbb{R})$ if and only if $\eta \in L_1^\omega(\mathbb{R})$ and*

$$C^{-1} \|\eta\|_{L_1^\omega} \le \|f\|_{E_\infty^\omega} \le C \|\eta\|_{L_1^\omega} + \eta(0).$$

Proof If $\eta \in L_1^\omega(\mathbb{R})$, then

$$\begin{aligned} \|f\|_{E_\infty^\omega} &\le \sum_{k=0}^{\infty} 2^{k(\omega+1)} \left\| \eta 1_{Q_k} \right\|_\infty \\ &= \sum_{k=1}^{\infty} 2^{k(\omega+1)} \eta(2^{k-1}\pi) + \eta(0) \le C \|\eta\|_{L_1^\omega} + \eta(0). \end{aligned}$$

For the converse, we use the function ν introduced in the proof of Theorem 4.1.9 to obtain

$$\begin{aligned} \|\eta\|_{L_1^\omega} \le \|\nu\|_{L_1^\omega} &= \sum_{k=0}^{\infty} a_{n_k} \int_{B(0,2^{n_k})\setminus B(0,2^{n_{k-1}})} (1+x)^\omega \, dx \\ &= C \sum_{k=0}^{\infty} 2^{n_k \omega} \left(2^{n_k} - 2^{n_{k-1}}\right) a_{n_k} \le C \|f\|_{E_\infty^\omega}, \end{aligned}$$

which proves the theorem. ∎

In this section, we investigate the restricted maximal operator

$$\sigma_\square f := \sup_{n \in \mathbb{R}_\tau^d} |\sigma_n f|,$$

where $\tau \ge 1$ is fixed. Recall that

$$\widetilde{K}_n(t) := \left(\prod_{j=1}^{d} n_j\right)^{-1} \left(1_{(-\pi,\pi)^d} K_n\right)\left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right).$$

Theorem 4.3.6 *If $\omega \geq 0$, $1 \leq p < \infty$, $1/p + 1/q = 1$ and*

$$\sup_{n \in \mathbb{R}^d_\tau} \left\| \widetilde{K}_n \right\|_{E_q^\omega(\mathbb{R}^d)} \leq C,$$

then

$$\sigma_\square f(x) \leq C \left(\sup_{n \in \mathbb{R}^d_\tau} \left\| \widetilde{K}_n \right\|_{E_q^\omega(\mathbb{R}^d)}\right) \mathcal{M}_p^{\omega,1} f(x)$$

for all $f \in L_p(\mathbb{T}^d)$ and $x \in \mathbb{T}^d$.

Proof We have

$$\begin{aligned}
&|\sigma_n f(x)| \\
&= \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} f(x-t) \left(1_{(-\pi,\pi)^d} K_n\right)(t)\, dt \right| \\
&\leq \frac{1}{(2\pi)^d} \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \int_{Q_{k_1}(n_1)} \cdots \int_{Q_{k_d}(n_d)} |f(x-t)| \left| \left(1_{(-\pi,\pi)^d} K_n\right)(t) \right| dt,
\end{aligned}$$

where

$$Q_i(n_j) := \{x \in \mathbb{R} : 2^{i-1}\pi/n_j \leq |x| < 2^i \pi/n_j\} \qquad (i \in \mathbb{N}_+)$$

and

$$Q_0(n_j) := (-\pi/n_j, \pi/n_j).$$

By Hölder's inequality,

$$\begin{aligned}
&|\sigma_n f(x)| \\
&\leq \frac{1}{(2\pi)^d} \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \left(\int_{Q_{k_1}(n_1)} \cdots \int_{Q_{k_d}(n_d)} |f(x-t)|^p \, dt \right)^{1/p} \\
&\quad \left(\int_{Q_{k_1}(n_1)} \cdots \int_{Q_{k_d}(n_d)} \left| \left(1_{(-\pi,\pi)^d} K_n\right)(t) \right|^q \, dt \right)^{1/q} \\
&= \frac{1}{(2\pi)^d} \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \left(\int_{Q_{k_1}(n_1)} \cdots \int_{Q_{k_d}(n_d)} |f(x-t)|^p \, dt \right)^{1/p}
\end{aligned}$$

$$\left(\prod_{j=1}^{d} n_j\right)^{-1/q} \left(\int_{Q_{k_1}} \cdots \int_{Q_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n\right)\left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right)\right|^q dt\right)^{1/q}. \quad (4.3.1)$$

Choose $s \in \mathbb{N}$ such that $2^{s-1} < \tau \le 2^s$. Since $n \in \mathbb{R}_\tau^d$, we conclude

$$\begin{aligned} &|\sigma_n f(x)| \\ &\le \frac{1}{(2\pi)^d} \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \left(\int_{-2^{k_1+s}\pi/n_1}^{2^{k_1+s}\pi/n_1} \cdots \int_{-2^{k_d+s}\pi/n_1}^{2^{k_d+s}\pi/n_1} |f(x-t)|^p \, dt\right)^{1/p} \\ &\quad \left(\prod_{j=1}^{d} n_j\right)^{-1/q} \left(\int_{Q_{k_1}} \cdots \int_{Q_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n\right)\left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right)\right|^q dt\right)^{1/q}. \quad (4.3.2) \end{aligned}$$

Let again

$$G(u) := \left(\int_{-u_1}^{u_1} \cdots \int_{-u_d}^{u_d} |f(x-t)|^p \, dt\right)^{1/p} \qquad (u \in \mathbb{R}_+^d).$$

Then

$$2^{-\omega(k_1+\ldots+k_d)p} n_1^d \frac{G^p(2^{k_1+s}\pi/n_1, \ldots, 2^{k_d+s}\pi/n_1)}{(2\pi)^d 2^{sd} 2^{k_1+\ldots+k_d}} \le (\mathcal{M}_p^{\omega,1})^p f(x)$$

and so

$$\begin{aligned} &|\sigma_n f(x)| \\ &\le C \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} G\left(\frac{2^{k_1+s}\pi}{n_1}, \ldots, \frac{2^{k_d+s}\pi}{n_1}\right) \\ &\quad \left(\prod_{j=1}^{d} n_j\right)^{-1/q} \left(\int_{Q_{k_1}} \cdots \int_{Q_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n\right)\left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right)\right|^q dt\right)^{1/q} \\ &\le C \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \left(\prod_{j=1}^{d} 2^{k_j(\omega+1/p)}\right) n_1^{-d/p} \mathcal{M}_p^{\omega,1} f(x) \\ &\quad \left(\prod_{j=1}^{d} n_j\right)^{-1/q} \left(\int_{Q_{k_1}} \cdots \int_{Q_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n\right)\left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right)\right|^q dt\right)^{1/q}. \end{aligned}$$

The fact $n \in \mathbb{R}_\tau^d$ implies

$$|\sigma_n f(x)| \le C \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \left(\prod_{j=1}^{d} 2^{k_j(\omega+1/p)} \right) \mathcal{M}_p^{\omega,1} f(x) \left(\prod_{j=1}^{d} n_j \right)^{-1}$$
$$\left(\int_{Q_{k_1}} \cdots \int_{Q_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n\right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right|^q dt \right)^{1/q}$$
$$= C \left\| \widetilde{K}_n \right\|_{E_q^\omega(\mathbb{R}^d)} \mathcal{M}_p^{\omega,1} f(x),$$

which shows the theorem. ∎

Taking into account Theorem 4.3.3, we have

Theorem 4.3.7 *If $\omega > 0$, $1 \le p < \infty$, $1/p + 1/q = 1$ and*

$$\sup_{n \in \mathbb{R}_\tau^d} \left\| \widetilde{K}_n \right\|_{E_q^\omega(\mathbb{R}^d)} \le C,$$

then

$$\sup_{\rho > 0} \rho \lambda(\sigma_\square f > \rho)^{1/p} \le C_p \left(\sup_{n \in \mathbb{R}_\tau^d} \left\| \widetilde{K}_n \right\|_{E_q^\omega(\mathbb{R}^d)} \right) \|f\|_p$$

for all $f \in L_p(\mathbb{T}^d)$. Moreover, for every $p < r \le \infty$,

$$\|\sigma_\square f\|_r \le C \left(\sup_{n \in \mathbb{R}_\tau^d} \left\| \widetilde{K}_n \right\|_{E_q^\omega(\mathbb{R}^d)} \right) \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

Corollary 4.3.8 *Suppose that $\omega > 0$, $1 \le p < \infty$, $1/p + 1/q = 1$ and*

$$\sup_{n \in \mathbb{R}_\tau^d} \left\| \widetilde{K}_n \right\|_{E_q^\omega(\mathbb{R}^d)} \le C.$$

If

$$\lim_{n \to \infty,\, n \in \mathbb{R}_\tau^d} \widehat{K_n}(k) = 1$$

for all $k \in \mathbb{Z}^d$, then

$$\lim_{n \to \infty,\, n \in \mathbb{R}_\tau^d} \sigma_n f = f \quad a.e.$$

for all $f \in L_p(\mathbb{T}^d)$.

For the rectangular θ-means

$$\sigma_n^\theta f(x) := \sum_{k_1 \in \mathbb{Z}} \cdots \sum_{k_d \in \mathbb{Z}} \theta \left(\frac{-k_1}{n_1}, \ldots, \frac{-k_d}{n_d} \right) \widehat{f}(k) e^{\imath k \cdot x},$$

we obtain

Theorem 4.3.9 *Suppose that $\omega \geq 0$, $1 \leq p < \infty$ and $1/p + 1/q = 1$. If $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ and $\widehat{\theta} \in E_q^\omega(\mathbb{R}^d)$, then*

$$\sigma_\square^\theta f(x) \leq C \left\| \widehat{\theta} \right\|_{E_q^\omega(\mathbb{R}^d)} \mathcal{M}_p^{\omega,1} f(x)$$

for all $f \in L_p(\mathbb{T}^d)$ and $x \in \mathbb{T}^d$.

This inequality can be proved as Theorem 4.3.6 (see also Theorem 4.2.11).

Theorem 4.3.10 *Suppose that $\omega > 0$, $1 \leq p < \infty$ and $1/p + 1/q = 1$. If $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ and $\widehat{\theta} \in E_q^\omega(\mathbb{R}^d)$, then*

$$\sup_{\rho > 0} \rho \lambda(\sigma_\square^\theta f > \rho)^{1/p} \leq C_p \left\| \widehat{\theta} \right\|_{E_q^\omega(\mathbb{R}^d)} \|f\|_p$$

for all $f \in L_p(\mathbb{T}^d)$. Moreover, for every $p < r \leq \infty$,

$$\left\| \sigma_\square^\theta f \right\|_r \leq C \left\| \widehat{\theta} \right\|_{E_q^\omega(\mathbb{R}^d)} \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

Corollary 4.3.11 *Suppose that $\omega > 0$, $1 \leq p < \infty$ and $1/p + 1/q = 1$. If $\theta(0) = 1$, $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ and $\widehat{\theta} \in E_q^\omega(\mathbb{R}^d)$, then*

$$\lim_{n \to \infty,\, n \in \mathbb{R}_\tau^d} \sigma_n^\theta f = f \quad a.e.$$

for all $f \in L_p(\mathbb{T}^d)$.

We introduce the third generalization of Lebesgue points as follows. Starting from the maximal function $\mathcal{M}_p^{\omega,1} f$, we introduce

$$U_{r,p}^{\omega,1} f(x) := \sup_{i \in \mathbb{N}^d, h > 0, 2^{i_k} h < r, k=1,\ldots d} 2^{-\omega \|i\|_1} \left(\frac{1}{(2h)^d 2^{\|i\|_1}} \int_{-2^{i_1} h}^{2^{i_1} h} \cdots \int_{-2^{i_d} h}^{2^{i_d} h} |f(x-t) - f(x)|^p \, dt \right)^{1/p}. \tag{4.3.3}$$

In case $p = 1$, we omit the notation p and write simply $U_r^{\omega,1} f$. In the two-dimensional case this definition reads as

$$U_{r,p}^{\omega,1} f(x_1, x_2) = \sup_{i_1, i_2 \in \mathbb{N}, h > 0, 2^{i_k} h < r, k=1,2} 2^{-\omega(i_1+i_2)} \left(\frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_1} h}^{2^{i_1} h} \int_{-2^{i_2} h}^{2^{i_2} h} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)|^p \, dt \right)^{1/p}.$$

Note that the definitions of the p-Lebesgue points and strong p-Lebesgue points (see Definitions 4.1.19 and 4.2.17) can be rewritten as

$$\lim_{r\to 0}\sup_{0<h<r}\left(\frac{1}{(2h)^d}\int_{-h}^{h}\cdots\int_{-h}^{h}|f(x-t)-f(x)|^p\,dt\right)^{1/p}=0$$

and

$$\lim_{r\to 0}\sup_{0<h_j<r,\,j=1,\ldots,d}\left(\frac{1}{\prod_{j=1}^{d}(2h_j)}\int_{-h_1}^{h_1}\cdots\int_{-h_d}^{h_d}|f(x-t)-f(x)|^p\,dt\right)^{1/p}=0,$$

respectively. Similarly to this definition, we introduce a new type of Lebesgue points.

Definition 4.3.12 For $1\le p<\infty$ and $\omega>0$, a point $x\in\mathbb{T}^d$ is called a (p,ω)-Lebesgue point of $f\in L_p(\mathbb{T}^d)$ if

$$\lim_{r\to 0}U_{r,p}^{\omega,1}f(x)=0.$$

If $\omega=0$, then the $(p,0)$-Lebesgue points are the same as the strong p-Lebesgue points. It is easy to see that every (p,ω_2)-Lebesgue point is a (p,ω_1)-Lebesgue point if $\omega_1>\omega_2>0$. Moreover, if $p<r$, then every (r,ω)-Lebesgue point is a (p,ω)-Lebesgue point. If f is continuous at x, then x is a (p,ω)-Lebesgue point of f for all $1\le p<\infty$ and $\omega>0$.

Theorem 4.3.13 *For $1\le p<\infty$ and $\omega>0$, almost every point $x\in\mathbb{T}^d$ is a (p,ω)-Lebesgue point of $f\in L_p(\mathbb{T}^d)$.*

Proof If f is a continuous function, then x is obviously a (p,ω)-Lebesgue point. By Theorem 4.3.3,

$$\begin{aligned}\rho^p\lambda\left(\sup_{r>0}U_{r,p}^{\omega,1}f>\rho\right)&\le\rho^p\lambda(\mathcal{M}_p^{\omega,1}f>\rho/2)+2\rho^p\lambda(|f|>\rho/2)\\&\le C\,\|f\|_p^p.\end{aligned}$$

Since the result holds for continuous functions and the continuous functions are dense in $L_p(\mathbb{T}^d)$, the theorem follows from the usual density argument of Theorem 1.3.7. ∎

Theorem 4.3.14 *Suppose that $\omega>0$, $1\le p<\infty$, $1/p+1/q=1$ and*

$$\sum_{k_1=0}^{\infty}\cdots\sum_{k_d=0}^{\infty}\left(\prod_{j=1}^{d}2^{k_j(\omega+1-1/q)}\right)\sup_{n\in\mathbb{R}^d_{\kappa,\tau}}\left\|\widetilde{K}_n 1_{Q_k}\right\|_q\le C. \tag{4.3.4}$$

If

$$\lim_{n\to\infty,\, n\in\mathbb{R}_\tau^d} \widehat{K_n}(0) = 1,$$

$\mathcal{M}_p^{\omega,1} f(x)$ *is finite and* x *is a* (p,ω)*-Lebesgue points of* $f \in L_p(\mathbb{T}^d)$*, then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}_\tau^d} \sigma_n f(x) = f(x).$$

Proof Choose again $s \in \mathbb{N}$ such that $2^{s-1} < \tau \le 2^s$. Since x is a (p,ω)-Lebesgue point of f, we can fix a number $r < 1$ such that

$$U_{r2^s\pi,p}^{\omega,1} f(x) < \epsilon,$$

where $U_{r,p}^{\omega,1} f$ was introduced in (4.3.3). Let us denote by r_0 the largest number i, for which $r/2 \le 2^i/n_1 < r$. We have

$$\begin{aligned} |\sigma_n f(x) - f(x)| &\le \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |f(x-t) - f(x)| \left| \left(1_{(-\pi,\pi)^d} K_n\right)(t) \right| \, dt \\ &\quad + \left| f(x) \left(\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K_n(t)\, dt - 1 \right) \right| \\ &= A_1(x) + A_2(x) + A_3(x), \end{aligned}$$

where

$$\begin{aligned} A_1(x) := \frac{1}{(2\pi)^d} &\sum_{k_1=0}^{r_0} \cdots \sum_{k_d=0}^{r_0} \\ &\int_{Q_{k_1}(n_1)} \cdots \int_{Q_{k_d}(n_d)} |f(x-t) - f(x)| \left| \left(1_{(-\pi,\pi)^d} K_n\right)(t) \right| \, dt, \end{aligned}$$

$$\begin{aligned} A_2(x) := \frac{1}{(2\pi)^d} &\sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=r_0+1}^{\infty} \cdots \sum_{k_{\pi_j}=r_0+1}^{\infty} \sum_{k_{\pi_{j+1}}=0}^{\infty} \cdots \sum_{k_{\pi_d}=0}^{\infty} \\ &\int_{Q_{k_1}(n_1)} \cdots \int_{Q_{k_d}(n_d)} |f(x-t) - f(x)| \left| \left(1_{(-\pi,\pi)^d} K_n\right)(t) \right| \, dt, \end{aligned}$$

and

$$A_3(x) := \left| f(x) \left(\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K_n(t)\, dt - 1 \right) \right| = \left| f(x) \left(\widehat{K_n}(0) - 1 \right) \right|.$$

Here $\{\pi_1, \ldots, \pi_d\}$ denotes a permutation of $\{1, \ldots, d\}$ and $1 \le j \le d$. Obviously,

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_\tau} A_3(x) = 0.$$

Similarly to (4.3.1) and (4.3.2), we deduce

$$\begin{aligned}
&A_1(x) \\
&\le \frac{1}{(2\pi)^d} \sum_{k_1=0}^{r_0} \cdots \sum_{k_d=0}^{r_0} \left(\int_{Q_{k_1}(n_1)} \cdots \int_{Q_{k_d}(n_d)} |f(x-t) - f(x)|^p \, dt \right)^{1/p} \\
&\quad \left(\prod_{j=1}^d n_j \right)^{-1/q} \left(\int_{Q_{k_1}} \cdots \int_{Q_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n\right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right|^q \, dt \right)^{1/q} \\
&\le \frac{1}{(2\pi)^d} \sum_{k_1=0}^{r_0} \cdots \sum_{k_d=0}^{r_0} \left(\int_{-2^{k_1+s}\pi/n_1}^{2^{k_1+s}\pi/n_1} \cdots \int_{-2^{k_d+s}\pi/n_1}^{2^{k_d+s}\pi/n_1} |f(x-t) - f(x)|^p \, dt \right)^{1/p} \\
&\quad \left(\prod_{j=1}^d n_j \right)^{-1/q} \left(\int_{Q_{k_1}} \cdots \int_{Q_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n\right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right|^q \, dt \right)^{1/q}. \qquad (4.3.5)
\end{aligned}$$

Setting

$$G(u) := \left(\int_{-u_1}^{u_1} \ldots \int_{-u_d}^{u_d} |f(x-t) - f(x)|^p \, dt \right)^{1/p} \qquad (u \in \mathbb{R}^d_+),$$

we conclude

$$2^{-\omega(k_1+\ldots+k_d)} n_1^{d/p} \frac{G(2^{k_1+s}\pi/n_1, \ldots, 2^{k_d+s}\pi/n_1)}{\left((2\pi)^d 2^{sd} 2^{k_1+\ldots+k_d}\right)^{1/p}} \le U^{\omega,1}_{r2^s\pi,p} f(x) \qquad (4.3.6)$$

because $n \in \mathbb{R}^d_\tau$ and so

$$\frac{2^{k_j+s}}{n_1} \le \frac{2^{r_0+s}}{n_1} < r2^s \qquad (j = 1, \ldots, d).$$

Hence

$$\begin{aligned}
&A_1(x) \\
&\le \frac{1}{(2\pi)^d} \sum_{k_1=0}^{r_0} \cdots \sum_{k_d=0}^{r_0} G(2^{k_1+s}\pi/n_1, \ldots, 2^{k_d+s}\pi/n_1) \\
&\quad \left(\prod_{j=1}^d n_j \right)^{-1/q} \left(\int_{Q_{k_1}} \cdots \int_{Q_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n\right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right|^q \, dt \right)^{1/q}
\end{aligned}$$

$$\leq C \sum_{k_1=0}^{r_0} \cdots \sum_{k_d=0}^{r_0} \left(\prod_{j=1}^{d} 2^{k_j(\omega+1/p)} \right) n_1^{-d/p} U_{r2^s\pi,p}^{\omega,1} f(x)$$
$$\left(\prod_{j=1}^{d} n_j \right)^{-1/q} \left(\int_{Q_{k_1}} \cdots \int_{Q_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n \right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right|^q \, dt \right)^{1/q}.$$

From this it follows immediately that

$$A_1(x) \leq C \sum_{k_1=0}^{r_0} \cdots \sum_{k_d=0}^{r_0} \left(\prod_{j=1}^{d} 2^{k_j(\omega+1/p)} \right) U_{r2^s\pi,p}^{\omega,1} f(x)$$
$$\left(\prod_{j=1}^{d} n_j \right)^{-1} \left(\int_{Q_{k_1}} \cdots \int_{Q_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n \right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right|^q \, dt \right)^{1/q}$$
$$\leq C \left\| \widetilde{K}_n \right\|_{E_q^\omega(\mathbb{R}^d)} \epsilon.$$

Similarly to (4.3.5) and (4.3.6), we can see that

$$A_2(x)$$
$$\leq \frac{1}{(2\pi)^d} \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=r_0+1}^{\infty} \cdots \sum_{k_{\pi_j}=r_0+1}^{\infty} \sum_{k_{\pi_{j+1}}=0}^{\infty} \cdots \sum_{k_{\pi_d}=0}^{\infty}$$
$$\left(\int_{-2^{k_1+s}\pi/n_1}^{2^{k_1+s}\pi/n_1} \cdots \int_{-2^{k_d+s}\pi/n_1}^{2^{k_d+s}\pi/n_1} |f(x-t) - f(x)|^p \, dt \right)^{1/p}$$
$$\left(\prod_{j=1}^{d} n_j \right)^{-1/q} \left(\int_{Q_{k_1}} \cdots \int_{Q_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n \right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right|^q \, dt \right)^{1/q}$$

and

$$2^{-\omega(k_1+\ldots+k_d)} n_1^{d/p} \frac{G(2^{k_1+s}\pi/n_1, \ldots, 2^{k_d+s}\pi/n_1)}{\left((2\pi)^d 2^{sd} 2^{k_1+\ldots+k_d} \right)^{1/p}} \leq \mathcal{M}_p^{\omega,1} f(x) + |f(x)|.$$

Since $\mathcal{M}_p^{\omega,1} f(x)$ is finite, we have

$$A_2(x)$$
$$\leq C \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=r_0+1}^{\infty} \cdots \sum_{k_{\pi_j}=r_0+1}^{\infty} \sum_{k_{\pi_{j+1}}=0}^{\infty} \cdots \sum_{k_{\pi_d}=0}^{\infty}$$

$$\left(\prod_{j=1}^{d} 2^{k_j(\omega+1/p)}\right) \left(\mathcal{M}_p^{\omega,1} f(x) + |f(x)|\right)$$

$$\left(\prod_{j=1}^{d} n_j\right)^{-1} \left(\int_{Q_{k_1}} \cdots \int_{Q_{k_d}} \left|\left(1_{(-\pi,\pi)^d} K_n\right)\left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right)\right|^q dt\right)^{1/q}$$

$$\leq C \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=r_0+1}^{\infty} \cdots \sum_{k_{\pi_j}=r_0+1}^{\infty} \sum_{k_{\pi_{j+1}}=0}^{\infty} \cdots \sum_{k_{\pi_d}=0}^{\infty}$$

$$\left(\prod_{j=1}^{d} 2^{k_j(\omega+1/p)}\right) \left(\int_{Q_{k_1}} \cdots \int_{Q_{k_d}} \left|\widetilde{K}_n(t)\right|^q dt\right)^{1/q} \left(\mathcal{M}_p^{\omega,1} f(x) + |f(x)|\right).$$

Since $r_0 \to \infty$ as $n_1 \to \infty$ and (4.3.4) holds, we conclude that

$$\lim_{n\to\infty} A_2(x) = 0,$$

which finishes the proof. ■

Note that (4.3.4) implies

$$\sup_{n\in\mathbb{R}_\tau^d} \left\|\widetilde{K}_n\right\|_{E_q^\omega(\mathbb{R}^d)} \leq C.$$

If $\widetilde{K}_n$ can be estimated by a function $g \in E_q^\omega(\mathbb{R}^d)$ which is independent of n, then (4.3.4) holds. This holds again for the Cesàro kernel $K_n^\alpha = K_{n_1}^\alpha \otimes \cdots \otimes K_{n_d}^\alpha$ and for the Riesz kernel $K_n^{\alpha,\gamma} = K_{n_1}^{\alpha,\gamma} \otimes \cdots \otimes K_{n_d}^{\alpha,\gamma}$. Indeed, taking into account (4.2.5) and (4.2.6), we can see that

$$\min\left\{1, |t|^{-\alpha-1}\right\} \in E_\infty^\omega(\mathbb{R}^d)$$

if $0 < \omega < \alpha \leq 1$ and

$$\min\left\{1, |t|^{-\min(\alpha,1)-1}\right\} \in E_\infty^\omega(\mathbb{R}^d)$$

if $0 < \omega < \min(\alpha, 1)$.

Corollary 4.3.15 *If $0 < \omega < \alpha \leq 1$, $\mathcal{M}^{\omega,1} f(x)$ is finite and x is a $(1, \omega)$-Lebesgue point of $f \in L_1(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}_\tau^d} \sigma_n^\alpha f(x) = f(x).$$

Moreover,

$$\sup_{\rho>0} \rho\lambda(\sigma_\square^\alpha > \rho) \leq C\|f\|_1 \qquad (f \in L_1(\mathbb{T}^d))$$

and, for every $1 < p \leq \infty$,

$$\left\|\sigma_{\square}^{\alpha} f\right\|_{p} \leq C_{p}\|f\|_{p} \qquad (f \in L_{p}(\mathbb{T}^{d})).$$

The same hold for the Riesz summation $\sigma_n^{\alpha,\gamma}$ *if* $0 < \omega < \min(\alpha, 1)$, $0 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$.

The proof of Theorem 4.3.14 shows also

Theorem 4.3.16 *Suppose that* $\omega > 0$, $1 \leq p < \infty$, $1/p + 1/q = 1$, $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ *and* $\widehat{\theta} \in E_q^{\omega}(\mathbb{R}^d)$. *If* $\theta(0) = 1$, $\mathcal{M}_p^{\omega,1} f(x)$ *is finite and* x *is a* (p, ω)*-Lebesgue point of* $f \in L_p(\mathbb{T}^d)$, *then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}_\tau^d} \sigma_n^{\theta} f(x) = f(x).$$

Corollary 4.3.17 *If the conditions of Theorem 4.3.14 or Theorem 4.3.16 are satisfied and if* $f \in L_1(\mathbb{T}^d)$ *is continuous at a point* x, *then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}_\tau^d} \sigma_n^{\theta} f(x) = f(x).$$

Taking into account Theorem 4.2.30, we obtain

Corollary 4.3.18 *Suppose that* $\theta = \theta_1 \otimes \cdots \otimes \theta_d \in W(C, \ell_1)(\mathbb{R}^d)$, $\theta(0) = 1$ *and* $\theta_j \in V_1^2(\mathbb{R})$ *for all* $j = 1, \ldots, d$. *If* $0 < \omega < 1$, $\mathcal{M}^{\omega,1} f(x)$ *is finite and* x *is a* $(1, \omega)$*-Lebesgue point of* $f \in L_1(\mathbb{T}^d)$, *then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}_\tau^d} \sigma_n^{\theta} f(x) = f(x).$$

Moreover,

$$\sup_{\rho>0} \rho\lambda(\sigma_{\square}^{\theta} > \rho) \leq C \left(\prod_{j=1}^{d} \left\|\theta_j\right\|_{E_\infty^{\omega}(\mathbb{R})}\right) \|f\|_1 \qquad (f \in L_1(\mathbb{T}^d))$$

and, for every $1 < p \leq \infty$,

$$\left\|\sigma_{\square}^{\theta} f\right\|_{p} \leq C_p \left(\prod_{j=1}^{d} \left\|\theta_j\right\|_{E_\infty^{\omega}(\mathbb{R})}\right) \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

All examples of Sect. 2.6.3 satisfy the condition $\theta \in V_1^2(\mathbb{R})$. This means that all results of Sect. 4.3.2, especially Corollary 4.3.18 hold if each θ_j denotes either the Cesàro summation or one of the examples of Sect. 2.6.3.

4.4 Restricted Rectangular Summability over a Cone-Like Set

In this section, we investigate the operators

$$\sigma_n f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n(t)\, dt$$

over a cone-like set, i.e., we assume that $n \in \mathbb{R}^d_{\kappa,\tau}$. Recall that $\mathbb{R}^d_{\kappa,\tau}$ was defined in Sect. 3.4 by

$$\mathbb{R}^d_{\kappa,\tau} := \{x \in \mathbb{R}^d_+ : \tau_j^{-1} \kappa_j(n_1) \le n_j \le \tau_j \kappa_j(n_1),\ j = 2, \ldots, d\},$$

where κ_1 is the identity function and, for all $j = 2, \ldots, d$, $\kappa_j : \mathbb{R}_+ \to \mathbb{R}_+$ are strictly increasing and continuous functions such that

$$\lim_{j \to \infty} \kappa_j = \infty \qquad \text{and} \qquad \lim_{j \to +0} \kappa_j = 0.$$

Instead of (3.4.1), we will suppose that there exist $c_j, \xi > 1$ such that

$$\kappa_j(\xi x) = c_j \kappa_j(x) \qquad (x > 0). \tag{4.4.1}$$

The higher dimensional rectangular Cesàro and Riesz kernels, K_n^α and $K_n^{\alpha,\gamma}$ will satisfy again the conditions of this section.

4.4.1 *Hardy-Littlewood Maximal Functions*

We generalize the definition of $\mathcal{M}_p^{\omega,1} f$ as follows.

Definition 4.4.1 For $\omega > 0$, $1 \le p < \infty$ and $f \in L_p(\mathbb{T}^d)$, the Hardy-Littlewood maximal function $\mathcal{M}_p^{\kappa,\omega} f$ is given by

$$\mathcal{M}_p^{\kappa,\omega} f(x) := \sup_{i \in \mathbb{N}^d, h > 0} \left(\prod_{j=1}^d \kappa_j(\xi^{i_j})^{-\omega} \right) \left(\frac{1}{\prod_{j=1}^d (2\kappa_j(\xi^{i_j} h)\pi)} \int_{-\kappa_1(\xi^{i_1} h)\pi}^{\kappa_1(\xi^{i_1} h)\pi} \cdots \int_{-\kappa_d(\xi^{i_d} h)\pi}^{\kappa_d(\xi^{i_d} h)\pi} |f(x-t)|^p\, dt \right)^{1/p}.$$

We show that the operator $\mathcal{M}_p^{\kappa,\omega}$ is of weak type (p, p) as was $\mathcal{M}_p^{\omega,1}$.

Theorem 4.4.2 *If* $1 \le p < \infty$*, then*

$$\sup_{\rho>0} \rho\lambda(\mathcal{M}_p^{\kappa,\omega} f > \rho)^{1/p} \le C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Moreover, if $p < r \le \infty$, then

$$\left\|\mathcal{M}_p^{\kappa,\omega} f\right\|_r \le C_r \|f\|_r \qquad (f \in L_r(\mathbb{R}^d)).$$

Proof Choosing

$$\mu(h) := \kappa_1(\xi^{i_1} h), \qquad \nu(h) := \kappa_2(\xi^{i_2} h)$$

in the definition of $M_p^{1,\mu,\nu} f$ (see Theorem 4.3.2), we get that

$$\begin{aligned}
&\rho^p \lambda(\mathcal{M}_p^{\kappa,\omega} f > \rho) \\
&\le \rho^p \lambda\Bigg(\bigcup_{i_1,i_2=0}^{\infty} \left(\prod_{j=1}^{2} \kappa_j(\xi^{i_j})^{-\omega} \right) \sup_{h>0} \Bigg(\frac{1}{\prod_{j=1}^{2} (2\kappa_j(\xi^{i_j} h))} \\
&\qquad \int_{-\kappa_1(\xi^{i_1} h)}^{\kappa_1(\xi^{i_1} h)} \int_{-\kappa_2(\xi^{i_2} h)}^{\kappa_2(\xi^{i_2} h)} |f(x_1 - t_1, x_2 - t_2)|^p \, dt \Bigg)^{1/p} > \rho \Bigg) \\
&\le \rho^p \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \lambda \left(M_p^{1,\mu,\nu} f > \left(\prod_{j=1}^{2} \kappa_j(\xi^{i_j})^{\omega} \right) \rho \right) \\
&\le C_p \sum_{i \in \mathbb{N}^d} \left(\prod_{j=1}^{2} \kappa_j(\xi^{i_j})^{-\omega p} \right) \|f\|_p^p \\
&\le C_p \sum_{i \in \mathbb{N}^d} \left(\prod_{j=1}^{2} c_j^{-\omega p i_j} \kappa_j(1)^{-\omega p} \right) \|f\|_p^p \le C_p \|f\|_p^p
\end{aligned}$$

because $c_j > 1$ $(j = 1, 2)$. ■

4.4.2 Lebesgue Points for the Summability over a Cone-Like Set

Using the functions κ_j, we modify slightly the norm of the Herz spaces $E_q^{\omega}(\mathbb{R}^d)$.

Definition 4.4.3 For $\omega \ge 0$ and $1 \le q \le \infty$, the weighted Herz space $E_q^{\kappa,\omega}(\mathbb{R}^d)$ contains all functions f for which

$$\|f\|_{E_q^{\kappa,\omega}} := \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \left(\prod_{j=1}^{d} \kappa_j(\xi^{k_j})^{\omega+1-1/q} \right) \left\| f 1_{Q_k} \right\|_q < \infty,$$

where

$$Q_k := Q_{1,k_1} \times \cdots \times Q_{d,k_d} \qquad (k \in \mathbb{N}^d),$$

$$Q_{j,0} := Q_{j,0}^{\kappa_j} := \big(-\kappa_j(1)\pi, \kappa_j(1)\pi\big) \qquad (j = 1, \ldots, d)$$

and

$$Q_{j,i} := Q_{j,i}^{\kappa_j} := \{x \in \mathbb{R} : \kappa_j(\xi^{i-1})\pi \le |x| < \kappa_j(\xi^i)\pi\} \qquad (i \in \mathbb{N}_+).$$

However, these spaces are equivalent to the Herz spaces $E_q^\omega(\mathbb{R}^d)$ studied in Sect. 4.3.2.

Theorem 4.4.4 *The spaces $E_q^\omega(\mathbb{R}^d)$ and $E_q^{\kappa,\omega}(\mathbb{R}^d)$ are equivalent for all $1 \le q \le \infty, \omega \ge 0$.*

Proof It is enough to show the result for one dimension. Then we denote the function κ_j and the corresponding constant c_j simply by κ and c, and the sets $Q_{j,k}^{\kappa_j}$ by Q_k^κ. For a fixed k, let ν be the smallest natural number l for which $\kappa(\xi^l) = c^l\kappa(1) \ge 2^k$ and μ be the largest natural number l for which $\kappa(\xi^l) \le 2^{k-1}$. Then

$$2^{k(\omega+1-1/q)} \left\| f 1_{Q_k} \right\|_q \le \sum_{j=\mu+1}^{\nu} \kappa(\xi^j)^{\omega+1-1/q} \left\| f 1_{Q_j^\kappa} \right\|_q,$$

which means that

$$\|f\|_{E_q^\omega} \le C \|f\|_{E_q^{\kappa,\omega}}.$$

The other side of the equivalence can be shown in the same way. ■

We will investigate a restricted maximal operator depending on the cone-like set:

$$\sigma_\kappa f := \sup_{n \in \mathbb{R}_{\kappa,\tau}^d} |\sigma_n f|.$$

Theorem 4.4.5 *If $\omega \ge 0$, $1 \le p < \infty$, $1/p + 1/q = 1$ and*

$$\sup_{n \in \mathbb{R}_{\kappa,\tau}^d} \left\| \widetilde{K}_n \right\|_{E_q^\omega(\mathbb{R}^d)} \le C,$$

then

$$\sigma_\kappa f(x) \le C \left(\sup_{n \in \mathbb{R}_{\kappa,\tau}^d} \left\| \widetilde{K}_n \right\|_{E_q^\omega(\mathbb{R}^d)} \right) \mathcal{M}_p^{\kappa,\omega} f(x)$$

for all $f \in L_p(\mathbb{T}^d)$ and $x \in \mathbb{T}^d$.

Proof Obviously,

$$
\begin{aligned}
|\sigma_n f(x)| &= \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} f(x-t) \left(1_{(-\pi,\pi)^d} K_n\right)(t)\, dt \right| \\
&\le \frac{1}{(2\pi)^d} \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \\
&\qquad \int_{Q_{1,k_1}(n_1)} \cdots \int_{Q_{d,k_d}(n_1)} |f(x-t)| \left| \left(1_{(-\pi,\pi)^d} K_n\right)(t) \right| dt,
\end{aligned}
$$

where

$$
Q_{j,0}(n_1) := \left(-\kappa_j(1/n_1)\pi, \kappa_j(1/n_1)\pi\right)
$$

and

$$
Q_{j,i}(n_1) := \{x \in \mathbb{R} : \kappa_j(\xi^{i-1}/n_1)\pi \le |x| < \kappa_j(\xi^i/n_1)\pi\} \qquad (i \in \mathbb{N}_+).
$$

By Hölder's inequality,

$$
\begin{aligned}
&|\sigma_n f(x)| \\
&\le \frac{1}{(2\pi)^d} \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \left(\int_{Q_{1,k_1}(n_1)} \cdots \int_{Q_{d,k_d}(n_1)} |f(x-t)|^p \, dt \right)^{1/p} \\
&\quad \left(\int_{Q_{1,k_1}(n_1)} \cdots \int_{Q_{d,k_d}(n_1)} \left| \left(1_{(-\pi,\pi)^d} K_n\right)(t) \right|^q \, dt \right)^{1/q} \\
&= \frac{1}{(2\pi)^d} \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \left(\int_{Q_{1,k_1}(n_1)} \cdots \int_{Q_{d,k_d}(n_1)} |f(x-t)|^p \, dt \right)^{1/p} \left(\prod_{j=1}^{d} n_j \right)^{-1/q} \\
&\quad \left(\int_{n_1 Q_{1,k_1}(n_1)} \cdots \int_{n_d Q_{d,k_d}(n_1)} \left| \left(1_{(-\pi,\pi)^d} K_n\right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right|^q \, dt \right)^{1/q}. \qquad (4.4.2)
\end{aligned}
$$

For

$$
G(u) := \left(\int_{-u_1}^{u_1} \cdots \int_{-u_d}^{u_d} |f(x-t)|^p \, dt \right)^{1/p} \qquad (u \in \mathbb{R}_+^d),
$$

we have

$$
\left(\prod_{j=1}^{d} \kappa_j(\xi^{k_j})^{-\omega p} \right) \frac{G^p(\kappa_1(\xi^{k_1}/n_1)\pi, \ldots, \kappa_d(\xi^{k_d}/n_1)\pi)}{\prod_{j=1}^{d} (2\kappa_j(\xi^{k_j}/n_1)\pi)} \le (\mathcal{M}_p^{\kappa,\omega})^p f(x).
$$

Thus

$$
\begin{aligned}
&|\sigma_n f(x)| \\
&\le C \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} G(\kappa_1(\xi^{k_1}/n_1)\pi, \ldots, \kappa_d(\xi^{k_d}/n_1)\pi) \left(\prod_{j=1}^{d} n_j \right)^{-1/q} \\
&\quad \left(\int_{n_1 Q_{1,k_1}(n_1)} \cdots \int_{n_d Q_{d,k_d}(n_1)} \left| \left(1_{(-\pi,\pi)^d} K_n\right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right|^q dt \right)^{1/q} \\
&\le C \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \left(\prod_{j=1}^{d} \kappa_j(\xi^{k_j})^{\omega} \right) \left(\prod_{j=1}^{d} \kappa_j(\xi^{k_j}/n_1) \right)^{1/p} \mathcal{M}_p^{\kappa,\omega} f(x) \left(\prod_{j=1}^{d} n_j \right)^{-1/q} \\
&\quad \left(\int_{n_1 Q_{1,k_1}(n_1)} \cdots \int_{n_d Q_{d,k_d}(n_1)} \left| \left(1_{(-\pi,\pi)^d} K_n\right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right|^q dt \right)^{1/q}.
\end{aligned}
$$

If $\xi^l \le n_1 < \xi^{l+1}$ for some $l \in \mathbb{Z}$, then by (4.4.1),

$$
\kappa_j(1) c_j^l = \kappa_j(\xi^l) \le \kappa_j(n_1) \le \kappa_j(\xi^{l+1}) = \kappa_j(1) c_j^{l+1}
$$

and

$$
\kappa_j\left(\frac{\xi^{k_j}}{n_1}\right) \le \kappa_j\left(\frac{\xi^{k_j}}{\xi^l}\right) = \frac{1}{c_j^l} \kappa_j(\xi^{k_j}) \le \kappa_j(1) c_j \frac{\kappa_j(\xi^{k_j})}{\kappa_j(n_1)} = \frac{\kappa_j(1)\kappa_j(\xi^{k_j+1})}{\kappa_j(n_1)}.
$$

Choose integers μ and ν such that

$$
\tau_j \kappa_j(1) \le c_j^{\mu} \qquad \text{and} \qquad \tau_j^{-1} \kappa_j(1) \ge c_j^{\nu}
$$

for all $j = 1, \ldots, d$. Using the definition of the cone-like set, we can see that

$$
\begin{aligned}
n_j \kappa_j\left(\frac{\xi^{k_j}}{n_1}\right) &\le \tau_j \kappa_j(n_1) \kappa_j\left(\frac{\xi^{k_j}}{n_1}\right) \le \tau_j \kappa_j(1) \kappa_j(\xi^{k_j+1}) \\
&\le c_j^{\mu} \kappa_j(\xi^{k_j+1}) = \kappa_j(\xi^{k_j+\mu+1}).
\end{aligned}
$$

On the other hand

$$
\kappa_j\left(\frac{\xi^{k_j}}{n_1}\right) \ge \kappa_j\left(\frac{\xi^{k_j}}{\xi^{l+1}}\right) = \frac{1}{c_j^{l+1}} \kappa_j(\xi^{k_j}) \ge \frac{\kappa_j(1)}{c_j} \frac{\kappa_j(\xi^{k_j})}{\kappa_j(n_1)} = \frac{\kappa_j(1)\kappa_j(\xi^{k_j-1})}{\kappa_j(n_1)}
$$

and

$$n_j \kappa_j \left(\frac{\xi^{k_j}}{n_1}\right) \geq \tau_j^{-1} \kappa_j(n_1) \kappa_j \left(\frac{\xi^{k_j}}{n_1}\right) \geq \tau_j^{-1} \kappa_j(1) \kappa_j(\xi^{k_j-1})$$
$$\geq c_j^{\nu} \kappa_j(\xi^{k_j-1}) = \kappa_j(\xi^{k_j+\nu-1}).$$

Setting

$$Q'_{j,0} := \left(-\kappa_j(\xi^{\mu+1})\pi, \kappa_j(\xi^{\mu+1})\pi\right)$$

and

$$Q'_{j,i} := \{x \in \mathbb{R} : \kappa_j(\xi^{i+\nu-2})\pi \leq |x| < \kappa_j(\xi^{i+\mu+1})\pi\} \qquad (i \in \mathbb{N}_+),$$

we conclude

$$|\sigma_n f(x)| \leq C \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \left(\prod_{j=1}^{d} \kappa_j(\xi^{k_j})^{\omega+1/p}\right)$$
$$\left(\prod_{j=1}^{d} \kappa_j(n_1)\right)^{-1/p} \mathcal{M}_p^{\kappa,\omega} f(x) \left(\prod_{j=1}^{d} n_j\right)^{-1/q}$$
$$\left(\int_{Q'_{1,k_1}} \cdots \int_{Q'_{d,k_d}} \left|\left(1_{(-\pi,\pi)^d} K_n\right)\left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right)\right|^q dt\right)^{1/q}$$
$$\leq C \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \left(\prod_{j=1}^{d} \kappa_j(\xi^{k_j})^{\omega+1/p}\right) \mathcal{M}_p^{\kappa,\omega} f(x)$$
$$\left(\int_{Q'_{1,k_1}} \cdots \int_{Q'_{d,k_d}} \left|\widetilde{K}_n(t)\right|^q dt\right)^{1/q}$$
$$\leq C \left\|\widetilde{K}_n\right\|_{E_q^{\omega}(\mathbb{R}^d)} \mathcal{M}_p^{\kappa,\omega} f(x),$$

which proves the theorem. ■

Theorem 4.4.2 implies

Theorem 4.4.6 *If* $\omega \geq 0$, $1 \leq p < \infty$, $1/p + 1/q = 1$ *and*

$$\sup_{n \in \mathbb{R}^d_{\kappa,\tau}} \left\|\widetilde{K}_n\right\|_{E_q^{\omega}(\mathbb{R}^d)} \leq C,$$

then

$$\sup_{\rho>0} \rho \lambda(\sigma_\kappa f > \rho)^{1/p} \leq C_p \left(\sup_{n \in \mathbb{R}^d_{\kappa,\tau}} \left\|\widetilde{K}_n\right\|_{E_q^{\omega}(\mathbb{R}^d)}\right) \|f\|_p$$

for all $f \in L_p(\mathbb{T}^d)$. *Moreover, for every* $p < r \leq \infty$,

$$\|\sigma_\kappa f\|_r \leq C \left(\sup_{n\in\mathbb{R}^d_{\kappa,\tau}} \left\| \widetilde{K}_n \right\|_{E_q^\omega(\mathbb{R}^d)} \right) \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

Corollary 4.4.7 *Suppose that* $\omega > 0$, $1 \leq p < \infty$, $1/p + 1/q = 1$ *and*

$$\sup_{n\in\mathbb{R}^d_{\kappa,\tau}} \left\| \widetilde{K}_n \right\|_{E_q^\omega(\mathbb{R}^d)} \leq C.$$

If

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_{\kappa,\tau}} \widehat{K_n}(k) = 1$$

for all $k \in \mathbb{Z}^d$*, then*

$$\lim_{n\to\infty,\, \mathbb{R}^d_{\kappa,\tau}} \sigma_n f = f \quad a.e.$$

for all $f \in L_p(\mathbb{T}^d)$.

For rectangular θ-means, we obtain the next theorems in the same way.

Theorem 4.4.8 *Suppose that* $\omega \geq 0$, $1 \leq p < \infty$ *and* $1/p + 1/q = 1$. *If* $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ *and* $\widehat{\theta} \in E_q^\omega(\mathbb{R}^d)$*, then*

$$\sigma_\kappa^\theta f(x) \leq C \left\| \widehat{\theta} \right\|_{E_q^\omega(\mathbb{R}^d)} \mathcal{M}_p^{\kappa,\omega} f(x)$$

for all $f \in L_p(\mathbb{T}^d)$ *and* $x \in \mathbb{T}^d$.

Theorem 4.4.9 *Suppose that* $\omega > 0$, $1 \leq p < \infty$ *and* $1/p + 1/q = 1$. *If* $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ *and* $\widehat{\theta} \in E_q^\omega(\mathbb{R}^d)$*, then*

$$\sup_{\rho>0} \rho \lambda(\sigma_\kappa^\theta f > \rho)^{1/p} \leq C_p \left\| \widehat{\theta} \right\|_{E_q^\omega(\mathbb{R}^d)} \|f\|_p$$

for all $f \in L_p(\mathbb{T}^d)$. *Moreover, for every* $p < r \leq \infty$,

$$\left\| \sigma_\kappa^\theta f \right\|_r \leq C \left\| \widehat{\theta} \right\|_{E_q^\omega(\mathbb{R}^d)} \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

Corollary 4.4.10 *Suppose that* $\omega > 0$, $1 \leq p < \infty$ *and* $1/p + 1/q = 1$. *If* $\theta(0) = 1$, $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ *and* $\widehat{\theta} \in E_q^\omega(\mathbb{R}^d)$*, then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_{\kappa,\tau}} \sigma_n^\theta f = f \quad a.e.$$

for all $f \in L_p(\mathbb{T}^d)$.

We generalize the concept of Lebesgue points as follows. Let

$$U_{r,p}^{\kappa,\omega} f(x) := \sup_{i\in\mathbb{N}^d, h>0, \xi^{i_j}h<r, j=1,\ldots d} \left(\prod_{j=1}^d \kappa_j(\xi^{i_j})^{-\omega}\right) \left(\frac{1}{\prod_{j=1}^d (2\kappa_j(\xi^{i_j}h)\pi)} \int_{-\kappa_1(\xi^{i_1}h)\pi}^{\kappa_1(\xi^{i_1}h)\pi} \cdots \int_{-\kappa_d(\xi^{i_d}h)\pi}^{\kappa_d(\xi^{i_d}h)\pi} |f(x-t)-f(x)|^p \, dt\right)^{1/p}.$$

Definition 4.4.11 For $1 \le p < \infty$ and $\omega > 0$, a point $x \in \mathbb{T}^d$ is called a (p, κ, ω)-Lebesgue point of $f \in L_p(\mathbb{T}^d)$ if

$$\lim_{r\to 0} U_{r,p}^{\kappa,\omega} f(x) = 0.$$

If κ is the identity function, then we get back the (p, ω)-Lebesgue points investigated in the previous section.

Theorem 4.4.12 *For $1 \le p < \infty$ and $\omega > 0$, almost every point $x \in \mathbb{T}^d$ is a (p, κ, ω)-Lebesgue point of $f \in L_p(\mathbb{T}^d)$.*

We omit the proof, since it is similar to that of Theorem 4.3.13. Our basic theorem about the convergence at (p, κ, ω)-Lebesgue points reads as follows.

Theorem 4.4.13 *Suppose that $\omega > 0$, $1 \le p < \infty$, $1/p + 1/q = 1$ and*

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \left(\prod_{j=1}^d \kappa_j(\xi^{k_j})^{\omega+1-1/q}\right) \sup_{n\in\mathbb{R}_{\kappa,\tau}^d} \left\| \widetilde{K}_n 1_{Q_k} \right\|_q \le C. \tag{4.4.3}$$

If

$$\lim_{n\to\infty,\, n\in\mathbb{R}_{\kappa,\tau}^d} \widehat{K_n}(0) = 1,$$

$\mathcal{M}_p^{\kappa,\omega} f(x)$ is finite and x is a (p, κ, ω)-Lebesgue points of $f \in L_p(\mathbb{T}^d)$, then

$$\lim_{n\to\infty,\, n\in\mathbb{R}_{\kappa,\tau}^d} \sigma_n f(x) = f(x).$$

Proof Since x is a (p, κ, ω)-Lebesgue point of f, we can fix a number $r < 1$ such that

$$U_{r,p}^{\kappa,\omega} f(x) < \epsilon.$$

Let us denote by r_0 the largest number i, for which $r/\xi \le \xi^i/n_1 < r$. We use again the decomposition

$$|\sigma_n f(x) - f(x)| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |f(x-t) - f(x)| \left| \left(1_{(-\pi,\pi)^d} K_n\right)(t) \right| dt + \left| f(x) \left(\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K_n(t)\, dt - 1 \right) \right| = A_1(x) + A_2(x) + A_3(x),$$

where

$$A_1(x) := \frac{1}{(2\pi)^d} \sum_{k_1=0}^{r_0} \cdots \sum_{k_d=0}^{r_0} \int_{Q_{1,k_1}(n_1)} \cdots \int_{Q_{d,k_d}(n_1)} |f(x-t) - f(x)| \left| \left(1_{(-\pi,\pi)^d} K_n\right)(t) \right| dt,$$

$$A_2(x) := \frac{1}{(2\pi)^d} \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=r_0+1}^{\infty} \cdots \sum_{k_{\pi_j}=r_0+1}^{\infty} \sum_{k_{\pi_{j+1}}=0}^{\infty} \cdots \sum_{k_{\pi_d}=0}^{\infty} \int_{Q_{1,k_1}(n_1)} \cdots \int_{Q_{d,k_d}(n_1)} |f(x-t) - f(x)| \left| \left(1_{(-\pi,\pi)^d} K_n\right)(t) \right| dt,$$

and

$$A_3(x) := \left| f(x) \left(\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K_n(t)\, dt - 1 \right) \right| = \left| f(x) \left(\widehat{K_n}(0) - 1 \right) \right|.$$

$\{\pi_1, \ldots, \pi_d\}$ denotes again a permutation of $\{1, \ldots, d\}$ and $1 \leq j \leq d$. Obviously,

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_\tau} A_3(x) = 0.$$

Taking into account (4.4.2), we can see that

$$A_1(x) \leq \frac{1}{(2\pi)^d} \sum_{k_1=0}^{r_0} \cdots \sum_{k_d=0}^{r_0} \left(\int_{Q_{1,k_1}(n_1)} \cdots \int_{Q_{d,k_d}(n_1)} |f(x-t) - f(x)|^p\, dt \right)^{1/p} \left(\prod_{j=1}^d n_j \right)^{-1/q} \left(\int_{n_1 Q_{1,k_1}(n_1)} \cdots \int_{n_d Q_{d,k_d}(n_1)} \left| \left(1_{(-\pi,\pi)^d} K_n\right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d} \right) \right|^q dt \right)^{1/q}$$

$$\leq C \sum_{k_1=0}^{r_0} \cdots \sum_{k_d=0}^{r_0} G(\kappa_1(\xi^{k_1}/n_1)\pi, \ldots, \kappa_d(\xi^{k_d}/n_1)\pi) \left(\prod_{j=1}^{d} n_j\right)^{-1/q}$$

$$\left(\int_{n_1 Q_{1,k_1}(n_1)} \cdots \int_{n_d Q_{d,k_d}(n_1)} \left| \left(1_{(-\pi,\pi)^d} K_n\right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right) \right|^q dt \right)^{1/q}, \qquad (4.4.4)$$

where

$$G(u) := \left(\int_{-u_1}^{u_1} \ldots \int_{-u_d}^{u_d} |f(x-t) - f(x)|^p \, dt\right)^{1/p} \qquad (u \in \mathbb{R}_+^d).$$

It comes from the definition of $U_{r,p}^{\kappa,\omega}$ that

$$\left(\prod_{j=1}^{d} \kappa_j(\xi^{k_j})^{-\omega}\right) \frac{G(\kappa_1(\xi^{k_1}/n_1)\pi, \ldots, \kappa_d(\xi^{k_d}/n_1)\pi)}{\left(\prod_{j=1}^{d}(2\kappa_j(\xi^{k_j}/n_1)\pi)\right)^{1/p}} \leq U_{r,p}^{\kappa,\omega} f(x), \qquad (4.4.5)$$

which implies

$$A_1(x)$$

$$\leq C \sum_{k_1=0}^{r_0} \cdots \sum_{k_d=0}^{r_0} \left(\prod_{j=1}^{d} \kappa_j(\xi^{k_j})^{\omega}\right) \left(\prod_{j=1}^{d} \kappa_j(\xi^{k_j}/n_1)\right)^{1/p} U_{r,p}^{\kappa,\omega} f(x) \left(\prod_{j=1}^{d} n_j\right)^{-1/q}$$

$$\left(\int_{n_1 Q_{1,k_1}(n_1)} \cdots \int_{n_d Q_{d,k_d}(n_1)} \left| \left(1_{(-\pi,\pi)^d} K_n\right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right) \right|^q dt \right)^{1/q}.$$

As in the proof of Theorem 4.4.5,

$$A_1(x) \leq C \sum_{k_1=0}^{r_0} \cdots \sum_{k_d=0}^{r_0} \left(\prod_{j=1}^{d} \kappa_j(\xi^{k_j})^{\omega+1/p}\right)$$

$$\left(\prod_{j=1}^{d} \kappa_j(n_1)\right)^{-1/p} U_{r,p}^{\kappa,\omega} f(x) \left(\prod_{j=1}^{d} n_j\right)^{-1/q}$$

$$\left(\int_{Q'_{1,k_1}} \cdots \int_{Q'_{d,k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n\right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right) \right|^q dt \right)^{1/q}$$

$$\leq C \sum_{k_1=0}^{r_0} \cdots \sum_{k_d=0}^{r_0} \left(\prod_{j=1}^{d} \kappa_j(\xi^{k_j})^{\omega+1/p}\right) U_{r,p}^{\kappa,\omega} f(x)$$

$$\left(\int_{Q'_{1,k_1}} \cdots \int_{Q'_{d,k_d}} \left|\widetilde{K}_n(t)\right|^q \, dt\right)^{1/q}$$
$$\leq C\epsilon \left\|\widetilde{K}_n\right\|_{E_q^\omega(\mathbb{R}^d)}.$$

In the same way as in (4.4.4), we get that

$$\begin{aligned}
&A_2(x) \\
&\leq C \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=r_0+1}^{\infty} \cdots \sum_{k_{\pi_j}=r_0+1}^{\infty} \sum_{k_{\pi_{j+1}}=0}^{\infty} \cdots \sum_{k_{\pi_d}=0}^{\infty} \\
&\quad G(\kappa_1(\xi^{k_1}/n_1)\pi, \ldots, \kappa_d(\xi^{k_d}/n_1)\pi) \left(\prod_{j=1}^d n_j\right)^{-1/q} \\
&\quad \left(\int_{n_1 Q_{1,k_1}(n_1)} \cdots \int_{n_d Q_{d,k_d}(n_1)} \left|\left(1_{(-\pi,\pi)^d} K_n\right)\left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right)\right|^q \, dt\right)^{1/q}.
\end{aligned}$$

Besides (4.4.5), we know also the inequality

$$\left(\prod_{j=1}^d \kappa_j(\xi^{k_j})^{-\omega}\right) \frac{G(\kappa_1(\xi^{k_1}/n_1)\pi, \ldots, \kappa_d(\xi^{k_d}/n_1)\pi)}{\left(\prod_{j=1}^d (2\kappa_j(\xi^{k_j}/n_1)\pi)\right)^{1/p}} \leq \mathcal{M}_p^{\kappa,\omega} f(x) + C|f(x)|.$$

As above, we get that

$$\begin{aligned}
&A_2(x) \\
&\leq C \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=r_0+1}^{\infty} \cdots \sum_{k_{\pi_j}=r_0+1}^{\infty} \sum_{k_{\pi_{j+1}}=0}^{\infty} \cdots \sum_{k_{\pi_d}=0}^{\infty} \\
&\quad \left(\prod_{j=1}^d \kappa_j(\xi^{k_j})^{\omega}\right) \left(\prod_{j=1}^d \kappa_j(\xi^{k_j}/n_1)\right)^{1/p} \left(\mathcal{M}_p^{\kappa,\omega} f(x) + |f(x)|\right) \left(\prod_{j=1}^d n_j\right)^{-1/q} \\
&\quad \left(\int_{n_1 Q_{1,k_1}(n_1)} \cdots \int_{n_d Q_{d,k_d}(n_1)} \left|\left(1_{(-\pi,\pi)^d} K_n\right)\left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right)\right|^q \, dt\right)^{1/q} \\
&\leq C \left(\mathcal{M}_p^{\kappa,\omega} f(x) + C|f(x)|\right) \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=r_0+1}^{\infty} \cdots \sum_{k_{\pi_j}=r_0+1}^{\infty} \sum_{k_{\pi_{j+1}}=0}^{\infty} \cdots \sum_{k_{\pi_d}=0}^{\infty} \\
&\quad \left(\prod_{j=1}^d \kappa_j(\xi^{k_j})^{\omega+1/p}\right) \left(\int_{Q'_{1,k_1}} \cdots \int_{Q'_{d,k_d}} \left|\widetilde{K}_n(t)\right|^q \, dt\right)^{1/q}.
\end{aligned}$$

Since $r_0 \to \infty$ as $n_1 \to \infty$, we deduce that

$$\lim_{n\to\infty} A_2(x) = 0.$$

The proof of the theorem is complete. ■

Obviously, (4.4.3) implies

$$\sup_{n\in\mathbb{R}^d_{\kappa,\tau}} \left\| \widetilde{K}_n \right\|_{E_q^\omega(\mathbb{R}^d)} \le C.$$

Since we basically work with the $E_q^\omega(\mathbb{R}^d)$ space, our results can be applied to all examples of Sect. 4.3.2, amongst others, to the Cesàro and Riesz summability.

Corollary 4.4.14 *If* $0 < \omega < \alpha \le 1$, $\mathcal{M}^{\kappa,\omega} f(x)$ *is finite and* x *is a* $(1, \kappa, \omega)$-*Lebesgue point of* $f \in L_1(\mathbb{T}^d)$, *then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_{\kappa,\tau}} \sigma_n^\alpha f(x) = f(x).$$

Moreover,

$$\sup_{\rho>0} \rho\lambda(\sigma_\kappa^\alpha > \rho) \le C\|f\|_1 \qquad (f \in L_1(\mathbb{T}^d))$$

and, for every $1 < p \le \infty$,

$$\left\|\sigma_\kappa^\alpha f\right\|_p \le C_p\|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

The same hold for the Riesz summation $\sigma_n^{\alpha,\gamma}$ *if* $0 < \omega < \min(\alpha, 1)$, $0 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$.

Theorem 4.4.15 *Suppose that* $\omega > 0$, $1 \le p < \infty$, $1/p + 1/q = 1$, $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ *and* $\widehat{\theta} \in E_q^\omega(\mathbb{R}^d)$. *If* $\theta(0) = 1$, $\mathcal{M}_p^{\kappa,\omega} f(x)$ *is finite and* x *is a* (p, κ, ω)-*Lebesgue point of* $f \in L_p(\mathbb{T}^d)$, *then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_{\kappa,\tau}} \sigma_n^\theta f(x) = f(x).$$

This theorem can be proved exactly as Theorem 4.4.13.

Corollary 4.4.16 *If the conditions of Theorem 4.4.13 or Theorem 4.4.15 are satisfied and if* $f \in L_1(\mathbb{T}^d)$ *is continuous at a point* x, *then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_{\kappa,\tau}} \sigma_n^\theta f(x) = f(x).$$

Corollary 4.4.17 *Suppose that $\theta = \theta_1 \otimes \cdots \otimes \theta_d \in W(C, \ell_1)(\mathbb{R}^d)$, $\theta(0) = 1$ and $\theta_j \in V_1^2(\mathbb{R})$ for all $j = 1, \ldots, d$. If $0 < \omega < 1$, $\mathcal{M}^{\kappa,\omega} f(x)$ is finite and x is a $(1, \kappa, \omega)$-Lebesgue point of $f \in L_1(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_{\kappa,\tau}} \sigma_n^\theta f(x) = f(x).$$

Moreover,

$$\sup_{\rho>0} \rho\lambda(\sigma_\kappa^\theta > \rho) \le C \left(\prod_{j=1}^d \left\|\theta_j\right\|_{E_\infty^\omega(\mathbb{R})}\right) \|f\|_1 \qquad (f \in L_1(\mathbb{T}^d))$$

and, for every $1 < p \le \infty$,

$$\left\|\sigma_\kappa^\theta f\right\|_p \le C_p \left(\prod_{j=1}^d \left\|\theta_j\right\|_{E_\infty^\omega(\mathbb{R})}\right) \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

We note again that all examples of Sect. 2.6.3 satisfy the condition $\theta \in V_1^2(\mathbb{R})$.

4.5 ℓ_∞-Summability

Now we consider Lebesgue points for the ℓ_∞-summability. We study the ℓ_∞-Cesàro means

$$\sigma_n^{\infty,\alpha} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n^{\infty,\alpha}(t)\, dt \qquad (n \in \mathbb{N})$$

and the ℓ_∞-θ-means $\sigma_n^{\infty,\theta} f$. Recall that the Cesàro kernel $K_n^{\infty,\alpha}$ was defined by

$$K_n^{\infty,\alpha}(t) := \frac{1}{A_{n-1}^\alpha} \sum_{k\in\mathbb{Z}^d,\, \|k\|_\infty \le n} A_{n-1-\|k\|_\infty}^\alpha e^{\imath k\cdot t}.$$

In this section, we cannot use the concept of Herz spaces, we will use other ideas.

4.5.1 Hardy-Littlewood Maximal Functions

In this section, we are going to generalize the maximal operator $\mathcal{M}_p^{\omega,1} f$ investigated in Sect. 4.3. Under a diagonal, we understand a diagonal of the cube $[0, \pi]^d$. Let us denote by $P_{2^{i_1}h,\ldots,2^{i_d}h}$ a parallelepiped, whose center is the origin and whose sides are parallel to the axes and/or to the diagonals and whose kth side length is

$2^{i_k+1}h$ if the kth side is parallel to an axis and $\sqrt{2}2^{i_k+1}h$ if the kth side is parallel to a diagonal ($i \in \mathbb{N}^d$, $h > 0$, $k = 1, \ldots, d$). More exactly, at least one side of $P_{2^{i_1}h,\ldots,2^{i_d}h}$ is parallel to one of the axes and the other sides are parallel to the axes and/or to the diagonals.

Definition 4.5.1 For $\omega > 0$, $1 \le p < \infty$ and $f \in L_p(\mathbb{T}^d)$, the Hardy-Littlewood maximal function $\mathcal{M}_p^\omega f$ is given by

$$\mathcal{M}_p^\omega f(x) := \sup_{P_{2^{i_1}h,\ldots,2^{i_d}h}, i\in\mathbb{N}^d, h>0} 2^{-\omega\|i\|_1} \left(\frac{1}{\left|P_{2^{i_1}h,\ldots,2^{i_d}h}\right|} \int_{P_{2^{i_1}h,\ldots,2^{i_d}h}} |f(x-t)|^p \, dt \right)^{1/p},$$

where the supremum is taken over all parallelepipeds $P_{2^{i_1}h,\ldots,2^{i_d}h}$ ($i \in \mathbb{N}^d$, $h > 0$) just defined.

For $p = 1$, we use the notation $\mathcal{M}^\omega f$. Obviously,

$$\mathcal{M}_p^{\omega_1} f \le \mathcal{M}_p^{\omega_2} f \qquad \text{for } \omega_1 > \omega_2 > 0 \text{ and } 1 \le p < \infty.$$

It is easy to see that

$$\mathcal{M}_p^\omega f(x) \ge \sup_{i\in\mathbb{N}^d, h>0} 2^{-\omega\|i\|_1} \left(\frac{1}{(2h)^d 2^{\|i\|_1}} \int_{-2^{i_1}h}^{2^{i_1}h} \int_{\delta_1 t_1 - 2^{i_2}h}^{\delta_1 t_1 + 2^{i_2}h} \cdots \int_{\delta_{d-1}(t_1-t_2-\cdots-t_{d-1})-2^{i_d}h}^{\delta_{d-1}(t_1-t_2-\cdots-t_{d-1})+2^{i_d}h} |f(x-t)|^p \, dt \right)^{1/p},$$

where $\delta_i \in \{0, 1\}$ ($i = 1, \ldots, d$). If we take the supremum only over all rectangles with sides parallel to the axes, we get back the definition of the maximal operator $\mathcal{M}_p^{\omega,1} f$ from Sect. 4.3.1. Thus

$$\mathcal{M}_p^\omega f \ge \mathcal{M}_p^{\omega,1} f.$$

In the two-dimensional case, besides $\mathcal{M}_p^{\omega,1} f$ defined in Sect. 4.3.1, we introduce

$$\mathcal{M}_p^{\omega,2} f(x_1, x_2) := \sup_{i_1,i_2\in\mathbb{N}, h>0} 2^{-\omega(i_1+i_2)} \left(\frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_1}h}^{2^{i_1}h} \int_{t_1-2^{i_2}h}^{t_1+2^{i_2}h} |f(x_1-t_1, x_2-t_2)|^p \, dt_2 \, dt_1 \right)^{1/p},$$

$$\mathcal{M}_p^{\omega,3} f(x_1,x_2) := \sup_{i_1,i_2\in\mathbb{N},h>0} 2^{-\omega(i_1+i_2)} \bigg(\frac{1}{4\cdot 2^{i_1+i_2}h^2} \int_{-2^{i_1}h}^{2^{i_1}h} \int_{-t_1-2^{i_2}h}^{-t_1+2^{i_2}h} |f(x_1-t_1,x_2-t_2)|^p \, dt_2\, dt_1 \bigg)^{1/p},$$

$$\mathcal{M}_p^{\omega,4} f(x_1,x_2) := \sup_{i_1,i_2\in\mathbb{N},h>0} 2^{-\omega(i_1+i_2)} \bigg(\frac{1}{4\cdot 2^{i_1+i_2}h^2} \int_{-2^{i_2}h}^{2^{i_2}h} \int_{t_2-2^{i_1}h}^{t_2+2^{i_1}h} |f(x_1-t_1,x_2-t_2)|^p \, dt_1\, dt_2 \bigg)^{1/p}$$

as well as

$$\mathcal{M}_p^{\omega,5} f(x_1,x_2) := \sup_{i_1,i_2\in\mathbb{N},h>0} 2^{-\omega(i_1+i_2)} \bigg(\frac{1}{4\cdot 2^{i_1+i_2}h^2} \int_{-2^{i_2}h}^{2^{i_2}h} \int_{-t_2-2^{i_1}h}^{-t_2+2^{i_1}h} |f(x_1-t_1,x_2-t_2)|^p \, dt_1\, dt_2 \bigg)^{1/p}.$$

Note that in $\mathcal{M}_p^{\omega,1} f$, we take the supremum over rectangles with sides parallel to the axes and in $\mathcal{M}_p^{\omega,j} f$ $(j=2,3,4,5)$, over parallelograms with at most one side parallel to one of the axes and with the other sides parallel to the diagonals of the square $[0,\pi]^2$. Then we have

$$\mathcal{M}_p^{\omega} f(x_1,x_2) = \sum_{j=1}^{5} \mathcal{M}_p^{\omega,j} f(x_1,x_2)$$

for all $\omega > 0$ and $1 \le p < \infty$. Similarly to $M_p^{1,\mu,\nu} f$, we introduce also

$$M_p^{2,\mu,\nu} f(x_1,x_2) := \sup_{h>0} \bigg(\frac{1}{4\mu(h)\nu(h)} \int_{-\mu(h)}^{\mu(h)} \int_{t_1-\nu(h)}^{t_1+\nu(h)} |f(x_1-t_1,x_2-t_2)|^p \, dt_2\, dt_1 \bigg)^{1/p},$$

$$M_p^{3,\mu,\nu} f(x_1,x_2) := \sup_{h>0} \bigg(\frac{1}{4\mu(h)\nu(h)} \int_{-\mu(h)}^{\mu(h)} \int_{-t_1-\nu(h)}^{-t_1+\nu(h)} |f(x_1-t_1,x_2-t_2)|^p \, dt_2\, dt_1 \bigg)^{1/p},$$

$$M_p^{4,\mu,\nu} f(x_1, x_2) := \sup_{h>0} \left(\frac{1}{4\mu(h)\nu(h)} \int_{-\nu(h)}^{\nu(h)} \int_{t_2-\mu(h)}^{t_2+\mu(h)} |f(x_1 - t_1, x_2 - t_2)|^p \, dt_1 \, dt_2 \right)^{1/p}$$

and

$$M_p^{5,\mu,\nu} f(x_1, x_2) := \sup_{h>0} \left(\frac{1}{4\mu(h)\nu(h)} \int_{-\nu(h)}^{\nu(h)} \int_{-t_2-\mu(h)}^{-t_2+\mu(h)} |f(x_1 - t_1, x_2 - t_2)|^p \, dt_1 \, dt_2 \right)^{1/p}.$$

Recall that $\mu(h)$ and $\nu(h)$ are two continuous functions of $h \geq 0$, strictly increasing to ∞ and 0 at $h = 0$. The next two theorems can be proved in the same way as in Sect. 4.3.1.

Theorem 4.5.2 *If $j = 1, \ldots, 5$ and $1 \leq p < \infty$, then*

$$\sup_{\rho>0} \rho \lambda (M_p^{j,\mu,\nu} f > \rho)^{1/p} \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Moreover, if $p < r \leq \infty$, then

$$\left\| M_p^{j,\mu,\nu} f \right\|_r \leq C_r \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)),$$

where the constants C_p and C_r are independent of μ and ν.

Theorem 4.5.3 *If $\omega > 0$ and $1 \leq p < \infty$, then*

$$\sup_{\rho>0} \rho \lambda (\mathcal{M}_p^{\omega} f > \rho)^{1/p} \leq C \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Moreover, if $p < r \leq \infty$, then

$$\left\| \mathcal{M}_p^{\omega} f \right\|_r \leq C_r \|f\|_r \qquad (f \in L_r(\mathbb{T}^d).$$

4.5.2 Lebesgue Points for the ℓ_∞-Summability

Here we introduce a stronger version of Lebesgue points than the (p, ω)-Lebesgue points. Similarly to Sect. 4.3.2, let

$$U^{\omega}_{r,p} f(x) := \sup_{\substack{P_{2^{i_1}h,\ldots,2^{i_d}h}, i\in\mathbb{N}^d, h>0 \\ 2^{i_k}h<r, k=1,\ldots d}} 2^{-\omega\|i\|_1} \left(\frac{1}{\left|P_{2^{i_1}h,\ldots,2^{i_d}h}\right|} \right)^{1/p} \left(\int_{P_{2^{i_1}h,\ldots,2^{i_d}h}} |f(x-t) - f(x)|^p \, dt \right)^{1/p},$$

where the supremum is taken over all parallelepipeds whose center is the origin and whose sides are parallel to the axes and/or to the diagonals as in the definition of $\mathcal{M}^{\omega}_p f$. Obviously,

$$U^{\omega}_{r,p} f(x) \geq \sup_{i\in\mathbb{N}^d, h>0, 2^{i_k}h<r, k=1,\ldots d} 2^{-\omega\|i\|_1} \left(\frac{1}{(2h)^d 2^{\|i\|_1}} \right)^{1/p} \times$$

$$\times \left(\int_{-2^{i_1}h}^{2^{i_1}h} \int_{\delta_1 t_1 - 2^{i_2}h}^{\delta_1 t_1 + 2^{i_2}h} \cdots \int_{\delta_{d-1}(t_1-t_2-\cdots-t_{d-1})-2^{i_d}h}^{\delta_{d-1}(t_1-t_2-\cdots-t_{d-1})+2^{i_d}h} |f(x-t) - f(x)|^p \, dt \right)^{1/p},$$

where $\delta_i = 0, 1$ ($i = 1, \ldots, d-1$). Taking the supremum in the definition of $U^{\omega}_{r,p} f$ over all parallelepipeds whose sides are parallel to the axes, we obtain the definition of $U^{\omega,1}_{r,p} f$ (see Definition 4.3.12). In case $p = 1$, we omit again the notation p and write simply $U^{\omega}_r f$. In the two-dimensional case, similarly to $\mathcal{M}^{\omega,j}_p f$, we can define $U^{\omega,j}_{r,p} f$ for $j = 2, 3, 4, 5$ as follows:

$$U^{\omega,2}_{r,p} f(x_1, x_2) := \sup_{i_1,i_2\in\mathbb{N}, h>0, 2^{i_k}h<r, k=1,2} 2^{-\omega(i_1+i_2)} \Bigg(\frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_1}h}^{2^{i_1}h} \int_{t_1-2^{i_2}h}^{t_1+2^{i_2}h} |f(x_1-t_1, x_2-t_2) - f(x_1, x_2)|^p \, dt_2 \, dt_1 \Bigg)^{1/p},$$

$$U^{\omega,3}_{r,p}(x_1, x_2) := \sup_{i_1,i_2\in\mathbb{N}, h>0, 2^{i_k}h<r, k=1,2} 2^{-\omega(i_1+i_2)} \Bigg(\frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_1}h}^{2^{i_1}h} \int_{-t_1-2^{i_2}h}^{-t_1+2^{i_2}h} |f(x_1-t_1, x_2-t_2) - f(x_1, x_2)|^p \, dt_2 \, dt_1 \Bigg)^{1/p},$$

$$U^{\omega,4}_{r,p} f(x_1, x_2) := \sup_{i_1,i_2\in\mathbb{N}, h>0, 2^{i_k}h<r, k=1,2} 2^{-\omega(i_1+i_2)} \Bigg(\frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_2}h}^{2^{i_2}h} \int_{t_2-2^{i_1}h}^{t_2+2^{i_1}h} |f(x_1-t_1, x_2-t_2) - f(x_1, x_2)|^p \, dt_1 \, dt_2 \Bigg)^{1/p}$$

and

$$U_{r,p}^{\omega,5} f(x_1, x_2) := \sup_{i_1, i_2 \in \mathbb{N}, h>0, 2^{i_k} h < r, k=1,2} 2^{-\omega(i_1+i_2)} \left(\frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_2} h}^{2^{i_2} h} \int_{-t_2-2^{i_1} h}^{-t_2+2^{i_1} h} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)|^p \, dt_1 \, dt_2 \right)^{1/p}.$$

Definition 4.5.4 For $1 \le p < \infty$ and $\omega > 0$, a point $x \in \mathbb{T}^d$ is called a strong (p, ω)-Lebesgue point of $f \in L_p(\mathbb{T}^d)$ if

$$\lim_{r \to 0} U_{r,p}^{\omega} f(x) = 0.$$

Recall that $x \in \mathbb{T}^d$ a (p, ω)-Lebesgue point of $f \in L_p(\mathbb{T}^d)$ if

$$\lim_{r \to 0} U_{r,p}^{\omega,1} f(x) = 0.$$

Since

$$U_{r,p}^{\omega,1} f \le U_{r,p}^{\omega} f \qquad (1 \le p < \infty, 0 < r < \infty),$$

Definition 4.5.4 is indeed stronger than the definition of (p, ω)-Lebesgue points. Note that every strong (p, ω_2)-Lebesgue point is a strong (p, ω_1)-Lebesgue point $(0 < \omega_2 < \omega_1 < \infty)$, because of

$$U_{r,p}^{\omega_1} f \le U_{r,p}^{\omega_2} f \qquad (0 < \omega_2 < \omega_1 < \infty, 1 \le p < \infty).$$

Moreover, if $p < r$, then every strong (r, ω)-Lebesgue point is a strong (p, ω)-Lebesgue point. If f is continuous at x, then x is a strong (p, ω)-Lebesgue point of f for all $1 \le p < \infty$ and $\omega > 0$. The proof of the next result is the same as that of Theorem 4.3.13.

Theorem 4.5.5 *For $1 \le p < \infty$ and $\omega > 0$, almost every point $x \in \mathbb{T}^d$ is a strong (p, ω)-Lebesgue point of $f \in L_p(\mathbb{T}^d)$.*

To be able to prove the main theorem of this section, we need the next lemma.

Lemma 4.5.6 *Suppose that $0 < \alpha \le 1$, $x \in \mathbb{T}^2$ and $\pi > x_1 > x_2 > 0$. Then*

$$\left| K_n^{\infty,\alpha}(x_1, x_2) \right| \le C n^2 \tag{4.5.1}$$

and

$$\left| K_n^{\infty,\alpha}(x_1, x_2) \right| \le C x_1^{-1} x_2^{-1}. \tag{4.5.2}$$

Moreover, if $x_1 - x_2 > 1/n$, then

$$\left|K_n^{\infty,\alpha}(x_1,x_2)\right| \le Cn^{-\alpha}x_1^{-1}x_2^{-1}(x_1-x_2)^{-\alpha} \tag{4.5.3}$$

and

$$\left|K_n^{\infty,\alpha}(x_1,x_2)\right| \le Cn^{1-\alpha}x_1^{-1}(x_1-x_2)^{-\alpha}. \tag{4.5.4}$$

These inequalities come easily from Lemma 2.2.19.

Theorem 4.5.7 *If* $0<\alpha<\infty$, $0<\omega<\min(\alpha,1)/d$, $\mathcal{M}^\omega f(x)$ *is finite and* x *is a strong* $(1,\omega)$*-Lebesgue point of* $f\in L_1(\mathbb{T}^d)$*, then*

$$\lim_{n\to\infty}\sigma_n^{\infty,\alpha}f(x)=f(x).$$

Proof By Lemma 2.2.8, we have to prove the theorem for $0<\alpha\le 1$. Let $0<\omega<\alpha/2$. Since (x_1,x_2) is a strong $(1,\omega)$-Lebesgue point of f, we can fix a number $r<1$ such that

$$U_r^\omega f(x_1,x_2)<\epsilon.$$

Let us denote the square $[0,r/2]\times[0,r/2]$ by $S_{r/2}$ and let $2/n<r/2$.

Since

$$\int_{\mathbb{T}^2}K_n^{\infty,\alpha}(x_1,x_2)\,dx=(2\pi)^2,$$

we have

$$\begin{aligned}&\left|\sigma_n^{\infty,\alpha}f(x_1,x_2)-f(x_1,x_2)\right|\\ &\quad\le\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}|f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|\left|K_n^{\infty,\alpha}(t_1,t_2)\right|dt.\end{aligned} \tag{4.5.5}$$

It is enough to integrate over the set

$$\{(t_1,t_2):0<t_2<t_1<\pi\}.$$

We decompose this set into the union of the same sets A_i $(i=1,\dots,5)$ as in the proof of Theorem 2.3.1 (see Fig. 4.1), where

$$\begin{aligned}A_1&:=\{(x_1,x_2):0<x_1\le 2/n,0<x_2<x_1<\pi\},\\ A_2&:=\{(x_1,x_2):2/n<x_1<\pi,0<x_2\le 1/n\},\\ A_3&:=\{(x_1,x_2):2/n<x_1<\pi,1/n<x_2\le x_1/2\},\\ A_4&:=\{(x_1,x_2):2/n<x_1<\pi,x_1/2<x_2\le x_1-1/n\},\\ A_5&:=\{(x_1,x_2):2/n<x_1<\pi,x_1-1/n<x_2<x_1\}.\end{aligned}$$

We will integrate the right-hand side of (4.5.5) over the sets

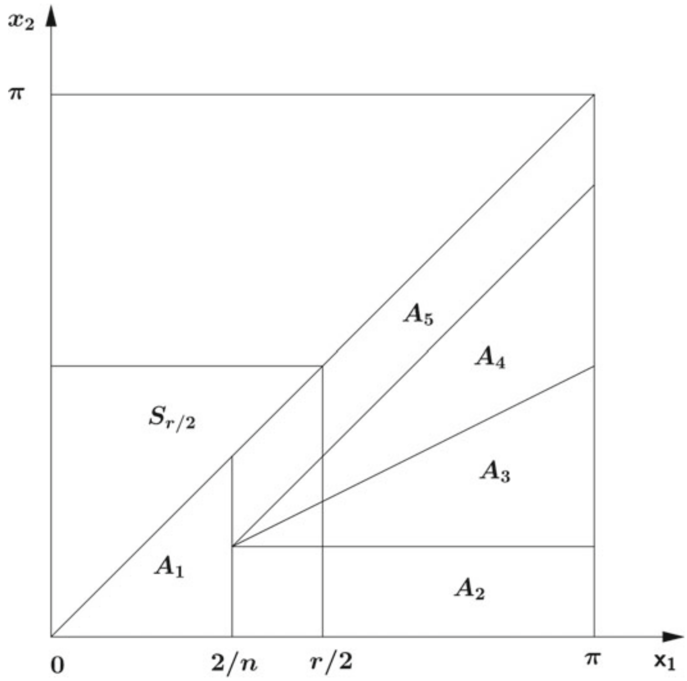

Fig. 4.1 The sets A_i

$$\bigcup_{i=1}^{5}(A_i \cap S_{r/2}) \qquad \text{and} \qquad \bigcup_{i=1}^{5}(A_i \cap S_{r/2}^c),$$

where S^c denotes the complement of the set S. Of course, $A_1 \subset S_{r/2}$. By (4.5.1),

$$\begin{aligned}\int_{A_1} &|f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left|K_n^{\infty,\alpha}(t_1, t_2)\right| dt \\ &\le Cn^2 \int_0^{2/n} \int_0^{2/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \, dt_1 \, dt_2 \\ &\le CU_r^{\omega,1} f(x_1, x_2) < C\epsilon.\end{aligned}$$

Let us denote by r_0 the largest number i, for which $r/2 \le 2^{i+1}/n < r$. By (4.5.4),

$$\begin{aligned}\int_{A_2 \cap S_{r/2}} &|f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left|K_n^{\infty,\alpha}(t_1, t_2)\right| dt \\ &\le C \sum_{i=1}^{r_0} n^{1-\alpha} \left(\frac{2^i}{n}\right)^{-1} \left(\frac{2^i}{n} - \frac{1}{n}\right)^{-\alpha}\end{aligned}$$

$$\int_{2^i/n}^{2^{i+1}/n} \int_0^{1/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \, dt_2 \, dt_1$$
$$\le C \sum_{i=1}^{r_0} 2^{(\omega-\alpha)i} 2^{-\omega i} \left(\frac{n^2}{2^i} \right)$$
$$\int_{2^i/n}^{2^{i+1}/n} \int_0^{1/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \, dt_2 \, dt_1$$
$$\le C \sum_{i=1}^{r_0} 2^{(\omega-\alpha)i} U_r^{\omega,1} f(x_1, x_2) < C\epsilon.$$

Since $t_1 - t_2 > t_1/2$ and $t_1 - t_2 > t_2$ on A_3, we obtain by (4.5.3) that

$$\left| K_n^{\infty,\alpha}(t_1, t_2) \right| \le C n^{-\alpha} t_1^{-1-\alpha/2} t_2^{-1-\alpha/2}.$$

Hence

$$\int_{A_3 \cap S_{r/2}} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left| K_n^{\infty,\alpha}(t_1, t_2) \right| dt$$
$$\le C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} n^{-\alpha} \left(\frac{2^i}{n} \right)^{-1-\alpha/2} \left(\frac{2^j}{n} \right)^{-1-\alpha/2}$$
$$\int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \, dt_1 \, dt_2$$
$$\le C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} 2^{-\omega(i+j)} \left(\frac{n^2}{2^{i+j}} \right)$$
$$\int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \, dt_1 \, dt_2$$
$$\le C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} U_r^{\omega,1} f(x_1, x_2) < C\epsilon.$$

Since $t_2 > t_1/2$ on A_4, (4.5.3) implies

$$\left| K_n^{\infty,\alpha}(t_1, t_2) \right| \le C n^{-\alpha} t_1^{-2} (t_1 - t_2)^{-\alpha}, \tag{4.5.6}$$

and so

$$\int_{A_4 \cap S_{r/2}} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left| K_n^{\infty,\alpha}(t_1, t_2) \right| dt$$
$$\leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} n^{-\alpha} \left(\frac{2^i}{n}\right)^{-2} \left(\frac{2^j}{n}\right)^{-\alpha}$$
$$\int_{2^i/n}^{2^{i+1}/n} \int_{t_1 - 2^{j+1}/n}^{t_1 - 2^j/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \, dt_2 \, dt_1$$
$$\leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-1)i} 2^{(\omega+1-\alpha)j} 2^{-\omega(i+j)} \left(\frac{n^2}{2^{i+j}}\right)$$
$$\int_{2^i/n}^{2^{i+1}/n} \int_{t_1 - 2^{j+1}/n}^{t_1 - 2^j/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \, dt_2 \, dt_1$$
$$\leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-1)i} 2^{(\omega+1-\alpha)j} U_r^{\omega,2} f(x_1, x_2)$$
$$\leq C \sum_{i=1}^{r_0} 2^{(2\omega-\alpha)i} U_r^{\omega,2} f(x_1, x_2) < C\epsilon.$$

We get from (4.5.2) that

$$\left| K_n^{\infty,\alpha}(t_1, t_2) \right| \leq C t_1^{-2}$$

on the set A_5. This implies

$$\int_{A_5 \cap S_{r/2}} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left| K_n^{\infty,\alpha}(t_1, t_2) \right| dt$$
$$\leq C \sum_{i=1}^{r_0} \left(\frac{2^i}{n}\right)^{-2} \int_{2^i/n}^{2^{i+1}/n} \int_{t_1 - 1/n}^{t_1} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \, dt_2 \, dt_1$$
$$\leq C \sum_{i=1}^{r_0} 2^{(\omega-1)i} 2^{-\omega i} \left(\frac{n^2}{2^i}\right)$$
$$\int_{2^i/n}^{2^{i+1}/n} \int_{t_1 - 1/n}^{t_1} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \, dt_2 \, dt_1$$
$$\leq C \sum_{i=1}^{r_0} 2^{(\omega-1)i} U_r^{\omega,2} f(x_1, x_2) < C\epsilon.$$

On the other hand, we get that

$$\begin{aligned}
\int_{A_2\cap S^c_{r/2}} & |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)| \left|K_n^{\infty,\alpha}(t_1,t_2)\right| dt \\
&\le C\sum_{i=r_0}^{\infty} 2^{(\omega-\alpha)i}\mathcal{M}^{\omega,1}f(x_1,x_2) + C\sum_{i=r_0}^{\infty} 2^{-\alpha i}|f(x_1,x_2)| \\
&\le C2^{(\omega-\alpha)r_0}\mathcal{M}^{\omega,1}f(x_1,x_2) + C2^{-\alpha r_0}|f(x_1,x_2)| \\
&\le C(nr)^{\omega-\alpha}\mathcal{M}^{\omega,1}f(x_1,x_2) + C(nr)^{-\alpha}|f(x_1,x_2)| \to 0
\end{aligned}$$

as $n\to\infty$. Similarly,

$$\begin{aligned}
\int_{A_3\cap S^c_{r/2}} & |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)| \left|K_n^{\infty,\alpha}(t_1,t_2)\right| dt \\
&\le C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)}\mathcal{M}^{\omega,1}f(x_1,x_2) + C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1} 2^{-\alpha(i+j)/2}|f(x_1,x_2)| \\
&\le C2^{(\omega-\alpha/2)r_0}\mathcal{M}^{\omega,1}f(x_1,x_2) + C2^{-\alpha r_0/2}|f(x_1,x_2)| \to 0
\end{aligned}$$

and

$$\begin{aligned}
\int_{A_4\cap S^c_{r/2}} & |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)| \left|K_n^{\infty,\alpha}(t_1,t_2)\right| dt \\
&\le C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1} 2^{(\omega-1)i}2^{(\omega+1-\alpha)j}\mathcal{M}^{\omega,2}f(x_1,x_2) + C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1} 2^{-i}2^{(1-\alpha)j}|f(x_1,x_2)| \\
&\le C2^{(2\omega-\alpha)r_0}\mathcal{M}^{\omega,2}f(x_1,x_2) + C2^{-\alpha r_0}|f(x_1,x_2)| \to 0,
\end{aligned}$$

as $n\to\infty$. In the last line, we have used that $0<\alpha<1$. The same holds for $\alpha=1$. Finally,

$$\begin{aligned}
\int_{A_5\cap S^c_{r/2}} & |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)| \left|K_n^{\infty,\alpha}(t_1,t_2)\right| dt \\
&\le C\sum_{i=r_0}^{\infty} 2^{(\omega-1)i}\mathcal{M}^{\omega,2}f(x_1,x_2) + C\sum_{i=r_0}^{\infty} 2^{-i}|f(x_1,x_2)| \\
&\le C2^{(\tau-1)r_0}\mathcal{M}^{\omega,2}f(x_1,x_2) + C2^{-r_0}|f(x_1,x_2)| \to 0,
\end{aligned}$$

as $n\to\infty$. Note that $A_1\cap S^c_{r/2}=\emptyset$. ■

Note that Belinsky [20] proved that the convergence does not hold for all p-Lebesgue points defined in Definition 4.1.19. Since by Theorems 4.5.5 and 4.5.3 almost every point is a strong $(1,\omega)$-Lebesgue point and the maximal operator $\mathcal{M}^\omega f$ is almost everywhere finite for $f\in L_1(\mathbb{T}^d)$, Theorem 4.5.7 implies the almost everywhere convergence

$$\lim_{n\to\infty} \sigma_n^{\infty,\alpha} f = f \qquad \text{a.e.}$$

if $f \in L_1(\mathbb{T}^d)$ (see Corollary 2.5.9).

In the next theorem, we use only the maximal operator $\mathcal{M}_p^{\omega,1} f$ and the (p,ω)-Lebesgue points as in Sect. 4.3.

Theorem 4.5.8 *Suppose that* $0 < \alpha < \infty$, $1 < p < \infty$, $1/p + 1/q = 1$ *and* $0 < \omega < \min(\alpha/d, 1/(2q))$. *If* $\mathcal{M}_p^{\omega,1} f(x)$ *is finite and* x *is a* (p,ω)*-Lebesgue point of* $f \in L_p(\mathbb{T}^d)$*, then*

$$\lim_{n\to\infty} \sigma_n^{\infty,\alpha} f(x) = f(x).$$

Proof Suppose that $0 < \alpha \le 1$. Since (x_1, x_2) is a (p,ω)-Lebesgue point of f, we fix again a number $0 < r < 1$ such that

$$U_{r,p}^{\omega,1} f(x_1, x_2) < \epsilon.$$

We can prove in the same way as in Theorem 4.5.7 that

$$\int_{A_i} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left| K_n^{\infty,\alpha}(t_1, t_2) \right| dt \to 0,$$

for $i = 1, 2, 3$, as $n \to \infty$. So we have to consider the sets A_4 and A_5, only. It is easy to see that

$$\begin{aligned}
&\int_{A_4 \cap S_{r/2}} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left| K_n^{\infty,\alpha}(t_1, t_2) \right| dt \\
&\le \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} \int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} \\
&\qquad |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left| K_n^{\infty,\alpha}(t_1, t_2) \right| 1_{A_4}(t_1, t_2)\, dt_2\, dt_1.
\end{aligned}$$

Hölder's inequality and (4.5.6) imply

$$\begin{aligned}
&\int_{A_4 \cap S_{r/2}} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left| K_n^{\infty,\alpha}(t_1, t_2) \right| dt \\
&\le \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} \left(\int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)|^p \, dt_2\, dt_1 \right)^{1/p} \\
&\quad \left(\int_{2^i/n}^{2^{i+1}/n} \int_{2^{i-1}/n}^{t_1 - 1/n} n^{-\alpha q} t_1^{-2q} (t_1 - t_2)^{-\alpha q} 1_{A_4}(t_1, t_2)\, dt_2\, dt_1 \right)^{1/q}.
\end{aligned}$$

If $q < 1/\alpha$, then

$$
\begin{aligned}
\int_{2^i/n}^{2^{i+1}/n} & \int_{2^{i-1}/n}^{t_1-1/n} n^{-\alpha q} t_1^{-2q} (t_1-t_2)^{-\alpha q} 1_{A_4}(t_1,t_2)\, dt_2\, dt_1 \\
& \le C n^{-\alpha q} \left(\frac{2^i}{n}\right)^{-\alpha q+1} \int_{2^i/n}^{2^{i+1}/n} t_1^{-2q}\, dt_1 \\
& \le C n^{-\alpha q} \left(\frac{2^i}{n}\right)^{-\alpha q+1} \left(\frac{2^i}{n}\right)^{1-2q} \quad dt_1 \\
& \le C \left(\frac{n}{2^i}\right)^{2q-2} 2^{-i\alpha q}
\end{aligned}
$$

and so

$$
\begin{aligned}
\int_{A_4\cap S_{r/2}} & |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)| \left|K_n^{\infty,\alpha}(t_1,t_2)\right| \, dt \\
& \le C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} 2^{(\omega-\alpha/2)(i+j)} \\
& \quad 2^{-\omega(i+j)} \left(\frac{n^2}{2^{i+j}} \int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|^p \, dt_2\, dt_1\right)^{1/p} \\
& \le C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} 2^{(\omega-\alpha/2)(i+j)} U_{r,p}^{\omega,1} f(x_1,x_2) < C_p \epsilon.
\end{aligned}
$$

An analogous inequality can also be proved for $q \ge 1/\alpha$. Indeed, choose a small number $0<\beta<1$ such that $\omega < (1-\beta)/2q$. Since $t_1 - t_2 < t_1/2$ on A_4 and $1 - \alpha q - \beta < 0$, we conclude

$$
\begin{aligned}
\int_{2^i/n}^{2^{i+1}/n} & \int_{2^{i-1}/n}^{t_1-1/n} n^{-\alpha q} t_1^{-2q} (t_1-t_2)^{-\alpha q} 1_{A_4}(t_1,t_2)\, dt_2\, dt_1 \\
& \le C \int_{2^i/n}^{2^{i+1}/n} \int_{2^{i-1}/n}^{t_1-1/n} n^{-\alpha q} t_1^{-2q+\beta} (t_1-t_2)^{-\alpha q-\beta} 1_{A_4}(t_1,t_2)\, dt_2\, dt_1 \\
& \le C n^{-\alpha q} \left(\frac{1}{n}\right)^{-\alpha q-\beta+1} \int_{2^i/n}^{2^{i+1}/n} t_1^{-2q+\beta}\, dt_1 \\
& \le C \left(\frac{n}{2^i}\right)^{2q-2} 2^{-i(1-\beta)}
\end{aligned}
$$

and

$$
\int_{A_4\cap S_{r/2}} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)| \left|K_n^{\infty,\alpha}(t_1,t_2)\right| \, dt
$$

$$\leq C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} 2^{(\omega-(1-\beta)/2q)(i+j)} 2^{-\omega(i+j)}$$
$$\left(\frac{n^2}{2^{i+j}} \int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)|^p \, dt_2 \, dt_1 \right)^{1/p}$$
$$\leq C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} 2^{(\omega-(1-\beta)/2q)(i+j)} U_{r,p}^{\omega,1} f(x_1, x_2) < C_p \epsilon.$$

For the set A_5, we obtain

$$\int_{A_5 \cap S_{r/2}} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left| K_n^{\infty,\alpha}(t_1, t_2) \right| \, dt$$
$$\leq \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} \left(\int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)|^p \, dt_2 \, dt_1 \right)^{1/p}$$
$$\left(\int_{2^i/n}^{2^{i+1}/n} \int_{t_1 - 1/n}^{t_1} t_1^{-2q} \, dt_2 \, dt_1 \right)^{1/q}.$$

We can compute that

$$\int_{2^i/n}^{2^{i+1}/n} \int_{t_1 - 1/n}^{t_1} t_1^{-2q} \, dt_2 \, dt_1 \leq C n^{-1} \left(\frac{2^i}{n} \right)^{-2q+1} = C \left(\frac{n}{2^i} \right)^{2q-2} 2^{-i}.$$

Then

$$\int_{A_5 \cap S_{r/2}} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left| K_n^{\infty,\alpha}(t_1, t_2) \right| \, dt$$
$$\leq \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} 2^{(\omega-1/2q)(i+j)} 2^{-\omega(i+j)}$$
$$\left(\frac{n^2}{2^{i+j}} \int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)|^p \, dt_2 \, dt_1 \right)^{1/p}$$
$$\leq C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} 2^{(\omega-1/(2q))(i+j)} U_{r,p}^{\omega,1} f(x_1, x_2) < C \epsilon.$$

Similarly, we can see that

$$\begin{aligned}
&\int_{A_4 \cap S_{r/2}^c} |f(x_1-t_1, x_2-t_2) - f(x_1,x_2)| \left| K_n^{\infty,\alpha}(t_1,t_2) \right| dt \\
&\quad \le C_p \sum_{i=r_0}^{\infty} \sum_{j=i-1}^{i} 2^{(\omega-\alpha/2)(i+j)} \mathcal{M}_p^{\omega,1} f(x_1,x_2) \\
&\qquad + C_p \sum_{i=r_0}^{\infty} \sum_{j=i-1}^{i} 2^{-\alpha(i+j)/2} |f(x_1,x_2)| \\
&\quad \le C_p 2^{(2\omega-\alpha)r_0} \mathcal{M}_p^{\omega,1} f(x_1,x_2) + C_p 2^{-\alpha r_0} |f(x_1,x_2)| \\
&\quad \le C(nr)^{2\omega-\alpha} \mathcal{M}_p^{\omega,1} f(x_1,x_2) + C(nr)^{-\alpha} |f(x_1,x_2)| \to 0
\end{aligned}$$

as $n \to \infty$ and $q < 1/\alpha$. If $q \ge 1/\alpha$, then

$$\begin{aligned}
&\int_{A_4 \cap S_{r/2}^c} |f(x_1-t_1, x_2-t_2) - f(x_1,x_2)| \left| K_n^{\infty,\alpha}(t_1,t_2) \right| dt \\
&\quad \le C_p \sum_{i=r_0}^{\infty} \sum_{j=i-1}^{i} 2^{(\omega-(1-\beta)/2q)(i+j)} \mathcal{M}_p^{\omega,1} f(x_1,x_2) \\
&\qquad + C_p \sum_{i=r_0}^{\infty} \sum_{j=i-1}^{i} 2^{-(1-\beta)(i+j)/2q} |f(x_1,x_2)| \\
&\quad \le C_p 2^{(2\omega-(1-\beta)/q)r_0} \mathcal{M}_p^{\omega,1} f(x_1,x_2) + C_p 2^{-(1-\beta)r_0/q} |f(x_1,x_2)| \to 0
\end{aligned}$$

as $n \to \infty$. Finally,

$$\begin{aligned}
&\int_{A_5 \cap S_{r/2}^c} |f(x_1-t_1, x_2-t_2) - f(x_1,x_2)| \left| K_n^{\infty,\alpha}(t_1,t_2) \right| dt \\
&\quad \le C_p \sum_{i=r_0}^{\infty} \sum_{j=i-1}^{i} 2^{(\omega-1/2q)(i+j)} \mathcal{M}_p^{\omega,1} f(x_1,x_2) \\
&\qquad + C_p \sum_{i=r_0}^{\infty} \sum_{j=i-1}^{i} 2^{-(i+j)/2q} |f(x_1,x_2)| \\
&\quad \le C_p 2^{(2\omega-1/q)r_0} \mathcal{M}_p^{\omega,1} f(x_1,x_2) + C_p 2^{-r_0/q} |f(x_1,x_2)| \to 0
\end{aligned}$$

as $n \to \infty$. The proof of the theorem is complete. ∎

Note that these results were proved in Weisz [345, 349]. Now we turn to the ℓ_∞-θ-means introduced by

$$\sigma_n^{\infty,\theta} f(x) := \sum_{k\in\mathbb{Z}^d} \theta\left(\frac{\|k\|_\infty}{n}\right) \widehat{f}(k) e^{\imath k\cdot x}$$

in Sect. 2.6.1. We suppose again that $\theta : \mathbb{R} \to \mathbb{R}$ satisfies (2.6.2) and (2.6.3).

Theorem 4.5.9 *Suppose that θ satisfies (2.6.2) and (2.6.3). If $0<\omega<1/d$, $\mathcal{M}^\omega f(x)$ is finite and x is a strong $(1,\omega)$-Lebesgue point of $f \in L_1(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty} \sigma_n^{\infty,\theta} f(x) = f(x).$$

Proof In Theorem 2.6.7, we have proved that

$$K_n^{\infty,\theta}(x) = \sum_{k=0}^{\infty} k\, \Delta_2\theta\left(\frac{k}{n}\right) K_k^\infty(x),$$

where K_k^∞ denotes the Fejér kernel. We have verified in (2.6.4) that

$$\sup_{n\ge 1} \sum_{k=0}^{\infty} k \left|\Delta_2\theta\left(\frac{k}{n}\right)\right| \le C < \infty.$$

Hence

$$\begin{aligned}\sigma_n^{\infty,\theta} f(x) - f(x) &= \int_{\mathbb{T}^d} \Big(f(x-t) - f(x)\Big) K_n^{\infty,\theta}(t)\, dt \\ &= \sum_{k=0}^{\infty} k\, \Delta_2\theta\left(\frac{k}{n}\right) \int_{\mathbb{T}^d} \Big(f(x-t) - f(x)\Big) K_k^\infty(t)\, dt.\end{aligned}$$

The proof can be finished using Theorem 4.5.7. ∎

This implies also the almost everywhere convergence of $\sigma_n^{\infty,\theta} f$ stated in Corollary 2.6.9. From Theorem 4.5.8, we obtain in the same way

Theorem 4.5.10 *Suppose that θ satisfies (2.6.2) and (2.6.3), $1 < p < \infty$, $1/p + 1/q = 1$ and $0 < \omega < \min(1/d, 1/(2q))$. If $\mathcal{M}_p^{\omega,1} f(x)$ is finite and x is a (p,ω)-Lebesgue point of $f \in L_p(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty} \sigma_n^{\infty,\theta} f(x) = f(x).$$

4.6 ℓ_1-Summability

Finally, we investigate Lebesgue points for the ℓ_1-Cesàro means

$$\sigma_n^{1,\alpha} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n^{1,\alpha}(t)\, dt \qquad (n \in \mathbb{N})$$

as well as the ℓ_1-θ-means $\sigma_n^{1,\theta} f$. The definition of the Cesàro kernel $K_n^{1,\alpha}$, i.e.,

$$K_n^{1,\alpha}(t) := \frac{1}{A_{n-1}^\alpha} \sum_{k \in \mathbb{Z}^d,\ \|k\|_1 \le n} A_{n-1-\|k\|_1}^\alpha e^{\imath k \cdot t},$$

can be found in Sect. 2.2. In this section, we use the same Hardy-Littlewood maximal functions $\mathcal{M}_p^\omega f$ and $\mathcal{M}_p^{\omega,1} f$ and the same (strong) (p,ω)-Lebesgue points as in Sect. 4.5. In what follows, we have to suppose that f is periodic with respect to π.

Instead of Lemma 2.2.14, we will use the next estimations.

Lemma 4.6.1 *Suppose that* $0 < \alpha \le 1$, $0 \le \beta \le 1$ *and* $\pi > x_1 > x_2 > 0$. *Then*

$$\left|K_n^{1,\alpha}(x_1, x_2)\right| \le C n^2, \tag{4.6.1}$$

$$\begin{aligned} \left|K_n^{1,\alpha}(x_1, x_2)\right| &\le C(x_1 - x_2)^{-1}(x_1 + x_2)^{-1} 1_{\{x_2 \le \pi/2\}} \\ &\quad + C(x_1 - x_2)^{-1}(\pi - x_2)^{-1} 1_{\{x_2 > \pi/2\}}. \end{aligned} \tag{4.6.2}$$

If $1/n < x_2 \le \pi/2$, *then*

$$\left|K_n^{1,\alpha}(x_1, x_2)\right| \le C n^{-\alpha}(x_1 - x_2)^{-1-\beta} x_2^{\beta-\alpha-1} \tag{4.6.3}$$

and

$$\left|K_n^{1,\alpha}(x_1, x_2)\right| \le C n^{1-\alpha} x_2^{-\alpha-1}. \tag{4.6.4}$$

If $\pi/2 < x_2 < \pi - 1/n$, *then*

$$\left|K_n^{1,\alpha}(x_1, x_2)\right| \le C n^{-\alpha}(x_1 - x_2)^{-1-\beta}(\pi - x_2)^{\beta-\alpha-1} \tag{4.6.5}$$

and

$$\left|K_n^{1,\alpha}(x_1, x_2)\right| \le C n^{1-\alpha}(\pi - x_2)^{-\alpha-1}. \tag{4.6.6}$$

Proof Inequalities (4.6.1) and (4.6.2) follow from Lemma 2.2.5 and (2.2.7), because $2\pi - x_1 - x_2 > \pi - x_2$, while (4.6.3) and (4.6.4) follow from (2.2.15) and (2.2.17). Finally, (4.6.5) and (4.6.6) can be proved as (2.2.16) and (2.2.18). ∎

The main theorem of this section reads as follows.

Theorem 4.6.2 *Suppose that* $0 < \alpha < \infty$, $0 < \omega < \min(\alpha, 1)/d$ *and* $\mathcal{M}^\omega f(x)$ *is finite. If* $f \in L_1(\mathbb{T}^d)$ *is periodic with respect to* π *and* x *is a strong* $(1,\omega)$*-Lebesgue point of* f, *then*

$$\lim_{n\to\infty} \sigma_n^{1,\alpha} f(x) = f(x).$$

Proof Again, it is enough to prove the theorem for $0 < \alpha \le 1$. Let $0 < \omega < \alpha/2$ and fix a number $r < 1$ such that

$$U_r^\omega f(x_1, x_2) < \epsilon.$$

Let

$$S_{r/2} := \left[-\frac{r}{2}, \frac{r}{2}\right] \times \left[-\frac{r}{2}, \frac{r}{2}\right], \qquad S'_{r/2} := \left[\pi - \frac{r}{2}, \pi + \frac{r}{2}\right] \times \left[\pi - \frac{r}{2}, \pi + \frac{r}{2}\right]$$

and $2/n < r/2$. We have

$$\begin{aligned}&\left|\sigma_n^{1,\alpha} f(x_1, x_2) - f(x_1, x_2)\right| \\ &\quad \le \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left|K_n^{1,\alpha}(t_1, t_2)\right| dt.\end{aligned}$$

We will integrate on the set

$$\{(t_1, t_2) : 0 < t_2 < t_1 < \pi\},$$

more exactly on

$$\bigcup_{i=1}^{5}(A_i \cap S_{r/2}), \qquad \bigcup_{i=1}^{5}(A_i \cap S_{r/2}^c), \qquad \bigcup_{i=6}^{10}(A_i \cap S'_{r/2}), \qquad \bigcup_{i=6}^{10}(A_i \cap (S'_{r/2})^c),$$

where the sets A_i $(i = 1, \ldots, 10)$ are defined by (see Fig. 4.2)

$$\begin{aligned}
A_1 &:= \{(x_1, x_2) : 0 < x_1 \le 2/n, 0 < x_2 < x_1 < \pi, x_2 \le \pi/2\},\\
A_2 &:= \{(x_1, x_2) : 2/n < x_1 < \pi, 0 < x_2 \le 1/n, x_2 \le \pi/2\},\\
A_3 &:= \{(x_1, x_2) : 2/n < x_1 < \pi, 1/n < x_2 \le x_1/2, x_2 \le \pi/2\},\\
A_4 &:= \{(x_1, x_2) : 2/n < x_1 < \pi, x_1/2 < x_2 \le x_1 - 1/n, x_2 \le \pi/2\},\\
A_5 &:= \{(x_1, x_2) : 2/n < x_1 < \pi, x_1 - 1/n < x_2 < x_1, x_2 \le \pi/2\}\\
A_6 &:= \{(x_1, x_2) : x_2 > \pi/2, \pi - 2/n \le x_2 < \pi, 0 < x_2 < x_1 < \pi\},\\
A_7 &:= \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2/n, \pi - 1/n < x_1 < \pi\},\\
A_8 &:= \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2/n, (\pi + x_2)/2 < x_1 \le \pi - 1/n\},\\
A_9 &:= \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2/n, x_2 + 1/n < x_1 \le (\pi + x_2)/2\},\\
A_{10} &:= \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2/n, x_2 < x_1 \le x_2 + 1/n\}.
\end{aligned}$$

Since $A_1 \subset S_{r/2}$ and $A_6 \subset S'_{r/2}$, we deduce by (4.6.1) that

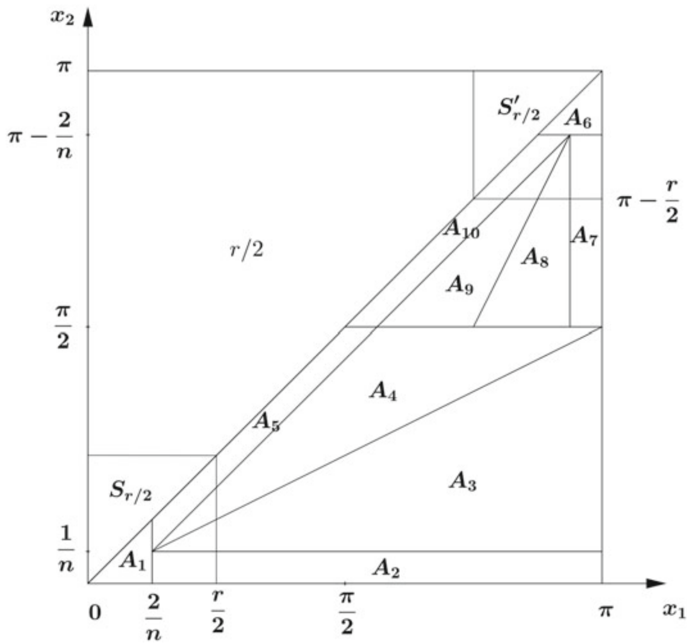

Fig. 4.2 The sets A_i

$$\begin{aligned}
&\int_{A_1} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left|K_n^{1,\alpha}(t_1, t_2)\right| dt \\
&\leq Cn^2 \int_0^{2/n} \int_0^{2/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \, dt_2 \, dt_1 \\
&\leq CU_r^{\omega,1} f(x_1, x_2) < C\epsilon
\end{aligned}$$

and

$$\begin{aligned}
&\int_{A_6} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left|K_n^{1,\alpha}(t_1, t_2)\right| dt \\
&\leq Cn^2 \int_{\pi-2/n}^{\pi} \int_{\pi-2/n}^{\pi} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \, dt_2 \, dt_1 \\
&\leq Cn^2 \int_{-2/n}^{0} \int_{-2/n}^{0} |f(x_1 - u_1 - \pi, x_2 - u_2 - \pi) - f(x_1, x_2)| \, du_2 \, du_1 \\
&\leq CU_r^{\omega,1} f(x_1, x_2) < C\epsilon.
\end{aligned}$$

Let us denote by r_0 the largest number i, for which $r/2 \leq 2^{i+1}/n < r$. By (4.6.2),

$$\begin{aligned}
&\int_{A_2 \cap S_{r/2}} |f(x_1-t_1, x_2-t_2) - f(x_1,x_2)| \left|K_n^{1,\alpha}(t_1,t_2)\right| dt \\
&\quad \le C \sum_{i=1}^{r_0} \left(\frac{2^i}{n} - \frac{1}{n}\right)^{-1} \left(\frac{2^i}{n}\right)^{-1} \\
&\qquad \int_{2^i/n}^{2^{i+1}/n} \int_0^{1/n} |f(x_1-t_1, x_2-t_2) - f(x_1,x_2)| \, dt_2 \, dt_1 \\
&\quad \le C \sum_{i=1}^{r_0} 2^{(\omega-1)i} 2^{-\omega i} \left(\frac{n^2}{2^i}\right) \\
&\qquad \int_{2^i/n}^{2^{i+1}/n} \int_0^{1/n} |f(x_1-t_1, x_2-t_2) - f(x_1,x_2)| \, dt_2 \, dt_1 \\
&\quad \le C \sum_{i=1}^{r_0} 2^{(\omega-1)i} U_r^{\omega,1} f(x_1,x_2) < C\epsilon.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{A_7 \cap S'_{r/2}} |f(x_1-t_1, x_2-t_2) - f(x_1,x_2)| \left|K_n^{1,\alpha}(t_1,t_2)\right| dt \\
&\quad \le C \sum_{i=1}^{r_0} \left(\frac{2^i}{n} - \frac{1}{n}\right)^{-1} \left(\frac{2^i}{n}\right)^{-1} \\
&\qquad \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} \int_{\pi-1/n}^{\pi} |f(x_1-t_1, x_2-t_2) - f(x_1,x_2)| \, dt_1 \, dt_2 \\
&\quad \le C \sum_{i=1}^{r_0} 2^{(\omega-1)i} 2^{-\omega i} \left(\frac{n^2}{2^i}\right) \\
&\qquad \int_{-2^{i+1}/n}^{-2^i/n} \int_{-1/n}^{0} |f(x_1-t_1-\pi, x_2-t_2-\pi) - f(x_1,x_2)| \, dt_1 \, dt_2 \\
&\quad \le C \sum_{i=1}^{r_0} 2^{(\omega-1)i} U_r^{\omega,1} f(x_1,x_2) < C\epsilon.
\end{aligned}$$

On the other hand, we get that

$$\begin{aligned}
&\int_{A_2 \cap S^c_{r/2}} |f(x_1-t_1, x_2-t_2) - f(x_1,x_2)| \left|K_n^{1,\alpha}(t_1,t_2)\right| dt \\
&+ \int_{A_7 \cap (S'_{r/2})^c} |f(x_1-t_1, x_2-t_2) - f(x_1,x_2)| \left|K_n^{1,\alpha}(t_1,t_2)\right| dt \\
&\quad \le C \sum_{i=r_0}^{\infty} 2^{(\omega-1)i} \mathcal{M}^{\omega,1} f(x_1,x_2) + C \sum_{i=r_0}^{\infty} 2^{-i} |f(x_1,x_2)|
\end{aligned}$$

$$
\begin{aligned}
&\le C 2^{(\omega-1)r_0} \mathcal{M}^{\omega,1} f(x_1,x_2) + C 2^{-r_0} |f(x_1,x_2)| \\
&\le C(nr)^{\omega-1} \mathcal{M}^{\omega,1} f(x_1,x_2) + C(nr)^{-1} |f(x_1,x_2)| \to 0
\end{aligned}
$$

as $n \to \infty$.

Since $t_1 - t_2 > t_1/2$ and $t_1 - t_2 > t_2$ on A_3, we obtain by (4.6.3) that

$$
\begin{aligned}
\left| K_n^{1,\alpha}(t_1,t_2) \right| &\le C n^{-\alpha} (t_1 - t_2)^{-1-\alpha/2} (t_1 - t_2)^{-\beta+\alpha/2} t_2^{\beta-\alpha-1} \\
&\le C n^{-\alpha} t_1^{-1-\alpha/2} t_2^{-1-\alpha/2},
\end{aligned}
$$

whenever $\beta > \alpha/2$. Hence

$$
\begin{aligned}
&\int_{A_3 \cap S_{r/2}} |f(x_1 - t_1, x_2 - t_2) - f(x_1,x_2)| \left| K_n^{1,\alpha}(t_1,t_2) \right| dt \\
&\quad \le C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} n^{-\alpha} \left(\frac{2^i}{n} \right)^{-1-\alpha/2} \left(\frac{2^j}{n} \right)^{-1-\alpha/2} \\
&\qquad \int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1,x_2)| \, dt_2 \, dt_1 \\
&\quad \le C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} 2^{-\omega(i+j)} \left(\frac{n^2}{2^{i+j}} \right) \\
&\qquad \int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1,x_2)| \, dt_2 \, dt_1 \\
&\quad \le C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} U_r^{\omega,1} f(x_1,x_2) < C\epsilon.
\end{aligned}
$$

On A_8, $t_1 - t_2 > (\pi - t_2)/2 \ge (\pi - t_1)/2$ and so (4.6.5) implies

$$
\begin{aligned}
\left| K_n^{1,\alpha}(t_1,t_2) \right| &\le C n^{-\alpha} (t_1 - t_2)^{-1-\alpha/2} (t_1 - t_2)^{-\beta+\alpha/2} (\pi - t_2)^{\beta-\alpha-1} \\
&\le C n^{-\alpha} (\pi - t_1)^{-1-\alpha/2} (\pi - t_2)^{-1-\alpha/2}
\end{aligned}
$$

if $\beta > \alpha/2$. From this it follows

$$
\begin{aligned}
&\int_{A_8 \cap S'_{r/2}} |f(x_1 - t_1, x_2 - t_2) - f(x_1,x_2)| \left| K_n^{1,\alpha}(t_1,t_2) \right| dt \\
&\quad \le C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} n^{-\alpha} \left(\frac{2^i}{n} \right)^{-1-\alpha/2} \left(\frac{2^j}{n} \right)^{-1-\alpha/2} \\
&\qquad \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} \int_{\pi-2^{j+1}/n}^{\pi-2^j/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1,x_2)| \, dt_1 \, dt_2
\end{aligned}
$$

$$\leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} 2^{-\omega(i+j)} \left(\frac{n^2}{2^{i+j}}\right)$$

$$\int_{-2^{i+1}/n}^{-2^i/n} \int_{-2^{j+1}/n}^{-2^j/n} |f(x_1 - t_1 - \pi, x_2 - t_2 - \pi) - f(x_1, x_2)| \, dt_1 \, dt_2$$

$$\leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} U_r^{\omega,1} f(x_1, x_2) < C\epsilon.$$

Similarly,

$$\int_{A_3 \cap S_{r/2}^c} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left|K_n^{1,\alpha}(t_1, t_2)\right| \, dt$$

$$+ \int_{A_8 \cap (S'_{r/2})^c} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left|K_n^{1,\alpha}(t_1, t_2)\right| \, dt$$

$$\leq C \sum_{i=r_0}^{\infty} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} \mathcal{M}^{\omega,1} f(x_1, x_2) + C \sum_{i=r_0}^{\infty} \sum_{j=0}^{i-1} 2^{-\alpha(i+j)/2} |f(x_1, x_2)|$$

$$\leq C 2^{(\omega-\alpha/2)r_0} \mathcal{M}^{\omega,1} f(x_1, x_2) + C 2^{-\alpha r_0/2} |f(x_1, x_2)| \to 0$$

as $n \to \infty$.

Since $t_2 > t_1/2$ on A_4, (4.6.3) with $\beta = \alpha/2$ implies

$$\begin{aligned} \left|K_n^{1,\alpha}(t_1, t_2)\right| &\leq C n^{-\alpha} (t_1 - t_2)^{-1-\alpha/2} t_2^{-1-\alpha/2} \\ &\leq C n^{-\alpha} t_1^{-1-\alpha/2} (t_1 - t_2)^{-1-\alpha/2} \end{aligned}$$

and so

$$\int_{A_4 \cap S_{r/2}} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left|K_n^{1,\alpha}(t_1, t_2)\right| \, dt$$

$$\leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} n^{-\alpha} \left(\frac{2^i}{n}\right)^{-1-\alpha/2} \left(\frac{2^j}{n}\right)^{-1-\alpha/2}$$

$$\int_{2^i/n}^{2^{i+1}/n} \int_{t_1 - 2^{j+1}/n}^{t_1 - 2^j/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \, dt_2 \, dt_1$$

$$\leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} 2^{-\omega(i+j)} \left(\frac{n^2}{2^{i+j}}\right)$$

$$\int_{2^i/n}^{2^{i+1}/n} \int_{t_1 - 2^{j+1}/n}^{t_1 - 2^j/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \, dt_2 \, dt_1$$

$$\leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} U_r^{\omega,2} f(x_1, x_2) < C\epsilon.$$

Inequality (4.6.5) with $\beta = \alpha/2$ yields

$$\left|K_n^{1,\alpha}(t_1, t_2)\right| \leq Cn^{-\alpha}(t_1 - t_2)^{-1-\alpha/2} (\pi - t_2)^{-1-\alpha/2}.$$

Thus

$$\begin{aligned}
&\int_{A_9 \cap S'_{r/2}} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left|K_n^{1,\alpha}(t_1, t_2)\right| dt \\
&\quad \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} n^{-\alpha} \left(\frac{2^i}{n}\right)^{-1-\alpha/2} \left(\frac{2^j}{n}\right)^{-1-\alpha/2} \\
&\qquad \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} \int_{t_2+2^j/n}^{t_2+2^{j+1}/n} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \, dt_1 \, dt_2 \\
&\quad \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} 2^{-\omega(i+j)} \left(\frac{n^2}{2^{i+j}}\right) \\
&\qquad \int_{-2^{i+1}/n}^{-2^i/n} \int_{t_2+2^j/n}^{t_2+2^{j+1}/n} |f(x_1 - t_1 - \pi, x_2 - t_2 - \pi) - f(x_1, x_2)| \, dt_1 \, dt_2 \\
&\quad \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} U_r^{\omega,4} f(x_1, x_2) < C\epsilon.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{A_4 \cap S^c_{r/2}} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left|K_n^{1,\alpha}(t_1, t_2)\right| dt \\
&\quad + \int_{A_9 \cap (S'_{r/2})^c} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left|K_n^{1,\alpha}(t_1, t_2)\right| dt \\
&\quad \leq C \sum_{i=r_0}^{\infty} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} \left(\mathcal{M}^{\omega,2} f(x_1, x_2) + \mathcal{M}^{\omega,4} f(x_1, x_2)\right) \\
&\qquad + C \sum_{i=r_0}^{\infty} \sum_{j=0}^{i-1} 2^{-\alpha(i+j)/2} |f(x_1, x_2)| \\
&\quad \leq C 2^{(\omega-\alpha/2)r_0} \mathcal{M}^{\omega} f(x_1, x_2) + C 2^{-\alpha r_0/2} |f(x_1, x_2)| \to 0
\end{aligned}$$

as $n \to \infty$.

Inequality (4.6.4) implies

$$\left|K_n^{1,\alpha}(t_1,t_2)\right| \le Cn^{1-\alpha} t_1^{-\alpha-1}$$

on the set A_5 and so

$$\begin{aligned}
&\int_{A_5\cap S_{r/2}} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)| \left|K_n^{1,\alpha}(t_1,t_2)\right| dt \\
&\quad \le C\sum_{i=1}^{r_0} n^{1-\alpha}\left(\frac{2^i}{n}\right)^{-\alpha-1} \int_{2^i/n}^{2^{i+1}/n}\int_{t_1-1/n}^{t_1} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|\, dt_1\, dt_2 \\
&\quad \le C\sum_{i=1}^{r_0} 2^{(\omega-\alpha)i}2^{-\omega i}\left(\frac{n^2}{2^i}\right) \\
&\qquad \int_{2^i/n}^{2^{i+1}/n}\int_{t_1-1/n}^{t_1} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|\, dt_1\, dt_2 \\
&\quad \le C\sum_{i=1}^{r_0} 2^{(\omega-1)i} U_r^{\omega,2} f(x_1,x_2) < C\epsilon.
\end{aligned}$$

In the same way, by (4.6.6),

$$\begin{aligned}
&\int_{A_{10}\cap S'_{r/2}} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)| \left|K_n^{1,\alpha}(t_1,t_2)\right| dt \\
&\quad \le C\sum_{i=1}^{r_0} n^{1-\alpha}\left(\frac{2^i}{n}\right)^{-1-\alpha} \\
&\qquad \int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{t_2}^{t_2+1/n} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|\, dt_1\, dt_2 \\
&\quad \le C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\omega-\alpha)i}2^{-\omega i}\left(\frac{n^2}{2^i}\right) \\
&\qquad \int_{-2^{i+1}/n}^{-2^i/n}\int_{t_2}^{t_2+1/n} |f(x_1-t_1-\pi,x_2-t_2-\pi)-f(x_1,x_2)|\, dt_1\, dt_2 \\
&\quad \le C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\omega-\alpha)i} U_r^{\omega,4} f(x_1,x_2) < C\epsilon.
\end{aligned}$$

Finally,

$$\int_{A_5 \cap S_{r/2}^c} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left|K_n^{1,\alpha}(t_1, t_2)\right| dt$$
$$+ \int_{A_{10} \cap (S'_{r/2})^c} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left|K_n^{1,\alpha}(t_1, t_2)\right| dt$$
$$\leq C \sum_{i=r_0}^{\infty} 2^{(\omega-\alpha)i} \left(\mathcal{M}^{\omega,2} f(x_1, x_2) + \mathcal{M}^{\omega,4} f(x_1, x_2)\right)$$
$$+ C \sum_{i=r_0}^{\infty} 2^{-\alpha i} |f(x_1, x_2)|$$
$$\leq C 2^{(\omega-\alpha)r_0} \mathcal{M}^{\omega} f(x_1, x_2) + C 2^{-\alpha r_0} |f(x_1, x_2)| \to 0$$

as $n \to \infty$, which finishes the proof. ∎

In this way, we obtain Corollary 2.5.9 for the ℓ_1-Cesàro means, i.e.,

$$\lim_{n\to\infty} \sigma_n^{1,\alpha} f = f \qquad \text{a.e.}$$

if $f \in L_1(\mathbb{T}^d)$. For $1 < p < \infty$, we get again a better result.

Theorem 4.6.3 *Suppose that* $0 < \alpha < \infty$, $1/(\min(\alpha, 1) < p < \infty$, $1/p + 1/q = 1$, $0 < \omega < (1 + q \min(\alpha, 1) - q)/2q$ *and* $\mathcal{M}_p^{\omega,1} f(x)$ *is finite. If* $f \in L_p(\mathbb{T}^2)$ *is periodic with respect to* π *and* x *is a* (p, ω)*-Lebesgue point of* f*, then*

$$\lim_{n\to\infty} \sigma_n^{1,\alpha} f(x) = f(x).$$

Proof We prove the theorem again for $0 < \alpha \leq 1$. Note that $1/\alpha < p < \infty$ implies $1 < q < 1/(1 - \alpha)$ and so $1 + \alpha q - q > 0$. Moreover,

$$\frac{1 + \alpha q - q}{2q} < \frac{\alpha}{2}.$$

Fix a number $0 < r < 1$ such that

$$U_{r,p}^{\omega,1} f(x_1, x_2) < \epsilon.$$

In Theorem 4.6.2, we have verified that

$$\int_{A_i} |f(x_1 - t_1, x_2 - t_2) - f(x_1, x_2)| \left|K_n^{1,\alpha}(t_1, t_2)\right| dt \to 0,$$

for $i = 1, 2, 3, 6, 7, 8$, as $n \to \infty$ and $\omega < \alpha/2$. So we need to consider the sets A_4, A_5, A_9 and A_{10}, only.

We apply (4.6.3) with $\beta = 0$ and that $t_2 > t_1/2$ on A_4 to obtain

$$\left|K_n^{1,\alpha}(t_1,t_2)\right| \le Cn^{-\alpha}(t_1-t_2)^{-1}t_1^{-\alpha-1}.$$

By Hölder's inequality,

$$\begin{aligned}
&\int_{A_4\cap S_{r/2}} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|\left|K_n^{1,\alpha}(t_1,t_2)\right| dt \\
&\quad \le \sum_{i=1}^{r_0}\sum_{j=i-1}^{i}\int_{2^i/n}^{2^{i+1}/n}\int_{2^j/n}^{2^{j+1}/n} \\
&\qquad |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|\left|K_n^{1,\alpha}(t_1,t_2)\right| 1_{A_4}(t_1,t_2)\,dt_2\,dt_1 \\
&\quad \le \sum_{i=1}^{r_0}\sum_{j=i-1}^{i}\left(\int_{2^i/n}^{2^{i+1}/n}\int_{2^j/n}^{2^{j+1}/n} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|^p\,dt_2\,dt_1\right)^{1/p} \\
&\qquad \left(\int_{2^i/n}^{2^{i+1}/n}\int_{2^{i-1}/n}^{t_1-1/n} n^{-\alpha q}(t_1-t_2)^{-q}t_1^{-q(1+\alpha)}1_{A_4}(t_1,t_2)\,dt_2\,dt_1\right)^{1/q}.
\end{aligned}$$

Since $1-q<0$, we have

$$\begin{aligned}
&\int_{2^i/n}^{2^{i+1}/n}\int_{2^{i-1}/n}^{t_1-1/n} n^{-\alpha q}(t_1-t_2)^{-q}t_1^{-q(1+\alpha)}1_{A_4}(t_1,t_2)\,dt_2\,dt_1 \\
&\quad \le Cn^{-\alpha q}\left(\frac{1}{n}\right)^{1-q}\int_{2^i/n}^{2^{i+1}/n} t_1^{-q(1+\alpha)}\,dt_1 \\
&\quad \le Cn^{-\alpha q}\left(\frac{1}{n}\right)^{1-q}\left(\frac{2^i}{n}\right)^{1-q(1+\alpha)} \\
&\quad \le C\left(\frac{n}{2^i}\right)^{2q-2}2^{-i(1+\alpha q-q)}
\end{aligned}$$

and so

$$\begin{aligned}
&\int_{A_4\cap S_{r/2}} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|\left|K_n^{1,\alpha}(t_1,t_2)\right| dt \\
&\le C_p\sum_{i=1}^{r_0}\sum_{j=i-1}^{i}2^{(\omega-(1+\alpha q-q)/2q)(i+j)} \\
&\quad 2^{-\omega(i+j)}\left(\frac{n^2}{2^{i+j}}\int_{2^i/n}^{2^{i+1}/n}\int_{2^j/n}^{2^{j+1}/n} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|^p\,dt_2\,dt_1\right)^{1/p} \\
&\le C_p\sum_{i=1}^{r_0}\sum_{j=i-1}^{i}2^{(\omega-(1+\alpha q-q)/2q)(i+j)}U_{r,p}^{\omega,1}f(x_1,x_2) < C_p\epsilon.
\end{aligned}$$

Let us use (4.6.5) with $\beta=0$ to get that

$$\int_{A_9\cap S'_{r/2}} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)| \left|K_n^{1,\alpha}(t_1,t_2)\right| dt$$
$$\le \sum_{i=1}^{r_0}\sum_{j=i-1}^{i} \left(\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{\pi-2^{j+1}/n}^{\pi-2^j/n} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|^p \, dt_1\, dt_2\right)^{1/p}$$
$$\left(\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{t_2+1/n}^{\pi-2^{i-1}/n} n^{-\alpha q}(t_1-t_2)^{-q}(\pi-t_2)^{-q(1+\alpha)} 1_{A_4}(t_1,t_2)\, dt_1\, dt_2\right)^{1/q}.$$

Therefore,

$$\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{t_2+1/n}^{\pi-2^{i-1}/n} n^{-\alpha q}(t_1-t_2)^{-q}(\pi-t_2)^{-q(1+\alpha)} 1_{A_4}(t_1,t_2)\, dt_1\, dt_2$$
$$\le Cn^{-\alpha q}\left(\frac{1}{n}\right)^{1-q}\int_{\pi-2^{i+1}/n}^{\pi-2^i/n} (\pi-t_2)^{-q(1+\alpha)}\, dt_2$$
$$\le Cn^{-\alpha q}\left(\frac{1}{n}\right)^{1-q}\left(\frac{2^i}{n}\right)^{1-q(1+\alpha)}$$
$$\le C\left(\frac{n}{2^i}\right)^{2q-2} 2^{-i(1+\alpha q-q)}$$

and

$$\int_{A_9\cap S'_{r/2}} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)| \left|K_n^{1,\alpha}(t_1,t_2)\right| dt$$
$$\le C_p\sum_{i=1}^{r_0}\sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)}2^{-\omega(i+j)}$$
$$\left(\frac{n^2}{2^{i+j}}\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{\pi-2^{j+1}/n}^{\pi-2^j/n} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|^p \, dt_2\, dt_1\right)^{1/p}$$
$$\le C_p\sum_{i=1}^{r_0}\sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} U_{r,p}^{\omega,1} f(x_1,x_2) < C_p\epsilon.$$

Similarly, we can see that

$$\int_{A_4\cap S^c_{r/2}} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)| \left|K_n^{1,\alpha}(t_1,t_2)\right| dt$$
$$+\int_{A_9\cap (S'_{r/2})^c} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)| \left|K_n^{1,\alpha}(t_1,t_2)\right| dt$$
$$\le C_p\sum_{i=r_0}^{\infty}\sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} \mathcal{M}_p^{\omega,1} f(x_1,x_2)$$

$$
\begin{aligned}
&+ C_p \sum_{i=r_0}^{\infty} \sum_{j=i-1}^{i} 2^{-(1+\alpha q-q)(i+j)/2q} |f(x_1,x_2)| \\
&\le C_p 2^{r_0(2\omega-(1+\alpha q-q)/q)} \mathcal{M}_p^{\omega,1} f(x_1,x_2) + C_p 2^{-r_0(1+\alpha q-q)/q} |f(x_1,x_2)| \\
&\le C(nr)^{2\omega-(1+\alpha q-q)/q} \mathcal{M}_p^{\omega,1} f(x_1,x_2) + C(nr)^{-(1+\alpha q-q)} |f(x_1,x_2)| \to 0
\end{aligned}
$$

as $n \to \infty$.

On A_5, $t_2 > t_1/2$ and so (4.6.4) implies

$$
\begin{aligned}
&\int_{A_5 \cap S_{r/2}} |f(x_1-t_1,x_2-t_2) - f(x_1,x_2)| \left|K_n^{1,\alpha}(t_1,t_2)\right| \, dt \\
&\le \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} \left(\int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x_1-t_1,x_2-t_2) - f(x_1,x_2)|^p \, dt_2 \, dt_1 \right)^{1/p} \\
&\quad \left(\int_{2^i/n}^{2^{i+1}/n} \int_{t_1-1/n}^{t_1} n^{q(1-\alpha)} t_1^{-q(\alpha+1)} \, dt_2 \, dt_1 \right)^{1/q}.
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
\int_{2^i/n}^{2^{i+1}/n} \int_{t_1-1/n}^{t_1} n^{q(1-\alpha)} t_1^{-q(\alpha+1)} \, dt_2 \, dt_1 &\le n^{-1} n^{q(1-\alpha)} \left(\frac{2^i}{n} \right)^{1-q(\alpha+1)} \\
&\le C \left(\frac{n}{2^i} \right)^{2q-2} 2^{-i(1+\alpha q-q)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_{A_5 \cap S_{r/2}} |f(x_1-t_1,x_2-t_2) - f(x_1,x_2)| \left|K_n^{1,\alpha}(t_1,t_2)\right| \, dt \\
&\le \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} 2^{-\omega(i+j)} \\
&\quad \left(\frac{n^2}{2^{i+j}} \int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x_1-t_1,x_2-t_2) - f(x_1,x_2)|^p \, dt_2 \, dt_1 \right)^{1/p} \\
&\le C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} U_{r,p}^{\omega,1} f(x_1,x_2) < C\epsilon.
\end{aligned}
$$

Let us use (4.6.6):

$$\int_{A_{10}\cap S'_{r/2}} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|\left|K_n^{1,\alpha}(t_1,t_2)\right| dt$$
$$\leq \sum_{i=1}^{r_0}\sum_{j=i-1}^{i}\left(\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{\pi-2^{j+1}/n}^{\pi-2^j/n} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|^p\, dt_1\, dt_2\right)^{1/p}$$
$$\left(\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{t_2}^{t_2+1/n} n^{q(1-\alpha)}(\pi-t_2)^{-q(\alpha+1)}\, dt_1\, dt_2\right)^{1/q}.$$

Then

$$\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{t_2}^{t_2+1/n} n^{q(1-\alpha)}(\pi-t_2)^{-q(\alpha+1)}\, dt_1\, dt_2 \leq n^{-1}n^{q(1-\alpha)}\left(\frac{2^i}{n}\right)^{1-q(\alpha+1)}$$
$$\leq C\left(\frac{n}{2^i}\right)^{2q-2} 2^{-i(1+\alpha q-q)}$$

and

$$\int_{A_{10}\cap S'_{r/2}} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|\left|K_n^{1,\alpha}(t_1,t_2)\right| dt$$
$$\leq \sum_{i=1}^{r_0}\sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)}2^{-\omega(i+j)}$$
$$\left(\frac{n^2}{2^{i+j}}\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{\pi-2^{j+1}/n}^{\pi-2^j/n} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|^p\, dt_2\, dt_1\right)^{1/p}$$
$$\leq C_p\sum_{i=1}^{r_0}\sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)}U_{r,p}^{\omega,1}f(x_1,x_2) < C\epsilon.$$

Finally,

$$\int_{A_5\cap S^c_{r/2}} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|\left|K_n^{1,\alpha}(t_1,t_2)\right| dt$$
$$+\int_{A_{10}\cap (S'_{r/2})^c} |f(x_1-t_1,x_2-t_2)-f(x_1,x_2)|\left|K_n^{1,\alpha}(t_1,t_2)\right| dt$$
$$\leq C_p\sum_{i=r_0}^{\infty}\sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)}\mathcal{M}_p^{\omega,1}f(x_1,x_2)$$
$$+C_p\sum_{i=r_0}^{\infty}\sum_{j=i-1}^{i} 2^{-(1+\alpha q-q)(i+j)/2q}|f(x_1,x_2)|$$
$$\leq C_p 2^{r_0(2\omega-(1+\alpha q-q)/q)}\mathcal{M}_p^{\omega,1}f(x_1,x_2)+C_p2^{-r_0(1+\alpha q-q)/q}|f(x_1,x_2)|\to 0$$

as $n \to \infty$. The proof of the theorem is complete. ■

Let us point out this result for $\alpha \geq 1$. Recall that for $\alpha = 1$, we get the ℓ_1-Fejér means.

Theorem 4.6.4 *Suppose that* $1 \leq \alpha < \infty$, $1 < p < \infty$, $1/p + 1/q = 1$, $0 < \omega < 1/2q$ *and* $\mathcal{M}_p^{\omega,1} f(x)$ *is finite. If* $f \in L_p(\mathbb{T}^d)$ *is periodic with respect to* π *and* x *is a* (p, ω)*-Lebesgue point of* f*, then*

$$\lim_{n\to\infty} \sigma_n^{1,\alpha} f(x) = f(x).$$

Recall that the ℓ_1-θ-means were introduced by

$$\sigma_n^{1,\theta} f(x) := \sum_{k\in\mathbb{Z}^d} \theta\left(\frac{\|k\|_1}{n}\right) \widehat{f}(k) e^{\imath k\cdot x}$$

in Sect. 2.6.1. The next two results can be proved as Theorems 4.5.9 and 4.5.10. For more details see the papers [325, 326].

Theorem 4.6.5 *Suppose that* θ *satisfies (2.6.2) and (2.6.3),* $0 < \omega < 1/d$ *and* $\mathcal{M}^\omega f(x)$ *is finite. If* $f \in L_1(\mathbb{T}^d)$ *is periodic with respect to* π *and* x *is a strong* $(1, \omega)$*-Lebesgue point of* f*, then*

$$\lim_{n\to\infty} \sigma_n^{1,\theta} f(x) = f(x).$$

Theorem 4.6.6 *Suppose that* θ *satisfies (2.6.2) and (2.6.3),* $1 < p < \infty$, $1/p + 1/q = 1$, $0 < \omega < 1/2q$ *and* $\mathcal{M}_p^{\omega,1} f(x)$ *is finite. If* $f \in L_p(\mathbb{T}^d)$ *is periodic with respect to* π *and* x *is a* (p, ω)*-Lebesgue point of* f*, then*

$$\lim_{n\to\infty} \sigma_n^{1,\theta} f(x) = f(x).$$