Chapter 3
Rectangular Summability of Higher Dimensional Fourier Series

In this chapter, we investigate the rectangular summability of d-dimensional Fourier series. We consider two types of convergence, the so-called restricted and unrestricted convergence. In the first case, $n \in \mathbb{N}^d$ is in a cone or a cone-like set and $n \to \infty$ while in the second case, we have $n \in \mathbb{N}^d$ and $\min(n_1, \ldots, n_d) \to \infty$, which is called Pringsheim's convergence. Similarly, we consider two types of maximal operators, the restricted one defined on a cone or cone-like set and the unrestricted one defined on $\mathbb{N}^d$. We prove similar results as for the ℓ_q-summability. In the restricted case, we use the Hardy space $H_p^\square(\mathbb{T}^d)$ and in the unrestricted case a new Hardy space $H_p(\mathbb{T}^d)$.

In the first section, we present the basic definitions for the rectangular summability and verify some estimations for the kernel functions. In the next section, we can find the $L_p(\mathbb{T}^d)$ convergence of the rectangular Cesàro and Riesz means. In Sect. 3.3, we investigate the restricted maximal operators of the rectangular Cesàro and Riesz means by taking the supremum over a cone. We show that these operators are bounded from the Hardy space $H_p^\square(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ for any $p > p_0$, where $p_0 < 1$ is depending again on the summation and on the dimension. As a consequence, we obtain the restricted almost everywhere convergence of the summability means. Similar results are also shown for cone-like sets.

We introduce the product Hardy spaces $H_p(\mathbb{T}^d)$ and present the atomic decomposition and a boundedness result for these spaces. Moreover, we show that the unrestricted maximal operator of the rectangular Cesàro and Riesz means is bounded from $H_p(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ for any $p > p_0$. This implies the almost everywhere convergence of the summability means in Pringsheim's sense. In the last section, we consider the rectangular θ-summability and prove similar results as mentioned above. We give a sufficient and necessary condition for the uniform and $L_1(\mathbb{T}^d)$ convergence of the rectangular θ-means.

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3.1 Summability Kernels

Definition 3.1.1 For $f \in L_1(\mathbb{T}^d)$ and $n \in \mathbb{N}^d$, the nth rectangular Fejér means $\sigma_n f$ of the Fourier series of f and the nth rectangular Fejér kernel K_n are introduced by

$$\sigma_n f(x) = \sum_{|k_1|\le n_1} \cdots \sum_{|k_d|\le n_d} \prod_{i=1}^{d} \left(1 - \frac{|k_i|}{n_i}\right) \widehat{f}(k) e^{\imath k\cdot x}$$

and

$$K_n(t) := \sum_{|k_1|\le n_1} \cdots \sum_{|k_d|\le n_d} \prod_{i=1}^{d} \left(1 - \frac{|k_i|}{n_i}\right) e^{\imath k\cdot t},$$

respectively.

Again, we generalize this definition as follows.

Definition 3.1.2 Let $f \in L_1(\mathbb{T}^d)$, $n \in \mathbb{N}^d$ and $\alpha \ge 0$. The nth rectangular Cesàro means $\sigma_n^\alpha f$ of the Fourier series of f and the nth rectangular Cesàro kernel K_n^α are introduced by

$$\sigma_n^\alpha f(x) := \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{|k_1|\le n_1} \cdots \sum_{|k_d|\le n_d} \prod_{i=1}^{d} A_{n_i-1-|k_i|}^\alpha \widehat{f}(k) e^{\imath k\cdot x}$$

and

$$K_n^\alpha(t) := \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{|k_1|\le n_1} \cdots \sum_{|k_d|\le n_d} \prod_{i=1}^{d} A_{n_i-1-|k_i|}^\alpha e^{\imath k\cdot t},$$

respectively.

The Cesàro means are also called rectangular (C, α)-means. If $\alpha = 1$, then these are the rectangular Fejér means and if $\alpha = 0$, then the rectangular partial sums (see Fig. 3.1).

Definition 3.1.3 For $f \in L_1(\mathbb{T}^d)$, $n \in \mathbb{N}^d$ and $0 < \alpha, \gamma < \infty$, the nth rectangular Riesz means $\sigma_n^{\alpha,\gamma} f$ of the Fourier series of f and the nth rectangular Riesz kernel $K_n^{\alpha,\gamma}$ are given by

$$\sigma_n^{\alpha,\gamma} f(x) := \sum_{|k_1|\le n_1} \cdots \sum_{|k_d|\le n_d} \prod_{i=1}^{d} \left(1 - \left(\frac{|k_i|}{n_i}\right)^\gamma\right)^\alpha \widehat{f}(k) e^{\imath k\cdot x}$$

and

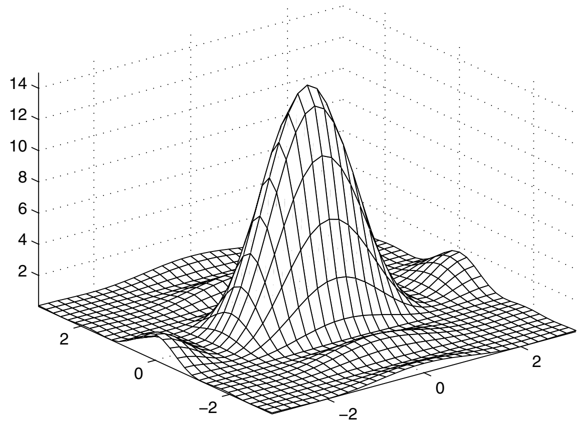

Fig. 3.1 The rectangular Fejér kernel K_n with $d = 2$, $n_1 = 3, n_2 = 5$

$$K_n^{\alpha,\gamma}(t) := \sum_{|k_1| \le n_1} \cdots \sum_{|k_d| \le n_d} \prod_{i=1}^{d} \left(1 - \left(\frac{|k_i|}{n_i}\right)^{\gamma}\right)^{\alpha} e^{\imath k \cdot t},$$

respectively.

For $\alpha = \gamma = 1$, we get back the rectangular Fejér means. The next results follow from

$$K_n^{\alpha} = K_{n_1}^{\alpha} \otimes \cdots \otimes K_{n_d}^{\alpha} \tag{3.1.1}$$

and

$$K_n^{\alpha,\gamma} = K_{n_1}^{\alpha,\gamma} \otimes \cdots \otimes K_{n_d}^{\alpha,\gamma}, \tag{3.1.2}$$

where $K_{n_j}^{\alpha}$ and $K_{n_j}^{\alpha,\gamma}$ are the corresponding one-dimensional kernels.

Lemma 3.1.4 *If* $0 \le \alpha, \gamma < \infty$ *and* $n \in \mathbb{N}^d$, *then*

$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K_n^{\alpha}(t)\, dt = 1$$

and

$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K_n^{\alpha,\gamma}(t)\, dt = 1.$$

Lemma 3.1.5 *If* $0 \le \alpha, \gamma < \infty$ *and* $n \in \mathbb{N}^d$, *then*

$$|K_n^{\alpha}(t)| \le C \prod_{i=1}^{d} n_i \quad \text{and} \quad |K_n^{\alpha,\gamma}(t)| \le C \prod_{i=1}^{d} n_i \quad (t \in \mathbb{T}^d).$$

Lemma 3.1.6 *For* $f \in L_1(\mathbb{T}^d)$, $n \in \mathbb{N}^d$ *and* $0 < \alpha, \gamma < \infty$,

$$\sigma_n^\alpha f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n^\alpha(t)\, dt$$

and

$$\sigma_n^{\alpha,\gamma} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n^{\alpha,\gamma}(t)\, dt.$$

The rectangular Cesàro means are the weighted arithmetic means of the rectangular partial sums.

Lemma 3.1.7 *For* $f \in L_1(\mathbb{T}^d)$, $\alpha > 0$ *and* $n \in \mathbb{N}^d$, *we have*

$$\sigma_n f(x) = \frac{1}{\prod_{i=1}^d n_i} \sum_{k_1=1}^{n_1-1} \cdots \sum_{k_d=1}^{n_d-1} s_k f(x),$$

$$\sigma_n^\alpha f(x) = \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_d=0}^{n_d-1} \prod_{i=1}^d A_{n_i-1-k_i}^{\alpha-1} s_k f(x)$$

and

$$K_n^\alpha(t) = \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_d=0}^{n_d-1} \prod_{i=1}^d A_{n_i-1-k_i}^{\alpha-1} D_k(t).$$

We will use the next estimation of the derivatives of the one-dimensional kernel functions.

Theorem 3.1.8 *For* $0 < \alpha \le r+1$, $n \in \mathbb{P}$ *and* $t \in \mathbb{T}$, $t \ne 0$,

$$\left| \left(K_n^\alpha\right)^{(r)}(t) \right| \le C n^{r+1} \qquad \text{and} \qquad \left| \left(K_n^\alpha\right)^{(r)}(t) \right| \le \frac{C}{n^{\alpha-r} |t|^{\alpha+1}}.$$

Proof Similar to Lemma 1.2.4 and Theorem 1.4.16, we have

$$|D_k^{(r)}| \le C k^{r+1} \qquad (k \in \mathbb{P}),$$

which implies the first inequality.

We have seen in Theorem 1.4.16 and Lemma 1.4.14 that

$$\begin{aligned}
K_n^\alpha(t) &= \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} \frac{\sin((k+1/2)t)}{\sin(t/2)} \\
&= \frac{1}{A_{n-1}^\alpha \sin(t/2)} \Im \left(\sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} e^{\imath(k+1/2)t} \right) \\
&= \frac{1}{A_{n-1}^\alpha \sin(t/2)} \Im \left(e^{\imath(n-1/2)t} \sum_{j=0}^{n-1} A_j^{\alpha-1} e^{-\imath jt} \right).
\end{aligned}$$

In this proof, we use the notation

$$u(\beta) := \sum_{k=0}^{n-1} A_k^\beta e^{-\imath kt}.$$

Abel rearrangement and Lemma 1.4.8 imply

$$\begin{aligned}
u(\beta) &= \sum_{k=0}^{n-2} \left(A_k^\beta - A_{k+1}^\beta \right) S_k + A_{n-1}^\beta S_{n-1} \\
&= -\sum_{k=0}^{n-2} A_{k+1}^{\beta-1} S_k + A_{n-1}^\beta S_{n-1} \\
&= -\sum_{k=1}^{n-1} A_k^{\beta-1} S_{k-1} + A_{n-1}^\beta S_{n-1},
\end{aligned}$$

where

$$S_k := \sum_{j=0}^{k} e^{-\imath jt} = \frac{1 - e^{-\imath(k+1)t}}{1 - e^{-\imath t}}.$$

Then

$$\begin{aligned}
u(\beta) &= -\sum_{k=1}^{n-1} A_k^{\beta-1} \frac{1 - e^{-\imath kt}}{1 - e^{-\imath t}} + A_{n-1}^\beta \frac{1 - e^{-\imath nt}}{1 - e^{-\imath t}} \\
&= \left(1 - e^{-\imath t}\right)^{-1} \left(\sum_{k=1}^{n-1} A_k^{\beta-1} e^{-\imath kt} - \sum_{k=1}^{n-1} A_k^{\beta-1} + A_{n-1}^\beta - A_{n-1}^\beta e^{-\imath nt} \right) \\
&= \left(1 - e^{-\imath t}\right)^{-1} u(\beta-1) - \left(1 - e^{-\imath t}\right)^{-1} A_{n-1}^\beta e^{-\imath nt}.
\end{aligned}$$

Iterating this result s-times ($s \in \mathbb{N}$),

$$\begin{aligned}
u(\beta) &= \left(1-e^{-\imath t}\right)^{-2} u(\beta-2) - \left(1-e^{-\imath t}\right)^{-2} A_{n-1}^{\beta-1} e^{-\imath nt} \\
&\quad - \left(1-e^{-\imath t}\right)^{-1} A_{n-1}^{\beta} e^{-\imath nt} \\
&= \dots \\
&= \left(1-e^{-\imath t}\right)^{-s} u(\beta-s) - e^{-\imath nt} \sum_{j=1}^{s} A_{n-1}^{\beta-j+1} \left(1-e^{-\imath t}\right)^{-j}.
\end{aligned}$$

Writing $\beta = \alpha - 1$ and using (1.4.11), we conclude

$$\begin{aligned}
K_n^{\alpha}(t) &= \frac{1}{A_{n-1}^{\alpha} \sin(t/2)} \Im\left(e^{\imath(n-1/2)t} u(\alpha-1)\right) \\
&= \frac{1}{A_{n-1}^{\alpha} \sin(t/2)} \Im\Bigg(e^{\imath(n-1/2)t} \left(1-e^{-\imath t}\right)^{-s} u(\alpha-1-s) \\
&\quad - e^{-\imath t/2} \sum_{j=1}^{s} A_{n-1}^{\alpha-j} \left(1-e^{-\imath t}\right)^{-j}\Bigg) \\
&= \frac{1}{A_{n-1}^{\alpha} \sin(t/2)} \Im\Bigg(e^{\imath(n-1/2)t} \left(1-e^{-\imath t}\right)^{-s} \sum_{k=0}^{n-1} A_k^{\alpha-1-s} e^{-\imath kt} \\
&\quad - e^{-\imath t/2} \sum_{j=1}^{s} A_{n-1}^{\alpha-j} \left(1-e^{-\imath t}\right)^{-j}\Bigg).
\end{aligned}$$

The equality

$$\begin{aligned}
K_n^{\alpha}(t) &= \frac{1}{A_{n-1}^{\alpha} \sin(t/2)} \Im\Bigg(e^{\imath(n-1/2)t} \left(1-e^{-\imath t}\right)^{-\alpha} \\
&\quad - \left(1-e^{-\imath t}\right)^{-s} \sum_{k=n}^{\infty} A_k^{\alpha-1-s} e^{-\imath(k-n+1/2)t} - e^{-\imath t/2} \sum_{j=1}^{s} A_{n-1}^{\alpha-j} \left(1-e^{-\imath t}\right)^{-j}\Bigg) \\
&=: I_1(t) + I_2(t) + I_3(t)
\end{aligned}$$

follows from (1.4.5). Suppose that $|t| \geq 1/n$. The rth derivative of I_1 can be estimated as

$$\begin{aligned}
\left|I_1^{(r)}(t)\right| &\leq \frac{C}{A_{n-1}^{\alpha}} \sum_{l=0}^{r} \frac{n^l}{|t|^{1+\alpha+r-l}} \\
&\leq C|t|^{-r-1} \sum_{l=0}^{r} (n|t|)^{l-\alpha} \\
&\leq C|t|^{-r-1} (n|t|)^{r-\alpha} = C n^{r-\alpha} |t|^{-\alpha-1}.
\end{aligned}$$

To estimate the second term, we choose $s > \alpha + r$. Then the r times termwise differentiated series in I_2 is absolutely convergent. Thus

$$\begin{aligned}\left|I_2^{(r)}(t)\right| &\le \frac{C}{A_{n-1}^{\alpha}} \sum_{l=0}^{r} \sum_{k=n}^{\infty} A_k^{\alpha-1-s} \frac{(k-n+1/2)^l}{|t|^{1+s+r-l}} \\ &\le \frac{C}{A_{n-1}^{\alpha}} \sum_{l=0}^{r} |t|^{-1-s-r+l} \sum_{k=n}^{\infty} k^{\alpha-1-s+l} \\ &\le C \sum_{l=0}^{r} |t|^{-1-s-r+l} n^{l-s} \\ &\le C|t|^{-r-1} \sum_{l=0}^{r} (n|t|)^{l-s} \\ &\le C|t|^{-r-1}(n|t|)^{r-s} \le C|t|^{-r-1}(n|t|)^{r-\alpha} = Cn^{r-\alpha}|t|^{-\alpha-1}.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|I_3^{(r)}(t)\right| &\le \frac{C}{A_{n-1}^{\alpha}} \sum_{j=1}^{s} A_{n-1}^{\alpha-j} \frac{1}{|t|^{1+j+r}} \\ &\le C|t|^{-r-1} \sum_{j=1}^{s} (n|t|)^{-j} \\ &\le C|t|^{-r-1}(n|t|)^{-1} \le C|t|^{-r-1}(n|t|)^{r-\alpha} = Cn^{r-\alpha}|t|^{-\alpha-1},\end{aligned}$$

because $0 < \alpha \le r+1$. Finally, if $|t| < 1/n$, then the first inequality of our theorem implies the second one. ■

The next lemma can be proved as Lemma 1.4.13.

Lemma 3.1.9 *For $\alpha > -1$ and $h > 0$, we have*

$$\sigma_n^{\alpha+h} f = \frac{1}{\prod_{i=1}^{d} A_{n_i-1}^{\alpha+h}} \sum_{k_1=1}^{n_1} \cdots \sum_{k_d=1}^{n_d} \prod_{i=1}^{d} A_{n_i-k_i}^{h-1} A_{k_i-1}^{\alpha} \sigma_k^{\alpha} f.$$

The same results hold if we choose different exponents α_i and γ_i in the products.

3.2 Norm Convergence of Rectangular Summability Means

The next results follow from (3.1.1), (3.1.2), Theorem 2.3.3 and from the one-dimensional theorems.

Theorem 3.2.1 *If* $0 < \alpha \le 1$, *then*

$$\sup_{n\in\mathbb{N}^d} \int_{\mathbb{T}^d} \left|K_n^\alpha(x)\right| \, dx \le C.$$

If $0 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$, *then*

$$\sup_{n\in\mathbb{N}^d} \int_{\mathbb{T}^d} \left|K_n^{\alpha,\gamma}(x)\right| \, dx \le C.$$

Theorem 3.2.2 *If* $1 \le p < \infty$, $0 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$, *then*

$$\sup_{n\in\mathbb{N}^d} \left\|\sigma_n^\alpha f\right\|_p \le C\|f\|_p$$

and

$$\sup_{n\in\mathbb{N}^d} \left\|\sigma_n^{\alpha,\gamma} f\right\|_p \le C\|f\|_p.$$

Moreover, for all $f \in L_p(\mathbb{T}^d)$,

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \qquad \text{in the } L_p(\mathbb{T}^d)\text{-norm}$$

and

$$\lim_{n\to\infty} \sigma_n^{\alpha,\gamma} f = f \qquad \text{in the } L_p(\mathbb{T}^d)\text{-norm.}$$

Here, the convergence is understood in Pringsheim's sense as in Theorem 2.1.8.

3.3 Almost Everywhere Restricted Summability over a Cone

In this section, we investigate the convergence of the rectangular Cesàro and Riesz summability means taken in a cone. For a given $\tau \ge 1$, we define a cone by

$$\mathbb{R}_\tau^d := \{x \in \mathbb{R}_+^d : \tau^{-1} \le x_i/x_j \le \tau, i, j = 1, \dots, d\}. \tag{3.3.1}$$

The choice $\tau = 1$ yields the diagonal. The definition of the Cesàro and Riesz means can be extended to distributions as follows.

Definition 3.3.1 Let $f \in D(\mathbb{T}^d)$, $n \in \mathbb{N}^d$ and $0 \le \alpha, \gamma < \infty$. The nth rectangular Cesàro means $\sigma_n^\alpha f$ and rectangular Riesz means $\sigma_n^{\alpha,\gamma} f$ of the Fourier series of f are given by

$$\sigma_n^\alpha f := f * K_n^\alpha$$

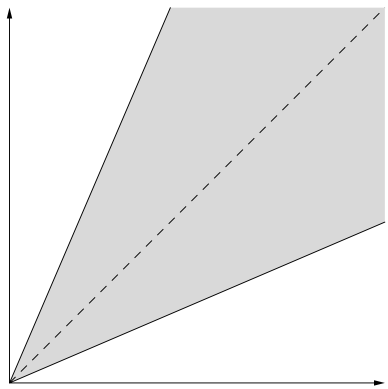

Fig. 3.2 The cone for $d = 2$

and

$$\sigma_n^{\alpha,\gamma} f := f * K_n^{\alpha,\gamma},$$

respectively.

Definition 3.3.2 We define the restricted maximal Cesàro and restricted maximal Riesz operator by

$$\sigma_\square^\alpha f := \sup_{n \in \mathbb{R}_\tau^d} |\sigma_n^\alpha f|$$

and

$$\sigma_\square^{\alpha,\gamma} f := \sup_{n \in \mathbb{R}_\tau^d} |\sigma_n^{\alpha,\gamma} f|,$$

respectively.

If $\alpha = 1$, we obtain the restricted maximal Fejér operator $\sigma_\square f$. As we can see on Fig. 3.2, in the restricted maximal operator the supremum is taken on a cone only. Marcinkiewicz and Zygmund [234] were the first who considered the restricted convergence. We show that the restricted maximal operator is bounded from $H_p^\square(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$.

The next result follows easily from Theorem 3.2.1.

Theorem 3.3.3 *If* $0 < \alpha \le 1$, *then*

$$\left\| \sigma_\square^\alpha f \right\|_\infty \le C \, \|f\|_\infty \qquad (f \in L_\infty(\mathbb{T}^d)).$$

If $0 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$, *then*

$$\left\| \sigma_\square^{\alpha,\gamma} f \right\|_\infty \le C \, \|f\|_\infty \qquad (f \in L_\infty(\mathbb{T}^d)).$$

Theorem 3.3.4 *If* $0 < \alpha \le 1$ *and*

$$\max\left\{\frac{d}{d+1}, \frac{1}{\alpha+1}\right\} < p \leq \infty,$$

then

$$\left\|\sigma_{\square}^{\alpha} f\right\|_{p} \leq C_{p}\|f\|_{H_{p}^{\square}} \qquad (f \in H_{p}^{\square}(\mathbb{T}^{d})).$$

Proof We have seen in Theorem 3.1.8 that

$$\left|K_{n_j}^{\alpha}(t)\right| \leq \frac{C}{n_j^{\alpha}|t|^{\alpha+1}} \qquad (t \neq 0) \tag{3.3.2}$$

and

$$\left|(K_{n_j}^{\alpha})'(t)\right| \leq \frac{C}{n_j^{\alpha-1}|t|^{\alpha+1}} \qquad (t \neq 0). \tag{3.3.3}$$

Let a be an arbitrary $H_{p}^{\square}$-atom with support $I = I_1 \times I_2$ and

$$2^{-K-1} < |I_1|/\pi = |I_2|/\pi \leq 2^{-K} \qquad (K \in \mathbb{N}).$$

We can suppose again that the center of I is zero. In this case,

$$[-\pi 2^{-K-2}, \pi 2^{-K-2}] \subset I_1, I_2 \subset [-\pi 2^{-K-1}, \pi 2^{-K-1}].$$

Choose $s \in \mathbb{N}$ such that $2^{s-1} < \tau \leq 2^{s}$. It is easy to see that if $n_1 \geq k$ or $n_2 \geq k$, then we have $n_1, n_2 \geq k2^{-s}$. Indeed, since (n_1, n_2) is in a cone, $n_1 \geq k$ implies $n_2 \geq \tau^{-1} n_1 \geq k2^{-s}$. By Theorem 2.4.19, it is enough to prove that

$$\int_{\mathbb{T}^2 \setminus 4(I_1 \times I_2)} \left|\sigma_{\square}^{\alpha} a(x_1, x_2)\right|^{p} \, dx_1 \, dx_2 \leq C_p. \tag{3.3.4}$$

First we integrate over $(\mathbb{T} \setminus 4I_1) \times 4I_2$. Obviously,

$$\begin{aligned}
&\int_{\mathbb{T}\setminus 4I_1} \int_{4I_2} \left|\sigma_{\square}^{\alpha} a(x_1, x_2)\right|^{p} \, dx_1 \, dx_2 \\
&\quad \leq \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{4I_2} \sup_{n_1, n_2 \geq 2^{K-s}} \left|\sigma_{n_1,n_2}^{\alpha} a(x_1, x_2)\right|^{p} \, dx_1 \, dx_2 \\
&\qquad + \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{4I_2} \sup_{n_1, n_2 < 2^{K}} \left|\sigma_{n_1,n_2}^{\alpha} a(x_1, x_2)\right|^{p} \, dx_1 \, dx_2 \\
&\quad =: (A) + (B).
\end{aligned}$$

We can suppose that $i > 0$. Using that

$$\int_{\mathbb{T}} \left| K_{n_2}^{\alpha}(x_2) \right| \, dx_2 \le C \qquad (n_2 \in \mathbb{N})$$

(see Corollary 1.5.3), (3.3.2) and the definition of the atom, we conclude

$$\begin{aligned}\left|\sigma_{n_1,n_2}^{\alpha} a(x_1, x_2)\right| &= \left| \int_{I_1} \int_{I_2} a(t_1, t_2) K_n^{\alpha}(x_1 - t_1) K_{n_2}^{\alpha}(x_2 - t_2) \, dt_1 \, dt_2 \right| \\ &\le C_p 2^{2K/p} \int_{I_1} \frac{1}{n_1^{\alpha} |x_1 - t_1|^{\alpha+1}} \, dt_1.\end{aligned}$$

For $x_1 \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ $(i \ge 1)$ and $t_1 \in I_1$, we have

$$\frac{1}{|x_1 - t_1|^{\nu}} \le \frac{1}{(\pi i 2^{-K} - \pi 2^{-K-1})^{\nu}} \le \frac{C 2^{K\nu}}{i^{\nu}} \qquad (\nu > 0). \tag{3.3.5}$$

From this, it follows that

$$\left|\sigma_{n_1,n_2}^{\alpha} a(x_1, x_2)\right| \le C_p 2^{2K/p + K\alpha} \frac{1}{n_1^{\alpha} i^{\alpha+1}}.$$

Since $n_1 \ge 2^K 2^{-s}$, we obtain

$$(A) \le C_p \sum_{i=1}^{2^K-1} 2^{-2K} 2^{2K + K\alpha p} \frac{1}{2^{K\alpha p} i^{(\alpha+1)p}} \le C_p \sum_{i=1}^{2^K-1} \frac{1}{i^{(\alpha+1)p}},$$

which is a convergent series if $p > 1/(\alpha+1)$.

To consider (B), let $I_1 = I_2 = (-\mu, \mu)$ and

$$A_1(x_1, v) := \int_{-\pi}^{x_1} a(t_1, v) \, dt_1, \qquad A_2(x_1, x_2) := \int_{-\pi}^{x_2} A_1(x_1, t_2) \, dt_2. \tag{3.3.6}$$

Then

$$|A_k(x_1, x_2)| \le C_p 2^{K(2/p - k)}. \tag{3.3.7}$$

Integrating by parts, we get that

$$\begin{aligned}&\int_{I_1} a(t_1, t_2) K_{n_1}^{\alpha}(x_1 - t_1) \, dt_1 \\ &\quad = A_1(\mu, t_2) K_{n_1}^{\alpha}(x_1 - \mu) - \int_{I_1} A_1(t_1, t_2) (K_{n_1}^{\alpha})'(x_1 - t_1) \, dt_1.\end{aligned} \tag{3.3.8}$$

Recall that the one-dimensional kernel $K_{n_2}^{\alpha}$ satisfies

$$\left| K_{n_2}^{\alpha} \right| \le C n_2 \qquad (n_2 \in \mathbb{N}).$$

For $x_1 \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$, the inequalities (3.3.2), (3.3.5) and (3.3.7) imply

$$\begin{aligned}
&\left| \int_{I_2} A_1(\mu, t_2) K_{n_1}^{\alpha}(x_1 - \mu) K_{n_2}^{\alpha}(x_2 - t_2)\, dt_2 \right| \\
&\quad \le C_p 2^{2K/p-K} 2^{-K} \frac{1}{n_1^{\alpha} |x_1 - \mu|^{\alpha+1}} n_2 \\
&\quad \le C_p 2^{2K/p+K\alpha-K} n_1^{1-\alpha} \frac{1}{i^{\alpha+1}}.
\end{aligned}$$

Moreover, by (3.3.3), (3.3.5) and (3.3.7),

$$\begin{aligned}
&\left| \int_{I_2} \int_{I_1} A_1(t_1, t_2) (K_{n_1}^{\alpha})'(x_1 - t_1) K_{n_2}^{\alpha}(x_2 - t_2)\, dt_2\, dt_1 \right| \\
&\quad \le C_p 2^{2K/p-K} \int_{I_1} \frac{1}{n^{\alpha-1} |x_1 - t_1|^{\alpha+1}}\, dt_1 \\
&\quad \le C_p 2^{2K/p+K\alpha-K} n_1^{1-\alpha} \frac{1}{i^{\alpha+1}}.
\end{aligned}$$

Consequently,

$$(B) \le C_p \sum_{i=1}^{2^K-1} 2^{-2K} 2^{2K+K\alpha p-Kp} 2^{K(1-\alpha)p} \frac{1}{i^{(\alpha+1)p}} \le C_p \sum_{i=1}^{2^K-1} \frac{1}{i^{(\alpha+1)p}} < \infty,$$

because $p > 1/(\alpha+1)$. Hence, we have proved that in this case

$$\int_{\mathbb{T}\setminus 4I_1} \int_{4I_2} \left| \sigma_{\square}^{\alpha} a(x_1, x_2) \right|^p \, dx_1\, dx_2 \le C_p.$$

Next, we integrate over $(\mathbb{T} \setminus 4I_1) \times (\mathbb{T} \setminus 4I_2)$:

$$\begin{aligned}
&\int_{\mathbb{T}\setminus 4I_1} \int_{\mathbb{T}\setminus 4I_2} \left| \sigma_{\square}^{\alpha} a(x_1, x_2) \right|^p \, dx_1\, dx_2 \\
&\quad \le \sum_{|i|=1}^{\infty} \sum_{|j|=1}^{\infty} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{\pi j 2^{-K}}^{\pi(j+1)2^{-K}} \sup_{n_1, n_2 \ge 2^{K-s}} \left| \sigma_{n_1,n_2}^{\alpha} a(x_1, x_2) \right|^p \, dx_1\, dx_2 \\
&\qquad + \sum_{|i|=1}^{\infty} \sum_{|j|=1}^{\infty} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{\pi j 2^{-K}}^{\pi(j+1)2^{-K}} \sup_{n_1, n_2 < 2^{K}} \left| \sigma_{n_1,n_2}^{\alpha} a(x_1, x_2) \right|^p \, dx_1\, dx_2 \\
&\quad =: (C) + (D).
\end{aligned}$$

We may suppose again that $i, j > 0$. For $x_1 \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ and $x_2 \in [\pi j 2^{-K}, \pi(j+1)2^{-K})$, we have by (3.3.2) and (3.3.5) that

$$\begin{aligned}\left|\sigma_{n_1,n_2}^{\alpha} a(x_1, x_2)\right| &\le C_p 2^{2K/p} \int_{I_1} \frac{1}{n_1^{\alpha}|x_1 - t_1|^{\alpha+1}}\, dt_1 \int_{I_2} \frac{1}{n_2^{\alpha}|x_2 - t_2|^{\alpha+1}}\, dt_2 \\ &\le C_p \frac{2^{2K/p+K\alpha+K\alpha}}{n_1^{\alpha} n_2^{\alpha} i^{\alpha+1} j^{\alpha+1}}.\end{aligned}$$

This implies that

$$\begin{aligned}(C) &\le C_p \sum_{i=1}^{2^K-1} \sum_{j=1}^{2^K-1} 2^{-2K} \frac{2^{2K+K\alpha p+K\alpha p}}{2^{K\alpha p+K\alpha p} i^{(\alpha+1)p} j^{(\alpha+1)p}} \\ &\le C_p \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{i^{(\alpha+1)p} j^{(\alpha+1)p}} < \infty.\end{aligned}$$

Using (3.3.8) and integrating by parts in both variables, we get that

$$\begin{aligned}&\int_{I_1} \int_{I_2} a(t_1, t_2) K_{n_1}^{\alpha}(x_1 - t_1) K_{n_2}^{\alpha}(x_2 - t_2)\, dt_1\, dt_2 \\ &= - \int_{I_2} A_2(\mu, t_2) K_{n_1}^{\alpha}(x_1 - \mu)(K_{n_2}^{\alpha})'(x_2 - t_2)\, dt_2 \\ &\quad + \int_{I_1} A_2(t_1, \mu)(K_{n_1}^{\alpha})'(x_1 - t_1) K_{n_2}^{\alpha}(x_2 - \mu)\, dt_1 \\ &\quad - \int_{I_1} \int_{I_2} A_2(t_1, t_2)(K_{n_1}^{\alpha})'(x_1 - t_1)(K_{n_2}^{\alpha})'(x_2 - t_2)\, dt_1\, dt_2 \\ &=: D_{n_1,n_2}^{1}(x_1, x_2) + D_{n_1,n_2}^{2}(x_1, x_2) + D_{n_1,n_2}^{3}(x_1, x_2).\end{aligned} \tag{3.3.9}$$

Note that $A(\mu, -\mu) = A(\mu, \mu) = 0$. Since $|K_{n_1}^{\alpha}| \le C n_1$ and (3.3.2) holds as well, we obtain

$$|K_{n_1}^{\alpha}(x_1)| \le C \frac{n_1^{\eta+\alpha(\eta-1)}}{|x_1|^{(\alpha+1)(1-\eta)}}$$

for all $0 \le \eta \le 1$. Moreover, the inequality

$$|(K_{n_2}^{\alpha})'| \le C n_2^2 \qquad (n_2 \in \mathbb{N})$$

and (3.3.3) imply

$$|(K_{n_2}^{\alpha})'(x_2)| \le C \frac{n_2^{2\zeta+(\alpha-1)(\zeta-1)}}{|x_2|^{(\alpha+1)(1-\zeta)}} = C \frac{n_2^{\zeta+1+\alpha(\zeta-1)}}{|x_2|^{(\alpha+1)(1-\zeta)}} \tag{3.3.10}$$

for all $0 \le \zeta \le 1$. We use inequalities (3.3.5) and (3.3.7) to obtain

$$
\begin{aligned}
\left|D^1_{n_1,n_2}(x_1,x_2)\right| &\le C_p 2^{2K/p-2K} \frac{n_1^{\eta+\alpha(\eta-1)}}{|x_1-\mu|^{(\alpha+1)(1-\eta)}} \int_{I_2} \frac{n_2^{\zeta+1+\alpha(\zeta-1)}}{|x_2-t_2|^{(\alpha+1)(1-\zeta)}}\, dt_2 \\
&\le C_p 2^{2K/p-3K} n_1^{\eta+\alpha(\eta-1)} \left(\frac{2^K}{i}\right)^{(\alpha+1)(1-\eta)} \\
&\quad n_2^{\zeta+1+\alpha(\zeta-1)} \left(\frac{2^K}{j}\right)^{(\alpha+1)(1-\zeta)},
\end{aligned}
\tag{3.3.11}
$$

whenever $x_1 \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$, $x_2 \in [\pi j 2^{-K}, \pi(j+1)2^{-K})$ and $0 \le \eta, \zeta \le 1$. If

$$\eta + \alpha(\eta-1) + \zeta + 1 + \alpha(\zeta-1) \ge 0,$$

then

$$\sup_{n_1,n_2<2^K} \left|D^1_{n_1,n_2}(x_1,x_2)\right| \le C_p 2^{2K/p} \frac{1}{i^{(\alpha+1)(1-\eta)}} \frac{1}{j^{(\alpha+1)(1-\mu)}}$$

because (n_1,n_2) is in a cone. Choosing

$$\eta := \zeta := \max\left\{\frac{2\alpha-1}{2(\alpha+1)}, 0\right\},$$

we can see that

$$
\begin{aligned}
&\int_{\mathbb{T}\setminus 4I_1} \int_{\mathbb{T}\setminus 4I_2} \sup_{n_1,n_2<2^K} \left|D^1_{n_1,n_2}(x_1,x_2)\right|^p \, dx_1\, dx_2 \\
&\quad \le C_p \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} 2^{-2K} 2^{2K} \frac{1}{i^{3p/2\wedge(\alpha+1)p}} \frac{1}{j^{3p/2\wedge(\alpha+1)p}},
\end{aligned}
$$

which is a convergent series. The analogous estimate for $\left|D^2_{n_1,n_2}(x_1,x_2)\right|$ can be similarly proved.

For $x_1 \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ and $x_2 \in [\pi j 2^{-K}, \pi(j+1)2^{-K})$, we conclude that

$$
\begin{aligned}
\left|D^3_{n_1,n_2}(x_1,x_2)\right| &\le C_p 2^{2K/p-2K} \int_{I_1} \frac{1}{n_1^{\alpha-1}|x_1-t_1|^{\alpha+1}}\, dt_1 \int_{I_2} \frac{1}{n_2^{\alpha-1}|x_2-t_2|^{\alpha+1}}\, dt_2 \\
&\le C_p \frac{2^{2K/p-2K+K\alpha+K\alpha} n_1^{1-\alpha} n_2^{1-\alpha}}{i^{\alpha+1} j^{\alpha+1}}.
\end{aligned}
$$

So

$$\int_{\mathbb{T}\setminus 4I_1}\int_{\mathbb{T}\setminus 4I_2}\sup_{n_1,n_2<2^K}\left|D^3_{n_1,n_2}(x_1,x_2)\right|^p\,dx_1\,dx_2$$
$$\leq C_p\sum_{i=1}^{2^K-1}\sum_{j=1}^{2^K-1}2^{-2K}\frac{2^{2K-2Kp+K\alpha p+K\alpha p}2^{K(2-\alpha-\alpha)p}}{i^{(\alpha+1)p}j^{(\alpha+1)p}}$$
$$\leq C_p\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{i^{(\alpha+1)p}}\frac{1}{j^{(\alpha+1)p}}<\infty$$

by the hypothesis. The integration over $4I_1\times(\mathbb{T}\setminus 4I_2)$ can be done as above. This finishes the proof of (3.3.4) as well as the theorem. ■

Remark 3.3.5 In the d-dimensional case, the constant $d/(d+1)$ appears if we investigate the corresponding term to D^1_n. More exactly, if we integrate the term

$$\int_{I_d}A(\mu,\cdots,\mu,t_d)K^\alpha_{n_1}(x_1-\mu)\cdots K^\alpha_{n_{d-1}}(x_{d-1}-\mu)(K^\alpha_{n_d})'(x_d-t_d)\,dt_d$$

over $(\mathbb{T}\setminus 4I_1)\times\cdots\times(\mathbb{T}\setminus 4I_d)$ similar to (3.3.11), then we get that $p>d/(d+1)$.

Corollary 3.3.6 *If* $0<\alpha\leq 1$ *and* $1<p<\infty$, *then*

$$\left\|\sigma^\alpha_\square f\right\|_p\leq C_p\|f\|_p\qquad(f\in L_p(\mathbb{T}^d)).$$

Let us turn to the Riesz means.

Theorem 3.3.7 *If* $0<\alpha<\infty$, $\gamma\in\mathbb{P}$ *and*

$$\max\left\{\frac{d}{d+1},\frac{1}{\alpha\wedge 1+1}\right\}<p\leq\infty,$$

then

$$\left\|\sigma^{\alpha,\gamma}_\square f\right\|_p\leq C_p\|f\|_{H^\square_p}\qquad(f\in H^\square_p(\mathbb{T}^d)).$$

Proof Let

$$\theta(s):=\begin{cases}(1-|s|^\gamma)^\alpha & \text{if } |s|\leq 1;\\ 0, & \text{if } |s|>1\end{cases}\qquad(s\in\mathbb{R}).$$

By the one-dimensional version of Corollary 2.2.28,

$$\left|\widehat{\theta}(t)\right|,\left|(\widehat{\theta})'(t)\right|\leq C|t|^{-\alpha-1}\qquad(t\neq 0).$$

Taking into account (2.2.34), we conclude that

$$\left|K^{\alpha,\gamma}_{n_j}(t)\right|\leq\frac{C}{n_j^\alpha|t|^{\alpha+1}}\qquad(t\neq 0)\tag{3.3.12}$$

and

$$\left|(K_{n_j}^{\alpha,\gamma})'(t)\right| \le \frac{C}{n_j^{\alpha-1}|t|^{\alpha+1}} \qquad (t \ne 0). \tag{3.3.13}$$

For $0 < \alpha \le 1$, the inequality can be proved as in Theorem 3.3.4. Now let $\alpha > 1$. Since

$$\left|\widehat{\theta}(t)\right|, \left|(\widehat{\theta})'(t)\right| \le C$$

trivially and since $|t|^{-\alpha-1} \le |t|^{-2}$ if $|t| \ge 1$, we conclude that

$$\left|\widehat{\theta}(t)\right|, \left|(\widehat{\theta})'(t)\right| \le C|t|^{-2} \qquad (t \ne 0).$$

Hence

$$\left|K_{n_j}^{\alpha,\gamma}(t)\right| \le \frac{C}{n_j|t|^2}, \qquad \left|(K_{n_j}^{\alpha,\gamma})'(t)\right| \le \frac{C}{|t|^2} \qquad (t \ne 0)$$

and the theorem can be proved as above. ∎

Corollary 3.3.8 *Suppose that* $0 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$. *If* $1 < p < \infty$, *then*

$$\left\|\sigma_\square^{\alpha,\gamma} f\right\|_p \le C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

As we have seen in Theorems 2.5.6 and 2.5.12, in the one-dimensional case, the operators $\sigma_\square^\alpha$ and $\sigma_\square^{\alpha,\gamma}$ are not bounded from $H_p^\square(\mathbb{T})$ to $L_p(\mathbb{T})$ if $0 < p \le 1/(\alpha+1)$ and $\alpha = 1$. Using interpolation, we obtain the weak type $(1, 1)$ inequality.

Corollary 3.3.9 *If* $0 < \alpha \le 1$, *then*

$$\sup_{\rho>0} \rho\lambda(\sigma_\square^\alpha f > \rho) \le C\|f\|_1 \qquad (f \in L_1(\mathbb{T}^d)).$$

If $0 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$, *then*

$$\sup_{\rho>0} \rho\lambda(\sigma_\square^{\alpha,\gamma} f > \rho) \le C\|f\|_1 \qquad (f \in L_1(\mathbb{T}^d)).$$

The density argument of Marcinkiewicz and Zygmund (Theorem 1.3.6) implies

Corollary 3.3.10 *Suppose that* $f \in L_1(\mathbb{T}^d)$. *If* $0 < \alpha \le 1$, *then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}_\tau^d} \sigma_n^\alpha f = f \qquad a.e.$$

If $0 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$, *then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}_\tau^d} \sigma_n^{\alpha,\gamma} f = f \qquad a.e.$$

This result was proved by Marcinkiewicz and Zygmund [234] for the two-dimensional Fejér means. The general version of Corollary 3.3.10 is due to the author [328, 329].

3.4 Almost Everywhere Restricted Summability over a Cone-Like Set

Now we generalize the results of Sect. 3.3 to so-called cone-like sets (see Fig. 3.3). Suppose that for all $j = 2, \ldots, d$, $\kappa_j : \mathbb{R}_+ \to \mathbb{R}_+$ are strictly increasing and continuous functions such that

$$\lim_{j \to \infty} \kappa_j = \infty \qquad \text{and} \qquad \lim_{j \to +0} \kappa_j = 0.$$

Moreover, suppose that there exist $c_{j,1}, c_{j,2}, \xi > 1$ such that

$$c_{j,1}\kappa_j(x) \le \kappa_j(\xi x) \le c_{j,2}\kappa_j(x) \qquad (x > 0). \tag{3.4.1}$$

Note that this is satisfied if κ_j is a power function. Let us define the numbers $\omega_{j,1}$ and $\omega_{j,2}$ via the formula

$$c_{j,1} = \xi^{\omega_{j,1}} \quad \text{and} \quad c_{j,2} = \xi^{\omega_{j,2}} \qquad (j = 2, \ldots, d). \tag{3.4.2}$$

For convenience, we extend the notations for $j = 1$ by $\kappa_1 := \mathcal{I}, c_{1,1} = c_{1,2} = \xi$. Here $\mathcal{I}$ denotes the identity function $\mathcal{I}(x) = x$. Let $\kappa = (\kappa_1, \ldots, \kappa_d)$ and $\tau = (\tau_1, \ldots, \tau_d)$ with $\tau_1 = 1$ and fixed $\tau_j \ge 1$ $(j = 2, \ldots, d)$. We define the cone-like set (with respect to the first dimension) by

$$\mathbb{R}^d_{\kappa,\tau} := \{x \in \mathbb{R}^d_+ : \tau_j^{-1}\kappa_j(n_1) \le n_j \le \tau_j\kappa_j(n_1), \, j = 2, \ldots, d\}.$$

Figure 3.3 shows a cone-like set for $d = 2$.

If $\kappa_j = \mathcal{I}$ for all $j = 2, \ldots, d$, then we get a cone investigated above. The condition on κ_j seems to be natural, because Gát [119] proved in the two-dimensional case that to each cone-like set with respect to the first dimension there exists a larger cone-like set with respect to the second dimension and conversely, if and only if (3.4.1) holds.

Here we have to consider a new Hardy space. We modify slightly the definition of $H_p^{\square}(\mathbb{T}^d)$. Fix $\psi \in S(\mathbb{R})$ such that $\int_{\mathbb{R}} \psi(x)dx \ne 0$. For $f \in D(\mathbb{T}^d)$, let

$$\psi_+^{\kappa}(f)(x) := \sup_{t \in (0,\infty)} \left| f * (\psi_t \otimes \psi_{\kappa_2(t)} \otimes \cdots \otimes \psi_{\kappa_d(t)})(x) \right|.$$

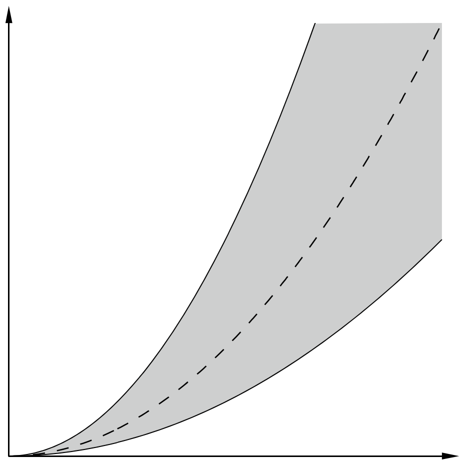

Fig. 3.3 Cone-like set for $d = 2$

Definition 3.4.1 For $0 < p < \infty$ the Hardy spaces $H_p^\kappa(\mathbb{T}^d)$ and weak Hardy spaces $H_{p,\infty}^\kappa(\mathbb{T}^d)$ consist of all distributions $f \in D(\mathbb{T}^d)$ for which

$$\|f\|_{H_p^\kappa} := \left\|\psi_+^\kappa(f)\right\|_p < \infty \qquad \text{and} \qquad \|f\|_{H_{p,\infty}^\kappa} := \left\|\psi_+^\kappa(f)\right\|_{p,\infty} < \infty.$$

We can prove all the theorems of Sect. 2.4 for $H_p^\kappa(\mathbb{T}^d)$. Among others,

$$\|f\|_{H_p^\kappa} \sim \left\|P_+^\kappa(f)\right\|_p \qquad (0 < p < \infty),$$

where P_t is the one-dimensional Poisson kernel and

$$P_+^\kappa(f)(x) := \sup_{t\in(0,\infty)} \left|f * (P_t \otimes P_{\kappa_2(t)} \otimes \cdots \otimes P_{\kappa_d(t)})(x)\right|.$$

If each $\kappa_j = \mathcal{I}$, we get back the Hardy spaces $H_p^\square(\mathbb{T}^d)$. We have to modify slightly the definition of atoms, too.

Definition 3.4.2 A bounded function a is an H_p^κ-atom if there exists a rectangle $I := I_1 \times \cdots \times I_d \subset \mathbb{T}^d$ with $|I_j| = \kappa_j(|I_1|^{-1})^{-1}$ such that

(i) $\operatorname{supp} a \subset I$,
(ii) $\|a\|_\infty \le |I|^{-1/p}$,
(iii) $\int_I a(x)x^k\,dx = 0$ for all multi-indices k with $|k| \le \lfloor d(1/p-1)\rfloor$.

The following two results can be proved as Theorems 2.4.18 and 2.4.19.

Theorem 3.4.3 *A distribution $f \in D(\mathbb{T}^d)$ is in $H_p^\kappa(\mathbb{T}^d)$ $(0 < p \le 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of H_p^κ-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty \qquad \textit{and} \qquad \sum_{k=0}^{\infty} \mu_k a_k = f \quad \textit{in } D(\mathbb{T}^d). \tag{3.4.3}$$

Moreover,

$$\|f\|_{H_p^\kappa} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f of the form (3.4.3).

Theorem 3.4.4 *For each $n \in \mathbb{N}^d$, let $K_n \in L_1(\mathbb{T}^d)$ and $V_n f := f * K_n$. Suppose that*

$$\int_{\mathbb{T}^d \setminus rI} |V_* a|^{p_0} \, d\lambda \le C_{p_0}$$

for all $H_{p_0}^\kappa$-atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p_0 \le 1$, where the rectangle I is the support of the atom. If V_* is bounded from $L_{p_1}(\mathbb{T}^d)$ to $L_{p_1}(\mathbb{T}^d)$ for some $1 < p_1 \le \infty$, then

$$\|V_* f\|_p \le C_p \|f\|_{H_p^\kappa} \qquad (f \in H_p^\kappa(\mathbb{T}^d))$$

for all $p_0 \le p \le p_1$.

Definition 3.4.5 For given κ, τ satisfying the above conditions, we define the restricted maximal Cesàro and restricted maximal Riesz operator by

$$\sigma_\kappa^\alpha f := \sup_{n \in \mathbb{R}^d_{\kappa,\tau}} |\sigma_n^\alpha f|$$

and

$$\sigma_\kappa^{\alpha,\gamma} f := \sup_{n \in \mathbb{R}^d_{\kappa,\tau}} |\sigma_n^{\alpha,\gamma} f|,$$

respectively.

The next theorem holds obviously.

Theorem 3.4.6 *If $0 < \alpha \le 1$, then*

$$\left\|\sigma_\kappa^\alpha f\right\|_\infty \le C \, \|f\|_\infty \qquad (f \in L_\infty(\mathbb{T}^d)).$$

If $0 < \alpha < \infty$ and $\gamma \in \mathbb{P}$, then

$$\left\|\sigma_\kappa^{\alpha,\gamma} f\right\|_\infty \le C \, \|f\|_\infty \qquad (f \in L_\infty(\mathbb{T}^d)).$$

Let H be an arbitrary subset of $\{1, \ldots, d\}$, $H \ne \emptyset$, $H \ne \{1, \ldots, d\}$ and $H^c := \{1, \ldots, d\} \setminus H$. Define

$$p_1 := \sup_{H\subset\{1,\ldots,d\}} \frac{\sum_{j\in H}\omega_{j,2}+\sum_{j\in H^c}\omega_{j,1}}{\sum_{j\in H}\omega_{j,2}+2\sum_{j\in H^c}\omega_{j,1}}, \tag{3.4.4}$$

where the numbers $\omega_{j,1}$ and $\omega_{j,2}$ are defined in (3.4.2).

Theorem 3.4.7 *If* $0<\alpha\le 1$ *and*

$$\max\left\{p_1, \frac{1}{\alpha+1}\right\} < p \le \infty,$$

then

$$\left\|\sigma_\kappa^\alpha f\right\|_p \le C_p \|f\|_{H_p^\kappa} \qquad (f \in H_p^\kappa(\mathbb{T}^d)).$$

Proof Since we will prove the result for $d=2$, we simplify the notation. Instead of $c_{2,1}, c_{2,2}$ and $\omega_{2,1}, \omega_{2,2}$, we will write c_1, c_2 and ω_1, ω_2, respectively. Let a be an arbitrary H_p^κ-atom with support $I=I_1\times I_2$, $|I_2|^{-1}=\kappa(|I_1|^{-1})$ and

$$2^{-K-1} < |I_1|/\pi \le 2^{-K}, \qquad \kappa(2^{K+1})^{-1} < |I_2|/\pi \le \kappa(2^K)^{-1}$$

for some $K\in\mathbb{N}$. We can suppose that the center of I is zero. In this case

$$[-\pi 2^{-K-2}, \pi 2^{-K-2}] \subset I_1 \subset [-\pi 2^{-K-1}, \pi 2^{-K-1}]$$

and

$$[-\pi\kappa(2^{K+1})^{-1}/2, \pi\kappa(2^{K+1})^{-1}/2] \subset I_2 \subset [-\pi\kappa(2^K)^{-1}/2, \pi\kappa(2^K)^{-1}/2].$$

To prove

$$\int_{\mathbb{T}^2\setminus 4(I_1\times I_2)} \left|\sigma_\kappa^\alpha a(x_1,x_2)\right|^p \,dx_1\,dx_2 \le C_p,$$

first we integrate over $(\mathbb{T}\setminus 4I_1)\times 4I_2$:

$$\begin{aligned}
&\int_{\mathbb{T}\setminus 4I_1}\int_{4I_2} |\sigma_\kappa^\alpha a(x_1,x_2)|^p\,dx_1\,dx_2 \\
&\quad\le \int_{\mathbb{T}\setminus 4I_1}\int_{4I_2} \sup_{n_1\ge 2^K,(n_1,n_2)\in\mathbb{R}^d_{\kappa,\tau}} \left|\sigma_{n_1,n_2}^\alpha a(x_1,x_2)\right|^p\,dx_1\,dx_2 \\
&\qquad+ \int_{\mathbb{T}\setminus 4I_1}\int_{4I_2} \sup_{n_1< 2^K,(n_1,n_2)\in\mathbb{R}^d_{\kappa,\tau}} \left|\sigma_{n_1,n_2}^\alpha a(x_1,x_2)\right|^p\,dx_1\,dx_2 \\
&\quad=: (A)+(B).
\end{aligned}$$

If $n_1\ge 2^K$ and $x\in[\pi i 2^{-K}, \pi(i+1)2^{-K})$ $(i\ge 1)$, then by (3.3.5),

$$
\begin{aligned}
\left|\sigma_{n_1,n_2}^{\alpha} a(x_1,x_2)\right| &= \left| \int_{I_1} \int_{I_2} a(t_1,t_2) K_{n_1}^{\alpha}(x_1-t_1) K_{n_2}^{\alpha}(x_2-t_2)\, dt_1\, dt_2 \right| \\
&\le C_p 2^{K/p} \kappa(2^K)^{1/p} \int_{I_1} \frac{1}{n_1^{\alpha}|x_1-t_1|^{\alpha+1}}\, dt_1 \\
&\le C_p 2^{K/p+K\alpha} \kappa(2^K)^{1/p} \frac{1}{n_1^{\alpha} i^{\alpha+1}} \\
&\le C_p 2^{K/p} \kappa(2^K)^{1/p} \frac{1}{i^{\alpha+1}}.
\end{aligned}
$$

From this, it follows that

$$
\begin{aligned}
(A) &\le \sum_{i=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{4I_2} \sup_{n_1 \ge 2^K} \left|\sigma_{n_1,n_2}^{\alpha} a(x_1,x_2)\right|^p\, dx_1\, dx_2 \\
&\le C_p \sum_{i=1}^{2^K-1} 2^{-K} \kappa(2^K)^{-1} 2^K \kappa(2^K) \frac{1}{i^{(\alpha+1)p}} \\
&= C_p \sum_{i=1}^{2^K-1} \frac{1}{i^{(\alpha+1)p}},
\end{aligned}
$$

which is a convergent series if $p > 1/(1+\alpha)$.

We estimate (B) by

$$
\begin{aligned}
(B) &\le \sum_{k=0}^{\infty} \int_{\mathbb{T}\setminus 4I_1} \int_{4I_2} \sup_{\frac{2^K}{\xi^{k+1}} \le n_1 < \frac{2^K}{\xi^k}, (n_1,n_2)\in \mathbb{R}^d_{\kappa,\tau}} \left|\sigma_{n_1,n_2}^{\alpha} a(x_1,x_2)\right|^p\, dx_1\, dx_2 \\
&\le \sum_{k=0}^{\infty} \left(\int_{\mathbb{T}\setminus\left[-\frac{\pi\xi^k}{2^K}, \frac{\pi\xi^k}{2^K}\right]} \int_{4I_2} + \int_{\left[-\frac{\pi\xi^k}{2^K}, \frac{\pi\xi^k}{2^K}\right]} \int_{4I_2} \right) \\
&\qquad \sup_{\frac{2^K}{\xi^{k+1}} \le n_1 < \frac{2^K}{\xi^k}, (n_1,n_2)\in \mathbb{R}^d_{\kappa,\tau}} \left|\sigma_{n_1,n_2}^{\alpha} a(x_1,x_2)\right|^p\, dx_1\, dx_2 \\
&=: (B_1) + (B_2).
\end{aligned}
$$

If $(n_1,n_2) \in \mathbb{R}^d_{\kappa,\tau}$ and $\frac{2^K}{\xi^{k+1}} \le n < \frac{2^K}{\xi^k}$, then $n_2 < \tau\kappa(\frac{2^K}{\xi^k})$. The inequality $|K_{n_2}^{\alpha}| \le C n_2$ and (3.3.2) imply

$$
\begin{aligned}
&\left|\sigma_{n_1,n_2}^{\alpha} a(x_1,x_2)\right| \\
&\quad \le C_p 2^{K/p} \kappa(2^K)^{1/p-1} n_2 \int_{I_1} \frac{1}{n_1^{\alpha}|x_1-t_1|^{\alpha+1}}\, dt_1 \\
&\quad \le C_p 2^{K/p-K} \kappa(2^K)^{1/p-1} \kappa\left(\frac{2^K}{\xi^k}\right) \left(\frac{2^K}{\xi^{k+1}}\right)^{-\alpha} |x_1 - \pi 2^{-K-1}|^{-\alpha-1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(B_1) &\le C_p \sum_{k=0}^{\infty} 2^{K(1-p-\alpha p)} \kappa(2^K)^{-p} \kappa\left(\frac{2^K}{\xi^k}\right)^p \xi^{k\alpha p} \\
&\qquad \int_{\mathbb{T}\setminus\left[-\frac{\pi\xi^k}{2^K}, \frac{\pi\xi^k}{2^K}\right]} |x_1 - \pi 2^{-K-1}|^{-(\alpha+1)p} \, dx_1 \\
&\le C_p \sum_{k=0}^{\infty} 2^{K(1-p-\alpha p)} \kappa(2^K)^{-p} \kappa\left(\frac{2^K}{\xi^k}\right)^p \xi^{k\alpha p} (\xi^k 2^{-K})^{-(\alpha+1)p+1}.
\end{aligned}
$$

Since $\kappa(x) \le c_1^{-1} \kappa(\xi x)$ by (3.4.1), we conclude

$$
(B_1) \le C_p \sum_{k=0}^{\infty} \kappa(2^K)^{-p} \kappa(2^K)^p c_1^{-kp} \xi^{k(1-p)} = C_p \sum_{k=0}^{\infty} \xi^{k(1-p-\omega_1 p)},
$$

which is convergent if $p > 1/(1+\omega_1)$. Note that

$$
\frac{1}{1+\omega_1} < \frac{1+\omega_1}{1+2\omega_1} \le p_1 < p.
$$

For (B_2), we obtain similarly that

$$
\begin{aligned}
\left|\sigma_{n_1,n_2}^{\alpha} a(x_1, x_2)\right| &\le C_p 2^{K/p-K} \kappa(2^K)^{1/p-1} n_1 n_2 \\
&\le C_p 2^{K/p-K} \kappa(2^K)^{1/p-1} \frac{2^K}{\xi^k} \kappa\left(\frac{2^K}{\xi^k}\right)
\end{aligned}
\tag{3.4.5}
$$

and, moreover,

$$
\begin{aligned}
(B_2) &\le C_p \sum_{k=0}^{\infty} \frac{\xi^k}{2^K} \kappa(2^K)^{-1} 2^K \kappa(2^K)^{1-p} \xi^{-kp} \kappa\left(\frac{2^K}{\xi^k}\right)^p \\
&\le C_p \sum_{k=0}^{\infty} \xi^{k(1-p)} c_1^{-kp},
\end{aligned}
$$

which was just considered. Hence, we have proved that

$$
\int_{\mathbb{T}\setminus 4I_1} \int_{4I_2} |\sigma_{\kappa}^{\alpha} a(x_1, x_2)|^p \, dx_1 \, dx_2 \le C_p \qquad (p_1 < p \le 1).
$$

The integral over $4I_1 \times (\mathbb{T} \setminus 4I_2)$ can be handled with a similar idea. Indeed, let us denote the terms corresponding to (A), (B), (B_1), (B_2) by (A'), (B'), (B_1'), (B_2'). If we take the integrals in (A') over

$$4I_1 \times \left[\pi j \kappa(2^K)^{-1}, \pi(j+1)\kappa(2^K)^{-1}\right] \qquad (j = 1, \ldots, \kappa(2^K)/2 - 1),$$

then we get in the same way that (A') is bounded if $p > 1/(1+\alpha)$. For (B_1'), we can see that

$$\begin{aligned}
(B_1') &= \sum_{k=0}^{\infty} \int_{4I_1} \int_{\mathbb{T}\setminus\left[-\pi\kappa\left(\frac{2^K}{\xi^k}\right)^{-1}, \pi\kappa\left(\frac{2^K}{\xi^k}\right)^{-1}\right]} \\
&\qquad \sup_{\frac{2^K}{\xi^{k+1}} \le n_1 < \frac{2^K}{\xi^k}, (n_1,n_2)\in\mathbb{R}^d_{\kappa,\tau}} \left|\sigma^{\alpha}_{n_1,n_2} a(x_1,x_2)\right|^p \, dx_1 \, dx_2 \\
&\le C_p \sum_{k=0}^{\infty} 2^K \kappa(2^K) 2^{-K} 2^{-Kp} \int_{\mathbb{T}\setminus\left[-\pi\kappa\left(\frac{2^K}{\xi^k}\right)^{-1}, \pi\kappa\left(\frac{2^K}{\xi^k}\right)^{-1}\right]} \\
&\qquad \sup_{\frac{2^K}{\xi^{k+1}} \le n_1 < \frac{2^K}{\xi^k}, (n_1,n_2)\in\mathbb{R}^d_{\kappa,\tau}} \left(n_1 \int_{I_2} \frac{1}{n_2^{\alpha}|x_2 - t_2|^{\alpha+1}} \, dt_2\right)^p \, dx_2.
\end{aligned}$$

Thus

$$\begin{aligned}
(B_1') &\le C_p \sum_{k=0}^{\infty} \xi^{-kp} \kappa(2^K)^{1-p} \kappa\left(\frac{2^K}{\xi^{k+1}}\right)^{-\alpha p} \\
&\qquad \int_{\mathbb{T}\setminus\left[-\pi\kappa\left(\frac{2^K}{\xi^k}\right)^{-1}, \pi\kappa\left(\frac{2^K}{\xi^k}\right)^{-1}\right]} |x_2 - \pi\kappa(2^K)^{-1}/2|^{-(\alpha+1)p} \, dx_2 \\
&\le C_p \sum_{k=0}^{\infty} \xi^{-kp} \kappa(2^K)^{1-p} \kappa\left(\frac{2^K}{\xi^k}\right)^{p-1} \\
&\le C_p \sum_{k=0}^{\infty} \xi^{-kp} c_2^{k(1-p)} \\
&= C_p \sum_{k=0}^{\infty} \xi^{k(\omega_2 - \omega_2 p - p)}
\end{aligned}$$

and this converges if $p > \omega_2/(1+\omega_2)$, which is less than

$$\frac{1+\omega_2}{2+\omega_2} \le p_1.$$

Using (3.4.5), we establish that

$$
\begin{aligned}
(B_2') &= \sum_{k=0}^{\infty} \int_{4I_1} \int_{\left[-\kappa\left(\frac{2^K}{\xi^k}\right)^{-1}, \kappa\left(\frac{2^K}{\xi^k}\right)^{-1}\right]} \\
&\qquad \sup_{\frac{2^K}{\xi^{k+1}} \le n_1 < \frac{2^K}{\xi^k}, (n_1,n_2)\in\mathbb{R}^d_{\kappa,\tau}} \left|\sigma^{\alpha}_{n_1,n_2} a(x_1,x_2)\right|^p \, dx_1 \, dx_2 \\
&\le C_p \sum_{k=0}^{\infty} 2^{-K}\kappa \left(\frac{2^K}{\xi^k}\right)^{-1} 2^K \kappa (2^K)^{1-p} \xi^{-kp} \kappa \left(\frac{2^K}{\xi^k}\right)^{p} \\
&\le C_p \sum_{k=0}^{\infty} \xi^{-kp} c_2^{k(1-p)}.
\end{aligned}
$$

Hence

$$
\int_{4I_1} \int_{\mathbb{T}\setminus 4I_2} \left|\sigma^{\alpha}_{\kappa} a(x_1,x_2)\right|^p \, dx_1 \, dx_2 \le C_p \qquad (p_1 < p \le 1).
$$

Integrating over $(\mathbb{T}\setminus 4I_1)\times(\mathbb{T}\setminus 4I_2)$, we decompose the integral as

$$
\begin{aligned}
&\int_{\mathbb{T}\setminus 4I_1} \int_{\mathbb{T}\setminus 4I_2} \left|\sigma^{\alpha}_{\kappa} a(x_1,x_2)\right|^p \, dx_1 \, dx_2 \\
&\quad \le \int_{\mathbb{T}\setminus 4I_1} \int_{\mathbb{T}\setminus 4I_2} \sup_{n_1 \ge 2^K, (n_1,n_2)\in\mathbb{R}^d_{\kappa,\tau}} \left|\sigma^{\alpha}_{n_1,n_2} a(x_1,x_2)\right|^p \, dx_1 \, dx_2 \\
&\qquad + \int_{\mathbb{T}\setminus 4I_1} \int_{\mathbb{T}\setminus 4I_2} \sup_{n_2 < 2^K, (n_1,n_2)\in\mathbb{R}^d_{\kappa,\tau}} \left|\sigma^{\alpha}_{n_1,n_2} a(x_1,x_2)\right|^p \, dx_1 \, dx_2 \\
&\quad =: (C) + (D).
\end{aligned}
$$

Notice that

$$
(C) \le \sum_{i=1}^{2^K-1} \sum_{j=1}^{\kappa(2^K)/2-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{\pi j \kappa(2^K)^{-1}}^{\pi(j+1)\kappa(2^K)^{-1}} \sup_{n_1 \ge 2^K} |\sigma^{\alpha}_{n_1,n_2} a(x_1,x_2)|^p \, dx_1 \, dx_2.
$$

For $x_1 \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ and $x_2 \in [\pi j \kappa(2^K)^{-1}, \pi(j+1)\kappa(2^K)^{-1})$, we have by (3.3.2) and (3.3.5) that

$$
\begin{aligned}
\left|\sigma^{\alpha}_{n_1,n_2} a(x_1,x_2)\right| &\le C_p 2^{K/p} \kappa(2^K)^{1/p} \int_{I_1} \frac{1}{n_1^{\alpha}|x_1-t_1|^{\alpha+1}} \, dt_1 \\
&\qquad \int_{I_2} \frac{1}{n_2^{\alpha}|x_2-t_2|^{\alpha+1}} \, dt_2 \\
&\le C_p \frac{2^{K/p+K\alpha}\kappa(2^K)^{1/p+\alpha}}{n_1^{\alpha} n_2^{\alpha} i^{\alpha+1} j^{\alpha+1}} \\
&\le C_p \frac{2^{K/p}\kappa(2^K)^{1/p}}{i^{\alpha+1} j^{\alpha+1}}.
\end{aligned} \tag{3.4.6}
$$

Then

$$(C) \le C_p \sum_{i=1}^{2^K-1} \sum_{j=1}^{\kappa(2^K)/2-1} \frac{1}{i^{(\alpha+1)p} j^{(\alpha+1)p}} < \infty$$

if $p > 1/(1+\alpha)$.

To consider (D) let us define $A_1(x_1, x_2)$, $A_2(x_1, x_2)$, $D^1_{n_1,n_2}(x_1, x_2)$, $D^2_{n_1,n_2}(x_1, x_2)$ and $D^3_{n_1,n_2}(x_1, x_2)$ as in (3.3.6) and (3.3.9), respectively, and let $I_1 = [-\mu, \mu]$, $I_2 = [-\nu, \nu]$. Then

$$|A_1(x_1, u)| \le 2^{K/p-K} \kappa(2^K)^{1/p}, \qquad |A_2(x_1, x_2)| \le 2^{K/p-K} \kappa(2^K)^{1/p-1}. \quad (3.4.7)$$

Obviously,

$$\begin{aligned}
&\int_{\mathbb{T}\setminus 4I_1} \int_{\mathbb{T}\setminus 4I_2} \sup_{n_1<2^K,(n_1,n_2)\in\mathbb{R}^d_{\kappa,\tau}} |D^1_{n_1,n_2}(x_1, x_2)|^p \, dx_1 \, dx_2 \\
&\le \sum_{k=0}^{\infty} \int_{\mathbb{T}\setminus 4I_1} \int_{\mathbb{T}\setminus 4I_2} \sup_{\frac{2^K}{\xi^{k+1}} \le n_1 < \frac{2^K}{\xi^k},(n_1,n_2)\in\mathbb{R}^d_{\kappa,\tau}} |D^1_{n_1,n_2}(x_1, x_2)|^p \, dx_1 \, dx_2 \\
&\le \sum_{k=0}^{\infty} \sum_{j=1}^{\kappa(2^K)/2-1} \int_{\mathbb{T}\setminus\left[-\frac{\pi\xi^k}{2^K}, \frac{\pi\xi^k}{2^K}\right]} \int_{\pi j \kappa(2^K)^{-1}}^{\pi(j+1)\kappa(2^K)^{-1}} \\
&\qquad \sup_{n_1<2^K,(n_1,n_2)\in\mathbb{R}^d_{\kappa,\tau}} |D^1_{n_1,n_2}(x_1, x_2)|^p \, dx_1 \, dx_2 \\
&\le \sum_{k=0}^{\infty} \sum_{j=1}^{\kappa(2^K)/2-1} \int_{\left[-\frac{\pi\xi^k}{2^K}, \frac{\pi\xi^k}{2^K}\right]} \int_{\pi j \kappa(2^K)^{-1}}^{\pi(j+1)\kappa(2^K)^{-1}} \\
&\qquad \sup_{n_1<2^K,(n_1,n_2)\in\mathbb{R}^d_{\kappa,\tau}} |D^1_{n_1,n_2}(x_1, x_2)|^p \, dx_1 \, dx_2 \\
&=: (D_1) + (D_2).
\end{aligned}$$

It follows from (3.3.5), (3.3.10) and (3.4.7) that

$$\begin{aligned}
&|D^1_{n_1,n_2}(x_1, x_2)| \\
&\le C_p 2^{K/p-K} \kappa(2^K)^{1/p-2} \frac{1}{n_1^\alpha |x_1-\mu|^{\alpha+1}} \frac{n_2^{\zeta+1+\alpha(\zeta-1)}}{|x_2-\nu|^{(\alpha+1)(1-\zeta)}} \\
&\le C_p 2^{K/p-K} \kappa(2^K)^{1/p-2+(\alpha+1)(1-\zeta)} \frac{\left(\frac{2^K}{\xi^{k+1}}\right)^{-\alpha}}{|x_1-\mu|^{\alpha+1}} \frac{\kappa\left(\frac{2^K}{\xi^k}\right)^{\zeta+1+\alpha(\zeta-1)}}{j^{(\alpha+1)(1-\zeta)}},
\end{aligned}$$

where $0 \le \zeta \le 1$. This leads to

$$(D_1) \le C_p \sum_{k=0}^{\infty} \sum_{j=1}^{\kappa(2^K)/2-1} \int_{\mathbb{T}\setminus\left[-\frac{\pi\xi^k}{2^K}, \frac{\pi\xi^k}{2^K}\right]} 2^{K(1-p-\alpha p)} \kappa(2^K)^{p(-2+(\alpha+1)(1-\zeta))} \xi^{k\alpha p}$$
$$|x_1 - \mu|^{-(\alpha+1)p} \frac{\kappa\left(\frac{2^K}{\xi^k}\right)^{p(2+(\alpha+1)(\zeta-1))}}{j^{p(\alpha+1)(1-\zeta)}} \, dx_1$$
$$\le C_p \sum_{k=0}^{\infty} \sum_{j=1}^{\kappa(2^K)/2-1} 2^{K(1-p-\alpha p)} \xi^{k\alpha p} (\xi^k 2^{-K})^{-(\alpha+1)p+1} \frac{c_1^{-kp(2+(\alpha+1)(\zeta-1))}}{j^{p(\alpha+1)(1-\zeta)}}$$
$$\le C_p \sum_{k=0}^{\infty} \sum_{j=1}^{\kappa(2^K)/2-1} \frac{\xi^{k(1-p-\omega_1 p(2+(\alpha+1)(\zeta-1)))}}{j^{p(\alpha+1)(1-\zeta)}},$$

which is convergent if

$$p > \frac{1}{1+\omega_1(2+(\alpha+1)(\zeta-1))} \quad \text{and} \quad p > \frac{1}{(\alpha+1)(1-\zeta)}.$$

After some computation, we can see that the optimal bound is reached if

$$\zeta = \frac{\alpha - \omega_1 + \alpha\omega_1}{1+\alpha+\omega_1+\alpha\omega_1},$$

which means that

$$p > \frac{1+\omega_1}{1+2\omega_1}.$$

Considering (D_2), we estimate as follows:

$$|D^1_{n_1,n_2}(x_1,x_2)| \le C_p 2^{K/p-K} \kappa(2^K)^{1/p-2} n_1 \frac{n_2^{\zeta+1+\alpha(\zeta-1)}}{|x_2-\nu|^{(\alpha+1)(1-\zeta)}}$$
$$\le C_p 2^{K/p} \kappa(2^K)^{1/p-2+(\alpha+1)(1-\zeta)} \xi^{-k} \frac{\kappa\left(\frac{2^K}{\xi^k}\right)^{\zeta+1+\alpha(\zeta-1)}}{j^{(\alpha+1)(1-\zeta)}}$$

and

$$(D_2) \le C_p \sum_{k=0}^{\infty} \sum_{j=1}^{\kappa(2^K)/2-1}$$
$$\int_{\left[-\frac{\pi\xi^k}{2^K}, \frac{\pi\xi^k}{2^K}\right]} 2^K \kappa(2^K)^{p(-2+(\alpha+1)(1-\zeta))} \xi^{-kp} \frac{\kappa\left(\frac{2^K}{\xi^k}\right)^{p(2+(\alpha+1)(\zeta-1))}}{j^{p(\alpha+1)(1-\zeta)}} \, dx_1$$

$$\leq C_p \sum_{k=0}^{\infty} \sum_{j=1}^{\kappa(2^K)/2-1} \frac{\xi^{k(1-p-\omega_1 p(2+(\alpha+1)(\zeta-1)))}}{j^{p(\alpha+1)(1-\zeta)}}$$
$$\leq C_p$$

as above.

The term $D^2_{n_1,n_2}$ can be handled similarly. We obtain

$$\int_{\mathbb{T}\setminus 4I_1} \int_{\mathbb{T}\setminus 4I_2} \sup_{n_1<2^K,(n_1,n_2)\in\mathbb{R}^d_{\kappa,\tau}} |D^2_{n_1,n_2}(x_1,x_2)|^p \, dx_1 \, dx_2 \leq C_p$$

if

$$p > \frac{1+\omega_2}{2+\omega_2}.$$

Using (3.3.3), we estimate $D^3_{n_1,n_2}$ in the same way as (C) in (3.4.6). Now the exponents of n_1 and n_2 are non-negative and so they can be estimated by 2^K and $\kappa(2^K)$ as in (3.4.6). This proves that

$$\int_{\mathbb{T}\setminus 4I_1} \int_{\mathbb{T}\setminus 4I_2} \left|\sigma^\alpha_\kappa a(x_1,x_2)\right|^p \, dx_1 \, dx_2 \leq C_p$$

which completes the proof. ■

Remark 3.4.8 In the d-dimensional case, the constant p_1 appears if we investigate the terms corresponding to $D^1_{n_1,n_2}$ and $D^2_{n_1,n_2}$. Indeed, let $\prod_{j=1}^d I_j$ be centered at 0 and the support of the atom a, A be the integral of a, $I_j =: [-\mu_j, \mu_j]$ and

$$\bar{t}_j := \begin{cases} \mu_j, & j \in H; \\ t_j, & j \in H^c, \end{cases}$$

$H \subset \{1,\ldots,d\}$, $H \neq \emptyset$, $H \neq \{1,\ldots,d\}$. If we integrate the term

$$\int_{\prod_{j\in H^c} I_j} A(\bar{t}_1,\ldots,\bar{t}_d) \prod_{j\in H} K^\alpha_{n_j}(x_j-\mu_j) \prod_{i\in H^c} (K^\alpha_{n_i})'(x_i-t_i)\, dt$$

over $\prod_{j=1}^d (\mathbb{T}\setminus 4I_j)$, then we get that

$$p > \frac{\sum_{j\in H}\omega_{j,2} + \sum_{j\in H^c}\omega_{j,1}}{\sum_{j\in H}\omega_{j,2} + 2\sum_{j\in H^c}\omega_{j,1}}.$$

Moreover, considering the integral

$$\int_{\prod_{j\in H}(\mathbb{T}\setminus 4I_j)} \int_{\prod_{j\in H^c} 4I_j} |\sigma_\kappa^\alpha a(x)|^p \, dx,$$

we obtain

$$p > \frac{\sum_{j\in H} \omega_{j,2}}{\sum_{j\in H} \omega_{j,2} + \sum_{j\in H^c} \omega_{j,1}}.$$

However, this bound is less than p_1.

Remark 3.4.9 If $\omega_{j,1} = \omega_{j,2} = 1$ for all $j = 1, \ldots, d$, then we obtain in Theorem 3.4.7 the bound

$$\max\left\{\frac{d}{d+1}, \frac{1}{\alpha+1}\right\}.$$

In particular, this holds if $\kappa_j = \mathcal{I}$ for all $j = 1, \ldots, d$, i.e., if we consider a cone. This bound was obtained for cones in Theorem 3.3.4.

Corollary 3.4.10 *If* $0 < \alpha \le 1$ *and* $1 < p < \infty$*, then*

$$\left\|\sigma_\kappa^\alpha f\right\|_p \le C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

We obtain similar results for the Riesz means (cf. Theorem 3.3.7). The details are left to the reader.

Theorem 3.4.11 *If* $0 < \alpha < \infty$*,* $\gamma \in \mathbb{P}$ *and*

$$\max\left\{p_1, \frac{1}{\alpha \wedge 1 + 1}\right\} < p \le \infty,$$

then

$$\left\|\sigma_\kappa^{\alpha,\gamma} f\right\|_p \le C_p \|f\|_{H_p^\kappa} \qquad (f \in H_p^\kappa(\mathbb{T}^d)).$$

Corollary 3.4.12 *Suppose that* $0 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$*. If* $1 < p < \infty$*, then*

$$\left\|\sigma_\kappa^{\alpha,\gamma} f\right\|_p \le C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Corollary 3.4.13 *If* $0 < \alpha \le 1$*, then*

$$\sup_{\rho>0} \rho\lambda(\sigma_\kappa^\alpha f > \rho) \le C\|f\|_1 \qquad (f \in L_1(\mathbb{T}^d)).$$

If $0 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$*, then*

$$\sup_{\rho>0} \rho\lambda(\sigma_\kappa^{\alpha,\gamma} f > \rho) \le C\|f\|_1 \qquad (f \in L_1(\mathbb{T}^d)).$$

Corollary 3.4.14 *Suppose that $f \in L_1(\mathbb{T}^d)$. If $0 < \alpha \le 1$, then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_{\kappa,\tau}} \sigma_n^\alpha f = f \qquad a.e.$$

If $0 < \alpha < \infty$ and $\gamma \in \mathbb{P}$, then

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_{\kappa,\tau}} \sigma_n^{\alpha,\gamma} f = f \qquad a.e.$$

In the two-dimensional case, Corollaries 3.4.13 and 3.4.14 were proved by Gát [119] for Fejér summability. In this case, he verified also that if the cone-like set $\mathbb{R}^d_{\kappa,\tau}$ is defined by $\tau_j(n_1)$ instead of τ_j and if $\tau_j(n_1)$ is not bounded, then Corollary 3.4.14 does not hold and the largest space for the elements of which we have almost everywhere convergence is $L \log L$. This means that under these conditions Theorem 3.4.7 cannot be true for any $p < 1$.

3.5 $H_p(\mathbb{T}^d)$ Hardy spaces

For the investigation of the unrestricted almost everywhere convergence of the rectangular summability means, we need a new type of Hardy spaces, the so-called product Hardy spaces.

Fix $\psi \in S(\mathbb{R})$ such that $\int_{\mathbb{R}} \psi(x)dx \neq 0$. We define the product radial maximal function, the product non-tangential maximal function and the hybrid maximal function of $f \in D(\mathbb{T}^d)$ by

$$\psi_+^*(f)(x) := \sup_{t_i\in(0,\infty), i=1,\ldots,d} \left|(f * (\psi_{t_1} \otimes \cdots \otimes \psi_{t_d}))(x)\right|,$$

$$\psi_\nabla^*(f)(x) := \sup_{t_i\in(0,\infty), |x_i-y_i|<t_i, i=1,\ldots,d} \left|(f * (\psi_{t_1} \otimes \cdots \otimes \psi_{t_d}))(y)\right|$$

and

$$\psi_{\sharp_i}^*(f)(x) := \sup_{t_k\in(0,\infty), k=1,\ldots,d; k\neq i} \left|(f * (\psi_{t_1} \otimes \cdots \otimes \psi_{t_{i-1}} \otimes \psi_{t_{i+1}} \otimes \cdots \otimes \psi_{t_d}))(x)\right|,$$

respectively, $(i = 1, \ldots, d)$.

Definition 3.5.1 For $0 < p < \infty$, the product Hardy spaces $H_p(\mathbb{T}^d)$, product weak Hardy spaces $H_{p,\infty}(\mathbb{T}^d)$ and the hybrid Hardy spaces $H_p^i(\mathbb{T}^d)$ $(i = 1, \ldots, d)$ consist of all distributions $f \in D(\mathbb{T}^d)$ for which

$$\|f\|_{H_p} := \left\|\psi_+^*(f)\right\|_p < \infty,$$

$$\|f\|_{H_{p,\infty}} := \left\|\psi_+^*(f)\right\|_{p,\infty} < \infty$$

and

$$\|f\|_{H_p^i} := \left\|\psi_{\sharp_i}^*(f)\right\|_p < \infty.$$

The Hardy spaces are independent of ψ_i, more exactly, different functions ψ_i give the same space with equivalent norms. For $f \in D(\mathbb{T}^d)$, let

$$P_+^*(f)(x) := \sup_{t_i \in (0,\infty), i=1,\ldots,d} \left|(f * (P_{t_1} \otimes \cdots \otimes P_{t_d}))(x)\right|,$$

$$P_\triangledown^*(f)(x) := \sup_{t_i \in (0,\infty), |x_i - y_i| < t_i, i=1,\ldots,d} \left|(f * (P_{t_1} \otimes \cdots \otimes P_{t_d}))(x)\right|$$

and

$$P_{\sharp_i}^*(f)(x) := \sup_{t_k \in (0,\infty), k=1,\ldots,d; k \neq i} \left|(f * (P_{t_1} \otimes \cdots \otimes P_{t_{i-1}} \otimes P_{t_{i+1}} \otimes \cdots \otimes P_{t_d}))(x)\right|,$$

respectively ($i = 1, \ldots, d$), where the Poisson kernel P_{t_i} was defined before Theorem 2.4.14. The next theorems were proved in Chang and Fefferman [54, 55], Gundy and Stein [155] or Weisz [346], so we omit the proofs.

Theorem 3.5.2 *Let $0 < p < \infty$. Fix $\psi \in S(\mathbb{R})$ such that $\int_{\mathbb{R}} \psi(x)dx \neq 0$. Then $f \in H_p(\mathbb{T}^d)$ if and only if $\psi_\triangledown^*(f) \in L_p(\mathbb{T}^d)$ or $P_+^*(f) \in L_p(\mathbb{T}^d)$ or $P_\triangledown^*(f) \in L_p(\mathbb{T}^d)$. We have the following equivalences of norms:*

$$\|f\|_{H_p^\square} \sim \|\psi_\triangledown^*(f)\|_p \sim \|P_+^*(f)\|_p \sim \|P_\triangledown^*(f)\|_p.$$

The same holds for the weak Hardy spaces:

$$\|f\|_{H_{p,\infty}^\square} \sim \|\psi_\triangledown^*(f)\|_{p,\infty} \sim \|P_+^*(f)\|_{p,\infty} \sim \|P_\triangledown^*(f)\|_{p,\infty}$$

and for the hybrid Hardy spaces:

$$\|f\|_{H_p^i} \sim \|P_{\sharp_i}^*(f)\|_p \qquad (i = 1, \ldots, d).$$

As we can see from the next theorem, in the theory of product Hardy spaces, the hybrid Hardy spaces $H_p^i(\mathbb{T}^d)$ will play the role of the $L_1(\mathbb{T}^d)$ spaces in some sense.

Theorem 3.5.3 *If $1 < p < \infty$ and $i = 1, \ldots, d$, then $H_p(\mathbb{T}^d) \sim H_p^i(\mathbb{T}^d) \sim L_p(\mathbb{T}^d)$ and*

$$\|f\|_p \leq \|f\|_{H_p^i} \leq \|f\|_{H_p} \leq C_p \|f\|_p.$$

For $p = 1$, $H_1(\mathbb{T}^d) \subset H_1^i(\mathbb{T}^d) \subset H_{1,\infty}^\square(\mathbb{T}^d) \cap L_1(\mathbb{T}^d)$ and

$$\|f\|_{H_1^i} \le \|f\|_{H_1} \qquad (f \in H_1(\mathbb{T}^d)),$$

$$\|f\|_{H_{1,\infty}} \le C\|f\|_{H_1^i} \qquad (f \in H_1^i(\mathbb{T}^d)).$$

Definition 3.5.4 The set $L(\log L)^{d-1}(\mathbb{T}^d)$ contains all measurable functions for which

$$\left\| |f|(\log^+ |f|)^{d-1} \right\|_1 < \infty.$$

Theorem 3.5.5 $H_1^i(\mathbb{T}^d) \supset L(\log L)^{d-1}(\mathbb{T}^d)$ *for all* $i = 1, \dots, d$ *and*

$$\|f\|_{H_1^i} \le C + C \left\| |f|(\log^+ |f|)^{d-1} \right\|_1 \qquad (f \in L(\log L)^{d-1}(\mathbb{T}^d)).$$

A straightforward generalization of the atoms would be the following:

(i) supp $a \subset I$, $I \subset \mathbb{T}^d$ is a rectangle,
(ii) $\|a\|_\infty \le |I|^{-1/p}$,
(iii) $\int_{\mathbb{T}} a(x) x_i^k \, dx_i = 0$, for all $i = 1, \dots, d$.

However, the space $H_p(\mathbb{T}^d)$ do not have atomic decomposition with respect to these atoms (see Weisz [327]). The atomic decomposition for $H_p(\mathbb{T}^d)$ is much more complicated. One reason of this is that the support of an atom is not a rectangle but an open set. Moreover, here we have to choose the atoms from $L_2(\mathbb{T}^d)$ instead of $L_\infty(\mathbb{T}^d)$.

First of all, we introduce some notations. By a dyadic interval we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k, n \in \mathbb{Z}$. A dyadic rectangle is the Cartesian product of d dyadic intervals. Suppose that $F \subset \mathbb{T}^d$ is an open set. Let $\mathcal{M}_1(F)$ denote those dyadic rectangles $R = I \times S \subset F$, $I \subset \mathbb{T}$ is a dyadic interval, $S \subset \mathbb{T}^{d-1}$ is a dyadic rectangle that are maximal in the first direction. In other words, if $I' \times S \supset R$ is a dyadic subrectangle of F (where $I' \subset \mathbb{T}$ is a dyadic interval) then $I = I'$. Define $\mathcal{M}_i(F)$ similarly. Denote by $\mathcal{M}(F)$ the maximal dyadic subrectangles of F in the above sense.

Recall that if $I \subset \mathbb{T}$ is an interval, then rI is the interval with the same center as I and with length $r|I|$ $(r \in \mathbb{N})$. For a rectangle $R = I_1 \times \ldots \times I_d \subset \mathbb{T}^d$ let $rR := rI_1 \times \ldots \times rI_d$. Instead of $2^r R$ we write R^r $(r \in \mathbb{N})$.

Definition 3.5.6 A function $a \in L_2(\mathbb{R}^d)$ is an H_p-atom $(0 < p \le 1)$ if

(i) supp $a \subset F$ for some open set $F \subset \mathbb{T}^d$ with finite measure,
(ii) $\|a\|_2 \le |F|^{1/2-1/p}$,
(iii) a can be decomposed further into the sum of "elementary particles" $a_R \in L_2(\mathbb{R}^d)$,

$$a = \sum_{R \in \mathcal{M}(F)} a_R,$$

satisfying

(a) supp $a_R \subset 5R$,

(b) for all $R \in \mathcal{M}(F)$, $i = 1, \ldots, d$ and almost every fixed $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d$,

$$\int_{\mathbb{T}} a_R(x) x_i^k \, dx_i = 0 \qquad (k = 0, \ldots, M(p) \geq \lfloor 2/p - 3/2 \rfloor),$$

(c) for every disjoint partition $\mathcal{P}_l$ $(l \in \mathbb{P})$ of $\mathcal{M}(F)$,

$$\left(\sum_{l \in \mathbb{P}} \left\| \sum_{R \in \mathcal{P}_l} a_R \right\|_2^2 \right)^{1/2} \leq |F|^{1/2-1/p}.$$

Theorem 3.5.7 *A distribution $f \in D(\mathbb{T}^d)$ is in $H_p(\mathbb{T}^d)$ $(0 < p \leq 1)$ if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of H_p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \mu_k a_k = f \quad \text{in } D(\mathbb{T}^d).$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f.

The result corresponding to Theorem 2.4.19 for the $H_p(\mathbb{T}^d)$ space is much more complicated. Since the definition of the H_p-atom is very complex, to obtain a usable condition about the boundedness of the operators, we have to introduce simpler atoms (see also the definition right after Theorem 3.5.5).

Definition 3.5.8 A function $a \in L_2(\mathbb{T}^d)$ is a simple H_p-atom or a rectangle H_p-atom if

(i) supp $a \subset R$ for a rectangle $R \subset \mathbb{T}^d$,
(ii) $\|a\|_2 \leq |R|^{1/2-1/p}$,
(iii) $\int_{\mathbb{T}} a(x) x_i^k \, dx_i = 0$ for $i = 1, \ldots, d$, $k = 0, \ldots, M(p) \geq \lfloor 2/p - 3/2 \rfloor$ and for almost every fixed x_j, $j = 1, \ldots, d$, $j \neq i$.

Note that $H_p(\mathbb{T}^d)$ cannot be decomposed into rectangle p-atoms, a counterexample can be found in Weisz [327]. However, the following result says that for an operator V to be bounded from $H_p(\mathbb{T}^2)$ to $L_p(\mathbb{T}^2)$ $(0 < p \leq 1)$, it is enough to check V on simple H_p-atoms and the boundedness of V on $L_2(\mathbb{T}^2)$. We omit the proof because it can be found for all dimensions in Weisz [332, 346] (see also Fefferman [98]).

Theorem 3.5.9 *Let $d = 2$, $0 < p_0 \le 1$, $K_n \in L_1(\mathbb{T}^2)$ and $V_n f := f * K_n$ $(n \in \mathbb{N}^2)$. Suppose that there exists $\eta > 0$ such that for every simple H_{p_0}-atom a and for every $r \ge 1$*

$$\int_{\mathbb{T}^2 \setminus R^r} |V_* a|^{p_0} \, d\lambda \le C_p 2^{-\eta r},$$

where R is the support of a. If V_* is bounded from $L_2(\mathbb{T}^2)$ to $L_2(\mathbb{T}^2)$, then

$$\|V_* f\|_p \le C_{p_0} \|f\|_{H_p} \qquad (f \in H_p(\mathbb{T}^2))$$

for all $p_0 \le p \le 2$.

Note that Theorem 2.4.16 holds also for $H_p(\mathbb{T}^d)$ spaces with a very similar proof.

Theorem 3.5.10 *If $K \in L_1(\mathbb{T}^d)$, $0 < p < \infty$ and*

$$\lim_{k\to\infty} f_k = f \quad \text{in the } H_p(\mathbb{T}^d)\text{-norm,}$$

then

$$\lim_{k\to\infty} f_k * K = f * K \quad \text{in } D(\mathbb{T}^d).$$

Corollary 3.5.11 *If $p_0 < 1$ in Theorem 3.5.9, then for all $f \in H_1^i(\mathbb{T}^2)$ and $i = 1, 2$,*

$$\sup_{\rho>0} \rho \, \lambda(|V_* f| > \rho) \le C \|f\|_{H_1^i}.$$

Proof Using the preceding theorem and interpolation, we conclude that the operator

$$V_* \quad \text{is bounded from} \quad H_{p,\infty}(\mathbb{T}^2) \quad \text{to} \quad L_{p,\infty}(\mathbb{T}^2)$$

when $p_0 < p < 2$. Thus, it holds also for $p = 1$. By Theorem 3.5.3,

$$\sup_{\rho>0} \rho \, \lambda(|V_* f| > \rho) = \|V_* f\|_{1,\infty} \le C \|f\|_{H_{1,\infty}} \le C \|f\|_{H_1^i}$$

for all $f \in H_1^i(\mathbb{T}^2)$, $i = 1, 2$. ∎

Note that for higher dimensions, we have to modify slightly Theorem 3.5.9, Corollary 3.5.11 as well as the definition of simple H_p-atoms (see Weisz [332, 346]).

3.6 Almost Everywhere Unrestricted Summability

For the almost everywhere unrestricted summability, we introduce the next maximal operators.

Definition 3.6.1 We define the unrestricted maximal Cesàro and unrestricted maximal Riesz operator by

$$\sigma_*^\alpha f := \sup_{n\in\mathbb{N}^d} |\sigma_n^\alpha f|$$

and

$$\sigma_*^{\alpha,\gamma} f := \sup_{n\in\mathbb{N}^d} |\sigma_n^{\alpha,\gamma} f|,$$

respectively.

For $\alpha = \gamma = 1$, the operator is called unrestricted maximal Fejér operator and denoted by $\sigma_* f$.

We will first prove that the operator σ_*^α is bounded from $L_p(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ $(1 < p \le \infty)$ and then that it is bounded from $H_p(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ $(1/(\alpha+1) < p \le 1)$. To this end, we introduce the next one-dimensional operators.

Definition 3.6.2 Let

$$\tau_n^\alpha f(x) := f * \left|K_n^\alpha\right|(x),$$

$$\tau_n^{\alpha,\gamma} f(x) := f * \left|K_n^{\alpha,\gamma}\right|(x)$$

and

$$\tau_*^\alpha f := \sup_{n\in\mathbb{N}} \left|\tau_n^\alpha f\right|,$$

$$\tau_*^{\alpha,\gamma} f := \sup_{n\in\mathbb{N}} \left|\tau_n^{\alpha,\gamma} f\right|.$$

Obviously,

$$|\sigma_n^\alpha f| \le \tau_n^\alpha |f| \quad (n\in\mathbb{N}) \qquad \text{and} \qquad \sigma_*^\alpha f \le \tau_*^\alpha |f|.$$

The same holds for the operators $\sigma_*^{\alpha,\gamma}$ and $\tau_*^{\alpha,\gamma}$. The next result can be proved similar to Theorem 3.3.4.

Theorem 3.6.3 *If* $0 < \alpha \le 1$ *and* $1/(\alpha+1) < p \le \infty$, *then*

$$\left\|\tau_*^\alpha f\right\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{T})).$$

Proof It is easy to see that

$$\left\|\tau_*^\alpha f\right\|_\infty \le C \|f\|_\infty \qquad (f \in L_\infty(\mathbb{T})).$$

Let a be an arbitrary H_p-atom with support $I \subset \mathbb{T}$ and

$$[-\pi 2^{-K-2}, \pi 2^{-K-2}] \subset I \subset [-\pi 2^{-K-1}, \pi 2^{-K-1}].$$

Then

$$\begin{aligned}\int_{\mathbb{T}\setminus 4I_1} |\tau_*^\alpha a(x)|^p\,dx &\le \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \sup_{n\ge 2^K} |\tau_n^\alpha a(x)|^p\,dx \\ &\quad + \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \sup_{n< 2^K} |\tau_n^\alpha a(x)|^p\,dx \\ &=: (A)+(B).\end{aligned}$$

Using (3.3.2) and (3.3.5), we can see that

$$\begin{aligned}\left|\tau_n^\alpha a(x)\right| &= \left|\int_I a(t)\left|K_n^\alpha(x-t)\right|\,dt\right| \\ &\le C_p 2^{K/p}\int_I \frac{1}{n^\alpha |x-t|^{\alpha+1}}\,dt \\ &\le C_p 2^{K/p}\frac{1}{i^{\alpha+1}}\end{aligned}$$

and

$$(A) \le C_p \sum_{i=1}^{2^K-1} 2^{-K}2^K \frac{1}{i^{(\alpha+1)p}} \le C_p$$

as in Theorem 3.3.4.

To estimate (B), observe that by (iii) of the definition of the atom,

$$\tau_n^\alpha a(x) = \int_I a(t)\left|K_n^\alpha(x-t)\right|\,dt = \int_I a(t)\Big(\left|K_n^\alpha(x-t)\right| - \left|K_n^\alpha(x)\right|\Big)\,dt.$$

Thus,

$$\left|\tau_n^\alpha a(x)\right| \le \int_I |a(t)|\Big|K_n^\alpha(x-t) - K_n^\alpha(x)\Big|\,dt.$$

Using Lagrange's mean value theorem and (3.3.3), we conclude

$$\begin{aligned}\Big|K_n^\alpha(x-t) - K_n^\alpha(x)\Big| &= \left|(K_n^\alpha)'(x-\xi)\right||t| \\ &\le \frac{C_p 2^{-K}}{n^{\alpha-1}|x-\xi|^{\alpha+1}} \le \frac{C_p 2^K}{i^{\alpha+1}},\end{aligned}$$

where $\xi\in I$ and $x\in[\pi i 2^{-K}, \pi(i+1)2^{-K})$. Consequently,

$$\left|\tau_n^\alpha a(x)\right| \le C_p 2^{K/p-K}\frac{2^K}{i^{\alpha+1}}$$

and

$$(B) \leq C_p \sum_{i=1}^{2^K-1} 2^{-K} 2^K \frac{1}{i^{(\alpha+1)p}} \leq C_p,$$

which proves the theorem. ∎

We can verify in the same way

Theorem 3.6.4 *If* $0 < \alpha < \infty$, $\gamma \in \mathbb{P}$ *and* $1/(\alpha \wedge 1 + 1) < p \leq \infty$, *then*

$$\left\| \tau_*^{\alpha,\gamma} f \right\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{T})).$$

The next result can be obtained by interpolation.

Corollary 3.6.5 *Suppose that* $1 < p \leq \infty$. *If* $0 < \alpha \leq 1$, *then*

$$\sup_{\rho>0} \rho \lambda(\tau_*^{\alpha} f > \rho) \leq C \|f\|_1 \qquad (f \in L_1(\mathbb{T}))$$

and

$$\left\| \tau_*^{\alpha} f \right\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{T})).$$

If $0 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$, *then*

$$\sup_{\rho>0} \rho \lambda(\tau_*^{\alpha,\gamma} f > \rho) \leq C \|f\|_1 \qquad (f \in L_1(\mathbb{T}))$$

and

$$\left\| \tau_*^{\alpha,\gamma} f \right\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{T})).$$

Now, we turn to the higher dimensional case and verify the $L_p(\mathbb{T}^d)$ boundedness of σ_*^{α} and $\sigma_*^{\alpha,\gamma}$.

Theorem 3.6.6 *Suppose that* $1 < p \leq \infty$. *If* $0 < \alpha < \infty$, *then*

$$\left\| \sigma_*^{\alpha} f \right\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

If $0 < \alpha < \infty$, $\gamma \in \mathbb{P}$ *and* $1 < p \leq \infty$, *then*

$$\left\| \sigma_*^{\alpha,\gamma} f \right\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Proof For $0 < \alpha \leq 1$, let us apply Corollary 3.6.5 to obtain

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \sup_{n_1,n_2\in\mathbb{N}} \left|\int_{\mathbb{T}}\int_{\mathbb{T}} f(t_1,t_2)K_{n_1}^{\alpha}(x_1-t_1)K_{n_2}^{\alpha}(x_2-t_2)\,dt_1\,dt_2\right|^p dx_1\,dx_2$$
$$\leq \int_{\mathbb{T}}\int_{\mathbb{T}} \sup_{n_2\in\mathbb{N}}$$
$$\left(\int_{\mathbb{T}}\left(\sup_{n_1\in\mathbb{N}}\left|\int_{\mathbb{T}} f(t_1,t_2)K_{n_1}^{\alpha}(x_1-t_1)\,dt_1\right|\right)\left|K_{n_2}^{\alpha}(x_2-t_2)\right|\,dt_2\right)^p dx_2\,dx_1$$
$$\leq C_p \int_{\mathbb{T}}\int_{\mathbb{T}} \sup_{n_1\in\mathbb{N}} \left|\int_{\mathbb{T}} f(t_1,x_2)K_{n_1}^{\alpha}(x_1-t_1)\,dt_1\right|^p dx_1\,dx_2$$
$$\leq C_p \int_{\mathbb{T}}\int_{\mathbb{T}} |f(x_1,x_2)|^p\,dx_1\,dx_2.$$

The inequality for $1<\alpha<\infty$ follows from Lemma 3.1.9. The result for $\sigma_*^{\alpha,\gamma}$ can be proved in the same way. ■

The next result is due to the author [331, 332].

Theorem 3.6.7 *If* $0<\alpha<\infty$ *and* $1/(\alpha+1)<p\leq\infty$, *then*

$$\left\|\sigma_*^{\alpha} f\right\|_p \leq C_p\|f\|_{H_p} \qquad (f\in H_p(\mathbb{T}^d)).$$

Proof By Theorem 3.1.8,

$$\left|\left(K_{n_j}^{\alpha}\right)^{(s)}(t)\right| \leq \frac{C}{n_j^{\alpha-s}|t|^{\alpha+1}} \tag{3.6.1}$$

for $0<\alpha\leq s+1, n_j\in\mathbb{P}$ and $t\in\mathbb{T}, t\neq 0$. Choose a simple H_p-atom a with support $R=I_1\times I_2$, where I_1 and I_2 are intervals with

$$2^{-K_i-1}<|I_i|/\pi\leq 2^{-K_i} \qquad (K_i\in\mathbb{N}, i=1,2)$$

and

$$[-\pi 2^{-K_i-2},\pi 2^{-K_i-2}]\subset I_i\subset[-\pi 2^{-K_i-1},\pi 2^{-K_i-1}].$$

We assume that $r\geq 2$ is an arbitrary integer. Theorem 3.6.6 implies that the operator σ_*^{α} is bounded from $L_2(\mathbb{T}^d)$ to $L_2(\mathbb{T}^d)$. By Theorem 3.5.9, we have to integrate $\left|\sigma_*^{\alpha}a\right|^p$ over

$$\mathbb{T}^2\setminus R^r = \left(\mathbb{T}\setminus I_1^r\right)\times I_2 \bigcup (\mathbb{T}\setminus I_1^r)\times(\mathbb{T}\setminus I_2)$$
$$\bigcup I_1\times(\mathbb{T}\setminus I_2^r)\bigcup(\mathbb{T}\setminus I_1)\times(\mathbb{T}\setminus I_2^r).$$

First, we integrate over $(\mathbb{T}\setminus I_1^r)\times I_2$:

$$
\begin{aligned}
&\int_{\mathbb{T}\setminus 4I_1}\int_{I_2}\left|\sigma_*^\alpha a(x_1,x_2)\right|^p\,dx_1\,dx_2\\
&\quad\le \int_{\mathbb{T}\setminus 4I_1}\int_{I_2}\sup_{n_1\ge 2^{K_1},n_2\in\mathbb{N}}\left|\sigma_n^\alpha a(x_1,x_2)\right|^p\,dx_1\,dx_2\\
&\qquad+\int_{\mathbb{T}\setminus 4I_1}\int_{I_2}\sup_{n_1< 2^{K_1},n_2\in\mathbb{N}}\left|\sigma_n^\alpha a(x_1,x_2)\right|^p\,dx_1\,dx_2\\
&\quad\le \sum_{|i_1|=2^{r-2}}^{2^{K_1}-1}\int_{\pi i_1 2^{-K_1}}^{\pi(i_1+1)2^{-K_1}}\int_{I_2}\sup_{n_1\ge 2^{K_1},n_2\in\mathbb{N}}\left|\sigma_n^\alpha a(x_1,x_2)\right|^p\,dx_1\,dx_2\\
&\qquad+\sum_{|i_1|=2^{r-2}}^{2^{K_1}-1}\int_{\pi i_1 2^{-K_1}}^{\pi(i_1+1)2^{-K_1}}\int_{I_2}\sup_{n_1< 2^{K_1},n_2\in\mathbb{N}}\left|\sigma_n^\alpha a(x_1,x_2)\right|^p\,dx_1\,dx_2\\
&\quad=: (A)+(B).
\end{aligned}
$$

Here we may suppose that $i_1 > 0$. For $k, l \in \mathbb{N}$ let $A_{0,0}(x) := a(x)$,

$$
A_{1,0}(x_1,x_2) := \int_{-\pi}^{x_1} a(t,x_2)\,dt \qquad A_{0,1}(x_1,x_2) := \int_{-\pi}^{x_2} a(x_1,u)\,du
$$

and

$$
A_{k,l}(x_1,x_2) := \int_{-\pi}^{x_1} A_{k-1,l}(t,x_2)\,dt = \int_{-\pi}^{x_2} A_{k,l-1}(x_1,u)\,du.
$$

By (iii) of the definition of the simple H_p-atom, we can show that supp $A_{k,l} \subset R$ and $A_{k,l}(x_1,x_2)$ is zero if x_1 is at the boundary of I_1 or x_2 is at the boundary of I_2 for $k, l = 0, \ldots, M(p)+1$ $(i = 1, 2)$, where $M(p) \ge \lfloor 2/p - 3/2 \rfloor$. Moreover, using (ii), we can compute that

$$
\left\|A_{k,l}\right\|_2 \le |I_1|^{k+1/2-1/p}|I_2|^{l+1/2-1/p} \qquad (k, l = 0, \ldots, M(p)+1). \tag{3.6.2}
$$

We may suppose that $M(p) \ge \alpha + 1$ and choose $N \in \mathbb{N}$ such that $N < \alpha \le N + 1$. For $x_1 \in [\pi i_1 2^{-K_1}, \pi(i_1+1)2^{-K_1})$, $t_1 \in [-\pi 2^{-K_1-1}, \pi 2^{-K_1-1})$, inequality (3.6.1) implies

$$
\left|(K_{n_1}^\alpha)^{(N)}(x_1-t_1)\right| \le \frac{C n_1^{N-\alpha} 2^{K_1(\alpha+1)}}{i_1^{\alpha+1}} \tag{3.6.3}
$$

and

$$
\left|(K_{n_1}^\alpha)^{(N+1)}(x_1-t_1)\right| \le \frac{C n_1^{N+1-\alpha} 2^{K_1(\alpha+1)}}{i_1^{\alpha+1}}. \tag{3.6.4}
$$

Integrating by parts, we can see that

$$
\begin{aligned}
\left|\sigma_n^\alpha a(x)\right| &= \left|\int_{I_1}\int_{I_2} A_{N,0}(t_1,t_2)(K_{n_1}^\alpha)^{(N)}(x_1-t_1)K_{n_2}^\alpha(x_2-t_2)\,dt_1\,dt_2\right| \\
&\le \frac{Cn_1^{N-\alpha}2^{K_1(\alpha+1)}}{i_1^{\alpha+1}}\int_{I_1}\left|\int_{I_2} A_{N,0}(t_1,t_2)K_{n_2}^\alpha(x_2-t_2)\,dt_2\right|\,dt_1
\end{aligned}
$$

whenever $x_1 \in [\pi i_1 2^{-K_1}, \pi(i_1+1)2^{-K_1})$. Hence, by Hölder's inequality and (3.6.3),

$$
\begin{aligned}
(A) \le C_p \sum_{i_1=2^{r-2}}^{2^{K_1}-1} 2^{-K_1}\frac{2^{K_1(N+1)p}}{i_1^{(\alpha+1)p}} \\
\int_{I_2}\left(\int_{I_1}\sup_{n_2\in\mathbb{N}}\left|\int_{I_2} A_{N,0}(t_1,t_2)K_{n_2}^\alpha(x_2-t_2)\,dt_2\right|\,dt_1\right)^p\,dx_2 \\
\le C_p|I_2|^{1-p}\sum_{i_1=2^{r-2}}^{2^{K_1}-1}\frac{2^{K_1((N+1)p-1)}}{i_1^{(\alpha+1)p}} \\
\left(\int_{I_2}\int_{I_1}\sup_{n_2\in\mathbb{N}}\left|\int_{I_2} A_{N,0}(t_1,t_2)K_{n_2}^\alpha(x_2-t_2)\,dt_2\right|\,dt_1\,dx_2\right)^p.
\end{aligned}
$$

Using again Hölder's inequality and the fact that σ_*^α is bounded on $L_2(\mathbb{T})$, we conclude

$$
\begin{aligned}
(A) \le C_p|I_2|^{1-p/2}\sum_{i_1=2^{r-2}}^{2^{K_1}-1}\frac{2^{K_1((N+1)p-1)}}{i_1^{(\alpha+1)p}} \\
\left(\int_{I_1}\left(\int_{I_2}\sup_{n_2\in\mathbb{N}}\left|\int_{I_2} A_{N,0}(t_1,t_2)K_{n_2}^\alpha(x_2-t_2)\,dt_2\right|^2\,dx_2\right)^{1/2}\,dt_1\right)^p \\
\le C_p|I_2|^{1-p/2}\sum_{i_1=2^{r-2}}^{2^{K_1}-1}\frac{2^{K_1((N+1)p-1)}}{i_1^{(\alpha+1)p}} \\
\left(\int_{I_1}\left(\int_{I_2}\left|A_{N,0}(t_1,x_2)\right|^2\,dx_2\right)^{1/2}\,dt_1\right)^p.
\end{aligned}
$$

Then (3.6.2) implies

$$
\begin{aligned}
(A) &\le C_p |I_2|^{1-p/2} \sum_{i_1=2^{r-2}}^{2^{K_1}-1} 2^{-K_1 p/2} \frac{2^{K_1((N+1)p-1)}}{i_1^{(\alpha+1)p}} \\
&\quad \left(\int_{I_1} \int_{I_2} \left| A_{N,0}(t_1, x_2) \right|^2 \, dx_2 \, dt_1 \right)^{p/2} \\
&\le C_p \sum_{i_1=2^{r-2}}^{2^{K_1}-1} \frac{1}{i_1^{(\alpha+1)p}} \le C_p 2^{-r((\alpha+1)p-1)}.
\end{aligned}
$$

To estimate (B), we use (3.6.4):

$$
\begin{aligned}
(B) &\le C_p \sum_{i_1=2^{r-2}}^{2^{K_1}-1} 2^{-K_1} \frac{2^{K_1(N+2)p}}{i_1^{(\alpha+1)p}} \\
&\quad \int_{I_2} \left(\int_{I_1} \sup_{n_2 \in \mathbb{N}} \left| \int_{I_2} A_{N+1,0}(t_1, t_2) K_{n_2}^{\alpha}(x_2 - t_2) \, dt_2 \right| \, dt_1 \right)^p \, dx_2 \\
&\le C_p |I_2|^{1-p/2} \sum_{i_1=2^{r-2}}^{2^{K_1}-1} \frac{2^{K_1((N+2)p-1)}}{i_1^{(\alpha+1)p}} \\
&\quad \left(\int_{I_1} \left(\int_{I_2} \sup_{n_2 \in \mathbb{N}} \left| \int_{I_2} A_{N+1,0}(t_1, t_2) K_{n_2}^{\alpha}(x_2 - t_2) \, dt_2 \right|^2 \, dx_2 \right)^{1/2} \, dt_1 \right)^p
\end{aligned}
$$

and

$$
\begin{aligned}
(B) &\le C_p |I_2|^{1-p/2} \sum_{i_1=2^{r-2}}^{2^{K_1}-1} \frac{2^{K_1((N+2)p-1)}}{i_1^{(\alpha+1)p}} \\
&\quad \left(\int_{I_1} \left(\int_{I_2} \left| A_{N+1,0}(t_1, x_2) \right|^2 \, dx_2 \right)^{1/2} \, dt_1 \right)^p \\
&\le C_p |I_2|^{1-p/2} \sum_{i_1=2^{r-2}}^{2^{K_1}-1} 2^{-K_1 p/2} \frac{2^{K_1((N+2)p-1)}}{i_1^{(\alpha+1)p}} \\
&\quad \left(\int_{I_1} \int_{I_2} \left| A_{N+1,0}(t_1, x_2) \right|^2 \, dx_2 \, dt_1 \right)^{p/2} \\
&\le C_p \sum_{i_1=2^{r-2}}^{2^{K_1}-1} \frac{1}{i_1^{(\alpha+1)p}} \le C_p 2^{-r((\alpha+1)p-1)}.
\end{aligned}
$$

Next, we integrate over $(\mathbb{T} \setminus I_1^r) \times (\mathbb{T} \setminus I_2)$:

$$
\begin{aligned}
&\int_{\mathbb{T}\setminus 4I_1}\int_{\mathbb{T}\setminus I_2} \left|\sigma_*^\alpha a(x_1,x_2)\right|^p \, dx_1\, dx_2 \\
&\quad \le \int_{\mathbb{T}\setminus 4I_1}\int_{\mathbb{T}\setminus I_2} \sup_{n_1\ge 2^{K_1}, n_2\ge 2^{K_2}} \left|\sigma_n^\alpha a(x_1,x_2)\right|^p \, dx_1\, dx_2 \\
&\qquad + \int_{\mathbb{T}\setminus 4I_1}\int_{\mathbb{T}\setminus I_2} \sup_{n_1\ge 2^{K_1}, n_2< 2^{K_2}} \left|\sigma_n^\alpha a(x_1,x_2)\right|^p \, dx_1\, dx_2 \\
&\qquad + \int_{\mathbb{T}\setminus 4I_1}\int_{\mathbb{T}\setminus I_2} \sup_{n_1< 2^{K_1}, n_2\ge 2^{K_2}} \left|\sigma_n^\alpha a(x_1,x_2)\right|^p \, dx_1\, dx_2 \\
&\qquad + \int_{\mathbb{T}\setminus 4I_1}\int_{\mathbb{T}\setminus I_2} \sup_{n_1< 2^{K_1}, n_2< 2^{K_2}} \left|\sigma_n^\alpha a(x_1,x_2)\right|^p \, dx_1\, dx_2 \\
&\quad =: (C)+(D)+(E)+(F).
\end{aligned}
$$

We will only consider the term (D):

$$
\begin{aligned}
(D) \le \sum_{|i_1|=2^{r-2}}^{2^{K_1}-1} \sum_{|i_2|=1}^{2^{K_2}-1} \int_{\pi i_1 2^{-K_1}}^{\pi(i_1+1)2^{-K_1}} \int_{\pi i_2 2^{-K_2}}^{\pi(i_2+1)2^{-K_2}} \\
\sup_{n_1\ge 2^{K_1}, n_2< 2^{K_2}} \left|\sigma_n^\alpha a(x_1,x_2)\right|^p \, dx_1\, dx_2,
\end{aligned}
$$

where we may suppose again that $i_1 > 0$ and $i_2 > 0$. Integrating by parts,

$$
\begin{aligned}
&\left|\sigma_n^\alpha a(x)\right| \\
&= \left| \int_{I_1}\int_{I_2} A_{N,N+1}(t_1,t_2)(K_{n_1}^\alpha)^{(N)}(x_1-t_1)(K_{n_2}^\alpha)^{(N+1)}(x_2-t_2)\, dt_1\, dt_2 \right| \\
&\le \frac{C 2^{K_1(N+1)} 2^{K_2(N+2)}}{i_1^{\alpha+1} i_2^{\alpha+1}} \int_{I_1}\int_{I_2} \left|A_{N,N+1}(t_1,t_2)\right| \, dt_1\, dt_2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
(D) &\le C_p \sum_{i_1=2^{r-2}}^{2^{K_1}-1} \sum_{i_2=1}^{2^{K_2}-1} 2^{-K_1} 2^{-K_2} \frac{2^{K_1(N+1)p} 2^{K_2(N+2)p}}{i_1^{(\alpha+1)p} i_2^{(\alpha+1)p}} \\
&\qquad \left(\int_{I_1}\int_{I_2} \left|A_{N,N+1}(t_1,t_2)\right| \, dt_1\, dt_2 \right)^p \\
&\le C_p \sum_{i_1=2^{r-2}}^{2^{K_1}-1} \sum_{i_2=1}^{2^{K_2}-1} 2^{-K_1 p/2} 2^{-K_2 p/2} \frac{2^{K_1((N+1)p-1)} 2^{K_2((N+2)p-1)}}{i_1^{(\alpha+1)p} i_2^{(\alpha+1)p}} \\
&\qquad \left(\int_{I_1}\int_{I_2} \left|A_{N,N+1}(t_1,t_2)\right|^2 \, dt_1\, dt_2 \right)^{p/2}
\end{aligned}
$$

$$\leq C_p \sum_{i_1=2^{r-2}}^{2^{K_1}-1} \sum_{i_2=1}^{2^{K_2}-1} \frac{1}{i_1^{(\alpha+1)p} i_2^{(\alpha+1)p}}$$
$$\leq C_p 2^{-r((\alpha+1)p-1)}.$$

All other integrals can be handled in the same way. Consequently,

$$\int_{\mathbb{T}^2 \setminus R^r} \left|\sigma_*^\alpha a(x_1, x_2)\right|^p \, dx_1 \, dx_2 \leq C_p 2^{-r((\alpha+1)p-1)},$$

which finishes the proof of the theorem. ■

Theorem 3.6.8 *If* $0 < \alpha < \infty$, $\gamma \in \mathbb{P}$ *and* $1/(\alpha+1) < p \leq \infty$, *then*

$$\left\|\sigma_*^{\alpha,\gamma} f\right\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{T}^d)).$$

Proof Similar to (3.3.13), for $s \in \mathbb{N}$, $n_j \in \mathbb{P}$ and $t \in \mathbb{T}$, $t \neq 0$, we have

$$\left|\left(K_{n_j}^{\alpha,\gamma}\right)^{(s)}(t)\right| \leq \frac{C}{n_j^{\alpha-s}|t|^{\alpha+1}}.$$

The theorem can be proved as Theorem 3.6.7. ■

Corollary 3.5.11 implies

Corollary 3.6.9 *Let* $f \in H_1^i(\mathbb{T}^d)$ *for some* $i = 1, \ldots, d$. *If* $0 < \alpha < \infty$, *then*

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\alpha f > \rho) \leq C\|f\|_{H_1^i}.$$

If $0 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$, *then*

$$\sup_{\rho>0} \rho\lambda(\sigma_*^{\alpha,\gamma} f > \rho) \leq C\|f\|_{H_1^i}.$$

By the density argument, we get here almost everywhere convergence for functions from the spaces $H_1^i(\mathbb{T}^d)$ instead of $L_1(\mathbb{T}^d)$. In some sense, the Hardy space $H_1^i(\mathbb{T}^d)$ plays the role of $L_1(\mathbb{T}^d)$ in higher dimensions.

Corollary 3.6.10 *Let* $f \in H_1^i(\mathbb{T}^d)$ *for some* $i = 1, \ldots, d$. *If* $0 < \alpha < \infty$, *then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \qquad \textit{a.e.}$$

If $0 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$, *then*

$$\lim_{n\to\infty} \sigma_n^{\alpha,\gamma} f = f \qquad \textit{a.e.}$$

The almost everywhere convergence is not true for all $f \in L_1(\mathbb{T}^d)$.

A counterexample, which shows that the almost everywhere convergence is not true for all integrable functions, is due to Gát [119]. Recall that

$$L_1(\mathbb{T}^d) \supset H_1^i(\mathbb{T}^d) \supset L(\log L)^{d-1}(\mathbb{T}^d) \supset L_p(\mathbb{T}^d) \qquad (1 < p \leq \infty).$$

3.7 Rectangular θ-Summability

In this section, we introduce some new function spaces and then we generalize the rectangular Cesàro and Riesz means. As we will see in Definition 3.7.4, instead of condition (2.6.2), we have to suppose here that $\theta : \mathbb{R}^d \to \mathbb{R}$ is a d-dimensional function and

$$\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \left| \theta\left(\frac{k_1}{n_1}, \ldots, \frac{k_d}{n_d}\right) \right| < \infty \tag{3.7.1}$$

for all $n \in \mathbb{P}^d$. We will see that it is more convenient to suppose that θ is in the Wiener algebra $W(C, \ell_1)(\mathbb{R}^d)$. All summability methods considered in the literature satisfy the condition $\theta \in W(C, \ell_1)(\mathbb{R}^d)$.

Definition 3.7.1 A measurable function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to the Wiener amalgam space $W(L_\infty, \ell_1)(\mathbb{R}^d)$ if

$$\|f\|_{W(L_\infty,\ell_1)} := \sum_{k \in \mathbb{Z}^d} \sup_{x \in [0,1)^d} |f(x+k)| < \infty.$$

The smallest closed subspace of $W(L_\infty, \ell_1)(\mathbb{R}^d)$ containing continuous functions is denoted by $W(C, \ell_1)(\mathbb{R}^d)$ and is called Wiener algebra.

Lemma 3.7.2 *If* $1 \leq p \leq \infty$, *then*

$$W(L_\infty, \ell_1)(\mathbb{R}^d) \subset L_p(\mathbb{R}^d) \qquad \textit{and} \qquad \|f\|_p \leq \|f\|_{W(L_\infty,\ell_1)}.$$

Moreover, $W(L_\infty, \ell_1)(\mathbb{R}^d)$ *is dense in* $L_p(\mathbb{R}^d)$ *for* $1 \leq p < \infty$.

Proof For $p = \infty$, the statement is trivial. If $1 \leq p < \infty$, then

$$\begin{aligned} \|f\|_p &= \left(\sum_{k \in \mathbb{Z}^d} \int_{k+[0,1)^d} |f(x)|^p \, dx \right)^{1/p} \\ &\leq \left(\sum_{k \in \mathbb{Z}^d} \sup_{x \in [0,1)^d} |f(x+k)|^p \right)^{1/p} \end{aligned}$$

$$\leq \sum_{k\in\mathbb{Z}^d} \sup_{x\in[0,1)^d} |f(x+k)|$$
$$= \|f\|_{W(L_\infty,\ell_1)} .$$

Since $W(L_\infty, \ell_1)(\mathbb{R}^d)$ contains the space of continuous functions with compact support, $W(L_\infty, \ell_1)(\mathbb{R}^d)$ is dense in $L_p(\mathbb{R}^d)$ if $1 \leq p < \infty$. ■

The Wiener amalgam spaces and Wiener algebra are used quite often in Gabor analysis, because they provide convenient and general classes of windows (see, e.g., Walnut [323] and Gröchenig [152]).

Theorem 3.7.3 *(a) If $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ then (3.7.1) holds.*
(b) If the one-dimensional function θ is continuous and $|\theta|$ can be estimated by an integrable function η which is non-decreasing on $(-\infty, c)$ and non-increasing on (c, ∞) then $\theta \in W(C, \ell_1)(\mathbb{R})$.
(c) There exists $\theta \notin W(C, \ell_1)(\mathbb{R})$ such that (3.7.1) holds.

Proof It is easy to see that

$$\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \left|\theta\left(\frac{k_1}{n_1}, \ldots, \frac{k_d}{n_d}\right)\right| \leq \sum_{l\in\mathbb{Z}^d} \left(\prod_{j=1}^{d} n_j\right) \sup_{x\in[0,1)^d} |\theta(x+l)|$$
$$= \left(\prod_{j=1}^{d} n_j\right) \|\theta\|_{W(C,\ell_1)} < \infty, \tag{3.7.2}$$

which shows (a). Under the conditions of (b), $\|\theta\|_{W(C,\ell_1)} \leq \|\eta\|_1$.

To see (c), let $\theta \geq 0$ be continuous and even on $\mathbb{R}$, $\theta(0) := 0$,

$$\theta(x) := 0 \quad \text{if} \quad j + \frac{1}{j+1} \leq x \leq j+1 \quad (j \in \mathbb{N})$$

and

$$\sup_{[j,j+1]} \theta = \frac{1}{j+1} \quad (j \in \mathbb{N}).$$

Then $\theta \in L_1(\mathbb{R})$,

$$\|\theta\|_{W(C,\ell_1)} = 2\sum_{k=0}^{\infty} \frac{1}{k+1} = \infty$$

and

$$\sum_{k=-\infty}^{\infty} \left|\theta\left(\frac{k}{n+1}\right)\right| \leq 2\sum_{j=0}^{n} \frac{1}{j+1}\frac{n+1}{j+1} < \infty \qquad (n \in \mathbb{N}).$$

This finishes the proof of Theorem 3.7.3. ■

Definition 3.7.4 Suppose that $\theta \in W(C, \ell_1)(\mathbb{R}^d)$. For $f \in L_1(\mathbb{T}^d)$ and $n \in \mathbb{N}^d$, the nth rectangular θ-means $\sigma_n^\theta f$ of the Fourier series of f and the nth rectangular θ-kernel K_n^θ are introduced by

$$\sigma_n^\theta f(x) := \sum_{k_1 \in \mathbb{Z}} \cdots \sum_{k_d \in \mathbb{Z}} \theta\left(\frac{-k_1}{n_1}, \ldots, \frac{-k_d}{n_d}\right) \widehat{f}(k) e^{\imath k \cdot x}$$

and

$$K_n^\theta(t) := \sum_{k_1 \in \mathbb{Z}} \cdots \sum_{k_d \in \mathbb{Z}} \theta\left(\frac{-k_1}{n_1}, \ldots, \frac{-k_d}{n_d}\right) e^{\imath k \cdot t},$$

respectively.

By Theorem 3.7.3, the θ-kernels K_n^θ and the θ-means $\sigma_n^\theta f$ are well defined. We suppose often that

$$\theta = \theta_1 \otimes \cdots \otimes \theta_d,$$

where $\theta_i \in W(C, \ell_1)(\mathbb{R})$ for all $i = 1, \ldots, d$. Then $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ and

$$K_n^\theta = K_{n_1}^{\theta_1} \otimes \cdots \otimes K_{n_d}^{\theta_d}.$$

Lemma 3.7.5 *Suppose that $\theta \in W(C, \ell_1)(\mathbb{R}^d)$. For $f \in L_1(\mathbb{T}^d)$ and $n \in \mathbb{N}^d$, we have*

$$\sigma_n^\theta f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n^\theta(t)\, dt.$$

The θ-means can also be written as a convolution of f and the Fourier transform of θ in the following way.

Theorem 3.7.6 *If $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ and $\widehat{\theta} \in L_1(\mathbb{R}^d)$, then*

$$\sigma_n^\theta f(x) = \left(\prod_{j=1}^d n_j\right) \int_{\mathbb{R}^d} f(x-t) \widehat{\theta}(n_1 t_1, \ldots, n_d t_d)\, dt$$

for almost every $x \in \mathbb{T}^d$ and for all $n \in \mathbb{N}^d$ and $f \in L_1(\mathbb{T}^d)$.

Proof If $f(t) = e^{\imath k \cdot t}$ ($k \in \mathbb{Z}^d, t \in \mathbb{T}^d$), then

$$\begin{aligned}\sigma_n^\theta f(x) &= \theta\left(\frac{-k_1}{n_1}, \ldots, \frac{-k_d}{n_d}\right) e^{\imath k \cdot x} \\ &= e^{\imath k \cdot x} \int_{\mathbb{R}^d} \left(\prod_{j=1}^d e^{-\imath k_j t_j / n_j}\right) \widehat{\theta}(t)\, dt\end{aligned}$$

$$= \left(\prod_{j=1}^{d} n_j \right) \int_{\mathbb{R}^d} e^{\imath k \cdot (x-t)} \widehat{\theta}(n_1 t_1, \dots, n_d t_d) \, dt.$$

Thus, the theorem holds also for trigonometric polynomials. The proof can be finished as in Theorem 2.2.30. ■

We extend again the definition of the rectangular θ-means to distributions.

Definition 3.7.7 Suppose that $\theta \in W(C, \ell_1)(\mathbb{R}^d)$. For $f \in D(\mathbb{T}^d)$ and $n \in \mathbb{N}^d$, the nth rectangular θ-means $\sigma_n^\theta f$ of the Fourier series of f are given by

$$\sigma_n^\theta f := f * K_n^\theta.$$

3.7.1 Feichtinger's Algebra $S_0(\mathbb{R}^d)$

Theorem 3.7.6 is a fundamental result, so the condition $\widehat{\theta} \in L_1(\mathbb{R}^d)$ is of great importance. In this subsection, we give some sufficient conditions for a function θ to satisfy $\widehat{\theta} \in L_1(\mathbb{R}^d)$. In contrary to the other sections, we do not prove all results here. Some of them are presented without proof. Several such conditions are already known. The next one can be found in Bachman, Narici and Beckenstein [15, p. 323].

Theorem 3.7.8 *If $\theta \in L_1(\mathbb{R})$ is bounded on a neighborhood of* 0 *and $\widehat{\theta} \geq 0$, then $\widehat{\theta} \in L_1(\mathbb{R})$.*

Obviously, θ is bounded on a neighborhood of 0 if $\theta \in L_\infty(\mathbb{R})$ or θ is continuous at 0. Moreover, if $\theta \in L_1(\mathbb{R})$ has compact support and $\theta \in \mathrm{Lip}(\alpha)$ for some $\alpha > 1/2$, then $\widehat{\theta} \in L_1(\mathbb{R})$ (see Natanson and Zuk [244, p. 176]).

Now we introduce a Banach space, called Feichtinger's algebra, the Fourier transforms of the elements of which are all integrable. This space was first considered in Feichtinger [100].

Definition 3.7.9 The short-time Fourier transform of $f \in L_2(\mathbb{R}^d)$ with respect to a window function $g \in L_2(\mathbb{R}^d)$ is defined by

$$S_g f(x, \omega) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-\imath \omega \cdot t} \, dt \qquad (x, \omega \in \mathbb{R}^d).$$

Definition 3.7.10 Let $g_0(x) := e^{-\pi \|x\|_2^2}$ be the Gauss function. We define the Feichtinger's algebra $S_0(\mathbb{R}^d)$ by

$$S_0(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) : \|f\|_{S_0} := \left\| S_{g_0} f \right\|_{L_1(\mathbb{R}^{2d})} < \infty \right\}.$$

Any other non-zero Schwartz function defines the same space and an equivalent norm. It is known that $S_0(\mathbb{R}^d)$ contains all Schwartz functions. Moreover, $S_0(\mathbb{R}^d)$ is isometrically invariant under translation, modulation and Fourier transform (see Feichtinger and Zimmermann [100, 106]). Actually, $S_0(\mathbb{R}^d)$ is the minimal Banach space having this property (see Feichtinger [100]). Furthermore, Feichtinger's algebra is a subspace of the Wiener algebra, the embedding $S_0(\mathbb{R}^d) \hookrightarrow W(C, \ell_1)(\mathbb{R}^d)$ is dense and continuous and

$$S_0(\mathbb{R}^d) \subsetneq W(C, \ell_1)(\mathbb{R}^d) \cap \mathcal{F}(W(C, \ell_1)(\mathbb{R}^d)),$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}(W(C, \ell_1)(\mathbb{R}^d))$ the set of Fourier transforms of the functions from $W(C, \ell_1)(\mathbb{R}^d)$ (see Feichtinger and Zimmermann [106], Losert [223] and Gröchenig [152]). Let us define the weight function

$$v_s(\omega) := \left(1 + \|\omega\|_2^2\right)^{d/2} \qquad (\omega \in \mathbb{R}^d, s \in \mathbb{R}).$$

Theorem 3.7.11 *(a) If $\theta \in S_0(\mathbb{R}^d)$, then $\widehat{\theta} \in S_0(\mathbb{R}^d) \subset L_1(\mathbb{R}^d)$.*
(b) If $\theta \in L_1(\mathbb{R}^d)$ and $\widehat{\theta}$ has compact support, then $\theta \in S_0(\mathbb{R}^d)$.
(c) If $\theta \in L_1(\mathbb{R}^d)$ has compact support and $\widehat{\theta} \in L_1(\mathbb{R}^d)$, then $\theta \in S_0(\mathbb{R}^d)$.
(d) If $\theta v_s, \widehat{\theta} v_s \in L_2(\mathbb{R}^d)$ for some $s > d$, then $\theta \in S_0(\mathbb{R}^d)$.
(e) If $\theta v_s, \widehat{\theta} v_s \in L_\infty(\mathbb{R}^d)$ for some $s > 3d/2$, then $\theta \in S_0(\mathbb{R}^d)$.

For more about Feichtinger's algebra see Feichtinger and Zimmermann [100, 106]).

Sufficient conditions can also be given with the help of Sobolev, fractional Sobolev and Besov spaces. We do not give a detailed description of these spaces. For the interested readers, we refer to Triebel [313], Runst and Sickel [267], Stein [289] and Grafakos [143]. The Sobolev space $W_p^k(\mathbb{R}^d)$ $(1 \le p \le \infty, k \in \mathbb{N})$ is defined by

$$W_p^k(\mathbb{R}^d) := \left\{\theta \in L_p(\mathbb{R}^d) : D^\alpha \theta \in L_p(\mathbb{R}^d), |\alpha| \le k\right\}$$

and endowed with the norm

$$\|\theta\|_{W_p^k} := \sum_{|\alpha| \le k} \|D^\alpha \theta\|_p\,,$$

where D denotes the distributional derivative.

This definition can be extended to every real s in the following way. The fractional Sobolev space $\mathcal{L}_p^s(\mathbb{R}^d)$ $(1 \le p \le \infty, s \in \mathbb{R})$ consists of all tempered distributions θ for which

$$\|\theta\|_{\mathcal{L}_p^s} := \left\|\mathcal{F}^{-1}\left((1 + |\cdot|^2)^{s/2} \widehat{\theta}\right)\right\|_p < \infty,$$

where $\mathcal{F}$ denotes the Fourier transform. It is known that

$$\mathcal{L}_p^s(\mathbb{R}^d) = W_p^k(\mathbb{R}^d) \quad \text{if} \quad s = k \in \mathbb{N} \quad \text{and} \quad 1 < p < \infty$$

with equivalent norms.

In order to define the Besov spaces, take a non-negative Schwartz function $\psi \in \mathcal{S}(\mathbb{R})$ with support $[1/2, 2]$ that satisfies

$$\sum_{k=-\infty}^{\infty} \psi(2^{-k}s) = 1 \quad \text{for all} \quad s \in \mathbb{R} \setminus \{0\}.$$

For $x \in \mathbb{R}^d$, let

$$\phi_k(x) := \psi(2^{-k}|x|) \quad \text{for} \quad k \geq 1 \quad \text{and} \quad \phi_0(x) = 1 - \sum_{k=1}^{\infty} \phi_k(x).$$

The Besov space $B_{p,r}^s(\mathbb{R}^d)$ $(0 < p, r \leq \infty, s \in \mathbb{R})$ is the space of all tempered distributions f for which

$$\|f\|_{B_{p,r}^s} := \left(\sum_{k=0}^{\infty} 2^{ksr} \left\| \left(\mathcal{F}^{-1}\phi_k\right) * f \right\|_p^r \right)^{1/r} < \infty.$$

The Sobolev, fractional Sobolev and Besov spaces are all quasi-Banach spaces, and if $1 \leq p, r \leq \infty$, then they are Banach spaces. All these spaces contain the Schwartz functions. The following facts are known: in the case $1 \leq p, r \leq \infty$, one has

$$W_p^m(\mathbb{R}^d),\, B_{p,r}^s(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d) \qquad \text{if} \quad s > 0, m \in \mathbb{N},$$

$$W_p^{m+1}(\mathbb{R}^d) \hookrightarrow B_{p,r}^s(\mathbb{R}^d) \hookrightarrow W_p^m(\mathbb{R}^d) \qquad \text{if} \quad m < s < m+1, \tag{3.7.3}$$

$$B_{p,r}^s(\mathbb{R}^d) \hookrightarrow B_{p,r+\epsilon}^s(\mathbb{R}^d),\, B_{p,\infty}^{s+\epsilon}(\mathbb{R}^d) \hookrightarrow B_{p,r}^s(\mathbb{R}^d) \qquad \text{if} \quad \epsilon > 0, \tag{3.7.4}$$

$$B_{p_1,1}^{d/p_1}(\mathbb{R}^d) \hookrightarrow B_{p_2,1}^{d/p_2}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d) \qquad \text{if} \quad 1 \leq p_1 \leq p_2 < \infty. \tag{3.7.5}$$

For two quasi-Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, the embedding $\mathbb{X} \hookrightarrow \mathbb{Y}$ means that $\mathbb{X} \subset \mathbb{Y}$ and $\|f\|_{\mathbb{Y}} \leq C\|f\|_{\mathbb{X}}$.

The connection between Besov spaces and Feichtinger's algebra is summarized in the next theorem.

Theorem 3.7.12 *We have*

(i) If $1 \leq p \leq 2$ and $\theta \in B_{p,1}^{d/p}(\mathbb{R}^d)$, then $\widehat{\theta} \in L_1(\mathbb{R}^d)$ and

$$\left\|\widehat{\theta}\right\|_1 \leq C \|\theta\|_{B_{p,1}^{d/p}}.$$

(ii) *If $s > d$, then $\mathcal{L}_1^s(\mathbb{R}^d) \hookrightarrow S_0(\mathbb{R}^d)$.*
(iii) *If d' denotes the smallest even integer which is larger than d and $s > d'$, then*

$$B_{1,\infty}^s(\mathbb{R}^d) \hookrightarrow W_1^{d'}(\mathbb{R}^d) \hookrightarrow S_0(\mathbb{R}^d).$$

Proof (i) was proved in Girardi and Weis [130] and (ii) in Okoudjou [250]. The first embedding of (iii) follows from (3.7.3) and (3.7.4). If k is even, then $W_1^k(\mathbb{R}^d) \hookrightarrow \mathcal{L}_1^k(\mathbb{R}^d)$ (see Stein [289, p. 160]). Then (ii) proves (iii). ∎

It follows from (i) and (3.7.3) that $\theta \in W_p^j(\mathbb{R}^d)$ $(j > d/p, j \in \mathbb{N})$ implies $\widehat{\theta} \in L_1(\mathbb{R}^d)$. If $j \geq d'$, then even $W_1^j(\mathbb{R}^d) \hookrightarrow S_0(\mathbb{R}^d)$ (see (iii)). Moreover, if $s > d'$ as in (iii), then

$$B_{1,\infty}^s(\mathbb{R}^d) \hookrightarrow B_{1,1}^d(\mathbb{R}^d) \hookrightarrow B_{p,1}^{d/p}(\mathbb{R}^d) \qquad (1 < p < \infty)$$

by (3.7.4) and (3.7.5). Theorem 3.7.12 says that $B_{1,\infty}^s(\mathbb{R}^d) \subset S_0(\mathbb{R}^d)$ $(s > d')$ and if we choose θ from the larger space $B_{p,1}^{d/p}(\mathbb{R}^d)$ $(1 \leq p \leq 2)$, then $\widehat{\theta}$ is still integrable.

The embedding $W_1^2(\mathbb{R}) \hookrightarrow S_0(\mathbb{R})$ follows from (iii). With the help of the usual derivative, we give another useful sufficient condition for a function to be in $S_0(\mathbb{R}^d)$. As usual, we denote by $C^k(\mathbb{R}^d)$ the set of k times continuously differentiable functions.

Definition 3.7.13 A function θ is in $V_1^k(\mathbb{R})$ if there are numbers $-\infty = a_0 < a_1 < \cdots < a_n < a_{n+1} = \infty$, where $n = n(\theta)$ depends on θ and

$$\theta \in C^{k-2}(\mathbb{R}), \qquad \theta \in C^k(a_i, a_{i+1}), \qquad \theta^{(j)} \in L_1(\mathbb{R})$$

for all $i = 0, \ldots, n$ and $j = 0, \ldots, k$. The norm of this space is defined by

$$\|\theta\|_{V_1^k} := \sum_{j=0}^{k} \left\|\theta^{(j)}\right\|_1 + \sum_{i=1}^{n} \left|\theta^{(k-1)}(a_i + 0) - \theta^{(k-1)}(a_i - 0)\right|,$$

where $\theta^{(k-1)}(a_i \pm 0)$ denotes the right and left limits of $\theta^{(k-1)}$.

These limits do exist and are finite because $\theta^{(k)} \in C(a_i, a_{i+1}) \cap L_1(\mathbb{R})$ implies

$$\theta^{(k-1)}(x) = \theta^{(k-1)}(a) + \int_a^x \theta^{(k)}(t)\,dt$$

for some $a \in (a_i, a_{i+1})$. Since $\theta^{(k-1)} \in L_1(\mathbb{R})$, we establish that

$$\lim_{x \to -\infty} \theta^{(k-1)}(x) = \lim_{x \to \infty} \theta^{(k-1)}(x) = 0.$$

Similarly, $\theta^{(j)} \in C_0(\mathbb{R})$ for $j = 0, \ldots, k-2$.

Of course, $W_1^2(\mathbb{R})$ and $V_1^2(\mathbb{R})$ are not identical. For $\theta \in V_1^2(\mathbb{R})$, we have $\theta' = D\theta$; however, $\theta'' = D^2\theta$ only if $\lim_{x\to a_i+0} \theta'(x) = \lim_{x\to a_i-0} \theta'(x)$ $(i = 1, \ldots, n)$.

Theorem 3.7.14 *We have* $V_1^2(\mathbb{R}) \hookrightarrow S_0(\mathbb{R})$.

Proof Integrating by parts, we have

$$\begin{aligned}
&S_{g_0}\theta(x,\omega)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \theta(t)\overline{g_0(t-x)}e^{-\imath\omega t}\,dt\\
&= \frac{1}{2\pi}\sum_{i=0}^{n}\int_{a_i}^{a_{i+1}} \theta(t)e^{-\pi(t-x)^2}e^{-\imath\omega t}\,dt\\
&= \frac{1}{2\pi}\sum_{i=0}^{n}\left[\theta(t)e^{-\pi(t-x)^2}\frac{e^{-\imath\omega t}}{-\imath\omega}\right]_{a_i}^{a_{i+1}}\\
&\quad - \frac{1}{2\pi}\sum_{i=0}^{n}\int_{a_i}^{a_{i+1}}\left(\theta'(t)e^{-\pi(t-x)^2} - 2\pi\theta(t)e^{-\pi(t-x)^2}(t-x)\right)\frac{e^{-\imath\omega t}}{-\imath\omega}\,dt.
\end{aligned}$$

Observe that the first sum is 0. In the second sum, we integrate by parts again to obtain

$$\begin{aligned}
S_{g_0}\theta(x,\omega) &= \frac{1}{2\pi}\sum_{i=0}^{n}\left[\left(\theta'(t)e^{-\pi(t-x)^2} - 2\pi\theta(t)e^{-\pi(t-x)^2}(t-x)\right)\frac{e^{-\imath\omega t}}{\omega^2}\right]_{a_i}^{a_{i+1}}\\
&\quad - \frac{1}{2\pi}\sum_{i=0}^{n}\int_{a_i}^{a_{i+1}}\Big(\theta''(t)e^{-\pi(t-x)^2} - 4\pi\theta'(t)e^{-\pi(t-x)^2}(t-x)\\
&\quad - 2\pi\theta(t)\left(-2\pi e^{-\pi(t-x)^2}(t-x)^2 + e^{-\pi(t-x)^2}\right)\Big)\frac{e^{-\imath\omega t}}{\omega^2}\,dt.
\end{aligned}$$

The first sum is equal to

$$\frac{1}{2\pi}\sum_{i=1}^{n}\Big(\theta'(a_i+0) - \theta'(a_i-0)\Big)e^{-\pi(a_i-x)^2}\frac{e^{-\imath\omega a_i}}{\omega^2}.$$

Hence

$$\int_{\mathbb{R}}\int_{\{|\omega|\ge 1\}} |S_{g_0}\theta(x,\omega)|\,dx\,d\omega \le C_s\|\theta\|_{V_1^2}.$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\{|\omega|<1\}} |S_{g_0}\theta(x,\omega)|\,dx\,d\omega &\le C_s\int_{\mathbb{R}}\int_{\{|\omega|<1\}}\int_{\mathbb{R}} |\theta(t)|g_0(t-x)\,dt\,dx\,d\omega\\
&\le C_s\|\theta\|_{V_1^2},
\end{aligned}$$

which finishes the proof of Theorem 3.7.14. ■

The next Corollary follows from the definition of $S_0(\mathbb{R}^d)$ and from Theorem 3.7.14.

Corollary 3.7.15 *If each* $\theta_j \in V_1^2(\mathbb{R})$ $(j = 1, \ldots, d)$*, then*

$$\theta = \theta_1 \otimes \cdots \otimes \theta_d \in S_0(\mathbb{R}^d).$$

3.7.2 Norm Convergence of the Rectangular θ-Means

First, we investigate the $L_2(\mathbb{T}^d)$-norm convergence of $\sigma_n^\theta f$ as $n \to \infty$ $(n \in \mathbb{N}^d)$ in Pringsheim's sense.

Theorem 3.7.16 *If* $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ *and* $\theta(0) = 1$*, then*

$$\lim_{n\to\infty} \sigma_n^\theta f = f \quad \text{in the } L_2(\mathbb{T}^d)\text{-norm for all } f \in L_2(\mathbb{T}^d).$$

Proof It is easy to see that the norm of the operator

$$\sigma_n^\theta : L_2(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$$

can be given by

$$\begin{aligned}
\sup_{f\in L_2(\mathbb{T}^d),\, \|f\|_2\le 1} \left\| f * K_n^\theta \right\|_2 &= \sup_{f\in L_2(\mathbb{T}^d),\, \|f\|_2\le 1} \left\| \widehat{f}\widehat{K_n^\theta} \right\|_2 \\
&= \sup_{\widehat{f}\in \ell_2(\mathbb{Z}^d),\, \|\widehat{f}\|_2\le 1} \left\| \widehat{f}\widehat{K_n^\theta} \right\|_2 \\
&= \left\| \widehat{K_n^\theta} \right\|_\infty \\
&= \sup_{k\in\mathbb{Z}^d} \left| \theta\left(\frac{-k_1}{n_1}, \ldots, \frac{-k_d}{n_d} \right) \right| \\
&\le C.
\end{aligned}$$

Thus, the norms of σ_n^θ $(n \in \mathbb{N}^d)$ are uniformly bounded. Since θ is continuous, the convergence holds for all trigonometric polynomials. The set of the trigonometric polynomials are dense in $L_2(\mathbb{T}^d)$, so the usual density theorem proves Theorem 3.7.16. ■

Now, we give a sufficient and necessary condition for the uniform and $L_1(\mathbb{T}^d)$ convergence $\sigma_n^\theta f \to f$.

Theorem 3.7.17 *If* $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ *and* $\theta(0) = 1$*, then the following conditions are equivalent:*

(i) $\widehat{\theta} \in L_1(\mathbb{R}^d)$,
(ii) $\sigma_n^\theta f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{N}^d$,
(iii) $\sigma_n^\theta f(x) \to f(x)$ *for all* $x \in \mathbb{T}^d$ *and* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{N}^d$,
(iv) $\sigma_n^\theta f \to f$ *in the* $L_1(\mathbb{T}^d)$*-norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{N}^d$,
(v) $\sigma_n^\theta f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{R}_\tau^d$,
(vi) $\sigma_n^\theta f(x) \to f(x)$ *for all* $x \in \mathbb{T}^d$ *and* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{R}_\tau^d$,
(vii) $\sigma_n^\theta f \to f$ *in the* $L_1(\mathbb{T}^d)$*-norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{R}_\tau^d$.

Recall the definition of R_τ^d from (3.3.1).

Proof We may suppose that $d = 1$, since the multi-dimensional case is similar. First, we verify the equivalence between (i), (ii), (iii) and (iv). If (i) holds, then by Theorem 3.7.6,

$$\left\|\sigma_n^\theta f\right\|_\infty \le \|f\|_\infty \left\|\widehat{\theta}\right\|_1 \qquad (f \in C(\mathbb{T}), n \in \mathbb{N})$$

and so the operators $\sigma_n : C(\mathbb{T}) \to C(\mathbb{T})$ are uniformly bounded. Since (ii) holds for all trigonometric polynomials and the set of the trigonometric polynomials are dense in $C(\mathbb{T})$, (ii) follows easily. (ii) implies (iii) trivially.

Suppose that (iii) is satisfied. We are going to prove (i). For a fixed $x \in \mathbb{T}$, the operators

$$U_n : C(\mathbb{T}) \to \mathbb{R}, \qquad U_n f := \sigma_n^\theta f(x) \qquad (n \in \mathbb{N})$$

are uniformly bounded by the Banach-Steinhaus theorem. We get by Lemma 3.7.5 that

$$\|U_n\| = \frac{1}{(2\pi)^d} \int_{\mathbb{T}} |K_n^\theta(x-t)|\, dt = \frac{1}{(2\pi)^d} \|K_n^\theta\|_1 \qquad (n \in \mathbb{N}).$$

Hence

$$\sup_{n \in \mathbb{N}} \|K_n^\theta\|_1 \le C.$$

Since K_n^θ is 2π-periodic, we have for $\alpha \le n/2$ that

$$\begin{aligned}
\int_{-2\alpha\pi}^{2\alpha\pi} \frac{1}{n} \left| \sum_{k=-\infty}^{\infty} \theta\left(\frac{-k}{n}\right) e^{\imath tk/n} \right| dt &\le \int_{-n\pi}^{n\pi} \frac{1}{n} \left| \sum_{k=-\infty}^{\infty} \theta\left(\frac{-k}{n}\right) e^{\imath tk/n} \right| dt \\
&= \int_{-\pi}^{\pi} \left| \sum_{k=-\infty}^{\infty} \theta\left(\frac{-k}{n}\right) e^{\imath kx} \right| dx \\
&= \int_{\mathbb{T}} |K_n^\theta(x)|\, dx \le C. \qquad (3.7.6)
\end{aligned}$$

For a fixed $t \in \mathbb{R}$, let

$$h_n(t) := \frac{1}{n} \sum_{k=-\infty}^{\infty} \theta\left(\frac{-k}{n}\right) e^{\imath tk/n}$$

and

$$\varphi_n(t,u) := \sum_{k=-\infty}^{\infty} \theta\left(\frac{-k}{n}\right) e^{\imath tk/n} 1_{[\frac{k}{n},\frac{k+1}{n})}(u).$$

It is easy to see that

$$\lim_{n\to\infty} \varphi_n(t,u) = \theta(-u)e^{\imath tu}.$$

Moreover,

$$|\varphi_n(t,u)| \le \sum_{l=-\infty}^{\infty} \sup_{x\in[0,1)} |\theta(x-l-1)|\, 1_{[l,l+1)}(u)$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \sup_{x\in[0,1)} |\theta(x-l-1)|\, 1_{[l,l+1)}(u)\, du &= \sum_{l=-\infty}^{\infty} \sup_{x\in[0,1)} |\theta(x-l-1)| \\ &= \|\theta\|_{W(C,\ell_1)}.\end{aligned}$$

Lebesgue's dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi_n(t,u)\, du = \int_{-\infty}^{\infty} \theta(-u)e^{\imath tu}\, du = (2\pi)^d \widehat{\theta}(t).$$

Obviously,

$$\int_{-\infty}^{\infty} \varphi_n(t,u)\, du = h_n(t)$$

and so

$$\lim_{n\to\infty} h_n(t) = (2\pi)^d \widehat{\theta}(t).$$

Of course, this holds for all $t \in \mathbb{R}$. We have by (3.7.2) that

$$|h_n(t)| \le \|\theta\|_{W(C,\ell_1)}.$$

Thus

$$\lim_{n\to\infty} \int_{-2\alpha\pi}^{2\alpha\pi} |h_n(t)|\, dt = (2\pi)^d \int_{-2\alpha\pi}^{2\alpha\pi} \left|\widehat{\theta}(t)\right|\, dt.$$

Inequality (3.7.6) yields that

$$\int_{-2\alpha\pi}^{2\alpha\pi} \left|\widehat{\theta}(t)\right|\, dt \le C \qquad \text{for all} \qquad \alpha > 0$$

and so

$$\int_{-\infty}^{\infty} \left|\widehat{\theta}(t)\right| \, dt \leq C,$$

which shows (i).

If $\widehat{\theta} \in L_1(\mathbb{R})$, then Theorem 3.7.6 implies

$$\left\|\sigma_n^\theta f\right\|_1 \leq \|f\|_1 \left\|\widehat{\theta}\right\|_1 \qquad (f \in L_1(\mathbb{T}), n \in \mathbb{N}).$$

Hence (iv) follows from (i) because the set of the trigonometric polynomials are dense in $L_1(\mathbb{T})$. The fact that (iv) implies (i) can be proved similarly as $(iii) \Rightarrow (i)$, since, by duality, the norm of the operator $\sigma_n^\theta : L_1(\mathbb{T}) \to L_1(\mathbb{T})$ is again

$$\left\|\sigma_n^\theta\right\| = \left\|K_n^\theta\right\|_1.$$

It is easy to see that the equivalence between (i), (v), (vi) and (vii) can be proved in the same way. ■

Note that the statement $(i) \Leftrightarrow (ii)$ was shown in the one-dimensional case by Natanson and Zuk [244] for θ having compact support. The situation in our general case is much more complicated and can be found in Feichtinger and Weisz [103]. One part of the preceding result can be generalized for $L_p(\mathbb{T}^d)$ spaces.

Theorem 3.7.18 *Assume that $\theta(0) = 1$, $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ and $\widehat{\theta} \in L_1(\mathbb{R}^d)$. If $1 \leq p < \infty$ and $f \in L_p(\mathbb{T}^d)$, then*

$$\sup_{n \in \mathbb{N}} \left\|\sigma_n^\theta f\right\|_p \leq C\|f\|_p$$

and

$$\lim_{n\to\infty} \sigma_n^\theta f = f \qquad \text{in the } L_p(\mathbb{T}^d)\text{-norm.}$$

Proof For simplicity, we show the theorem for $d = 1$. Using Theorem 3.7.6, we conclude

$$\begin{aligned}\sigma_n^\theta f(x) - f(x) &= n \int_{\mathbb{R}} \Big(f(x-t) - f(x)\Big)\widehat{\theta}(nt)\, dt \\ &= \int_{\mathbb{R}} \left(f\left(x - \frac{t}{n}\right) - f(x)\right) \widehat{\theta}(t)\, dt\end{aligned}$$

and

$$\left\|\sigma_n^\theta f - f\right\|_p = \int_{\mathbb{R}} \left\|f\left(\cdot - \frac{t}{n}\right) - f(\cdot)\right\|_p \left|\widehat{\theta}(t)\right| \, dt.$$

The theorem follows from the Lebesgue dominated convergence theorem. ■

Since $\theta \in S_0(\mathbb{R}^d)$ implies $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ and $\widehat{\theta} \in S_0(\mathbb{R}^d) \subset L_1(\mathbb{R}^d)$, the next corollary follows from Theorems 3.7.17 and 3.7.18.

Corollary 3.7.19 *If $\theta \in S_0(\mathbb{R}^d)$ and $\theta(0) = 1$, then*

(i) $\sigma_n^\theta f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{N}^d$,
(ii) $\sigma_n^\theta f \to f$ *in the* $L_1(\mathbb{T}^d)$*-norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{N}^d$,
(iii) $\sigma_n^\theta f \to f$ *in the* $L_p(\mathbb{T}^d)$*-norm for all* $f \in L_p(\mathbb{T}^d)$ $(1 < p < \infty)$ *as* $n \to \infty$ *and* $n \in \mathbb{N}^d$.

The next corollary follows from the fact that $\theta \in S_0(\mathbb{R}^d)$ is equivalent to $\widehat{\theta} \in L_1(\mathbb{R}^d)$, provided that θ has compact support (see, e.g., Feichtinger and Zimmermann [106]).

Corollary 3.7.20 *If $\theta \in C(\mathbb{R}^d)$ has compact support and $\theta(0) = 1$, then the following conditions are equivalent:*

(i) $\theta \in S_0(\mathbb{R}^d)$,
(ii) $\sigma_n^\theta f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{N}^d$,
(iii) $\sigma_n^\theta f(x) \to f(x)$ *for all* $x \in \mathbb{T}^d$ *and* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{N}^d$,
(iv) $\sigma_n^\theta f \to f$ *in the* $L_1(\mathbb{T}^d)$*-norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$,
(v) $\sigma_n^\theta f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{R}_\tau^d$,
(vi) $\sigma_n^\theta f(x) \to f(x)$ *for all* $x \in \mathbb{T}^d$ *and* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{R}_\tau^d$,
(vii) $\sigma_n^\theta f \to f$ *in the* $L_1(\mathbb{T}^d)$*-norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$ *and* $n \in \mathbb{R}_\tau^d$.

3.7.3 Almost Everywhere Convergence of the Rectangular θ-Means

Definition 3.7.21 For given κ, τ satisfying the conditions given in Sect. 3.4, we define the restricted maximal θ-operators by

$$\sigma_\square^\theta f := \sup_{n \in \mathbb{R}_\tau^d} \left|\sigma_n^\theta f\right|, \qquad \sigma_\kappa^\theta f := \sup_{n \in \mathbb{R}_{\kappa,\tau}^d} \left|\sigma_n^\theta f\right|.$$

The unrestricted maximal θ-operator is defined by

$$\sigma_*^\theta f := \sup_{n \in \mathbb{N}^d} \left|\sigma_n^\theta f\right|.$$

In this subsection, we suppose that

$$\theta(0) = 1, \qquad \theta = \theta_1 \otimes \cdots \otimes \theta_d, \qquad \theta_j \in W(C, \ell_1)(\mathbb{R}), \qquad j = 1, \ldots, d. \tag{3.7.7}$$

For the restricted convergence, we suppose in addition that

$$\mathcal{I}\theta_j \in W(C, \ell_1)(\mathbb{R}), \qquad j = 1, \ldots, d. \tag{3.7.8}$$

Here $\mathcal{I}$ denotes the identity function, so

$$\mathcal{I}(x) = x \qquad \text{and} \qquad (\mathcal{I}\theta_j)(x) = x\theta_j(x).$$

Similar to (2.6.6), assume that $\widehat{\theta}_j$ is $(N+1)$-times differentiable $(N \geq 0)$ and there exists

$$N < \beta_j \leq N+1$$

such that

$$\left| \left(\widehat{\theta}_j\right)^{(i)}(x) \right| \leq C|x|^{-\beta_j - 1} \qquad (x \neq 0) \tag{3.7.9}$$

for $i = N, N+1$ and all $j = 1, \ldots, d$.

Theorem 3.7.22 *Assume that* (3.7.7), (3.7.8) *and* (3.7.9) *are satisfied with* $N = 0$. *If*

$$\max\left\{\frac{d}{d+1}, \frac{1}{\beta_j + 1}, j = 1, \ldots, d\right\} < p \leq \infty,$$

then

$$\left\|\sigma_{\square}^{\alpha} f\right\|_p \leq C_p \|f\|_{H_p^{\square}} \qquad (f \in H_p^{\square}(\mathbb{T}^d)).$$

Moreover,

$$\sup_{\rho > 0} \rho\, \lambda(\sigma_{\square}^{\theta} f > \rho) \leq C\|f\|_1 \qquad (f \in L_1(\mathbb{T}^d)).$$

Proof Inequality (3.7.2) implies that

$$\left|K_{n_j}^{\theta_j}\right| \leq C n_j \qquad (n_j \in \mathbb{N}).$$

Similarly,

$$\sum_{k=-\infty}^{\infty} \left| \frac{k}{n_j} \theta_j \left(\frac{k}{n_j}\right) \right| \leq n_j \left\|\mathcal{I}\theta_j\right\|_{W(C,\ell_1)} < \infty \qquad (n_j \in \mathbb{N}),$$

from which we get immediately that

$$\left| \left(K_{n_j}^{\theta_j}\right)' \right| \leq C n_j^2 \qquad (n_j \in \mathbb{N}).$$

By Theorem 3.7.6,

$$K_{n_j}^{\theta_j}(x) = 2\pi n_j \sum_{k=-\infty}^{\infty} \widehat{\theta}_j \left(n_j(x + 2k\pi)\right) \qquad (x \in \mathbb{T})$$

as in (2.2.34). From this, it follows that

$$\left|K_{n_j}^{\theta_j}(x)\right| \leq \frac{C}{n_j^{\beta_j}|x|^{\beta_j+1}} \qquad (x \neq 0)$$

and

$$\left|\left(K_{n_j}^{\theta_j}\right)'(x)\right| \leq \frac{C}{n_j^{\beta_j-1}|x|^{\beta_j+1}} \qquad (x \neq 0).$$

The proof can be finished as in Theorem 3.3.4. ■

Corollary 3.7.23 *Assume that* (3.7.7)*,* (3.7.8) *and* (3.7.9) *are satisfied with* $N = 0$*. If* $f \in L_1(\mathbb{T}^d)$*, then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}_\tau^d} \sigma_n^\theta f = f \qquad a.e.$$

Combining the proofs of Theorems 3.7.22 and 3.4.7, we obtain

Theorem 3.7.24 *Assume that* (3.7.7)*,* (3.7.8) *and* (3.7.9) *are satisfied with* $N = 0$*. If*

$$\max\left\{p_1, \frac{1}{\beta_j+1}, j = 1, \ldots, d\right\} < p \leq \infty,$$

then

$$\left\|\sigma_\kappa^\theta f\right\|_p \leq C_p \|f\|_{H_p^\kappa} \qquad (f \in H_p^\kappa(\mathbb{T}^d)).$$

Moreover,

$$\sup_{\rho>0} \rho\, \lambda(\sigma_\kappa^\theta f > \rho) \leq C\|f\|_1 \qquad (f \in L_1(\mathbb{T}^d)).$$

We recall that p_1 was defined in (3.4.4).

Corollary 3.7.25 *Assume that* (3.7.7)*,* (3.7.8) *and* (3.7.9) *are satisfied with* $N = 0$*. If* $f \in L_1(\mathbb{T}^d)$*, then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}_{\kappa,\tau}^d} \sigma_n^\theta f = f \qquad a.e.$$

For the unrestricted convergence, we can allow more general conditions for θ. The next theorem can be shown as Theorems 2.6.7 and 3.6.7.

Theorem 3.7.26 *If each* θ_j *satisfies (2.6.2) and (2.6.3), then*

$$\left\|\sigma_*^\theta f\right\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{T}^d))$$

for $1/2 < p \leq \infty$*. If* (3.7.7)*,* (3.7.8) *and* (3.7.9) *are satisfied, then the preceding inequality holds for*

$$\max\left\{\frac{1}{\beta_j+1}, j = 1, \ldots, d\right\} < p \leq \infty.$$

In both cases

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\theta f > \rho) \leq C\|f\|_{H_1^i} \qquad (f \in H_1^i(\mathbb{T}^d))$$

for all $i = 1, \dots, d$.

Corollary 3.7.27 *Under the conditions of Theorem 3.7.26,*

$$\lim_{n\to\infty} \sigma_n^\theta f = f \qquad a.e.$$

for all $f \in H_1^i(\mathbb{T}^d)$ and $i = 1, \dots, d$.

Note that these results are proved in Weisz [332, 333, 335].

3.7.4 Some Summability Methods

It is easy to see that $\theta \in V_1^2(\mathbb{R}) \subset S_0(\mathbb{R})$ for all examples 2.6.13–2.6.20 of Sect. 2.6.3 and Example 2.6.21 (the Riesz summation) with $1 \leq \alpha < \infty$. Moreover, in Example 2.6.21, $\theta \in S_0(\mathbb{R})$ for all $0 < \alpha < \infty$. In the next examples, θ has d variables and $\theta \in S_0(\mathbb{R}^d)$.

Example 3.7.28 (Riesz summation]). Let

$$\theta(t) = \begin{cases} (1 - \|t\|_2^\gamma)^\alpha & \text{if } \|t\|_2 \leq 1; \\ 0 & \text{if } \|t\|_2 > 1 \end{cases} \qquad (t \in \mathbb{R}^d)$$

for some $(d-1)/2 < \alpha < \infty$, $\gamma \in \mathbb{P}$ (see Fig. 3.4).

Example 3.7.29 (Weierstrass summation). Let

$$\theta(t) = e^{-\|t\|_2^2/2} \qquad \text{or} \qquad \theta(t) = e^{-\|t\|_2} \qquad (t \in \mathbb{R}^d)$$

(see Fig. 3.5). In the first case $\widehat{\theta}(x) = e^{-\|x\|_2^2/2}$ and in the second one, $\widehat{\theta}(x) = c_d/(1 + \|x\|_2^2)^{(d+1)/2}$ for some $c_d \in \mathbb{R}$ (see Stein and Weiss [293, p. 6.]).

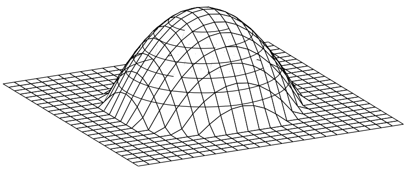

Fig. 3.4 Riesz summability function with $d = 2$, $\alpha = 1$, $\gamma = 2$

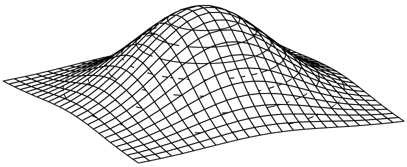

Fig. 3.5 Weierstrass summability function $\theta(t) = e^{-\|t\|_2^2/2}$

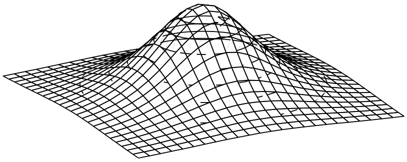

Fig. 3.6 Picard-Bessel summability function with $d = 2$

Example 3.7.30 (Picard and Bessel summations). Let

$$\theta_0(t) = \frac{1}{(1 + \|t\|_2^2)^{(d+1)/2}} \qquad (t \in \mathbb{R}^d)$$

(see Fig. 3.6). Here $\widehat{\theta_0}(x) = c_d e^{-\|x\|_2}$ for some $c_d \in \mathbb{R}^d$.

Lemma 3.7.31 *Let $\theta \in W(C, \ell_1)(\mathbb{R})$, $\mathcal{I}\theta \in W(C, \ell_1)(\mathbb{R})$ and θ be even and twice differentiable on the interval $(0, c)$, where $[-c, c]$ is the support of θ $(0 < c \leq \infty)$. Suppose that*

$$\lim_{x \to c-0} x\theta(x) = 0, \quad \lim_{x \to +0} \theta' \in \mathbb{R}, \quad \lim_{x \to c-0} \theta' \in \mathbb{R} \quad \textit{and} \quad \lim_{x \to \infty} x\theta'(x) = 0.$$

If θ' and $\max(\mathcal{I}, 1)\theta''$ are integrable, then

$$\left|\widehat{\theta}(x)\right| \leq \frac{C}{x^2}, \qquad \left|\left(\widehat{\theta}\right)'(x)\right| \leq \frac{C}{x^2} \qquad (x \neq 0),$$

i.e., (3.7.9) *hold with $N = 0$ and $\beta_j = 0$.*

Proof By integrating by parts, we have

$$\begin{aligned}
\widehat{\theta}(x) &= \frac{2}{2\pi} \int_0^c \theta(t) \cos tx \, dt \\
&= \frac{1}{\pi x} \int_0^c \theta'(t) \sin tx \, dt \\
&= \frac{-1}{\pi x^2} [\theta'(t) \cos tx]_0^c + \frac{1}{\pi x^2} \int_0^c \theta''(t) \cos tx \, dt.
\end{aligned}$$

Similarly,

$$\begin{aligned}(\widehat{\theta})'(x) &= \frac{2}{2\pi}\int_0^c t\theta(t)\cos tx\,dt\\ &= \frac{1}{\pi x}\int_0^c (t\theta(t))'\sin tx\,dt\\ &= \frac{-1}{\pi x^2}[(t\theta(t))'\cos tx]_0^c + \frac{1}{\pi x^2}\int_0^c (t\theta(t))''\cos tx\,dt,\end{aligned}$$

which proves the lemma. ∎

Note that all examples 2.6.13–2.6.21 satisfy Lemma 3.7.31, (3.7.7), (3.7.8) and (3.7.9). Thus, all results of Sects. 3.7.2 and 3.7.3 hold.