Chapter 2
ℓ_q-Summability of Higher Dimensional Fourier Series

Here, we study the theory of multi-dimensional Fourier series. In the first section, we introduce different versions of the partial sums of the d-dimensional Fourier series and the corresponding Dirichlet kernels, i.e., the cubic, triangular, circular and rectangular partial sums and Dirichlet kernels. We show that the cubic, triangular and rectangular partial sums converge in the $L_p(\mathbb{T}^d)$-norm to the function $(1 < p < \infty)$. The multi-dimensional version of Carleson's theorem is also considered.

The summability of Fourier series can be generalized for higher dimensions basically in two ways. In this chapter, we study the ℓ_q-summability of higher dimensional Fourier series. As in the literature, we investigate the three cases $q = 1$, $q = 2$ and $q = \infty$. The other type of summability, the so-called rectangular summability will be investigated in the next chapter. For each type, we investigate the Cesàro and Riesz summation. In Sect. 2.2, we present the basic definitions of the ℓ_q-summability and prove some estimations for the ℓ_q-Cesàro and Riesz kernels. In the next section, we prove that the ℓ_q-Cesàro means and ℓ_q-Riesz means of $f \in L_p(\mathbb{T}^d)$ $(1 \le p < \infty)$ converge to f in the $L_p(\mathbb{T}^d)$-norm.

In Sect. 2.4, we prove the basic results for Fourier series of distributions. We introduce the Hardy spaces $H_p^\square(\mathbb{T}^d)$ and present the atomic decomposition of these spaces. We verify also sufficient conditions for an operator to be bounded from $H_p^\square(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$. Applying this result, we show that the maximal operator of the ℓ_q-Cesàro and Riesz means are bounded from $H_p^\square(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ for any $p > p_0$, where $p_0 < 1$ is depending on the summation and on the dimension. This result implies the almost everywhere convergence of the summability means. In Sect. 2.6, we introduce a general summability method, the so-called θ-summability generated by a single function θ and prove similar results for the ℓ_q-θ-means. In the last section, as special cases, we present some summability methods, such as the de La Vallée-Poussin, Jackson-de La Vallée-Poussin, Rogosinski, Weierstrass, Picard and Bessel summations.

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2.1 Higher Dimensional Partial Sums

In this section, we generalize the results of Sect. 1.2, we introduce four types of partial sums of the d-dimensional trigonometric Fourier series and study their $L_p(\mathbb{T}^d)$-norm and almost everywhere convergence of a function $f \in L_p(\mathbb{T}^d)$.

We introduce the following notations. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ set

$$u \cdot x := \sum_{k=1}^{d} u_k x_k, \qquad \|x\|_p := \left(\sum_{k=1}^{d} |x_k|^p \right)^{1/p} \qquad (1 \leq p < \infty)$$

and

$$\|x\|_\infty := \sup_{k=1,\ldots,d} |x_k|, \qquad |x| := \|x\|_2 .$$

Definition 2.1.1 The functions

$$e^{\imath k \cdot x} = \prod_{j=1}^{d} e^{\imath k_j x_j}$$

are called d-dimensional trigonometric system, where $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, $x = (x_1, \ldots, x_d) \in \mathbb{T}^d$.

Definition 2.1.2 For an integrable function $f \in L_1(\mathbb{T}^d)$, its kth d-dimensional Fourier coefficient is defined by

$$\widehat{f}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-\imath k \cdot x} \, dx \qquad (k \in \mathbb{Z}^d).$$

The formal trigonometric series

$$\sum_{k \in \mathbb{Z}^d} \widehat{f}(k) e^{\imath k \cdot x} \qquad (x \in \mathbb{T}^d)$$

is called the d-dimensional Fourier series of f.

We will generalize the one-dimensional partial sums in Definition 1.2.2 for higher dimensional functions in two ways. In the first generalization, we take the sum over the indices $\|k\|_q \leq n$ instead of $k = -n, \ldots, n$, where $1 \leq q \leq \infty$. These sums are called ℓ_q-partial sums. In the second generalization, we take the sum in each dimension, i.e., over the indices $|k_1| \leq n_1, \ldots, |k_d| \leq n_d$. Here, we call the sums rectangular partial sums. The most natural choices $q = 2$, $q = 1$, $q = \infty$ and the rectangular partial sums are investigated in several papers and books (for $q = 2$, see e.g. Stein and Weiss [290, 293], Davis and Chang [76], Grafakos [143, 145, 146], Lu and Yan [229], Feichtinger and Weisz [103, 104], for $q = 1$, Berens, Li and Xu

[30–32, 356], Weisz [336, 337], for $q = \infty$, Marcinkiewicz [233], Zhizhiashvili [366], Weisz [332, 342, 346], for the rectangular sums, Zygmund [367] and Weisz [332, 342, 346]).

Definition 2.1.3 For $f \in L_1(\mathbb{T}^d)$, $1 \le q \le \infty$ and $n \in \mathbb{N}$, the nth ℓ_q-partial sum $s_n^q f$ of the Fourier series of f and the nth ℓ_q-Dirichlet kernel D_n^q are given by

$$s_n^q f(x) := \sum_{k \in \mathbb{Z}^d,\, \|k\|_q \le n} \widehat{f}(k) e^{\imath k \cdot x}$$

and

$$D_n^q(u) := \sum_{k \in \mathbb{Z}^d,\, \|k\|_q \le n} e^{\imath k \cdot u},$$

respectively.

The next lemma follows easily from the definition.

Lemma 2.1.4 *For all $n \in \mathbb{N}$, $1 \le q \le \infty$ and $t \in \mathbb{T}^d$, we have*

$$|D_n^q(t)| \le C n^d.$$

The partial sums are called triangular if $q = 1$, circular if $q = 2$ and cubic if $q = \infty$ (see Figs. 2.1, 2.2, 2.3 and 2.4).

Definition 2.1.5 For $f \in L_1(\mathbb{T}^d)$ and $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$, the nth rectangular partial sum $s_n f$ of the Fourier series of f and the nth rectangular Dirichlet kernel D_n are given by

$$s_n f(x) := \sum_{|k_1| \le n_1} \cdots \sum_{|k_d| \le n_d} \widehat{f}(k) e^{\imath k \cdot x}$$

and

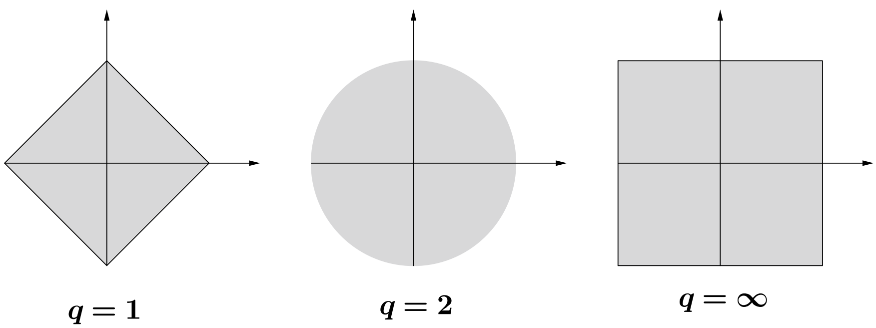

Fig. 2.1 Regions of the ℓ_q-partial sums for $d = 2$

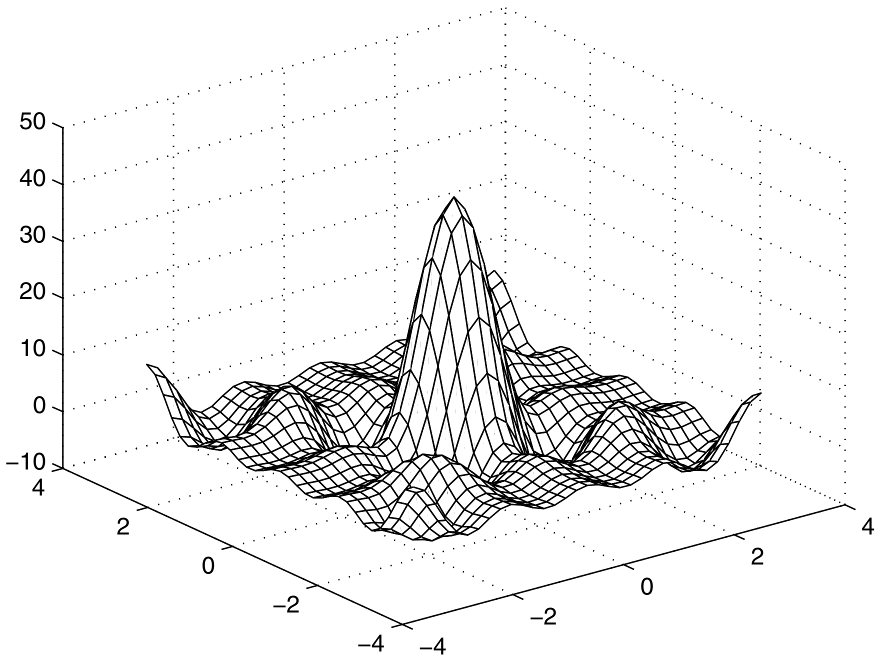

Fig. 2.2 The Dirichlet kernel D_n^q with $d = 2, q = 1, n = 4$

$$D_n(u) := \sum_{|k_1| \le n_1} \cdots \sum_{|k_d| \le n_d} e^{\imath k \cdot u},$$

respectively.

Similar to (1.2.1), we obtain

Lemma 2.1.6 *For $f \in L_1(\mathbb{T}^d)$ and $n \in \mathbb{N}$,*

$$s_n^q f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x - t) D_n^q(t)\, dt$$

and

$$s_n f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x - t) D_n(t)\, dt.$$

It is clear that

$$D_n(u) = D_{n_1}(u_1) \cdots D_{n_d}(u_d),$$

where D_{n_j} is the one-dimensional Dirichlet kernel (see Fig. 2.5).

Definition 2.1.7 For some $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$, the function

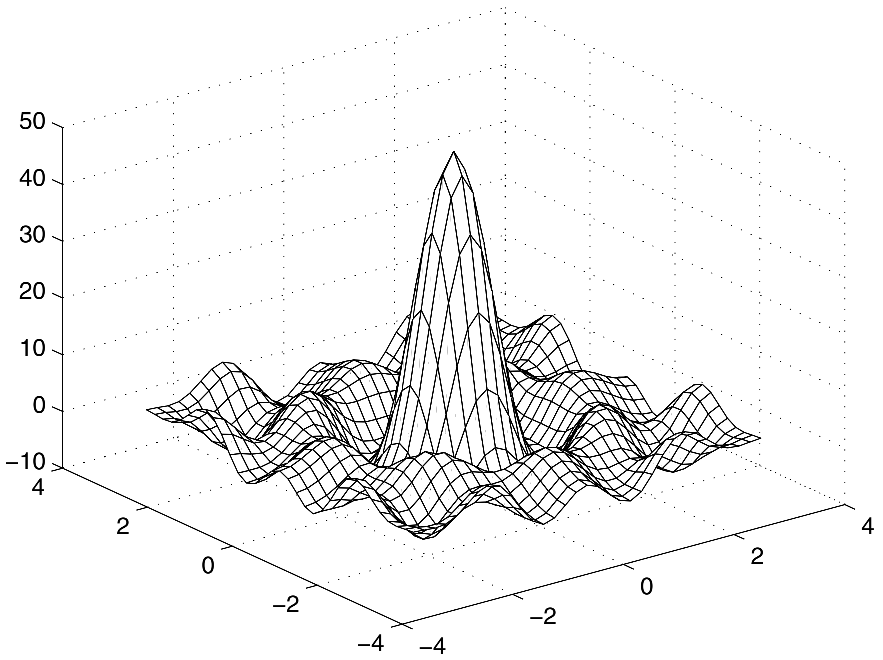

Fig. 2.3 The Dirichlet kernel D_n^q with $d = 2, q = 2, n = 4$

$$\sum_{k_1=-n_1}^{n_1} \cdots \sum_{k_d=-n_d}^{n_d} c_k e^{\imath k \cdot x} \qquad (x \in \mathbb{T}^d)$$

is said to be a trigonometric polynomial.

By iterating the one-dimensional result, we get easily the L_p-norm convergence for the rectangular partial sums.

Theorem 2.1.8 *If $f \in L_p(\mathbb{T}^d)$ for some $1 < p < \infty$, then*

$$\sup_{n \in \mathbb{N}^d} \|s_n f\|_p \leq C_p \|f\|_p$$

and

$$\lim_{n \to \infty} s_n f = f \qquad \text{in the } L_p(\mathbb{T}^d)\text{-norm}.$$

Here, $n \to \infty$ means the Pringsheim convergence, i.e., $\min(n_1, \ldots, n_d) \to \infty$.

Proof By Theorem 1.2.10,

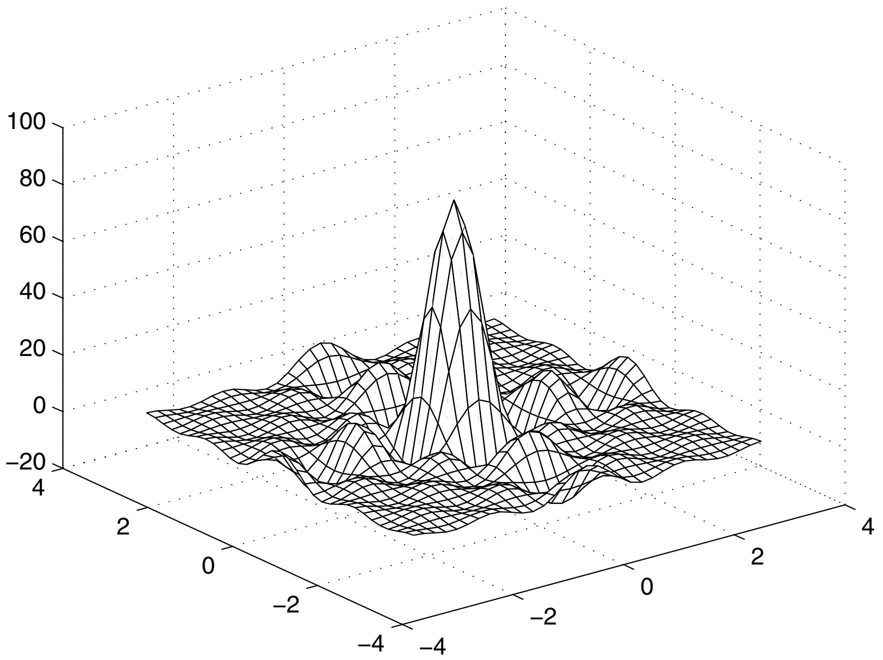

Fig. 2.4 The Dirichlet kernel D_n^q with $d = 2, q = \infty, n = 4$

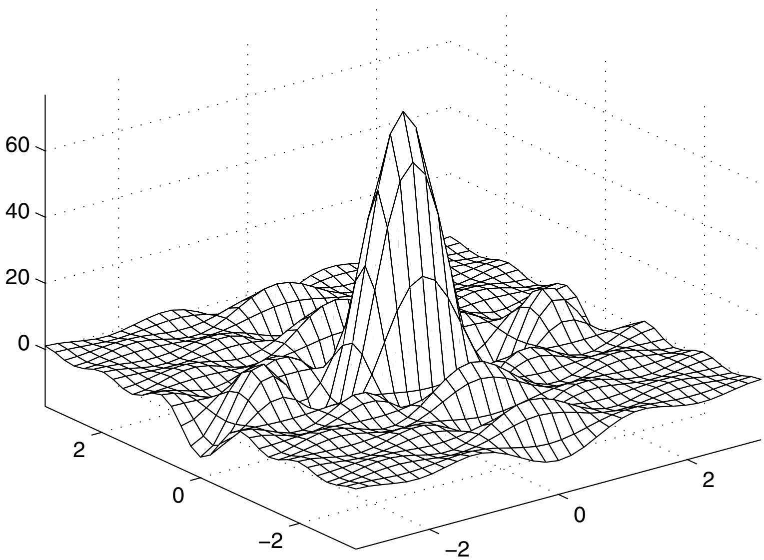

Fig. 2.5 The rectangular Dirichlet kernel with $d = 2, n_1 = 3, n_2 = 5$

$$\begin{aligned}
&\int_{\mathbb{T}} |s_n f(x)|^p \, dx_1 \\
&\quad = \int_{\mathbb{T}} \left| \int_{\mathbb{T}} \left(\int_{\mathbb{T}} f(t) D_{n_2}(x_2 + t_2) \, dt_2 \right) D_{n_1}(x_1 + t_1) \, dt_1 \right|^p \, dx_1 \\
&\quad \le C_p \int_{\mathbb{T}} \left| \int_{\mathbb{T}} f(t) D_{n_2}(x_2 + t_2) \, dt_2 \right|^p \, dt_1.
\end{aligned}$$

Again by the same theorem,

$$\begin{aligned}
\int_{\mathbb{T}} \int_{\mathbb{T}} |s_n f(x)|^p \, dx_1 \, dx_2 &\le C_p \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \int_{\mathbb{T}} f(t) D_{n_2}(x_2 + t_2) \, dt_2 \right|^p \, dx_2 \, dt_1 \\
&\le C_p \int_{\mathbb{T}} \int_{\mathbb{T}} |f(t)|^p \, dt_2 \, dt_1,
\end{aligned}$$

which gives the desired inequality of Theorem 2.1.8. The convergence is a consequence of this inequality and of the density of trigonometric polynomials. ■

In the next theorem, we present the norm convergence of the triangular and cubic partial sums. We omit the proof since it can be found at several places of the literature (see e.g., Fefferman [93], Grafakos [143] or Weisz [346]).

Theorem 2.1.9 *If $q = 1, \infty$ and $f \in L_p(\mathbb{T}^d)$ for some $1 < p < \infty$, then*

$$\sup_{n \in \mathbb{N}} \left\| s_n^q f \right\|_p \le C_p \|f\|_p$$

and

$$\lim_{n \to \infty} s_n^q f = f \qquad \text{in the } L_p(\mathbb{T}^d)\text{-norm.}$$

If $q = 2$, then the same result is valid for $p = 2$.

Since the characteristic function of the unit ball is not an $L_p(\mathbb{R}^d)$ $(1 < p \neq 2 < \infty, d \ge 2)$ multiplier (see Fefferman [95] or Grafakos [143, p. 743] or Lu and Yan [229, p. 743]), we have

Theorem 2.1.10 *If $d \ge 2$, $q = 2$ and $1 < p \neq 2 < \infty$, then the preceding theorem is not true.*

The analogue of Carleson's theorem holds also for the triangular and cubic partial sums in higher dimensions (see Fefferman [93, 94] and Grafakos [143, p. 231]), but it does not hold for the circular and rectangular partial sums.

Definition 2.1.11 We denote by

$$s_*^q f := \sup_{n \in \mathbb{N}} \left| s_n^q f \right|$$

the maximal operator of the ℓ_q-partial sums.

Theorem 2.1.12 *If $q = 1, \infty$ and $f \in L_p(\mathbb{T}^d)$ for some $1 < p < \infty$, then*

$$\left\| s_*^q f \right\|_p \le C_p \|f\|_p$$

and if $1 < p \le \infty$, then

$$\lim_{n\to\infty} s_n^q f = f \qquad a.e.$$

Theorem 2.1.12 does not hold for circular partial sums (see Stein and Weiss [293, p. 268]).

Theorem 2.1.13 *If $q = 2$ and $p < 2d/(d+1)$, then there exists a function $f \in L_p(\mathbb{T}^d)$ whose circular partial sums $s_n^q f$ diverge almost everywhere.*

This means that for a general function in $L_p(\mathbb{T}^d)$ $(p < 2)$ almost everywhere convergence of the circular partial sums is not true if the dimension is sufficiently large. It is an open problem, whether Theorem 2.1.12 holds for $p = 2$ and for circular partial sums. A counterexample, which proves the next result, can be found in Fefferman [94].

Theorem 2.1.14 *There exists a continuous function f such that for the rectangular partial sums $s_n f$,*

$$\lim_{n\to\infty} s_n f(x) = f(x)$$

does not hold for any $x \in \mathbb{T}^d$.

The generalization of Theorem 1.2.13 for higher dimensions was proved by Antonov [8].

Theorem 2.1.15 *If $q = \infty$ and*

$$\int_{\mathbb{T}^d} |f(x)|(\log^+ |f(x)|)^d \log^+ \log^+ \log^+ |f(x)|\, dx < \infty,$$

then

$$\lim_{n\to\infty} s_n^q f = f \qquad a.e.$$

2.2 The ℓ_q-Summability Kernels

As in the one-dimensional case, Theorems 2.1.8, Theorem 2.1.9 and the inequality in Theorem 2.1.12 do not hold for $p = 1$ and $p = \infty$. Using a summability method, we can extend the theorems to $p = 1$ and $p = \infty$ again. Now we introduce the ℓ_q-summability means and kernels and show some results for the kernels. We concentrate on the two-dimensional kernels.

Definition 2.2.1 For $f \in L_1(\mathbb{T}^d)$, $1 \le q \le \infty$ and $n \in \mathbb{N}$, the nth ℓ_q-Fejér means $\sigma_n^q f$ of the Fourier series of f and the nth ℓ_q-Fejér kernel K_n^q are introduced by

$$\sigma_n^q f(x) := \sum_{k\in\mathbb{Z}^d,\ \|k\|_q \le n} \left(1 - \frac{\|k\|_q}{n}\right) \widehat{f}(k) e^{\imath k\cdot x}$$

and

$$K_n^q(t) := \sum_{k\in\mathbb{Z}^d,\ \|k\|_q \le n} \left(1 - \frac{\|k\|_q}{n}\right) e^{\imath k\cdot t},$$

respectively.

We generalize this definition as we did for the one-dimensional Fourier series and introduce the ℓ_q-Cesàro means.

Definition 2.2.2 Let $f \in L_1(\mathbb{T}^d)$, $n \in \mathbb{N}$, $\alpha \ge 0$ and $q = 1$ or $q = \infty$. The nth ℓ_q-Cesàro means $\sigma_n^{q,\alpha} f$ of the Fourier series of f and the nth ℓ_q-Cesàro kernel $K_n^{q,\alpha}$ are introduced by

$$\sigma_n^{q,\alpha} f(x) := \frac{1}{A_{n-1}^{\alpha}} \sum_{k\in\mathbb{Z}^d,\ \|k\|_q \le n} A_{n-1-\|k\|_q}^{\alpha} \widehat{f}(k) e^{\imath k\cdot x}$$

and

$$K_n^{q,\alpha}(t) := \frac{1}{A_{n-1}^{\alpha}} \sum_{k\in\mathbb{Z}^d,\ \|k\|_q \le n} A_{n-1-\|k\|_q}^{\alpha} e^{\imath k\cdot t},$$

respectively.

We also call the Cesàro means ℓ_q-(C,α)-means. For $\alpha = 1$, we get back the ℓ_q-Fejér means and for $\alpha = 0$, the ℓ_q-partial sums. We introduce also a second generalization of the Fejér summation. For the circular summability (i.e., for $q = 2$), we will investigate rather this generalization.

Definition 2.2.3 For $f \in L_1(\mathbb{T}^d)$, $1 \le q \le \infty$, $n \in \mathbb{N}$ and $0 < \alpha, \gamma < \infty$, the nth ℓ_q-Riesz means $\sigma_n^{q,\alpha,\gamma} f$ of the Fourier series of f and the nth ℓ_q-Riesz kernel $K_n^{q,\alpha,\gamma}$ are given by

$$\sigma_n^{q,\alpha,\gamma} f(x) := \sum_{k\in\mathbb{Z}^d,\ \|k\|_q \le n} \left(1 - \left(\frac{\|k\|_q}{n}\right)^{\gamma}\right)^{\alpha} \widehat{f}(k) e^{\imath k\cdot x}$$

and

$$K_n^{q,\alpha,\gamma}(t) := \sum_{k\in\mathbb{Z}^d,\ \|k\|_q \le n} \left(1 - \left(\frac{\|k\|_q}{n}\right)^{\gamma}\right)^{\alpha} e^{\imath k\cdot t},$$

respectively.

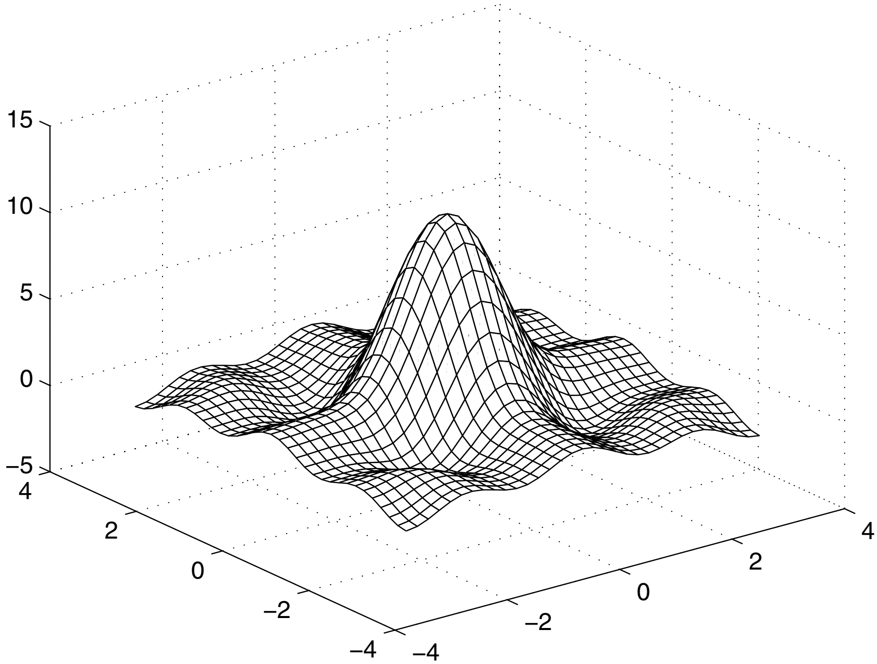

Fig. 2.6 The Fejér kernel K_n^q with $d = 2, q = 1, n = 4$

We will always suppose that $0 \le \alpha < \infty$, $1 \le \gamma < \infty$. If $\alpha = \gamma = 1$, we get back the ℓ_q-Fejér means. In the case $q = 2$, let $\gamma \in \mathbb{N}$. If $\alpha = 0$, we get the partial sums and if $q = \gamma = 2$, $\alpha > 0$, the means are called Bochner-Riesz means. The cubic summability (when $q = \infty$) is also called Marcinkiewicz summability (see Figs. 2.6, 2.7, 2.8, 2.9 and 2.10).

The following two lemmas follow the definition.

Lemma 2.2.4 *Let* $0 \le \alpha, \gamma < \infty$ *and* $n \in \mathbb{N}$. *If* $q = 1$ *or* $q = \infty$, *then*

$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K_n^{q,\alpha}(t)\, dt = 1.$$

If $1 \le q \le \infty$, *then*

$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K_n^{q,\alpha,\gamma}(t)\, dt = 1.$$

Lemma 2.2.5 *Under the same conditions as in Lemma 2.2.4,*

$$|K_n^{q,\alpha}(t)| \le Cn^d \qquad \textit{and} \qquad |K_n^{q,\alpha,\gamma}(t)| \le Cn^d \qquad (t \in \mathbb{T}^d).$$

Proof We have

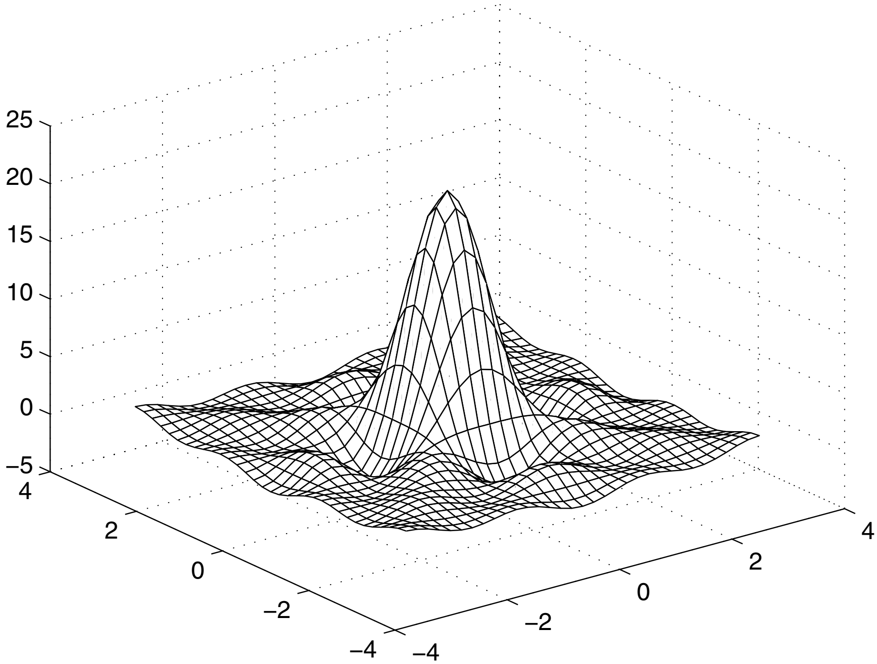

Fig. 2.7 The Fejér kernel K_n^q with $d = 2, q = \infty, n = 4$

$$|K_n^{q,\alpha}(t)| \leq \frac{1}{A_{n-1}^{\alpha}} \sum_{k\in\mathbb{Z}^d,\ \|k\|_q\leq n} A_{n-1-\|k\|_q}^{\alpha} \leq C \frac{1}{A_{n-1}^{\alpha}} \sum_{j=0}^{n} A_{n-1-j}^{\alpha} j^{d-1} \leq Cn^d.$$

The second inequality can be shown in the same way. ■

One can easily see that

Lemma 2.2.6 *Let $f \in L_1(\mathbb{T}^d)$, $n \in \mathbb{N}$ and $0 < \alpha, \gamma < \infty$. If $q = 1$ or $q = \infty$, then*

$$\sigma_n^{q,\alpha} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n^{q,\alpha}(t)\, dt.$$

If $1 \leq q \leq \infty$, then

$$\sigma_n^{q,\alpha,\gamma} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n^{q,\alpha,\gamma}(t)\, dt.$$

Lemma 2.2.7 *For $f \in L_1(\mathbb{T}^d)$, $\alpha > 0$, $q = 1, \infty$ and $n \in \mathbb{N}$, we have*

$$\sigma_n^{q,\alpha} f(x) = \frac{1}{A_{n-1}^{\alpha}} \sum_{j=0}^{n-1} A_{n-1-j}^{\alpha-1} s_j^q f(x)$$

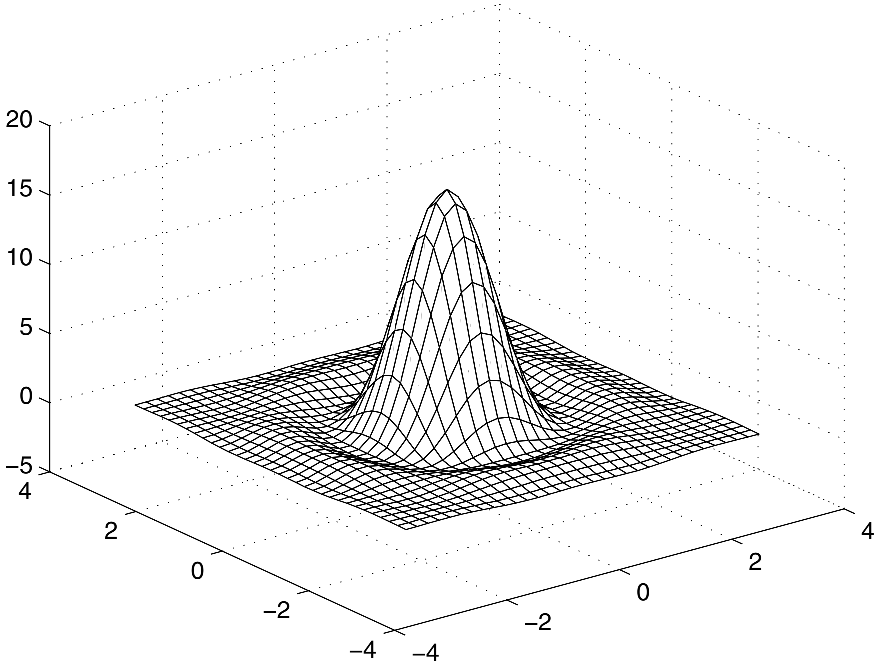

Fig. 2.8 The Fejér kernel K_n^q with $d = 2$, $q = 2$, $n = 4$

and

$$K_n^{q,\alpha}(t) = \frac{1}{A_{n-1}^{\alpha}} \sum_{j=0}^{n-1} A_{n-1-j}^{\alpha-1} D_j^q(t).$$

Proof Since $\|k\|_q$ is an integer, Lemma 1.4.8 implies that

$$\begin{aligned} K_n^{q,\alpha}(t) &= \frac{1}{A_{n-1}^{\alpha}} \sum_{k\in\mathbb{Z}^d,\ \|k\|_q\le n} A_{n-1-\|k\|_q}^{\alpha} e^{\imath k\cdot t} \\ &= \frac{1}{A_{n-1}^{\alpha}} \sum_{k\in\mathbb{Z}^d,\ \|k\|_q\le n} \sum_{j=\|k\|_q}^{n-1} A_{n-1-j}^{\alpha-1} e^{\imath k\cdot t} \\ &= \frac{1}{A_{n-1}^{\alpha}} \sum_{j=0}^{n-1} A_{n-1-j}^{\alpha-1} D_j^q(t), \end{aligned}$$

which shows the lemma. ∎

Obviously, the ℓ_q-Fejér means are the arithmetic means of the ℓ_q-partial sums when $q = 1, \infty$:

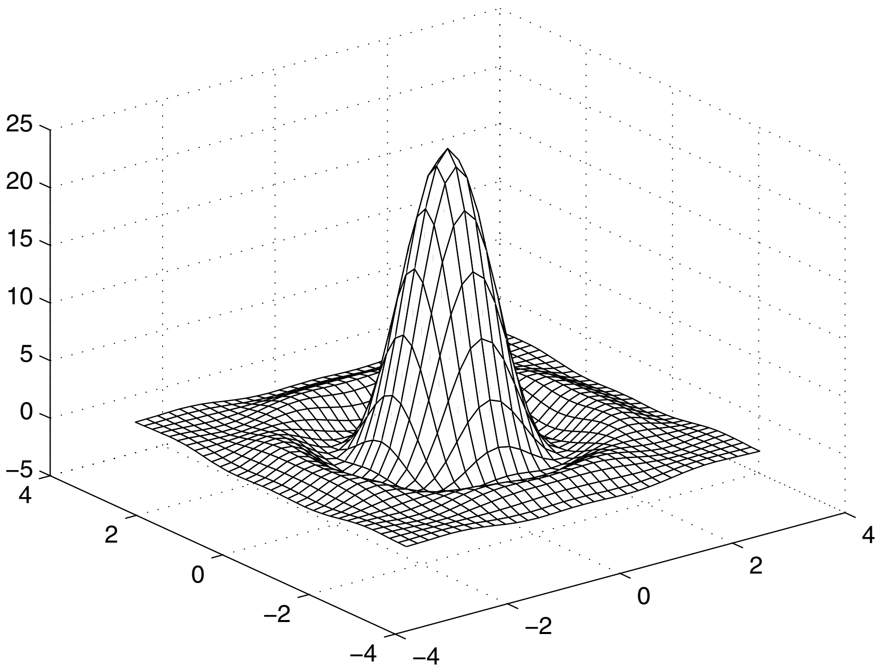

Fig. 2.9 The Bochner-Riesz kernel $K_n^{q,\alpha}$ with $d = 2$, $q = 2$, $n = 4$, $\alpha = 1$, $\gamma = 2$

$$\sigma_n^q f(x) = \frac{1}{n} \sum_{k=0}^{n-1} s_k^q f(x).$$

Similar to Lemma 1.4.13, we have

Lemma 2.2.8 *For $\alpha > -1$, $q = 1, \infty$ and $h > 0$, we have*

$$\sigma_n^{q,\alpha+h} f = \frac{1}{A_{n-1}^{\alpha+h}} \sum_{k=1}^{n} A_{n-k}^{h-1} A_{k-1}^{\alpha} \sigma_k^{q,\alpha} f.$$

The proofs of the results presented later are very different for the cases $q = 1, 2, \infty$ because the kernel functions are very different. In the next subsections, we give some estimations for the kernels. Since we will prove later the results basically for $d = 2$, we present these estimations in the two-dimensional case.

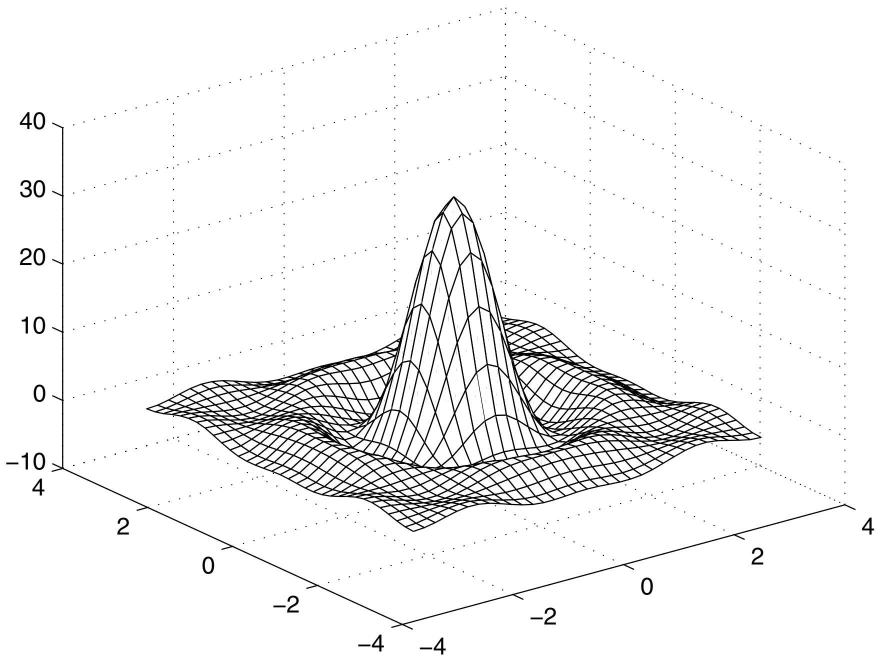

Fig. 2.10 The Bochner-Riesz kernel $K_n^{q,\alpha}$ with $d = 2, q = 2, n = 4, \alpha = 1/2, \gamma = 2$

2.2.1 Kernel Functions for q = 1

For the triangular Dirichlet kernel, we need the notion of the divided difference, which is usually used in numerical analysis.

Definition 2.2.9 The nth divided difference of a one-dimensional function f at the (pairwise distinct) knots $x_1, \ldots, x_n \in \mathbb{R}$ is introduced inductively as

$$[x_1]f := f(x_1), \qquad [x_1, \ldots, x_n]f := \frac{[x_1, \ldots, x_{n-1}]f - [x_2, \ldots, x_n]f}{x_1 - x_n}.$$

One can see that the difference is a symmetric function of the nodes. The following theorem is proved in DeVore and Lorentz [82, p. 120]), so we omit the proof.

Theorem 2.2.10 *We have*

$$[x_1, \ldots, x_n]f = \sum_{k=1}^{n} \frac{f(x_k)}{\prod_{j=1, j\neq k}^{n}(x_k - x_j)}. \tag{2.2.1}$$

If f is $(n-1)$-times continuously differentiable on $[a, b]$ and $x_i \in [a, b]$, then there exists $\xi \in [a, b]$ such that

$$[x_1, \ldots, x_n]f = \frac{f^{(n-1)}(\xi)}{(n-1)!}. \tag{2.2.2}$$

To give an explicit form of the triangular Dirichlet kernel, we will need the following trigonometric identities.

Lemma 2.2.11 *For all $n \in \mathbb{N}$ and $0 \le x, y \le \pi$,*

$$\sum_{k=0}^{n} \epsilon_k \cos(ky) \sin((n-k+1/2)x) = \sin(x/2) \frac{\cos(x/2)\cos((n+1/2)x) - \cos(y/2)\cos((n+1/2)y)}{\cos x - \cos y} \tag{2.2.3}$$

and

$$\sum_{k=0}^{n} \epsilon_k \cos(ky) \cos((n-k+1/2)x) = \cos(x/2) \frac{\sin(y/2)\sin((n+1/2)y) - \sin(x/2)\sin((n+1/2)x)}{\cos x - \cos y}, \tag{2.2.4}$$

where $\epsilon_0 := 1/2$ and $\epsilon_k := 1$, $k \ge 1$.

Proof By trigonometric identities,

$$\begin{aligned}
\sum_{k=0}^{n} \epsilon_k \cos(ky) \sin((n-k+1/2)x)
&= \sin((n+1/2)x) \sum_{k=0}^{n} \epsilon_k \cos(ky)\cos(kx) \\
&\quad - \cos((n+1/2)x) \sum_{k=0}^{n} \epsilon_k \cos(ky)\sin(kx) \\
&= \frac{1}{2}\sin((n+1/2)x) \sum_{k=0}^{n} \Big(\epsilon_k \cos(k(x-y)) + \epsilon_k \cos(k(x+y)\Big) \\
&\quad - \frac{1}{2}\cos((n+1/2)x) \sum_{k=0}^{n} \Big(\epsilon_k \sin(k(x-y)) + \epsilon_k \sin(k(x+y))\Big).
\end{aligned}$$

Similarly to (1.2.2), we can show that

$$\sum_{k=0}^{n} \epsilon_k \sin(kx) = \frac{\cos(x/2) - \cos((n+1/2)x)}{2\sin(x/2)}.$$

Using this and (1.2.2), we conclude

$$
\begin{aligned}
&\sum_{k=0}^{n} \epsilon_k \cos(ky) \sin((n-k+1/2)x) \\
&\quad = \frac{1}{4} \sin((n+1/2)x) \left(\frac{\sin((n+1/2)(x-y))}{\sin((x-y)/2)} + \frac{\sin((n+1/2)(x+y))}{\sin((x+y)/2)} \right) \\
&\qquad - \frac{1}{4} \cos((n+1/2)x) \left(\frac{\cos((x-y)/2) - \cos((n+1/2)(x-y))}{\sin((x-y)/2)} \right. \\
&\qquad \left. + \frac{1}{4} \frac{\cos((x+y)/2) - \cos((n+1/2)(x+y))}{\sin((x+y)/2)} \right).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\sin((n+1/2)x) \sin((n+1/2)(x-y)) \\
&\qquad + \cos((n+1/2)x) \cos((n+1/2)(x-y)) = \cos((n+1/2)y)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sin((n+1/2)x) \sin((n+1/2)(x+y)) \\
&\qquad + \cos((n+1/2)x) \cos((n+1/2)(x+y)) = \cos((n+1/2)y),
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
&\sum_{k=0}^{n} \epsilon_k \cos(ky) \sin((n-k+1/2)x) \\
&\quad = \frac{1}{4} \frac{\cos((n+1/2)y) - \cos((n+1/2)x) \cos((x-y)/2)}{\sin((x-y)/2)} \\
&\qquad + \frac{1}{4} \frac{\cos((n+1/2)y) - \cos((n+1/2)x) \cos((x+y)/2)}{\sin((x+y)/2)} \\
&\quad = \frac{1}{4} \frac{\cos((n+1/2)y) \Big(\sin((x+y)/2) + \sin((x-y)/2) \Big)}{\sin((x-y)/2) \sin((x+y)/2)} \\
&\qquad - \frac{1}{4} \frac{\cos((n+1/2)x)}{\sin((x-y)/2) \sin((x+y)/2)} \times \\
&\qquad \times \Big(\cos((x-y)/2) \sin((x+y)/2) + \cos((x+y)/2) \sin((x-y)/2) \Big).
\end{aligned}
$$

Using again some trigonometric identities, we get that

$$
\begin{aligned}
&\sum_{k=0}^{n} \epsilon_k \cos(ky) \sin((n-k+1/2)x) \\
&\quad = \frac{1}{2} \frac{2 \cos((n+1/2)y) \sin(x/2) \cos(y/2)}{\cos y - \cos x}
\end{aligned}
$$

$$-\frac{1}{2}\frac{\cos((n+1/2)x)\sin x}{\cos y - \cos x}$$
$$= \sin(x/2)\frac{\cos(x/2)\cos((n+1/2)x) - \cos(y/2)\cos((n+1/2)y)}{\cos x - \cos y}.$$

Formula (2.2.4) can be shown in the same way. ■

Define the function G_n by

$$G_n(\cos x) := (-1)^{[(d-1)/2]} 2\cos(x/2)(\sin x)^{d-2}\mathrm{soc}\,((n+1/2)x),$$

where the function soc is defined by

$$\mathrm{soc}\, x := \begin{cases} \cos x, & \text{if } d \text{ is even;} \\ \sin x, & \text{if } d \text{ is odd.} \end{cases}$$

The following representation of the triangular Dirichlet kernel was proved by Herriot [165] and Berens and Xu [30, 356].

Lemma 2.2.12 *For $x \in \mathbb{T}^d$,*

$$D_n^1(x) = [\cos x_1, \ldots, \cos x_d] G_n$$
$$= (-1)^{[(d-1)/2]} 2 \sum_{k=1}^{d} \frac{\cos(x_k/2)(\sin x_k)^{d-2}\mathrm{soc}\,((n+1/2)x_k)}{\prod_{j=1, j\neq k}^{d}(\cos x_k - \cos x_j)}. \tag{2.2.5}$$

Proof We will prove this lemma for all dimensions because the main idea of the proof is induction with respect to the dimension. First, we note that the second equality follows from the definition of G_n and from the property of the divided difference described in (2.2.1). In this proof, let us denote the Dirichlet kernel by $D_{d,n}^1(x) := D_n^1(x)$. We have seen in (1.2.2) that in the one-dimensional case

$$D_{1,n}^1(x) = D_n^1(x) = \frac{\sin((n+1/2)x)}{\sin(x/2)}$$
$$= 2\cos(x/2)(\sin x)^{-1}\sin((n+1/2)x),$$

thus (2.2.5) holds for $d = 1$. Suppose the lemma is true for integers up to d and let d be even. It is easy to see that

$$D_{d+1,n}^1(x) = 2^{d+1} \sum_{j\in\mathbb{N}^d,\, \|j\|_1 \le n} \epsilon_{j_1}\cos(j_1x_1)\cdots\epsilon_{j_{d+1}}\cos(j_{d+1}x_{d+1})$$
$$= 2\sum_{l=0}^{n} \epsilon_l \cos(l x_{d+1}) D_{d,n-l}(x_1, \ldots, x_d)$$

$$= (-1)^{[(d-1)/2]} 4 \sum_{k=1}^{d} \frac{\cos(x_k/2)(\sin x_k)^{d-2}}{\prod_{j=1, j\neq k}^{d} (\cos x_k - \cos x_j)}$$
$$\sum_{l=0}^{n} \epsilon_l \cos(l x_{d+1}) \cos((n-l+1/2)x_k),$$

where $\epsilon_0 := 1/2$ and $\epsilon_l := 1$, $l \geq 1$. Using (2.2.4), we obtain

$$D_{d+1,n}^{1}(x) = -(-1)^{[(d-1)/2]} 4 \sum_{k=1}^{d} \frac{\cos(x_k/2)(\sin x_k)^{d-2}}{\prod_{j=1, j\neq k}^{d+1} (\cos x_k - \cos x_j)}$$
$$\cos(x_k/2)\sin(x_k/2)\sin((n+1/2)x_k)$$
$$+ (-1)^{[(d-1)/2]} 4 \sum_{k=1}^{d} \frac{\cos(x_k/2)(\sin x_k)^{d-2}}{\prod_{j=1, j\neq k}^{d+1} (\cos x_k - \cos x_j)}$$
$$\cos(x_k/2)\sin(x_{d+1}/2)\sin((n+1/2)x_{d+1})$$
$$= -(-1)^{[(d-1)/2]} 2 \left(\sum_{k=1}^{d} \frac{\cos(x_k/2)(\sin x_k)^{d-1}\sin((n+1/2)x_k)}{\prod_{j=1, j\neq k}^{d+1} (\cos x_k - \cos x_j)} \right.$$
$$- \sin(x_{d+1}/2)\sin((n+1/2)x_{d+1}) \times$$
$$\left. \times \sum_{k=1}^{d} \frac{(1+\cos x_k)(\sin x_k)^{d-2}}{\prod_{j=1, j\neq k}^{d+1} (\cos x_k - \cos x_j)} \right). \tag{2.2.6}$$

Since d is even, the function $h(t) := (1+t)(1-t^2)^{(d-2)/2}$ is a polynomial of degree $d-1$. Then, by (2.2.2),

$$0 = [\cos x_1, \ldots, \cos x_{d+1}]h$$
$$= \sum_{k=1}^{d} \frac{(1+\cos x_k)(\sin x_k)^{d-2}}{\prod_{j=1, j\neq k}^{d+1} (\cos x_k - \cos x_j)} + \frac{(1+\cos x_{d+1})(\sin x_{d+1})^{d-2}}{\prod_{j=1, j\neq d+1}^{d+1} (\cos x_{d+1} - \cos x_j)}.$$

This and (2.2.6) imply

$$D_{d+1,n}^{1}(x) = -(-1)^{[(d-1)/2]} 2 \left(\sum_{k=1}^{d} \frac{\cos(x_k/2)(\sin x_k)^{d-1}\sin((n+1/2)x_k)}{\prod_{j=1, j\neq k}^{d+1} (\cos x_k - \cos x_j)} \right.$$
$$\left. + \sin(x_{d+1}/2)\sin((n+1/2)x_{d+1}) \frac{(1+\cos x_{d+1})(\sin x_{d+1})^{d-2}}{\prod_{j=1, j\neq d+1}^{d+1} (\cos x_{d+1} - \cos x_j)} \right)$$
$$= (-1)^{[d/2]} 2 \sum_{k=1}^{d+1} \frac{\cos(x_k/2)(\sin x_k)^{d-1}\sin((n+1/2)x_k)}{\prod_{j=1, j\neq k}^{d+1} (\cos x_k - \cos x_j)},$$

which proves the result if d is even. If d is odd, the lemma can be proved similarly. ■

As a special case, for $d = 2$, we get the next corollary.

Corollary 2.2.13 *For $x \in \mathbb{T}^2$, we have*

$$\begin{aligned} D_n^1(x_1, x_2) &= [\cos x_1, \cos x_2] G_n \\ &= 2 \frac{\cos(x_1/2)\cos((n+1/2)x_1) - \cos(x_2/2)\cos((n+1/2)x_2)}{\cos x_1 - \cos x_2}. \end{aligned}$$

In what follows, we may suppose that $x \in \mathbb{T}^2$ and $\pi > x_1 > x_2 > 0$. We denote the characteristic function of a set H by 1_H, i.e.,

$$1_H(x) := \begin{cases} 1, & \text{if } x \in H; \\ 0, & \text{if } x \notin H. \end{cases}$$

Lemma 2.2.14 *If $0 < \alpha \leq 1$ and $\pi > x_1 > x_2 > 0$, then*

$$\begin{aligned} \left|K_n^{1,\alpha}(x_1, x_2)\right| &\leq C(x_1 - x_2)^{-1}(x_1 + x_2)^{-1} 1_{\{x_2 \leq \pi/2\}} \\ &\quad + C(x_1 - x_2)^{-1}(2\pi - x_1 - x_2)^{-1} 1_{\{x_2 > \pi/2\}}, \end{aligned} \tag{2.2.7}$$

$$\begin{aligned} 1_{\{x_2 \leq \pi/2\}} &\left|K_n^{1,\alpha}(x_1, x_2)\right| \\ &\leq C n^{-\alpha}(x_1 - x_2)^{-1}(x_1 + x_2)^{-1} x_2^{-\alpha} 1_{\{x_2 \leq \pi/2\}} \\ &\quad + C n^{-1}(x_1 - x_2)^{-1}(x_1 + x_2)^{-1} x_2^{-1} 1_{\{x_2 \leq \pi/2\}}, \end{aligned} \tag{2.2.8}$$

$$\begin{aligned} 1_{\{x_2 > \pi/2\}} &\left|K_n^{1,\alpha}(x_1, x_2)\right| \\ &\leq C n^{-\alpha}(x_1 - x_2)^{-1}(2\pi - x_1 - x_2)^{-1} x_2^{-\alpha} 1_{\{x_2 > \pi/2\}} \\ &\quad + C n^{-1}(x_1 - x_2)^{-1}(2\pi - x_1 - x_2)^{-1} x_2^{-1} 1_{\{x_2 > \pi/2\}}, \end{aligned} \tag{2.2.9}$$

$$\begin{aligned} 1_{\{x_2 \leq \pi/2\}} \left|K_n^{1,\alpha}(x_1, x_2)\right| &\leq C n^{1-\alpha}(x_1 + x_2)^{-1} x_2^{-\alpha} 1_{\{x_2 \leq \pi/2\}} \\ &\quad + C(x_1 + x_2)^{-1} x_2^{-1} 1_{\{x_2 \leq \pi/2\}} \end{aligned} \tag{2.2.10}$$

and

$$\begin{aligned} 1_{\{x_2 > \pi/2\}} \left|K_n^{1,\alpha}(x_1, x_2)\right| &\leq C n^{1-\alpha}(2\pi - x_1 - x_2)^{-1} x_2^{-\alpha} 1_{\{x_2 > \pi/2\}} \\ &\quad + C(2\pi - x_1 - x_2)^{-1} x_2^{-1} 1_{\{x_2 > \pi/2\}}. \end{aligned} \tag{2.2.11}$$

Proof By the trigonometric identity,

$$\cos a - \cos b = -2\sin((a-b)/2)\sin((a+b)/2),$$

Corollary 2.2.13 can be rewritten as

$$D_k^1(x_1,x_2) = -\frac{\cos(x_1/2)\cos((k+1/2)x_1) - \cos(x_2/2)\cos((k+1/2)x_2)}{\sin((x_1-x_2)/2)\sin((x_1+x_2)/2)}. \tag{2.2.12}$$

We will use that

$$\sin(x_1 \pm x_2)/2 \sim x_1 \pm x_2 \quad \text{if} \quad x_2 \le \pi/2$$

and

$$\sin(x_1 - x_2)/2 \sim x_1 - x_2, \quad \sin(x_1 + x_2)/2 \sim 2\pi - x_1 - x_2 \quad \text{if} \quad x_2 > \pi/2.$$

By Lemma 2.2.7 and (2.2.12), we can see that

$$\begin{aligned} K_n^{1,\alpha}(x_1,x_2) &= \frac{1}{A_{n-1}^{\alpha}} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} \\ &\quad \frac{\cos(x_2/2)\cos((k+1/2)x_2) - \cos(x_1/2)\cos((k+1/2)x_1)}{\sin((x_1-x_2)/2)\sin((x_1+x_2)/2)} \\ &\le 2(x_1-x_2)^{-1}(x_1+x_2)^{-1} 1_{\{x_2 \le \pi/2\}} \\ &\quad + 2(x_1-x_2)^{-1}(2\pi - x_1 - x_2)^{-1} 1_{\{x_2 > \pi/2\}}, \end{aligned} \tag{2.2.13}$$

which is exactly (2.2.7).

Suppose that $x_2 \le \pi/2$. By (2.2.13) and Lemma 1.4.14,

$$\begin{aligned} &\left|K_n^{1,\alpha}(x_1,x_2)\right| \\ &\quad \le (x_1-x_2)^{-1}(x_1+x_2)^{-1}\frac{1}{A_{n-1}^{\alpha}} \\ &\qquad \left(\left|\sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1}\cos((k+1/2)x_2)\right| + \left|\sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1}\cos((k+1/2)x_1)\right|\right) \\ &\quad \le Cn^{-\alpha}(x_1-x_2)^{-1}(x_1+x_2)^{-1}x_2^{-\alpha} + Cn^{-1}(x_1-x_2)^{-1}(x_1+x_2)^{-1}x_2^{-1}, \end{aligned}$$

which shows (2.2.8).

Lagrange's mean value theorem and (2.2.12) imply that there exists $x_1 > \xi > x_2$, such that

$$D_k^1(x_1,x_2) = -\frac{H_k'(\xi)(x_1-x_2)}{\sin((x_1-x_2)/2)\sin((x_1+x_2)/2)},$$

where

$$H_k(t) = \cos(t/2)\cos((k+1/2)t).$$

Then (2.2.10) follows from

$$\begin{aligned}\left|K_n^{1,\alpha}(x_1,x_2)\right| &\le \frac{1}{A_{n-1}^{\alpha}}\left|\sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1}\frac{H_k'(\xi)(x_1-x_2)}{\sin((x_1-x_2)/2)\sin((x_1+x_2)/2)}\right| \\ &\le Cn(x_1-x_2)(x_1-x_2)^{-1}(x_1+x_2)^{-1}(n^{-\alpha}x_2^{-\alpha}+n^{-1}x_2^{-1}).\end{aligned}$$

The inequalities (2.2.9) and (2.2.11) for $x_2 > \pi/2$ can be proved in the same way. ■

The next estimations of the kernel function come easily from Lemma 2.2.14.

Lemma 2.2.15 *If* $0 < \alpha \le 1$, $0 \le \beta \le 1$ *and* $\pi > x_1 > x_2 > 0$, *then*

$$\begin{aligned}\left|K_n^{1,\alpha}(x_1,x_2)\right| &\le C(x_1-x_2)^{-3/2}x_2^{-1/2}1_{\{x_2\le\pi/2\}} \\ &\quad + C(x_1-x_2)^{-3/2}(\pi-x_1)^{-1/2}1_{\{x_2>\pi/2\}},\end{aligned} \tag{2.2.14}$$

$$\begin{aligned}&1_{\{x_2\le\pi/2\}}\left|K_n^{1,\alpha}(x_1,x_2)\right| \\ &\quad\le Cn^{-\alpha}(x_1-x_2)^{-1-\beta}x_2^{\beta-\alpha-1}1_{\{x_2\le\pi/2\}} \\ &\quad\quad + Cn^{-1}(x_1-x_2)^{-1-\beta}x_2^{\beta-2}1_{\{x_2\le\pi/2\}},\end{aligned} \tag{2.2.15}$$

$$\begin{aligned}&1_{\{x_2>\pi/2\}}\left|K_n^{1,\alpha}(x_1,x_2)\right| \\ &\quad\le Cn^{-\alpha}(x_1-x_2)^{-1-\beta}(\pi-x_1)^{\beta-\alpha-1}1_{\{x_2>\pi/2\}} \\ &\quad\quad + Cn^{-1}(x_1-x_2)^{-1-\beta}(\pi-x_1)^{\beta-2}1_{\{x_2>\pi/2\}},\end{aligned} \tag{2.2.16}$$

$$1_{\{x_2\le\pi/2\}}\left|K_n^{1,\alpha}(x_1,x_2)\right| \le Cn^{1-\alpha}x_2^{-\alpha-1}1_{\{x_2\le\pi/2\}} + Cx_2^{-2}1_{\{x_2\le\pi/2\}} \tag{2.2.17}$$

and

$$\begin{aligned}1_{\{x_2>\pi/2\}}\left|K_n^{1,\alpha}(x_1,x_2)\right| &\le Cn^{1-\alpha}(\pi-x_1)^{-\alpha-1}1_{\{x_2>\pi/2\}} \\ &\quad + C(\pi-x_1)^{-2}1_{\{x_2>\pi/2\}}.\end{aligned} \tag{2.2.18}$$

Proof The basic facts

$$x_1+x_2 > x_1-x_2, \qquad x_1+x_2 > x_2$$

and

$$2\pi-x_1-x_2 > x_1-x_2, \qquad 2\pi-x_1-x_2 > \pi-x_1$$

together with (2.2.7) imply

$$\left|K_n^{1,\alpha}(x_1,x_2)\right| \le 2(x_1-x_2)^{-3/2}x_2^{-1/2}1_{\{x_2\le\pi/2\}} + 2(x_1-x_2)^{-3/2}(\pi-x_1)^{-1/2}1_{\{x_2>\pi/2\}},$$

which shows (2.2.14). Since $0\le\beta\le 1$, (2.2.8) implies

$$\left|K_n^{1,\alpha}(x_1,x_2)\right| \le Cn^{-\alpha}(x_1-x_2)^{-1-\beta}x_2^{\beta-\alpha-1} + Cn^{-1}(x_1-x_2)^{-1-\beta}x_2^{\beta-2}$$

if $x_2\le\pi/2$. The other inequalities can be shown similarly. ■

Lemma 2.2.16 *If* $0<\alpha\le 1$ *and* $\pi>x_1>x_2>0$*, then*

$$1_{\{x_2\le\pi/2\}}\left|K_n^{1,\alpha}(x_1,x_2)\right| \le C(x_1-x_2)^{\alpha-1}x_2^{-\alpha-1}1_{\{x_2\le\pi/2\}} + Cx_2^{-2}1_{\{x_2\le\pi/2\}} \tag{2.2.19}$$

and

$$1_{\{x_2>\pi/2\}}\left|K_n^{1,\alpha}(x_1,x_2)\right| \le C(x_1-x_2)^{\alpha-1}(\pi-x_1)^{-\alpha-1}1_{\{x_2>\pi/2\}} + C(\pi-x_1)^{-2}1_{\{x_2>\pi/2\}}. \tag{2.2.20}$$

Proof If $\beta=0$ and $n\ge(x_1-x_2)^{-1}$, then (2.2.15) implies (2.2.19). On the other hand, (2.2.19) follows from (2.2.17) if $n<(x_1-x_2)^{-1}$. ■

In the next lemma, we estimate the partial derivatives of the kernel function.

Lemma 2.2.17 *If* $0<\alpha\le 1$, $0\le\beta\le 1$ *and* $\pi>x_1>x_2>0$*, then for* $j=1,2$*,*

$$1_{\{x_2\le\pi/2\}}\left|\partial_j K_n^{1,\alpha}(x_1,x_2)\right| \le Cn^{1-\alpha}(x_1-x_2)^{-1-\beta}x_2^{\beta-\alpha-1}1_{\{x_2\le\pi/2\}} + C(x_1-x_2)^{-1-\beta}x_2^{\beta-2}1_{\{x_2\le\pi/2\}} \tag{2.2.21}$$

and

$$1_{\{x_2>\pi/2\}}\left|\partial_j K_n^{1,\alpha}(x_1,x_2)\right| \le Cn^{1-\alpha}(x_1-x_2)^{-1-\beta}(\pi-x_1)^{\beta-\alpha-1}1_{\{x_2>\pi/2\}} + C(x_1-x_2)^{-1-\beta}(\pi-x_1)^{\beta-2}1_{\{x_2>\pi/2\}}. \tag{2.2.22}$$

Proof Let $x_2\le\pi/2$. By Lagrange's mean value theorem and (2.2.12),

$$
\begin{aligned}
&\partial_1 D_k^1(x_1, x_2) \\
&= \frac{1}{2}\Big(\sin(x_1/2)\cos((k+1/2)x_1) + \cos(x_1/2)(2k+1)\sin((k+1/2)x_1)\Big) \\
&\quad \sin((x_1-x_2)/2)^{-1}\sin((x_1+x_2)/2)^{-1} \\
&\quad + \frac{1}{2}(x_1-x_2)\Big(\sin((x_1-x_2)/2)^{-2}\sin((x_1+x_2)/2)^{-1}\cos((x_1-x_2)/2) \\
&\quad + \sin((x_1-x_2)/2)^{-1}\sin((x_1+x_2)/2)^{-2}\cos((x_1+x_2)/2)\Big) H_k'(\xi),
\end{aligned}
$$

where $y < \xi < x$ is a suitable number. Using the methods above,

$$
\begin{aligned}
\left|\partial_1 K_n^{1,\alpha}(x_1, x_2)\right| &= \frac{1}{A_{n-1}^{\alpha}} \left| \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} \partial_1 D_k^1(x_1, x_2) \right| \\
&\le C(x_1-x_2)^{-1}(x_1+x_2)^{-1}(n^{1-\alpha}x_2^{-\alpha} + x_2^{-1}) \\
&\quad + C(x_1+x_2)^{-2}(n^{1-\alpha}x_2^{-\alpha} + x_2^{-1}) \\
&\le C(x_1-x_2)^{-1-\beta}(n^{1-\alpha}x_2^{\beta-\alpha-1} + x_2^{\beta-2}),
\end{aligned}
$$

which proves (2.2.21). The case $x_2 > \pi/2$, i.e., (2.2.22), can be shown similarly. ■

2.2.2 Kernel Functions for $q = \infty$

Lemma 2.2.18 *For $x \in \mathbb{T}^d$,*

$$
D_n^{\infty}(x) = \prod_{i=1}^{d} D_n^{\infty}(x_i) = \prod_{i=1}^{d} \frac{\sin((n+1/2)x_i)}{\sin(x_i/2)}.
$$

Proof The proof follows from the definition of the cubic Dirichlet kernels and from Lemma 1.2.3. ■

To estimate the cubic Cesàro kernels, we may suppose again that $x \in \mathbb{T}^2$ and $\pi > x_1 > x_2 > 0$.

Lemma 2.2.19 *If $0 < \alpha \le 1$, $x \in \mathbb{T}^2$ and $\pi > x_1 > x_2 > 0$, then*

$$
\left|K_n^{\infty,\alpha}(x_1, x_2)\right| \le C x_1^{-1} x_2^{-1}, \tag{2.2.23}
$$

$$
\begin{aligned}
\left|K_n^{\infty,\alpha}(x_1, x_2)\right| &\le C n^{-\alpha} x_1^{-1} x_2^{-1} (x_1-x_2)^{-\alpha} \\
&\quad + C n^{-1} x_1^{-1} x_2^{-1} (x_1-x_2)^{-1}
\end{aligned} \tag{2.2.24}
$$

and

$$\left|K_n^{\infty,\alpha}(x_1,x_2)\right| \le Cn^{1-\alpha}x_1^{-1}(x_1-x_2)^{-\alpha} + Cx_1^{-1}(x_1-x_2)^{-1}. \tag{2.2.25}$$

Proof The first inequality, (2.2.23) follows easily from Lemma 1.4.8 and from

$$\begin{aligned}K_n^{\infty,\alpha}(x_1,x_2) &= \frac{1}{A_{n-1}^{\alpha}}\sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} D_k^{\infty}(x_1,x_2)\\ &= \frac{1}{A_{n-1}^{\alpha}}\sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1}\frac{\sin((k+1/2)x_1)}{\sin(x_1/2)}\frac{\sin((k+1/2)x_2)}{\sin(x_2/2)}.\end{aligned}$$

The trigonometric identity

$$\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b)) \tag{2.2.26}$$

yields

$$\begin{aligned}&\left|K_n^{\infty,\alpha}(x_1,x_2)\right|\\ &= \frac{1}{2A_{n-1}^{\alpha}}\left|\sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1}\frac{\cos((k+1/2)(x_1-x_2)) - \cos((k+1/2)(x_1+x_2))}{\sin(x_1/2)\sin(x_2/2)}\right|.\end{aligned}$$

Observe that $\sin(x_i/2) \sim x_i$,

$$\sin(x_1 \pm x_2)/2 \sim x_1 \pm x_2 \quad \text{if} \quad x_2 \le \pi/2$$

and

$$\sin(x_1 - x_2)/2 \sim x_1 - x_2, \quad \sin(x_1 + x_2)/2 \sim 2\pi - x_1 - x_2 \quad \text{if} \quad x_2 > \pi/2.$$

Using the facts $x_1 + x_2 > x_1 - x_2$, $2\pi - x_1 - x_2 > x_1 - x_2$ and and Lemma 1.4.14, we conclude that

$$\begin{aligned}&\left|K_n^{\infty,\alpha}(x_1,x_2)\right|\\ &= \frac{C}{2A_{n-1}^{\alpha}}\frac{1}{|\sin(x_1/2)\sin(x_2/2)|}\Big(\frac{1}{|\sin(x_1-x_2)/2|^{\alpha}} + \frac{n^{\alpha-1}}{|\sin(x_1-x_2)/2|}\\ &\quad + \frac{1}{|\sin(x_1+x_2)/2|^{\alpha}} + \frac{n^{\alpha-1}}{|\sin(x_1+x_2)/2|}\Big)\\ &\le Cn^{-\alpha}x_1^{-1}x_2^{-1}(x_1-x_2)^{-\alpha} + Cn^{-1}x_1^{-1}x_2^{-1}(x_1-x_2)^{-1},\end{aligned} \tag{2.2.27}$$

which is (2.2.24). Using Lagrange's theorem in (2.2.27) and Lemma 1.4.15, there exists $x_1 - x_2 < \xi < x_1 + x_2$ such that

$$\left|K_n^{\infty,\alpha}(x_1,x_2)\right| = \frac{1}{A_{n-1}^{\alpha}}\left|\sum_{k=0}^{n-1}\frac{A_{n-1-k}^{\alpha-1}(k+1/2)x_2\sin((k+1/2)\xi)}{\sin(x_1/2)\sin(x_2/2)}\right|$$
$$\le Cn^{1-\alpha}x_1^{-1}(x_1-x_2)^{-\alpha} + Cx_1^{-1}(x_1-x_2)^{-1}.$$

This finishes the proof of the lemma. ■

Lemma 2.2.20 *If* $0<\alpha\le 1$ *and* $\pi > x_1 > x_2 > 0$, *then*

$$\left|K_n^{\infty,\alpha}(x_1,x_2)\right| \le Cx_2^{\alpha-1}(x_1-x_2)^{-\alpha-1} + C(x_1-x_2)^{-2}. \tag{2.2.28}$$

Proof The inequality follows from (2.2.24) if $n \ge x_2^{-1}$ and from (2.2.25) if $n < x_2^{-1}$. ■

The partial derivatives of the cubic Cesàro kernels can be estimated as follows.

Lemma 2.2.21 *If* $0<\alpha\le 1$, $j=1,2$ *and* $\pi > x_1 > x_2 > 0$, *then*

$$\left|\partial_j K_n^{\infty,\alpha}(x_1,x_2)\right| \le Cn^{1-\alpha}x_1^{-1}x_2^{-1}(x_1-x_2)^{-\alpha}.$$

Proof By Lagrange's mean value theorem and (2.2.26),

$$\begin{aligned}
&\partial_1 D_k^{\infty}(x_1,x_2)\\
&= \frac{1}{2}(k+1/2)\Big(\sin((k+1/2)(x_1+x_2)) - \sin((k+1/2)(x_1-x_2))\Big)\\
&\quad \sin(x_1/2)^{-1}\sin(x_2/2)^{-1}\\
&\quad + \frac{1}{4}\Big(\cos((k+1/2)(x_1-x_2)) - \cos((k+1/2)(x_1+x_2))\Big)\\
&\quad \cos(x_1/2)\sin(x_1/2)^{-2}\sin(x_2/2)^{-1}\\
&= \frac{1}{2}(k+1/2)\Big(\sin((k+1/2)(x_1+x_2)) - \sin((k+1/2)(x_1-x_2))\Big)\\
&\quad \sin(x_1/2)^{-1}\sin(x_2/2)^{-1}\\
&\quad + \frac{1}{4}(k+1/2)x_2\sin((k+1/2)\xi)\cos(x_1/2)\sin(x_1/2)^{-2}\sin(x_2/2)^{-1},
\end{aligned}$$

where $x_1-x_2<\xi<x_1+x_2$ is a suitable number. Similarly as above,

$$\begin{aligned}
\left|\partial_1 K_n^{\infty,\alpha}(x_1,x_2)\right| &= \frac{1}{A_{n-1}^{\alpha}}\left|\sum_{k=0}^{n-1}A_{n-1-k}^{\alpha-1}\partial_1 D_k^{\infty}(x_1,x_2)\right|\\
&\le Cn^{1-\alpha}x_1^{-1}x_2^{-1}(x_1-x_2)^{-\alpha} + Cx_1^{-1}x_2^{-1}(x_1-x_2)^{-1}\\
&\quad + Cn^{1-\alpha}x_1^{-2}(x_1-x_2)^{-\alpha} + Cx_1^{-2}(x_1-x_2)^{-1}\\
&\le Cn^{1-\alpha}x_1^{-1}x_2^{-1}(x_1-x_2)^{-\alpha} + Cx_1^{-1}x_2^{-1}(x_1-x_2)^{-1},
\end{aligned}$$

which proves the lemma. ∎

2.2.3 Kernel Functions for $q = 2$

As we mentioned before, for $q = 2$, we will consider the Riesz summability. To this, we have to introduce some special functions. For the sake of completeness, we prove some elementary properties for these functions. First, we introduce the gamma function by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt \qquad (x > 0).$$

Integration by parts yields

$$\Gamma(x) = \Big[\frac{t^x e^{-t}}{x}\Big]_0^\infty + \frac{1}{x}\int_0^\infty t^x e^{-t}\, dt = \frac{1}{x}\Gamma(x+1) \qquad (x > 0).$$

Since $\Gamma(1) = 1$, we have

$$\Gamma(x+1) = x\Gamma(x) \quad (x > 0) \qquad \text{and} \qquad \Gamma(n) = (n-1)!. \tag{2.2.29}$$

After a substitution, we can see that

$$\Gamma\Big(\frac{1}{2}\Big) = \int_0^\infty t^{-1/2} e^{-t}\, dt = 2\int_0^\infty e^{-u^2}\, du = \sqrt{\pi}.$$

The beta function is defined by

$$B(x, y) := \int_0^1 s^{x-1}(1-s)^{y-1}\, ds = \int_0^1 s^{y-1}(1-s)^{x-1}\, ds,$$

where $x, y > 0$. The relationship between the beta and gamma function reads as follows:

$$\Gamma(x+y)B(x, y) = \Gamma(x)\Gamma(y). \tag{2.2.30}$$

Indeed, substituting $s = u/(1+u)$, we obtain

$$\begin{aligned}
\Gamma(x+y)B(x, y) &= \Gamma(x+y)\int_0^1 s^{y-1}(1-s)^{x-1}\, ds \\
&= \Gamma(x+y)\int_0^\infty u^{y-1}\Big(\frac{1}{1+u}\Big)^{x+y} du \\
&= \int_0^\infty \int_0^\infty u^{y-1}\Big(\frac{1}{1+u}\Big)^{x+y} v^{x+y-1} e^{-v}\, dv du.
\end{aligned}$$

The substitution $v = t(1+u)$ in the inner integral yields

$$\begin{aligned}\Gamma(x+y)B(x,y) &= \int_0^\infty \int_0^\infty u^{y-1} t^{x+y-1} e^{-t(1+u)}\, dt\, du \\ &= \int_0^\infty t^x e^{-t} \int_0^\infty (ut)^{y-1} e^{-tu}\, du\, dt \\ &= \int_0^\infty t^{x-1} e^{-t} \Gamma(y)\, dt \\ &= \Gamma(x)\Gamma(y),\end{aligned}$$

which shows (2.2.30).

Definition 2.2.22 For $k > -1/2$, the Bessel functions are defined by

$$J_k(t) := \frac{(t/2)^k}{\Gamma(k+1/2)\Gamma(1/2)} \int_{-1}^1 e^{\imath ts} (1-s^2)^{k-1/2}\, ds \qquad (t \in \mathbb{R}).$$

Using the Euler formulas, we can see that the Bessel functions are real-valued.

Lemma 2.2.23 *We have*

$$J_k'(t) = kt^{-1} J_k(t) - J_{k+1}(t) \qquad (t \neq 0).$$

Proof By integrating by parts and by (2.2.29), we conclude

$$\begin{aligned}\frac{d}{dt}(t^{-k} J_k(t)) &= \frac{\imath 2^{-k}}{\Gamma(k+1/2)\Gamma(1/2)} \int_{-1}^1 e^{\imath ts} s (1-s^2)^{k-1/2}\, ds \\ &= \frac{\imath 2^{-k}}{(2k+1)\Gamma(k+1/2)\Gamma(1/2)} \Bigg(-\Big[e^{\imath ts} (1-s^2)^{k+1/2} \Big]_{-1}^1 \\ &\quad + \int_{-1}^1 \imath t e^{\imath ts} (1-s^2)^{k+1/2}\, ds \Bigg) \\ &= \frac{-2^{-k-1} t}{(k+1/2)\Gamma(k+1/2)\Gamma(1/2)} \int_{-1}^1 e^{\imath ts} (1-s^2)^{k+1/2}\, ds \\ &= -t^{-k} J_{k+1}(t),\end{aligned}$$

which proves the desired result. ■

Lemma 2.2.24 *For $k > -1/2$ and $t > 0$,*

$$J_k(t) \le C_k t^k \qquad \textit{and} \qquad J_k(t) \le C_k t^{-1/2},$$

where C_k is independent of t.

Proof Since $1 - s^2 \geq 1 - |s|$ for $|s| \leq 1$, the first estimate follows from the definition of J_k. The second one follows from the first one if $0 < t \leq 1$. So we may assume that $t > 1$. Let us integrate the complex valued function

$$e^{\imath tz}(1 - z^2)^{k-1/2} \qquad (z \in \mathbb{C})$$

over the boundary of the rectangle whose lower side is $[-1, 1]$ and whose height is $R > 0$. By Cauchy's theorem,

$$\begin{aligned} 0 = \imath \int_R^0 e^{\imath t(-1+\imath s)}(s^2 + 2\imath s)^{k-1/2}\, ds + \int_{-1}^1 e^{\imath ts}(1 - s^2)^{k-1/2}\, ds \\ + \imath \int_0^R e^{\imath t(1+\imath s)}(s^2 - 2\imath s)^{k-1/2}\, ds + \epsilon(R), \end{aligned}$$

where $\epsilon(R) \to 0$ as $R \to \infty$. Hence, taking the limit as $R \to \infty$,

$$\begin{aligned} \int_{-1}^1 e^{\imath ts}(1 - s^2)^{k-1/2}\, ds &= \imath e^{-\imath t} \int_0^\infty e^{-ts}(s^2 + 2\imath s)^{k-1/2}\, ds \\ &\quad - \imath e^{\imath t} \int_0^\infty e^{-ts}(s^2 - 2\imath s)^{k-1/2}\, ds \\ &=: I_1 + I_2. \end{aligned}$$

Observe that

$$(s^2 + 2\imath s)^{k-1/2} = (2\imath s)^{k-1/2} + \phi(s),$$

where

$$|\phi(s)| \leq Cs^{k+1/2} \qquad \text{if } 0 < s \leq 1 \text{ or } s > 1 \text{ and } k \leq 3/2$$

and

$$|\phi(s)| \leq Cs^{2k-1} \qquad \text{if } s > 1 \text{ and } k > 3/2.$$

Indeed, it follows from Lagrange's mean value theorem that

$$|\phi(s)| = \left|(2\imath s)^{k-1/2}\right| \left| \left(\frac{s}{2\imath} + 1\right)^{k-1/2} - 1 \right| \leq C_k s^{k+1/2} \left| \frac{\xi}{2\imath} + 1 \right|^{k-3/2},$$

where $0 < \xi < s$. Hence

$$\left|s^2 + 2\imath s\right|^{k-1/2} \leq C_k s^{k-1/2} + |\phi(s)|$$

and

$$
\begin{aligned}
|I_1| &\le \int_0^\infty e^{-ts}\left(C_k s^{k-1/2} + |\phi(s)|\right) ds \\
&= C_k \int_0^\infty e^{-ts} s^{k-1/2} + \int_0^1 e^{-ts}|\phi(s)|\, ds + \int_1^\infty e^{-ts}|\phi(s)|\, ds \\
&= I_{1,1} + I_{1,2} + I_{1,3}.
\end{aligned}
$$

By the substitution $ts = u$ and by the definition of the gamma function,

$$
\begin{aligned}
I_{1,1} &= C_k t^{-1} \int_0^\infty e^{-u}(u/t)^{k-1/2}\, du \\
&= C_k t^{-k-1/2} \int_0^\infty e^{-u} u^{k-1/2}\, du = C_k \Gamma(k+1/2) t^{-k-1/2}.
\end{aligned}
$$

The same substitution implies

$$
\begin{aligned}
I_{1,2} &\le \int_0^1 e^{-ts} s^{k+1/2}\, ds \le t^{-k-3/2} \int_0^\infty e^{-u} u^{k+1/2}\, ds \\
&= \Gamma(k+3/2) t^{-k-3/2} \le C_k t^{-k-1/2}.
\end{aligned} \tag{2.2.31}
$$

If $k \le 3/2$, then

$$
I_{1,3} \le \Gamma(k+3/2) t^{-k-3/2} \le C_k t^{-k-1/2}
$$

as in (2.2.31). Similarly, for $k > 3/2$,

$$
\begin{aligned}
I_{1,3} &\le \int_1^\infty e^{-ts} s^{2k-1}\, ds \le t^{-2k} \int_0^\infty e^{-u} u^{2k-1}\, ds \\
&= \Gamma(2k) t^{-2k} \le C_k t^{-k-1/2}.
\end{aligned}
$$

The integral I_2 can be estimated in the same way. ■

Lemma 2.2.25 *If $k > -1/2$, $l > -1$ and $t > 0$, then*

$$
J_{k+l+1}(t) = \frac{t^{l+1}}{2^l \Gamma(l+1)} \int_0^1 J_k(ts) s^{k+1} (1-s^2)^l\, ds.
$$

Proof Taking into account (2.2.30), we get that

$$
\begin{aligned}
J_k(t) &= \frac{2(t/2)^k}{\Gamma(k+1/2)\Gamma(1/2)} \int_0^1 \cos(ts)(1-s^2)^{k-1/2}\, ds \\
&= \sum_{j=0}^\infty (-1)^j \frac{2(t/2)^k t^{2j}}{(2j)!\Gamma(k+1/2)\Gamma(1/2)} \int_0^1 s^{2j}(1-s^2)^{k-1/2}\, ds
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty}(-1)^j \frac{(t/2)^k t^{2j}}{(2j)!\Gamma(k+1/2)\Gamma(1/2)} \int_0^1 u^{j-1/2}(1-u)^{k-1/2}\,du \\
&= \sum_{j=0}^{\infty}(-1)^j \frac{(t/2)^k t^{2j}}{(2j)!\Gamma(k+1/2)\Gamma(1/2)} B(j+1/2, k+1/2) \\
&= \frac{(t/2)^k}{\Gamma(1/2)} \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1/2)}{\Gamma(j+k+1)} \frac{t^{2j}}{(2j)!}.
\end{aligned} \tag{2.2.32}
$$

Thus

$$
\begin{aligned}
&\int_0^1 J_k(ts)s^{k+1}(1-s^2)^l\,ds \\
&= \int_0^1 \left(\frac{(ts/2)^k}{\Gamma(1/2)} \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1/2)}{\Gamma(j+k+1)} \frac{(ts)^{2j}}{(2j)!} \right) s^{k+1}(1-s^2)^l\,ds \\
&= \frac{(t/2)^k}{\Gamma(1/2)} \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1/2)}{\Gamma(j+k+1)} \frac{t^{2j}}{(2j)!} \int_0^1 s^{2k+2j+1}(1-s^2)^l\,ds.
\end{aligned}
$$

Substituting $s^2 = u$ and using (2.2.30) and (2.2.32), we conclude

$$
\begin{aligned}
&\int_0^1 J_k(ts)s^{k+1}(1-s^2)^l\,ds \\
&= \frac{(t/2)^k}{\Gamma(1/2)} \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1/2)}{2\Gamma(j+k+1)} \frac{t^{2j}}{(2j)!} \int_0^1 u^{k+j}(1-u)^l\,du \\
&= \frac{(t/2)^k}{\Gamma(1/2)} \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1/2)}{2\Gamma(j+k+1)} \frac{t^{2j}}{(2j)!} B(k+j+1, l+1) \\
&= \frac{2^l\Gamma(l+1)}{t^{l+1}} \frac{(t/2)^{k+l+1}}{\Gamma(1/2)} \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+1/2)}{\Gamma(k+l+j+2)} \frac{t^{2j}}{(2j)!} \\
&= \frac{2^l\Gamma(l+1)}{t^{l+1}} J_{k+l+1}(t),
\end{aligned}
$$

which proves the lemma. ∎

Now we can turn back to the circular Riesz means.

Definition 2.2.26 For $f \in L_1(\mathbb{R}^d)$, the Fourier transform is defined by

$$
\widehat{f}(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(t)e^{-\imath x \cdot t}\,dt \qquad (x \in \mathbb{R}^d).
$$

Define

$$\theta(s) := \begin{cases} (1 - |s|^2)^\alpha & \text{if } |s| \le 1; \\ 0 & \text{if } |s| > 1 \end{cases} \qquad (s \in \mathbb{R})$$

and

$$\theta_0(x) := \theta(\|x\|_2) \qquad (x \in \mathbb{R}^d).$$

θ_0 is called a radial function. No we use another method than for $q = 1$ or $q = \infty$. We will express the Riesz means in terms of the Fourier transform of θ_0. As we will see in the next lemma, $\widehat{\theta_0}$ can be computed with the help of the Bessel functions.

Theorem 2.2.27 *If $\alpha > 0$ and $x \in \mathbb{R}^d$, then*

$$\widehat{\theta_0}(x) = \frac{1}{(2\pi)^{d/2}} 2^\alpha \Gamma(\alpha + 1) \|x\|_2^{-d/2-\alpha} J_{d/2+\alpha}(\|x\|_2).$$

Proof The function $\theta_0 \in L_1(\mathbb{R}^d)$ because

$$\int_{\mathbb{R}^d} |\theta_0(x)|\, dx \le C \int_0^\infty |\theta(r)| r^{d-1}\, dr < \infty.$$

Using the notation $r = \|x\|_2$, $x = rx'$, $s = \|u\|_2$ and $u = su'$, we get that

$$\begin{aligned} \widehat{\theta_0}(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \theta_0(u) e^{-\imath x \cdot u}\, du \\ &= \frac{1}{(2\pi)^d} \int_0^\infty \theta(s) \left(\int_{\Sigma_{d-1}} e^{-\imath r s x' \cdot u'}\, du' \right) s^{d-1}\, ds, \end{aligned} \tag{2.2.33}$$

where Σ_{d-1} denotes the sphere. In the inner integral, we integrate first over the parallel

$$P_\delta := \{u' \in \Sigma_{d-1} : x' \cdot u' = \cos\delta\}$$

orthogonal to x' obtaining a function of $0 \le \delta \le \pi$, which we then integrate over $[0, \pi]$. If ω_{d-2} denotes the surface area of Σ_{d-2}, then the measure of P_δ is

$$\omega_{d-2} (\sin\delta)^{d-2} = \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)} (\sin\delta)^{d-2}.$$

Hence

$$\begin{aligned} \int_{\Sigma_{d-1}} e^{-\imath r s x' \cdot u'}\, du' &= \int_0^\pi e^{-\imath r s \cos\delta} \omega_{d-2} (\sin\delta)^{d-2}\, d\delta \\ &= \omega_{d-2} \int_{-1}^1 e^{\imath r s \xi} (1 - \xi^2)^{(d-3)/2}\, d\xi \end{aligned}$$

$$= \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)} \frac{\Gamma(d/2-1/2)\Gamma(1/2)}{(rs/2)^{d/2-1}} J_{d/2-1}(rs)$$
$$= (2\pi)^{d/2}(rs)^{-d/2+1} J_{d/2-1}(rs).$$

Taking into account this and (2.2.33), we conclude

$$\begin{aligned}\widehat{\theta_0}(x) &= \frac{1}{(2\pi)^{d/2}} r^{-d/2+1} \int_0^\infty \theta(s) J_{d/2-1}(rs) s^{d/2}\, ds \\ &= \frac{1}{(2\pi)^{d/2}} \|x\|_2^{-d/2+1} \int_0^1 J_{d/2-1}(\|x\|_2 s) s^{d/2} (1-s^2)^\alpha \, ds.\end{aligned}$$

Applying Lemma 2.2.25 with $k = d/2 - 1$, $l = \alpha$, we see that

$$\widehat{\theta_0}(x) = \frac{1}{(2\pi)^{d/2}} \|x\|_2^{-d/2+1} J_{d/2+\alpha}(\|x\|_2) \|x\|_2^{-\alpha-1} 2^\alpha \Gamma(\alpha+1),$$

which shows the theorem. ∎

Theorem 2.2.27, Lemma 2.2.23 and 2.2.24 imply that $\widehat{\theta_0}(x)$ as well as all of its derivatives can be estimated by $\|x\|_2^{-d/2-\alpha-1/2}$.

Corollary 2.2.28 *For all $i_1, \ldots, i_d \geq 0$ and $\alpha > 0$,*

$$|\partial_1^{i_1} \cdots \partial_d^{i_d} \widehat{\theta_0}(x)| \leq C \|x\|_2^{-d/2-\alpha-1/2} \qquad (x \neq 0).$$

The same result holds for

$$\theta(s) := \begin{cases} (1-|s|^\gamma)^\alpha & \text{if } |s| \leq 1; \\ 0 & \text{if } |s| > 1 \end{cases} \qquad (s \in \mathbb{R})$$

and

$$\theta_0(x) := \theta(\|x\|_2) \qquad (x \in \mathbb{R}^d),$$

whenever $\gamma \in \mathbb{P}$ (see Lu [224, p. 132]). From now on, we assume that $\gamma \in \mathbb{P}$. The next result is an easy consequence of Corollary 2.2.28.

Corollary 2.2.29 *$\widehat{\theta_0} \in L_1(\mathbb{R}^d)$ if*

$$\frac{d-1}{2} < \alpha < \infty.$$

Now we are ready to express the Riesz means using the Fourier transform of θ_0.

Theorem 2.2.30 *If $n \in \mathbb{N}$, $f \in L_1(\mathbb{T}^d)$, $(d-1)/2 < \alpha < \infty$ and $\gamma \in \mathbb{P}$, then*

$$\sigma_n^{2,\alpha,\gamma} f(x) = n^d \int_{\mathbb{R}^d} f(x-t) \widehat{\theta_0}(nt)\, dt$$

for almost every $x \in \mathbb{T}^d$.

Proof If $f(t) = e^{\imath k \cdot t}$ $(k \in \mathbb{Z}^d, t \in \mathbb{T}^d)$, then

$$\begin{aligned}\sigma_n^{2,\alpha,\gamma} f(x) &= \theta_0\left(\frac{-k}{n}\right) e^{\imath k \cdot x} \\ &= e^{\imath k \cdot x} \int_{\mathbb{R}^d} e^{-\imath k \cdot t/n} \widehat{\theta_0}(t)\,dt \\ &= n^d \int_{\mathbb{R}^d} e^{\imath k \cdot (x-t)} \widehat{\theta_0}(nt)\,dt.\end{aligned}$$

The theorem holds also for trigonometric polynomials. Let f be an arbitrary element from $L_1(\mathbb{T}^d)$ and (f_k) be a sequence of trigonometric polynomials such that $f_k \to f$ in the $L_1(\mathbb{T}^d)$-norm. It follows from Lemma 2.2.6 and from the fact that $K_n^{2,\alpha,\gamma} \in L_1(\mathbb{T}^d)$ that

$$\lim_{n\to\infty} \sigma_n^{2,\alpha,\gamma} f_k = \sigma_n^{2,\alpha,\gamma} f$$

in the $L_1(\mathbb{T}^d)$ norm.

On the other hand, since $\widehat{\theta_0} \in L_1(\mathbb{R}^d)$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_k(x-t)\widehat{\theta_0}(nt)\,dt = \int_{\mathbb{R}^d} f(x-t)\widehat{\theta_0}(nt)\,dt$$

in the $L_1(\mathbb{T}^d)$-norm. ∎

Lemma 2.2.31 *If* $n \in \mathbb{N}$, $(d-1)/2 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$, *then*

$$K_n^{2,\alpha,\gamma}(t) = (2\pi)^d n^d \sum_{k\in\mathbb{Z}^d} \widehat{\theta_0}(n(t+2k\pi)). \tag{2.2.34}$$

Proof Since f is periodic, Theorem 2.2.30 implies that

$$\begin{aligned}\sigma_n^{2,\alpha,\gamma} f(x) &= n^d \sum_{k\in\mathbb{Z}^d} \int_{2k\pi+\mathbb{T}^d} f(x-t)\widehat{\theta_0}(nt)\,dt \\ &= n^d \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{T}^d} f(x-t)\widehat{\theta_0}(n(t+2k\pi))\,dt.\end{aligned}$$

The result follows from Lemma 2.2.6. ∎

2.3 Norm Convergence of the ℓ_q-Summability Means

In this section, we will prove that the Cesàro and Riesz means, $\sigma_n^{q,\alpha} f$ and $\sigma_n^{q,\alpha,\gamma} f$ are uniformly bounded on the $L_p(\mathbb{T}^d)$ spaces and they converge to the original function f in norm when $1 \leq p < \infty$, $q = 1, 2$ or $q = \infty$. Having the results of Sect. 2.2, we are ready to prove that the $L_1(\mathbb{T}^d)$-norms of the kernel functions are uniformly bounded. We start with the triangular and cubic Cesàro summability.

Theorem 2.3.1 *If* $0 < \alpha \leq 1$ *and* $q = 1$ *or* $q = \infty$, *then*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{T}^d} \left|K_n^{q,\alpha}(x)\right| \, dx \leq C.$$

***Proof of Theorem 2.3.1* for** $q = 1$. It is enough to integrate the kernel function over the set

$$\{(x_1, x_2) : 0 < x_2 < x_1 < \pi\}.$$

Let us decompose this set into the union $\cup_{i=1}^{10} A_i$, where

$$\begin{aligned}
A_1 &:= \{(x_1, x_2) : 0 < x_1 \leq 2/n, 0 < x_2 < x_1 < \pi, x_2 \leq \pi/2\},\\
A_2 &:= \{(x_1, x_2) : 2/n < x_1 < \pi, 0 < x_2 \leq 1/n, x_2 \leq \pi/2\},\\
A_3 &:= \{(x_1, x_2) : 2/n < x_1 < \pi, 1/n < x_2 \leq x_1/2, x_2 \leq \pi/2\},\\
A_4 &:= \{(x_1, x_2) : 2/n < x_1 < \pi, x_1/2 < x_2 \leq x_1 - 1/n, x_2 \leq \pi/2\},\\
A_5 &:= \{(x_1, x_2) : 2/n < x_1 < \pi, x_1 - 1/n < x_2 < x_1, x_2 \leq \pi/2\}\\
A_6 &:= \{(x_1, x_2) : x_2 > \pi/2, \pi - 2/n \leq x_2 < \pi, 0 < x_2 < x_1 < \pi\},\\
A_7 &:= \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2/n, \pi - 1/n < x_1 < \pi\},\\
A_8 &:= \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2/n, (\pi + x_2)/2 < x_1 \leq \pi - 1/n\},\\
A_9 &:= \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2/n, x_2 + 1/n < x_1 \leq (\pi + x_2)/2\},\\
A_{10} &:= \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2/n, x_2 < x_1 \leq x_2 + 1/n\}.
\end{aligned}$$

The sets A_i can be seen on Fig. 2.11.

By Lemma 2.2.5, we can see that

$$\int_{A_1} \left|K_n^{1,\alpha}(x_1, x_2)\right| \, dx + \int_{A_6} \left|K_n^{1,\alpha}(x_1, x_2)\right| \, dx \leq C.$$

Inequality (2.2.14) implies

$$\int_{A_2} \left|K_n^{1,\alpha}(x_1, x_2)\right| \, dx \leq C \int_{2/n}^{\pi} \int_0^{1/n} (x_1 - 1/n)^{-3/2} x_2^{-1/2} \, dx_2 \, dx_1 \leq C$$

and

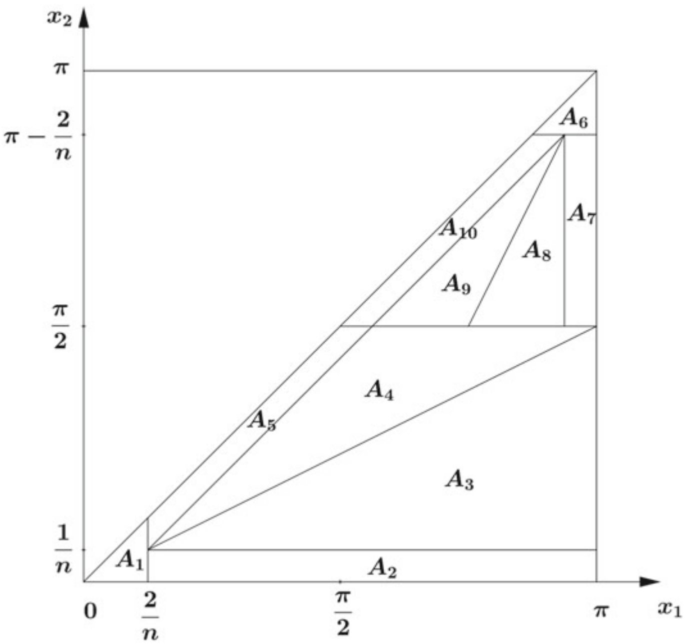

Fig. 2.11 The sets A_i

$$\int_{A_7} \left|K_n^{1,\alpha}(x_1,x_2)\right| \, dx \le C \int_{\pi/2}^{\pi-2/n} \int_{\pi-1/n}^{\pi} (\pi - 1/n - x_2)^{-3/2} (\pi - x_1)^{-1/2} \, dx_1 \, dx_2 \le C.$$

Observe that $x_1 - x_2 \ge x_1/2$ on the set A_3. Choosing β such that $0 < \beta < \alpha$, we get from (2.2.15) that

$$\int_{A_3} \left|K_n^{1,\alpha}(x_1,x_2)\right| \, dx \le Cn^{-\alpha} \int_{2/n}^{\pi} \int_{1/n}^{x_1/2} x_1^{-1-\beta} x_2^{\beta-\alpha-1} \, dx_2 \, dx_1 + Cn^{-1} \int_{2/n}^{\pi} \int_{1/n}^{x_1/2} x_1^{-1-\beta} x_2^{\beta-2} \, dx_2 \, dx_1 \le C.$$

Similarly, $x_1 - x_2 > (\pi - x_2)/2$ on the set A_8 and so, by (2.2.16),

$$\int_{A_8} \left|K_n^{1,\alpha}(x_1,x_2)\right| dx$$
$$\le Cn^{-\alpha}\int_{\pi/2}^{\pi-2/n}\int_{(\pi+x_2)/2}^{\pi-1/n}(\pi-x_2)^{-1-\beta}(\pi-x_1)^{\beta-\alpha-1}\,dx_1\,dx_2$$
$$+ Cn^{-1}\int_{\pi/2}^{\pi-2/n}\int_{(\pi+x_2)/2}^{\pi-1/n}(\pi-x_2)^{-1-\beta}(\pi-x_1)^{\beta-2}\,dx_1\,dx_2 \le C.$$

We have $x_2 > x_1/2$ on A_4, hence (2.2.15) implies

$$\int_{A_4} \left|K_n^{1,\alpha}(x_1,x_2)\right| dx$$
$$\le Cn^{-\alpha}\int_{2/n}^{\pi}\int_{x_1/2}^{x_1-1/n}(x_1-x_2)^{-1-\beta}x_1^{\beta-\alpha-1}\,dx_2\,dx_1$$
$$+ Cn^{-1}\int_{2/n}^{\pi}\int_{x_1/2}^{x_1-1/n}(x_1-x_2)^{-1-\beta}x_1^{\beta-2}\,dx_2\,dx_1 \le C.$$

Similarly, $\pi - x_1 \ge (\pi - x_2)/2$ on the set A_9. Thus

$$\int_{A_9} \left|K_n^{1,\alpha}(x_1,x_2)\right| dx$$
$$\le Cn^{-\alpha}\int_{\pi/2}^{\pi-2/n}\int_{x_2+1/n}^{(\pi+x_2)/2}(x_1-x_2)^{-1-\beta}(\pi-x_2)^{\beta-\alpha-1}\,dx_1\,dx_2$$
$$+ Cn^{-1}\int_{\pi/2}^{\pi-2/n}\int_{x_2+1/n}^{(\pi+x_2)/2}(x_1-x_2)^{-1-\beta}(\pi-x_2)^{\beta-2}\,dx_1\,dx_2 \le C.$$

Finally, by (2.2.19),

$$\int_{A_5} \left|K_n^{1,\alpha}(x_1,x_2)\right| dx \le C\int_{1/n}^{\pi}\int_{x_2}^{x_2+1/n}(x_1-x_2)^{\alpha-1}x_2^{-\alpha-1}\,dx_1dx_2$$
$$+ C\int_{1/n}^{\pi}\int_{x_2}^{x_2+1/n}x_2^{-2}\,dx_1dx_2 \le C$$

and

$$\int_{A_{10}} \left|K_n^{1,\alpha}(x_1,x_2)\right| dx \le C\int_{\pi/2}^{\pi-1/n}\int_{x_1-1/n}^{x_1}(x_1-x_2)^{\alpha-1}(\pi-x_1)^{-\alpha-1}\,dx_2dx_1$$
$$+ C\int_{\pi/2}^{\pi-1/n}\int_{x_1-1/n}^{x_1}(\pi-x_1)^{-2}\,dx_2dx_1 \le C$$

which completes the proof of the theorem. ■

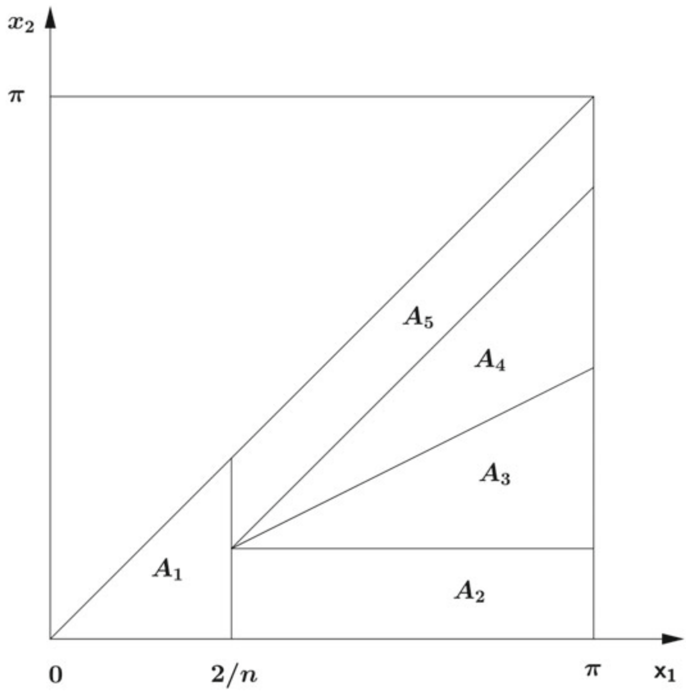

Fig. 2.12 The sets A_i

Proof of Theorem 2.3.1 **for** $q = \infty$. We integrate again over the set

$$\{(x_1, x_2) : 0 < x_2 < x_1 < \pi\}$$

and decompose this set into the union $\cup_{i=1}^{5} A_i$, where

$$\begin{aligned}
A_1 &:= \{(x_1, x_2) : 0 < x_1 \le 2/n, 0 < x_2 < x_1 < \pi\}, \\
A_2 &:= \{(x_1, x_2) : 2/n < x_1 < \pi, 0 < x_2 \le 1/n\}, \\
A_3 &:= \{(x_1, x_2) : 2/n < x_1 < \pi, 1/n < x_2 \le x_1/2\}, \\
A_4 &:= \{(x_1, x_2) : 2/n < x_1 < \pi, x_1/2 < x_2 \le x_1 - 1/n\}, \\
A_5 &:= \{(x_1, x_2) : 2/n < x_1 < \pi, x_1 - 1/n < x_2 < x_1\}
\end{aligned}$$

(see Fig. 2.12).

First of all,

$$\int_{A_1} \left| K_n^{\infty,\alpha}(x_1, x_2) \right| \, dx \le C.$$

By (2.2.25),

$$\left|K_n^{\infty,\alpha}(x_1,x_2)\right| \le Cn^{1-\alpha}(x_1-x_2)^{-1-\alpha} + C(x_1-x_2)^{-2}$$

and so

$$\int_{A_2} \left|K_n^{\infty,\alpha}(x_1,x_2)\right| dx_1\,dx_2 \le Cn^{1-\alpha}\int_{2/n}^{\pi}\int_0^{1/n}(x_1-1/n)^{-1-\alpha}\,dx_2\,dx_1 + C\int_{2/n}^{\pi}\int_0^{1/n}(x_1-1/n)^{-2}\,dx_2\,dx_1 \le C.$$

Since $x_1 - x_2 \ge x_1/2$ on A_3, we get from (2.2.24) that

$$\begin{aligned}\left|K_n^{\infty,\alpha}(x_1,x_2)\right| &\le Cn^{-\alpha}x_1^{-1-\alpha}x_2^{-1} + Cn^{-1}x_1^{-2}x_2^{-1}\\ &\le Cn^{-\alpha}x_1^{-1-\alpha+\beta}x_2^{-1-\beta} + Cn^{-1}x_1^{-2+\beta}x_2^{-1-\beta}\end{aligned} \tag{2.3.1}$$

for any $0 < \beta < \alpha$. Thus

$$\int_{A_3} \left|K_n^{\infty,\alpha}(x_1,x_2)\right| dx_1\,dx_2 \le Cn^{-\alpha}\int_{2/n}^{\pi}\int_{1/n}^{x_1/2} x_1^{-1-\alpha+\beta}x_2^{-1-\beta}\,dx_2\,dx_1 + Cn^{-1}\int_{2/n}^{\pi}\int_{1/n}^{x_1/2} x_1^{-2+\beta}x_2^{-1-\beta}\,dx_2\,dx_1 \le C.$$

Since $x_2 > x_1/2$ and $x_2 > x_1 - x_2$ on A_4, we get from (2.2.24) that

$$\begin{aligned}\left|K_n^{\infty,\alpha}(x_1,x_2)\right| &\le Cn^{-\alpha}x_1^{-1}x_2^{-1}(x_1-x_2)^{-\alpha} + Cn^{-1}x_1^{-1}x_2^{-1}(x_1-x_2)^{-1}\\ &\le Cn^{-\alpha}x_1^{-1-\beta}(x_1-x_2)^{-1-\alpha+\beta}\\ &\quad + Cn^{-1}x_1^{-1-\beta}(x_1-x_2)^{-2+\beta}\end{aligned} \tag{2.3.2}$$

for any $0 < \beta < \alpha$. Then

$$\begin{aligned}&\int_{A_4} \left|K_n^{\infty,\alpha}(x_1,x_2)\right| dx_1\,dx_2\\ &\le Cn^{-\alpha}\int_{2/n}^{\pi}\int_{x_1/2}^{x_1-1/n} x_1^{-1-\beta}(x_1-x_2)^{-1-\alpha+\beta}\,dx_2\,dx_1\\ &\quad + Cn^{-1}\int_{2/n}^{\pi}\int_{x_1/2}^{x_1-1/n} x_1^{-1-\beta}(x_1-x_2)^{-2+\beta}\,dx_2\,dx_1 \le C.\end{aligned}$$

Finally, $x_2 > x_1/2$ also on A_5 and so (2.2.23) implies

$$\int_{A_5} \left|K_n^{\infty,\alpha}(x_1,x_2)\right| dx_1\,dx_2 \le C\int_{2/n}^{\pi}\int_{x_1-1/n}^{x_1} x_1^{-2}\,dx_2\,dx_1 \le C,$$

which finishes the proof. ■

Now we continue with the circular Riesz summability.

Theorem 2.3.2 *If $q = 2$, $\alpha > (d-1)/2$ and $\gamma \in \mathbb{P}$, then*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{T}^d} \left|K_n^{2,\alpha,\gamma}(x)\right| \, dx \le C.$$

Proof Taking into account Lemma 2.2.31, we can see that

$$\int_{\mathbb{T}^d} |K_n^{2,\alpha,\gamma}(x)| \, dx \le (2\pi)^d n^d \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{T}^d} |\widehat{\theta_0}(n(x+2k\pi))| \, dx = (2\pi)^d \left\|\widehat{\theta_0}\right\|_1.$$

Now the theorem follows easily from Corollary 2.2.28. ■

These imply easily

Theorem 2.3.3 *If $1 \le p < \infty$, $0 < \alpha < \infty$ and $q = 1$ or $q = \infty$, then*

$$\sup_{n\in\mathbb{N}} \left\|\sigma_n^{q,\alpha} f\right\|_p \le C\|f\|_p$$

and

$$\lim_{n\to\infty} \sigma_n^{q,\alpha} f = f \qquad \text{in the } L_p(\mathbb{T}^d)\text{-norm for all } f \in L_p(\mathbb{T}^d).$$

Proof For $0 < \alpha \le 1$, we use Minkowski's inequality and Theorem 2.3.1 to obtain

$$\begin{aligned} \left\|\sigma_n^{q,\alpha} f\right\|_p &\le \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \|f(\cdot - t)\|_p K_n^{q,\alpha}(t) \, dt \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \|f\|_p K_n^{q,\alpha}(t) \, dt \\ &\le C\|f\|_p. \end{aligned}$$

For $1 < \alpha < \infty$, we can use Lemma 2.2.8. The convergence follows easily from this because the trigonometric polynomials are dense in $L_p(\mathbb{T}^d)$. ■

The next theorem can be proved in the same way.

Theorem 2.3.4 *If $1 \le p < \infty$, $q = 2$, $(d-1)/2 < \alpha < \infty$ and $\gamma \in \mathbb{P}$, then*

$$\sup_{n\in\mathbb{N}} \left\|\sigma_n^{q,\alpha,\gamma} f\right\|_p \le C\|f\|_p$$

and

$$\lim_{n\to\infty} \sigma_n^{q,\alpha,\gamma} f = f \qquad \text{in the } L_p(\mathbb{T}^d)\text{-norm for all } f \in L_p(\mathbb{T}^d).$$

Theorems 2.3.3 and 2.3.4 were proved in Berens, Li and Xu [30], Oswald [253] and Weisz [337, 338, 341] for $q = 1, \infty$ and in Bochner [36] and Stein and Weiss [293] for $q = 2$.

The situation is more complicated and not completely solved if $q = 2$ and $\alpha \leq (d-1)/2$. It is clear by the Banach-Steinhaus theorem that $\lim_{n\to\infty} \sigma_n^{q,\alpha,\gamma} f = f$ in the $L_p(\mathbb{T}^d)$-norm for all $f \in L_p(\mathbb{T}^d)$ if and only if the operators $\sigma_n^{q,\alpha,\gamma}$ are uniformly bounded from $L_p(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$. We note that each operator $\sigma_n^{q,\alpha,\gamma}$ is bounded on $L_p(\mathbb{T}^d)$ because $K_n^{q,\alpha,\gamma} \in L_1(\mathbb{T}^d)$. For more about the norm convergence of the Bochner-Riesz means (i.e., $q = 2$, $\gamma = 2$) see Grafakos [143].

2.4 $H_p^\square(\mathbb{T}^d)$ Hardy Spaces

To prove almost everywhere convergence of the Cesàro and Riesz means, we will need the concept of Hardy spaces and their atomic decomposition. Before studying Hardy spaces, we have to introduce the concept of distributions.

Let $C^\infty(\mathbb{T}^d)$ denote the set of all infinitely differentiable functions on $\mathbb{T}^d$. Then $f \in C^\infty(\mathbb{T}^d)$ implies

$$\sup_{x\in\mathbb{T}^d} \left|\partial^k f(x)\right| < \infty \qquad \text{for all } k = (k_1, \ldots, k_d) \in \mathbb{N}^d,$$

where $\partial^k = \partial_1^{k_1} \cdots \partial_d^{k_d}$.

Definition 2.4.1 Let $n \in \mathbb{N}$, $f_n, f \in C^\infty(\mathbb{T}^d)$. We say that

$$\lim_{n\to\infty} f_n = f \qquad \text{in } C^\infty(\mathbb{T}^d)$$

if

$$\lim_{n\to\infty} \left\|\partial^k f_n - \partial^k f\right\|_\infty = 0 \qquad \text{for all } k \in \mathbb{N}^d.$$

Definition 2.4.2 A map $u : C^\infty(\mathbb{T}^d) \to \mathbb{C}$ is called distribution if it is linear and continuous, more exactly,

$$u(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 u(f_1) + \alpha_2 u(f_2)$$

for all $f_1, f_2 \in C^\infty(\mathbb{T}^d)$ and $\alpha_1, \alpha_2 \in \mathbb{C}$ and

$$\lim_{n\to\infty} u(f_n) = u(f) \quad \text{if} \quad \lim_{n\to\infty} f_n = f \quad \text{in} \quad C^\infty(\mathbb{T}^d).$$

The set of distributions are denoted by $D(\mathbb{T}^d)$.

If $g \in L_p(\mathbb{T}^d)$ $(1 \leq p \leq \infty)$, then

$$u_g(f) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f g \, d\lambda \qquad (f \in C^\infty(\mathbb{T}^d))$$

is a distribution. Indeed, if $\lim_{n\to\infty} f_n = f$ in $C^\infty(\mathbb{T}^d)$, then $\lim_{n\to\infty} f_n = f$ in $L_{p'}(\mathbb{T}^d)$ as well. Applying Hölder's inequality,

$$\begin{aligned} \left|u_g(f_n) - u_g(f)\right| &\le \int_{\mathbb{T}^d} |f_n(x) - f(x)| \, |g(x)| \, dx \\ &\le \|f_n - f\|_{p'} \|g\|_p \to 0, \end{aligned}$$

as $n \to \infty$. So every function from $L_p(\mathbb{T}^d)$ $(1 \le p \le \infty)$ can be identified with a distribution $u \in D(\mathbb{T}^d)$ in the previous way.

Proposition 2.4.3 *A linear functional u on $C^\infty(\mathbb{T}^d)$ is a distribution if and only if there exist $C > 0$ and $m \in \mathbb{N}$ such that*

$$|u(f)| \le C \sup_{|k| \le m} \left\| \partial^k f \right\|_\infty$$

for all $f \in C^\infty(\mathbb{T}^d)$.

Proof It is evident that the inequality ensures the continuity of u, thus u is a distribution. Conversely, suppose that u is a distribution and the inequality is not true. Then there exists $f_n \in C^\infty(\mathbb{T}^d)$ such that

$$|u(f_n)| > n \sup_{|k| \le n} \left\| \partial^k f_n \right\|_\infty .$$

Since the right-hand side is not 0, we may define

$$g_n := \frac{f_n}{n \sup_{|k| \le n} \left\| \partial^k f_n \right\|_\infty}.$$

Then $g_n \in C^\infty(\mathbb{T}^d)$ and

$$\sup_{|k| \le n} \left\| \partial^k g_n \right\|_\infty = \frac{1}{n},$$

which means that $g_n \to 0$ in $C^\infty(\mathbb{T}^d)$. On the other hand,

$$u(g_n) = \frac{u(f_n)}{n \sup_{|k| \le n} \left\| \partial^k f_n \right\|_\infty} > 1.$$

This contradicts to the continuity of u, i.e., to $u(g_n) \to 0$ as $n \to \infty$. ■

Definition 2.4.4 The least integer m for which Proposition 2.4.3 holds is called the order of u.

Definition 2.4.5 The distributions u_n tend to the distribution u in the sense of distributions or in $D(\mathbb{T}^d)$ if

$$\lim_{n\to\infty} u_n(f) \to u(f) \quad \text{for all} \quad f \in C^\infty(\mathbb{T}^d).$$

The next definition extends the Fourier coefficients to distributions.

Definition 2.4.6 Let

$$e_n(x) := e^{\imath n\cdot x} \qquad (n \in \mathbb{Z}^d, x \in \mathbb{T}^d).$$

For a distribution $u \in D(\mathbb{T}^d)$, the nth Fourier coefficient is defined by

$$\widehat{u}(n) := u(e_{-n}) \qquad (n \in \mathbb{Z}^d).$$

The Fourier series, the partial sums and the summability means of u are defined in the same way as in Definitions 2.1.2, 2.1.3, 2.1.5, 2.2.2 and 2.2.3.

Theorem 2.4.7 *If $u \in C^\infty(\mathbb{T}^d)$ is of order m, then*

$$\widehat{u}(n) = O(|n|^m) \qquad \text{as } |n| \to \infty. \tag{2.4.1}$$

Moreover, for $1 \le q \le \infty$ and $N \in \mathbb{N}$,

$$s_N^q u = \sum_{n\in\mathbb{Z}^d,\ \|n\|_q\le N} \widehat{u}(n)e_n \to u \qquad \text{in } D(\mathbb{T}^d) \text{ as } N \to \infty.$$

Conversely, if $c_n = O(|n|^m)$, then

$$s_N^q := \sum_{n\in\mathbb{Z}^d,\ \|n\|_q\le N} c_n e_n$$

converge to u in $D(\mathbb{T}^d)$ as $N \to \infty$ and $\widehat{u}(n) = c_n$. The same holds for the rectangular partial sums s_N.

Proof Equality (2.4.1) follows immediately from the inequality of Proposition 2.4.3 if we take therein $f = e_{-k}$. For $f \in C^\infty(\mathbb{T}^d)$,

$$s_N^q u(f) = \sum_{n\in\mathbb{Z}^d,\ \|n\|_q\le N} \widehat{u}(n)\widehat{f}(-n) = u\left(\sum_{n\in\mathbb{Z}^d,\ \|n\|_q\le N} \widehat{f}(-n)e_{-n}\right).$$

It is easy to see that $\widehat{f}(n) = O(|n|^{-k})$ for any $k \in \mathbb{N}$. Hence

$$\lim_{N\to\infty} \sum_{n\in\mathbb{Z}^d,\ \|n\|_q\le N} \widehat{f}(n)e_n = f$$

in $C^\infty(\mathbb{T}^d)$ and so

$$\lim_{N\to\infty} s_N^q u(f) = u(f).$$

Conversely, if $c_n = O(|n|^m)$, then

$$s_N^q(f) = \sum_{n\in\mathbb{Z}^d,\ \|n\|_q\le N} c_n \widehat{f}(-n) \tag{2.4.2}$$

for all $f \in C^\infty(\mathbb{T}^d)$. Since the series on the right-hand side is absolutely convergent, let

$$u(f) := \lim_{N\to\infty} s_N^q(f) = \sum_{n\in\mathbb{Z}^d} c_n \widehat{f}(-n).$$

Then u is linear and we can show easily that u is continuous as well. Writing $f = e_{-n}$ in (2.4.2), we can see that $\widehat{u}(n) = c_n\ (n \in \mathbb{Z}^d)$. ■

Definition 2.4.8 The convolution of two functions $f, g \in L_1(\mathbb{T}^d)$ is defined by

$$(f * g)(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(x-t)g(t)\,dt \qquad (x \in \mathbb{T}^d).$$

It is easy to see that

$$(f * g)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(t)g(x-t)\,dt \qquad (x \in \mathbb{T}^d).$$

Using Minkowski's inequality, we obtain Young's inequality. More exactly, for $f \in L_r(\mathbb{T}^d)$, $g \in L_1(\mathbb{T}^d)$ and $1 \le r \le \infty$, we have

$$\|f * g\|_r \le \|f\|_r \|g\|_1.$$

Lemma 2.4.9 *If $f, g \in L_1(\mathbb{T}^d)$, then $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$*

Proof We have,

$$\begin{aligned}
\widehat{f * g}(n) &= \frac{1}{(2\pi)^{2d}}\int_{\mathbb{T}^d}\left(\int_{\mathbb{T}^d} f(x-t)g(t)\,dt\right) e^{-\imath n\cdot x}\,dx \\
&= \frac{1}{(2\pi)^{2d}}\int_{\mathbb{T}^d}\left(\int_{\mathbb{T}^d} f(x-t)\,e^{-\imath n\cdot(x-t)}\,dx\right) g(t)\,e^{-\imath n\cdot t}\,dt \\
&= \frac{1}{(2\pi)^{2d}}\int_{\mathbb{T}^d}\left(\int_{\mathbb{T}^d} f(u)\,e^{-\imath n\cdot u}\,du\right) g(t)\,e^{-\imath n\cdot t}\,dt \\
&= \widehat{f}(n)\cdot\widehat{g}(n),
\end{aligned}$$

which finishes the proof of the lemma. ■

Now we are able to define the convolution of a distribution and function.

Definition 2.4.10 The convolution of $f \in D(\mathbb{T}^d)$ and $g \in L_1(\mathbb{T}^d)$ is defined by

$$f * g := \sum_{n \in \mathbb{N}^d} \widehat{f}(n)\widehat{g}(n)e_n \qquad \text{in } D(\mathbb{T}^d).$$

Since $\widehat{g}$ is bounded, the series is convergent by Theorem 2.4.7. Similarly, we can also define the convolution $f \in D(\mathbb{T}^d)$ and $\psi \in L_1(\mathbb{R}^d)$.

Definition 2.4.11 For $f \in D(\mathbb{T}^d)$ and $\psi \in L_1(\mathbb{R}^d)$ let

$$f * \psi := \sum_{n \in \mathbb{N}^d} \widehat{f}(n)\widehat{\psi}(n)e_n \qquad \text{in } D(\mathbb{T}^d),$$

where $\widehat{\psi}$ denotes the Fourier transform of $\psi \in L_1(\mathbb{R}^d)$.

Similar to Lemma 2.4.9,

$$f * \psi(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(x-u)\psi(u)\,du$$

if $f \in D(\mathbb{T}^d)$ and $\psi \in L_1(\mathbb{R}^d)$. For $t \in (0, \infty)$ and $x \in \mathbb{R}^d$, let

$$\psi_t(\xi) := t^{-d}\psi(\xi/t).$$

It is easy to see that for $f \in D(\mathbb{T}^d)$ and $\psi \in L_1(\mathbb{R}^d)$, we have

$$f * \psi_t = \sum_{n \in \mathbb{N}^d} \widehat{f}(n)\widehat{\psi}(tn)e_n \qquad \text{in } D(\mathbb{T}^d). \tag{2.4.3}$$

To define the Hardy spaces, we need the concept of Schwartz functions.

Definition 2.4.12 The function $f \in C^\infty(\mathbb{R}^d)$ is called a Schwartz function if for all $\alpha, \beta \in \mathbb{N}^d$,

$$\sup_{x \in \mathbb{R}^d} \left| x^\alpha \partial^\beta f(x) \right| = C_{\alpha,\beta} < \infty,$$

where $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$. The set of Schwartz functions are denoted by $S(\mathbb{R}^d)$.

Then $f \in D(\mathbb{T}^d)$ and $\psi \in S(\mathbb{R}^d)$ implies that (2.4.3) converges absolutely in each point as well and so $f * \psi_t \in L_\infty(\mathbb{T}^d)$.

Fix $\psi \in S(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \psi(x)dx \neq 0$. We define the radial maximal function and the non-tangential maximal function of $f \in D(\mathbb{T}^d)$ associated to ψ by

$$\psi^*_{\square,+}(f)(x) := \sup_{t \in (0,\infty)} |f * \psi_t(x)|$$

and

$$\psi^*_{\square,\nabla}(f)(x) := \sup_{t\in(0,\infty),|y-x|<t} |f * \psi_t(y)|,$$

respectively. For $N \in \mathbb{N}$, let

$$\mathcal{F}_N(\mathbb{R}^d) := \left\{ \psi \in S(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d, \|\beta\|_1 \le N} (1+|x|)^{N+d} |\partial^\beta \psi(x)| \le 1 \right\},$$

where $\|\beta\|_1 = \beta_1 + \cdots + \beta_d$. For any $N \in \mathbb{N}$, the radial grand maximal function and the non-tangential grand maximal function of $f \in D(\mathbb{T}^d)$ are defined by

$$f^*_{\square,+}(x) := \sup_{\psi\in\mathcal{F}_N(\mathbb{R}^d)} \sup_{t\in(0,\infty)} |f * \psi_t(y)|$$

and

$$f^*_{\square,\nabla}(x) := \sup_{\psi\in\mathcal{F}_N(\mathbb{R}^d)} \sup_{t\in(0,\infty),|y-x|<t} |f * \psi_t(y)|,$$

respectively. We fix a positive integer $N > \lfloor d(1/p-1) \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$.

Definition 2.4.13 For $0 < p < \infty$ the Hardy spaces $H_p^\square(\mathbb{T}^d)$ and weak Hardy spaces $H_{p,\infty}^\square(\mathbb{T}^d)$ consist of all distributions $f \in D(\mathbb{T}^d)$ for which

$$\|f\|_{H_p^\square} := \left\| \psi^*_{\square,+}(f) \right\|_p < \infty$$

and

$$\|f\|_{H_{p,\infty}^\square} := \left\| \psi^*_{\square,+}(f) \right\|_{p,\infty} < \infty.$$

We will see in the next theorem that the Hardy spaces are independent of ψ and N, more exactly, different functions ψ and different integers N give the same space with equivalent norms.

The d-dimensional periodic Poisson kernel is introduced by

$$P_t(x) := \sum_{k\in\mathbb{Z}^d} e^{-t\|k\|_2} e^{\imath k\cdot x} \qquad (x \in \mathbb{T}^d, t > 0).$$

Notice that $P_t \in L_1(\mathbb{T}^d)$. In the one-dimensional case, we get back the usual Poisson kernel

$$P_t(x) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{\imath kx} = \frac{1-r^2}{1+r^2-2r\cos x} \qquad (x \in \mathbb{T}),$$

where $r := e^{-t}$. For $f \in D(\mathbb{T}^d)$, let

$$P^*_{\Box,+}(f)(x) := \sup_{t\in(0,\infty)} |f * P_t(x)|$$

and

$$P^*_{\Box,\nabla}(f)(x) := \sup_{t\in(0,\infty),|y-x|<t} |f * P_t(y)|.$$

Theorem 2.4.14 *Let $0 < p < \infty$. Fix $\psi \in S(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \psi(x)dx \neq 0$ and fix a positive integer $N > \lfloor d(1/p-1)\rfloor$. Then $f \in H_p^\Box(\mathbb{T}^d)$ if and only if $\psi^*_{\Box,\nabla}(f) \in L_p(\mathbb{T}^d)$ or $f^*_{\Box,+} \in L_p(\mathbb{T}^d)$ or $f^*_{\Box,\nabla} \in L_p(\mathbb{T}^d)$ or $P^*_{\Box,+}(f) \in L_p(\mathbb{T}^d)$ or $P^*_{\Box,\nabla}(f) \in L_p(\mathbb{T}^d)$. We have the following equivalences of norms:*

$$\|f\|_{H_p^\Box} \sim \|\psi^*_{\Box,\nabla}(f)\|_p \sim \|f^*_{\Box,+}\|_p \sim \|f^*_{\Box,\nabla}\|_p \sim \|P^*_{\Box,+}(f)\|_p \sim \|P^*_{\Box,\nabla}(f)\|_p.$$

The same holds for the weak Hardy spaces:

$$\|f\|_{H_{p,\infty}^\Box} \sim \|\psi^*_{\Box,\nabla}(f)\|_{p,\infty} \sim \|f^*_{\Box,+}\|_{p,\infty}$$
$$\sim \|f^*_{\Box,\nabla}\|_{p,\infty} \sim \|P^*_{\Box,+}(f)\|_{p,\infty} \sim \|P^*_{\Box,\nabla}(f)\|_{p,\infty}.$$

Note that $\sim$ denotes the equivalence of norms and spaces, more exactly we write that $A \sim B$ if there exist positive constants c_1 and c_2 such that $c_1 A \le B \le c_2 A$.

Theorem 2.4.15 *If $1 < p < \infty$, then $H_p^\Box(\mathbb{T}^d) \sim L_p(\mathbb{T}^d)$ and*

$$\|f\|_p \le \|f\|_{H_p^\Box} \le C_p \|f\|_p .$$

For $p = 1$, $H_1^\Box(\mathbb{T}^d) \subset L_1(\mathbb{T}^d) \subset H_{1,\infty}^\Box(\mathbb{T}^d)$ and

$$\|f\|_1 \le \|f\|_{H_1^\Box} \qquad (f \in H_1^\Box(\mathbb{T}^d)),$$

$$\|f\|_{H_{1,\infty}^\Box} \le C\|f\|_1 \qquad (f \in L_1(\mathbb{T}^d)).$$

We omit the proofs of these theorems because they are very similar to the proofs of the corresponding theorems for $H_p(\mathbb{R}^d)$, which can be found in several books and papers (e.g., in Stein [290], Grafakos [143], Lu [224], Stein [289], Stein and Weiss [293], Uchiyama [320], Fefferman and Stein [96], Weisz [346]).

We define the reflection and translation operators by

$$\check{h}(x) := h(-x), \qquad T_x h(t) := h(t-x).$$

Theorem 2.4.16 *If $K \in L_1(\mathbb{T}^d)$, $0 < p < \infty$ and*

$$\lim_{k\to\infty} f_k = f \quad \textit{in the } H_p^\Box(\mathbb{T}^d)\textit{-norm},$$

then

$$\lim_{k\to\infty} f_k * K = f * K \quad in\ D(\mathbb{T}^d).$$

Proof Observe that for $f \in H_p(\mathbb{T}^d)$ and $h \in C^\infty(\mathbb{T}^d)$,

$$\begin{aligned} f * h(x) &= \sum_{n\in\mathbb{N}^d} \widehat{f}(n)\widehat{h}(n)e_n(x) \\ &= \sum_{n\in\mathbb{N}^d} \widehat{f}(n)\widehat{T_{-x}h}(n) \\ &= \sum_{n\in\mathbb{N}^d} \widehat{f}(n)e_n(T_x\check{h}) \\ &= f(T_x\check{h}), \end{aligned}$$

Thus

$$\left|f(\check{h})\right| = |f * h(0)| \le \left(\sup_{x\in\mathbb{T}^d, \|\beta\|_1\le N} (1+|x|)^{N+d}|\partial^\beta h(x)|\right) f_\nabla^*(y),$$

where $|y| < 1$ and $N > \lfloor d(1/p-1)\rfloor$. Then

$$\begin{aligned} \left|f(\check{h})\right| &\le C\left(\sup_{x\in\mathbb{T}^d, \|\beta\|_1\le N} |\partial^\beta h(x)|\right) \inf_{|y|<1} f_\nabla^*(y) \\ &\le C\left(\sup_{x\in\mathbb{T}^d, \|\beta\|_1\le N} |\partial^\beta h(x)|\right)\left(\int_{\mathbb{T}^d} f_\nabla^*(y)^p\,dy\right)^{1/p} \\ &\le C\left(\sup_{\|\beta\|_1\le N} |\partial^\beta \check{h}|\right)\|f\|_{H_p}, \end{aligned}$$

which implies that the order of f is at most N and that

$$\lim_{k\to\infty} f_k = f \quad \text{in } D(\mathbb{T}^d).$$

By Theorem 2.4.7 and by the definition of the convolution,

$$\begin{aligned} (f_k - f) * K(h) &= \sum_{n\in\mathbb{N}^d} \left(\widehat{f_k} - \widehat{f}\right)(n)\widehat{K}(n)e_n(h) \\ &= \sum_{n\in\mathbb{N}^d} \left(\widehat{f_k} - \widehat{f}\right)(n)\widehat{K}(n)\widehat{h}(-n), \end{aligned}$$

where $h \in C^\infty(\mathbb{T}^d)$ is arbitrary. Observe that the orders of f_k and f are at most N, $\widehat{K}$ is bounded and $\left|\widehat{h}(n)\right| \le C|n|^{-l}$ for any $l \in \mathbb{N}$. Then for all $\epsilon > 0$ there exists

$m \in \mathbb{N}^d$ such that

$$\left| \left(\sum_{n\in\mathbb{N}^d} - \sum_{|n|\le m} \right) \left(\widehat{f_k} - \widehat{f}\right)(n)\widehat{K}(n)\widehat{h}(-n) \right| \le \epsilon.$$

On the other hand, since

$$\lim_{k\to\infty} (f_k - f)(e_{-n}) = 0,$$

we conclude that

$$\left| \sum_{|n|\le m} \left(\widehat{f_k} - \widehat{f}\right)(n)\widehat{K}(n)\widehat{h}(-n) \right| \le \sum_{|n|\le m} \left| \left(\widehat{f_k} - \widehat{f}\right)(e_{-n}) \right| \to 0$$

as $k \to \infty$, which finishes the proof. ■

The atomic decomposition provides a useful characterization of Hardy spaces. First, we introduce the concept of an atom.

Definition 2.4.17 A bounded function a is an $H_p^\square$-atom if there exists a cube $I \subset \mathbb{T}^d$ such that

(i) $\operatorname{supp} a \subset I$,
(ii) $\|a\|_\infty \le |I|^{-1/p}$,
(iii) $\int_I a(x)x^k \, dx = 0$ for all multi-indices $k = (k_1, \ldots, k_d)$ with $|k| \le \lfloor d(1/p - 1)\rfloor$.

In the definition, the cubes can be replaced by balls and (ii) by

(ii') $\|a\|_q \le |I|^{1/q-1/p}$ $\quad (0 < p < q \le \infty, q > 1)$.

We could suppose that the integral in (iii) is zero for all multi-indices k for which $|k| \le N$, where $N \ge \lfloor d(1/p - 1)\rfloor$. The best possible choice of such numbers N is $\lfloor d(1/p - 1)\rfloor$. Hardy spaces have atomic decompositions. In other words, every function from the Hardy space can be decomposed into the sum of atoms (see e.g. Latter [195], Lu [224], Coifman and Weiss [62], Wilson [353, 354], Stein [290], Grafakos [143] and Weisz [346]).

Theorem 2.4.18 *A distribution $f \in D(\mathbb{T}^d)$ is in $H_p^\square(\mathbb{T}^d)$ $(0 < p \le 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of $H_p^\square$-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \mu_k a_k = f \quad \text{in } D(\mathbb{T}^d). \tag{2.4.4}$$

Moreover,

$$\|f\|_{H_p^{\square}} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f of the form (2.4.4).

The following result gives a sufficient condition for an operator to be bounded from $H_p^{\square}(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$. If $I \subset \mathbb{T}$ is an interval, then rI denotes the interval with the same center as I and with length $r|I|$ $(r \in \mathbb{N})$. For a rectangle

$$R = I_1 \times \cdots \times I_d \subset \mathbb{T}^d, \qquad \text{let} \qquad rR = rI_1 \times \cdots \times rI_d.$$

Instead of $2^r R$ we write R^r $(r \in \mathbb{N})$. For operators $V_n : L_1(\mathbb{T}^d) \to L_1(\mathbb{T}^d)$, we define the maximal operator

$$V_* f := \sup_{n \in \mathbb{N}^d} |V_n f|.$$

Theorem 2.4.19 *For each $n \in \mathbb{N}^d$, let $K_n \in L_1(\mathbb{T}^d)$ and $V_n f := f * K_n$. Suppose that*

$$\int_{\mathbb{T}^d \setminus rI} |V_* a|^{p_0} \, d\lambda \le C_{p_0}$$

for all $H_{p_0}^{\square}$-atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p_0 \le 1$, where the cube I is the support of the atom. If V_* is bounded from $L_{p_1}(\mathbb{T}^d)$ to $L_{p_1}(\mathbb{T}^d)$ for some $1 < p_1 \le \infty$, then

$$\|V_* f\|_p \le C_p \|f\|_{H_p^{\square}} \qquad (f \in H_p^{\square}(\mathbb{T}^d)) \tag{2.4.5}$$

for all $p_0 \le p \le p_1$.

Proof Observe that, under the conditions of Theorem 2.4.19, the L_{p_0}-norms of $V_* a$ are uniformly bounded for all $H_{p_0}^{\square}$-atoms a. Indeed,

$$\begin{aligned}
\int_{\mathbb{T}^d} |V_* a|^{p_0} \, d\lambda &= \int_{rI} |V_* a|^{p_0} \, d\lambda + \int_{\mathbb{T}^d \setminus rI} |V_* a|^{p_0} \, d\lambda \\
&\le \left(\int_{rI} |V_* a|^{p_1} \, d\lambda \right)^{p_0/p_1} |rI|^{1-p_0/p_1} + C_{p_0} \\
&\le C_{p_0} \left(\int_{rI} |a|^{p_1} \, d\lambda \right)^{p_0/p_1} |I|^{1-p_0/p_1} + C_{p_0} \\
&\le C_{p_0} \left(|I|^{-p_1/p_0} |I^r| \right)^{p_0/p_1} |I|^{1-p_0/p_1} + C_{p_0} \\
&= C_{p_0}.
\end{aligned}$$

There is an atomic decomposition such that

$$f = \sum_{k=0}^{\infty} \mu_k a_k \quad \text{in the } H_{p_0}^{\square}\text{-norm} \quad \text{and} \quad \left(\sum_{k=0}^{\infty} |\mu_k|^{p_0} \right)^{1/p_0} \leq C_{p_0} \|f\|_{H_{p_0}^{\square}},$$

where the convergence holds also in the $H_1^{\square}(\mathbb{T}^d)$-norm and in the $L_1(\mathbb{T}^d)$-norm if $f \in H_1^{\square}(\mathbb{T}^d)$. Since $V_n : L_1(\mathbb{T}^d) \to L_1(\mathbb{T}^d)$ is bounded, we have

$$V_n f = \sum_{k=0}^{\infty} \mu_k V_n a_k$$

and

$$|V_* f| \leq \sum_{k=0}^{\infty} |\mu_k| |V_* a_k|$$

for $f \in H_1^{\square}(\mathbb{T}^d)$. Thus

$$\|V_* f\|_{p_0}^{p_0} \leq \sum_{k=0}^{\infty} |\mu_k|^{p_0} \|V_* a_k\|_{p_0}^{p_0} \leq C_{p_0} \|f\|_{H_{p_0}^{\square}}^{p_0} \qquad (f \in H_1^{\square}(\mathbb{T}^d)). \tag{2.4.6}$$

Obviously, the same inequality holds for the operators V_n. This and interpolation proves the theorem if $p_0 = 1$. Assume that $p_0 < 1$. Since $H_1^{\square}(\mathbb{T}^d)$ is dense in $L_1(\mathbb{T}^d)$ as well as in $H_{p_0}^{\square}(\mathbb{T}^d)$, we can extend uniquely the operators V_n and V_* such that (2.4.6) holds for all $f \in H_{p_0}^{\square}(\mathbb{T}^d)$. Let us denote these extended operators by V_n' and V_*'. Then $V_n f = V_n' f$ and $V_* f = V_*' f$ for all $f \in H_1^{\square}(\mathbb{T}^d)$. We get by interpolation from (2.4.6) that the operator

$$V_*' \quad \text{is bounded from} \quad H_{p,\infty}^{\square}(\mathbb{T}^d) \quad \text{to} \quad L_{p,\infty}(\mathbb{T}^d) \tag{2.4.7}$$

when $p_0 < p < p_1$. For the basic definitions and theorems on interpolation theory, see Bergh and Löfström [33], Bennett and Sharpley [28] or Weisz [346]. Since $p_0 < 1$, the boundedness in (2.4.7) holds especially for $p = 1$, and so Theorem 2.4.15 implies that V_*' is of weak type (1, 1):

$$\sup_{\rho>0} \rho \, \lambda(|V_*' f| > \rho) = \|V_*' f\|_{1,\infty} \leq C \|f\|_{H_{1,\infty}^{\square}} \leq C \|f\|_1 \tag{2.4.8}$$

for all $f \in L_1(\mathbb{R}^d)$. Obviously, the same holds for V_n'. Since V_n is bounded on $L_1(\mathbb{T}^d)$, if $f_k \in H_1^{\square}(\mathbb{T}^d)$ such that $\lim_{k\to\infty} f_k = f$ in the L_1-norm, then

$$\lim_{k\to\infty} V_n f_k = V_n f \qquad \text{in the } L_1(\mathbb{T}^d)\text{-norm}.$$

Inequality (2.4.8) implies that

$$\lim_{k\to\infty} V_n f_k = V_n' f \qquad \text{in the } L_{1,\infty}(\mathbb{T}^d)\text{-norm},$$

hence $V_n f = V_n' f$ for all $f \in L_1(\mathbb{T}^d)$. Similarly, for a fixed $N \in \mathbb{N}$, the operator

$$V_{N,*} f := \sup_{|n|\le N} |V_n f|$$

satisfies (2.4.8) for all $f \in H_1^\square(\mathbb{T}^d)$ and its extension $V_{N,*}'$ for all $f \in L_1(\mathbb{T}^d)$. Then

$$\begin{aligned}\sup_{\rho>0} \rho\,\lambda(|V_{N,*}' f - V_{N,*} f| > \rho) &\le \sup_{\rho>0} \rho\,\lambda(|V_{N,*}' f - V_{N,*}' f_k| > \rho/2)\\ &\quad + \sup_{\rho>0} \rho\,\lambda(|V_{N,*} f_k - V_{N,*} f| > \rho/2)\\ &\le \sup_{\rho>0} \rho\,\lambda(|V_{N,*}'(f - f_k)| > \rho/2)\\ &\quad + \sum_{n=0}^{N} \sup_{\rho>0} \rho\,\lambda(|V_n(f_k - f)| > \rho/2N)\\ &\le C\,\|f - f_k\| \to 0\end{aligned}$$

as $k \to \infty$. This shows the equality

$$V_{N,*}' f = V_{N,*} f \qquad \text{for all } f \in L_1(\mathbb{T}^d).$$

Moreover, for a fixed ρ,

$$\begin{aligned}&\lambda(|V_*' f - V_{N,*} f| > \rho)\\ &\quad\le \lambda(|V_*' f - V_*' f_k| > \rho/3) + \lambda(|V_* f_k - V_{N,*} f_k| > \rho/3)\\ &\qquad + \lambda(|V_{N,*} f_k - V_{N,*} f| > \rho/3)\\ &\quad\le \lambda(V_*'(f - f_k) > \rho/3) + \lambda(V_* f_k - V_{N,*} f_k > \rho/3)\\ &\qquad + \lambda(V_{N,*}(f_k - f) > \rho/3)\\ &\quad\le \frac{C}{\rho}\|f - f_k\|_1 + \lambda(V_* f_k - V_{N,*} f_k > \rho/3)\\ &\quad< \epsilon\end{aligned}$$

if k and N are large enough. Hence $\lim_{N\to\infty} V_{N,*} f = V_*' f$ in measure for all $f \in L_1(\mathbb{T}^d)$. On the other hand, $\lim_{N\to\infty} V_{N,*} f = V_* f$ a.e., which implies that

$$V_* f = V_*' f \quad \text{for all} \quad f \in L_1(\mathbb{T}^d).$$

Consequently, (2.4.8) holds also for V_* and (2.4.6) for all $f \in H_{p_0}^\square(\mathbb{T}^d) \cap L_1(\mathbb{T}^d)$.

Assume that $p < 1$, $f_k \in H_p^\square(\mathbb{T}^d) \cap L_1(\mathbb{T}^d)$ $(k \in \mathbb{N})$ and that $\lim_{k\to\infty} f_k = f$ in the $H_p^\square(\mathbb{T}^d)$-norm. By Theorem 2.4.16,

$$\lim_{k\to\infty} V_n f_k = V_n f \qquad \text{in } D(\mathbb{T}^d)$$

for all $n \in \mathbb{N}^d$. Since by (2.4.5), $V_n f_k$ is convergent in the $L_p(\mathbb{T}^d)$-norm as $k \to \infty$, we can identify the distribution $V_n f$ with the $L_p(\mathbb{T}^d)$-limit $\lim_{k\to\infty} V_n f_k$. Hence the same holds for $V_{N,*} f$:

$$V_{N,*} f = \lim_{k\to\infty} V_{N,*} f_k \qquad \text{in the } L_p(\mathbb{T}^d)\text{-norm.}$$

Moreover,

$$\begin{aligned}
&\|V'_* f - V_{N,*} f\|_p \\
&\quad \le \|V'_* f - V'_* f_k\|_p + \|V_* f_k - V_{N,*} f_k\|_p + \|V_{N,*} f_k - V_{N,*} f\|_p \\
&\quad \le C_p \|f - f_k\|_{H_p^\square} + \|V_* f_k - V_{N,*} f_k\|_p + \|V_{N,*} f_k - V_{N,*} f\|_p \\
&\quad < \epsilon
\end{aligned}$$

if k and N are large enough. Thus

$$\lim_{N\to\infty} V_{N,*} f = V'_* f \qquad \text{in the } L_p(\mathbb{T}^d)\text{-norm}$$

and, on the other hand,

$$\lim_{N\to\infty} V_{N,*} f = V_* f \qquad \text{a.e.,}$$

which implies that $V_* f = V'_* f$ for all $f \in H_p^\square(\mathbb{T}^d)$. Consequently, (2.4.5) holds for all $f \in H_p^\square(\mathbb{T}^d)$. ■

Unfortunately, for a general linear operator V, the uniform boundedness of the L_{p_0}-norms of Va is not enough for the boundedness $V : H_{p_0}^\square(\mathbb{T}^d) \to L_{p_0}(\mathbb{T}^d)$ (see [41, 42, 235, 236, 259]). The next weak version of Theorem 2.4.19 can be proved similarly (see also the proof in Weisz [346]).

Theorem 2.4.20 *For each $n \in \mathbb{N}^d$, let $K_n \in L_1(\mathbb{T}^d)$ and $V_n f := f * K_n$. Suppose that*

$$\sup_{\rho>0} \rho^p \lambda\Big(\{|V_* a| > \rho\} \cap \{\mathbb{T}^d \setminus rI\}\Big) \le C_p$$

for all $H_p^\square$-atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p < 1$. If V_* is bounded from $L_{p_1}(\mathbb{T}^d)$ to $L_{p_1}(\mathbb{T}^d)$ $(1 < p_1 \le \infty)$, then

$$\|V_* f\|_{p,\infty} \le C_p \|f\|_{H_p^\square} \qquad (f \in H_p^\square(\mathbb{T}^d)).$$

The weak type $(1, 1)$ inequality follows from inequality (2.4.8).

Corollary 2.4.21 *For each $n \in \mathbb{N}^d$, let $K_n \in L_1(\mathbb{T}^d)$ and $V_n f := f * K_n$. Suppose that*

$$\int_{\mathbb{T}^d \setminus rI} |V_* a|^{p_0} \, d\lambda \le C_{p_0}$$

for all $H_{p_0}^\square$-atoms a and for some fixed $r \in \mathbb{N}$ and $0 < p_0 < 1$, where the cube I is the support of the atom. If V_* is bounded from $L_{p_1}(\mathbb{T}^d)$ to $L_{p_1}(\mathbb{T}^d)$ for some $1 < p_1 \le \infty$, then for all $f \in L_1(\mathbb{T}^d)$,

$$\sup_{\rho>0} \rho \, \lambda(|V_* f| > \rho) \le C \|f\|_1.$$

Proof By Theorem 2.4.19 and interpolation,

$$V_* \quad \text{is bounded from} \quad H_{p,\infty}^\square(\mathbb{T}^d) \quad \text{to} \quad L_{p,\infty}(\mathbb{T}^d)$$

when $p_0 < p < p_1$. Since $p_0 < 1$, this holds also for $p = 1$. Thus, by Theorem 2.4.15:

$$\sup_{\rho>0} \rho \, \lambda(|V_* f| > \rho) = \|V_* f\|_{1,\infty} \le C \|f\|_{H_{1,\infty}^\square} \le C \|f\|_1$$

for all $f \in L_1(\mathbb{T}^d)$. ■

Theorem 2.4.19 and Corollary 2.4.21 can be regarded also as an alternative tool to the Calderon-Zygmund decomposition lemma for proving weak type (1, 1) inequalities. In many cases, this method can be applied better and more simply than the Calderon-Zygmund decomposition lemma.

2.5 Almost Everywhere Convergence of the ℓ_q-Summability Means

Since the kernels $K_n^{q,\alpha}$ and $K_n^{q,\alpha,\gamma}$ are integrable, the definition of the Fejér and Riesz means can be extended to distributions.

Definition 2.5.1 Let $f \in D(\mathbb{T}^d)$, $1 \le q \le \infty$, $n \in \mathbb{N}$ and $0 \le \alpha, \gamma < \infty$. The nth ℓ_q-Cesàro means $\sigma_n^{q,\alpha} f$ and ℓ_q-Riesz means $\sigma_n^{q,\alpha,\gamma} f$ of the Fourier series of f are given by

$$\sigma_n^{q,\alpha} f := f * K_n^{q,\alpha}$$

and

$$\sigma_n^{q,\alpha,\gamma} f := f * K_n^{q,\alpha,\gamma},$$

respectively.

Definition 2.5.2 We define the maximal Cesàro and maximal Riesz operator by

$$\sigma_*^{q,\alpha} f := \sup_{n\in\mathbb{N}} |\sigma_n^{q,\alpha} f|$$

and

$$\sigma_*^{q,\alpha,\gamma} f := \sup_{n\in\mathbb{N}} |\sigma_n^{q,\alpha,\gamma} f|,$$

respectively.

If $\alpha = 1$, we obtain the maximal Fejér operator and write it simply as $\sigma_*^q f := \sigma_*^{q,1} f$. We will prove that the Cesàro and Riesz maximal operators, $\sigma_*^{q,\alpha} f$ and $\sigma_*^{q,\alpha,\gamma} f$ are bounded from the Hardy space $H_p^\square(\mathbb{T}^d)$ to the Lebesgue space $L_p(\mathbb{T}^d)$ when $q = 1, 2$ or $q = \infty$ and p is greater than a critical index $p_0 < 1$ which is depending on q, d and α. If p is equal to this critical index, then weak type inequality holds. As a consequence, we obtain the almost everywhere convergence of the ℓ_q-Cesàro and Riesz means to the original function. We start again with the triangular and cubic Cesàro summability.

2.5.1 Almost Everywhere Convergence for q = 1 and q = ∞

Proposition 2.5.3 *If* $0 < \alpha \le 1$ *and* $q = 1$ *or* $q = \infty$, *then*

$$\left\|\sigma_*^{q,\alpha} f\right\|_\infty \le C \left\|f\right\|_\infty \qquad (f \in L_\infty(\mathbb{T}^d)).$$

Proof The proof follows easily from the fact that the $L_1(\mathbb{T}^d)$-norms of the kernel functions are uniformly bounded (see Theorem 2.3.1) and from Lemma 2.2.8. ■

In what follows we use the notation $a \wedge b := \min(a, b)$.

Theorem 2.5.4 *Suppose that* $q = 1, \infty$ *and* $0 < \alpha < \infty$. *If*

$$p_0 := \frac{d}{d + \alpha \wedge 1} < p \le \infty,$$

then

$$\left\|\sigma_*^{q,\alpha} f\right\|_p \le C_p \left\|f\right\|_{H_p^\square} \qquad (f \in H_p^\square(\mathbb{T}^d)). \tag{2.5.1}$$

Corollary 2.5.5 *If* $q = 1, \infty$, $0 < \alpha < \infty$ *and* $1 < p < \infty$, *then*

$$\|\sigma_*^{q,\alpha} f\|_p \le C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Proof This follows from Theorem 2.5.4 and from the fact that $H_p^\square(\mathbb{T}^d) \sim L_p(\mathbb{T}^d)$ for $1 < p \le \infty$. ■

Proof of Theorem 2.5.4 **for** $q = 1$. By Lemma 2.2.8, we may suppose again that $0 < \alpha \le 1$. It is enough to show that

$$\begin{aligned}
&\int_{\mathbb{T}^2} \left|\sigma_*^{1,\alpha} a(x_1, x_2)\right|^p \, dx_1 \, dx_2 \\
&\quad = \int_{\mathbb{T}^2} \sup_{n\ge1} \left| \int_I a(t_1, t_2) K_n^{1,\alpha}(x_1 - t_1, x_2 - t_2) \, dt_1 \, dt_2 \right|^p \, dx_1 \, dx_2 \\
&\quad \le C_p
\end{aligned} \tag{2.5.2}$$

for every $H_p^{\square}$-atom a, where $2/(2+\alpha) < p < 1$ and I is the support of the atom. By Theorem 2.4.19 and Proposition 2.5.3, this will imply (2.5.1). Without loss of generality, we can suppose that a is a $H_p^{\square}$-atom with support $I = I_1 \times I_2$ and

$$[-2^{-K-2}, 2^{-K-2}] \subset I_j \subset [-2^{-K-1}, 2^{-K-1}] \qquad (j = 1, 2)$$

for some $K \in \mathbb{N}$. By symmetry, we may assume that $\pi > x_1 - t_1 > x_2 - t_2 > 0$, and so, instead of (2.5.2), it is enough to show that

$$\begin{aligned}
&\int_{\mathbb{T}^2} \sup_{n\ge1} \left| \int_I a(t_1, t_2) K_n^{1,\alpha}(x_1 - t_1, x_2 - t_2) 1_{A_i}(x_1 - t_1, x_2 - t_2) \, dt_1 \, dt_2 \right|^p \, dx_1 \, dx_2 \\
&\quad \le C_p
\end{aligned}$$

for all $i = 1, \ldots, 10$, where

$$\begin{aligned}
A_1 &:= \{(x_1, x_2) : 0 < x_1 \le 2^{-K+5}, 0 < x_2 < x_1 < \pi, x_2 \le \pi/2\}, \\
A_2 &:= \{(x_1, x_2) : 2^{-K+5} < x_1 < \pi, 0 < x_2 \le 2^{-K+2}, x_2 \le \pi/2\}, \\
A_3 &:= \{(x_1, x_2) : 2^{-K+5} < x_1 < \pi, 2^{-K+2} < x_2 \le x_1/2, x_2 \le \pi/2\}, \\
A_4 &:= \{(x_1, x_2) : 2^{-K+5} < x_1 < \pi, x_1/2 < x_2 \le x_1 - 2^{-K+2}, x_2 \le \pi/2\}, \\
A_5 &:= \{(x_1, x_2) : 2^{-K+5} < x_1 < \pi, x_1 - 2^{-K+2} < x_2 < x_1, x_2 \le \pi/2\} \\
A_6 &:= \{(x_1, x_2) : x_2 > \pi/2, \pi - 2^{-K+5} \le x_2 < \pi, 0 < x_2 < x_1 < \pi\}, \\
A_7 &:= \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2^{-K+5}, \pi - 2^{-K+2} < x_1 < \pi\}, \\
A_8 &:= \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2^{-K+5}, (\pi + x_2)/2 < x_1 \le \pi - 2^{-K+2}\}, \\
A_9 &:= \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2^{-K+5}, x_2 + 2^{-K+2} < x_1 \le (\pi + x_2)/2\}, \\
A_{10} &:= \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2^{-K+5}, x_2 < x_1 \le x_2 + 2^{-K+2}\}.
\end{aligned}$$

These sets are similar to those in Theorem 2.3.1 (see Fig. 2.11). If $0 < x_1 - t_1 \le 2^{-K+5}$, then $-2^{-K-1} < x_1 \le 2^{-K+6}$ and the same holds for x_2. If $\pi - 2^{-K+5} \le x_2 - t_2 < \pi$, then $\pi - 2^{-K+6} < x_2 \le \pi + 2^{-K-1}$ and the same is true for x_1. By the definition of the $H_p^{\square}$-atom and by Theorem 2.3.1,

$$\begin{aligned}
&\int_{\mathbb{T}^2} \sup_{n\ge 1} \left| \int_I a(t_1,t_2) K_n^{1,\alpha}(x_1-t_1,x_2-t_2) 1_{A_1}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right|^p \, dx_1\, dx_2 \\
&\le 2^{2K} \int_{\mathbb{T}^2} \sup_{n\ge 1} \left| \int_I \left| K_n^{1,\alpha}(x_1-t_1,x_2-t_2)\right| 1_{A_1}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right|^p \, dx_1\, dx_2 \\
&\le C_p 2^{2K} 2^{-2K} \le C_p
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{T}^2} \sup_{n\ge 1} \left| \int_I a(t_1,t_2) K_n^{1,\alpha}(x_1-t_1,x_2-t_2) 1_{A_6}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right|^p \, dx_1\, dx_2 \\
&\qquad \le C_p 2^{2K} 2^{-2K} \le C_p.
\end{aligned}$$

On the set A_2, we have $2^{-K+5} < x_1 - t_1 < \pi$ and $0 < x_2 - t_2 \le 2^{-K+2}$, thus

$$2^{-K+4} < x_1 < \pi + 2^{-K-1} \qquad \text{and} \qquad -2^{-K-1} < x_2 \le 2^{-K+3}.$$

Using (2.2.14), we conclude

$$\begin{aligned}
&\left| \int_I a(t_1,t_2) K_n^{1,\alpha}(x_1-t_1,x_2-t_2) 1_{A_2}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right| \\
&\quad \le C_p 2^{2K/p} \int_I (x_1-t_1-x_2+t_2)^{-3/2}(x_2-t_2)^{-1/2} \\
&\qquad 1_{A_2}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \\
&\quad \le C_p 2^{2K/p} 1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}} 1_{\{-2^{-K-1}<x_2\le 2^{-K+3}\}} \\
&\qquad \int_I (x_1-2^{-K+3})^{-3/2}(x_2-t_2)^{-1/2}\, dt_1\, dt_2 \\
&\quad \le C_p 2^{2K/p-3K/2} 1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}} \\
&\qquad 1_{\{-2^{-K-1}<x_2\le 2^{-K+3}\}} (x_1-2^{-K+3})^{-3/2}
\end{aligned} \tag{2.5.3}$$

and

$$\begin{aligned}
&\int_{\mathbb{T}^2} \sup_{n\ge 1} \left| \int_I a(t_1,t_2) K_n^{1,\alpha}(x_1-t_1,x_2-t_2) 1_{A_2}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right|^p \, dx_1\, dx_2 \\
&\quad \le C_p 2^{2K-3Kp/2} \int_{2^{-K+4}}^{\pi+2^{-K-1}} \int_{-2^{-K-1}}^{2^{-K+3}} (x_1-2^{-K+3})^{-3p/2}\, dx_1\, dx_2 \\
&\quad \le C_p.
\end{aligned}$$

Here we have used that $p > 2/3$. Similarly, on A_7, $\pi/2 < x_2 - t_2 < \pi - 2^{-K+5}$ and $\pi - 2^{-K+2} < x_1 - t_1 < \pi$, thus

$$\pi/2 - 2^{-K-1} < x_2 < \pi - 2^{-K+4} \qquad \text{and} \qquad \pi - 2^{-K+3} < x_1 < \pi + 2^{-K-1}.$$

By (2.2.14),

$$\begin{aligned}
&\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_7}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right| \\
&\quad\le C_p 2^{2K/p}\int_I (x_1-t_1-x_2+t_2)^{-3/2}(\pi-x_1+t_1)^{-1/2} \\
&\qquad 1_{A_7}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2 \\
&\quad\le C_p 2^{2K/p} 1_{\{\pi/2-2^{-K-1}<x_2<\pi-2^{-K+4}\}} 1_{\{\pi-2^{-K+3}<x_1<\pi+2^{-K-1}\}} \\
&\qquad \int_I (\pi-2^{-K+3}-x_2)^{-3/2}(\pi-x_1+t_1)^{-1/2}\,dt_1\,dt_2 \\
&\quad\le C_p 2^{2K/p-3K/2} 1_{\{\pi/2-2^{-K-1}<x_2<\pi-2^{-K+4}\}} \\
&\qquad 1_{\{\pi-2^{-K+3}<x_1<\pi+2^{-K-1}\}}(\pi-2^{-K+3}-x_2)^{-3/2}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{T}^2}\sup_{n\ge 1}\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_7}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right|^p\,dx_1\,dx_2 \\
&\quad\le C_p 2^{2K-3Kp/2}\int_{\pi/2-2^{-K-1}}^{\pi-2^{-K+4}}\int_{\pi-2^{-K+3}}^{\pi+2^{-K-1}} (\pi-2^{-K+3}-x_2)^{-3p/2}\,dx_2\,dx_1 \\
&\quad\le C_p.
\end{aligned}$$

We may suppose that the center of I is zero, in other words $I := (-\nu,\nu)\times(-\nu,\nu)$. Let

$$A_1(u,t_2) := \int_{-\nu}^{u} a(t_1,t_2)\,dt_1 \qquad \text{and} \qquad A_2(u,v) := \int_{-\nu}^{v} A_1(u,t_2)\,dt_2.$$

Observe that

$$|A_k(u,v)| \le C_p 2^{K(2/p-k)} \qquad (k=1,2).$$

Integrating by parts, we can see that

$$\begin{aligned}
&\int_{I_1} a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_3\cup A_8}(x_1-t_1,x_2-t_2)\,dt_1 \\
&\quad= A_1(\nu,t_2)K_n^{1,\alpha}(x_1-\nu,x_2-t_2)1_{A_3\cup A_8}(x_1-\nu,x_2-t_2) \\
&\qquad+\int_{-\nu}^{\nu} A_1(t_1,t_2)\partial_1 K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_3\cup A_8}(x_1-t_1,x_2-t_2)\,dt_1,
\end{aligned}$$

because $A_1(-\nu,t_2)=0$. Let us integrate the first term again by parts and use that

$$A_2(\nu,\nu) = \int_{I_1}\int_{I_2} a(t_1,t_2)\,dt_1\,dt_2 = 0$$

to obtain

$$\begin{aligned}
&\int_{I_1}\int_{I_2} a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_3\cup A_8}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2 \\
&\quad = \int_{-\nu}^{\nu} A_2(\nu,t_2)\partial_2 K_n^{1,\alpha}(x_1-\nu,x_2-t_2)1_{A_3\cup A_8}(x_1-\nu,x_2-t_2)\,dt_2 \\
&\qquad + \int_{I_1}\int_{I_2} A_1(t_1,t_2)\partial_1 K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_3\cup A_8}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2.
\end{aligned}$$

Note that

$$x_1-t_1-x_2+t_2 > (x_1-t_1)/2 \tag{2.5.4}$$

on the set A_3 and

$$x_1-t_1-x_2+t_2 > (\pi-x_2+t_2)/2 \tag{2.5.5}$$

on the set A_8. If $n \le 2^K$, we get from Lemma 2.2.17 and (2.5.4) that

$$\begin{aligned}
&\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_3}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right| \\
&\quad \le C_p n^{1-\gamma}2^{2K/p-2K} \\
&\qquad \int_{I_2}(x_1-\nu)^{-1-\beta}(x_2-t_2)^{\beta-\gamma-1}1_{A_3}(x_1-\nu,x_2-t_2)\,dt_2 \\
&\qquad + C_p n^{1-\gamma}2^{2K/p-K} \\
&\qquad\quad \int_I (x_1-t_1)^{-1-\beta}(x_2-t_2)^{\beta-\gamma-1}1_{A_3}(x_1-t_1,x_2-t_2)\,dt_1dt_2 \\
&\quad \le C_p 2^{2K/p-2K-K\gamma}1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}}1_{\{2^{-K+1}<x_2\le x_1/2+2^{-K}\}} \\
&\qquad (x_1-2^{-K-1})^{-1-\beta}(x_2-2^{-K-1})^{\beta-\gamma-1},
\end{aligned} \tag{2.5.6}$$

where $0 \le \beta \le 1$, $\gamma = \alpha$ or $\gamma = 1$. On A_8, we use (2.5.5) to obtain

$$\begin{aligned}
&\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_8}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right| \\
&\quad \le C_p n^{1-\gamma}2^{2K/p-2K} \\
&\qquad \int_{I_2}(\pi-x_2+t_2)^{-1-\beta}(\pi-x_1+\nu)^{\beta-\gamma-1}1_{A_8}(x_1-\nu,x_2-t_2)\,dt_2 \\
&\qquad + C_p n^{1-\gamma}2^{2K/p-K} \\
&\qquad\quad \int_I (\pi-x_2+t_2)^{-1-\beta}(\pi-x_1+t_1)^{\beta-\gamma-1}1_{A_8}(x_1-t_1,x_2-t_2)\,dt_1dt_2 \\
&\quad \le C_p 2^{2K/p-2K-K\gamma}1_{\{\pi/2-2^{-K-1}<x_2<\pi-2^{-K+4}\}}1_{\{(\pi+x_2)/2-2^{-K}<x_1<\pi-2^{-K+1}\}}
\end{aligned}$$

$$(\pi - x_2 - 2^{-K-1})^{-1-\beta}(\pi - x_1 - 2^{-K-1})^{\beta-\gamma-1}. \tag{2.5.7}$$

Similarly, if $n > 2^K$, then we get from (2.2.15) and (2.5.4) that

$$\begin{aligned}
&\left| \int_I a(t_1,t_2) K_n^{1,\alpha}(x_1-t_1, x_2-t_2) 1_{A_3}(x_1-t_1, x_2-t_2)\, dt_1\, dt_2 \right| \\
&\leq C_p n^{-\gamma} 2^{2K/p} \int_I (x_1-t_1)^{-1-\beta}(x_2-t_2)^{\beta-\gamma-1} 1_{A_3}(x_1-t_1, x_2-t_2)\, dt_1\, dt_2 \\
&\leq C_p 2^{2K/p-2K-K\gamma} 1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}} 1_{\{2^{-K+1}<x_2\leq x_1/2+2^{-K}\}} \\
&\quad (x_1 - 2^{-K-1})^{-1-\beta}(x_2 - 2^{-K-1})^{\beta-\gamma-1}
\end{aligned} \tag{2.5.8}$$

and, by (2.5.5),

$$\begin{aligned}
&\left| \int_I a(t_1,t_2) K_n^{1,\alpha}(x_1-t_1, x_2-t_2) 1_{A_8}(x_1-t_1, x_2-t_2)\, dt_1\, dt_2 \right| \\
&\leq C_p n^{-\gamma} 2^{2K/p} \\
&\quad \int_I (\pi - x_2 + t_2)^{-1-\beta}(\pi - x_1 + t_1)^{\beta-\gamma-1} 1_{A_8}(x_1-t_1, x_2-t_2)\, dt_1\, dt_2 \\
&\leq C_p 2^{2K/p-2K-K\gamma} 1_{\{\pi/2-2^{-K-1}<x_2<\pi-2^{-K+4}\}} 1_{\{(\pi+x_2)/2-2^{-K}<x_1<\pi-2^{-K+1}\}} \\
&\quad (\pi - x_2 - 2^{-K-1})^{-1-\beta}(\pi - x_1 - 2^{-K-1})^{\beta-\gamma-1}.
\end{aligned} \tag{2.5.9}$$

Choosing $\beta = \gamma/2$, we conclude

$$\begin{aligned}
&\int_{\mathbb{T}^2} \sup_{n\geq 1} \left| \int_I a(t_1,t_2) K_n^{1,\alpha}(x_1-t_1, x_2-t_2) 1_{A_3}(x_1-t_1, x_2-t_2)\, dt_1\, dt_2 \right|^p dx_1 dx_2 \\
&\leq C_p 2^{2K-2Kp-K\gamma p} \\
&\quad \int_{2^{-K+4}}^{\pi+2^{-K-1}} \int_{2^{-K+1}}^{x_1/2+2^{-K}} (x_1 - 2^{-K-1})^{-p(1+\gamma/2)}(x_2 - 2^{-K-1})^{-p(1+\gamma/2)}\, dx_2 dx_1 \\
&\leq C_p 2^{2K-2Kp-K\gamma p} 2^{-K(1-p(1+\gamma/2))} 2^{-K(1-p(1+\gamma/2))} \\
&\leq C_p
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{T}^2} \sup_{n\geq 1} \left| \int_I a(t_1,t_2) K_n^{1,\alpha}(x_1-t_1, x_2-t_2) 1_{A_8}(x_1-t_1, x_2-t_2)\, dt_1\, dt_2 \right|^p dx_1 dx_2 \\
&\leq C_p 2^{2K-2Kp-K\gamma p} \int_{\pi/2-2^{-K-1}}^{\pi-2^{-K+4}} \int_{(\pi+x_2)/2-2^{-K}}^{\pi-2^{-K+1}} \\
&\quad (\pi - x_2 - 2^{-K-1})^{-p(1+\gamma/2)}(\pi - x_1 - 2^{-K-1})^{-p(1+\gamma/2)}\, dx_1 dx_2 \\
&\leq C_p 2^{2K-2Kp-K\gamma p} 2^{-K(1-p(1+\gamma/2))} 2^{-K(1-p(1+\gamma/2))}
\end{aligned}$$

$$\leq C_p,$$

whenever $p > 2/(2+\gamma)$. Recall that $\gamma = \alpha$ or $\gamma = 1$.

Since

$$x_2 - t_2 > (x_1 - t_1)/2 \quad \text{on } A_4$$

and

$$\pi - x_1 + t_1 > (\pi - x_2 + t_2)/2 \quad \text{on } A_9,$$

Lemma 2.2.17 implies

$$\begin{aligned}
&\left| \int_I a(t_1,t_2) K_n^{1,\alpha}(x_1-t_1, x_2-t_2) 1_{A_4}(x_1-t_1, x_2-t_2)\, dt_1\, dt_2 \right| \\
&\leq C_p n^{1-\gamma} 2^{2K/p-2K} \int_{I_2} (x_1 - \nu - x_2 + t_2)^{-1-\beta} \\
&\qquad (x_1 - t_1)^{\beta-\gamma-1} 1_{A_4}(x_1-\nu, x_2-t_2)\, dt_2 \\
&\quad + C_p n^{1-\gamma} 2^{2K/p-K} \int_I (x_1 - t_1 - x_2 + t_2)^{-1-\beta} \\
&\qquad (x_1 - t_1)^{\beta-\gamma-1} 1_{A_4}(x_1-t_1, x_2-t_2)\, dt_1 dt_2 \\
&\leq C_p 2^{2K/p-2K-K\gamma} 1_{\{2^{-K+4} < x_1 < \pi + 2^{-K-1}\}} 1_{\{x_1/2 - 2^{-K} < x_2 \leq x_1 - 2^{-K+1}\}} \\
&\quad (x_1 - x_2 - 2^{-K})^{-1-\beta} (x_1 - 2^{-K-1})^{\beta-\gamma-1}
\end{aligned} \tag{2.5.10}$$

and

$$\begin{aligned}
&\left| \int_I a(t_1,t_2) K_n^{1,\alpha}(x_1-t_1, x_2-t_2) 1_{A_9}(x_1-t_1, x_2-t_2)\, dt_1\, dt_2 \right| \\
&\leq C_p n^{1-\gamma} 2^{2K/p-2K} \int_{I_2} (x_1 - \nu - x_2 + t_2)^{-1-\beta} \\
&\qquad (\pi - x_2 + \nu)^{\beta-\gamma-1} 1_{A_9}(x_1-\nu, x_2-t_2)\, dt_2 \\
&\quad + C_p n^{1-\gamma} 2^{2K/p-K} \int_I (x_1 - t_1 - x_2 + t_2)^{-1-\beta} \\
&\qquad (\pi - x_2 + t_2)^{\beta-\gamma-1} 1_{A_9}(x_1-t_1, x_2-t_2)\, dt_1 dt_2 \\
&\leq C_p 2^{2K/p-2K-K\gamma} 1_{\{\pi/2 - 2^{-K-1} < x_2 < \pi - 2^{-K+4}\}} 1_{\{x_2 + 2^{-K+1} < x_1 < (\pi + x_2)/2 + 2^{-K}\}} \\
&\quad (x_1 - x_2 - 2^{-K})^{-1-\beta} (\pi - x_2 - 2^{-K-1})^{\beta-\gamma-1},
\end{aligned} \tag{2.5.11}$$

whenever $n \leq 2^K$. If $n > 2^K$, then by (2.2.15),

$$
\begin{aligned}
&\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_4}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right| \\
&\quad\le C_p n^{-\gamma}2^{2K/p}\int_I (x_1-t_1-x_2+t_2)^{-1-\beta} \\
&\qquad\qquad (x_1-t_1)^{\beta-\gamma-1}1_{A_4}(x_1-t_1,x_2-t_2)\,dt_1dt_2 \\
&\quad\le C_p 2^{2K/p-2K-K\gamma}1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}}1_{\{x_1/2-2^{-K}<x_2\le x_1-2^{-K+1}\}} \\
&\qquad (x_1-x_2-2^{-K})^{-1-\beta}(x_1-2^{-K-1})^{\beta-\gamma-1}
\end{aligned}
\tag{2.5.12}
$$

and

$$
\begin{aligned}
&\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_9}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right| \\
&\quad\le C_p n^{-\gamma}2^{2K/p}\int_I (x_1-t_1-x_2+t_2)^{-1-\beta} \\
&\qquad\qquad (\pi-x_2+t_2)^{\beta-\gamma-1}1_{A_9}(x_1-t_1,x_2-t_2)\,dt_1dt_2 \\
&\quad\le C_p 2^{2K/p-2K-K\gamma}1_{\{\pi/2-2^{-K-1}<x_2<\pi-2^{-K+4}\}}1_{\{x_2+2^{-K+1}<x_1<(\pi+x_2)/2+2^{-K}\}} \\
&\qquad (x_1-x_2-2^{-K})^{-1-\beta}(\pi-x_2-2^{-K-1})^{\beta-\gamma-1}.
\end{aligned}
\tag{2.5.13}
$$

Choosing again $\beta=\gamma/2$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{T}^2}\sup_{n\ge 1}\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_4}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right|^p\,dx_1dx_2 \\
&\le C_p 2^{2K-2Kp-K\gamma p} \\
&\quad \int_{2^{-K+4}}^{\pi+2^{-K-1}}\int_{x_1/2-2^{-K}}^{x_1-2^{-K+1}}(x_1-x_2-2^{-K})^{-p(1+\gamma/2)}(x_1-2^{-K-1})^{-p(1+\gamma/2)}\,dx_2dx_1 \\
&\le C_p 2^{2K-2Kp-K\gamma p}2^{-K(1-p(1+\gamma/2))}2^{-K(1-p(1+\gamma/2))} \\
&\le C_p
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{T}^2}\sup_{n\ge 1}\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_9}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right|^p\,dx_1dx_2 \\
&\le C_p 2^{2K-2Kp-K\gamma p}\int_{\pi/2-2^{-K-1}}^{\pi-2^{-K+4}}\int_{x_2+2^{-K+1}}^{(\pi+x_2)/2+2^{-K}} \\
&\quad (x_1-x_2-2^{-K})^{-p(1+\gamma/2)}(\pi-x_2-2^{-K-1})^{-p(1+\gamma/2)}\,dx_1dx_2 \\
&\le C_p 2^{2K-2Kp-K\gamma p}2^{-K(1-p(1+\gamma/2))}2^{-K(1-p(1+\gamma/2))} \\
&\le C_p,
\end{aligned}
$$

whenever $p>2/(2+\gamma)$.

Finally, inequality (2.2.19) imply

$$
\begin{aligned}
&\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_5}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right| \\
&\quad\le C_p 2^{2K/p}\int_I (x_1-t_1-x_2+t_2)^{\gamma-1} \\
&\qquad (x_2-t_2)^{-\gamma-1}1_{A_5}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2 \\
&\quad\le C_p 2^{2K/p-K\gamma}1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}} \\
&\qquad 1_{\{x_1-2^{-K+3}<x_2\le x_1+2^{-K}\}}\int_{I_2}(x_2-t_2)^{-\gamma-1}\,dt_2 \\
&\quad\le C_p 2^{2K/p-K\gamma-K}1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}} \\
&\qquad 1_{\{x_1-2^{-K+3}<x_2\le x_1+2^{-K}\}}(x_2-2^{-K-1})^{-\gamma-1}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_{10}}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right| \\
&\quad\le C_p 2^{2K/p}\int_I (x_1-t_1-x_2+t_2)^{\gamma-1} \\
&\qquad (\pi-x_1+t_1)^{-\gamma-1}1_{A_5}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2 \\
&\quad\le C_p 2^{2K/p-K\gamma}1_{\{\pi/2-2^{-K-1}<x_2<\pi-2^{-K+4}\}} \\
&\qquad 1_{\{x_2-2^{-K}<x_1<x_2+2^{-K+3}\}}\int_{I_2}(\pi-x_1+t_1)^{-\gamma-1}\,dt_2 \\
&\quad\le C_p 2^{2K/p-K\gamma-K}1_{\{\pi/2-2^{-K-1}<x_2<\pi-2^{-K+4}\}} \\
&\qquad 1_{\{x_2-2^{-K}<x_1<x_2+2^{-K+3}\}}(\pi-x_1-2^{-K-1})^{-\gamma-1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_{\mathbb{T}^2}\sup_{n\ge 1}\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_5}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right|^p\,dx_1dx_2 \\
&\quad\le C_p 2^{2K-K\gamma p-Kp}\int_{2^{-K+4}}^{\pi+2^{-K-1}}\int_{x_1-2^{-K+3}}^{x_1+2^{-K}}(x_2-2^{-K-1})^{-p(\gamma+1)}\,dx_2dx_1 \\
&\quad\le C_p 2^{2K-K\gamma p-Kp}\int_{2^{-K+3}}^{\pi+2^{-K+5}}\int_{x_2-2^{-K}}^{x_2+2^{-K+3}}(x_2-2^{-K-1})^{-p(\gamma+1)}\,dx_1dx_2 \\
&\quad\le C_p
\end{aligned}
$$

and

$$\int_{\mathbb{T}^2} \sup_{n\ge 1} \left| \int_I a(t_1,t_2) K_n^{1,\alpha}(x_1-t_1, x_2-t_2) 1_{A_{10}}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right|^p dx_1 dx_2$$
$$\le C_p 2^{2K-K\gamma p-Kp} \int_{\pi/2-2^{-K-1}}^{\pi-2^{-K+4}} \int_{x_2-2^{-K}}^{x_2+2^{-K+3}} (\pi - x_1 - 2^{-K-1})^{-p(\gamma+1)}\, dx_1 dx_2$$
$$\le C_p 2^{2K-K\gamma p-Kp} \int_{\pi/2-2^{-K+1}}^{\pi-2^{-K+3}} \int_{x_1+2^{-K}}^{x_1-2^{-K+3}} (\pi - x_1 - 2^{-K-1})^{-p(\gamma+1)}\, dx_2 dx_1$$
$$\le C_p,$$

whenever $p > 1/(1+\gamma)$, which finishes the proof of the theorem. ■

Proof of Theorem 2.5.4 **for** $q=\infty$. We assume again that $\alpha \le 1$ and a is a cube $H_p^{\square}$-atom with support $I = I_1 \times I_2$,

$$[-2^{-K-2}, 2^{-K-2}] \subset I_j \subset [-2^{-K-1}, 2^{-K-1}] \qquad (j=1,2)$$

for some $K \in \mathbb{Z}$. As before, it is enough to show that

$$\int_{\mathbb{T}^2} \sup_{n\ge 1} \left| \int_I a(t_1,t_2) K_n^{\infty,\alpha}(x_1-t_1, x_2-t_2) 1_{A_i}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right|^p dx_1\, dx_2$$
$$\le C_p$$

for all $i = 1,2,3,4,5$, where $\pi > x_1 - t_1 > x_2 - t_2 > 0$ and

$$A_1 := \{(x_1,x_2) : 0 < x_1 \le 2^{-K+5}, 0 < x_2 < x_1 < \pi\},$$
$$A_2 := \{(x_1,x_2) : 2^{-K+5} < x_1 < \pi, 0 < x_2 \le 2^{-K+2}\},$$
$$A_3 := \{(x_1,x_2) : 2^{-K+5} < x_1 < \pi, 2^{-K+2} < x_2 \le x_1/2\},$$
$$A_4 := \{(x_1,x_2) : 2^{-K+5} < x_1 < \pi, x_1/2 < x_2 \le x_1 - 2^{-K+2}\},$$
$$A_5 := \{(x_1,x_2) : 2^{-K+5} < x_1 < \pi, x_1 - 2^{-K+2} < x_2 < x_1\}$$

(see Fig. 2.12). The estimation on the set A_1 is the same as before in the proof for $q=1$. Inequality (2.2.28) implies

$$\left| \int_I a(t_1,t_2) K_n^{\infty,\alpha}(x_1-t_1, x_2-t_2) 1_{A_2}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right|$$
$$\le C_p 2^{2K/p} \int_I (x_1 - t_1 - x_2 + t_2)^{-1-\gamma} (x_2-t_2)^{\gamma-1}$$
$$1_{A_2}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2$$
$$\le C_p 2^{2K/p} 1_{\{2^{-K+4} < x_1 < \pi + 2^{-K-1}\}} 1_{\{-2^{-K-1} < x_2 \le 2^{-K+3}\}}$$
$$\int_I (x_1 - 2^{-K+3})^{-1-\gamma} (x_2-t_2)^{\gamma-1}\, dt_1\, dt_2$$

$$
\begin{aligned}
&\le C_p 2^{2K/p-K-K\gamma} 1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}} \\
&\quad 1_{\{-2^{-K-1}<x_2\le 2^{-K+3}\}}(x_1-2^{-K+3})^{-1-\gamma},
\end{aligned}
$$

where $\gamma = \alpha$ or $\gamma = 1$ in the whole proof. Furthermore,

$$
\begin{aligned}
&\int_{\mathbb{T}^2} \sup_{n\ge 1} \left| \int_I a(t_1,t_2) K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2) 1_{A_2}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right|^p dx_1\, dx_2 \\
&\le C_p 2^{2K-Kp-K\gamma p} \int_{2^{-K+4}}^{\pi+2^{-K-1}} \int_{-2^{-K-1}}^{2^{-K+3}} (x_1-2^{-K+3})^{-p(1+\gamma)}\, dx_2 dx_1 \\
&\le C_p,
\end{aligned}
$$

provided that $p > 1/(1+\gamma)$. For any $0 < \beta < \alpha$, we get from (2.3.1) that

$$
\begin{aligned}
&\left| \int_I a(t_1,t_2) K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2) 1_{A_3}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right| \\
&\le C_p 2^{2K/p} n^{-\gamma} \int_I (x_1-t_1)^{-1-\gamma+\beta}(x_2-t_2)^{-1-\beta} \\
&\quad 1_{A_3}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \\
&\le C_p 2^{2K/p-2K-K\gamma} 1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}} 1_{\{2^{-K+1}<x_2\le x_1/2+2^{-K}\}} \\
&\quad (x_1-2^{-K-1})^{-1-\gamma+\beta}(x_2-2^{-K-1})^{-1-\beta},
\end{aligned}
$$

whenever $n > 2^K$. Lemma 2.2.21 and (2.5.4) imply that

$$
\left| \partial_j K_n^{\infty,\alpha}(x_1,x_2) \right| \le C n^{1-\alpha} x_1^{-1-\alpha+\beta} x_2^{-1-\beta}
$$

on A_3, where $j = 1, 2$. Similar to the proof for $q = 1$, we get by integration by parts that

$$
\begin{aligned}
&\left| \int_I a(t_1,t_2) K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2) 1_{A_3}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right| \\
&\le \left| \int_{-\nu}^{\nu} A_2(\nu,t_2) \partial_2 K_n^{\infty,\alpha}(x_1-\nu,x_2-t_2) 1_{A_3}(x_1-\nu,x_2-t_2)\, dt_2 \right| \\
&\quad + \left| \int_{I_1}\int_{I_2} A_1(t_1,t_2) \partial_1 K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2) 1_{A_3}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right| \\
&\le C_p n^{1-\alpha} 2^{2K/p-2K} \int_{I_2} (x_1-\nu)^{-1-\alpha+\beta}(x_2-t_2)^{-1-\beta} 1_{A_3}(x_1-t_1,x_2-t_2)\, dt_2 \\
&\quad + C_p n^{1-\alpha} 2^{2K/p-K} \int_I (x_1-t_1)^{-1-\alpha+\beta}(x_2-t_2)^{-1-\beta} \\
&\qquad 1_{A_3}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \\
&\le C_p 2^{2K/p-2K-K\alpha} 1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}} 1_{\{2^{-K+1}<x_2\le x_1/2+2^{-K}\}}
\end{aligned}
$$

$$(x_1 - 2^{-K-1})^{-1-\alpha+\beta}(x_2 - 2^{-K-1})^{-1-\beta}$$

if $n \leq 2^K$. Thus

$$\begin{aligned}
&\int_{\mathbb{T}^2} \sup_{n\geq 1} \left| \int_I a(t_1,t_2) K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2) 1_{A_3}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right|^p dx_1\, dx_2 \\
&\leq C_p 2^{2K-2Kp-K\gamma p} \\
&\qquad \int_{2^{-K+4}}^{\pi+2^{-K-1}} \int_{2^{-K+1}}^{x_1/2+2^{-K}} (x_1 - 2^{-K-1})^{(-1-\gamma+\beta)p}(x_2 - 2^{-K-1})^{-(1+\beta)p}\, dx_2 dx_1 \\
&\leq C_p 2^{2K-2Kp-K\gamma p} 2^{-K(1-(\gamma-\beta+1)p)} 2^{-K(1-(1+\beta)p)} \\
&\leq C_p,
\end{aligned}$$

whenever $p > 1/(1+\beta)$ and $p > 1/(\gamma-\beta+1)$. $\beta = \gamma/2$ implies $p > \frac{2}{2+\gamma}$.

Using (2.3.2), we see that

$$\begin{aligned}
&\left| \int_I a(t_1,t_2) K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2) 1_{A_4}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right| \\
&\quad \leq C_p n^{-\gamma} 2^{2K/p} \int_I (x_1-t_1-x_2+t_2)^{-1-\gamma+\beta}(x_1-t_1)^{-1-\beta} \\
&\qquad 1_{A_4}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \\
&\quad \leq C_p 2^{2K/p-2K-K\gamma} 1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}} 1_{\{x_1/2-2^{-K}<x_2\leq x_1-2^{-K+1}\}} \\
&\qquad (x_1-x_2-2^{-K})^{-1-\gamma+\beta}(x_1-2^{-K-1})^{-1-\beta},
\end{aligned} \tag{2.5.14}$$

where $n > 2^K$ and $0 < \beta < \alpha$. Since $x_2 > x_1/2$ and $x_2 > x_1 - x_2$ on A_4, Lemma 2.2.21 implies

$$\left|\partial_j K_n^{\infty,\alpha}(x_1,x_2)\right| \leq C n^{1-\alpha} x_1^{-1-\beta}(x_1-x_2)^{-1-\alpha+\beta},$$

where $j = 1, 2$. For $n \leq 2^K$, we get by integration by parts that

$$\begin{aligned}
&\left| \int_I a(t_1,t_2) K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2) 1_{A_4}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right| \\
&\leq \left| \int_{-\nu}^{\nu} A_2(\nu,t_2) \partial_2 K_n^{\infty,\alpha}(x_1-\nu,x_2-t_2) 1_{A_4}(x_1-\nu,x_2-t_2)\, dt_2 \right| \\
&\quad + \left| \int_{I_1}\int_{I_2} A_1(t_1,t_2) \partial_1 K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2) 1_{A_4}(x_1-t_1,x_2-t_2)\, dt_1\, dt_2 \right| \\
&\leq C_p n^{1-\alpha} 2^{2K/p-2K} \int_{I_2} (x_1-\nu)^{-1-\beta}(x_1-t_1-x_2+t_2)^{-1-\alpha+\beta} \\
&\qquad 1_{A_4}(x_1-t_1,x_2-t_2)\, dt_2
\end{aligned}$$

$$+ C_p n^{1-\alpha} 2^{2K/p-K} \int_I (x_1-t_1)^{-1-\beta}(x_1-t_1-x_2+t_2)^{-1-\alpha+\beta}$$
$$1_{A_4}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2$$
$$\le C_p 2^{2K/p-2K-K\alpha} 1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}} 1_{\{x_1/2-2^{-K}<x_2\le x_1-2^{-K+1}\}}$$
$$(x_1-x_2-2^{-K})^{-1-\alpha+\beta}(x_1-2^{-K-1})^{-1-\beta}. \tag{2.5.15}$$

From this it follows that

$$\int_{\mathbb{T}^2} \sup_{n\ge 1} \left| \int_I a(t_1,t_2) K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2) 1_{A_4}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2 \right|^p dx_1\,dx_2$$
$$\le C_p 2^{2K-2Kp-K\gamma p} \int_{2^{-K+4}}^{\pi+2^{-K-1}} \int_{x_1/2-2^{-K}}^{x_1-2^{-K+1}}$$
$$(x_1-x_2-2^{-K})^{(-1-\gamma+\beta)p}(x_1-2^{-K-1})^{-(1+\beta)p}\,dx_2 dx_1$$
$$\le C_p 2^{2K-2Kp-K\gamma p} 2^{-K(1-(\gamma-\beta+1)p)} 2^{-K(1-(1+\beta)p)}$$
$$\le C_p,$$

whenever $\beta=\gamma/2$ and $p>\frac{2}{2+\gamma}$.

Finally, since $x_2>x_1/2$ also on A_5,

$$\left| \int_I a(t_1,t_2) K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2) 1_{A_5}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2 \right|$$
$$\le C_p 2^{2K/p} \int_I (x_1-t_1)^{-2} 1_{A_5}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2$$
$$\le C_p 2^{2K/p-2K} 1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}} 1_{\{x_1-2^{-K+3}<x_2\le x_1+2^{-K}\}} (x_1-2^{-K-1})^{-2}$$

and so

$$\int_{\mathbb{T}^2} \sup_{n\ge 1} \left| \int_I a(t_1,t_2) K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2) 1_{A_4}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2 \right|^p dx_1\,dx_2$$
$$\le C_p 2^{2K-2Kp} \int_{2^{-K+4}}^{\pi+2^{-K-1}} \int_{x_1-2^{-K+3}}^{x_1+2^{-K}} (x_1-2^{-K-1})^{-2p}\,dx_2 dx_1$$
$$\le C_p.$$

This completes the proof. ■

If p is smaller than or equal to the critical index, then this theorem is not true (see Oswald [253] and Stein, Taibleson and Weiss [292]). More exactly, we have

Theorem 2.5.6 *If $q=\infty$ and $\alpha=1$, then the operator $\sigma_*^{q,\alpha}$ is not bounded from $H_p^{\square}(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ if p is smaller than or equal to the critical index $d/(d+1)$.*

However, if p is equal to the critical index, then we can verify a weak type inequality.

Theorem 2.5.7 *Suppose that* $q = 1, \infty$ *and* $0 < \alpha < \infty$. *If*

$$p_0 := \frac{d}{d + \alpha \wedge 1}$$

and $f \in H_{p_0}^{\square}(\mathbb{T}^d)$, *then*

$$\left\| \sigma_*^{q,\alpha} f \right\|_{p_0,\infty} = \sup_{\rho>0} \rho \lambda(\sigma_*^{q,\alpha} f > \rho)^{1/p_0} \le C \left\| f \right\|_{H_{p_0}^{\square}}. \tag{2.5.16}$$

***Proof of Theorem 2.5.7* for** $q = 1$. We may suppose again that $0 < \alpha \le 1$. We use Theorem 2.4.20 and prove that

$$\sup_{\rho>0} \rho^{2/(2+\alpha)} \lambda(\sigma_*^{1,\alpha} a > \rho) \le C$$

for all $H_{2/(2+\alpha)}^{\square}$-atoms a. In other words, we have to show that

$$\lambda\left(\sup_{n\ge1} \left| \int_I a(t_1,t_2) K_n^{1,\alpha}(x_1-t_1, x_2-t_2) 1_{A_i}(x_1-t_1, x_2-t_2)\, dt_1\, dt_2 \right| > \rho \right) \\ \le C\rho^{-2/(2+\alpha)}$$

for $i = 1, \ldots, 10$ and $\rho > 0$. Since

$$\rho^{2/(2+\alpha)} \lambda(|g| > \rho) \le \int_{\mathbb{T}^d} |g|^{2/(2+\alpha)}, \tag{2.5.17}$$

the desired inequality follows from the proof of Theorem 2.5.4 for $i = 1, 6, 5, 10$. The same holds for $i = 2, 7$ if $\alpha < 1$. So for $i = 2, 7$, we suppose that $\alpha = 1$.

For $i = 2$ and $p = 2/3$, we have seen in (2.5.3) that

$$\left| \int_I a(t_1,t_2) K_n^{1,\alpha}(x_1-t_1, x_2-t_2) 1_{A_2}(x_1-t_1, x_2-t_2)\, dt_1\, dt_2 \right| \\ \le C 2^{3K/2} 1_{\{2^{-K+4} < x_1 < \pi + 2^{-K-1}\}} 1_{\{-2^{-K-1} < x_2 \le 2^{-K+3}\}} (x_1 - 2^{-K+3})^{-3/2}.$$

If this is greater than ρ, then

$$1_{\{2^{-K+4} < x_1 < \pi + 2^{-K-1}\}} (x_1 - 2^{-K+3}) < C\rho^{-2/3} 2^K 1_{\{-2^{-K-1} < x_2 \le 2^{-K+3}\}}$$

and

$$2^{-K+4} < x_1 < C\rho^{-2/3} 2^K + 2^{-K+4}.$$

Consequently,

$$\lambda\left(\sup_{n\ge 1}\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_2}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right|>\rho\right)$$
$$\le \int_{\mathbb{T}^2} 1_{\left\{1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}}(x_1-2^{-K+3})<C\rho^{-2/3}2^K 1_{\{-2^{-K-1}<x_2\le 2^{-K+3}\}}\right\}}\,dx_1\,dx_2$$
$$\le C\rho^{-2/3}2^K\int_{\mathbb{T}} 1_{\{-2^{-K-1}<x_2\le 2^{-K+3}\}}\,dx_2$$
$$\le C\rho^{-2/3}.$$

Similarly,

$$\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_7}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right|$$
$$\le C2^{3K/2}1_{\{\pi/2-2^{-K-1}<x_2<\pi-2^{-K+4}\}}$$
$$1_{\{\pi-2^{-K+3}<x_1<\pi+2^{-K-1}\}}(\pi-2^{-K+3}-x_2)^{-3/2}.$$

If this is greater than ρ, then

$$1_{\{\pi/2-2^{-K-1}<x_2<\pi-2^{-K+4}\}}(\pi-2^{-K+3}-x_2)$$
$$<C\rho^{-2/3}2^K 1_{\{\pi-2^{-K+3}<x_1<\pi+2^{-K-1}\}}.$$

Let us denote the set of (x_1,x_2) for which the preceding inequality holds by H_7. If $(x_1,x_2)\in H_7$, then

$$\pi-2^{-K+3}-C\rho^{-2/3}2^K<x_2<\pi-2^{-K+3}.$$

Furthermore,

$$\lambda\left(\sup_{n\ge 1}\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_7}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right|>\rho\right)$$
$$\le \int_{\mathbb{T}^2} 1_{H_7}(x_1,x_2)\,dx_2\,dx_1$$
$$\le C\rho^{-2/3}2^K\int_{\mathbb{T}} 1_{\{\pi-2^{-K+3}<x_1<\pi+2^{-K-1}\}}\,dx_1$$
$$\le C\rho^{-2/3}.$$

For $i=3,8,4,9$, we may suppose that $\gamma=\alpha$ and $p=2/(2+\alpha)$. We get by (2.5.6) and (2.5.8) that

$$
\begin{aligned}
&\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_3}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right| \\
&\quad\le C1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}}1_{\{2^{-K+1}<x_2\le x_1/2+2^{-K}\}} \\
&\qquad (x_1-2^{-K-1})^{-1-\beta}(x_2-2^{-K-1})^{\beta-\alpha-1}.
\end{aligned}
$$

If this is greater than ρ, then

$$
\begin{aligned}
&1_{\{2^{-K+1}<x_2\le x_1/2+2^{-K}\}}(x_2-2^{-K-1}) \\
&\quad< C\rho^{-\frac{1}{1+\alpha-\beta}}1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}}(x_1-2^{-K-1})^{-\frac{1+\beta}{1+\alpha-\beta}}.
\end{aligned}
$$

Note that $x_1/2+2^{-K}<x_1$. Choosing β such that $-\frac{1+\beta}{1+\alpha-\beta}+1<0$, i.e., $\alpha/2<\beta\le 1$, we obtain

$$
\begin{aligned}
&\lambda\left(\sup_{n\ge 1}\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_3}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right|>\rho\right) \\
&\le \int_{2^{-K+4}}^{\rho^{-1/(2+\alpha)}+2^{-K-1}} \frac{x_1}{2}+2^{-K}\,dx_1 \\
&\quad + C\rho^{-\frac{1}{1+\alpha-\beta}}\int_{\rho^{-1/(2+\alpha)}+2^{-K-1}}^{\pi}(x_1-2^{-K-1})^{-\frac{1+\beta}{1+\alpha-\beta}}\,dx_1 \\
&\le C\rho^{-2/(2+\alpha)}+C\rho^{-\frac{1}{1+\alpha-\beta}}\rho^{\frac{-1}{2+\alpha}(-\frac{1+\beta}{1+\alpha-\beta}+1)} \\
&= C\rho^{-2/(2+\alpha)}.
\end{aligned}
$$

Similarly, by (2.5.7) and (2.5.9),

$$
\begin{aligned}
&\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_8}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right| \\
&\quad\le C1_{\{\pi/2-2^{-K-1}<x_2<\pi-2^{-K+4}\}}1_{\{(\pi+x_2)/2-2^{-K}<x_1<\pi-2^{-K+1}\}} \\
&\qquad (\pi-x_2-2^{-K-1})^{-1-\beta}(\pi-x_1-2^{-K-1})^{\beta-\alpha-1}.
\end{aligned}
$$

If this is greater than ρ, then

$$
\begin{aligned}
&1_{\{(\pi+x_2)/2-2^{-K}<x_1<\pi-2^{-K+1}\}}(\pi-x_1-2^{-K-1}) \\
&\quad< C\rho^{-\frac{1}{1+\alpha-\beta}}1_{\{\pi/2-2^{-K-1}<x_2<\pi-2^{-K+4}\}}(\pi-x_2-2^{-K-1})^{-\frac{1+\beta}{1+\alpha-\beta}}.
\end{aligned}
$$

Here $(\pi-x_2)/2+2^{-K}<\pi-x_2$. Choosing β as before, we obtain

$$
\begin{aligned}
&\lambda\left(\sup_{n\ge 1}\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_8}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right|>\rho\right) \\
&\le \int_{\pi-\rho^{-1/(2+\alpha)}-2^{-K-1}}^{\pi-2^{-K+4}} \frac{\pi-x_2}{2}+2^{-K}\,dx_2
\end{aligned}
$$

$$
\begin{aligned}
&+ C\rho^{-\frac{1}{1+\alpha-\beta}} \int_{-\pi}^{\pi-\rho^{-1/(2+\alpha)}-2^{-K-1}} (\pi - x_2 - 2^{-K-1})^{-\frac{1+\beta}{1+\alpha-\beta}} \, dx_2 \\
&\le C\rho^{-2/(2+\alpha)} + C\rho^{-\frac{1}{1+\alpha-\beta}} \rho^{\frac{-1}{2+\alpha}(-\frac{1+\beta}{1+\alpha-\beta}+1)} \\
&= C\rho^{-2/(2+\alpha)}.
\end{aligned}
$$

For A_4, we get from (2.5.10) and (2.5.12) that

$$
\begin{aligned}
&\left| \int_I a(t_1, t_2) K_n^{1,\alpha}(x_1 - t_1, x_2 - t_2) 1_{A_4}(x_1 - t_1, x_2 - t_2) \, dt_1 \, dt_2 \right| \\
&\quad \le C 1_{\{2^{-K+4} < x_1 < \pi + 2^{-K-1}\}} 1_{\{x_1/2 - 2^{-K} < x_2 \le x_1 - 2^{-K+1}\}} \\
&\qquad (x_1 - x_2 - 2^{-K})^{-1-\beta} (x_1 - 2^{-K-1})^{\beta-\alpha-1}.
\end{aligned}
$$

If this is greater than ρ, then

$$
\begin{aligned}
&1_{\{x_1/2 - 2^{-K} < x_2 \le x_1 - 2^{-K+1}\}} (x_1 - x_2 - 2^{-K}) \\
&\qquad < C\rho^{-\frac{1}{1+\beta}} 1_{\{2^{-K+4} < x_1 < \pi + 2^{-K-1}\}} (x_1 - 2^{-K-1})^{\frac{\beta-\alpha-1}{1+\beta}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\lambda \left(\sup_{n \ge 1} \left| \int_I a(t_1, t_2) K_n^{1,\alpha}(x_1 - t_1, x_2 - t_2) 1_{A_4}(x_1 - t_1, x_2 - t_2) \, dt_1 \, dt_2 \right| > \rho \right) \\
&\le \int_{2^{-K+4}}^{\rho^{-1/(2+\alpha)}+2^{-K-1}} x_1 \, dx_1 \\
&\quad + C\rho^{-\frac{1}{1+\beta}} \int_{\rho^{-1/(2+\alpha)}+2^{-K-1}}^{\pi} (x_1 - 2^{-K-1})^{\frac{\beta-\alpha-1}{1+\beta}} \, dx_1 \\
&\le C\rho^{-2/(2+\alpha)} + C\rho^{-\frac{1}{1+\beta}} \rho^{\frac{-1}{2+\alpha}(\frac{\beta-\alpha-1}{1+\beta}+1)} \\
&= C\rho^{-2/(2+\alpha)}.
\end{aligned}
$$

Here we have chosen β such that $\frac{\beta-\alpha-1}{1+\beta} + 1 < 0$, i.e., $0 < \beta < \alpha/2$.

Finally, by (2.5.11) and (2.5.13),

$$
\begin{aligned}
&\left| \int_I a(t_1, t_2) K_n^{1,\alpha}(x_1 - t_1, x_2 - t_2) 1_{A_9}(x_1 - t_1, x_2 - t_2) \, dt_1 \, dt_2 \right| \\
&\quad \le C 1_{\{\pi/2 - 2^{-K-1} < x_2 < \pi - 2^{-K+4}\}} 1_{\{x_2 + 2^{-K+1} < x_1 < (\pi + x_2)/2 + 2^{-K}\}} \\
&\qquad (x_1 - x_2 - 2^{-K})^{-1-\beta} (\pi - x_2 - 2^{-K-1})^{\beta-\alpha-1}
\end{aligned}
$$

and

$$1_{\{x_2+2^{-K+1}<x_1<(\pi+x_2)/2+2^{-K}\}}(x_1 - x_2 - 2^{-K})$$
$$< C\rho^{-\frac{1}{1+\beta}} 1_{\{\pi/2-2^{-K-1}<x_2<\pi-2^{-K+4}\}}(\pi - x_2 - 2^{-K-1})^{\frac{\beta-\alpha-1}{1+\beta}}.$$

This implies that

$$x_2 + 2^{-K+1} < x_1 < (\pi - x_2 - 2^{-K-1})^{\frac{\beta-\alpha-1}{1+\beta}} + x_2 + 2^{-K+1}$$

and so

$$\begin{aligned}
&\lambda\left(\sup_{n\ge 1}\left|\int_I a(t_1,t_2)K_n^{1,\alpha}(x_1-t_1,x_2-t_2)1_{A_9}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right| > \rho\right)\\
&\le \int_{\pi-\rho^{-1/(2+\alpha)}-2^{-K-1}}^{\pi-2^{-K+4}} \frac{\pi - x_2}{2} + 2^{-K}\,dx_2\\
&\quad + C\rho^{-\frac{1}{1+\beta}}\int_{-\pi}^{\pi-\rho^{-1/(2+\alpha)}-2^{-K-1}} (\pi - x_2 - 2^{-K-1})^{\frac{\beta-\alpha-1}{1+\beta}}\,dx_2\\
&\le C\rho^{-2/(2+\alpha)} + C\rho^{-\frac{1}{1+\beta}}\rho^{\frac{-1}{2+\alpha}(\frac{\beta-\alpha-1}{1+\beta}+1)}\\
&= C\rho^{-2/(2+\alpha)}
\end{aligned}$$

with the same β as for A_4, i.e., $0 < \beta < \alpha/2$. The proof of the theorem is complete. ■

***Proof of Theorem 2.5.7* for** $q = \infty$. Similar to the proof for $q = 1$, we have to show that

$$\lambda\left(\sup_{n\ge 1}\left|\int_I a(t_1,t_2)K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2)1_{A_i}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right| > \rho\right)$$
$$\le C\rho^{-2/(2+\alpha)}$$

for $\alpha \le 1$, $i = 1,\dots,5$, for all $H_{2/(2+\alpha)}^{\Box}$-atoms a and $\rho > 0$. For $i = 1,2,5$, this inequality follows from (2.5.17) and the proof of Theorem 2.5.4. For $i = 3,4$, we may suppose that $\gamma = \alpha$ and $p = 2/(2+\alpha)$. We have seen in (2.5.6) and (2.5.8) that

$$\left|\int_I a(t_1,t_2)K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2)1_{A_3}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right|$$
$$\le C1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}}1_{\{2^{-K+1}<x_2\le x_1/2+2^{-K}\}}$$
$$(x_1 - 2^{-K-1})^{-1-\alpha+\beta}(x_2 - 2^{-K-1})^{-1-\beta}.$$

If this is greater than ρ, then

$$
\begin{aligned}
&1_{\{2^{-K+1}<x_2\le x_1/2+2^{-K}\}}(x_2-2^{-K-1}) \\
&\qquad < C\rho^{-\frac{1}{1+\beta}} 1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}}(x_1-2^{-K-1})^{-\frac{1+\alpha-\beta}{1+\beta}}.
\end{aligned}
$$

Since $x_1/2+2^{-K} < x_1$ and β can be chosen such that $-\frac{1+\alpha-\beta}{1+\beta}+1<0$, i.e., $0<\beta<\alpha/2$, we obtain

$$
\begin{aligned}
&\lambda\left(\sup_{n\ge 1}\left|\int_I a(t_1,t_2)K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2)1_{A_3}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right|>\rho\right) \\
&\le \int_{2^{-K+4}}^{\rho^{-1/(2+\alpha)}+2^{-K-1}} \frac{x_1}{2}+2^{-K}\,dx_1 \\
&\qquad + C\rho^{-\frac{1}{1+\beta}}\int_{\rho^{-1/(2+\alpha)}+2^{-K-1}}^{\pi}(x_1-2^{-K-1})^{-\frac{1+\alpha-\beta}{1+\beta}}\,dx_1 \\
&\le C\rho^{-2/(2+\alpha)} + C\rho^{-\frac{1}{1+\beta}}\rho^{\frac{-1}{2+\alpha}(-\frac{1+\alpha-\beta}{1+\beta}+1)} \\
&= C\rho^{-2/(2+\alpha)}.
\end{aligned}
$$

Similarly, by (2.5.14) and (2.5.15),

$$
\begin{aligned}
&\left|\int_I a(t_1,t_2)K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2)1_{A_4}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right| \\
&\qquad \le C_p 2^{2K/p-2K-K\gamma}1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}}1_{\{x_1/2-2^{-K}<x_2\le x_1-2^{-K+1}\}} \\
&\qquad\quad (x_1-x_2-2^{-K})^{-1-\alpha+\beta}(x_1-2^{-K-1})^{-1-\beta},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&1_{\{x_1/2-2^{-K}<x_2\le x_1-2^{-K+1}\}}(x_1-x_2-2^{-K}) \\
&\qquad < C\rho^{-\frac{1}{1+\alpha-\beta}}1_{\{2^{-K+4}<x_1<\pi+2^{-K-1}\}}(x_1-2^{-K-1})^{-\frac{1+\beta}{1+\alpha-\beta}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\lambda\left(\sup_{n\ge 1}\left|\int_I a(t_1,t_2)K_n^{\infty,\alpha}(x_1-t_1,x_2-t_2)1_{A_4}(x_1-t_1,x_2-t_2)\,dt_1\,dt_2\right|>\rho\right) \\
&\le \int_{2^{-K+4}}^{\rho^{-1/(2+\alpha)}+2^{-K-1}} x_1\,dx_1 \\
&\qquad + C\rho^{-\frac{1}{1+\alpha-\beta}}\int_{\rho^{-1/(2+\alpha)}+2^{-K-1}}^{\pi}(x_1-2^{-K-1})^{-\frac{1+\beta}{1+\alpha-\beta}}\,dx_1 \\
&\le C\rho^{-2/(2+\alpha)} + C\rho^{-\frac{1}{1+\alpha-\beta}}\rho^{\frac{-1}{2+\alpha}(-\frac{1+\beta}{1+\alpha-\beta}+1)} \\
&= C\rho^{-2/(2+\alpha)},
\end{aligned}
$$

where $-\frac{1+\beta}{1+\alpha-\beta}+1<0$, i.e., $\alpha/2<\beta<\alpha$. This completes the proof of the theorem. ■

Of course, (2.5.16) cannot be true for $p<p_0$, i.e., $\sigma_*^{q,\alpha}$ is not bounded from $H_p^\square(\mathbb{T}^d)$ to the weak $L_{p,\infty}(\mathbb{T}^d)$ space for $p<p_0$. If the operator was bounded, then by interpolation (2.5.1) would hold for $p=p_0$, which contradicts Theorem 2.5.6.

Oswald [253] proved a similar theorem to Theorem 2.5.4 for the Riesz means of the Fourier transforms and for $q=\infty$. Theorems 2.5.4 and 2.5.7 can be found in Weisz [330, 339]. For a detailed proof of the multi-dimensional version, see [337, 338, 341, 344].

Marcinkiewicz [233] verified for two-dimensional Fourier series that the cubic (i.e., $q=\infty$) Fejér means of a function $f\in L\log L(\mathbb{T}^2)$ converge almost everywhere to f as $n\to\infty$. Later Zhizhiashvili [364, 366] extended this result to all $f\in L_1(\mathbb{T}^2)$ and to Cesàro means and Berens, Li and Xu [30] to $q=1$. The general convergence result can be found in [330, 337–339, 341].

The next corollary follows easily from Theorem 2.5.4.

Corollary 2.5.8 *Suppose that $q=1,\infty$ and $0<\alpha<\infty$. If $f\in L_1(\mathbb{T}^d)$, then*

$$\sup_{\rho>0}\rho\,\lambda(\sigma_*^{q,\alpha}f>\rho)\le C\|f\|_1.$$

The density argument of Marcinkiewicz and Zygmund implies

Corollary 2.5.9 *Suppose that $q=1,\infty$ and $0<\alpha<\infty$. If $f\in L_1(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty}\sigma_n^{q,\alpha}f=f \qquad a.e.$$

Proof Since the trigonometric polynomials are dense in $L_1(\mathbb{T}^d)$, the corollary follows from Theorem 1.3.6 and Corollary 2.5.8. ■

2.5.2 *Almost Everywhere Convergence for q = 2*

Theorem 2.5.10 *Suppose that $q=2$, $(d-1)/2<\alpha<\infty$ and $\gamma\in\mathbb{P}$. If*

$$p_0:=\frac{d}{d/2+\alpha+1/2}<p<\infty,$$

and $f\in H_p^\square(\mathbb{T}^d)$, then

$$\left\|\sigma_*^{q,\alpha,\gamma}f\right\|_p\le C_p\|f\|_{H_p^\square}.$$

Proof Let us choose $N\in\mathbb{N}$ such that $N<\alpha-(d-1)/2\le N+1$. As we mentioned in Sect. 2.4, we may suppose that the support of an atom a is a ball B with radius β, $2^{-K-1} < \beta \le 2^{-K}$ ($K \in \mathbb{N}$). Moreover, we may suppose that the center of B is zero, i.e., $B = B(0, \beta)$. Obviously,

$$\begin{aligned}
&\int_{\mathbb{T}^d \setminus (rB)} |\sigma_*^{2,\alpha,\gamma} a(x)|^p \, dx \\
&\le \sum_{i=4\lfloor d^{1/2} \rfloor - 1}^{\lfloor d^{1/2} 2^K \pi \rfloor - 1} \int_{B(0,(i+2)2^{-K}) \setminus B(0,(i+1)2^{-K}) \cap \mathbb{T}^d} \sup_{n \ge d^{1/2} 2^{K+1}} |\sigma_n^{2,\alpha,\gamma} a(x)|^p \, dx \\
&\quad + \sum_{i=4\lfloor d^{1/2} \rfloor - 1}^{\lfloor d^{1/2} 2^K \pi \rfloor - 1} \int_{B(0,(i+2)2^{-K}) \setminus B(0,(i+1)2^{-K}) \cap \mathbb{T}^d} \sup_{n < d^{1/2} 2^{K+1}} |\sigma_n^{2,\alpha,\gamma} a(x)|^p \, dx \\
&=: (A) + (B),
\end{aligned}$$

where $r = 8d^{1/2}$. Note that if $K \le 3$, then the integral is equal to 0.

We use Taylor's formula for $g_k(t) = \widehat{\theta_0}(n(x - 2k\pi - t))$:

$$g_k(t) = \sum_{l=0}^{N-1} \sum_{\|i\|_1 = l} \partial_1^{i_1} \ldots \partial_d^{i_d} g_k(0) \prod_{j=1}^d \frac{t_j^{i_j}}{i_j!} + \sum_{\|i\|_1 = N} \partial_1^{i_1} \ldots \partial_d^{i_d} g_k(\nu t) \prod_{j=1}^d \frac{t_j^{i_j}}{i_j!}$$

for some $0 < \nu < 1$. Here

$$\partial_1^{i_1} \ldots \partial_d^{i_d} g_k(t) = (-1)^{\|i\|_1} n^{\|i\|_1} \partial_1^{i_1} \ldots \partial_d^{i_d} \widehat{\theta_0}(n(x - 2k\pi - t)).$$

Using this with $t - 2k\pi$ instead of t, Theorem 2.2.30 and the definition of the atom, we obtain

$$\begin{aligned}
\sigma_n^{2,\alpha,\gamma} a(x) &= \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} n^d \int_{B+2k\pi} a(t) \widehat{\theta_0}(n(x - t)) \, dt \\
&= \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} n^d \int_{B+2k\pi} a(t) \\
&\qquad \left(\widehat{\theta_0}(n(x - t)) - \sum_{l=0}^{N-1} \sum_{\|i\|_1 = l} \partial_1^{i_1} \ldots \partial_d^{i_d} g_k(0) \prod_{j=1}^d \frac{(t_j - 2k_j\pi)^{i_j}}{i_j!} \right) dt \\
&= \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} n^d \sum_{\|i\|_1 = N} (-1)^{\|i\|_1} n^{\|i\|_1} \int_{B+2k\pi} a(t) \\
&\qquad \partial_1^{i_1} \cdots \partial_d^{i_d} \widehat{\theta_0}\Big(n(x - 2k\pi) - n v_k (t - 2k\pi)\Big) \prod_{j=1}^d \frac{(t_j - 2k_j\pi)^{i_j}}{i_j!} \, dt,
\end{aligned}$$

where $0 < v_k < 1$. Then, by Corollary 2.2.28,

$$\left|\sigma_n^{2,\alpha,\gamma} a(x)\right| \le C_p \sum_{k\in\mathbb{Z}^d} n^{(d-1)/2+N-\alpha} 2^{Kd/p} 2^{-KN} \int_{B+2k\pi} \|x-2k\pi - v_k(t-2k\pi)\|_2^{-d/2-\alpha-1/2}\, dt. \tag{2.5.18}$$

Moreover,

$$\begin{aligned}\sup_{n\ge d^{1/2}2^{K+1}} \left|\sigma_n^{2,\alpha,\gamma} a(x)\right| &\le C_p \sum_{k\in\mathbb{Z}^d} 2^{K((d-1)/2-\alpha)} 2^{Kd/p} \int_{B+2k\pi} \|x-2k\pi - v_k(t-2k\pi)\|_2^{-d/2-\alpha-1/2}\, dt \\ &=: A_1(x) + A_2(x),\end{aligned}$$

where

$$A_1(x) := 2^{K((d-1)/2-\alpha)} 2^{Kd/p} \int_{B+2k\pi} \|x - v_0 t\|_2^{-d/2-\alpha-1/2}\, dt$$

and

$$A_2(x) := \sum_{k\in\mathbb{Z}^d, k\ne 0} 2^{K((d-1)/2-\alpha)} 2^{Kd/p} \int_{B+2k\pi} \|x-2k\pi - v_k(t-2k\pi)\|_2^{-d/2-\alpha-1/2}\, dt.$$

If $k=0$, $u\in B$ and $x\in B(0,(i+2)2^{-K})\setminus B(0,(i+1)2^{-K})\cap\mathbb{T}^d$ for some $i = 4\lfloor d^{1/2}\rfloor - 1, \ldots, \lfloor d^{1/2}2^K\pi\rfloor - 1$, then

$$\|x-u\|_2 \ge \|x\|_2 - \|u\|_2 \ge i2^{-K}.$$

In case $k\ne 0$, $u\in B+2k\pi$ and $x\in B(0,(i+2)2^{-K})\setminus B(0,(i+1)2^{-K})\cap\mathbb{T}^d$, one can see that

$$\|x-u\|_2 \ge \|k\|_2/4.$$

Then

$$\begin{aligned}A_1(x) &\le C_p 2^{K((d-1)/2-\alpha)} 2^{Kd/p} \int_B (i2^{-K})^{-d/2-\alpha-1/2}\, dt \\ &\le C_p 2^{Kd/p} i^{-d/2-\alpha-1/2}\end{aligned}$$

and

$$
\begin{aligned}
A_2(x) &\le C_p \sum_{k\in\mathbb{Z}^d, k\neq 0} 2^{K((d-1)/2-\alpha)} 2^{Kd/p} \int_{B+2k\pi} \|k\|_2^{-d/2-\alpha-1/2}\, dt \\
&\le C_p \sum_{k\in\mathbb{Z}^d, k\neq 0} 2^{K(-d/2-1/2-\alpha)} 2^{Kd/p} \|k\|_2^{-d/2-1/2-\alpha} \\
&\le C_p \sum_{j=1}^{\infty} 2^{K(-d/2-1/2-\alpha)} 2^{Kd/p} j^{(-d/2-1/2-\alpha)} j^{d-1} \\
&\le C_p
\end{aligned}
$$

for $p \ge d/(d/2+\alpha+1/2)$. Hence,

$$
(A) \le C_p \sum_{i=4\lfloor d^{1/2}\rfloor-1}^{\lfloor d^{1/2}2^K\pi\rfloor-1} 2^{-Kd} i^{d-1} 2^{Kd} i^{p(-d/2-\alpha-1/2)} + C_p \sum_{i=4\lfloor d^{1/2}\rfloor-1}^{\lfloor d^{1/2}2^K\pi\rfloor-1} 2^{-Kd} i^{d-1} \le C_p
$$

if $p > d/(d/2+\alpha+1/2)$.

Applying Taylor's formula for $N+1$ instead of N, we get similar to (2.5.18) that

$$
\left|\sigma_n^{2,\alpha,\gamma} a(x)\right| \le C_p \sum_{k\in\mathbb{Z}^d} n^{(d-1)/2+(N+1)-\alpha} 2^{Kd/p} 2^{-K(N+1)} \int_{B+2k\pi} \|x-2k\pi - v_k(t-2k\pi)\|_2^{-d/2-\alpha-1/2}\, dt
$$

and

$$
\sup_{n<d^{1/2}2^{K+1}} \left|\sigma_n^{2,\alpha,\gamma} a(x)\right| \le C_p \sum_{k\in\mathbb{Z}^d} 2^{K((d-1)/2-\alpha)} 2^{Kd/p} \int_{B+2k\pi} \|x-2k\pi - v_k(t-2k\pi)\|_2^{-d/2-\alpha-1/2}\, dt.
$$

The inequality

$$
(B) \le C_p
$$

can be shown as above. ■

Corollary 2.5.11 *Suppose that* $q=2$, $(d-1)/2 < \alpha < \infty$ *and* $\gamma \in \mathbb{P}$. *If* $1 < p < \infty$, *then*

$$
\|\sigma_*^{q,\alpha,\gamma} f\|_p \le C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).
$$

Theorem 2.5.10 was proved by Stein, Taibleson and Weiss [292] and Lu [224]. The author generalized it for other summability methods in Weisz [332, 334]. The theorem is not true if p is smaller than or equal to the critical index $d/(d/2+\alpha+1/2)$ (see Stein, Taibleson and Weiss [292]).

Theorem 2.5.12 *If $q = 2$, $(d-1)/2 < \alpha < \infty$ and $\gamma \in \mathbb{P}$, then the operator $\sigma_*^{q,\alpha,\gamma}$ is not bounded from $H_p^\Box(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ if p is smaller than or equal to the critical index $d/(d/2+\alpha+1/2)$.*

If p is equal to the critical index, then we have again a weak type inequality.

Theorem 2.5.13 *Suppose that $q = 2$, $(d-1)/2 < \alpha < \infty$ and $\gamma \in \mathbb{P}$. If*

$$p_0 := \frac{d}{d/2+\alpha+1/2}$$

and $f \in H_{p_0}^\Box(\mathbb{T}^d)$, then

$$\left\| \sigma_*^{q,\alpha,\gamma} f \right\|_{p_0,\infty} = \sup_{\rho>0} \rho \lambda(\sigma_*^{q,\alpha,\gamma} f > \rho)^{1/p_0} \le C \left\| f \right\|_{H_{p_0}^\Box}.$$

Proof We will use Theorem 2.4.20. Let us introduce the set

$$E_\rho := \left\{ i \ge 4\lfloor d^{1/2} \rfloor - 1 : i^{-d/2-\alpha-1/2} > C^{-1} \rho 2^{-Kd/p} \right\},$$

where $p = d/(d/2+\alpha+1/2)$. Observe that

$$\rho^p \, \lambda\Big(\{A_1 > \rho\} \cap \{\mathbb{T}^d \setminus (rB)\} \Big) \le C \rho^p \sum_{i \in E_\rho} i^{d-1} 2^{-Kd}.$$

If k is the largest integer for which $k^{-d/2-\alpha-1/2} > C^{-1} \rho 2^{-Kd/p}$, then

$$\rho^p \, \lambda\Big(\{A_1 > \rho\} \cap \{\mathbb{T}^d \setminus (rB)\} \Big) \le \rho^p 2^{-Kd} k^d \le C.$$

The same inequality for (A_2) is trivial. We can estimate $\sup_{n < d^{1/2} 2^{K+1}} |\sigma_n^{2,\alpha,\gamma} a(x)|$ similarly, which shows the theorem. ■

Corollary 2.5.14 *Suppose that $q = 2$, $(d-1)/2 < \alpha < \infty$ and $\gamma \in \mathbb{P}$. If $f \in L_1(\mathbb{T}^d)$, then*

$$\sup_{\rho>0} \rho \, \lambda(\sigma_*^{q,\alpha,\gamma} f > \rho) \le C \|f\|_1.$$

As in the previous subsection, this implies

Corollary 2.5.15 *Suppose that $q = 2$, $(d-1)/2 < \alpha < \infty$ and $\gamma \in \mathbb{P}$. If $f \in L_1(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty} \sigma_n^{q,\alpha,\gamma} f = f \qquad a.e.$$

2.6 ℓ_q-Summability Defined by a Function θ

Now we generalize the ℓ_q-Fejér and Riesz means investigated above. We introduce a general summability method, the so-called θ-summability generated by a given one-dimensional function θ.

We suppose that $\theta : \mathbb{R} \to \mathbb{R}$ and

$$\sum_{k\in\mathbb{Z}^d} \left|\theta\left(\frac{\|k\|_q}{n}\right)\right| < \infty \tag{2.6.1}$$

for all $n \in \mathbb{N}$. If θ has compact support, then this holds obviously. As we will see in Sect. 2.6.1, (2.6.2) implies (2.6.1).

Definition 2.6.1 Suppose that θ satisfies (2.6.1). For $f \in L_1(\mathbb{T}^d)$, $1 \le q \le \infty$ and $n \in \mathbb{N}$, the nth ℓ_q-θ-means $\sigma_n^{q,\theta} f$ of the Fourier series of f and the nth ℓ_q-θ kernel $K_n^{q,\theta}$ are defined by

$$\sigma_n^{q,\theta} f(x) := \sum_{k\in\mathbb{Z}^d} \theta\left(\frac{\|k\|_q}{n}\right) \widehat{f}(k) e^{\imath k\cdot x}$$

and

$$K_n^{q,\theta}(t) := \sum_{k\in\mathbb{Z}^d} \theta\left(\frac{\|k\|_q}{n}\right) e^{\imath k\cdot t},$$

respectively.

Lemma 2.6.2 *Suppose that θ satisfies (2.6.1). For $f \in L_1(\mathbb{T}^d)$ and $n \in \mathbb{N}$,*

$$\sigma_n^{q,\theta} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n^{q,\theta}(t)\, dt.$$

The definition of the ℓ_q-θ-means can be extended to distributions as usual.

Definition 2.6.3 Suppose that θ satisfies (2.6.1). For $f \in D(\mathbb{T}^d)$, $1 \le q \le \infty$ and $n \in \mathbb{N}$, the nth ℓ_q-θ-means $\sigma_n^{q,\theta} f$ of the Fourier series of f are given by

$$\sigma_n^{q,\theta} f := f * K_n^{q,\theta}.$$

Definition 2.6.4 We define the maximal θ-operator by

$$\sigma_*^{q,\theta} f := \sup_{n\in\mathbb{N}} \left|\sigma_n^{q,\theta} f\right|.$$

Note that $K_n^{q,\theta}$ is bounded and integrable. If $\theta(t) = \max((1-|t|^\gamma)^\alpha, 0)$, then we get back the Riesz (or in special case $\alpha = \gamma = 1$, the Fejér) means. θ-summability was considered in many papers and books, such as Butzer and Nessel [47], Trigub and Belinsky [319], Natanson and Zuk [244], Bokor, Schipp, Szili and Vértesi [38, 272, 274, 300, 301], and Feichtinger and Weisz [103, 104, 332, 337, 338, 342, 346].

2.6.1 Triangular and Cubic Summability

For $q = 1$ or ∞, instead of (2.6.1), we suppose that

$$\begin{cases} \text{the support of } \theta \text{ is } [-c, c] \ (0 < c \leq \infty), \\ \theta \text{ is even and continuous, } \theta(0) = 1, \\ \sum_{k=0}^{\infty} k^d \left| \Delta_1 \theta(\frac{k}{n}) \right| < \infty, \\ \lim_{t \to \infty} t^d \theta(t) = 0, \end{cases} \tag{2.6.2}$$

where

$$\Delta_1 \theta \left(\frac{k}{n} \right) := \theta \left(\frac{k}{n} \right) - \theta \left(\frac{k+1}{n} \right)$$

is the first difference. If the support of θ is not compact, then we say that $c = \infty$. Abel rearrangement implies

$$\sum_{j \in \mathbb{Z}^d} \left| \theta \left(\frac{\|j\|_q}{n} \right) \right| \leq C \sum_{k=0}^{\infty} k^{d-1} \left| \theta \left(\frac{k}{n} \right) \right| \leq C \sum_{k=0}^{\infty} k^d \left| \Delta_1 \theta \left(\frac{k}{n} \right) \right| < \infty,$$

thus (2.6.1) holds.

Lemma 2.6.5 *Suppose that θ satisfies (2.6.2). For $f \in L_1(\mathbb{T}^d)$, $q = 1, \infty$ and $n \in \mathbb{N}$, we have*

$$\sigma_n^{q,\theta} f(x) = \sum_{j=0}^{\infty} \Delta_1 \theta \left(\frac{j}{n} \right) s_j^q f(x).$$

and

$$K_n^{q,\theta}(t) = \sum_{j=0}^{\infty} \Delta_1 \theta \left(\frac{j}{n} \right) D_j^q(t)$$

Proof The proof follows from

$$K_n^{q,\theta}(t) = \sum_{k \in \mathbb{Z}^d} \sum_{j \geq \|k\|_q} \Delta_1 \theta \left(\frac{j}{n} \right) e^{\imath k \cdot t} = \sum_{j=0}^{\infty} \Delta_1 \theta \left(\frac{j}{n} \right) D_j^q(t)$$

■

We need also the following condition:

$$\begin{cases} \theta \text{ is twice continuously differentiable on} (0, c), \\ \theta'' \neq 0 \text{ except at finitely many points and finitely many intervals}, \\ \lim_{t \to 0+0} t\theta'(t) \text{ is finite}, \\ \lim_{t \to c-0} t\theta'(t) \text{ is finite}, \\ \lim_{t \to \infty} t\theta'(t) = 0. \end{cases} \tag{2.6.3}$$

The norm convergence follows easily from Theorem 2.6.7.

Theorem 2.6.6 *Assume that $q = 1$ or $q = \infty$ and (2.6.2) and (2.6.3) are satisfied. If $1 \leq p < \infty$, then*

$$\sup_{n \in \mathbb{N}} \left\| \sigma_n^{q,\theta} f \right\|_p \leq C \|f\|_p$$

and

$$\lim_{n \to \infty} \sigma_n^{q,\theta} f = f \qquad \text{in the } L_p(\mathbb{T}^d)\text{-norm for all } f \in L_p(\mathbb{T}^d).$$

For the almost everywhere convergence, we introduce some notations. Let $\mathbb{X}$ and $\mathbb{Y}$ be two complete quasi-normed spaces of measurable functions, $L_\infty(\mathbb{T}^d)$ be continuously embedded into $\mathbb{X}$ and $L_\infty(\mathbb{T}^d)$ be dense in $\mathbb{X}$. Suppose that if $0 \leq f \leq g$, $f, g \in \mathbb{Y}$, then $\|f\|_{\mathbb{Y}} \leq \|g\|_{\mathbb{Y}}$. If $f_n, f \in \mathbb{Y}$, $f_n \geq 0$ $(n \in \mathbb{N})$ and $f_n \nearrow f$ a.e. as $n \to \infty$, then assume that $\|f - f_n\|_{\mathbb{Y}} \to 0$. Recall that σ_*^q denotes the maximal Fejér operator.

Theorem 2.6.7 *Assume that $q = 1$ or $q = \infty$ and (2.6.2) and (2.6.3) are satisfied. If $\sigma_*^q : \mathbb{X} \to \mathbb{Y}$ is bounded, i.e.,*

$$\|\sigma_*^q f\|_{\mathbb{Y}} \leq C\|f\|_{\mathbb{X}} \qquad (f \in \mathbb{X} \cap L_\infty(\mathbb{T}^d)),$$

then $\sigma_*^{q,\theta}$ is also bounded,

$$\|\sigma_*^{q,\theta} f\|_{\mathbb{Y}} \leq C\|f\|_{\mathbb{X}} \qquad (f \in \mathbb{X}).$$

Proof By Abel rearrangement,

$$\sum_{k=0}^{m} \Delta_1\theta\left(\frac{k}{n}\right) D_k^q(x) = \sum_{k=0}^{m-1} \Delta_2\theta\left(\frac{k}{n}\right) kK_k^q(x) + \Delta_1\theta\left(\frac{m}{n}\right) mK_m^q(x),$$

where

$$\Delta_2\theta\left(\frac{k}{n}\right) := \Delta_1\theta\left(\frac{k}{n}\right) - \Delta_1\theta\left(\frac{k+1}{n}\right)$$

is the second difference and K_m^q denotes the Fejér kernel. Observe that for a fixed x, we have that $K_m^q(x)$ is uniformly bounded in m. By Lagrange's mean value theorem there exists $m < \xi(m) < m + 1$, such that

$$m \Delta_1 \theta\left(\frac{m}{n}\right) = -\frac{m}{n}\theta'\left(\frac{\xi(m)}{n}\right)$$

and this converges to zero if $m \to \infty$. Thus,

$$K_n^{q,\theta}(x) = \sum_{k=0}^{\infty} k\, \Delta_2\theta\left(\frac{k}{n}\right) K_k^q(x).$$

Now we prove that

$$\sup_{n\geq 1}\sum_{k=0}^{\infty} k \left|\Delta_2\theta\left(\frac{k}{n}\right)\right| \leq C < \infty. \tag{2.6.4}$$

If $\theta'' \geq 0$ on the interval $(i/n, (j+2)/n)$, then θ is convex on this interval and this yields that

$$\Delta_2\theta\left(\frac{k}{n}\right) \geq 0 \quad \text{for} \quad i \leq k \leq j.$$

Hence

$$\begin{aligned}\sum_{k=i}^{j} k \left|\Delta_2\theta\left(\frac{k}{n}\right)\right| &= \sum_{k=i}^{j} k\, \Delta_2\theta\left(\frac{k}{n}\right) \\ &= \theta\left(\frac{i}{n}\right) + (i-1)\,\Delta_1\theta\left(\frac{i}{n}\right) - \\ &\qquad j\,\Delta_1\theta\left(\frac{j+1}{n}\right) - \theta\left(\frac{j+1}{n}\right).\end{aligned}$$

Applying again Lagrange's mean value theorem, we have

$$(i-1)\left|\Delta_1\theta\left(\frac{i}{n}\right)\right| = \frac{i-1}{n}\left|\theta'\left(\frac{\xi(i)}{n}\right)\right| = \frac{i-1}{\xi(i)}\left|\frac{\xi(i)}{n}\theta'\left(\frac{\xi(i)}{n}\right)\right| \leq C,$$

where $i < \xi(i) < i+1$. Here, we used the fact that the function $x \mapsto |x\theta'(x)|$ is bounded, which follows from (2.6.3). If $\theta'' = 0$ at an isolated point u or if θ'' is not twice continuously differentiable at u, $u \in (k/n, (k+1)/n)$, then the boundedness of $k\left|\Delta_2\theta\left(\frac{k}{n}\right)\right|$ can be seen in the same way. Since there are only finitely many intervals and isolated points satisfying the above properties, we have shown (2.6.4).

Hence

$$\begin{aligned}\sigma_n^{q,\theta} f(x) &= \int_{\mathbb{T}^d} f(t) K_n^{q,\theta}(x-t)\,dt \\ &= \sum_{k=0}^{\infty}\int_{\mathbb{T}^d} k\,\Delta_2\theta\left(\frac{k}{n}\right) f(t) K_k^q(x-t)\,dt\end{aligned}$$

for all $f \in L_\infty(\mathbb{T}^d)$. Thus

$$\sigma_*^{q,\theta} f \le C \sigma_*^q f \qquad (f \in L_\infty(\mathbb{T}^d))$$

and so

$$\left\| \sigma_*^{q,\theta} f \right\|_{\mathbb{Y}} \le C \left\| f \right\|_{\mathbb{X}} \qquad (f \in \mathbb{X} \cap L_\infty(\mathbb{T}^d)).$$

By a usual density argument, we finish the proof of the theorem. ■

It is easy to see that $\mathbb{X}$ can be chosen to be the Hardy space $H_p^\square(\mathbb{T}^d)$ and $\mathbb{Y}$ to be the space $L_p(\mathbb{T}^d)$ or $L_{p,\infty}(\mathbb{T}^d)$ $(0 < p \le \infty)$. Theorems 2.6.7 and 2.5.4 imply

Theorem 2.6.8 *Assume that $q = 1$ or $q = \infty$ and (2.6.2) and (2.6.3) are satisfied. If*

$$\frac{d}{d+1} < p \le \infty,$$

then

$$\left\| \sigma_*^{q,\theta} f \right\|_p \le C_p \left\| f \right\|_{H_p^\square} \qquad (f \in H_p^\square(\mathbb{T}^d))$$

and, for $f \in H_{d/(d+1)}^\square(\mathbb{T}^d)$,

$$\left\| \sigma_*^{q,\theta} f \right\|_{d/(d+1),\infty} = \sup_{\rho>0} \rho \lambda(\sigma_*^{q,\theta} f > \rho)^{(d+1)/d} \le C \left\| f \right\|_{H_{d/(d+1)}^\square}.$$

Moreover,

$$\sup_{\rho>0} \rho\, \lambda(\sigma_*^{q,\theta} f > \rho) \le C \|f\|_1 \qquad (f \in L_1(\mathbb{T}^d)).$$

Corollary 2.6.9 *Assume that $q = 1$ or $q = \infty$ and (2.6.2) and (2.6.3) are satisfied. If $f \in L_1(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty} \sigma_n^{q,\theta} f = f \qquad a.e.$$

2.6.2 Circular Summability

If $q = 2$, then we have to assume other additional conditions instead of (2.6.2) and (2.6.3). Recall that

$$\theta_0(x) = \theta(\|x\|_2).$$

Let

$$\theta_0 \in L_1(\mathbb{R}^d) \qquad \text{and} \qquad \widehat{\theta_0} \in L_1(\mathbb{R}^d). \tag{2.6.5}$$

Assume that $\widehat{\theta_0}$ is $(N + 1)$-times differentiable $(N \ge 0)$ and there exists

$$d + N - 1 < \beta \leq d + N$$

such that

$$\left|\partial_1^{i_1} \cdots \partial_d^{i_d} \widehat{\theta_0}(x)\right| \leq C\|x\|_2^{-\beta-1} \qquad (x \neq 0), \tag{2.6.6}$$

whenever $i_1 + \cdots + i_d = N$ or $i_1 + \cdots + i_d = N + 1$. If $\beta = d + N$, then it is enough to suppose (2.6.6) for $i_1 + \cdots + i_d = N + 1$.

We recall that the Riesz summability, i.e., if $\theta(t) = \max((1 - |t|^\gamma)^\alpha, 0)$, satisfy (2.6.5) and (2.6.6) with $\beta = d/2 + \alpha - 1/2$ (see Corollary 2.2.28).

The norm convergence can be proved as Theorem 2.3.2.

Theorem 2.6.10 *Assume that $q = 2$, $\theta(0) = 1$ and (2.6.1) and (2.6.5) are satisfied. If $1 \leq p < \infty$, then*

$$\sup_{n \in \mathbb{N}} \left\|\sigma_n^{q,\theta} f\right\|_p \leq C\|f\|_p$$

and

$$\lim_{n\to\infty} \sigma_n^{q,\theta} f = f \qquad \text{in the } L_p(\mathbb{T}^d)\text{-norm for all } f \in L_p(\mathbb{T}^d).$$

We can prove the next theorem similar to Theorem 2.5.10. The details are left to the reader.

Theorem 2.6.11 *Assume that $q = 2$ and (2.6.1), (2.6.5) and (2.6.6) are satisfied. If*

$$\frac{d}{\beta + 1} < p \leq \infty,$$

then

$$\left\|\sigma_*^{q,\theta} f\right\|_p \leq C_p \|f\|_{H_p^\square} \qquad (f \in H_p^\square(\mathbb{T}^d))$$

and, for $f \in H_{d/(\beta+1)}^\square(\mathbb{T}^d)$,

$$\left\|\sigma_*^{q,\theta} f\right\|_{d/(\beta+1),\infty} = \sup_{\rho>0} \rho\lambda(\sigma_*^{q,\theta} f > \rho)^{(\beta+1)/d} \leq C \|f\|_{H_{d/(\beta+1)}^\square}.$$

Moreover,

$$\sup_{\rho>0} \rho\, \lambda(\sigma_*^{q,\theta} f > \rho) \leq C\|f\|_1 \qquad (f \in L_1(\mathbb{T}^d)).$$

Corollary 2.6.12 *Assume that $q = 2$, $\theta(0) = 1$ and (2.6.1), (2.6.5) and (2.6.6) are satisfied. If $f \in L_1(\mathbb{T}^d)$, then*

$$\lim_{n\to\infty} \sigma_n^{q,\theta} f = f \qquad a.e.$$

We note again, that (2.6.2) implies (2.6.1).

2.6.3 Some Summability Methods

Now we give some examples for the θ-summation.

Example 2.6.13 (Fejér summation). Let

$$\theta(t) = \begin{cases} 1 - |t| & \text{if } |t| \le 1; \\ 0 & \text{if } |t| > 1. \end{cases}$$

Example 2.6.14 (de La Vallée-Poussin summation). Let

$$\theta(t) = \begin{cases} 1 & \text{if if } |t| \le 1/2; \\ -2|t| + 2 & \text{if } 1/2 < |t| \le 1; \\ 0 & \text{if } |t| > 1. \end{cases}$$

Example 2.6.15 (Jackson-de La Vallée-Poussin summation). Let

$$\theta(t) = \begin{cases} 1 - 3t^2/2 + 3|t|^3/4 & \text{if } |t| \le 1; \\ (2 - |t|)^3/4 & \text{if } 1 < |t| \le 2; \\ 0 & \text{if } |t| > 2. \end{cases}$$

Example 2.6.16 Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m$ and $\beta_0, \ldots, \beta_m$ $(m \in \mathbb{N})$ be real numbers, $\beta_0 = 1$, $\beta_m = 0$. Suppose that θ is even, $\theta(\alpha_j) = \beta_j$ $(j = 0, 1, \ldots, m)$, $\theta(t) = 0$ for $t \ge \alpha_m$, θ is a polynomial on the interval $[\alpha_{j-1}, \alpha_j]$ $(j = 1, \ldots, m)$.

Example 2.6.17 (Rogosinski summation). Let

$$\theta(t) = \begin{cases} \cos \pi t/2 & \text{if } |t| \le 1 + 2j; \\ 0 & \text{if } |t| > 1 + 2j \end{cases} \qquad \text{for some } j \in \mathbb{N}.$$

Example 2.6.18 (Weierstrass summation). Let

$$\theta(t) = e^{-|t|^\gamma} \quad \text{for some} \quad 1 \le \gamma < \infty.$$

Note that if $\gamma = 1$, then we obtain the Abel means.

Example 2.6.19 Let

$$\theta(t) = e^{-(1+|t|^q)^\gamma} \quad \text{for some} \quad 1 \le q < \infty, 0 < \gamma < \infty.$$

Example 2.6.20 (Picard and Bessel summations). Let

$$\theta(t) = (1 + |t|^\gamma)^{-\alpha} \quad \text{for some} \quad 0 < \alpha < \infty, 1 \le \gamma < \infty, \alpha\gamma > d.$$

Example 2.6.21 (Riesz summation). Let

$$\theta(t) = \begin{cases} (1-|t|^\gamma)^\alpha & \text{if } |t| \le 1; \\ 0 & \text{if } |t| > 1 \end{cases}$$

for some $0 < \alpha, \gamma < \infty$.

It is easy to see that all of these examples satisfy (2.6.2) and (2.6.3).

Theorem 2.6.22 *Suppose that θ is one of the Examples 2.6.13–2.6.21. Then Theorems 2.6.6, 2.6.8 and Corollary 2.6.9 hold.*

One can show [334, 343] that Example 2.6.21 with $\alpha > (d-1)/2$, $\gamma \in \mathbb{P}$ and $\beta = d/2 + \alpha - 1/2$, Example 2.6.18 with $0 < \gamma < \infty$ and $\beta = d + N$, Example 2.6.19 with $0 < \gamma, q < \infty$ and $\beta = d + N$ and Example 2.6.20 with $\beta = d + N$ satisfy (2.6.2), (2.6.5) and (2.6.6).

Theorem 2.6.23 *Suppose that θ is one of the Examples 2.6.18, 2.6.19, 2.6.20 or 2.6.21 with the parameter β just defined. Then Theorems 2.6.10, 2.6.11 and Corollary 2.6.12 hold.*