Chapter 4
Classical and Weak Solutions, Relative Energy

The concept of weak (distributional) solution for general systems of conservation/balance laws was introduced in Sect. 1.1.1 and discussed in Chaps. 2, 3. Here, we revisit the topic in detail for the Euler and Navier–Stokes systems. In the former case, we show the class is possibly not closed with respect to the available weak topologies induced by *a priori* bounds; whence extension of this concept to a larger class of objects is of interest. A similar problem for the Navier–Stokes system is more subtle. Although, as we have seen in Chap. 3, the set of weak solutions is closed for certain models, the necessary estimates may not be easily available at the level of a numerical scheme. This is the main reason why to extend the class of weak solutions to some models of viscous fluids as well.

Given a time interval $(0, T)$ and a spatial domain $\Omega \subset R^d$, we say that a solution is *classical* if it is continuous in the closed set $[0, T] \times \overline{\Omega}$, and if all relevant partial derivatives exist and are continuous in the open set $(0, T) \times \Omega$. If the boundary conditions involve derivatives, then those must be continuously extendable from $(0, T) \times \Omega$ to $(0, T) \times \overline{\Omega}$. Thus this issue is inseparable from the geometric properties and, in particular, the smoothness of the boundary $\partial\Omega$. In particular, if the boundary conditions involve the outer normal vector, then the latter must exist at any point of $\partial\Omega$. The equations as well as the boundary and initial conditions are satisfied pointwise.

In the literature, the notion of *smooth* solution and/or *smooth* domain is frequently used. Very often, "smooth" in this context does not mean of class C^∞ or even analytic but should be interpreted as "sufficiently smooth" or "as smooth as necessary". We try to avoid this a bit dubious and misleading terminology. The term *strong* solution will be used in the situation when all required derivatives, expressed in terms of the theory of distributions, can be interpreted as (locally) integrable functions.

In this chapter, we focus on the relation between weak and strong solutions. In particular, we introduce the concept of *relative energy* functional. This quantity is derived from the total energy E of the system, and, if the latter is a convex function of the state variables, represents a *Bregman distance* with respect to the (convex)

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total energy E, see e.g. Sprung [185]. In particular, the relative energy can be used to measure the distance between a weak and strong solution starting from different (or identical) initial data. The relative energy vanishes if the data are the same for both solutions which is the desired weak-strong **(WS)** property. The relative energy may be seen as an alternative to the relative *entropy* functional introduced in the context of nonlinear conservation laws by Dafermos [59]. The approach based on the relative entropy requires integrability of the (total) energy flux. Therefore the method based on relative energy rather than entropy works efficiently in the context of weak and even more general dissipative solutions for the Euler and Navier–Stokes system, for which the available *a priori* bounds are not strong enough to render the total energy flux integrable.

Convexity of the total energy with respect to a suitable set of state variables plays a crucial role here that is intimately related to the property of *thermodynamic stability* of the fluid system. The interpretation of the notion of *thermodynamic stability* can be twofold: (1) certain material constants as compressibility and specific heat at constant volume are nonnegative, (2) equilibrium states are (linearly) stable, cf. Bechtel etal. [14]. We provide a unified approach identifying the relative energy with the Bregman distance associated to total energy.

4.1 Weak and Strong Solutions to the Euler System

The Euler system has been introduced in Chap. 2. As we have seen, there are several possible choices of basic field variables and, accordingly, the formulation of the field equations:

- **Standard (or primitive) variables**. The mass density ϱ, the (absolute) temperature ϑ, the velocity $\boldsymbol{u}$.
- **Conservative variables**. The mass density ϱ, the (total) energy E, the momentum $\boldsymbol{m}$.
- **Conservative-entropy variables**. The mass density ϱ, the (total) entropy S, the momentum $\boldsymbol{m}$.

Of course, there are other (infinitely many) possibilities how to choose the set of independent field variables.

Recall that

$$\boldsymbol{m} = \varrho \boldsymbol{u}, \ E = \frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho} + \varrho e, \ S = \varrho s,$$

where the internal energy e, the pressure p, and the entropy s satisfy Gibbs equation

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right). \tag{4.1}$$

In (4.1), the symbol D denotes a differential (gradient) with respect to either $[\varrho, \vartheta]$, $[\varrho, S]$ or any other choice of the independent parameters.

The field equations written in the standard variables read:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \boldsymbol{u}) = 0 \text{ in } (0, T) \times \Omega; \tag{4.2}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p(\varrho, \vartheta) = 0 \text{ in } (0, T) \times \Omega; \tag{4.3}$$

$$\partial_t \left(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \varrho e(\varrho, \vartheta) \right) + \operatorname{div}_x \left[\left(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \varrho e(\varrho, \vartheta) + p(\varrho, \vartheta) \right) \boldsymbol{u} \right] = 0 \text{ in } (0, T) \times \Omega. \tag{4.4}$$

Furthermore we consider the impermeability boundary conditions

$$\boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \tag{4.5}$$

or, alternatively, the periodic boundary conditions $\Omega = \mathbb{T}^d$.

In some real world applications, for instance in meteorology, it is more convenient to express the pressure $p = p(\varrho, s)$ as a function of the density and the entropy and to rewrite the Euler system in the form

$$\partial_t \varrho + \operatorname{div}_x(\varrho \boldsymbol{u}) = 0 \text{ in } (0, T) \times \Omega; \tag{4.6}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p(\varrho, s) = 0 \text{ in } (0, T) \times \Omega; \tag{4.7}$$

$$\partial_t s + \nabla_x s \cdot \boldsymbol{u} = 0 \text{ in } (0, T) \times \Omega. \tag{4.8}$$

The entropy s satisfies the transport equation (4.8) and it is easy to see that

$$\partial_t Z(s) + \nabla_x Z(s) \cdot \boldsymbol{u} = 0 \text{ in } (0, T) \times \Omega \tag{4.9}$$

as soon Z is a continuously differentiable function. In particular, if $Z' > 0$, we may replace s by Z and write $p = p(\varrho, Z)$ obtaining a new problem in terms of $[\varrho, \boldsymbol{u}, Z]$. The limit case $Z \to$ const gives rise to the isentropic (barotropic) Euler system with $p = p(\varrho)$,

$$\partial_t \varrho + \operatorname{div}_x(\varrho \boldsymbol{u}) = 0 \text{ in } (0, T) \times \Omega; \tag{4.10}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p(\varrho) = 0 \text{ in } (0, T) \times \Omega. \tag{4.11}$$

Although the systems (4.2)–(4.4) and (4.6)–(4.9) are completely equivalent in the framework of classical solutions, for which $\varrho > 0$, $\vartheta > 0$, they give rise to qualitatively different *weak* formulations of the Euler system. We will discuss this issue later in this chapter.

4.1.1 Classical Solutions to the Euler System

In this section, we suppose that $\Omega \subset R^d$ is a bounded domain with a boundary of class C^1. In particular, the outer normal vector exists at any point $x \in \partial\Omega$.

Definition 4.1 **(Classical solution to the Euler system)** We say that a trio $[\varrho, \vartheta, \boldsymbol{u}]$ is a *classical solution* of the Euler system (4.2)–(4.4), with the impermeability boundary condition (4.5), if

$$\begin{aligned}
&\varrho \in C([0,T] \times \overline{\Omega}) \cap C^1((0,T) \times \Omega) \cap W^{1,\infty}((0,T) \times \Omega),\\
&\vartheta \in C([0,T] \times \overline{\Omega}) \cap C^1((0,T) \times \Omega) \cap W^{1,\infty}((0,T) \times \Omega),\\
&\boldsymbol{u} \in C([0,T] \times \overline{\Omega}; R^d) \cap C^1((0,T) \times \Omega; R^d) \cap W^{1,\infty}((0,T) \times \Omega; R^d);
\end{aligned}$$

$$0 < \underline{\varrho} \le \varrho(t,x),\ 0 < \underline{\vartheta} \le \vartheta(t,x) \text{ for any } t \in [0,T],\ x \in \overline{\Omega},$$
$$\boldsymbol{u}(t,x) \cdot \boldsymbol{n}(x) = 0 \text{ for any } t \in [0,T],\ x \in \partial\Omega;$$

and the equations (4.2)–(4.4) hold.

Remark 4.1 Here we have tacitly assumed that both $p = p(\varrho, \vartheta)$ and $e = e(\varrho, \vartheta)$ are continuously differentiable for ϱ and ϑ bounded below away from zero.

Remark 4.2 (Strong solutions) We speak about strong solutions if $[\varrho, \vartheta, \boldsymbol{u}]$ are required to be only (globally) Lipschitz continuous.

Remark 4.3 (Periodic boundary conditions, Lipschitz domain) The above definition can be modified in an obvious way to accommodate the periodic boundary conditions $\Omega = \mathbb{T}^d$, where, of course the issue of boundary regularity is irrelevant. We recall that $\mathbb{T}^d$ can be viewed as a smooth manifold over R^d without boundary.

The requirement of the existence of the outer normal vector at *any* boundary point of Ω is rather restrictive. Recall that Lipschitz domains admit an normal vector for a.a. boundary point $x \in \Omega$, where the latter is endowed with the standard $(d-1)$-Hausdorff measure. The above definition extends to this case in a direct manner. The impermeability or zero normal trace condition (4.5) can be also reformulated in a weak form

$$\int_\Omega \boldsymbol{u}(t,\cdot) \cdot \nabla_x \phi \ \mathrm{d}x + \int_\Omega \phi \mathrm{div}_x \boldsymbol{u}(t,\cdot) \ \mathrm{d}x = 0 \ \text{ for any } \ \phi \in C^1_c(R^d),\ t \in [0,T]. \tag{4.12}$$

Obviously, any classical solution of the Euler system written in the form (4.2)–(4.4) is also a classical solution of the entropy formulation (4.6)–(4.8) as long as the thermodynamic functions are interrelated through Gibbs' equation (4.1).

As we observed in Sect. 2.1.2, classical solutions develop shock singularities in a finite time for a fairly general class of initial data. They obviously violate the existence condition **(E)** postulated in the introductory section of this part. Their application in the numerical analysis is therefore limited to the regime, where their life span exceeds the time desired for prediction. A relevant example is the Euler system in the low Mach number regime, considered often in meteorology, where the fluid velocity is largely dominated by the speed of sound and the occurrence of shocks is not expected. Still we are very far from a rigorous proof of such a statement, in particular in the physically relevant 3*D*-case.

4.1.2 Weak (Distributional) Solutions to the Barotropic Euler System

We revisit our discussion concerning weak (distributional) solutions starting with the barotropic Euler system (4.10), (4.11), where the theory is quite simple, elegant, and self-contained. Formally, the definition is obtained via multiplying the field equations (4.10), (4.11) on a suitable *test* function and declaring the resulting identity to be a proper definition of the weak solution. Following these lines, we obtain, exactly as in Sect. 2.1.3,

$$\left[\int_\Omega \varrho \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho \partial_t \varphi + \varrho \boldsymbol{u} \cdot \nabla_x \varphi \right] \mathrm{d}x \, \mathrm{d}t \tag{4.13}$$

for any $0 \leq \tau \leq T$, and any $\varphi \in C^1([0,T] \times \overline{\Omega})$;

$$\left[\int_\Omega \varrho \boldsymbol{u} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x \, \mathrm{d}t \tag{4.14}$$

for any $0 \leq \tau \leq T$, and any $\boldsymbol{\varphi} \in C^1([0,T] \times \overline{\Omega}; R^d)$, $\boldsymbol{\varphi} \cdot \boldsymbol{n}|_{\partial\Omega} = 0$.

Given the recent state of the art discussed in Chap. 2, such a definition complies with the existence principle **(E)** as well as with the compatibility principle **(C)**. Specifically, the weak solutions satisfying "*only*" (4.13), (4.14) exist globally in time for any (continuous) initial data, cf. [1]. However, the problem is desperately ill-posed even in the class of smooth initial data (the reader may consult the literature collected in Sect. 2.4). In particular, the weak-strong uniqueness principle **(WS)** is violated unless suitable *admissibility conditions* are imposed. To identify the class of suitable admissible solutions, we append (4.13), (4.14) by the *energy inequality*

$$\left[\int_\Omega \left(\frac{1}{2} \varrho |\boldsymbol{u}|^2 + P(\varrho) \right) \ \mathrm{d}x \right]_{t=0}^{t=\tau} \leq 0 \tag{4.15}$$

for any $0 \leq \tau \leq T$, where P is the pressure potential satisfying

$$P'(\varrho)\varrho - P(\varrho) = p(\varrho).$$

It is easy to check the P is uniquely determined by p modulo a linear function of ϱ. On the other hand, as the boundary is impermeable, the total mass

$$M = \int_\Omega \varrho(t, x) \, \mathrm{d}x$$

is a constant of motion; whence (4.15) remains unchanged for any affine perturbation of P.

Boundedness of the total energy/mass in terms of the initial data is basically the only available source of *a priori* (stability) estimates for the Euler system. The weak point of the formulation (4.13)–(4.15) in terms of the standard variables is that the velocity $\boldsymbol{u}$ is not controlled by the energy on the (hypothetical) vacuum region where $\varrho = 0$. It is therefore more convenient to consider the conservative variables $[\varrho, \boldsymbol{m} = \varrho \boldsymbol{u}]$ which give rise to the concept of weak solution for the barotropic Euler system introduced in Definition 2.5 that we now reproduce for convenience:

Definition 4.2 **(Weak solution of barotropic Euler system)** A pair $[\varrho, \boldsymbol{m}]$ is an *admissible weak solution* of the barotropic Euler system (4.10), (4.11) with the impermeability boundary condition (4.5) in $(0, T) \times \Omega$ if the following holds:

- (**weak continuity**) the quantities $\varrho, \boldsymbol{m}$ belong to the class

$$\begin{aligned} &\varrho \in C_{\mathrm{weak}}([0, T]; L^\gamma(\Omega)), \ \varrho \geq 0 \text{ a.a. in } (0, T) \times \Omega \\ &\boldsymbol{m} \in C_{\mathrm{weak}}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)) \end{aligned} \tag{4.16}$$

  for some $\gamma > 1$;
- (**equation of continuity**) the integral identity

$$\left[\int_\Omega \varrho \varphi \, \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho \partial_t \varphi + \boldsymbol{m} \cdot \nabla_x \varphi \right] \, \mathrm{d}x \, \mathrm{d}t \tag{4.17}$$

  holds for any $0 \leq \tau \leq T$, and any test function $\varphi \in C^1([0, T] \times \overline{\Omega})$;

- (**momentum equation**) the integral identity

$$\left[\int_{\Omega} \boldsymbol{m}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^{\tau}\int_{\Omega}\left[\boldsymbol{m}\cdot\partial_t\boldsymbol{\varphi} + \left(1_{\varrho>0}\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\varrho}\right):\nabla_x\boldsymbol{\varphi} + p(\varrho)\mathrm{div}_x\boldsymbol{\varphi}\right]\,\mathrm{d}x\,\mathrm{d}t \tag{4.18}$$

  holds for any $0\le\tau\le T$, and any test function $\boldsymbol{\varphi}\in C^1([0,T]\times\overline{\Omega};R^d)$, $\boldsymbol{\varphi}\cdot\boldsymbol{n}|_{\partial\Omega}=0$;
- (**energy inequality**) the integral inequality

$$\left[\int_{\Omega}\left(\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho}+P(\varrho)\right)\psi\,\mathrm{d}x\right]_{t=0}^{t=\tau} \le \int_0^{\tau}\int_{\Omega}\left(\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho}+P(\varrho)\right)\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t \tag{4.19}$$

  holds for any test function $\psi\in C^1[0,T]$, $\psi\ge 0$.

Remark 4.4 (Initial data) In the above, we have assumed

$$\varrho(0,\cdot)=\varrho_0,\ \boldsymbol{m}(0,\cdot)=\boldsymbol{m}_0,$$

where ϱ_0, $\boldsymbol{m}_0$ are the initial data, with the initial energy

$$\int_{\Omega}\left[\frac{1}{2}\frac{|\boldsymbol{m}_0|^2}{\varrho_0}+P(\varrho_0)\right]\,\mathrm{d}x<\infty.$$

Here, the kinetic energy $\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho}$ is interpreted as a convex l.s.c. function via formula (2.60).

Keeping in mind the canonical example of the isentropic pressure $p(\varrho)=a\varrho^{\gamma}$, we suppose

$$\begin{aligned} &p\in C^1[0,\infty)\cap C^2(0,\infty),\ p(0)=0,\ p'(\varrho)>0 \text{ for } \varrho>0,\\ &0<\liminf_{\varrho\to\infty}\frac{p'(\varrho)}{\varrho^{\gamma-1}}\le\limsup_{\varrho\to\infty}\frac{p'(\varrho)}{\varrho^{\gamma-1}}<\infty \end{aligned} \tag{4.20}$$

for some $\gamma>1$. Seeing that

$$P''(\varrho)=\frac{p'(\varrho)}{\varrho} \text{ for } \varrho>0$$

we easily deduce that the pressure potential P is a strictly convex function of ϱ, and

$$|p(\varrho)| \lesssim \Big(c + P(\varrho)\Big) \text{ for a suitable } c > 0. \tag{4.21}$$

This means that all nonlinearities appearing in the momentum equation (4.19) are controlled in the L^1 norm by the total energy. More precisely, we have the following result.

Proposition 4.1 (Uniform stability estimates for barotropic Euler system) *Let $\Omega \subset R^d$ be a bounded domain. Let the pressure $p = p(\varrho)$ satisfy (4.20) with $\gamma > 1$. Suppose that $[\varrho, \boldsymbol{m}]$ is an admissible weak solution of the Euler system in the sense of Definition 4.2, with the initial data $[\varrho_0, \boldsymbol{m}_0]$ such that*

$$\int_\Omega \left[\frac{1}{2} \frac{|\boldsymbol{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \mathrm{d}x = E_0 < \infty. \tag{4.22}$$

Then

$$\begin{aligned} \sup_{t \in [0,T]} \left\| 1_{\varrho > 0} \frac{\boldsymbol{m} \otimes \boldsymbol{m}}{\varrho}(t, \cdot) \right\|_{L^1(\Omega; R^{d \times d})} &\leq c(E_0), \\ \sup_{t \in [0,T]} \|\varrho(t, \cdot)\|_{L^\gamma(\Omega)} \leq c(E_0), \quad \sup_{t \in [0,T]} \|p(\varrho)(t, \cdot)\|_{L^1(\Omega)} &\leq c(E_0), \\ \sup_{t \in [0,T]} \|\boldsymbol{m}(t, \cdot)\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)} &\leq c(E_0). \end{aligned} \tag{4.23}$$

Remark 4.5 Recall that the total energy is defined as a convex function of $[\varrho, \boldsymbol{m}]$ via (2.60). In particular, hypothesis (4.22) entails nonnegativity of the initial density as well as a compatibility condition of the initial momentum on the vacuum zone,

$$\varrho_0 \geq 0, \text{ and } \varrho_0 = 0 \Rightarrow \boldsymbol{m}_0 = 0 \text{ a.a. in } \Omega.$$

Proof Taking $\psi \equiv 1$ in the energy inequality (4.19) we obtain

$$\int_\Omega \left[\frac{1}{2} \frac{|\boldsymbol{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \, \mathrm{d}x \leq E_0 \text{ for any } \tau \in [0, T]. \tag{4.24}$$

Note that the above inequality is indeed valid for *any* $\tau \in [0, T]$ as the energy is convex and both ϱ and $\boldsymbol{m}$ are weakly continuous as functions of the time.

In view of the hypothesis (4.20), the bound (4.24) gives rise to all estimates claimed in (4.23). Indeed to see the last bound in (4.23) write

$$\boldsymbol{m} = 1_{\varrho > 0} \frac{\boldsymbol{m}}{\sqrt{\varrho}} \sqrt{\varrho};$$

whence, by Hölder's inequality,

$$\|\boldsymbol{m}\|_{L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d)} \leq \left\| 1_{\varrho>0} \frac{\boldsymbol{m}}{\sqrt{\varrho}} \right\|_{L^2(\Omega;R^d)} \|\sqrt{\varrho}\|_{L^{2\gamma}(\Omega)} \leq c(E_0).$$

As pointed out in Remark 4.5, boundedness of the kinetic energy given by (2.60) implies that $\boldsymbol{m} = 0$ a.a. on the vacuum set $\{\varrho = 0\}$. □

Proposition 4.1 may be seen as the first step towards the stability property **(S)**. The set of all weak solutions emanating from a bounded set of initial data remains bounded uniformly in time. This definitely implies stability in suitable L^p-topologies, however, oscillations and/or concentrations may still appear and the set of weak solutions in the sense of Definition 4.2 is not likely to be closed (sequentially stable). We discuss this issue in detail in the concluding part of this chapter.

4.1.3 Relative Energy for the Barotropic Euler System

Relative energy is a simple but extremely useful tool in the analysis of nonlinear systems. It may be seen as a variant of the concept of *Bregman distance (divergence)* known in convex analysis, where the underlying convex potential is the total energy. Closely related is the concept of *relative entropy* frequently used in the analysis of conservation laws. For technical reasons that become evident in the forthcoming part of this chapter, relative energy seems better adapted to the rather poor stability bounds available for the Euler system. The possibility to compute effectively the *time evolution* of the relative energy on the sole basis of the weak formulation of the Euler system is the key ingredient of the proof of weak-strong uniqueness **(WS)** property. The relative energy for the barotropic Euler system is formally defined as

$$E\left(\varrho, \boldsymbol{m} \middle| \widetilde{\varrho}, \widetilde{\boldsymbol{u}}\right) = \frac{1}{2}\left(\frac{|\boldsymbol{m}|^2}{\varrho} - 2\boldsymbol{m}\cdot\widetilde{\boldsymbol{u}} + \varrho|\widetilde{\boldsymbol{u}}|^2\right) + P(\varrho) - P'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - P(\widetilde{\varrho}). \tag{4.25}$$

Here $[\varrho, \boldsymbol{m}]$ represents an admissible weak solution of the barotropic Euler system, while $\widetilde{\varrho}, \widetilde{\boldsymbol{u}}$ play a role of "test" functions. The first expression is rather awkward and should be interpreted as follows: As the energy of the weak solution $[\varrho, \boldsymbol{m}]$ is finite, we have

$$\frac{1}{2}\left(\frac{|\boldsymbol{m}|^2}{\varrho} - 2\boldsymbol{m}\cdot\widetilde{\boldsymbol{u}} + \varrho|\widetilde{\boldsymbol{u}}|^2\right) = \frac{1}{2}1_{\varrho>0}\left(\frac{|\boldsymbol{m}|^2}{\varrho} - 2\boldsymbol{m}\cdot\widetilde{\boldsymbol{u}} + \varrho|\widetilde{\boldsymbol{u}}|^2\right) = 1_{\varrho>0}\frac{1}{2}\varrho|\boldsymbol{u} - \widetilde{\boldsymbol{u}}|^2, \tag{4.26}$$

where we have set $\boldsymbol{u} = \frac{\boldsymbol{m}}{\varrho}$ on the set where $\varrho > 0$. Thus if P is convex, meaning $p' \geq 0$, and $[\varrho, \boldsymbol{m}]$ represent an admissible weak solution, then $E(\varrho, \boldsymbol{m}|\widetilde{\varrho}, \widetilde{\boldsymbol{u}}) \geq 0$ a.a. in $(0, T) \times \Omega$.

Denoting

$$E(\varrho, \boldsymbol{m}) = \frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho} + P(\varrho)$$

we can be interpret the relative energy as the *Bregman distance*

$$E\left(\varrho, \boldsymbol{m} \,\middle|\, \widetilde{\varrho}, \widetilde{\boldsymbol{u}}\right) = \mathfrak{B}^{\xi}_{E}\left(\varrho, \boldsymbol{m} \middle| \widetilde{\varrho}, \widetilde{\boldsymbol{m}}\right) = E(\varrho, \boldsymbol{m}) - \xi \cdot (\varrho - \widetilde{\varrho}, \boldsymbol{m} - \widetilde{\boldsymbol{m}}) - E(\widetilde{\varrho}, \widetilde{\boldsymbol{m}}),$$
$$\widetilde{\boldsymbol{m}} = \widetilde{\varrho}\widetilde{\boldsymbol{u}},\ \xi \in \partial E(\widetilde{\varrho}, \widetilde{\boldsymbol{m}}),$$

associated to the convex potential $E = E[\varrho, \boldsymbol{m}]$, see e.g. Sprung [185]. Bregman distance is not symmetric, therefore not a proper metric. However, the following holds:

$$E\left(\varrho, \boldsymbol{m} \,\middle|\, \widetilde{\varrho}, \widetilde{\boldsymbol{u}}\right) \geq 0 \text{ and } E\left(\varrho, \boldsymbol{m} \,\middle|\, \widetilde{\varrho}, \widetilde{\boldsymbol{u}}\right) = 0 \ \Leftrightarrow \ \varrho = \widetilde{\varrho},\ \boldsymbol{m} = \widetilde{\varrho}\widetilde{\boldsymbol{u}}.$$

Remark 4.6 Strictly speaking, the relative energy should be written entirely in conservative variables,

$$E\left(\varrho, \boldsymbol{m}\middle|\widetilde{\varrho}, \widetilde{\boldsymbol{m}}\right) = \frac{1}{2}\left(\frac{|\boldsymbol{m}|^2}{\varrho} - 2\boldsymbol{m}\cdot\frac{\widetilde{\boldsymbol{m}}}{\widetilde{\varrho}} + \frac{\varrho}{\widetilde{\varrho}^2}|\widetilde{\boldsymbol{m}}|^2\right) + P(\varrho) - P'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - P(\widetilde{\varrho}),$$

which is consistent with its interpretation as Bregman distance. Considering the test functions in the standard variables, however, is more suitable in the applications, notably in the proof of weak-strong uniqueness.

Our goal is to compute

$$\left[\int_{\Omega} E\left(\varrho, \boldsymbol{m} \,\middle|\, \widetilde{\varrho}, \widetilde{\boldsymbol{u}}\right)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \left[\int_{\Omega}\left[\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho} + P(\varrho)\right]\,\mathrm{d}x\right]_{t=0}^{t=\tau} - \left[\int_{\Omega}\boldsymbol{m}\cdot\widetilde{\boldsymbol{u}}\,\mathrm{d}x\right]_{t=0}^{t=\tau}$$
$$+ \left[\int_{\Omega}\varrho\left(\frac{1}{2}|\widetilde{\boldsymbol{u}}|^2 - P'(\widetilde{\varrho})\right)\,\mathrm{d}x\right]_{t=0}^{t=\tau} + \left[\int_{\Omega}p(\widetilde{\varrho})\,\mathrm{d}x\right]_{t=0}^{t=\tau}.$$

Remarkably, all integrals on the right-hand side can be evaluated by means of the weak formulation as soon as $\varphi = \left(\frac{1}{2}|\widetilde{\boldsymbol{u}}|^2 - P'(\widetilde{\varrho})\right)$ can be taken as a test function in (4.17), and $\boldsymbol{\varphi} = \widetilde{\boldsymbol{u}}$ as a test function in (4.18). To this end, we impose an extra hypothesis

$$\widetilde{\boldsymbol{u}} \in C^1([0, T] \times \overline{\Omega}; R^d),\ \widetilde{\boldsymbol{u}} \cdot \boldsymbol{n}|_{\partial\Omega} = 0,\ \widetilde{\varrho} \in C^1([0, T] \times \overline{\Omega}),\ \widetilde{\varrho} > 0 \text{ in } [0, T] \times \overline{\Omega}. \tag{4.27}$$

Accordingly, we have

$$\left[\int_\Omega \left[\frac{1}{2} \frac{|\boldsymbol{m}|^2}{\varrho} + P(\varrho) \right] \, \mathrm{d}x \right]_{t=0}^{t=\tau} \leq 0, \tag{4.28}$$

$$\left[\int_\Omega \boldsymbol{m} \cdot \widetilde{\boldsymbol{u}} \, \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\boldsymbol{m} \cdot \partial_t \widetilde{\boldsymbol{u}} + \left(1_{\varrho>0} \frac{\boldsymbol{m} \otimes \boldsymbol{m}}{\varrho} \right) : \nabla_x \widetilde{\boldsymbol{u}} + p(\varrho) \mathrm{div}_x \widetilde{\boldsymbol{u}} \right] \, \mathrm{d}x \, \mathrm{d}t, \tag{4.29}$$

and

$$\begin{aligned} \left[\int_\Omega \varrho \left(\frac{1}{2} |\widetilde{\boldsymbol{u}}|^2 - P'(\widetilde{\varrho}) \right) \, \mathrm{d}x \right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega \left[\varrho \widetilde{\boldsymbol{u}} \cdot \partial_t \widetilde{\boldsymbol{u}} + \boldsymbol{m} \cdot \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\boldsymbol{u}} \right] \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_0^\tau \int_\Omega \left[\varrho P''(\widetilde{\varrho}) \partial_t \widetilde{\varrho} + P''(\widetilde{\varrho}) \boldsymbol{m} \cdot \nabla_x \widetilde{\varrho} \right] \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_0^\tau \int_\Omega \left[\varrho \widetilde{\boldsymbol{u}} \cdot \partial_t \widetilde{\boldsymbol{u}} + 1_{\varrho>0} \frac{\boldsymbol{m}}{\varrho} \cdot \varrho \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\boldsymbol{u}} \right] \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_0^\tau \int_\Omega \left[\varrho P''(\widetilde{\varrho}) \partial_t \widetilde{\varrho} + P''(\widetilde{\varrho}) \boldsymbol{m} \cdot \nabla_x \widetilde{\varrho} \right] \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

Regrouping several terms we obtain, after a straightforward manipulation, the *relative energy inequality* for the barotropic Euler system,

$$\begin{aligned} \left[\int_\Omega E\left(\varrho, \boldsymbol{m} \,\middle|\, \widetilde{\varrho}, \widetilde{\boldsymbol{u}} \right) \, \mathrm{d}x \right]_{t=0}^{t=\tau} &\leq - \int_0^\tau \int_\Omega 1_{\varrho>0} \varrho \nabla_x \widetilde{\boldsymbol{u}} \cdot \left(\frac{\boldsymbol{m}}{\varrho} - \widetilde{\boldsymbol{u}} \right) \cdot \left(\frac{\boldsymbol{m}}{\varrho} - \widetilde{\boldsymbol{u}} \right) \, \mathrm{d}x \, \mathrm{d}t \\ &- \int_0^\tau \int_\Omega \left[p(\varrho) - p'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - p(\widetilde{\varrho}) \right] \mathrm{div}_x \widetilde{\boldsymbol{u}} \, \mathrm{d}x \, \mathrm{d}t \\ &+ \int_0^\tau \int_\Omega \frac{1}{\widetilde{\varrho}} (\varrho \widetilde{\boldsymbol{u}} - \boldsymbol{m}) \cdot \left[\partial_t (\widetilde{\varrho} \widetilde{\boldsymbol{u}}) + \mathrm{div}_x \, (\widetilde{\varrho} \widetilde{\boldsymbol{u}} \otimes \widetilde{\boldsymbol{u}}) + \nabla_x p(\widetilde{\varrho}) \right] \mathrm{d}x \, \mathrm{d}t \\ &+ \int_0^\tau \int_\Omega \left[\left(1 - \frac{\varrho}{\widetilde{\varrho}} \right) p'(\widetilde{\varrho}) + \frac{1}{\widetilde{\varrho}} \widetilde{\boldsymbol{u}} \cdot (\boldsymbol{m} - \varrho \widetilde{\boldsymbol{u}}) \right] \left[\partial_t \widetilde{\varrho} + \mathrm{div}_x (\widetilde{\varrho} \widetilde{\boldsymbol{u}}) \right] \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{4.30}$$

Now, it is clear how the weak-strong uniqueness property **(WS)** can be deduced from (4.30). On condition that the problem admits a strong solution $\widetilde{\varrho}$, $\widetilde{\varrho} > 0$, and $\widetilde{\boldsymbol{m}} = \widetilde{\varrho}\widetilde{\boldsymbol{u}}$ belonging to the class (4.27), the quantities $\widetilde{\varrho}, \widetilde{\boldsymbol{u}}$ can be used as test functions in (4.30). As $\widetilde{\varrho}$ and $\widetilde{\boldsymbol{u}}$ represent a strong solution, the last two integrals vanish while the first two can be "absorbed" by the left-hand side via a Gronwall type argument. We postpone the proof to the end of this section. In Chap. 5 we actually show a more general result.

Going back to Definition 4.1, we observe that the class of strong solutions is slightly larger than (4.27). It is therefore desirable to extend validity of the relative energy inequality (4.30) to a larger class of functions. Given the rather limited available integrability of the weak solutions, the optimal result in this direction is to allow $\widetilde{\varrho}$ and $\widetilde{\boldsymbol{u}}$ to be only Lipschitz continuous. We need the following auxiliary result.

Lemma 4.1 *Let $\Omega \subset R^d$ be a bounded domain of class C^2. Suppose that*

$$\widetilde{\varrho} \in W^{1,\infty}((0,T) \times \Omega), \ \inf_{t \in (0,T), x \in \Omega} \widetilde{\varrho}(t,x) > 0,$$
$$\widetilde{\boldsymbol{u}} \in W^{1,\infty}((0,T) \times \Omega; R^d), \ \widetilde{\boldsymbol{u}} \cdot \boldsymbol{n}|_{\partial\Omega} = 0.$$

Then there exist sequences

$$\{\varrho_n\}_{n=1}^{\infty}, \ \varrho_n \in C^1([0,T] \times \overline{\Omega}), \ \inf_{x \in \Omega} \varrho_n(x) > 0 \text{ uniformly for } n = 1, 2, \dots,$$
$$\{\boldsymbol{u}_n\}_{n=1}^{\infty}, \ \boldsymbol{u}_n \in C^1([0,T] \times \overline{\Omega}; R^d), \ \boldsymbol{u}_n \cdot \boldsymbol{n}|_{\partial\Omega} = 0 \text{ for all } n = 1, 2, \dots$$

such that

$$\varrho_n \to \widetilde{\varrho} \text{ in } W^{1,p}((0,T) \times \Omega) \text{ for any } 1 \le p < \infty$$
$$\text{and weakly-(*) in } W^{1,\infty}((0,T) \times \Omega);$$
$$\boldsymbol{u}_n \to \widetilde{\boldsymbol{u}} \text{ in } W^{1,p}((0,T) \times \Omega; R^d) \text{ for any } 1 \le p < \infty,$$
$$\text{and weakly-(*) in } W^{1,\infty}((0,T) \times \Omega; R^d).$$

Moreover,

$$\|\nabla_x \varrho_n\|_{L^\infty((0,T)\times\Omega;R^d)} \le c\left(\Omega; \|\nabla_x \widetilde{\varrho}\|_{L^\infty((0,T)\times\Omega;R^d)}\right),$$
$$\|\nabla_x \boldsymbol{u}_n\|_{L^\infty((0,T)\times\Omega;R^{d\times d})} \le c\left(\Omega; \|\nabla_x \widetilde{\boldsymbol{u}}\|_{L^\infty((0,T)\times\Omega;R^{d\times d})}\right) \tag{4.31}$$

uniformly for $n \to \infty$.

Proof **Step 1:**

As $\widetilde{\varrho}$ and $\widetilde{\boldsymbol{u}}$ are globally Lipschitz on $[0,T] \times \overline{\Omega}$, they can be extended to the whole space R^{d+1} in such a way that

$$\widetilde{\varrho} \in W^{1,\infty}(R^{d+1}), \ \widetilde{\boldsymbol{u}} \in W^{1,\infty}(R^{d+1}; R^d).$$

Using the standard regularization procedure, we may construct a sequence $\varrho_n \in C^\infty(R^{d+1})$ such that

$$\|\varrho_n\|_{W^{1,\infty}(K)} \leq c(K) \text{ uniformly for } n \to \infty$$

$$\varrho_n \to \widetilde{\varrho} \text{ in } C(K),\ \partial_t \varrho_n \to \partial_t \widetilde{\varrho},\ \nabla_x \varrho_n \to \nabla_x \widetilde{\varrho} \text{ a.a. in } K$$

for any compact $K \subset R^d$, which completes the proof for $\widetilde{\varrho}$. As a matter of fact, this procedure can carried over on arbitrary bounded domain as the extension theorem holds.

Step 2:

Obviously, the same treatment can be applied to $\widetilde{\boldsymbol{u}}$, however, we have to preserve the property of zero normal trace for the approximate sequence. First, we observe that regularization in time can be done in the same way as above. To simplify the proof, we shall therefore suppose that $\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{u}}(x)$ is a function of x only.

As the boundary is of class C^2, there is an open neighborhood $\mathcal{U}$ of $\partial\Omega$ such that the distance function

$$d(x) = \text{dist}[x, \partial\Omega] \text{ is of class } C^2(\overline{\mathcal{U}}).$$

Next, write

$$\widetilde{\boldsymbol{u}} = \boldsymbol{v} + \boldsymbol{w}, \text{ where } \boldsymbol{v} \in W^{1,\infty} \cap C_c(\Omega; R^d),\ \boldsymbol{w} \in W^{1,\infty} \cap C_c(\overline{\Omega} \cap \mathcal{U}; R^d).$$

The function $\boldsymbol{v}$ can be approximated in the same way as in **Step 1**.

Finally, we write $\boldsymbol{w}$ as

$$\boldsymbol{w} = [\boldsymbol{w} \cdot \nabla_x d]\nabla_x d - \nabla_x d \times (\nabla_x d \times \boldsymbol{w}).$$

Note that $\nabla_x d$ is a unit vector of class $C^1(\mathcal{U}; R^d)$,

$$\nabla_x d(x) = -\boldsymbol{n}(x) \text{ for } x \in \partial\Omega. \tag{4.32}$$

Thus applying the approximation procedure of **Step 1** to $\boldsymbol{w}$, we obtain a sequence $\{\boldsymbol{w}_n\}_{n=1}^\infty$ of class C^1,

$$\nabla_x d \times (\nabla_x d \times \boldsymbol{w}_n) \to \nabla_x d \times (\nabla_x d \times \boldsymbol{w}) \text{ in } W^{1,p}(\Omega; R^d) \text{ for any } 1 \leq p < \infty,$$
$$\text{and weakly-(*) in } W^{1,\infty}(\Omega; R^d).$$

Obviously, $\nabla_x d \times (\nabla_x d \times \boldsymbol{w}_n) \cdot \boldsymbol{n}|_{\partial\Omega} = 0$ for any $n = 1, 2, \ldots$.

As the normal component of $\boldsymbol{w}$ vanishes on $\partial\Omega$, we have

$$w^N = (\boldsymbol{w} \cdot \nabla_x d) \text{ in } W_0^{1,\infty}(\mathcal{U} \cap \Omega).$$

Consequently, there is a sequence of smooth functions $W_n^N \in C_c^\infty(\mathcal{U} \cap \Omega)$ such that

$$w_n^N \to W^N \text{ in } W_0^{1,p}(\mathcal{U} \cap \Omega) \text{ and weakly-(*) in} W_0^{1,\infty}(\mathcal{U} \cap \Omega)$$

We therefore conclude

$$w_n^N \nabla_x d \to [\boldsymbol{w} \cdot \nabla_x d] \nabla_x d \text{ in } W_0^{1,p}(\mathcal{U} \cap \Omega; R^d) \text{ and weakly-(*) in} W_0^{1,\infty}(\mathcal{U} \cap \Omega; R^d).$$

□

4.1.3.1 Relative Energy Inequality

In view of Lemma 4.1, we may summarize the previous discussion in the following statement.

Theorem 4.1 (Relative energy inequality for barotropic Euler system) *Let $\Omega \subset R^d$, $d = 2, 3$, be a bounded domain of class C^2. Let ϱ, $\boldsymbol{m}$ be an admissible weak solution of the barotropic Euler system (4.10), (4.11) in $(0, T) \times \Omega$, with the impermeability boundary condition (4.5), in the sense of Definition 4.1, where $p \in C^1[0, \infty) \cap C^2(0, \infty)$.*

Let $\widetilde{\varrho}$, $\widetilde{\boldsymbol{u}}$ be (test) functions belonging to the class

$$\begin{aligned} &\widetilde{\varrho} \in W^{1,\infty}((0, T) \times \Omega), \quad \inf_{(t,x) \in (0,T) \times \Omega} \widetilde{\varrho}(t, x) > 0, \\ &\widetilde{\boldsymbol{u}} \in W^{1,\infty}((0, T) \times \Omega; R^d), \ \widetilde{\boldsymbol{u}} \cdot \boldsymbol{n}|_{\partial\Omega} = 0. \end{aligned} \tag{4.33}$$

Then the relative energy inequality

$$\begin{aligned} \left[\int_\Omega E\left(\varrho, \boldsymbol{m} \,\middle|\, \widetilde{\varrho}, \widetilde{\boldsymbol{u}}\right) \,\mathrm{d}x \right]_{t=0}^{t=\tau} &\leq - \int_0^\tau \int_\Omega 1_{\varrho > 0} \varrho \nabla_x \widetilde{\boldsymbol{u}} \cdot \left(\frac{\boldsymbol{m}}{\varrho} - \widetilde{\boldsymbol{u}} \right) \cdot \left(\frac{\boldsymbol{m}}{\varrho} - \widetilde{\boldsymbol{u}} \right) \,\mathrm{d}x \,\mathrm{d}t \\ &- \int_0^\tau \int_\Omega \left[p(\varrho) - p'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - p(\widetilde{\varrho}) \right] \mathrm{div}_x \widetilde{\boldsymbol{u}} \,\mathrm{d}x \,\mathrm{d}t \\ &+ \int_0^\tau \int_\Omega \frac{1}{\widetilde{\varrho}} (\varrho \widetilde{\boldsymbol{u}} - \boldsymbol{m}) \cdot \left[\partial_t (\widetilde{\varrho} \widetilde{\boldsymbol{u}}) + \mathrm{div}_x \left(\widetilde{\varrho} \widetilde{\boldsymbol{u}} \otimes \widetilde{\boldsymbol{u}} \right) + \nabla_x p(\widetilde{\varrho}) \right] \,\mathrm{d}x \,\mathrm{d}t \\ &+ \int_0^\tau \int_\Omega \left[\left(1 - \frac{\varrho}{\widetilde{\varrho}} \right) p'(\widetilde{\varrho}) + \frac{1}{\widetilde{\varrho}} \widetilde{\boldsymbol{u}} \cdot (\boldsymbol{m} - \varrho \widetilde{\boldsymbol{u}}) \right] \left[\partial_t \widetilde{\varrho} + \mathrm{div}_x (\widetilde{\varrho} \widetilde{\boldsymbol{u}}) \right] \,\mathrm{d}x \,\mathrm{d}t. \end{aligned} \tag{4.34}$$

holds for any $0 \leq \tau \leq T$.

Remark 4.7 The previous result can be extended to unbounded domains, and, obviously, to the case of periodic boundary conditions $\Omega = \mathbb{T}^d$.

Remark 4.8 As a matter of fact, the total energy need not be a nonincreasing function as required in (4.19). For the relative energy inequality to hold, it is enough that

$$\int_{\Omega} \left[\frac{1}{2} \frac{|\boldsymbol{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \, \mathrm{d}x \leq \int_{\Omega} \left[\frac{1}{2} \frac{|\boldsymbol{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \, \mathrm{d}x$$

for any $0 \leq \tau \leq T$.

4.1.3.2 Weak-Strong Uniqueness

The weak-strong uniqueness property **(WS)** is a straightforward corollary of the relative energy inequality established in Theorem 4.1. As a matter of fact, we establish a more general result later in Chap. 5.

Theorem 4.2 (Weak-strong uniqueness) *Let $\Omega \subset R^d$, $d = 2, 3$, be a bounded domain of class C^2. Let ϱ, $\boldsymbol{m}$ be an admissible weak solution of the barotropic Euler system* (4.10), (4.11) *in $(0, T) \times \Omega$, with the impermeability boundary condition* (4.5)*, in the sense of Definition 4.2. Let the pressure p satisfy the growth condition* (4.20). *Let $\widetilde{\varrho}$, $\widetilde{\boldsymbol{u}}$ be a strong solution of the same problem belonging to the class*

$$\widetilde{\varrho} \in W^{1,\infty}((0, T) \times \Omega), \quad \inf_{(t,x) \in (0,T) \times \Omega} \widetilde{\varrho}(t, x) > 0,$$
$$\widetilde{\boldsymbol{u}} \in W^{1,\infty}((0, T) \times \Omega; R^d), \ \widetilde{\boldsymbol{u}} \cdot \boldsymbol{n}|_{\partial\Omega} = 0,$$

and such that

$$\varrho(0, \cdot) = \widetilde{\varrho}(0, \cdot), \ \boldsymbol{m}(0, \cdot) = \widetilde{\varrho}(0, \cdot)\widetilde{\boldsymbol{u}}(0, \cdot).$$

Then

$$\varrho = \widetilde{\varrho}, \ \boldsymbol{m} = \widetilde{\varrho}\widetilde{\boldsymbol{u}} \text{ in } (0, T) \times \Omega.$$

Proof The proof is an easy application of the relative energy inequality (4.34) in combination with the standard Gronwall type argument. Indeed plugging the strong solution in (4.34) we have only to observe that

$$\begin{aligned} &\left| \int_{\Omega} 1_{\varrho > 0} \varrho \nabla_x \widetilde{\boldsymbol{u}} \cdot \left(\frac{\boldsymbol{m}}{\varrho} - \widetilde{\boldsymbol{u}} \right) \cdot \left(\frac{\boldsymbol{m}}{\varrho} - \widetilde{\boldsymbol{u}} \right) \, \mathrm{d}x \right| \\ &+ \left| \int_{\Omega} \left[p(\varrho) - p'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - p(\widetilde{\varrho}) \right] \mathrm{div}_x \widetilde{\boldsymbol{u}} \, \mathrm{d}x \right| \\ &\lesssim c \left(\|\nabla_x \widetilde{\boldsymbol{u}}\|_{L^\infty}, \|\widetilde{\varrho}\|_{L^\infty} \right) \int_{\Omega} E\left(\varrho, \boldsymbol{m} \,\middle|\, \widetilde{\varrho}, \widetilde{\boldsymbol{u}} \right) \, \mathrm{d}x. \end{aligned} \tag{4.35}$$

As the pressure satisfies (4.20), the pressure potential P is strictly convex; whence

$$|p(\varrho) - p'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - p(\widetilde{\varrho})| \leq c\,(\min \widetilde{\varrho}, \max \widetilde{\varrho}) \Big(P(\varrho) - P'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - P(\widetilde{\varrho}) \Big) \tag{4.36}$$

whenever

$$\varrho \in \left[\frac{1}{2} \min_{(0,T)\times\Omega} \widetilde{\varrho}, 2 \max_{(0,T)\times\Omega} \widetilde{\varrho} \right].$$

Next, by the same token,

$$\inf_{0 \leq \varrho \leq \frac{1}{2} \min_{(0,T)\times\Omega} \widetilde{\varrho}} \Big(P(\varrho) - P'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - P(\widetilde{\varrho}) \Big) > 0$$

and, by virtue of (4.21), the inequality (4.36) holds also in the regime $\varrho \geq 2 \max_{(0,T)\times\Omega} \widetilde{\varrho}$. Clearly (4.36) implies (4.35). □

Remark 4.9 (Domain regularity) The assumption on regularity of the spatial domain Ω may seem rather restrictive, in particular in numerical applications, where the underlying domain is typically a polygon. Note, however, that *existence* of a smooth solution requires similar restrictions. Fortunately, the problem is irrelevant in the case of periodic boundary conditions.

4.1.3.3 Relative Energy—Summary

With future applications of the method in mind, we conclude by collecting the principal steps of the proof of the weak-strong **(WS)** uniqueness principle:

- **Conservative state variables**. Identify a suitable set *conservative* state variables, the time evolution of which can be expressed in terms of the weak formulation. These quantities enjoy certain kind of continuity in the time variable.
- **Convex energy**. Express the total energy as a *convex* function of these state variables.
- **Relative energy**. Relative energy is the (integrated) *Bregman distance* associated to the energy between a weak solutions and suitable test functions. Its time evolution can be identified by means of the energy balance and the weak formulation of the field equations. The energy balance is *indispensable* at this stage; whence only admissible weak solutions are eligible.
- **Weak-strong uniqueness principle**. Use the strong solution as a test function in the relative energy inequality.

4.1.4 Weak (Distributional) Solutions to the Complete Euler System

The weak solutions to the complete Euler system have been introduced in Sect. 2.1.3. Unfortunately, however, the bounds provided by the energy balance Eq. (4.4), are not strong enough to control the energy flux

$$\left(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \varrho e\right)\boldsymbol{u} + p\boldsymbol{u}$$

in the space of integrable functions. This is a serious obstacle when dealing with sequences of approximate solutions since the limit in the energy equation is problematic. From this point of view, it seems more convenient to work with *entropy* rather than the *energy* balance. Indeed the renormalized entropy equation (4.9), written in the form

$$\partial_t\left(\varrho Z(s)\right) + \mathrm{div}_x\left(\varrho Z(s)\boldsymbol{u}\right) = 0, \tag{4.37}$$

requires only ϱ and $\boldsymbol{m} = \varrho\boldsymbol{u}$ to be integrable as soon as Z is a bounded function. We can therefore consider the system consisting of the equation of continuity (4.6), the momentum equation (4.7), and the entropy balance (4.37). Similarly to the barotropic case, the system can be appended by the total energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\left[\frac{1}{2}\varrho|\boldsymbol{u}|^2 + \varrho e\right]\,\mathrm{d}x \leq 0, \text{ with } e = e(\varrho, s),$$

as an *admissibility condition*. This approach is frequently used in meteorological models that describe fluids in the low Mach number regime, where occurrence of shock waves or other singularities is not expected; whence all formulations are equivalent.

In order to accommodate more general regimes of fluid motion, in particular those when the entropy is not conserved, we propose the following problem as a basis for the weak formulation

$$\partial_t\varrho + \mathrm{div}_x\boldsymbol{m} = 0, \tag{4.38}$$

$$\partial_t\boldsymbol{m} + \mathrm{div}_x\left(\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\varrho}\right) + \nabla_x p = 0, \tag{4.39}$$

$$\partial_t(\varrho Z(s)) + \mathrm{div}_x\left(Z(s)\boldsymbol{m}\right) \geq 0,\ Z'(s) \geq 0, \tag{4.40}$$

together with the admissibility condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\left[\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho} + \varrho e\right]\,\mathrm{d}x \leq 0, \tag{4.41}$$

where $p = p(\varrho, s)$, $e = e(\varrho, s)$ are determined by appropriate EOS. Of course, the energy balance (4.41) is conditioned by the impermeability of the boundary

$$\boldsymbol{m} \cdot \boldsymbol{n}|_{\partial\Omega} = 0. \tag{4.42}$$

The resulting system may be seen underdetermined as we have replaced the entropy *equation* (4.37) by *inequality* (4.40). Moreover, the total energy is no longer conserved which is at odds with the First law of thermodynamics encoded in the complete Euler system. However, as we shall see below, the compatibility property **(C)** as well as the weak-strong uniqueness principle **(WS)** remain in force.

The exact definition of *generalized weak solution* to the complete Euler system presented below can be seen as the first attempt of relaxation of the concept of "standard" weak solution. We formulate the problems in terms of ϱ, $\boldsymbol{m}$, and s for definiteness. Of course, other settings are possible and the modification of the definition is straightforward.

Definition 4.3 (Generalized weak solution to complete Euler system) A trio $[\varrho, \boldsymbol{m}, s]$ is called *generalized weak solution* of the complete Euler system (4.2)–(4.4) with the impermeability boundary condition (4.5) in $(0, T) \times \Omega$ if the following holds:

- (**weak continuity**) the quantities ϱ, $\boldsymbol{m}$ belong to the class

$$\begin{aligned} &\varrho \in C_{\rm weak}([0,T]; L^\gamma(\Omega)),\ \varrho \geq 0 \text{ a.a. in } (0,T)\times\Omega, \\ &\boldsymbol{m} \in C_{\rm weak}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)), \\ &\varrho Z(s) = S_1^Z + S_2^Z, \text{ where } S_1^Z \in C_{\rm weak}([0,T]; L^\gamma(\Omega)), \\ &\tau \mapsto \int_\Omega S_2^Z(\tau,\cdot)\phi \, {\rm d}x \text{ nondecreasing for any } \phi \in C^1(\overline{\Omega}),\ \phi \geq 0, \end{aligned} \tag{4.43}$$

  for some $\gamma > 1$, and for any $Z \in C^1(R)$, $Z' \geq 0$, Z concave, $Z(s) \leq \overline{Z}$ for any s;
- (**constitutive equations**) the pressure $p = p(\varrho, s)$ and the internal energy $e = e(\varrho, s)$ are determined by a given EOS,

$$p \in L^1((0,T)\times\Omega),\ \varrho e \in L^1((0,T)\times\Omega);$$

- (**equation of continuity**) the integral identity

$$\left[\int_\Omega \varrho\varphi \, {\rm d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho\partial_t\varphi + \boldsymbol{m}\cdot\nabla_x\varphi \right] {\rm d}x\,{\rm d}t \tag{4.44}$$

  holds for any $0 \leq \tau \leq T$, and any test function $\varphi \in C^1([0,T]\times\overline{\Omega})$;

- (**momentum equation**) the integral identity

$$\left[\int_\Omega \boldsymbol{m}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega \left[\boldsymbol{m}\cdot\partial_t\boldsymbol{\varphi} + \left(1_{\varrho>0}\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\varrho}\right) : \nabla_x\boldsymbol{\varphi} + p\mathrm{div}_x\boldsymbol{\varphi}\right]\ \mathrm{d}x\,\mathrm{d}t \tag{4.45}$$

holds for any $0 \le \tau \le T$, and any test function $\boldsymbol{\varphi} \in C^1([0,T]\times\overline{\Omega}; R^d)$, $\boldsymbol{\varphi}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$;
- (**entropy inequality**) the integral inequality

$$\left[\int_\Omega \varrho Z(s)\varphi\ \mathrm{d}x\right]_{t=0}^{t=\tau+} \ge \int_0^\tau\int_\Omega \left[\varrho Z(s)\partial_t\varphi + Z(s)\boldsymbol{m}\cdot\nabla_x\varphi\right]\ \mathrm{d}x\,\mathrm{d}t \tag{4.46}$$

holds for any $0 \le \tau < T$, any test function $\varphi \in C^1([0,T]\times\overline{\Omega})$, $\varphi \ge 0$, and any $Z \in C^1(R)$, $Z' \ge 0$, Z concave, $Z(s) \le \overline{Z}$ for any s;
- (**energy inequality**) the integral inequality

$$\left[\int_\Omega \left(\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho} + \varrho e\right)\ \mathrm{d}x\right]_{t=0}^{t=\tau} \le 0 \tag{4.47}$$

holds for a.a. $0 < \tau < T$.

Remark 4.10 The upper bound $\tau+$ in (4.46) can be replaced by $\tau-$ for any $0 < \tau \le T$. Note that, in view of (4.43) the one sided limits $\int_\Omega \varrho Z(s)\varphi(\tau\pm)\ \mathrm{d}x$ exist for any test function $\varphi \in C^1([0,T]\times\overline{\Omega})$.

We are ready to show the compatibility property **(C)**.

Proposition 4.2 (Compatibility) *Suppose that $\Omega \subset R^d$, $d = 2,3$ is a bounded domain with C^2 boundary. Let $[\varrho, \boldsymbol{m}, s]$ be an admissible weak solution of the Euler system in the sense of Definition 4.3. Let $\vartheta = \vartheta(\varrho, s)$ be given by the (implicit) constitutive relation $s(\varrho,\vartheta) = s$. Suppose that*

$$\mathrm{ess}\inf_{(0,T)\times\Omega}\varrho > 0,\ \ \mathrm{ess}\inf_{(0,T)\times\Omega}\vartheta > 0$$

and that ϱ, $\boldsymbol{u} = \frac{\boldsymbol{m}}{\varrho}$, ϑ belong to the regularity class of classical solutions specified in Definition 4.1.

Then $[\varrho,\vartheta,\boldsymbol{u}]$ is a classical solution in the sense of Definition 4.1.

Remark 4.11 Given $\varrho > 0$ and $s = s(\varrho,\vartheta)$, the temperature is uniquely determined by the value of the entropy s.

Proof As $\varrho > 0$ we may introduce the velocity $\boldsymbol{u}$ as well as the temperature ϑ (cf. Remark 4.11). Now it is standard to observe that the equation of continuity as well as the momentum equation are satisfied in the classical way:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \boldsymbol{u}) = 0, \tag{4.48}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p = 0. \tag{4.49}$$

Next, integrating (4.44) by parts and using (4.48), we obtain

$$\int_0^T \int_{\partial\Omega} \varphi \varrho \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{dS}_x \, \mathrm{d}t = 0$$

for any $\varphi \in C_c^1((0,T) \times \overline{\Omega})$. This yields

$$\boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} = 0. \tag{4.50}$$

Multiplying (4.49) on $\boldsymbol{u}$ and using (4.48) we derive the kinetic energy balance

$$\partial_t \left(\frac{1}{2} \varrho |\boldsymbol{u}|^2 \right) + \operatorname{div}_x \left(\frac{1}{2} \varrho |\boldsymbol{u}|^2 \boldsymbol{u} \right) + \operatorname{div}_x(p\boldsymbol{u}) = p \operatorname{div}_x \boldsymbol{u}. \tag{4.51}$$

Furthermore, by virtue of (4.48) and Gibbs' equation (4.1),

$$\frac{1}{\varrho} \partial_t \varrho + \frac{1}{\varrho} \boldsymbol{u} \nabla_x \varrho = -\operatorname{div}_x \boldsymbol{u}, \ \vartheta \frac{\partial s}{\partial \varrho} = \frac{\partial e}{\partial \varrho} - \frac{p}{\varrho^2};$$

whence

$$p \operatorname{div}_x \boldsymbol{u} = -\frac{p}{\varrho} \partial_t \varrho - \frac{p}{\varrho} \boldsymbol{u} \nabla_x \varrho = \varrho \vartheta \frac{\partial s}{\partial \varrho} \partial_t \varrho - \varrho \frac{\partial e}{\partial \varrho} \partial_t \varrho + \varrho \vartheta \frac{\partial s}{\partial \varrho} \boldsymbol{u} \cdot \nabla_x \varrho - \varrho \frac{\partial e}{\partial \varrho} \boldsymbol{u} \cdot \nabla_x \varrho. \tag{4.52}$$

Finally, we deduce from the equation of continuity (4.48) and the entropy inequality (4.46) that

$$\varrho \vartheta \left(\frac{\partial s}{\partial \varrho} \partial_t \varrho + \frac{\partial s}{\partial \vartheta} \partial_t \vartheta \right) + \varrho \vartheta \boldsymbol{u} \cdot \left(\frac{\partial s}{\partial \varrho} \nabla_x \varrho + \frac{\partial s}{\partial \vartheta} \nabla_x \vartheta \right) = \varrho \vartheta \partial_t s + \varrho \vartheta \boldsymbol{u} \cdot \nabla_x s \geq 0. \tag{4.53}$$

Combining (4.51)–(4.53) we may infer that

$$\partial_t \left(\frac{1}{2} \varrho |\boldsymbol{u}|^2 + \varrho e \right) + \operatorname{div}_x \left(\frac{1}{2} \varrho |\boldsymbol{u}|^2 \boldsymbol{u} + \varrho e \boldsymbol{u} \right) + \operatorname{div}_x(p\boldsymbol{u}) \geq 0.$$

This inequality, integrated over Ω and the compared with the energy inequality (4.47), yields the total energy balance (4.4). Of course, this step requires the impermeability boundary condition (4.50). □

4.1.5 Lower Bound on the Entropy

As shown in Proposition 2.1, an interesting consequence of the renormalized entropy inequality is a version of *minimum principle* for the entropy. The result transfer directly to the present setting.

Theorem 4.3 (Minimum entropy principle) *Let* $[\varrho, \boldsymbol{m}, s]$ *be a generalized weak solution to the complete Euler system in* $(0, T) \times \Omega$ *in the sense of Definition 4.3. Suppose that*

$$\varrho Z(s)(0, \cdot) = \varrho_0 Z(s_0), \ \textit{where} \int_\Omega \varrho_0 \, \mathrm{d}x > 0, \ s_0(x) \geq \underline{s} \ \textit{for a.a.}\ x \in \Omega.$$

Then

$$s(t, x) \geq \underline{s} \ \textit{a.a. in the set} \ \left\{ (t, x) \in (0, T) \times \Omega \ \middle| \ \varrho(t, x) > 0 \right\}.$$

4.1.6 Relative Energy for the Complete Euler System

Using the general principles introduced in Sect. 4.1.3, we identify the relative energy for the complete Euler system. We emphasize once more that the quantity we obtain is indeed a relative *energy* and not *entropy*. We give several definitions in terms of the standard, conservative, and entropy-conservative variables. We will finally identify the *entropy-conservative variables* as the only choice of phase variables that renders the total energy convex and agrees with all general principles stated at the end of Sect. 4.1.3. The key property to besides Gibbs' relation to be satisfied by the equation of state is the **hypothesis of thermodynamic stability** formulated in the *standard* variables as follows

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0. \tag{4.54}$$

The physical meaning of the former condition is positive *compressibility*, while the latter expresses *positivity of the specific heat at constant volume* of the fluid in question.

4.1.6.1 Relative Energy in the Standard Variables

For given $\widetilde{\vartheta} > 0$, we introduce the *ballistic energy* functional,

$$H_{\widetilde{\vartheta}}(\varrho, \vartheta) = \varrho \left(e(\varrho, \vartheta) - \widetilde{\vartheta} s(\varrho, \vartheta) \right).$$

Expressed in terms of the standard variables, the relative energy reads

$$E\left(\varrho, \vartheta, \boldsymbol{m} \Big| \widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\boldsymbol{u}}\right) = \frac{1}{2}\varrho \left| \frac{\boldsymbol{m}}{\varrho} - \widetilde{\boldsymbol{u}} \right|^2 + H_{\widetilde{\vartheta}}(\varrho, \vartheta) - \frac{\partial H_{\widetilde{\vartheta}}(\widetilde{\varrho}, \widetilde{\vartheta})}{\partial \varrho}(\varrho - \widetilde{\varrho}) - H_{\widetilde{\vartheta}}(\widetilde{\varrho}, \widetilde{\vartheta}). \tag{4.55}$$

Furthermore, we write

$$\begin{aligned} H_{\widetilde{\vartheta}}(\varrho, \vartheta) &- \frac{\partial H_{\widetilde{\vartheta}}(\widetilde{\varrho}, \widetilde{\vartheta})}{\partial \varrho}(\varrho - \widetilde{\varrho}) - H_{\widetilde{\vartheta}}(\widetilde{\varrho}, \widetilde{\vartheta}) \\ &= H_{\widetilde{\vartheta}}(\varrho, \vartheta) - H_{\widetilde{\vartheta}}(\varrho, \widetilde{\vartheta}) + H_{\widetilde{\vartheta}}(\varrho, \widetilde{\vartheta}) - \frac{\partial H_{\widetilde{\vartheta}}(\widetilde{\varrho}, \widetilde{\vartheta})}{\partial \varrho}(\varrho - \widetilde{\varrho}) - H_{\widetilde{\vartheta}}(\widetilde{\varrho}, \widetilde{\vartheta}). \end{aligned}$$

As e and s are interrelated through Gibbs' equation (2.5), a direct manipulation yields

$$\frac{\partial H_{\widetilde{\vartheta}}(\varrho, \vartheta)}{\partial \vartheta} = \frac{\varrho}{\vartheta}(\vartheta - \widetilde{\vartheta}) \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta}$$

and

$$\frac{\partial^2 H_{\widetilde{\vartheta}}(\varrho, \widetilde{\vartheta})}{\partial \varrho^2} = \frac{1}{\varrho} \frac{\partial p(\varrho, \widetilde{\vartheta})}{\partial \varrho}.$$

Consequently,

$$\vartheta \mapsto H_{\widetilde{\vartheta}}(\varrho, \vartheta) - H_{\widetilde{\vartheta}}(\varrho, \widetilde{\vartheta}) + H_{\widetilde{\vartheta}}(\varrho, \widetilde{\vartheta})$$

is a nonnegative function attaining strict minimum at $\vartheta = \widetilde{\vartheta}$ for any fixed ϱ, and

$$\varrho \mapsto H_{\widetilde{\vartheta}}(\varrho, \widetilde{\vartheta}) - \frac{\partial H_{\widetilde{\vartheta}}(\widetilde{\varrho}, \widetilde{\vartheta})}{\partial \varrho}(\varrho - \widetilde{\varrho}) - H_{\widetilde{\vartheta}}(\widetilde{\varrho}, \widetilde{\vartheta}) \text{ is strictly convex.}$$

In particular,

$$E\left(\varrho, \vartheta, \boldsymbol{m} \Big| \widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\boldsymbol{u}}\right) = 0 \text{ only if } \varrho = \widetilde{\varrho},\ \vartheta = \widetilde{\vartheta},\ \boldsymbol{m} = \widetilde{\varrho}\widetilde{\boldsymbol{u}}$$

whenever $\widetilde{\varrho} > 0$. Thus, similarly to its counterpart introduced in Sect. 4.1.3 for the barotropic Euler system, the relative energy represents a "distance" between $[\varrho, \vartheta, \boldsymbol{m}]$ and $[\widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\varrho}\widetilde{\boldsymbol{u}}]$. Note, however, that the relative energy is definitely not convex with respect to the standard variables $[\varrho, \vartheta, \boldsymbol{m}]$.

4.1.6.2 Relative Energy in the Conservative Variables

Now, we pass to the conservative variables. Note that formula (4.55) is rather awkward, containing the derivatives of the ballistic energy. Seeing that

$$\widetilde{\varrho}\frac{\partial(e-\widetilde{\vartheta}s)(\widetilde{\varrho},\widetilde{\vartheta})}{\partial\varrho}=\frac{p(\widetilde{\varrho},\widetilde{\vartheta})}{\widetilde{\varrho}}$$

we may rewrite the relative energy in the form

$$\begin{aligned}E\left(\varrho,\vartheta,\boldsymbol{m}\middle|\widetilde{\varrho},\widetilde{\vartheta},\widetilde{\boldsymbol{u}}\right)=&\left(\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho}+\varrho e(\varrho,\vartheta)\right)-\left(\frac{1}{2}\widetilde{\varrho}|\widetilde{\boldsymbol{u}}|^2+\widetilde{\varrho}e(\widetilde{\varrho},\widetilde{\vartheta})\right)\\&+\frac{1}{2}\widetilde{\varrho}|\widetilde{\boldsymbol{u}}|^2-\boldsymbol{m}\cdot\widetilde{\boldsymbol{u}}+\frac{1}{2}\varrho|\widetilde{\boldsymbol{u}}|^2-\widetilde{\vartheta}\left(\varrho s(\varrho,\vartheta)-\widetilde{\varrho}s(\widetilde{\varrho},\widetilde{\vartheta})\right)\\&-\left(e(\widetilde{\varrho},\widetilde{\vartheta})-\widetilde{\vartheta}s(\widetilde{\varrho},\widetilde{\vartheta})+\frac{p(\widetilde{\varrho},\widetilde{\vartheta})}{\widetilde{\varrho}}\right)(\varrho-\widetilde{\varrho}).\end{aligned}\tag{4.56}$$

Next, we recall the definition of the conservative variables,

$$\varrho,\ \boldsymbol{m},\ E=\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho}+\varrho e.$$

Writing $p=p(\varrho,e)$, $s=s(\varrho,e)$ we may use Gibbs' equation (2.5) to compute

$$\frac{\partial s}{\partial e}(\varrho,e)=\frac{1}{\vartheta},\ \ \frac{\partial s}{\partial\varrho}(\varrho,e)=-\frac{p}{\vartheta\varrho^2}.\tag{4.57}$$

Let

$$S=S(\varrho,\boldsymbol{m},E)=\varrho s\left(\varrho,\frac{1}{\varrho}\left(E-\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho}\right)\right)$$

be the total entropy. With help of (4.57) we compute

$$\begin{aligned}\frac{\partial S(\varrho,\boldsymbol{m},E)}{\partial\varrho}&=s-\frac{p}{\vartheta\varrho}-\frac{E}{\vartheta\varrho}+\frac{1}{\vartheta}\frac{|\boldsymbol{m}|^2}{\varrho^2}=\frac{1}{\vartheta}\left(\vartheta s-\frac{p}{\varrho}-e+\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho^2}\right),\\\nabla_{\boldsymbol{m}}S(\varrho,\boldsymbol{m},E)&=-\frac{1}{\varrho\vartheta}\boldsymbol{m},\\\frac{\partial S(\varrho,\boldsymbol{m},E)}{\partial E}&=\frac{1}{\vartheta}.\end{aligned}$$

Setting

$$\tilde{E}=\frac{1}{2}\widetilde{\varrho}|\widetilde{\boldsymbol{u}}|^2+\widetilde{\varrho}e(\widetilde{\varrho},\widetilde{\vartheta}),\ \ \widetilde{\boldsymbol{m}}=\widetilde{\varrho}\widetilde{\boldsymbol{u}},$$

we rewrite the relative energy in the conservative variables

$$
\begin{aligned}
&E\left(\varrho, \boldsymbol{m}, E \middle| \widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\boldsymbol{u}}\right)\\
&= E - \widetilde{\vartheta} S(\varrho, \boldsymbol{m}, E) - \boldsymbol{m} \cdot \widetilde{\boldsymbol{u}} + \frac{1}{2}\varrho|\widetilde{\boldsymbol{u}}|^2 + p(\widetilde{\varrho}, \widetilde{\vartheta})\\
&- \left(e(\widetilde{\varrho}, \widetilde{\vartheta}) - \widetilde{\vartheta} s(\widetilde{\varrho}, \widetilde{\vartheta}) + \frac{p(\widetilde{\varrho}, \widetilde{\vartheta})}{\widetilde{\varrho}} \right) \varrho\\
&= -\widetilde{\vartheta}\Big[S(\varrho, \boldsymbol{m}, E) - S(\widetilde{\varrho}, \widetilde{\boldsymbol{m}}, \widetilde{E})\\
&- \frac{\partial S(\widetilde{\varrho}, \widetilde{\boldsymbol{m}}, \widetilde{E})}{\partial \varrho}(\varrho - \widetilde{\varrho}) - \nabla_{\boldsymbol{m}} S(\widetilde{\varrho}, \widetilde{\boldsymbol{m}}, \widetilde{E}) \cdot (\boldsymbol{m} - \widetilde{\boldsymbol{m}}) - \frac{\partial S(\widetilde{\varrho}, \widetilde{\boldsymbol{m}}, \widetilde{E})}{\partial E}(E - \widetilde{E}) \Big].
\end{aligned}
\tag{4.58}
$$

Identity (4.58) reveals the intimate relation between the relative energy and relative entropy that differ by a multiplicative factor $\widetilde{\vartheta}$. It also shows that the thermodynamic stability hypothesis (4.54) may be expressed in term of *concavity* of the total entropy S with respect to the conservative variables $[\varrho, \boldsymbol{m}, E]$.

4.1.6.3 Relative Energy in the Conservative-Entropy Variables

The relative energy, expressed in terms of the conservative-entropy variables—the density ϱ, the momentum $\boldsymbol{m}$, and the total entropy $S = \varrho s$—fits in the general framework introduced in the preceding section and may be seen as the Bregman distance associated to the total energy. Indeed returning to (4.58) we obtain

$$
\begin{aligned}
&E\left(\varrho, \boldsymbol{m}, S \middle| \widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\boldsymbol{u}}\right)\\
&= E - \widetilde{\vartheta} S - \boldsymbol{m} \cdot \widetilde{\boldsymbol{u}} + \frac{1}{2}\varrho|\widetilde{\boldsymbol{u}}|^2 + p(\widetilde{\varrho}, \widetilde{\vartheta}) - \left(e(\widetilde{\varrho}, \widetilde{\vartheta}) - \widetilde{\vartheta} s(\widetilde{\varrho}, \widetilde{\vartheta}) + \frac{p(\widetilde{\varrho}, \widetilde{\vartheta})}{\widetilde{\varrho}} \right) \varrho\\
&= \frac{1}{2}\varrho \left| \frac{\boldsymbol{m}}{\varrho} - \widetilde{\boldsymbol{u}} \right|^2 + \varrho e(\varrho, S) - \left(e(\widetilde{\varrho}, \widetilde{\vartheta}) - \widetilde{\vartheta} s(\widetilde{\varrho}, \widetilde{\vartheta}) + \frac{p(\widetilde{\varrho}, \widetilde{\vartheta})}{\widetilde{\varrho}} \right)(\varrho - \widetilde{\varrho})\\
&- \widetilde{\vartheta}(S - \widetilde{S}) - \widetilde{\varrho} e(\widetilde{\varrho}, \widetilde{\vartheta}),
\end{aligned}
$$

where we have denoted $\widetilde{S} = \widetilde{\varrho} s(\widetilde{\varrho}, \widetilde{\vartheta})$.

Using Gibbs' relation (2.5) we check easily that

$$
\frac{\partial\left(\varrho e(\varrho, S)\right)}{\partial \varrho} = e(\varrho, S) - \vartheta \frac{S}{\varrho} + \frac{p(\varrho, S)}{\varrho},
$$

and

$$
\frac{\partial\left(\varrho e(\varrho, S)\right)}{\partial S} = \vartheta(\varrho, S).
$$

Consequently, we may infer that

$$
\begin{aligned}
&E\left(\varrho, \boldsymbol{m}, S \middle| \widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{S}\right) \\
&= \frac{1}{2}\varrho \left| \frac{\boldsymbol{m}}{\varrho} - \widetilde{\boldsymbol{u}} \right|^2 + \varrho e(\varrho, S) - \frac{\partial (\widetilde{\varrho} e(\widetilde{\varrho}, \widetilde{S}))}{\partial \varrho} (\varrho - \widetilde{\varrho}) \\
&- \frac{\partial (\widetilde{\varrho} e(\widetilde{\varrho}, \widetilde{S}))}{\partial S} (\widetilde{\varrho}, \widetilde{S})(S - \widetilde{S}) - \widetilde{\varrho} e(\widetilde{\varrho}, \widetilde{S}),
\end{aligned}
\tag{4.59}
$$

where we have replaced $\widetilde{\vartheta}$ by $\widetilde{S}$. We may infer that, similarly to the barotropic case, the relative energy expressed in terms of the conservative-entropy variables is the Bregman distance associated to the total energy

$$
E(\varrho, \boldsymbol{m}, S) = \frac{1}{2} \frac{|\boldsymbol{m}|^2}{\varrho} + \varrho e(\varrho, S).
$$

Of course, to see that the relative energy is Bregman distance, we have to rewrite it in terms of the conservative-entropy variables $[\widetilde{\varrho}, \widetilde{\boldsymbol{m}}, \widetilde{S}]$.

4.1.6.4 Thermodynamic Stability

A direct comparison of (4.55), (4.58), and (4.59) reveals equivalent formulation of the **hypothesis of thermodynamic stability**, namely:

- **Standard variables**.

$$
\begin{aligned}
&\frac{p(\varrho, \vartheta)}{\partial \varrho} > 0 \text{ (positive compressibility)}, \\
&\frac{e(\varrho, \vartheta)}{\partial \vartheta} > 0 \text{ (positive specific heat at constant volume)}
\end{aligned}
$$

- **Conservative variables**.

$$
(\varrho, \boldsymbol{m}, E) \mapsto S(\varrho, \boldsymbol{m}, E) \text{ concave}
$$

- **Conservative-entropy variables**.

$$
(\varrho, \boldsymbol{m}, S) \mapsto E(\varrho, \boldsymbol{m}, S) \equiv \frac{1}{2} \frac{|\boldsymbol{m}|^2}{\varrho} + \varrho e(\varrho, S) \text{ convex}
$$

4.1.7 *Relative Energy Inequality for the Complete Euler System*

Finally, we derive a relative energy inequality for the complete Euler system. Similarly to Theorem 4.1, we use only the weak formulation of the problem specified in Definition 4.3. We start by rewriting (4.56) in the form

$$
\begin{aligned}
&E\left(\varrho, \boldsymbol{m}, s \middle| \widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\boldsymbol{u}}\right) \\
&= \left[\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho} + \varrho e(\varrho, s)\right] - \widetilde{\vartheta}\varrho\Big[s - \widetilde{s}(\widetilde{\varrho}, \widetilde{\vartheta})\Big] - \boldsymbol{m}\cdot\widetilde{\boldsymbol{u}} + \frac{1}{2}\varrho|\widetilde{\boldsymbol{u}}|^2 \\
&+ p(\widetilde{\varrho}, \widetilde{\vartheta}) - \left(\widetilde{e}(\widetilde{\varrho}, \widetilde{\vartheta}) + \frac{\widetilde{p}(\widetilde{\varrho}, \widetilde{\vartheta})}{\widetilde{\varrho}}\right)\varrho.
\end{aligned}
$$

As we have seen above, there are several choices of phase variables, all of them being essentially equivalent. Here we have opted for $[\varrho, \boldsymbol{m}, s]$ for the weak solutions, while the test functions $[\widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\boldsymbol{u}}]$ correspond to the standard variables. Note that the above formula formally coincides with its isentropic (barotropic) counterpart (4.25), with $s = \text{const}$, and

$$
p = p(\varrho),\ \varrho e = P(\varrho),\ P'(\varrho)\varrho - P(\varrho) = p(\varrho).
$$

We have deliberately used the different symbols p and $\widetilde{p}$, e and $\widetilde{e}$, s and $\widetilde{s}$ to distinguish between the thermodynamic functions related to the weak solution expressed in terms of ϱ and s and those written in terms of the standard variables $\widetilde{\varrho}$ and $\widetilde{\vartheta}$.

Pursuing step by step the arguments of Sect 4.1.3, we may calculate the time increments

$$
\left[\int_{\Omega} E\left(\varrho, s, \boldsymbol{m} \middle| \widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\boldsymbol{u}}\right)(t, \cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau}
$$

in terms of the weak formulation (4.44)–(4.46) as long as the quantities $[\widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\boldsymbol{u}}]$ are sufficiently regular to be used as test functions. Similarly to (4.27) we therefore require

$$
\begin{aligned}
&\widetilde{\boldsymbol{u}} \in C^1([0, T] \times \overline{\Omega}; R^d),\ \widetilde{\boldsymbol{u}}\cdot\boldsymbol{n}|_{\partial\Omega} = 0, \\
&\widetilde{\varrho},\ \widetilde{\vartheta} \in C^1([0, T] \times \overline{\Omega}),\ \widetilde{\varrho} > 0,\ \widetilde{\vartheta} > 0 \text{ in } [0, T] \times \overline{\Omega}.
\end{aligned}
\tag{4.60}
$$

In virtue of (4.47),

$$
\left[\int_{\Omega}\left[\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho} + \varrho e\right]\ \mathrm{d}x\right]_{t=0}^{t=\tau} \le 0
$$

for a.a. $\tau \in (0, T)$. Next, considering $\widetilde{\boldsymbol{u}}$ as a test function in the momentum equation (4.45), we obtain

$$\left[\int_{\Omega} \boldsymbol{m} \cdot \widetilde{\boldsymbol{u}} \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega} \left[\boldsymbol{m} \cdot \partial_t \widetilde{\boldsymbol{u}} + \left(1_{\varrho > 0} \frac{\boldsymbol{m} \otimes \boldsymbol{m}}{\varrho} \right) : \nabla_x \widetilde{\boldsymbol{u}} + p \mathrm{div}_x \widetilde{\boldsymbol{u}} \right] \ \mathrm{d}x \, \mathrm{d}t.$$

Similarly, it follows from the weak formulation of the equation of continuity (4.44) that

$$\left[\int_{\Omega} \varrho \left(\frac{1}{2} |\widetilde{\boldsymbol{u}}|^2 - \widetilde{e}(\widetilde{\varrho}, \widetilde{\vartheta}) - \frac{\widetilde{p}(\widetilde{\varrho}, \widetilde{\vartheta})}{\widetilde{\varrho}} \right) \ \mathrm{d}x \right]_{t=0}^{t=\tau}$$
$$= \int_0^\tau \int_{\Omega} \left[\varrho \widetilde{\boldsymbol{u}} \cdot \partial_t \widetilde{\boldsymbol{u}} + 1_{\varrho > 0} \frac{\boldsymbol{m}}{\varrho} \cdot \varrho \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\boldsymbol{u}} \right] \ \mathrm{d}x \, \mathrm{d}t$$
$$- \int_0^\tau \int_{\Omega} \left[\varrho \partial_t \left(\widetilde{e}(\widetilde{\varrho}, \widetilde{\vartheta}) + \frac{\widetilde{p}(\widetilde{\varrho}, \widetilde{\vartheta})}{\widetilde{\varrho}} \right) + \boldsymbol{m} \cdot \nabla_x \left(\widetilde{e}(\widetilde{\varrho}, \widetilde{\vartheta}) + \frac{\widetilde{p}(\widetilde{\varrho}, \widetilde{\vartheta})}{\widetilde{\varrho}} \right) \right] \ \mathrm{d}x \, \mathrm{d}t.$$

Finally, we use the entropy inequality (4.46) with $\widetilde{\vartheta} > 0$ as test function to deduce

$$\left[\int_{\Omega} \varrho Z(s) \widetilde{\vartheta} \ \mathrm{d}x \right]_{t=0}^{t=\tau} \geq \int_0^\tau \int_{\Omega} \left[\varrho Z(s) \partial_t \widetilde{\vartheta} + Z(s) \boldsymbol{m} \cdot \nabla_x \widetilde{\vartheta} \right] \ \mathrm{d}x \, \mathrm{d}t$$

for any Z as in (4.46). Moreover, in view of (4.44),

$$\left[\int_{\Omega} \varrho \widetilde{\vartheta} \widetilde{s}(\widetilde{\varrho}, \widetilde{\vartheta}) \ \mathrm{d}x \right]_{t=0}^{t=\tau}$$
$$= \int_0^\tau \int_{\Omega} \widetilde{s}(\widetilde{\varrho}, \widetilde{\vartheta}) \left[\varrho \partial_t \widetilde{\vartheta} + \boldsymbol{m} \cdot \nabla_x \widetilde{\vartheta} \right] \ \mathrm{d}x \, \mathrm{d}t$$
$$+ \int_0^\tau \int_{\Omega} \widetilde{\vartheta} \left[\varrho \partial_t \widetilde{s}(\widetilde{\varrho}, \widetilde{\vartheta}) + \boldsymbol{m} \cdot \nabla_x \widetilde{s}(\widetilde{\varrho}, \widetilde{\vartheta}) \right] \ \mathrm{d}x \, \mathrm{d}t.$$

Thus, introducing a modified relative energy

$$
\begin{aligned}
E_Z\left(\varrho, s, \boldsymbol{m} \middle| \widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\boldsymbol{u}}\right) &= \left[\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho} + \varrho e(\varrho, s)\right] - \widetilde{\vartheta}\varrho\Big[Z(s) - \widetilde{s}(\widetilde{\varrho}, \widetilde{\vartheta})\Big] - \boldsymbol{m}\cdot\widetilde{\boldsymbol{u}} \\
&+ \frac{1}{2}\varrho|\widetilde{\boldsymbol{u}}|^2 + \widetilde{p}(\widetilde{\varrho}, \widetilde{\vartheta}) - \left(\widetilde{e}(\widetilde{\varrho}, \widetilde{\vartheta}) + \frac{\widetilde{p}(\widetilde{\varrho}, \widetilde{\vartheta})}{\widetilde{\varrho}}\right)\varrho,
\end{aligned}
\tag{4.61}
$$

and summing up the preceding calculations, we arrive at a *relative energy inequality* in the form

$$
\begin{aligned}
&\left[\int_\Omega E_Z\left(\varrho, s, \boldsymbol{m} \middle| \widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\boldsymbol{u}}\right) \ \mathrm{d}x\right]_{t=0}^{t=\tau} \\
&\leq -\int_0^\tau\int_\Omega 1_{\varrho>0}\frac{(\varrho\widetilde{\boldsymbol{u}} - \boldsymbol{m})\otimes(\varrho\widetilde{\boldsymbol{u}} - \boldsymbol{m})}{\varrho} : \nabla_x\widetilde{\boldsymbol{u}} \ \mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\tau\int_\Omega \Big[p(\varrho, s) - \widetilde{p}(\widetilde{\varrho}, \widetilde{\vartheta})\Big]\mathrm{div}_x\widetilde{\boldsymbol{u}} \ \mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau\int_\Omega (\varrho\widetilde{\boldsymbol{u}} - \boldsymbol{m})\cdot\left[\partial_t\widetilde{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}\cdot\nabla_x\widetilde{\boldsymbol{u}} + \frac{1}{\widetilde{\varrho}}\nabla_x\widetilde{p}(\widetilde{\varrho}, \widetilde{\vartheta})\right] \ \mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\tau\int_\Omega \left[\varrho Z(s)\partial_t\widetilde{\vartheta} + Z(s)\boldsymbol{m}\cdot\nabla_x\widetilde{\vartheta}\right] \ \mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau\int_\Omega \left[\varrho\widetilde{s}(\widetilde{\varrho}, \widetilde{\vartheta})\partial_t\widetilde{\vartheta} + \widetilde{s}(\widetilde{\varrho}, \widetilde{\vartheta})\boldsymbol{m}\cdot\nabla_x\widetilde{\vartheta}\right] \ \mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau\int_\Omega \left[(\widetilde{\varrho} - \varrho)\frac{1}{\widetilde{\varrho}}\partial_t\widetilde{p}(\widetilde{\varrho}, \widetilde{\vartheta}) + (\widetilde{\varrho} - \varrho)\frac{1}{\widetilde{\varrho}}\widetilde{\boldsymbol{u}}\cdot\nabla_x\widetilde{p}(\widetilde{\varrho}, \widetilde{\vartheta})\right] \ \mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{4.62}
$$

where we have used Gibbs' equation (4.1) to handle the terms $p(\widetilde{\varrho}, \widetilde{\vartheta})$, $e(\widetilde{\varrho}, \widetilde{\vartheta})$ and $s(\widetilde{\varrho}, \widetilde{\vartheta})$.

Finally, using Lemma 4.1 we may extend the class of admissible test functions similarly to Theorem 4.1. We are ready to formulate the relative energy inequality for the complete Euler system.

Theorem 4.4 (Relative energy inequality for complete Euler system) *Let $\Omega \subset R^d$, $d = 2, 3$, be a bounded domain of class C^2. Let $[\varrho, \boldsymbol{m}, s]$ be an admissible weak solution of the complete Euler system (4.2)–(4.4), with the impermeability boundary condition (4.5) in $(0, T) \times \Omega$ in the sense of Definition 4.3. Suppose that the thermodynamic functions $p = p(\varrho, \vartheta)$, $e = e(\varrho, \vartheta)$, $s = s(\varrho, \vartheta)$ are of class $C^2(0, \infty)^2$. Let $\widetilde{\varrho}$, $\widetilde{\vartheta}$, $\widetilde{\boldsymbol{u}}$ be (test) functions belonging to the class*

$$\widetilde{\varrho},\ \widetilde{\vartheta} \in W^{1,\infty}((0, T) \times \Omega),\ \inf_{(t,x)\in(0,T)\times\Omega} \widetilde{\varrho}(t, x) > 0,\ \inf_{(t,x)\in(0,T)\times\Omega} \widetilde{\vartheta}(t, x) > 0,$$
$$\widetilde{\boldsymbol{u}} \in W^{1,\infty}((0, T) \times \Omega; R^d),\ \widetilde{\boldsymbol{u}} \cdot \boldsymbol{n}|_{\partial\Omega} = 0. \tag{4.63}$$

Let Z be as in (4.46), and let E_Z be the relative energy defined through (4.61).

Then the relative energy inequality

$$\begin{aligned}
&\left[\int_\Omega E_Z\left(\varrho, s, \boldsymbol{m}\Big|\widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\boldsymbol{u}}\right)\ \mathrm{d}x\right]_{t=0}^{t=\tau}\\
&\leq -\int_0^\tau\int_\Omega 1_{\varrho>0}\frac{(\varrho\widetilde{\boldsymbol{u}} - \boldsymbol{m}) \otimes (\varrho\widetilde{\boldsymbol{u}} - \boldsymbol{m})}{\varrho} : \nabla_x\widetilde{\boldsymbol{u}}\ \mathrm{d}x\,\mathrm{d}t\\
&-\int_0^\tau\int_\Omega \left[p - \widetilde{p}\right]\mathrm{div}_x\widetilde{\boldsymbol{u}}\ \mathrm{d}x\,\mathrm{d}t\\
&+\int_0^\tau\int_\Omega (\varrho\widetilde{\boldsymbol{u}} - \boldsymbol{m}) \cdot \left[\partial_t\widetilde{\boldsymbol{u}} + \widetilde{\boldsymbol{u}} \cdot \nabla_x\widetilde{\boldsymbol{u}} + \frac{1}{\widetilde{\varrho}}\nabla_x\widetilde{p}\right]\ \mathrm{d}x\,\mathrm{d}t\\
&-\int_0^\tau\int_\Omega \left[\varrho Z(s)\partial_t\widetilde{\vartheta} + Z(s)\boldsymbol{m} \cdot \nabla_x\widetilde{\vartheta}\right]\ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega \left[\varrho\widetilde{s}\partial_t\widetilde{\vartheta} + \widetilde{s}\boldsymbol{m} \cdot \nabla_x\widetilde{\vartheta}\right]\ \mathrm{d}x\,\mathrm{d}t\\
&+\int_0^\tau\int_\Omega \left[(\widetilde{\varrho} - \varrho)\frac{1}{\widetilde{\varrho}}\partial_t\widetilde{p} + (\widetilde{\varrho} - \varrho)\frac{1}{\widetilde{\varrho}}\widetilde{\boldsymbol{u}} \cdot \nabla_x\widetilde{p}\right]\ \mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{4.64}$$

holds for a.a. $\tau \in (0,T)$. *Here we have denoted* $p = p(\varrho, s)$ *the pressure related to the weak solution, while* $\widetilde{p} = p(\widetilde{\varrho}, \widetilde{\vartheta})$, $\widetilde{s} = s(\widetilde{\varrho}, \widetilde{\vartheta})$ *denote thermodynamic functions written in terms of* $\widetilde{\varrho}$ *and* $\widetilde{\vartheta}$.

Relation (4.64) simplifies considerably if written in the conservative-entropy variables $[\varrho, \boldsymbol{m}, S]$, and, accordingly, $[\widetilde{\varrho}, \widetilde{\boldsymbol{u}}, \widetilde{S}]$. Indeed we first observe that

$$
\begin{aligned}
&\int_0^\tau \int_\Omega \left[(\widetilde{\varrho} - \varrho) \frac{1}{\widetilde{\varrho}} \partial_t \widetilde{p} + (\widetilde{\varrho} - \varrho) \frac{1}{\widetilde{\varrho}} \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{p} \right] \, \mathrm{d}x \, \mathrm{d}t \\
&\quad = \int_0^\tau \int_\Omega \left[(\widetilde{\varrho} - \varrho) \frac{1}{\widetilde{\varrho}} \frac{\partial p(\widetilde{\varrho}, \widetilde{S})}{\partial \varrho} \left(\partial_t \widetilde{\varrho} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\varrho} \right) \right] \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \left[(\widetilde{\varrho} - \varrho) \frac{1}{\widetilde{\varrho}} \frac{\partial p(\widetilde{\varrho}, \widetilde{S})}{\partial S} \left(\partial_t \widetilde{S} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{S} \right) \right] \, \mathrm{d}x \, \mathrm{d}t \\
&\quad = \int_0^\tau \int_\Omega \left[(\varrho - \widetilde{\varrho}) \frac{\partial p(\widetilde{\varrho}, \widetilde{S})}{\partial \varrho} \mathrm{div}_x \widetilde{\boldsymbol{u}} \right] \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \left[(\widetilde{\varrho} - \varrho) \frac{1}{\widetilde{\varrho}} \frac{\partial p(\widetilde{\varrho}, \widetilde{S})}{\partial \varrho} \left(\partial_t \widetilde{\varrho} + \mathrm{div}_x (\widetilde{\varrho} \widetilde{\boldsymbol{u}}) \right) \right] \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \left[(\widetilde{\varrho} - \varrho) \frac{1}{\widetilde{\varrho}} \frac{\partial p(\widetilde{\varrho}, \widetilde{S})}{\partial S} \left(\partial_t \widetilde{S} + \mathrm{div}_x (\widetilde{\boldsymbol{u}} \widetilde{S}) \right) \right] \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_0^\tau \int_\Omega \left[(\widetilde{\varrho} - \varrho) \frac{\widetilde{S}}{\widetilde{\varrho}} \frac{\partial p(\widetilde{\varrho}, \widetilde{S})}{\partial S} \mathrm{div}_x \widetilde{\boldsymbol{u}} \right] \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
$$

Consequently, the inequality (4.64) can be written as

$$
\begin{aligned}
&\left[\int_{\Omega} E_Z\left(\varrho, S, \boldsymbol{m} \middle| \widetilde{\varrho}, \widetilde{S}, \widetilde{\boldsymbol{u}}\right) \ \mathrm{d}x\right]_{t=0}^{t=\tau} \\
&\leq -\int_0^\tau \int_\Omega 1_{\varrho>0} \frac{(\varrho\widetilde{\boldsymbol{u}} - \boldsymbol{m}) \otimes (\varrho\widetilde{\boldsymbol{u}} - \boldsymbol{m})}{\varrho} : \nabla_x \widetilde{\boldsymbol{u}} \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left[p(\varrho, S) - (\varrho - \widetilde{\varrho})\frac{\partial p(\widetilde{\varrho}, \widetilde{S})}{\partial \varrho} - (S - \widetilde{S})\frac{\partial p(\widetilde{\varrho}, \widetilde{S})}{\partial S} - p(\widetilde{\varrho}, \widetilde{S})\right] \mathrm{div}_x \widetilde{\boldsymbol{u}} \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega (\varrho\widetilde{\boldsymbol{u}} - \boldsymbol{m}) \cdot \left[\partial_t \widetilde{\boldsymbol{u}} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\boldsymbol{u}} + \frac{1}{\widetilde{\varrho}} \nabla_x \widetilde{p}\right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[(\widetilde{\varrho} - \varrho)\frac{1}{\widetilde{\varrho}} \frac{\partial p(\widetilde{\varrho}, \widetilde{S})}{\partial \varrho} \left(\partial_t \widetilde{\varrho} + \mathrm{div}_x(\widetilde{\varrho}\widetilde{\boldsymbol{u}})\right)\right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[(\widetilde{\varrho} - \varrho)\frac{1}{\widetilde{\varrho}} \frac{\partial p(\widetilde{\varrho}, \widetilde{S})}{\partial S} \left(\partial_t \widetilde{S} + \mathrm{div}_x(\widetilde{\boldsymbol{u}}\widetilde{S})\right)\right] \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left[\varrho Z\left(\frac{S}{\varrho}\right) \partial_t \widetilde{\vartheta} + Z\left(\frac{S}{\varrho}\right) \boldsymbol{m} \cdot \nabla_x \widetilde{\vartheta}\right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\frac{\varrho}{\widetilde{\varrho}} \widetilde{S} \left(\partial_t \widetilde{\vartheta} + \frac{\boldsymbol{m}}{\varrho} \cdot \nabla_x \widetilde{\vartheta}\right)\right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\left(\frac{\varrho}{\widetilde{\varrho}} \widetilde{S} - S\right) \frac{\partial p(\widetilde{\varrho}, \widetilde{S})}{\partial S} \mathrm{div}_x \widetilde{\boldsymbol{u}}\right] \ \mathrm{d}x \, \mathrm{d}t,
\end{aligned}
\tag{4.65}
$$

with $\widetilde{S} = \widetilde{\varrho} s(\widetilde{\varrho}, \widetilde{\vartheta})$.

Finally, we let $Z(s) \nearrow s$ obtaining

$$
\begin{aligned}
&\int_0^\tau \int_\Omega \left[\varrho Z\left(\frac{S}{\varrho}\right) \partial_t \widetilde{\vartheta} + Z\left(\frac{S}{\varrho}\right) \boldsymbol{m} \cdot \nabla_x \widetilde{\vartheta}\right] \ \mathrm{d}x \, \mathrm{d}t \\
&\quad \to \int_0^\tau \int_\Omega \left[S \partial_t \widetilde{\vartheta} + S \frac{\boldsymbol{m}}{\varrho} \cdot \nabla_x \widetilde{\vartheta}\right] \ \mathrm{d}x \, \mathrm{d}t,
\end{aligned}
$$

and

$$
E_Z\left(\varrho, S, \boldsymbol{m} \;\middle|\; \widetilde{\varrho}, \widetilde{S}, \widetilde{\boldsymbol{u}}\right) \to E\left(\varrho, S, \boldsymbol{m} \;\middle|\; \widetilde{\varrho}, \widetilde{S}, \widetilde{\boldsymbol{u}}\right).
$$

Consequently,

$$
\begin{aligned}
&- \int_0^\tau \int_\Omega \left[\varrho Z \left(\frac{S}{\varrho} \right) \partial_t \widetilde{\vartheta} + Z \left(\frac{S}{\varrho} \right) \boldsymbol{m} \cdot \nabla_x \widetilde{\vartheta} \right] \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\frac{\varrho}{\widetilde{\varrho}} \widetilde{S} \left(\partial_t \widetilde{\vartheta} + \frac{\boldsymbol{m}}{\varrho} \cdot \nabla_x \widetilde{\vartheta} \right) \right] \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\left(\frac{\varrho}{\widetilde{\varrho}} \widetilde{S} - S \right) \frac{\partial p(\widetilde{\varrho}, \widetilde{S})}{\partial S} \mathrm{div}_x \widetilde{\boldsymbol{u}} \right] \,\mathrm{d}x \,\mathrm{d}t \\
&\to \int_0^\tau \int_\Omega \left[\left(\frac{\varrho}{\widetilde{\varrho}} \widetilde{S} - S \right) \left(\partial_t \widetilde{\vartheta} + \frac{\boldsymbol{m}}{\varrho} \cdot \nabla_x \widetilde{\vartheta} \right) \right] \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\left(\frac{\varrho}{\widetilde{\varrho}} \widetilde{S} - S \right) \frac{\partial p(\widetilde{\varrho}, \widetilde{S})}{\partial S} \mathrm{div}_x \widetilde{\boldsymbol{u}} \right] \,\mathrm{d}x \,\mathrm{d}t,
\end{aligned}
$$

where, furthermore,

$$
\begin{aligned}
&\int_0^\tau \int_\Omega \left[\left(\frac{\varrho}{\widetilde{\varrho}} \widetilde{S} - S \right) \left(\partial_t \widetilde{\vartheta} + \frac{\boldsymbol{m}}{\varrho} \cdot \nabla_x \widetilde{\vartheta} \right) \right] \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \left[\left(\frac{\varrho}{\widetilde{\varrho}} \widetilde{S} - S \right) \frac{\partial p(\widetilde{\varrho}, \widetilde{S})}{\partial S} \mathrm{div}_x \widetilde{\boldsymbol{u}} \right] \,\mathrm{d}x \,\mathrm{d}t \\
&\quad = \int_0^\tau \int_\Omega \left[\left(\frac{\varrho}{\widetilde{\varrho}} \widetilde{S} - S \right) \left(\partial_t \widetilde{\vartheta} + \widetilde{\boldsymbol{u}} \cdot \nabla_x \widetilde{\vartheta} + \frac{\partial p(\widetilde{\varrho}, \widetilde{S})}{\partial S} \mathrm{div}_x \widetilde{\boldsymbol{u}} \right) \right] \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \left(\frac{\varrho}{\widetilde{\varrho}} \widetilde{S} - S \right) \left(\frac{\boldsymbol{m}}{\varrho} - \widetilde{\boldsymbol{u}} \right) \cdot \nabla_x \widetilde{\vartheta} \,\mathrm{d}x \,\mathrm{d}t
\end{aligned}
$$

Summing up the previous computation we may rewrite (4.65) in the form

$$
\begin{aligned}
&\left[\int_\Omega E\left(\varrho, S, \boldsymbol{m}\Big|\widetilde{\varrho}, \widetilde{S}, \widetilde{\boldsymbol{u}}\right)\ \mathrm{d}x\right]_{t=0}^{t=\tau} \\
&\leq -\int_0^\tau\int_\Omega 1_{\varrho>0}\frac{(\varrho\widetilde{\boldsymbol{u}}-\boldsymbol{m})\otimes(\varrho\widetilde{\boldsymbol{u}}-\boldsymbol{m})}{\varrho} : \nabla_x\widetilde{\boldsymbol{u}}\ \mathrm{d}x\,\mathrm{d}t \\
&-\int_0^\tau\int_\Omega\left[p(\varrho,S)-(\varrho-\widetilde{\varrho})\frac{\partial p(\widetilde{\varrho},\widetilde{S})}{\partial\varrho}-(S-\widetilde{S})\frac{\partial p(\widetilde{\varrho},\widetilde{S})}{\partial S}-p(\widetilde{\varrho},\widetilde{S})\right]\mathrm{div}_x\widetilde{\boldsymbol{u}}\ \mathrm{d}x\,\mathrm{d}t \\
&+\int_0^\tau\int_\Omega(\varrho\widetilde{\boldsymbol{u}}-\boldsymbol{m})\cdot\left[\partial_t\widetilde{\boldsymbol{u}}+\widetilde{\boldsymbol{u}}\cdot\nabla_x\widetilde{\boldsymbol{u}}+\frac{1}{\widetilde{\varrho}}\nabla_x\widetilde{p}\right]\ \mathrm{d}x\,\mathrm{d}t \\
&+\int_0^\tau\int_\Omega\left[(\widetilde{\varrho}-\varrho)\frac{1}{\widetilde{\varrho}}\frac{\partial p(\widetilde{\varrho},\widetilde{S})}{\partial\varrho}\left(\partial_t\widetilde{\varrho}+\mathrm{div}_x(\widetilde{\varrho}\widetilde{\boldsymbol{u}})\right)\right]\ \mathrm{d}x\,\mathrm{d}t \\
&+\int_0^\tau\int_\Omega\left[(\widetilde{\varrho}-\varrho)\frac{1}{\widetilde{\varrho}}\frac{\partial p(\widetilde{\varrho},\widetilde{S})}{\partial S}\left(\partial_t\widetilde{S}+\mathrm{div}_x(\widetilde{\boldsymbol{u}}\widetilde{S})\right)\right]\ \mathrm{d}x\,\mathrm{d}t \\
&+\int_0^\tau\int_\Omega\left[\left(\frac{\varrho}{\widetilde{\varrho}}\widetilde{S}-S\right)\left(\partial_t\widetilde{\vartheta}+\widetilde{\boldsymbol{u}}\cdot\nabla_x\widetilde{\vartheta}+\frac{\partial p(\widetilde{\varrho},\widetilde{S})}{\partial S}\mathrm{div}_x\widetilde{\boldsymbol{u}}\right)\right]\ \mathrm{d}x\,\mathrm{d}t \\
&+\int_0^\tau\int_\Omega\left(\frac{\varrho}{\widetilde{\varrho}}\widetilde{S}-S\right)\left(\frac{\boldsymbol{m}}{\varrho}-\widetilde{\boldsymbol{u}}\right)\cdot\nabla_x\widetilde{\vartheta}\ \mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{4.66}
$$

Similarly to Sect. 4.1.3, the relative energy inequality (4.65), or in the form (4.66), can be used to show the weak-strong uniqueness principle. Indeed the relative energy

$$E\left(\varrho,\boldsymbol{m},S\ \Big|\ \widetilde{\varrho},\widetilde{\boldsymbol{u}},\widetilde{S}\right)\approx E\left(\varrho,\boldsymbol{m},S\ \Big|\ \widetilde{\varrho},\widetilde{\boldsymbol{m}}=\widetilde{\varrho}\widetilde{\boldsymbol{u}},\widetilde{S}\right)$$

corresponds to the *Bregman distance* associated to the total energy $E(\varrho,\boldsymbol{m},S)$ – a convex function of the conservative-entropy variables $[\varrho,\boldsymbol{m},S]$. In comparison with the barotropic case, the proof of the weak-strong uniqueness is more involved, and we postpone it to Chap. 5, where a more general class of weak solutions is treated.

4.2 Weak and Strong Solutions to the Navier–Stokes System, Relative Energy

The concepts of weak and strong solution to the Navier–Stokes(–Fourier) system have been introduced and discussed in Sect. 3. Here, we focus on deriving the relative energy inequality similarly to the Euler system. As a matter of fact, the relative energy functional remains the same as for the inviscid fluid models.

4.2.1 Relative Energy for the Navier–Stokes System

We restrict ourselves to the *barotropic* Navier–Stokes system :

$$\partial_t \varrho + \mathrm{div}_x(\varrho \boldsymbol{u}) = 0, \tag{4.67}$$

$$\partial_t(\varrho \boldsymbol{u}) + \mathrm{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}. \tag{4.68}$$

For definiteness, we consider the non-slip boundary condition

$$\boldsymbol{u}|_{\partial\Omega} = 0. \tag{4.69}$$

As the viscous stress is in many cases a function of the velocity gradient, it is more convenient to work in the frame of standard variables. The energy balance (inequality) takes the form

$$\int_\Omega \left[\frac{1}{2}\varrho |\boldsymbol{u}|^2 + P(\varrho) \right](\tau, \cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S} : \nabla_x \boldsymbol{u}\ \mathrm{d}x\,\mathrm{d}t \leq \int_\Omega \left[\frac{1}{2}\varrho_0 |\boldsymbol{u}_0|^2 + P(\varrho_0) \right] \mathrm{d}x \tag{4.70}$$

where $\varrho_0, \boldsymbol{u}_0$ are the initial data and P the pressure potential, $P'(\varrho)\varrho - P(\varrho) = p(\varrho)$. For the moment, we deliberately leave open the specific choice of the rheological relation for the viscous stress $\mathbb{S}$, keeping in mind only the Second law of thermodynamics requiring

$$\mathbb{S} : \nabla_x \boldsymbol{u} \geq 0.$$

The relative energy, written in terms of standard variables ϱ and $\boldsymbol{u}$ reads

$$E\left(\varrho, \boldsymbol{u} \,\middle|\, \widetilde{\varrho}, \widetilde{\boldsymbol{u}}\right) = \frac{1}{2}\varrho|\boldsymbol{u} - \widetilde{\boldsymbol{u}}|^2 + P(\varrho) - P'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - P(\widetilde{\varrho}) \tag{4.71}$$

cf. (4.25). Note that E *is not* a convex function of $[\varrho, \boldsymbol{u}]$.

The weak solutions to the Navier–Stokes system were introduced in Definition 3.1. The relative entropy inequality can be now derived mimicking the procedure applied to the Euler system in Sect. 4.1.3. Indeed the only two steps to be modified are:

(**i**) the energy inequality (4.28) that should read

$$\left[\int_{\Omega} \left[\frac{1}{2} \varrho |\boldsymbol{u}|^2 + P(\varrho) \right] \, \mathrm{d}x \right]_{t=0}^{t=\tau} + \int_0^\tau \int_{\Omega} \mathbb{S} : \nabla_x \boldsymbol{u} \, \mathrm{d}x \, \mathrm{d}t \leq 0,$$

(**ii**) the relation (4.29) that should be replaced by

$$\left[\int_{\Omega} \varrho \boldsymbol{u} \cdot \widetilde{\boldsymbol{u}} \, \mathrm{d}x \right]_{t=0}^{t=\tau}$$
$$= \int_0^\tau \int_{\Omega} \left[\varrho \boldsymbol{u} \cdot \partial_t \widetilde{\boldsymbol{u}} + (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) : \nabla_x \widetilde{\boldsymbol{u}} + p(\varrho) \mathrm{div}_x \widetilde{\boldsymbol{u}} - \mathbb{S} : \nabla_x \widetilde{\boldsymbol{u}} \right] \, \mathrm{d}x \, \mathrm{d}t,$$

where

$$\widetilde{\boldsymbol{u}} \in C^1([0, T] \times \overline{\Omega}; R^d), \ \widetilde{\boldsymbol{u}}|_{\partial\Omega} = 0$$

is a "test function". Putting together all the remaining integrals exactly as in Sect. 4.1.3, we deduce the *relative energy inequality* in the form

$$\begin{aligned}
&\left[\int_{\Omega} E\left(\varrho, \boldsymbol{u} \,\middle|\, \widetilde{\varrho}, \widetilde{\boldsymbol{u}} \right) \, \mathrm{d}x \right]_{t=0}^{t=\tau} + \int_0^\tau \int_{\Omega} (\mathbb{S} - \widetilde{\mathbb{S}}) : (\nabla_x \boldsymbol{u} - \nabla_x \widetilde{\boldsymbol{u}}) \, \mathrm{d}x \, \mathrm{d}t \\
&\leq - \int_0^\tau \int_{\Omega} \varrho \nabla_x \widetilde{\boldsymbol{u}} \cdot (\boldsymbol{u} - \widetilde{\boldsymbol{u}}) \cdot (\boldsymbol{u} - \widetilde{\boldsymbol{u}}) \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_{\Omega} \left[p(\varrho) - p'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - p(\widetilde{\varrho}) \right] \mathrm{div}_x \widetilde{\boldsymbol{u}} \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_{\Omega} \frac{\varrho}{\widetilde{\varrho}} (\widetilde{\boldsymbol{u}} - \boldsymbol{u}) \cdot \left[\partial_t (\widetilde{\varrho} \widetilde{\boldsymbol{u}}) + \mathrm{div}_x \, (\widetilde{\varrho} \widetilde{\boldsymbol{u}} \otimes \widetilde{\boldsymbol{u}}) + \nabla_x p(\widetilde{\varrho}) - \mathrm{div}_x \widetilde{\mathbb{S}} \right] \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_{\Omega} \left[\left(1 - \frac{\varrho}{\widetilde{\varrho}} \right) p'(\widetilde{\varrho}) + \frac{\varrho}{\widetilde{\varrho}} \widetilde{\boldsymbol{u}} \cdot (\boldsymbol{u} - \widetilde{\boldsymbol{u}}) \right] \left[\partial_t \widetilde{\varrho} + \mathrm{div}_x (\widetilde{\varrho} \widetilde{\boldsymbol{u}}) \right] \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_{\Omega} \left(\frac{\varrho}{\widetilde{\varrho}} - 1 \right) (\widetilde{\boldsymbol{u}} - \boldsymbol{u}) \cdot \mathrm{div}_x \widetilde{\mathbb{S}} \, \mathrm{d}x \, \mathrm{d}t
\end{aligned} \tag{4.72}$$

for any pair of test functions $\widetilde{\varrho}, \widetilde{\boldsymbol{u}}$ in the class

$$\begin{aligned}&\widetilde{\varrho} \in C^1([0,T] \times \overline{\Omega}), \widetilde{\varrho} > 0,\ \widetilde{\boldsymbol{u}} \in C^1([0,T] \times \overline{\Omega}; R^d),\ \widetilde{\boldsymbol{u}}|_{\partial\Omega} = 0,\\ &\text{and } \widetilde{\mathbb{S}} \in C^1([0,T] \times \overline{\Omega}; R^{d\times d}_{\rm sym}),\end{aligned} \tag{4.73}$$

cf. (4.30).

Remark 4.12 The tensor $\widetilde{\mathbb{S}}$ has been added "artificially" to relative energy inequality. Of course, we consider $\widetilde{S} = \widetilde{\mathbb{S}}(\nabla_x \widetilde{\boldsymbol{u}})$ in future applications to the weak-strong uniqueness problem.

In comparison with the relative energy inequality (4.34) for the barotropic Euler system, the relation (4.72) contains an extra term

$$\int_0^\tau \int_\Omega \left(\frac{\varrho}{\widetilde{\varrho}} - 1 \right) (\widetilde{\boldsymbol{u}} - \boldsymbol{u}) \cdot {\rm div}_x \widetilde{\mathbb{S}}\ {\rm d}x\, {\rm d}t.$$

Note carefully that this integral *cannot* be controlled by the relative energy on the vacuum zone $\{\varrho = 0\}$ as

$$\left(\frac{\varrho}{\widetilde{\varrho}} - 1 \right) (\widetilde{\boldsymbol{u}} - \boldsymbol{u}) \cdot {\rm div}_x \widetilde{\mathbb{S}} \approx (\widetilde{\boldsymbol{u}} - \boldsymbol{u}) \text{ for } \varrho \to 0.$$

Although vacuum is not expected to appear *spontaneously* in viscous fluid flows, a rigorous proof is not yet available. In order to show the weak-strong uniqueness principle, the term

$$\int_0^\tau \int_\Omega (\mathbb{S} - \widetilde{\mathbb{S}}) : (\nabla_x \boldsymbol{u} - \nabla_x \widetilde{\boldsymbol{u}})\ {\rm d}x\, {\rm d}t$$

must be used. Details can be found in Chap. 5, where a large class of generalized solutions is considered.

4.3 Conclusion, Bibliographical Remarks

We have introduced the concept of weak (distributional) solution and the relative energy to both the Euler and the Navier–Stokes system. These are objects solving the problem in the sense of generalized derivatives (distributions); satisfying automatically the compatibility principle **(C)**. If supplemented by a suitable form of the energy balance, they also satisfy the weak-strong uniqueness principle **(WS)**. These results are based on the concept of relative energy and the associated relative energy inequality for the weak solutions. In particular, the relative energy can be interpreted as the *Bregman distance* associated to the total energy of the system, cf. e.g. Sprung [185]. Results of this type go back to the pioneering paper by Dafermos [59] in the context of general conservation laws with involutions and are intimately related to the thermodynamic stability of the fluid system, see e.g. Bechtel et al. [14]. Since then the method of relative energy/entropy has found numerical applications, in particular in problems of weak-strong uniqueness for models of viscous and inviscid fluids, see e.g. Brenier et al. Székelyhidi [26], Germain [117], Gwiazda et al. [126], Mellet and Vasseur [164], the survey paper of Wiedemann [199], and the references therein. Here, we have postponed the rigorous proofs to the next chapter, where even more general class of *dissipative* solutions will be introduced.

There are two main issues that make analysis of numerical schemes in the framework of *weak solutions* quite delicate:

- Despite the numerous examples provided by the method of convex integration, see e.g., Chiodaroli et al. [49, 51], the global-in-time existence of *admissible* weak solutions for the Euler system is still an open problem. Although the weak solutions are known to exist for the Navier–Stokes(–Fourier) system, the existence proof is rather delicate, limited to severe restrictions imposed on constitutive relations, and difficult to implement in the proof of convergence of a numerical scheme, unless the latter is of very special type, cf. Karper [139, 140].
- The set of all weak solutions emanating from given initial data for the Euler system is not sequentially closed, meaning the property **(S)** listed in the introduction to Part II.

To demonstrate that the class of weak solutions to the barotropic Euler system fails to comply with the stability property **(S)**, we report the following result proved in [25, Proposition 2.1].

Theorem 4.5 *Let $\Omega \subset R^d$, $d = 2, 3$ be a bounded domain. Let $\varrho_0 \in L^\infty(\Omega)$, $\varrho_0 > 0$ be given.*

Then there exists a sequence of weak solutions $[\varrho_n, \boldsymbol{m}_n]$ to the Euler system (4.10), (4.11) in $(0, T) \times \Omega$, with the impermeability condition (4.5), such that $\varrho_n = \varrho_n(x)$ depends only on the spatial variable, and

$$\varrho_n \to \varrho_0 \textit{ weakly-(*) in } L^\infty(\Omega),\ \boldsymbol{m}_n \to 0 \textit{ weakly-(*) in } L^\infty((0,T) \times \Omega; R^N),$$

$$\liminf_{n\to\infty} \int_\Omega |\varrho_n - \varrho_0| \ \mathrm{d}x > 0.$$

Note that the limit ϱ_0 is arbitrary, while the momentum limit $\boldsymbol{m} \equiv 0$. Thus the limit is a solution of the Euler system only if $\varrho_0 = \overline{\varrho}$ is a (positive) constant. In particular, a weak limit of a sequence of weak solutions to the barotropic Euler system may not be a weak solution.

In the following chapter, we extend the class of weak solutions to more general objects commonly known as *measure-valued* or *dissipative* solutions. We show that the dissipative solutions comply with all the requirements introduced above. In particular, they exist globally in time for any physically relevant initial data, they are compatible with classical solutions, they obey the weak-strong uniqueness principle, and the solution set associated to given initial data is compact.