Chapter 9
Almost Periodic Functions with Values in a Non-locally Convex Space

In this section, $(X, +, \cdot, ||\cdot||)$ will be a p-Fréchet space with $0 < p < 1$ (over the field $\Phi = \mathbb{R}$ or $\mathbb{C}$). Also, denote $D(x, y) = ||x - y||$.

Similarly to [22], p. 137, a trigonometric polynomial of degree $\leq n$ with coefficients (and values) in the p-Fréchet space X, is defined as a finite sum of the form $T_n(t) = \sum_{k=1}^{n} c_k e^{i\lambda_n t}$, where $c_k \in X$, $k = 1, \ldots, n$.

Also, recall that $f : \mathbb{R} \to X$ is said to be continuous on $x_0 \in \mathbb{R}$ if $\forall \epsilon > 0$, there exists $\delta > 0$ such that $||f(x) - f(x_0)|| < \epsilon$, whenever $x \in \mathbb{R}$, $|x - x_0| < \delta$. From the triangle inequality satisfied by the p-norm $||\cdot||$, it easily follows the inequality $|\ ||x|| - ||y||\ | \leq ||x - y||$, which immediately implies that if f is continuous in x_0 as above, then the real-valued function $||f(t)||$ is also continuous at x_0.

1 Definitions and Properties

In this section, starting from a Bohr-kind definition for the almost periodicity, we develop a theory of almost periodic functions with values in a p-Fréchet space, $0 < p < 1$, similar to that for functions with values in a Banach space.

The following three points in Definition 3.1 represent the basic concepts in the theory of almost periodic functions with values in the p-Fréchet space X.

Definition 9.1 Let $f : \mathbb{R} \to X$ be continuous on $\mathbb{R}$.

(i) We say that **f is almost periodic** if $\forall \epsilon > 0$, there exists $l(\epsilon) > 0$ such that any interval of length $l(\epsilon)$ of the real line contains at least one point ξ with

$$||f(t + \xi) - f(t)|| < \epsilon, \ \forall t \in \mathbb{R}.$$

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(ii) We say that **f is normal** if for any sequence $F_n : \mathbb{R} \to X$ of the form $F_n(x) = f(x + h_n)$, $n \in \mathbb{N}$, where $(h_n)_n$ is a sequence of real numbers, one can extract a subsequence of $(F_n)_n$, converging uniformly on $\mathbb{R}$ (i.e. $\forall (h_n)_n, \exists \left(F_{n_k}\right), \exists F : \mathbb{R} \to X$ (which may depend on $(h_n)_n$), such that $\lim_{k\to\infty} ||F_{n_k}(x) - F(x)|| = 0$, uniformly with respect to $x \in \mathbb{R}$).

(iii) We say that **f has the approximation property**, if $\forall \epsilon > 0$, there exists some trigonometric polynomial T with coefficients in X, such that $||f(x) - T(x)|| < \epsilon, \forall x \in \mathbb{R}$.

Let us denote $AP(X) = \{f : \mathbb{R} \to X;\ f \text{ is almost periodic}\}$. The next two theorems show that $AP(X)$ is a subclass of uniformly continuous bounded functions.

Remark 9.2 We can reformulate 9.1(i), as follows: $f : \mathbb{R} \to X$ is called almost periodic if for every $\varepsilon > 0$, there exists a relatively dense set $\{\tau\}_\varepsilon$, such that

$$\sup_{t\in\mathbb{R}} ||f(t+\tau) - f(t)|| \le \varepsilon, \text{ for all } \tau \in \{\tau\}_\varepsilon.$$

Also, each $\tau \in \{\tau\}_\varepsilon$ is called ε-almost period of f.

Remark 9.3 Theorems 10.5, 8.5, 8.6, and 8.10 and Remark 8.8 remain valid in p-Fréchet spaces, $0 < p < 1$.

Theorem 9.4 *If f has the approximation property, then it is almost periodic.*

Proof A function $f : \mathbb{R} \to \mathbb{X}$ is said to be τ-periodic if $f(t+\tau) = f(t)$ for all $t \in \mathbb{R}$. Obviously, any trigonometric polynomial with values in $\mathbb{X}$ is almost periodic. This with Theorem 8.6 completes the proof. □

Remark 9.5 Let us denote $AP(X) = \{f : \mathbb{R} \to X;\ f \text{ is B-almost periodic}\}$, and for $f \in AP(X)$, let us define $\|f\|_b = \sup\{\|f(t)\|;\ t \in \mathbb{R}\}$. By Theorem 3.2, we get $\|f\|_b < +\infty$. It easily follows that $||\cdot||_b$ is also a p-norm on the space

$$BC(\mathbb{R}, X) = \{f : \mathbb{R} \to X \text{ is continuous and bounded on } \mathbb{R}\}.$$

In addition, since (X, D), where D is defined by $D(x, y) = \|x - y\|$, is a complete metric space, by standard reasonings, it follows that $BC(\mathbb{R}, X)$ becomes complete metric space with respect to the metric $D_b(f, g) = ||f - g||_b$, that is, $(BC(\mathbb{R}, X), ||\cdot||_b)$ becomes a p-Fréchet space.

Then, Theorems 3.2 and 3.5 show that $AP(X)$ is a closed subset of $BC(\mathbb{R}, X)$, i.e. $(AP(X), D_b)$ is a complete metric space, and therefore $(AP(X), ||\cdot||_b)$ becomes a p-Fréchet space. By similar reasonings with those in the proofs of Theorems 6.9 and 6.10 in [22], pp. 142–143 (where we define on X^m the p-norm $||x||_m = \sum_{k=1}^{m} ||x_k||$ and the metric $D_m(x, y) = \sum_{i=1}^{m} D(x_i, y_i)$, $\forall x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in X^m$), we can state the following compactness criterion.

Theorem 9.6 *The necessary and sufficient condition that a family $\mathcal{A} \subset AP(\mathbb{X})$ be relatively compact is that the following properties hold true:*

(i) *$\mathcal{A}$ is equicontinuous;*
(ii) *$\mathcal{A}$ is equi-almost periodic;*
(iii) *for any $t \in \mathbb{R}$, the set of values of functions from $\mathcal{A}$ be relatively compact in $\mathbb{X}$.*

In what follows, we consider the concept of Bochner's transform. Thus, Bochner's transform of f in $BC(\mathbb{R}, X)$ is denoted by $B(f) := \tilde{f}$ and is defined by $\tilde{f} : \mathbb{R} \to BC(\mathbb{R}, X)$, $\tilde{f}(s) \in BC(\mathbb{R}, X)$, $\tilde{f}(s)(t) = f(t+s)$, for all $t \in \mathbb{R}$.

The properties of the Bochner transform can be summarized in the following theorem:

Theorem 9.7 *(i)* $||\tilde{f}(s)||_b = ||f(\cdot + s)||_b = ||\tilde{f}(0)||_b$, *for all* $s \in \mathbb{R}$;
(ii) $||\tilde{f}(s+\tau) - \tilde{f}(s)||_b = sup\{||f(t+\tau) - f(t)||; t \in \mathbb{R}\} = ||\tilde{f}(\tau) - \tilde{f}(0)||_b$, *for all* $s, \tau \in \mathbb{R}$;
(iii) *f is B-almost periodic if and only if $\tilde{f}$ is B-almost periodic, with the same set of ε-almost periods $\{\tau\}_\varepsilon$;*
(iv) *$\tilde{f}$ is B-almost periodic, if and only if there exists a relatively dense sequence in $\mathbb{R}$, denoted by $\{s_n; n \in \mathbb{N}\}$, such that the set of functions $\{\tilde{f}(s_n); n \in \mathbb{N}\}$ is relatively compact in the complete metric space $(BC(\mathbb{R}, \mathbb{X}), D_b)$;*
(v) *$\tilde{f}$ is B-almost periodic, if and only $\tilde{f}(\mathbb{R})$ is relatively compact in the complete metric space $(BC(\mathbb{R}, \mathbb{X}), D_b)$;*
(vi) *(Bochner's criterion) f is B-almost periodic if and only if $\tilde{f}(\mathbb{R})$ is relatively compact in the complete metric space $(BC(\mathbb{R}, \mathbb{X}), D_b)$.*

Proof It is absolutely similar to the proof for Banach space valued functions, see e.g. [3, pp. 7–9].

Now, we are in position to prove the following sufficient condition for B-almost periodicity in p-Fréchet spaces, $0 < p < 1$. □

Theorem 9.8 *Let $f \in BC(\mathbb{R}, \mathbb{X})$. Let us suppose that there exists a relatively dense set of real numbers (s_n), such that*

(i) *the set $\{f(s_n); n \in \mathbb{N}\}$ is relatively compact in the metric space (X, D) and*
(ii) *for any $n, m \in \mathbb{N}$, the relation*

$$||f(s_n) - f(s_m)|| \geq c||f(\cdot + s_n) - f(\cdot + s_m)||_b$$

holds with $c > 0$ independent of n, m.

Then, f is almost periodic.

Proof The inequality in statement together with Theorem 9.7, obviously implies

$$D[f(s_n), f(s_m)] = ||f(s_n) - f(s_m)|| \geq c||\tilde{f}(s_n) - \tilde{f}(s_m)||_b = cD_b[\tilde{f}(s_n), \tilde{f}(s_m)].$$

Since by hypothesis, the set $\{f(s_n); n \in \mathbb{N}\}$ is relatively compact in the metric space (X, D), it has a convergent subsequence $(f(s_n'))_n$, which therefore is a

Cauchy sequence in the complete metric space (X, D), so it is convergent. The above inequality implies that $(\tilde{f}(s'_n))_n$ is also a Cauchy sequence in the complete metric space $(C(\mathbb{R}, \mathbb{X}), D_b)$, so it is convergent. Combined with Theorem 3.11(iv), it follows that $\tilde{f}$ is almost periodic, which combined with Theorem 3.11(iii), implies that f is almost periodic. The theorem is proved. □

2 Weakly Almost Periodic Functions

In what follows, we will consider the concept of weakly almost periodicity, at least in the cases of l^p and H^p spaces, $0 < p < 1$. Indeed, according to Remark 9.13(1) in Sect. 2, the dual spaces $(l^p)^*$ and $(H^p)^*$ are non-null. In addition, since $\{e_i, i \in \mathbb{N}\}$, with $e_i = (\delta_{i,n})_n \in l^p$, $\delta_{i,n}$, Kronecker's symbol, is a basis in l^p (see e.g. [5], p. 20), and since any $e_i^* : l^p \to \mathbb{R}$ is linear and continuous (see e.g. [5], p. 12, Theorem 1.8), it easily follows that $\{p_\varphi; \varphi \in (l^p)^*\}$, with $p_\varphi(x) = |\varphi(x)|$, for all $x \in l^p$, defines a sufficient family of seminorms on l^p, which evidently induces a weak topology on l^p, namely a locally convex Hausdorff topology on l^p.

Also, since according to e.g. [37, p. 35], the point evaluations $\varphi_z(f) = f(z)$, $z \in \mathbb{D}$, satisfy $\varphi_z \in (H^p)^*$, for all $z \in \mathbb{D}$, again it easily follows that $\{p_\varphi(x); \varphi \in (H^p)^*\}$ with $p_\varphi(x) = |\varphi(x)|$, for all $x \in H^p$, defines a sufficient family of seminorms on H^p, which evidently induces a locally convex Hausdorff (weak) topology on H^p.

Definition 9.9 Let $X = l^p$ or $X = H^p$ with $0 < p < 1$. A function $f : \mathbb{R} \to X$ is called weakly almost periodic, if $f : \mathbb{R} \to X$ is continuous and almost periodic, considering X endowed with the (weak) locally convex topology as above (see e.g. [2, pp. 159–160], or [7, 8]).

Remark 9.10 Obviously that Definition 9.9 has no sense for the p-Fréchet space $L^p[0, 1]$, $0 < p < 1$, whose dual is $\{0\}$.

Theorem 9.11 *Let $X = l^p$ or $X = H^p$, $0 < p < 1$. The necessary and sufficient condition that the function $f : \mathbb{R} \to X$ be weakly almost periodic is that for any $\varphi \in X^*$, the numerical function $h : \mathbb{R} \to \mathbb{R}$, defined by $h(t) = \varphi[f(t)]$, be almost periodic.*

Proof It is similar to the proof for Banach space-valued functions (see Theorem 6.1.7, p. 160 in [2]). □

Theorem 9.12 *Let $X = l^p$ or $X = H^p$, $0 < p < 1$. The necessary and sufficient condition that the function $f : \mathbb{R} \to X$ be almost periodic is that f be weakly almost periodic and that $f(\mathbb{R})$ be relatively compact.*

Proof Since for any $\varphi \in X^*$ and all $t, \tau \in \mathbb{R}$, we have

$$|\varphi[f(t+\tau)] - \varphi[f(r)]| \le |||\varphi||| \cdot ||f(t+\tau) - f(t)||^{1/p},$$

the usual (strong) almost periodicity and continuity of f immediately imply that f is weakly almost periodic and weakly continuous. This together with Theorem 3.6 immediately proves the necessity of theorem.

To prove the sufficiency, we would need the analogue for p-Fréchet space of the following result of Philips for Banach space (see the proof of Theorem 6.18, pp. 160–161 in [2]): from any bounded sequence $(\varphi_n)_n$ in X^* defined on a relatively compact subset $A \subset X$, one can extract a convergent subsequence on A. □

Remark 9.13

(1) In the case when X is a p-Fréchet space endowed with the p-norm $||\cdot||$, $0 < p < 1$, in [3, p. 102] (see also [1, p. 158]), the integral was introduced as follows. First, for $a = a_0 < a_1 < \ldots < a_n = b$, a partition of $[a, b]$, a step function on $[a, b]$ is of the form $s(x) = \sum_{k=0}^{n-1} y_k \cdot \chi_{[a_k, a_{k+1})}(x)$ (where $\chi_{[a_k, a_{k+1})}$ is the characteristic function of $[a_k, a_{k+1})$ and $y_k \in X$, $k = 0, 1, \ldots, n-1$), and its integral on $[a, b]$ is defined by $\int_a^b s(x)dx = \sum_{k=0}^{n-1} y_k(a_{k+1} - a_k)$. Then, since any continuous function $f : [a, b] \to X$ is uniformly continuous on $[a, b]$, it is easy to show that it is the uniform limit on $[a, b]$ of the sequences of step functions $s_n(x)$, $n \in \mathbb{N}$ defined by

$$s_n(x) = \sum_{k=0}^{n-1} f(a_k) \cdot \chi_{[a_k, a_{k+1})}(x),\ a_k = a + k\frac{b-a}{n},\ k = 0, 1, \ldots, n-1,$$

and the integral of f will be defined by $\int_a^b f(x)dx \in X$, where

$$\lim_{n\to\infty} \left\| \int_a^b f(x)dx - \int_a^b s_n(x)dx \right\| = 0.$$

(It is easy to see that the above $\int_a^b f(x)dx$ does not depend on the sequence of step functions uniformly convergent to f.) Unfortunately, the fundamental theorem of calculus stated in [3, Theorem 2, pp. 104–105] (see also [10], pp. 161–162) seems to be not valid in general, since for a continuous function $f : [a, b] \to X$, for the integral $F(t) = \int_a^t f(x)dx$, we have

$$\left\| \frac{F(t+h) - F(t)}{h} - f(t) \right\| = \left\| \frac{\int_t^{t+h} [f(x) - f(t)]dx}{h} \right\|,$$

but we do not get, in general, the estimate

$$\left\| \int_t^{t+h} [f(x) - f(t)]dx \right\| \leq |h|^p ||f(u) - f(t)||,$$

with u between t and $t+h$, as claimed in [10], p. 162 (which would imply that $\lim_{h \to 0} \left\| \frac{F(t+h) - F(t)}{h} - f(t) \right\| = 0$). As a first consequence, it follows that the implication "f' uniformly continuous and f B-almost periodic imply f' B-almost periodic" does not hold, although in the case of Banach space-valued functions, it is valid (see e.g. [3, Theorem VI, p. 6]).

(2) One could also adopt the more particular definition (of Riemann-type) for the integral on $[a, b]$ of a function $f : [a, b] \to X$, as unique limit of all the Riemann sums $\sum_{k=0}^{n-1} f(\xi_k)(a_{k+1} - a_k)$, with $\xi_k \in [a_k, a_{k+1}]$. Unfortunately, for this kind of integral too, the property $||\lambda x|| = |\lambda|^p ||x||$, where $0 < p < 1$, produces a bad estimate for the difference between the Riemann sums attached to two functions $f, g : [a, b] \to X$, namely

$$\begin{aligned} &\left\| \sum_{k=0}^{n-1} f(\xi_k)(a_{k+1} - a_k) - \sum_{k=0}^{n-1} g(\xi_k)(a_{k+1} - a_k) \right\| \\ &\leq \sum_{k=0}^{n-1} (a_{k+1} - a_k)^p \, \| f(\xi_k) - g(\xi_k) \| \\ &\not\leq \sum_{k=0}^{n-1} (a_{k+1} - a_k) ||f(\xi_k) - g(\xi_k)|| \end{aligned}$$

(in fact, since $0 < p < 1$, for $a_{k+1} - a_k \leq 1$, we have $(a_{k+1} - a_k)^p \geq (a_{k+1} - a_k)$, which is the case for n sufficiently large). This fact that has the similar effect as for the first integral, namely the fundamental theorem of calculus for this second integral also does not hold.

(3) From Remarks 9.13(1) and (2), it is evident that for a continuous $f : [a, b] \to X$, the inequality

$$\left\| \int_a^b f(t)dt \right\| \leq \int_a^b ||f(t)||dt$$

does not hold.

Now, if we introduce (as in the case of Banach space-valued functions) the mean value

$$M(f) = \lim_{T \to +\infty} \frac{1}{T} \cdot \int_0^T f(t)\, dt \in X,$$

where the limit is considered in the metric space (X, D) (i.e. there exists $M(f) \in X$ with $\lim_{T \to +\infty} D\left(M(f), \frac{1}{T} \cdot \int_0^T f(t)\, dt\right) = 0$), then because of Remark 9.13, it seems that $M(f)$ does not exist for any $f \in AP(X)$, since in the proof for the case of Banach space-valued functions, the inequality mentioned in Remark 9.13(3) is essential. This has as an effect the fact that, in general, one cannot attach Fourier series to a function $f \in AP(X)$ and the fact that the almost periodicity of f does not imply the approximation property mentioned in Definition 3.1(iii).

(4) In [6], a theory of the semigroups of linear and continuous operators is developed. As one of the applications, it was obtained that the initial value problem in the p-Fréchet space X, $0 < p < 1$,

$$\frac{du(t)}{dt} = A[u(t)],\ t \geq 0, u(0) = x \in X$$

(where $A : X \to X$ is linear and continuous) has as the unique solution $u(t) = T(t)(x)$, with $T(t)(x) = \lim_{n \to \infty} \left(I - \frac{t}{n} A\right)^{-n}(x)$, the limit being in the p-norm in X. On the other hand, taking into account Remarks 9.13(1) and 9.13(2), it follows that the inhomogeneous Cauchy problem

$$\frac{du(t)}{dt} = A[u(t)] + f(t), t \geq 0, u(0) = x \in X,$$

in general, seems to not have mild solution, in the sense that, even if we can define it as usual, it does not satisfy the differential equation.

(5) It is easy to construct almost periodic functions $f : \mathbb{R} \to X$ for which there exists $M(f)$ and the fundamental theorem of calculus holds. Indeed, any f of the form $c \cdot g$, where $c \in X$ and $g : \mathbb{R} \to \mathbb{R}$ is almost periodic, satisfies these two requirements.

Remark 9.14 The results in this section are contributions from Gal and N'Guérékata [30].

3 Applications

Firstly, we illustrate the idea of propagation of almost periodicity from the input data to the solutions of a simple differential equation in a p-Fréchet space $(X, ||\cdot||)$. In this sense, we present the following.

Theorem 9.15 *Let $f : \mathbb{R} \to \mathbb{R}$ be a usual almost periodic function and $c \in X$. Then, the function $y : \mathbb{R} \to X$ given by*

$$y(t) = c \cdot \int_{-\infty}^{t} e^{u-t} f(u)du, \ t \in \mathbb{R},$$

is B-almost periodic and satisfies the differential equation

$$y'(t) + y(t) = c \cdot f(t),$$

for all $t \in \mathbb{R}$.

Here, $y'(t)$ is defined as usual, that is, the limit in the metric $D(x, y) = ||x - y||$, given by

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}.$$

Proof Let us denote $F(t) = \int_{-\infty}^{t} e^{u-t} f(u)du$, $t \in \mathbb{R}$. By the classical theory, F is a usual almost periodic function. Then, by

$$||c \cdot F(t) - c \cdot F(t + \tau)|| = |F(t) - F(t + \xi)|^p \cdot ||c||,$$

it is immediate that $y(t) = c \cdot F(t)$ is B-almost periodic in the sense of Definition 3.1(i).

Since

$$F'(t) = f(t) - \int_{-\infty}^{t} e^{u-t} f(u), \forall t \in \mathbb{R},$$

it easily follows that $y'(t) = c \cdot \left[f(t) - \int_{-\infty}^{t} e^{u-t} f(u)du \right]$ and that $y(t)$ satisfies the differential equation, which proves the theorem. □

Bibliographical Notes The materials in this chapter are due to Gal and N'Guérékata [30]. As one can see, there are some open problems that need further investigations.