Chapter 8
Almost Periodic Functions with Values in a Locally Convex Space

1 Almost Periodic Functions

Definition 8.1 Let $E = E(\tau)$ be a complete Hausdorff locally convex space. A function $f : \mathbb{R} \to E$ is said to be almost periodic if for every neighborhood (of the origin) U, there exists a real number $l > 0$ such that every interval $[a, a + l]$ contains at least one point s such that

$$f(t - s) - f(t) \in U, \ \forall t \in \mathbb{R}.$$

The numbers s depend on U and are called U-translation numbers, or U-almost periods of the function f.

Remark 8.2 In the case where E is a Banach space $\mathbb{X}$ with norm $\|\cdot\|$, Definition 8.1 can be rewritten as:
$f : \mathbb{R} \to X$ is said to be almost periodic if for every $\varepsilon > 0$, there exists a real number $l > 0$ such that every interval $[a, a + l]$ contains at least one point s such that

$$\sup_{t \in \mathbb{R}} \|f(t - s) - f(t)\| < \varepsilon.$$

The numbers s are called the ε-almost periods of f.

Remark 8.3

(i) From Definition 8.1, we observe that for each neighborhood U, the set of all U-translation numbers is relatively dense in $\mathbb{R}$.
(ii) It is obvious that every continuous periodic function $f : \mathbb{R} \to E$ is almost periodic.

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We now present some elementary properties of almost periodic functions taking values in locally convex spaces.

Theorem 8.4 *If* $f, f_1, f_2 : \mathbb{R} \to E$ *are almost periodic and* λ *is a scalar, then the following functions are also almost periodic:*

(i) $f_1 + f_2$;
(ii) λf;
(iii) $\check{f}$ *defined by* $\check{f}(t) = f(-t)$ *for every* $t \in \mathbb{R}$.

Proof (i) and (ii) are obvious.

Let us prove (iii). Take U an arbitrary neighborhood of the origin. By almost periodicity of f, there exists $l > 0$ such that every interval $[a, a+l]$ contains at least a point s such that

$$f(t-s) - f(t) \in U, \ \forall t \in \mathbb{R}.$$

If we put $r = -t$, we get

$$\check{f}(r-s) - \check{f}(r) = f(-r+s) - f(-r) = f(t+s) - f(t).$$

Therefore $\check{f}(r-s) - \check{f}(r) \in U$ for every $r \in \mathbb{R}$, which proves almost periodicity of $\check{f}$ with $-s$ as U-translation numbers. □

We will denote by $AP(E)$ the space of all almost periodic functions $f : \mathbb{R} \to E$.

The following two results are easy to prove (cf. [51, 54]):

Theorem 8.5 *Let* $f \in AP(E)$. *Then* f *is uniformly continuous on* $\mathbb{R}$.

Theorem 8.6 *Let* $f_n \in AP(E)$, $n = 1, 2, \ldots$ *and suppose that* $f_n \to f$ *uniformly in* $t \in \mathbb{R}$. *Then* $f \in AP(E)$.

Theorem 8.7 *If* $f \in AP(E)$, *then its range* $\{f(t) \ / \ t \in \mathbb{R}\}$ *is totally bounded in* E.

Proof Let U be a neighborhood and V a symmetric neighborhood such that $V + V \subset U$; let $l = l(V)$ as in Definition 8.1. By the continuity of f, the set $\{f(t) \ / \ t \in [0, l]\}$ is compact in E. But in a locally convex space, every compact set is totally bounded; therefore there exists $x_1, x_2, \ldots, x_n \in E$ such that for every $t \in [0, l]$, we have

$$f(t) \in \cup_{j=1}^{n} (x_j + V).$$

Take now an arbitrary $t \in \mathbb{R}$ and consider $s \in [-t, -t + l]$ a V-translation number of the function f. Then we have

$$f(t+s) - f(t) \in V.$$

Choose x_k among $x_1, \ldots, x_n$ such that

$$f(t+s) \in x_k + V.$$

Let us write $f(t) - x_k = (f(t) - f(t+s)) + (f(t+s) - x_k)$. Then we have $f(t) - x_k \in V + V$, and therefore $f(t) - x_k \in U$, or $f(t) \in x_k + U$. Since t is an arbitrary real number, we conclude that

$$\{f(t) \;/\; t \in \mathbb{R}\} \subset \cup_{j=1}^{n}(x_j + U).$$

The proof is complete. □

Remark 8.8 If $f \in AP(E)$ with E a Fréchet space, then its range is relatively compact in E, since in every complete metric space, relative compactness and totally boundedness are equivalent notions. We conclude in this case that every sequence $(f(t_n))$ contains a convergent subsequence $(f(t_{n_k}))$.

Theorem 8.9 *Let E be a Fréchet space and $f \in AP(E)$. Then for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that $(f(t+s_n))$ is uniformly convergent in $t \in \mathbb{R}$.*

Proof Let (s_n) be a sequence of real numbers and consider the sequence of functions $f_{s_n} : \mathbb{R} \to E$ defined by $f_{s_n}(t) = f(t+s_n)$, $n = 1, 2, \ldots$ Let $S = (\eta_n)$ be a countable dense set in $\mathbb{R}$. By Remark 8.8, we can extract from $(f(\eta_1 + s_n))$ a convergent subsequence, since the set $\{f(t) \;/\; t \in \mathbb{R}\}$ is relatively compact in E.

Let $(f_{s_{1,n}})$ be the subsequence of (f_n) which converges at η_1.We apply the same argument to the sequence $(f_{s_{1,n}})$ to choose a subsequence $(f_{s_{2,n}})$ which converges at η_2. We continue the process and consider the diagonal sequence $(f_{s_{n,n}})$ which converges at η_n in S.

Call this last sequence (f_{r_n}). Now let us show that it is uniformly convergent on $\mathbb{R}$: that is, for every neighborhood U, there exists $N = N(U)$ such that

$$f(t+r_n) - f(t+r_m) \in U$$

for every $t \in \mathbb{R}$, if $n, m > N$.

Consider now an arbitrary neighborhood U and a symmetric neighborhood V such that $V + V + V + V + V \subset U$. Let $l = l(V) > 0$ as in Definition 8.1. Since f is uniformly continuous on $\mathbb{R}$ (Theorem 8.5), we can find $\delta = \delta(V) > 0$ such that

$$f(t) - f(t') \in V$$

for every $t, t' \in \mathbb{R}$ with $|t - t'| < \delta$.

Let us divide the interval $[0, l]$ into ν subintervals of lengths smaller than δ and choose in each interval a point of S, obtaining $S_0 = \{\xi_1, \ldots, \xi_\nu\}$. Since S_0 is a finite set, (f_{r_n}) is uniformly convergent over S_0; therefore there exists a natural number $N = N(V)$ such that

$$f(\xi_i + r_n) - f(\xi_i + r_m) \in V$$

for every $i = 1, \dots, \nu$, and for $n, m > N$.

Let $t \in \mathbb{R}$ be arbitrary and $s \in [-t, -t + l]$ such that $f(t + s) - f(t) \in V$. Let us choose ξ_i such that $|t + s - \xi_i| < \delta$; then

$$f(t + s + r_n) - f(\xi_i + r_n) \in V$$

for every n.

Let us write

$$\begin{aligned} f(t + r_n) - f(t + r_m) = (f(t + r_n) - f(r + r_n + s)) + (f(r + r_n + s) \\ - f(\xi_i + r_n)) + (f(\xi_i + r_n) - f(\xi_i + r_m)) \\ + (f(\xi_i + r_m) - f(t + r_m + s)) + (f(t + r_m + s) \\ - f(t + r_m)). \end{aligned}$$

Then it appears

$$f(t + r_n) - f(t + r_m) \in V + V + V + V + V \subset U$$

if $n, m > N$, which proves the uniform convergence of $(f(t + r_n))$. □

We are now ready to establish the following important result called also the **Bochner's criterion**:

Theorem 8.10 *Let E be a Fréchet space. Then $f \in AP(E)$ if and only if for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that $(f(t+s_n))$ converges uniformly in $t \in \mathbb{R}$.*

Proof The condition is necessary by Theorem 8.9.

Now we need to prove that it is sufficient. Suppose by contradiction that $f \notin AP(E)$. Then there exists a neighborhood U such that for every real number $l > 0$, there exists an interval of length l which contains no U-translation number of f, or there exists an interval $[-a, -a + l]$ such that for every $s \in [-a, -a + l]$, there exists $t = t_s$ such that $f(t + s) - f(t) \notin U$.

Let us consider $s_1 \in \mathbb{R}$ and an interval (a_1, b_1) with $b_1 - a_1 > 2|s_1|$ which contains no U-translation number of f. Now let $s_2 = \frac{(a_1-b_1)}{2}$; then $s_2-s_1 \in (a_1, b_1)$ and therefore $s_2 - s_1$ cannot be a U-translation number of f.

Let us consider another interval (a_2, b_2) with $b_2 - a_2 > 2(|s_1| + |s_2|)$, which contains no U-translation number of f. Let $s_3 = \frac{(a_2-b_2)}{2}$; then $s_3 - s_1, s_3 - s_2 \in (a_2, b_2)$ and therefore $s_3 - s_1$ and $s_3 - s_2$ cannot be U-translation numbers of f.

We proceed and obtain a sequence (s_n) of real numbers such that no $s_m - s_n$ is a U-translation number of f, that is

$$f(t + s_m - s_n) - f(t) \notin U.$$

Putting $\sigma = t - s_n$, we get

$$f(\sigma + s_m) - f(\sigma + s_n) \notin U. \tag{1.1}$$

Suppose there exists a subsequence (s'_n) of (s_n) such that $(f(t + s'_n))$ converges uniformly in $t \in \mathbb{R}$. Then for every neighborhood V, there exists a natural number $N = N(V)$ such that, if $n, m > N$ (we may take $m > n$), we have

$$f(t + s'_m) - f(t + s'_n) \in V$$

for every $t \in \mathbb{R}$. This contradicts (2.1) and so establishes the sufficiency of the condition.

The proof is complete. □

Theorem 8.11 *Let $f \in AP(E)$. Then the following hold true:*

(i) $Af(t) \in AP(E)$ for every linear bounded operator A on E.
(ii) $\nu f \in AP(E)$ where $\nu : \mathbb{R} \to \Phi$ is almost automorphic.

Proof Trivial, cf for instance [51, 54]. □

Using the Bochner's criterion, one can easily prove the following:

Theorem 8.12 *Let E be a Fréchet space and $f_1, f_2 \in AP(E)$. Then the function $F : \mathbb{R} \to E \times E$ defined by $F(t) = (f_1(t), f_2(t))$ is also almost periodic.*

Corollary 8.13 *Let $f_1, f_2 \in AP(E)$ where E is a Fréchet space. Then for every neighborhood U, f_1 and f_2 have common U-translation numbers.*

Proof Let U be a neighborhood in E. Then by Theorem 8.12 the function $f(t) = (f_1(t), f_2(t)) \in AP(E \times E)$. Consider now s a U-translation number of f; then $f(t+s) - f(t) \in U \times U$ for every $t \in \mathbb{R}$, and therefore $f_i(t+s) - f_i(t) \in U, \quad i = 1, 2$ for every $t \in \mathbb{R}$. s is a U-translation number for f_1 and f_2. □

Theorem 8.14 *Let E be a Fréchet space. Then $AP(E)$ is also a Fréchet space.*

Proof Consider $BC(\mathbb{R}, E)$ the linear space of all bounded and continuous functions $\mathbb{R} \to E$ and denote by (p_n), $n \in \mathbb{N}$, the family of seminorms which generates the topology of E. Without loss of generality we may assume that $p_n \leq p_{n+1}$, pointwise for $n \in \mathbb{N}$. Define

$$q_n(f) = \sup_{t \in \mathbb{R}} p_n(f(t)), \; n \in \mathbb{N}.$$

Obviously (q_n) forms a family of seminorms on $BC(\mathbb{R}, E)$. Moreover, it is clear that $q_n \leq q_{n+1}$ for $n \in \mathbb{N}$. Define the pseudonorm

$$|f| := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{q_n(f)}{1 + q_n(f)}, \quad f \in BC(\mathbb{R}, E).$$

Obviously $BC(\mathbb{R}, E)$ with the above defined pseudonorm is a Fréchet space. It is also a closed linear subspace of $BC(\mathbb{R}, E)$. This completes the proof. □

2 Weakly Almost Periodic Functions

Definition 8.15 Let E be a complete Hausdorff locally convex space. A weakly continuous function $f : \mathbb{R} \to E$ is said to be weakly almost periodic if the numerical function $F(t) = (x^* f)(t)$ is almost periodic for every $x^* \in E^*$ the dual space of E.

We will denote by $WAP(E)$ the set of all weakly almost periodic functions $\mathbb{R} \to E$.

Remark 8.16

(i) Every weakly almost periodic function is weakly bounded.
(ii) Every almost periodic function is weakly almost periodic.

Theorem 8.17 *Let $f \in WAP(E) \cap C(\mathbb{R}, E)$. Assume that the set $\{F(t) \ / \ t \in \mathbb{R}\}$ be weakly bounded where the function $F : \mathbb{R} \to E$ is defined by $F(t) = \int_0^t f(s)ds$. Then $F \in WAP(E)$.*

Proof We first observe that the integral exists in E since f is (strongly) continuous on $\mathbb{R}$. Take $x^* \in E^*$, so $x^* f \in AP(\mathbb{R})$. By the continuity of x^*, $(x^* F)(t) = x^* \int_0^t f(s)ds = \int_0^t (x^* f)(s)ds$ which is bounded by assumption and so is almost periodic. The Theorem is proved. □

Theorem 8.18 *Let E be a Fréchet space and $f : \mathbb{R} \to E$. Then $f \in AP(E)$ if and only if $f \in WAP(E)$ and its range is relatively compact.*

Proof The condition is necessary by Remarks 8.8 and 8.16. Let us show by contradiction that it is sufficient.

Suppose there exists $t_0 \in \mathbb{R}$ such that f is discontinuous at t_0, so we can find a neighborhood U and two sequences of real numbers (s'_n) and (s''_n) such that

$$\lim_{n\to\infty} s'_n = 0 = \lim_{n\to\infty} s''_n$$

and

$$f(t_0 + s'_n) - f(t_0 + s''_n) \notin U, \ \forall n \in \mathbb{N}. \tag{2.1}$$

By the relative compactness of $\{f(t) \ / \ t \in \mathbb{R}\}$, we can extract (r'_n) and (r''_n) from (s'_n) and (s''_n) respectively, such that

$$\lim_{n\to\infty} f(t_0 + r'_n) = a_1 \in E$$

and

$$\lim_{n\to\infty} f(t_0 + r_n'') = a_2 \in E.$$

Consequently, $a_1 - a_2 \notin U$ by (2.1), and using the Hahn–Banach Theorem (Proposition 1.41 Chap. 1), we can find $x^* \in E^*$ such that $x^*(a_1 - a_2) \neq 0$, hence $x^*(a_1) \neq x^*(a_2)$. By the continuity of x^*, we have

$$x^*(a_1) = \lim_{n\to\infty} (x^* f)(t_0 + r_n') = \lim_{n\to\infty} (x^* f)(t_0 + r_n'') = x^*(a_2)$$

which is a contradiction. So we conclude that f is continuous on $\mathbb{R}$. □

To prove the almost periodicity of f we need the following:

Lemma 8.19 *Let E be a Fréchet space and $\phi \in AP(E)$. Let (s_n) be a sequence of real numbers such that $\lim_{n\to\infty} \phi(s_n + \eta_k)$ exists for each $k = 1, 2, \ldots$ where the set (η_k) is dense in $\mathbb{R}$. Then the sequence $(\phi(t + s_n))$ is uniformly convergent in $t \in \mathbb{R}$.*

Proof (of Lemma 8.19) Suppose by contradiction that $(\phi(t + s_n))$ is not uniformly convergent in $t \in \mathbb{R}$. Then there exists a neighborhood U such that for every $N = 1, 2, \ldots$ there exists $n_N, m_N > N$ and $t_N \in \mathbb{R}$ such that

$$\phi(t_N + s_{n_N}) - \phi(t_N + s_{m_N}) \notin U.$$

By the Bochner's criterion (Theorem 8.10), we can extract two sequences $(s'_{n_N}) \subset (s_{n_N})$ and $(s'_{m_N}) \subset (s_{m_N})$ such that

$$\lim_{N\to\infty} \phi(t + s'_{n_N}) = g_1(t) \text{ uniformly in } t \in \mathbb{R}$$

and

$$\lim_{N\to\infty} \phi(t + s'_{m_N}) = g_2(t) \text{ uniformly in } t \in \mathbb{R}.$$

Let V be a symmetric neighborhood such that $V + V + V \subset U$. Then there exists $N_0 = N_0(V)$ such that if $N > n_0$, we have

$$\phi\left(t_N + s'_{n_N}\right) - g_1(t_N) \in V$$

and

$$\phi\left(t_N + s'_{m_N}\right) - g_2(t_N) \in V.$$

We deduce that $g_1(t_N) - g_2(t_N) \notin V$, otherwise we should have

$$\phi\left(t_N + s'_{n_N}\right) - \phi\left(t_N + s'_{m_N}\right) \in U$$

which contradicts (2.1).

Indeed if $g_1(t_N) - g_2(t_N) \in V$, then by writing

$$\begin{aligned}\phi\left(t_N + s'_{n_N}\right) - \phi\left(t_N + s'_{m_N}\right) = \phi\left(t_N + s'_{n_N}\right) - g_1(t_N)\\ + g_1(t_N) - g_2(t_N)\\ + g_2(t_N) - \phi\left(t_N + s'_{m_N}\right)\end{aligned}$$

we obtain

$$\phi(t_N + s'_{n_N}) - \phi(t_N + s'_{m_N}) \in V + V + V \subset U.$$

Thus we have found a symmetric neighborhood V with the property that if N is large enough, there exists $t_N \in \mathbb{R}$ such that

$$g_1(t_N) - g_2(t_N) \notin V.$$

But this is impossible, because if we take a subsequence (ξ_k) of (η_k) with $\xi_k \to t_N$, then we would obtain

$$\lim_{N\to\infty} \phi\left(\xi_k + s'_{n_N}\right) = \lim_{N\to\infty} \phi\left(\xi_k + s'_{m_N}\right)$$

for every k.

Therefore $g_1(\xi_k) = g_2(\xi_k)$ for every k. By the continuity of g_1 and g_2, $g_1(t_N) = g_2(t_N)$, thus $g_1(t_N) - g_2(t_N)$ belongs to every neighborhood.

The lemma is proved. □

Proof (of Theorem 8.18 (continued)) Consider a sequence of real numbers (h_n) and a sequence of rational numbers (η_r). By the relative compactness of $\{f(t)\ /\ t \in \mathbb{R}\}$, we can extract a subsequence (h_n) (we do not change the notation) such that for each $r = 1, 2, \ldots$

$$\lim_{n\to\infty} f(\eta_r + h_n) = x_r$$

exists in E. Now the sequence $(f(\eta_r + h_n))$ is uniformly convergent in η_r, or we could find a neighborhood U and three subsequences $(\xi_r) \subset (\eta_r)$, $(h'_r) \subset (h_r)$, and $(h''_r) \subset (h_r)$ with

$$f(\xi_r + h'_r) - f(\xi_r + h''_r) \notin U. \tag{2.2}$$

By the relative compactness of $\{f(t) \ / \ t \in \mathbb{R}\}$, we may say that

$$\lim_{r\to\infty} f\left(\xi_r + h_r'\right) = b' \in E$$

$$\lim_{r\to\infty} f\left(\xi_r + h_r''\right) = b'' \in E.$$

Then, using (2.2), we get

$$b' - b'' \notin U.$$

By the Hahn–Banach Theorem, there exists $x^* \in E^*$ such that

$$x^*(b') \neq x^*(b'').$$

Now $x^* f \in AP(\mathbb{R})$, therefore it is uniformly continuous over $\mathbb{R}$.

Let us consider the functions (φ_n) defined on $\mathbb{R}$ by

$$\varphi_n(t) := (x^* f)(t + h_n), \quad n = 1, 2, \ldots$$

The equality

$$\varphi_n(t+s) - \varphi_n(t) = (x^* f)(t + s + h_n) - (x^* f)(t + h_n)$$

shows the almost periodicity of each $\varphi_n, \quad n = 1, 2, \ldots$, if s is seen as a U-translation number of $(x^* f)(t)$. Also the sequence of functions (φ_n) is equicontinuous over $\mathbb{R}$ because $(x^* f)(t)$ is uniformly continuous on $\mathbb{R}$.

Since

$$\lim_{n\to\infty} f(\eta_r + h_n) = x_r,$$

we get

$$\lim_{n\to\infty} (x^* f)(n_r + h_n) = x^* x_r$$

for every $r = 1, 2, \ldots$ Therefore by Lemma 8.19, $((x^* f)(\eta_r + h_n))$ is uniformly convergent in t.

Consider now the sequences $(\xi_r + h_r')$ and $(\xi_r + h_r'')$. By the Bochner's criterion, we can extract a subsequence from each sequence, respectively, such that, using the same notations, $((x^* f)(t + \xi_r + h_r'))$ and $((x^* f)(t + \xi_r + h_r''))$ are uniformly convergent in $t \in \mathbb{R}$.

Let us now prove that

$$\lim_{r\to\infty} (x^* f)(t + \xi_r + h_r') = \lim_{r\to\infty} (x^* f)(t + \xi_r + h_r'').$$

Write $(x^*f)(t+\xi_r+h'_r)-(x^*f)(t+\xi_r+h''_r)$ as follows:

$$\begin{aligned}&(x^*f)(t+\xi_r+h'_r)-(x^*f)(t+\xi_r+h''_r)\\&\quad=(x^*f)(t+\xi_r+h'_r)-(x^*f)(t+\xi_r+h_r)\\&\qquad+(x^*f)(t+\xi_r+h_r)-(x^*f)(t+\xi_r+h''_r)\end{aligned}$$

and consider the following inequality (IN):

$$\begin{aligned}&|(x^*f)(t+\xi_r+h'_r)-(x^*f)(t+\xi_r+h''_r)|\\&\quad\le|(x^*f)(t+\xi_r+h'_r)-(x^*f)(t+\xi_r+h_r)|\\&\qquad+|(x^*f)(t+\xi_r+h_r)-(x^*f)(t+\xi_r+h''_r)|\end{aligned}$$

which holds true for $r=1,2,\ldots$

Let $\varepsilon>0$ be given. Since $((x^*f)(t+h_r))$ is uniformly convergent in t, we can choose η_ε such that for $r,s>\eta_\varepsilon$, we obtain

$$|(x^*f)(t+h_s)-(x^*f)(t+h_r)|<\frac{\varepsilon}{2},\quad\forall t\in\mathbb{R}.$$

So, replacing t by $t+\xi_r$ gives,

$$|(x^*f)(t+\xi_r+h_s)-(x^*f)(t+\xi_r+h_r)|<\frac{\varepsilon}{2},$$

and consequently

$$|(x^*f)(t+\xi_r+h'_r)-(x^*f)(t+\xi_r+h_r)|<\frac{\varepsilon}{2},$$

$$|(x^*f)(t+\xi_r+h''_r)-(x^*f)(t+\xi_r+h_r)|<\frac{\varepsilon}{2}.$$

The inequality (IN) above gives

$$|(x^*f)(t+\xi_r+h'_r)-(x^*f)(t+\xi_r+h''_r)|<\varepsilon,\quad\forall t$$

which proves that

$$\lim_{r\to\infty}(x^*f)(t+\xi_r+h'_r)=\lim_{r\to\infty}(x^*f)(t+\xi_r+h''_r)$$

which contradicts $x^*(b')\neq x^*(b'')$ obtained earlier and uniform continuity of $(f(\eta_r+h_n))$ as well.

If $i,j>N$, we have

$$f(\eta_r+h_i)-f(\eta_r+j)\in U.$$

This proves that $f \in AP(E)$ by the Bochner's criterion. □

Theorem 8.20 *Let E be a Fréchet space. If* $f \in AP(E)$ *and* $\{F(t) \ / \ t \in \mathbb{R}\}$ *is relatively compact in E where* $F(t) = \int_0^t f(s)ds$*, then* $F \in AP(E)$.

Proof This is immediate by Theorems 8.17 and 8.18. □

Theorem 8.21 *Let E be a complete locally convex space and* $f \in AP(E)$*. If the derivative* f' *exists and is uniformly continuous on* $\mathbb{R}$*, then* $f' \in AP(E)$.

Proof This is similar to the proof of the almost automorphic case (Theorem 4.1). We consider the sequence of almost periodic functions $(n(f(t+\frac{1}{n}) - f(t)))$ and let $U = U(\varepsilon;\ p_i,\ 1 \le i \le k)$ be a neighborhood. Since $f'(t)$ is uniformly continuous on $\mathbb{R}$, we can choose $\delta = \delta(U) > 0$ such that

$$f'(t_1) - f(t_2) \in U$$

for every t_1, t_2 such that $|t_1 - t_2| < \delta$. Let us write

$$f'(t) - n\left(f\left(t + \frac{1}{n}\right) - f(t)\right) = n\int_0^{\frac{1}{n}} (f'(t) - f'(t+s))ds.$$

Then if $N = N(U) > \frac{1}{n}$ and $n > N$, we would obtain

$$p_i\left[f'(t) - n\left(f\left(t + \frac{1}{n}\right) - f(t)\right)\right] \le n\int_0^{\frac{1}{n}} p_i\left[(f'(t) - f'(t+s))\right]ds < \varepsilon$$

for every seminorm p_i and every $t \in \mathbb{R}$. That shows that the sequence of almost periodic functions $\left(n\left(f(t+\frac{1}{n}) - f(t)\right)\right)$ converges uniformly to $f'(t)$ on $\mathbb{R}$. By Theorem 8.6, it follows that $f' \in AP(E)$. □

Theorem 8.22 *If* $f : \mathbb{R} \to E$ *(E a Fréchet space) is weakly bounded, then it is bounded.*

Proof For f to be weakly bounded means $\sup_{t\in\mathbb{R}} |x^* f(t)| < \infty$ for every $x^* \in E^*$. Suppose $f(\mathbb{R})$ is not bounded. Then there would exist a seminorm p such that $p(f(t_n)) \to \infty$ as $n \to \infty$ for some sequence of real numbers (t_n).

Let E_p be the completion of the normed space $E/kerp$ in the norm p. So E_p is a Banach space and $\tilde{f}(t_n) = f(t_n)/kerp$ is unbounded in E_p. Consequently there exists $\varphi \in E_p^*$ such that $|\varphi(\tilde{f}(t_n))| \to \infty$ as $n \to \infty$.

The natural map $J : E \to E_p$ is continuous, so its adjoint $J^* : E_p^* \to E^*$ is continuous. Finally setting $\psi = J^*(\varphi) \in E^*$, we have

$$|\psi(f(t_n))| = |J^*(\varphi)(f(t_n))| = |\varphi(\tilde{f}(t_n))| \to \infty$$

as $n \to \infty$. This completes the proof. □

Theorem 8.23 *Let E be a Fréchet space, $f \in WAP(E)$ and $A \in L(E)$ a bounded linear operator on E. Then $Af \in WAP(E)$.*

Proof Obvious. We leave it to the reader. □

Proposition 8.24 *Let E be a complete locally convex space and $f \in AP(E)$. Then for every sequence of real numbers (s_n), there exists a subsequence (s'_n) such that for every neighborhood U,*

$$f(t + s'_n) - f(t + s'_m) \in U$$

for every $t \in \mathbb{R}$ and every n, m.

Proof Let $U = U(\varepsilon;\ p_i,\ 1 \le i \le n)$ and $V = V(\frac{\varepsilon}{3};\ p_i,\ 1 \le i \le n)$ be a symmetric neighborhood such that $V + V + V \subset U$. By the definition of almost periodicity, there exists a number $l = l(V) > 0$ (depending also on U) such that every compact interval of length l contains a number τ such that

$$f(t + \tau) - f(t) \in V$$

for every $t \in \mathbb{R}$.

Consider now a given sequence of real numbers (s_n). For each s_n, we can find τ_n and σ_n such that $s_n = \tau_n + \sigma_n$ with τ_n a V-translation number of f and $\sigma_n \in [0, l]$. In fact it suffices to take $\tau_n \in [s_n - l, s_n]$ and then $\sigma_n = s_n - \tau_n$.

Since f is uniformly continuous, there exists $\delta = \delta(\varphi)$ such that

$$f(t') - f(t'') \in V$$

for all t', t'' with $|t' - t''| < 2\delta$.

Note that $\sigma_n \in [0, l]$ for all n. Hence by the Bolzano–Weierstrass Theorem, the sequence (σ_n) has a convergent subsequence, say (σ_{n_k}). Let $\sigma = \lim_{k\to\infty} \sigma_{n_k}$, which shows that $\sigma \in [0, l]$.

Now consider the subsequence of (σ_{n_k}) (we use the same notation) with

$$\sigma - \delta \le \sigma_{n_k} \le \sigma + \delta, \quad k = 1, 2, \dots$$

and let (s_{n_k}) be the corresponding subsequence of (s_n) with

$$s_{n_k} = \tau_{n_k} + \sigma_{n_k}, \quad k = 1, 2, \dots$$

Let us prove that

$$f(t + s_{n_k}) - f(t + s_{n_j}) \in U$$

for all t and all k, j. For this, let us write

$$f(t+s_{n_k}) - f(t+s_{n_j}) = f(t+\tau_{n_k}+\sigma_{n_k}) - f(t+\sigma_{n_k}) + f(t+\sigma_{n_k}) \\ - f(t+\sigma_{n_j}) + f(t+\sigma_{n_j}) - f(t+\tau_{n_j}+\sigma_{n_j}).$$

Because τ_{n_k} and τ_{n_j} are V-translation numbers of f, we have

$$f(t+\tau_{n_k}+\sigma_{n_k}) - f(t+\sigma_{n_k}) \in V$$

and

$$f(t+\sigma_{n_j}) - f(t+\tau_{n_j}+\sigma_{n_j}) \in V$$

for every t and every k, j. Also

$$f(t+\sigma_{n_k}) - f(t+\sigma_{n_j}) \in V$$

for every t and every k, j, since

$$|(t+\sigma_{n_k}) - (t+\sigma_{n_j})| = |\sigma_{n_k} - \sigma_{n_j}| \leq |\sigma_{n_k} - \sigma| + |\sigma - \sigma_{n_j}| \leq 2\delta.$$

The result is complete if we set $s'_k = s_{n_k}, \quad k = 1, 2, \ldots$ □

Theorem 8.25 *Let E be a Fréchet space and $(T(t))_{t\in\mathbb{R}}$ be an equicontinuous C_0-group of linear operators with $\{T(t)x;\ t \in \mathbb{R}\}$ relatively compact in E for every $x \in E$. Assume also that $f : \mathbb{R} \to E$ is a function with a relatively compact range in E. Then $\{T(t)f(t) :\ t \in \mathbb{R}\}$ is relatively compact in E.*

Proof Let (t''_n) be a sequence of real numbers. Since the range of $f(t)$ is relatively compact in E, we can extract a subsequence $(t'_n) \subset (t''_n)$ such that

$$\lim_{n\to\infty} f(t'_n) = x, \text{ exists in } E.$$

Further, by the assumption on $T(t)$, we can find a subsequence $(t_n) \subset (t'_n)$ such that $(T(t_n)x)$ is convergent, thus a Cauchy sequence in E.

Let us write

$$T(t_n)f(t_n) - T(t_m)f(t_m) = [T(t_n) - T(t_m)][f(t_n) - x] + [(T(t_n) - T(t_m))x] \\ + T(t_m)[f(t_n) - f(t_m)].$$

For an arbitrary seminorm p we have

$$\begin{aligned} p(T(t_n)f(t_n) - T(t_m)f(t_m)) \leq\ & p([T(t_n) - T(t_m)][f(t_n) - x]) \\ & + p([(T(t_n) - T(t_m))x]) \\ & + p(T(t_m)[f(t_n) - f(t_m)]). \end{aligned}$$

Using the equicontinuity of $T(t)$, we can find a seminorm q such that

$$p(T(t_m)[f(t_n) - f(t_m)]) \leq q(f(t_n) - f(t_m))$$

and

$$p([T(t_n) - T(t_m)][f(t_n) - x]) \leq 2q(f(t_n) - x).$$

Now choose n large enough so that

$$q(f(t_n) - f(t_m)) < \frac{\varepsilon}{3} q(f(t_n) - x) < \frac{\varepsilon}{3}$$

and

$$q([T(t_n) - T(t_m)]x) < \frac{\varepsilon}{3}$$

$$p(T(t_n)f(t_n) - T(t_m)) < \varepsilon,$$

which shows that $(T(t_n)f(t_n))$ is a Cauchy sequence, thus convergent. The theorem is proved. □

Theorem 8.26 *Let E be a Fréchet space and consider an equicontinuous C_0-group of linear operators $(T(t))_{t\in\mathbb{R}}$ such that $T(t)x : \mathbb{R} \to E$ is almost periodic for every $x \in E$. Suppose also that $f \in AP(E)$. Then $T(t)f(t) \in AP(E)$.*

Proof Consider $U = U(\varepsilon;\ p_i,\ 1 \leq i \leq n)$ be a given neighborhood of the origin. Because of the equicontinuity of $T(t)$, one can find, for each semi-norm p_i, a seminorm q_i such that

$$p_i(T(t)x) \leq q_i(x)$$

for every $t \in \mathbb{R}$ and $x \in E$. Consider also the symmetric neighborhood

$$V = V\left(\frac{\varepsilon}{4};\ p_i, q_i,\ 1 \leq i \leq n\right).$$

Then $V + V + V + V \subset U$. Since $\{f(t) :\ t \in \mathbb{R}\}$ is totally bounded, there exists $t_1, \ldots, t_\nu$ such that

$$f(t) \in \bigcup_{k=1}^{\nu} (f(t_k) + V)$$

for every $t \in \mathbb{R}$.

Consider now the almost periodic functions

$$f(t),\ T(t)(f(t_k)),\quad k = 1, 2, \ldots, \nu.$$

These are the same V-translation numbers by Corollary 8.13; therefore we can find a number $l = l(V) > 0$ such that any interval $[a, a+l]$ contains at least one number s such that

$$f(t+s) - f(s) \in V \quad \text{for every} \quad t \in \mathbb{R}, \tag{2.3}$$

$$T(t+s)f(t_k) - T(t)f(t_k) \in V \quad \text{for every } t \in \mathbb{R} \tag{2.4}$$

and for every $k = 1, 2, \ldots, \nu$.

Take now an arbitrary $t \in \mathbb{R}$. Then there exists $(1 \leq j \leq \nu)$ such that

$$f(t) \in f(t_j) + V. \tag{2.5}$$

Write

$$\begin{aligned} T(t+s)f(t+s) - T(t)f(t) = & \; T(t+s)(f(t+s) - f(t)) \\ & + T(t+s)(f(t) - f(t_j)) \\ & + T(t+s)f(t_j) - T(t)f(t_j) \\ & + T(t)(f(t_j) - f(t)). \end{aligned}$$

For every seminorm p_i, we can find a seminorm q_i such that

$$\begin{aligned} p_i[T(t+s)f(t+s) - T(t)f(t)] \leq & \; q_i(f(t+s) - f(t)) \\ & + q_i(f(t) - f(t_j)) + p_i(T(t+s)f(t_j) \\ & - T(t)f(t_j)) + q_i(f(t_j) - f(t)) \\ & < \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} \\ = & \; \varepsilon \end{aligned}$$

using (2.3), (2.4), and (2.5) above. Thus we have

$$T(t+s)f(t+s) - T(t)f(t) \in U$$

for every $t \in \mathbb{R}$, which establishes the almost periodicity of $T(t)f(t)$. □

Definition 8.27 A Fréchet space E is said to be perfect if every bounded function $f : \mathbb{R} \to E$ with an almost periodic derivative f' is necessarily almost periodic.

Example 8.28 Denote by s the linear space of all real sequences

$$s := \{s = (x_n) \; / \; x_n \in \mathbb{R}, \; n = 1, 2, \ldots\}.$$

For each $n \in \mathbb{N}$, define $p_n(x) := |x_n|, \; x \in s$. Obviously p_n is a seminorm defined on s. Now define $q_n := p_1 \vee p_2 \vee \ldots \vee p_n$ for $n \in \mathbb{N}$. We have $q_n \leq q_{n+1}$ for $n \in \mathbb{N}$. The space s considered with the family of seminorms (q_n) is a Fréchet space. Moreover, it can be proved (cf. [1] 17.7 p. 210) that each closed and bounded subset of s is compact. Thus, in particular, s is not a Banach space. Moreover in view of Theorem 8.20, s is perfect.

Definition 8.29 A function $f \in C(\mathbb{R}, \mathbb{X})$ is called periodic if there exists $l > 0$ such that

$$f(t+l) = f(t), \quad \forall t \in \mathbb{R}.$$

Here, l is called a period of f. We denote the collection of all such functions by $P(\mathbb{X})$. For $f \in P(\mathbb{X})$, we call l_0 the fundamental period if l_0 is the smallest period of f.

Remark 8.30 Similar to the proof in [22, p. 1], it is not difficult to show that if $f \in P(\mathbb{X})$ is not constant, then f has the fundamental period.

Theorem 8.31 ([72]) *Let X be a Banach space with norm $\|\cdot\|$, then $P(\mathbb{X})$ is a set of first category in $AP(\mathbb{X})$.*

Proof For $n = 1, 2, \ldots$, we denote

$$P_n = \{f \in C(\mathbb{R}, \mathbb{X}) \ : \ \exists l \in [n, n+1] \text{ such that } f(t+1) = f(t), \ \forall t \in \mathbb{R}\}.$$

Then, it is easy to see that

$$P(\mathbb{X}) = \bigcup_{n=1}^{\infty} P_n.$$

We divide the remaining proof into two steps.

Step 1 Every P_n is a closed subset of $AP(\mathbb{X})$.

Let $f \in AP(\mathbb{X}) \backslash P_n$. Then, for every $l \in [n, n+1]$, there exists $t_l \in \mathbb{R}$ such that $f(t_l + l) \neq f(t_l)$. Denote

$$\varepsilon_l := \frac{1}{4}\|f(t_l + l) - f(t_l)\| > 0, \quad l \in [n, n+1].$$

In addition, due to the continuity of f, for every $l \in [n, n+1]$, there exists $\delta_l > 0$ such that

$$\|f(t_l + s) - f(t_l)\| \geq 3\varepsilon_l, \quad \forall s \in (l - \delta_l, l + \delta_l). \tag{2.6}$$

Obviously, we have

$$[n, n+1] \subset \bigcup_{l \in [n, n+1]} (l - \delta_l, l + \delta_l).$$

Then, by the Heine-Borel theorem, there exists $l_1, \ldots l_k \in [n, n+1]$ such that

$$[n, n+1] \subset \bigcup_{i=1}^{k} (l_i - \delta_{l_i}, l_i + \delta_{l_i}),$$

where k is a fixed positive integer. Letting $\varepsilon = \min_{1 \le i \le k} \{\varepsilon_{l_i}\}$, and

$$N(f, \varepsilon) := \{g \in AP(\mathbb{X}) : \|g - f\|_{AP(X)} < \varepsilon\},$$

for every $g \in N(f, \varepsilon)$, we claim that $g \notin P_n$. In fact, for every $l \in [n, n+1]$, there exists $i \in \{1, \ldots, k\}$ such that

$$l \in (l_i - \delta_{l_i}, l_i + \delta_{l_i}).$$

Then, by (4.3), we have

$$\|f(t_{l_i} + l) - f(t_{l_i})\| \ge 3\varepsilon_{l_i} \ge 3\varepsilon,$$

which yields that

$$\begin{aligned}\|g(t_{l_i} + l) - g(t_{l_i})\| \ge \|f(t_{l_i} + l) - f(t_{l_i})\| - \|f(t_{l_i} + l) - g(t_{l_i} + l)\| \\ - \|f(t_{l_i}) - g(t_{l_i})\| \ge 3\varepsilon - \varepsilon - \varepsilon = \varepsilon > 0,\end{aligned}$$

where $\|g - f\|_{AP(\mathbb{R})} < \varepsilon$ was used. So, we know that $N(f, \varepsilon) \subset AP(\mathbb{X}) \backslash P_n$, which means that P_n is a closed subset of $AP(\mathbb{X})$.

Step 2 Every P_n has an empty interior.

It suffices to prove that for every $f \in P_n$ and $\delta > 0$, $N(f, \delta) \bigcap (AP(\mathbb{X}) \backslash P_n) \neq \emptyset$. Now let $f \in P_n$ and $\delta > 0$. In the following, we discuss by two cases:

Case I f is constant.

We denote

$$f_\delta(t) = \frac{\cos t + \cos(\sqrt{2}t)}{3} \cdot \delta \cdot x_0 + f(t), \quad t \in \mathbb{R}$$

where $x_0 \in \mathbb{X}$ is some constant with $\|x_0\| = 1$. Then $f_\delta \in N(f, \delta)$, and $f_\delta \notin P_n$ since f_δ is not periodic.

Case II f is not constant.

Let f be a fundamental period l_0. We denote

$$f_\delta(t) = f(t) + f\left(\frac{t}{\pi}\right) \cdot \frac{\delta}{M_f}, \quad t \in \mathbb{R},$$

where $M_f = \sup_{t \in \mathbb{R}} \|f(t)\|$. Obviously, $f_\delta \in N(f, \delta)$. Also, we claim that $f_\delta \notin P_n$. In fact, if this is not true, then there exists $T \in [n, n+1]$ such that

$$f_\delta(t+T) = f_\delta(t), \quad t \in \mathbb{R},$$

i.e.

$$f(t+T) + f\left(\frac{t+T}{\pi}\right) \cdot \frac{\delta}{M_f} = f(t) + f\left(\frac{t}{\pi}\right) \cdot \frac{\delta}{M_f}, \quad t \in \mathbb{R}.$$

Let

$$F_1(t) = f(t+T) - f(t), \quad F_2(t) = \frac{\delta}{M_f}\left[f\left(\frac{t}{\pi}\right) - f\left(\frac{t+T}{\pi}\right)\right], \quad t \in \mathbb{R}.$$

Then $F_1(t) \equiv F_2(t)$. If $F_1(t) \equiv F_2(t) \equiv C$, where C is a fixed constant, then

$$f(t+T) = f(t) + C, \quad t \in \mathbb{R},$$

which yields

$$C = \frac{f(kT) - f(0)}{k} \to 0, \ k \to \infty,$$

since f is bounded. Thus, we have

$$f(t+T) = f(t), \quad f\left(\frac{t}{\pi}\right) = f\left(\frac{t+T}{\pi}\right), \quad t \in \mathbb{R}.$$

Noting that l_0 is the fundamental period of f and πl_0 is the fundamental period of $f\left(\frac{\cdot}{\pi}\right)$, there exist two positive integers p, q such that

$$pl_0 = T = q\pi l_0,$$

i.e. $\pi = \frac{p}{q}$, which is a contradiction. If $F_1 = F_2$ is not constant, then by Remark 8.30, we can assume that T_0 is the fundamental period of F_1 and F_2. Noting that l_0 is a period of F_1 and πl_0 is a period of F_2, similar to the above proof, we can also show that π is a rational number, which is a contradiction.

In conclusion, $P(\mathbb{X})$ is countable unions of closed subsets with empty interior. So $P(\mathbb{X})$ is a set of first category. □

Theorem 8.32 ([72]) *Let $\mathbb{X}$ be a Banach space. Then $AP(\mathbb{X})$ is a set of first category in $AA(\mathbb{X})$.*

Proof It suffices to note that $AP(\mathbb{X})$ is a proper closed subspace of $AA(\mathbb{X})$ equipped with the supnorm. Therefore it is of first category in $AA(\mathbb{X})$. □

3 Almost Periodicity of the Function $f(t, x)$

Definition 8.33 Let E be a Fréchet space. A function $f \in C(\mathbb{R} \times E, E)$ is said to be almost periodic in $t \in \mathbb{R}$ for each $x \in E$ if for each neighborhood of the origin U, there exists a real number $l > 0$ such that every interval $[a, a + l]$ contains at least a point τ such that

$$f(t + \tau, x) - f(t, x) \in U, \quad \text{for each } t \in \mathbb{R} \text{ and each } x \in E.$$

In view of the Bochner's criterion, this definition is equivalent to the following: $f \in C(\mathbb{R} \times E, E)$ is almost periodic in $t \in \mathbb{R}$ for each $x \in E$ if and only if for every sequence of real numbers (s'_n) there exists a subsequence $(s_n) \subset (s'_n)$ such that $(f(t + s_n, x))$ converges uniformly in $t \in \mathbb{R}$ and $x \in E$.

Theorem 8.34 *Let $f : \mathbb{R} \times E \to E$ be almost periodic in $t \in \mathbb{R}$ for each $x \in E$, and assume that f satisfies a Lipschitz condition in x uniformly in t, that is $d(f(t, x) - f(t, y)) \leq Ld(x, y)$ for all $t \in \mathbb{R}$ and $x, y \in E$, where d is a metric on E. Let $\phi : \mathbb{R} \to E$ be almost periodic. Then the Nemytskii's operator $\mathcal{N}$ defined by $\mathcal{N}(\cdot) := f(\cdot, \phi(\cdot))$ is almost periodic.*

Proof Trivial. We leave it to the reader. □

4 Equi-Asymptotically Almost Periodic Functions

In this section, we introduce the notion of equi-asymptotically almost periodicity (cf. [24]), and present some basic and interesting properties for equi-asymptotically almost periodic functions.

Definition 8.35 Let $\mathbb{X}$ be a Banach space. A set $F \subset C(\mathbb{R}, \mathbb{X})$ is called equi-asymptotically almost periodic if for every $\varepsilon > 0$, there exist a constant $M(\varepsilon) > 0$ and a relatively dense set $T(F, \varepsilon) \subset \mathbb{R}$ such that

$$\|f(t + \tau) - f(t)\| < \varepsilon,$$

for all $f \in F$, $t \in \mathbb{R}$ with $|t| > M(\varepsilon)$ and $\tau \in T(F, \varepsilon)$ with $|t + \tau| > M(\varepsilon)$.

Theorem 8.36 *Let $F \subset AAP(\mathbb{R}, \mathbb{X})$. Then the following assertions are equivalent:*

(i) *F is precompact in $AAP(\mathbb{R}, \mathbb{X})$.*
(ii) *F satisfy the following three conditions:*

  (a) *for every $t \in \mathbb{R}$, $\{f(t) : f \in F\}$ is precompact in $\mathbb{X}$.*
  (b) *F is equi-uniformly continuous.*
  (c) *F is equi-asymptotically almost periodic.*

(iii) *G is precompact in $AP(\mathbb{R}, X)$ (in short AP) and H is precompact in $C_0(\mathbb{R}, X)$ (in short C_0), where*

$$G = \{f_{AP} : f \in F\} \text{ and } H = \{f_{C_0} : f \in F\}.$$

Proof

(i) ⇒ (ii) Let F be precompact in $AAP(\mathbb{R}, \mathbb{X})$. Then, obviously, for every $t \in \mathbb{R}$, $\{f(t) : f \in F\}$ is precompact in X. In addition, for every $\varepsilon > 0$, there exist $f_1, f_2, \ldots, f_k \in F$ such that for every $f \in F$,

$$\min_{1 \le i \le k} \|f - f_i\| < \varepsilon,$$

where k is a positive integer dependent on ε. Combining this with the fact that $(f_i)_{i=1}^k$ is equi-uniformly continuous and equi-asymptotically almost periodic, we know that (b) and (c) hold.

(ii) ⇒ (iii) Let $(g_n) \subset G$. For every n, there exist $f_n \in F$ and $h_n \in H$ such that $f_n = g_n + h_n$. By (a) and (b), applying Arzela–Ascoli theorem and choosing diagonal sequence, we can get a subsequence of (f_n), which we still denote by (f_n) for convenience, such that $(f_n(t))$ is uniformly convergent on every compact subsets of $\mathbb{R}$.

Since (f_n) is equi-asymptotically almost periodic, for every $\varepsilon > 0$, there exists $l(\varepsilon), M(\varepsilon) > 0$ such that for every $t \in \mathbb{R}$ with $|t| > M(\varepsilon)$, there is a

$$\tau_t \in [M(\varepsilon) + 1 - t, M(\varepsilon) + 1 - t + l(\varepsilon)]$$

satisfying

$$\|f_n(t + \tau_t) - f_n(t)\| < \frac{\varepsilon}{3} \tag{4.1}$$

for all $n \in \mathbb{N}$. Noting that $(f_n(t))$ is uniformly convergent on

$$[-M(\varepsilon) - l(\varepsilon) - 1, M(\varepsilon) + l(\varepsilon) + 1],$$

for the above $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m \ge n \ge N$ and $t \in [-M(\varepsilon) - l(\varepsilon) - 1, M(\varepsilon) + l(\varepsilon) + 1]$,

$$\|f_m(t) - f_n(t)\| < \frac{\varepsilon}{3}. \tag{4.2}$$

Combining (4.1) and (4.2), for all $m \ge n \ge N$ and $t \in \mathbb{R}$ with $|t| > M(\varepsilon)$, we have

$$\|f_m(t) - f_n(t)\| \le \|f_m(t) - f_m(t + \tau_t)\| + \|f_m(t + \tau_t) - f_n(t + \tau_t)\|$$

$$+\|f_n(t+\tau_t)-f_n(t)\| \le \varepsilon,$$

which and (4.2) yield that $(f_n(t))$ is uniformly convergent on $\mathbb{R}$. In view of

$$\{g_m(t)-g_n(t) : t \in \mathbb{R}\} \subset \overline{\{f_m(t)-f_n(t) : t \in \mathbb{R}\}}$$

for all $m, n \in \mathbb{N}$, we conclude that $(g_n(t))$ is also uniformly convergent on $\mathbb{R}$, i.e. (g_n) is convergent in $AP(\mathbb{R}, X)$. So G is precompact in $AP(\mathbb{R}, X)$. In addition, it follows from the above proof that F is precompact, and thus H is also precompact.

(iii) ⇒ (i) The proof is straightforward.

□

Remark 8.37 Theorem 8.36 can be seen as an extension of the corresponding compactness criteria for the subsets of $AP(\mathbb{R}, \mathbb{X})$ (cf. e.g., [22]).

Definition 8.38 $F \subset C_0(\mathbb{R}, X)$ is called equi-C_0 if

$$\lim_{|t|\to\infty} \sup_{f\in F} \|f(t)\| = 0.$$

Theorem 8.39 *The following two assertions are equivalent:*

(I) *F is equi-asymptotically almost periodic;*
(II) *G is equi-almost periodic and H is equi-C_0, where*

$$G = \{f_{AP} : f \in F\} \text{ and } H = \{f_{C_0} : f \in F\}.$$

Proof The proof from (II) to (I) is straightforward. We will only give the proof from (I) to (II) by using the idea in the proof of [71, p. 24, Theorem 2.5].

Since F is equi-asymptotically almost periodic, for every $k \in \mathbb{N}$, there exist a constant $M_k > 0$ and a relatively dense set $T(F, k) \subset \mathbb{R}$ such that

$$\|f(t+\tau)-f(t)\| < \frac{1}{k}, \tag{4.3}$$

for all $f \in F$, $t \in \mathbb{R}$ with $|t| > M_k$ and $\tau \in T(F, k)$ with $|t+\tau| > M_k$. Moreover, for every $f \in F \subset AAP(\mathbb{R}, \mathbb{X})$, noting that f is uniformly continuous, for the above $k \in \mathbb{N}$, there exists $\delta_k^f > 0$ such that

$$\|f(t_1)-f(t_2)\| < \frac{1}{k} \tag{4.4}$$

for all $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \delta_k^f$.

Now, for every $t \in \mathbb{R}$ and $k \in \mathbb{N}$, we choose $\tau_k^t \in T(F, k)$ with $t + \tau_k^t > M_k$. Also, we denote

$$g_k^f(t) = f(t + \tau_k^t), \quad t \in \mathbb{R},\ k \in \mathbb{N},\ f \in F.$$

Next, we divide the remaining proof into eight steps.

Step 1 For every $f \in F$, there holds

$$\left\| g_k^f(t_1) - g_k^f(t_2) \right\| < \frac{5}{k} \tag{4.5}$$

for all $k \in \mathbb{N}$, and $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \delta_k^f$.
In fact, by (4.3) and (4.4), we have

$$\begin{aligned}
\|g_k^f(t_1) - g_k^f(t_2)\| &= \|f(t_1 + \tau_k^{t_1}) - f(t_2 + \tau_k^{t_2})\| \\
&\le \|f(t_1 + \tau_k^{t_1}) - f(t_1 + \tau_k^{t_1} + \tau)\| \\
&\quad + \|f(t_1 + \tau_k^{t_1} + \tau) - f(t_2 + \tau_k^{t_1} + \tau)\| \\
&\quad + \|f(t_2 + \tau_k^{t_1} + \tau) - f(t_2 + \tau)\| \\
&\quad + \|f(t_2 + \tau) - f(t_2 + \tau + \tau_k^{t_2})\| \\
&\quad + \|f(t_2 + \tau + \tau_k^{t_2}) - f(t_2 + \tau_k^{t_2})\| < \frac{5}{k},
\end{aligned}$$

where $\tau \in T(F, k)$ satisfying

$$\min\left\{t_1 + \tau_k^{t_1} + \tau, t_2 + \tau_k^{t_1} + \tau, t_2 + \tau, t_2 + \tau + \tau_k^{t_2}\right\} > M_k.$$

Step 2 For every $k \in \mathbb{N}$, there holds

$$\|g_k^f(t + \tau) - g_k^f(t)\| < \frac{5}{k} \tag{4.6}$$

for all $f \in F$, $\tau \in T(F, k)$, and $t \in \mathbb{R}$.
In fact, by using (4.3), we have

$$\begin{aligned}
\|g_k^f(t + \tau) - g_k^f(t)\| &= \|f(t + \tau + \tau_k^{t+\tau}) - f(t + \tau_k^t)\| \\
&\le \|f(t + \tau + \tau_k^{t+\tau}) - f(t + \tau + \tau_k^{t+\tau} + \tau')\| \\
&\quad + \|f(t + \tau + \tau_k^{t+\tau} + \tau') - f(t + \tau_k^{t+\tau} + \tau')\| \\
&\quad + \|f(t + \tau_k^{t+\tau} + \tau') - f(t + \tau')\| \\
&\quad + \|f(t + \tau') - f(t + \tau' + \tau_k^t)\| \\
&\quad + \|f(t + \tau' + \tau_k^t) - f(t + \tau_k^t)\| < \frac{5}{k},
\end{aligned}$$

where $\tau' \in T(F, k)$ satisfying

$$\min\left\{t + \tau + \tau_k^{t+\tau} + \tau', t + \tau_k^{t+\tau} + \tau', t + \tau', t + \tau' + \tau_k^t\right\} > M_k.$$

Step 3 For every $n \in \mathbb{N}$, there holds

$$\left\| g_m^f(t) - g_n^f(t) \right\| < \frac{4}{n} \tag{4.7}$$

for all $f \in F$, $t \in \mathbb{R}$, and $m, n \in \mathbb{N}$ with $m \geq n$.

In fact, without loss of generality, we can assume that $M_{k+1} \geq M_k$ for all $k \in \mathbb{N}$. Then, by using (4.3), we have

$$\begin{aligned}
&\left\| g_m^f(t) - g_n^f(t) \right\| \\
&\quad = \| f(t + \tau_m^t) - f(t + \tau_n^t) \| \\
&\quad \leq \| f(t + \tau_m^t) - f(t + \tau_m^t + \tau) \| + \| f(t + \tau_m^t + \tau) - f(t + \tau_m^t + \tau + \tau_n^t) \| \\
&\qquad + \| f(t + \tau_m^t + \tau + \tau_n^t) - f(t + \tau + \tau_n^t) \| + \| f(t + \tau + \tau_n^t) - f(t + \tau_n^t) \| \\
&\qquad < \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \frac{1}{n} \leq \frac{4}{n},
\end{aligned}$$

where $\tau \in T(F, n)$ satisfying

$$\min\left\{t + \tau_m^t + \tau, t + \tau_m^t + \tau + \tau_n^t, t + \tau + \tau_n^t\right\} > M_m.$$

Step 4 Let

$$g^f(t) = \lim_{n \to \infty} g_n^f(t), \quad t \in \mathbb{R}, \ f \in F.$$

By Step 3, we know that for every $f \in F$, g^f is well-defined. Moreover, it follows from Step 3 that for every $n \in \mathbb{N}$, there holds

$$\| g^f(t) - g_n^f(t) \| \leq \frac{4}{n} \tag{4.8}$$

for all $f \in F$, R, and $n \in \mathbb{N}$.

Step 5 For every $f \in F$, g^f is uniformly continuous on $\mathbb{R}$.

In fact, by (4.5) and (4.8), we have

$$\begin{aligned}
\| g^f(t_1) - g^f(t_2) \| &\leq \| g^f(t_1) - g_n^f(t_1) \| + \| g_n^f(t_1) - g_n^f(t_2) \| \\
&\quad + \| g_n^f(t_2) - g^f(t_2) \| \leq \frac{4}{n} + \frac{5}{n} + \frac{4}{n} = \frac{13}{n},
\end{aligned}$$

for all $n \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \delta_n^f$.

Step 6 $\{g^f\}_{f\in F}$ is equi-almost periodic.

By (4.6) and (4.8), for every $n \in \mathbb{N}$, we get

$$\|g^f(t+\tau) - g^f(t)\| \le \|g^f(t+\tau) - g_n^f(t+\tau)\| + \|g_n^f(t+\tau) - g_n^f(t)\| + \|g_n^f(t) - g^f(t)\| \le \frac{4}{n} + \frac{5}{n} + \frac{4}{n} = \frac{13}{n},$$

for all $f \in F$, $\tau \in T(F, n)$, and $t \in \mathbb{R}$. Then, it follows that $\{g^f\}_{f\in F}$ is equi-almost periodic.

Step 7 $\{h^f\}_{f\in F}$ is equi-C_0, where $h^f(t) = f(t) - g^f(t)$ for all $f \in F$ and $t \in \mathbb{R}$. In fact, firstly, by Step 5, $h^f \in C(\mathbb{R}, X)$ for every $f \in F$; secondly, for every $n \in \mathbb{N}$, by (4.8) and the definition of τ_n^t, we have

$$\begin{aligned}
\|h^f(t)\| &= \|f(t) - g^f(t)\| \\
&\le \|f(t) - g_n^f(t)\| + \|g_n^f(t) - g^f(t)\| \\
&\le \|f(t) - f(t+\tau_n^t)\| + \frac{4}{n} \\
&\le \frac{1}{n} + \frac{4}{n} = \frac{5}{n},
\end{aligned}$$

for all $f \in F$, $t \in \mathbb{R}$ with $|t| > M_n$. Thus, $\{h^f\}_{f\in F}$ is equi-C_0.

Step 8 It follows from the above proof that $G = \{g^f\}_{f\in F}$ and $H = \{h^f\}_{f\in F}$. This completes the proof.

□

Bibliographical Notes Section 1 in this chapter is in [51, 54]. Theorem 8.32 is in [72] with a different proof. Section 3 is a work by Ding et al. [24].