Chapter 3
Almost Automorphy of the Function $f(t, x)$

1 The Nemytskii's Operator

Definition 3.1 A continuous function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is said to be almost automorphic if $f(t, x)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $x \in K$, where K is any bounded subset of $\mathbb{X}$. In other words for every sequence of real numbers (s'_n) there exists a subsequence (s_n) such that

$$g(t, x) = \lim_{n\to\infty} f(t + s_n, x)$$

is well-defined in $t \in \mathbb{R}$ for all K and

$$\lim_{n\to\infty} g(t - s_n, x) = f(t, x)$$

for all $t \in \mathbb{R}$ and $x \in K$.

We denote by $AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ the set of all such functions.

Theorem 3.2 *If $f, f_1, f_2 \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, then we have*

(i) $f_1 + f_2 \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$.
(ii) $\lambda f \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$*, for any scalar* λ.

Proof Obvious. □

Theorem 3.3 *If $f \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, then*

$$\sup_{t\in\mathbb{R}} \|f(t, x)\| = \sup_{t\in\mathbb{R}} \|g(t, x)\| = C_x < \infty$$

for x in any bounded set $K \subset \mathbb{X}$ where g is the function in Definition 3.1.

G. M. N'Guérékata, *Almost Periodic and Almost Automorphic Functions in Abstract Spaces*, https://doi.org/10.1007/978-3-030-73718-4_3

Proof It is analogous to the proof of Remark 2.6. □

Theorem 3.4 *If $f \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ is lipschitzian in x uniformly in $t \in \mathbb{R}$, then the function g as in Definition 3.1 is also lipschitzian with the same Lipschitz constant.*

Proof Let L be a Lipschitz constant for the function f, i.e.

$$\|f(t, x) - f(t, y)\| < L\|x - y\|$$

for x, y in any bounded subset K of $\mathbb{X}$ uniformly in $t \in \mathbb{R}$.

Let $t \in \mathbb{R}$ be arbitrary and $\varepsilon > 0$ and K a bounded set in $\mathbb{X}$ be given. Then for any sequence of real numbers (s'_n), there exists a subsequence (s_n) such that

$$\|f(t + s_n, x) - g(t, x)\| < \frac{\varepsilon}{2}$$

and

$$\|g(t - s_n, x) - f(t, x)\| < \frac{\varepsilon}{2}$$

for n sufficiently large and uniformly in $x \in K$.

Let us write for $x, y \in K$

$$g(t, x) - g(t, y) = g(t, x) - f(t + s_n, x) + f(t + s_n, x) - f(t + s_n, y) + f(t + s_n, y) - g(t, y).$$

For n sufficiently large we get

$$\|g(t, x) - g(t, y)\| < \varepsilon + L\|x - u\|.$$

And since ε is arbitrary we obtain

$$\|g(t, x) - g(t, y)\| \leq \varepsilon$$

uniformly for $x, y \in K$, which completes the proof. □

Theorem 3.5 ([39]) *Let $f \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and assume that $f(t, \cdot)$ is uniformly continuous on each bounded set $K \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$; in other words, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in K$ with $\|x - y\| < \delta$, then $\|f(t, x) - f(t, y)\| < \varepsilon$ for all $t \in \mathbb{R}$. Let $\varphi \in AA(\mathbb{X})$.*

Then the Nemytskii operator $\mathcal{N} : \mathbb{R} \to \mathbb{X}$ defined by $\mathcal{N}(\cdot) := f(\cdot, \varphi(\cdot))$ is in $AA(\mathbb{X})$.

Proof Let (s'_n) be a sequence of real numbers. Then there exists a subsequence $(s_n) \subset (s'_n)$ such that

(i) $\lim_{n\to\infty} f(t+s_n,x) = g(t,x)$, for each $t\in\mathbb{R}$ and $x\in\mathbb{X}$,
(ii) $\lim_{n\to\infty} g(t-s_n,x) = f(t,x)$, for each $t\in\mathbb{R}$ and $x\in\mathbb{X}$,
(iii) $\lim_{n\to\infty} \varphi(t+s_n) = \gamma(t)$ for each $t\in\mathbb{R}$,
(iv) $\lim_{n\to\infty} \gamma(t+s_n) = \varphi(t)$ for each $t\in\mathbb{R}$.

Let us define $G:\mathbb{R}\to\mathbb{X}$ by $G(t) = g(t,\gamma(t))$. Then we obtain

$$\lim_{n\to\infty} \mathcal{N}(t+s_n) = G(t)$$

and

$$\lim_{n\to\infty} G(t-s_n) = \mathcal{N}(t)$$

for each $t\in\mathbb{R}$.

Consider the inequality

$$\begin{aligned}\|\mathcal{N}(t+s_n) - G(t)\| \le\ & \|f(t+s_n,\varphi(t+s_n)) - f(t+s_n,\gamma(t))\| \\ & + \|f(t+s_n,\gamma(t)) - g(t,\gamma(t))\|.\end{aligned}$$

Since $\varphi \in AA(\mathbb{X})$, then φ and γ are bounded. Let us choose $K\in\mathbb{X}$ such that $\varphi(t),\gamma(t)\in K$ for all $t\in\mathbb{R}$. In view of (iii) and the uniform continuity of $f(t,x)$ in $x\in K$, we will have

$$\lim_{n\to\infty} \|f(t+s_n,\varphi(t+s_n)) - f(t+s_n,\gamma(t))\| = 0.$$

Now by (i), we get

$$\lim_{n\to\infty} \|f(t+s_n,\gamma(t)) - g(t,\gamma(t))\| = 0,$$

which proves that for each $t\in\mathbb{R}$

$$\lim_{n\to\infty} \mathcal{N}(t+s_n) = G(t).$$

Similarly, we can prove that

$$\lim_{n\to\infty} G(t-s_n) = \mathcal{N}(t)$$

for each $t\in\mathbb{R}$. The proof is now complete. □

Theorem 3.6 ([39]) *Let $f \in AAA(\mathbb{R}^+\times\mathbb{X},\mathbb{X})$ with principal term $g(t,x)$ and corrective term $h(t,x)$. Assume that $g(t,x)$ is uniformly continuous on any bounded set $K\subset\mathbb{X}$ uniformly for $t\in\mathbb{R}$. Assume also that $\varphi\in AAA(\mathbb{X})$. Then the Nemytskii operator $\mathcal{N}:\mathbb{R}\to\mathbb{X}$ defined by $\mathcal{N}(\cdot) := f(\cdot,\varphi(\cdot))$ is in $AAA(\mathbb{X})$.*

Proof Let $\alpha(t)$ and $\beta(t)$ be the principal and corrective terms of $\varphi(t)$, respectively. Let us write

$$f(t, \varphi(t)) = g(t, \alpha(t)) + f(t, \varphi(t)) - g(t, \alpha(t)) = g(t, \alpha(t)) + g(t, \varphi(t)) \\ - g(t, \alpha(t)) + h(t, \varphi(t)).$$

In view of Theorem 3.5, $g(t, \alpha(t)) \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$.

On the other hand, the uniform continuity of $g(t, \varphi(t))$ implies that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|g(t, \varphi(t)) - g(t, \alpha(t))\| < \varepsilon$$

if $\varphi(t), \alpha(t) \in K$ for any $t \in \mathbb{R}^+$ and a given bounded set $K \subset \mathbb{X}$ and $\|\varphi(t) - \alpha(t)\| < \delta$. Moreover since $\beta(t) \in C_0(\mathbb{R}, \mathbb{X})$, there exists $T > 0$ such that

$$\|\varphi(t) - \alpha(t)\| = \|\beta(t)\| < \delta,$$

for $t > T$. Consequently, we get

$$\lim_{t\to\infty} \|g(t, \varphi(t)) - g(t, \alpha(t))\| = 0.$$

We know also that

$$\lim_{t\to\infty} \|h(t, \varphi(t))\| = 0.$$

This proves that

$$g(t, \varphi(t)) - g(t, \alpha(t)) + h(t, \varphi(t)) \in C_0(\mathbb{R}^+, \mathbb{X}),$$

and consequently

$$\mathcal{N}(\cdot) := f(\cdot, \varphi(\cdot)) \in AAA(\mathbb{X})$$

□

Bibliographical Notes Most of this chapter are contained in the first edition of this book. It is noted that C. Lizama and J.G Mesquita [41–43] and Milcé et al. [46, 47, 50, 62, 63] studied almost automorphy on time scales and its application to dynamic equations on time scales. This is another growing field which needs further investigation.