Chapter 10
The Equation x'(t)=A(t)x(t)+f(t)

1 The Equation x'(t)=A(t)x(t)+f(t)

Case I Let us first assume that $\mathbb{X}$ is of **finite dimension**, say $\mathbb{X} = \mathbb{C}^n$.

Let us consider the inhomogeneous linear evolution equations of the form

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t), \quad t \in \mathbb{R},\ x(t) \in \mathbb{X}, \tag{1.1}$$

where $A(\cdot)$ is a τ-periodic (unbounded) linear operator-valued function and $f \in AA(\mathbb{X})$.

Theorem 10.1 ([40, 55]) *Every bounded solution on the whole real line of Eq. (1.1) is in $AA(\mathbb{X})$.*

Proof First, we note that by Floquet theory of periodic ordinary differential equations and by Proposition 2.11 [40], without loss of generality, we may assume that A is independent of t.

Next, we will show that the problem can be reduced to the one-dimensional case. In fact, if A is independent of t, by a change of variable if necessary, we may assume that A is of Jordan normal form. In this direction, we can go further with the assumption that A has only one Jordan box; that is, we have to prove the theorem for equations of the form

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

Let us consider the last equation involving $x_n(t)$. We have

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$$\dot{x}_n(t) = \lambda x_n(t) + f_n(t), \quad t \in \mathbb{R},\ x(t) \in \mathbb{C}^n.$$

If $\Re\lambda \neq 0$, then we can easily check that either

$$y(t) = \int_{-\infty}^{t} e^{\lambda(t-\xi)} f(\xi)d\xi \quad (\Re\lambda < 0)$$

or

$$z(t) = \int_{t}^{\infty} e^{\lambda(t-\xi)} f(\xi)d\xi \quad (\Re\lambda > 0)$$

is a unique bounded solution of Eq. (1.1). Moreover, by Proposition 2.11 [40] in both cases, $y(t)$ and $z(t)$ are in $AA(\mathbb{X})$. Hence, x_n is in $AA(\mathbb{X})$.

If $\Re\lambda = 0$, then $\lambda = i\eta$ for $\eta \in \mathbb{R}$. By assumption, there is a constant c such that the function

$$x_n(t) := ce^{i\eta t} + \int_0^t e^{i\eta(t-\xi)} f(\xi)d\xi$$

is bounded on $\mathbb{R}$. This yields the boundedness of $\int_0^t e^{-i\eta\xi} f(\xi)d\xi$ on $\mathbb{R}$. Hence, $\int_0^t e^{-i\eta\xi} f(\xi)d\xi$ is in $AA(\mathbb{X})$. Finally, this yields that x_n is in $AA(\mathbb{X})$.

Let us consider next the equation involving x_{n-1} and x_n. Since x_n is in $AA(\mathbb{X})$, by repeating the above argument, we can show that x_{n-1} is also in $AA(\mathbb{X})$. Continuing this process, we can show that all $x_k(\cdot)$ are in $AA(\mathbb{X})$. The proof is complete. □

Case II Let us now consider Eq. (1.1) in an **infinite dimensional Banach space** $\mathbb{X}$ where $f \in AA(\mathbb{X})$, and $A(t)$ generates a 1-periodic evolutionary process $(U(t,s))_{t\geq s}$ in $\mathbb{X}$, that is, a two-parameter family of bounded linear operators that satisfies the following conditions:

(i) $U(t,t) = I$ for all $t \in \mathbb{R}$,
(ii) $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r$,
(iii) The map $(t,s) \mapsto U(t,s)x$ is continuous for every fixed $x \in \mathbb{X}$,
(iv) $U(t+1, s+1) = U(t,s)$ for all $t \geq s$ (1-periodicity),
(v) $\|U(t,s)\| \leq Ne^{\omega(t-s)}$ for some positive $N,\ \omega$ independent of $t \geq s$.

We emphasize that the above choice of the period of the equations is merely for the simplification of the notation but does not mean a restriction. We refer the reader to [6, 17, 36] for more information on the applications of this concept of evolutionary processes to partial differential equations.

Definition 10.2 An $\mathbb{X}$-valued continuous function u on $\mathbb{R}$ is said to be a mild solution of Eq. (1.1) if

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi)d\xi, \quad \forall t \geq s; t,s \in \mathbb{R}. \tag{1.2}$$

Lemma 10.3 ([40]) *Let u be a bounded mild solution of Eq. (1.1) on $\mathbb{R}$ and f be almost automorphic. Then, u is almost automorphic if and only if the sequence $(u(n))_{n\in\mathbb{Z}}$ is almost automorphic.*

Proof

Necessity: Obviously, if u is almost automorphic, the sequence $(u(n))_{n\in\mathbb{Z}}$ is almost automorphic.

Sufficiency: Let the sequence $(u(n))_{n\in\mathbb{Z}}$ be almost automorphic. We now prove that u is almost automorphic. The proof is divided into several steps:

Step 1: We first suppose that (n'_k) is a given sequence of integers. Then, there exist a subsequence (n_k) and a sequence $(v(n))$ such that

$$\lim_{k\to\infty} u(n+n_k) = v(n); \qquad \lim_{k\to\infty} v(n-n_k) = u(n), \quad \forall n \in \mathbb{Z} \tag{1.3}$$

$$\lim_{k\to\infty} f(t+n_k) = g(t); \qquad \lim_{k\to\infty} g(t-n_k) = f(t), \quad \forall t \in \mathbb{R}. \tag{1.4}$$

For every fixed $t \in \mathbb{R}$, let us denote by $[t]$ the integer part of t. Then, define

$$v(\eta) := U(\eta,[t])v([t]) + \int_{[t]}^{\eta} U(\eta,\xi)g(\xi)d\xi, \qquad \eta \in [[t],[t]+1).$$

In this way, we can define v on the whole line $\mathbb{R}$. Now, we show that

$$\lim_{k\to\infty} u(t+n_k) = v(t).$$

In fact,

$$\begin{aligned}
\lim_{k\to\infty} \|u(t+n_k)-v(t)\| &\le \lim_{k\to\infty} \|U(t+n_k,[t]+n_k)u([t]+n_k)-U(t,[t])v([t])\| \\
&\quad + \lim_{k\to\infty}\int_{[t]}^{t} \|U(t,\eta)\|\|f(\eta+n_k)-g(\eta)\|d\eta \\
&= \lim_{k\to\infty} \|U(t,[t])u([t]+n_k)-U(t,[t])v([t])\| \\
&\quad + \lim_{k\to\infty}\int_{[t]}^{t} \|U(t,\eta)\|\|f(\eta+n_k)-g(\eta)\|d\eta = 0.
\end{aligned}$$

Similarly, we can show that

$$\lim_{k\to\infty} \|v(t-n_k)-u(t)\| = 0.$$

Step 2: Now, we consider the general case where $(s'_k)_{k\in\mathbb{Z}}$ may not be an integer sequence. The main lines are similar to those in Step 1 combined with the strong continuity of the process and the precompactness of the range of the function f.

Set $n'_k = [s'_k]$ for every k. Since $(t_k)_{k\in\mathbb{Z}}$, where $t_k := s'_k - [s'_k]$, is a sequence in $[0, 1)$, we can choose a subsequence (n_k) from $\{n'_k\}$ such that $\lim_{k\to\infty} t_k = t_0 \in [0, 1]$ and (1.3) holds for a function v, as shown in Step 1.

Let us first consider the case $0 < t_0 + t - [t_0 + t]$. We show that

$$\lim_{k\to\infty} u(t_k + t + n_k) = \lim_{k\to\infty} u(t_0 + t + n_k) = v(t_0 + t). \tag{1.5}$$

In fact, for sufficiently large k, from the above assumption, we have $[t_0+t] = [t_k+t]$. Using the 1-periodicity of the process $(U(t, s))_{t\geq s}$, we have

$$\|u(t_k + t + n_k) - u(t_0 + t + n_k)\| \leq A(k) + B(k), \tag{1.6}$$

where $A(k)$ and $B(k)$ are defined and estimated as below. By the 1-periodicity of the process $(U(t, s))_{t\geq s}$, we have

$$\begin{aligned} A(k) :=& \|U(t_k + t + n_k, [t_k + t] + n_k)u([t_k + t] + n_k) \\ &-U(t_0 + t + n_k, [t_0 + t] + n_k)u([t_0 + t] + n_k)\| \\ =& \|U(t_k + t, [t_0 + t])u([t_0 + t] + n_k) - U(t_0 + t, [t_0 + t])u([t_0 + t] + n_k)\|. \end{aligned}$$

Using the strong continuity of the process $(U(t, s))_{t\geq s}$ and the precompactness of the range of the sequence $(u(n))_{n\in\mathbb{Z}}$, we have $\lim_{k\to\infty} A(k) = 0$. Next, we define

$$B(k) := \| \int_{[t_k+t]+n_k}^{t_k+t+n_k} U(t_k+t+n_k, \eta) f(\eta)d\eta - \int_{[t_0+t]+n_k}^{t_0+t+n_k} U(t_0+t+n_k, \eta) f(\eta)d\eta\|.$$

By the 1-periodicity of the process $(U(t, s))_{t\geq s}$ and $[t_0 + t] = [t_k + t]$, we have

$$\begin{aligned} B(k) =& \left\| \int_0^{t_k+t-[t_k+t]} U(t_k + t + n_k, [t_0 + t] + n_k + \theta) f([t_0 + t] + n_k + \theta)d\theta \right. \\ &\left. - \int_0^{t_0+t-[t_0+t]} U(t_0 + t + n_k, [t_0 + t] + n_k + \theta) f([t_0 + t] + n_k + \theta)d\theta \right\| \\ =& \left\| \int_0^{t_k+t-[t_0+t]} U(t_k + t - [t_0 + t], \theta) f([t_0 + t] + n_k + \theta)d\theta \right. \\ &\left. - \int_0^{t_0+t-[t_0+t]} U(t_0 + t - [t_0 + t], \theta) f([t_0 + t] + n_k + \theta)d\theta \right\|. \end{aligned}$$

From the strong continuity of the process $(U(t, s))_{t\geq s}$ and the precompactness of the range of f, it follows that $\lim_{k\to\infty} B(k) = 0$. So, in view of Step 1, we see that (1.5) holds.

Next, we consider the case when $t_0+t-[t_0+t]=0$, that is, t_0+t is an integer. If $t_k+t \geq t_0+t$, we can repeat the above argument. So, we omit the details. Now, suppose that $t_k+t < t_0+t$. Then,

$$\|u(t_k+t+n_k)-u(t_0+t+n_k)\| \leq C(k)+D(k), \tag{1.7}$$

where $C(k)$ and $D(k)$ are defined and estimated as below:

$$\begin{aligned} C(k) &:= \|U(t_k+t+n_k,[t_k+t]+n_k)u([t_k+t]+n_k) \\ &\quad -U(t_0+t+n_k,t_0+t-1+n_k)u(t_0+t-1+n_k)\| \\ &= \|U(t_k+t,t_0+t-1)u(t_0+t-1+n_k) \\ &\quad -U(t_0+t,t_0+t-1)u(t_0+t-1+n_k)\|. \end{aligned}$$

Now, using the strong continuity of the process $(U(t,s))_{t\geq s}$ and the precompactness of the range of the sequence $(u(n))_{n\in\mathbb{Z}}$, we obtain $\lim_{k\to\infty} C(k)=0$.

As for $D(k)$, we have

$$\begin{aligned} D(k) &:= \left\| \int_{[t_k+t]+n_k}^{t_k+t+n_k} U(t_k+t+n_k,\eta)f(\eta)d\eta \right. \\ &\quad \left. - \int_{[t_0+t]+n_k-1}^{t_0+t+n_k} U(t_0+t+n_k,\eta)f(\eta)d\eta \right\| \\ &= \left\| \int_{[t_0+t]+n_k-1}^{t_k+t+n_k} U(t_k+t+n_k,\eta)f(\eta)d\eta \right. \\ &\quad \left. - \int_{[t_0+t]+n_k-1}^{t_0+t+n_k} U(t_0+t+n_k,\eta)f(\eta)d\eta \right\| \\ &= \left\| \int_0^{t_k+1-t_0} U(t_k+t,t_0+t-1+\theta)f(t_0+t+n_k-1+\theta)d\theta \right. \\ &\quad \left. - \int_0^1 U(t_0+t,t_0+t-1+\theta)f(t_0+t+n_k-1+\theta)d\theta \right\|. \end{aligned}$$

From the strong continuity of the process $(U(t,s))_{t\geq s}$ and the precompactness of the range of f, it follows that $\lim_{k\to\infty} D(k)=0$. This finishes the proof of the lemma. □

Theorem 10.4 ([40]) *Let $A(t)$ in Eq. (1.1) generate a 1-periodic strongly continuous evolutionary process, and let f be almost automorphic. Assume further that the space $\mathbb{X}$ does not contain any subspace isomorphic to c_0, and the part of spectrum of the monodromy operator $U(1,0)$ on the unit circle is countable. Then, every bounded mild solution of Eq. (1.1) on the real line is almost automorphic.*

Proof The theorem is an immediate consequence of the results above. In fact, we need only to prove the sufficiency. Let us consider the discrete equation

$$u(n+1) = U(n+1,n)u(n) + \int_n^{n+1} U(n+1,\xi)f(\xi)d\xi, \quad n \in \mathbb{Z}.$$

From the 1-periodicity of the process $(U(t,s))_{t\geq s}$, this equation can be rewritten in the form

$$u(n+1) = Bu(n) + y_n, \quad n \in \mathbb{Z}, \tag{1.8}$$

where

$$B := U(1,0);\ y_n := \int_n^{n+1} U(n+1,\xi)f(\xi)d\xi, \quad n \in \mathbb{Z}.$$

We are going to show that the sequence $(y_n)_{n\in\mathbb{Z}}$ defined as above is almost automorphic. In fact, since f is automorphic, for any given sequence (n'_k), there are a subsequence (n_k) and a measurable function g such that $\lim_{k\to\infty} f(t+n_k) = g(t)$ and $\lim_{m\to\infty} g(t-n_m) = f(t)$ for every $t \in \mathbb{R}$. Therefore, if we set

$$w_n = \lim_{k\to\infty} \int_{n+n_k}^{n+n_k+1} U(n+n_k,\xi)f(\xi)d\xi, \quad n \in \mathbb{Z},$$

then, by the 1-periodicity of $(U(t,s))_{t\geq s}$ and the Lebesgue Dominated Convergence Theorem, we have

$$w_n = \lim_{k\to\infty} \int_n^{n+1} U(n,\eta)f(n_k+\eta)d\eta = \int_n^{n+1} U(n,\eta)g(\eta)d\eta.$$

Therefore, $\lim_{k\to\infty} y_{n+n_k} = w_n$ for every $n \in \mathbb{Z}$. Similarly, we can show that $\lim_{k\to\infty} w_{n-n_k} = y_n$.

By Lemma 2.11 [48], since $(u(n))$ is a bounded solution of (1.8), $\mathbb{X}$ does not contain any subspace isomorphic to c_0, and the part of spectrum of $U(1,0)$ on the unit circle is countable, $(u(n))$ is almost automorphic. By Lemma 10.3, this yields that the solution u itself is almost automorphic. □

Now, let us consider Eq. (1.1) where $A(t) = A$.

Theorem 10.5 ([70]) *Suppose that A generates an asymptotically stable C_0-semigroup $(T(t))_{t\geq 0}$, that is,*

$$\lim_{t\to\infty} T(t)x = 0, \quad \textit{for every}\ \ x \in \mathbb{X},$$

and $f \in AA(\mathbb{X})$*. If* $x(t)$ *is a mild solution of Eq. (1.1) with a relatively compact range in* $\mathbb{X}$*, then* $x \in AA(\mathbb{X})$*.*

Proof $x(t)$ will admit the representation

$$x(t) = T(t-a)x(a) + \int_a^t T(t-\xi)f(\xi)d\xi$$

for each $a \in \mathbb{R}$ and $t \geq a$.

Let (s'_n) be a sequence of real numbers. Since $f \in AA(\mathbb{X})$, we can find a subsequence $(s_n) \subset (s'_n)$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well-defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t-s_n) = f(t)$$

for each $t \in \mathbb{R}$.

Suppose now that the sequence $(x(t_0+s_n))$ is not convergent for some $t_0 \in \mathbb{R}$. Then, there exist some $\alpha > 0$ and two subsequences (σ'_n) and (σ''_n) of (s_n) such that

$$\|x(t_0+\sigma'_n) - x(t_0+\sigma''_n)\| > \alpha \tag{1.9}$$

for $n = 1, 2, \ldots$.

We have, for $a \leq t_0$

$$\begin{aligned} x(t_0+\sigma'_n) - x(t_0+\sigma''_n) = T(t_0-a)[x(t_0+\sigma'_n) - x(t_0+\sigma''_n)] \\ + \textstyle\int_a^{t_0} T(t_0-\xi)[f(\xi+\sigma'_n) - f(\xi-\sigma''_n)]d\xi. \end{aligned}$$

Let $K = \overline{\{x(t) \;/\; t \in \mathbb{R}\}}$ be the closure of the range of $x(t)$; by assumption K is compact in $\mathbb{X}$.

Since $\lim_{t\to\infty} T(t)x = 0$ for every $x \in \mathbb{X}$, it is easy to observe that $\lim_{t\to\infty} T(t)x = 0$ uniformly in any compact subset of $\mathbb{X}$. Thus we can choose some $a < 0$ such that

$$\|T(t_0-a)x(a+\sigma'_n)\| < \frac{\varepsilon}{3}$$

and

$$\|T(t_0-a)x(a+\sigma''_n)\| < \frac{\varepsilon}{3}$$

for all $n = 1, 2, \ldots$ Now, fix a and put

$$F_n(\xi) := T(t-\xi)[f(\xi+s_n') - f(\xi+s_n'')]$$

with $a \le \xi \le t_0$. Since $\sup_{t\in\mathbb{R}} \|f(t)\| < \infty$, and $\|T(t_0)\| \le M$ for some $M > 0$, we get

$$\|F_n(\xi)\| \le \|T(t_0-\xi)\|(\|f(\xi+s_n')\| + \|f(\xi+s_n'')\|) \le L$$

for some $L < \infty$.

Also, we observe that $\lim_{n\to\infty} F_n(\xi) = 0$ in the strong sense for every ξ since $\lim_{n\to\infty} f(t+s_n)$ exists for every t, and (σ_n') and (σ_n'') both are subsequences of (s_n). Finally, $F_n(\xi)$ is measurable for each $n = 1, 2, \ldots$

Using Lebesgue's Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_a^t T(t-\xi)[f(\xi+\sigma_n') - f(\xi+\sigma_n'')]d\xi = 0.$$

This implies the existence of some positive integer N such that

$$\|x(t_0+\sigma_n') - x(t_0+\sigma_n'')\| < \varepsilon \ \text{ if } \ n > N,$$

which contradicts (1.9).

Consequently, we deduce that the sequence $(x(t+s_n))$ is convergent in $\mathbb{X}$ for $t \in \mathbb{R}$.

Let $y(t) := \lim_{n\to\infty} x(t+s_n), \quad t \in \mathbb{R}$. It follows that

$$y(t) = T(t-a)y(a) + \int_a^t T(t-\xi)g(\xi)d\xi$$

for every $a \in \mathbb{R}$ and $t \ge a$. Moreover, $\Gamma := \{y(t) \ / \ t \in \mathbb{R}\} \subset K$. And consequently, Γ is relatively compact in $\mathbb{X}$. We may assume that

$$\lim_{n\to\infty} y(t-s_n) = u(t)$$

pointwise on $\mathbb{R}$.

Using the same argument as above, we can get

$$u(t) = T(t-a)u(a) + \int_a^t T(t-\xi)g(\xi)d\xi$$

for every $a \in \mathbb{R}$ and $t \geq a$. We also have

$$\{u(t)\ /\ t \in \mathbb{R}\} \subset \overline{\{y(t)\ /\ t \in \mathbb{R}\}} \subset K.$$

It remains to prove that $u(t) = x(t), t \in \mathbb{R}$.

Let us write $y(t - s_n) - x(t) = T(t - a)y(a - s_n) - T(t - a)x(a) + \int_a^t T(t - \xi)(g(\xi - s_n) - f(s))d\xi$.

Fix $t \in \mathbb{R}$, and let $\varepsilon > 0$. Since K is compact, one may choose $a < 0$ large enough such that

$$\|T(t - a)y(a - s_n)\| < \frac{\varepsilon}{3}, \quad \forall n$$

and

$$\|T(t - a)x(a)\| < \frac{\varepsilon}{3}.$$

The second inequality is based on the assumption that $\lim_{t\to\infty} T(t)x = 0$. Now, fix a, and let

$$H_n(\xi) := T(t - \xi)(g(\xi - s_n) - f(\xi)).$$

It is clear that each $H_n(\xi),\ n = 1, 2, \ldots$, is bounded in norm since $\sup_{t\geq 0} \|T(t)\| < \infty$ and $\sup_{t\in\mathbb{R}} \|f(t)\| < M < \infty$. By Lebesgue's Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_a^t T(t - \xi)(g(\xi - s_n) - f(\xi))d\xi = 0.$$

We then obtain

$$\|y(t - s_n) - x(t)\| < \varepsilon,$$

if $n > N$ for some given positive integer N. This implies that $x(t) = u(t)$ for each $t \in \mathbb{R}$. The proof is complete. □

Theorem 10.6 (N'Guérékata [55]) *Assume that A is bounded and $f \in AA(\mathbb{X})$. Let $x(t)$ be a (strong) solution of Eq. (1.1) with a relatively compact range in $\mathbb{X}$. Assume also that there exists a finite dimensional subspace $\mathbb{X}_1$ of $\mathbb{X}$ with the properties:*

(i) $e^{tA}u \in \mathbb{X}_1, \quad \forall u \in \mathbb{X}_1$;
(ii) $Ax(0) \in \mathbb{X}_1$;
(iii) $(e^{tA} - I)f(s) \in \mathbb{X}_1 \quad \forall t, s \in \mathbb{R}$.

Then, $x \in AA(\mathbb{X})$.

Proof Consider the projection $P : \mathbb{X} \to \mathbb{X}_1$. Then, we have $\mathbb{X} = \mathbb{X}_1 \oplus N(P)$, where $N(P)$ is the null space of P. Note that $Q = I - P$ is the projection on $N(P)$. Both P and Q are bounded linear operators.

Let $x(t)$ be a solution of Eq. (1.1). Then, we can write

$$x(t) = x_1(t) + y(t), \quad t \in \mathbb{R},$$

where $x_1(t) = Px(t) \in \mathbb{X}_1$ and $y(t) = Qx(t) \in N(P)$.

Since the range of $x(t)$ is relatively compact in $\mathbb{X}$, so are the ranges of $x_1(t)$ and $y(t)$ as we can easily observe. Also,

$$x'(t) = x_1'(t) + y(t) = Ax_1(t) + Ay(t) + Pf(t) + Qf(t), \quad t \in \mathbb{R}. \tag{1.10}$$

$x(t)$ has the integral representation

$$\begin{aligned} x(t) &= e^{tA}x(0) + \textstyle\int_0^t e^{(t-\xi)A} f(\xi)d\xi \\ &= e^{tA}x(0) + \textstyle\int_0^t f(\xi)d\xi + \int_0^t (e^{(t-\xi)A} - I) f(\xi)d\xi. \end{aligned}$$

Using assumption (iii), we can deduce that $\int_0^t (e^{(t-\xi)A} - I) f(\xi)d\xi \in \mathbb{X}_1$; then, applying Q to both sides of the last equation above, we get

$$y(t) = Qe^{tA}x(0) + Q\int_0^t f(\xi)d\xi = Qe^{tA}x(0) + \int_0^t Qf(\xi)d\xi.$$

Thus,

$$y'(t) = QAe^{tA}x(0) + Qf(t) = Qf(t)$$

since $Ax(0) \in \mathbb{X}_1$, so $e^{tA}Ax(0) \in \mathbb{X}_1$ by (ii).

Now, $Qf(t) : \mathbb{R} \to \mathbb{X}$ is an almost automorphic function since Q is a bounded linear operator. Hence, $y'(t) \in AA(\mathbb{X})$. Thus, $y(t) \in AA(\mathbb{X})$ since its range is relatively compact in $\mathbb{X}$ in view of Theorem 4.3.

Now, if we apply P to both sides of Eq. (1.10), we get in $\mathbb{X}_1$ the following equation:

$$x_1'(t) = PAx_1(t) + PAy(t) + P^3 f(t) + PAf(t), \quad t \in \mathbb{R}.$$

We observe that the function $g(t) := PAy(t) + P^3 f(t) + PAf(t)$ is almost automorphic.

Now, the operator PA restricted to the subspace $\mathbb{X}_1$ is a matrix and the function $x_1(t)$ is bounded since its range is relatively compact. So, we deduce that it is almost automorphic in view of Theorem 4.5.

Finally, $x(t) \in AA(\mathbb{X})$ as the sum of two almost automorphic functions.

The proof is complete. □

Now we consider in a general Banach space $\mathbb{X}$, the equation:

$$x'(t) = (A + B)x(t), \quad t \in \mathbb{R} \tag{1.11}$$

and the associated inhomogeneous one

$$x'(t) = (A + B)x(t) + f(t), \quad t \in \mathbb{R}. \tag{1.12}$$

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We make the following assumptions:

(i) A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ such that $T(t)x : \mathbb{R} \to \mathbb{X}$ is almost automorphic for each $x \in \mathbb{X}$
(ii) There exists a finite dimensional subspace $\mathbb{X}_1$ of $\mathbb{X}$ such that $D(A) \cap \mathbb{X}_1$ is dense in $\mathbb{X}$.
(iii) The projection $P : \mathbb{X} \to \mathbb{X}_1$ commutes with A.
(iv) B is a continuous linear operator such that $B(\mathbb{X}) = \mathbb{X}_1$.

Theorem 10.7 (N'Guérékata [55]) *Under assumptions (i)–(iv), every bounded solution of Eq. (1.11) is almost automorphic.*

Proof We recall that P is a bounded linear operator and has the property

$$\mathbb{X} = \mathbb{X}_1 \oplus N(P),$$

where $N(P)$ is the kernel of P. We set $Q = I - P$.

Now, if $x(t)$ is a bounded solution of Eq. (1.11), then it can be decomposed as

$$x(t) = x_1(t) + x_2(t) \quad t \in \mathbb{R},$$

where $x_1(t) = Px(t) \in \mathbb{X}_1$ and $x_2(t) = Qx(t) \in N(P)$ are also bounded.

First, let us show that $x_2(t) \in AA(\mathbb{X})$. We have

$$\begin{aligned} x_2'(t) = \tfrac{d}{dt}Qx(t) = Q\tfrac{d}{dt}x(t) &= Q(A+B)x(t) \\ &= QAx(t), \quad \text{since } \; QBx(t) = 0 \\ &= AQx(t), \quad \text{since } \; A \; \text{and} \; Q \; \text{commute} \\ &= Ax_2(t). \end{aligned}$$

Thus, we can write $x_2(t) = T(t)x_2(0), \quad t \in \mathbb{R}$, which shows that $x_2(t) \in AA(\mathbb{X})$.

Now, if we apply P to Eq. (1.11) and use the commutativity of A and P, we obtain

$$x_1'(t) = (A + PB)x_1(t) + PBx_2(t),$$

where $g(t) := PBx_2(t) \in AA(\mathbb{X})$.

It is clear that $A + PB = A + B$ is a linear operator restricted to $D(A) \cap \mathbb{X}_1 \equiv \mathbb{X}_1$ because of assumption (ii). Since $x_1(t)$ is bounded, it is almost automorphic (Theorem 4.5). Finally, $x(t) \in AA(\mathbb{X})$ as the sum of two almost automorphic functions. The proof is complete. □

Theorem 10.8 *Assume that assumptions (i)–(iv) above are satisfied and* $f \in AA(\mathbb{X})$. *Then, every solution of Eq. (1.12) with a relatively compact range is almost automorphic.*

Proof We start the proof as in Theorem 10.7 with the same notations. Consider a solution $x(t)$ of Eq. (1.12) with a relatively compact range in $\mathbb{X}$, and let

$$x(t) = x_1(t) + x_2(t), \quad t \in \mathbb{R},$$

as above. Observe that the range of $x_2(t)$ is also relatively compact in $\mathbb{X}$. It is easy to check that it satisfies the following equation in $N(P)$:

$$x_2'(t) = Ax_2(t) + Qf(t), \quad t \in \mathbb{R}.$$

The function $Qf(t) : \mathbb{R} \to N(P)$ is almost automorphic since Q is bounded. We deduce that $x_2(t) \in AA(\mathbb{X})$ in view of Theorem (4.3).

In applying P to Eq. (1.12), we obtain in the finite dimensional space $\mathbb{X}_1$ the equation

$$x_1'(t) = (A + PB)x_1(t) + g(t),$$

where $g(t) := PBx_2(t) + Pf(t)$ is an almost automorphic function $\mathbb{R} \to \mathbb{X}_1$. As in Theorem 10.7, $A + PB = A + B$ on $D(A) \cap X_1$. Now, since $x_1(t)$ has a relatively compact range and thus it is bounded in the finite dimensional space $\mathbb{X}_1$, we conclude in view of Theorem 4.5 that it is almost automorphic. Finally, $x(t)$ is almost automorphic as the sum of two almost automorphic functions. The proof is now complete. □

Bibliographical Notes This section is essentially based on the contributions by Zaki, Zaidman, and N'Guérékata. Theorem 10.5 is a slight generalization of a result by Zaki [70]. The results by N'Guérékata and Pankov in [61] provide a nontrivial interplay between Bohr almost periodicity, Besicovich almost periodicity, and almost automorphy of solutions of some almost periodic elliptic equations.

The interaction between the spectral theory of functions and mild solutions of evolution equations with almost automorphic forcing terms was initiated by

Diagana, N'Guérékata, and Nguyen Van Minh in [23] using the concept of uniform spectrum; cf. also [48, 49]. That interaction brings more operator theory, harmonic analysis' ideas, and complex functions to these differential equations. The reader can obtain a complete presentation of results in this direction in the recent book by N'Guérékata [59].