Gaston M. N'Guérékata

Almost Periodic and Almost Automorphic Functions in Abstract Spaces

Second Edition

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This book is dedicated to my brother
the late
Maître François N'Guérékata

Preface to the First Edition

The aim of this monograph is to present for the first time a unified and homogeneous exposition of the theory of almost automorphic functions and its application to the fast growing field of differential equations in abstract spaces (Banach and Hilbert spaces).

It is based essentially on the work of M. Zaki, S. Zaidman and the author during the last three decades.

The concept of almost automorphy is a generalization of almost periodicity. It has been introduced in the literature by S. Bochner in relation to some aspects of differential equations [11–13], and [14]. Almost automorphic functions are characterized by the following property:

Given any sequence of real numbers (s'_n), we can extract a subsequence (s_n) such that

$$\lim_{n\to\infty}\lim_{m\to\infty} f(t+s_n-s_m)=f(t)$$

for each real number t. The convergence is simply pointwise while one requires uniform convergence for almost periodicity.

In his important publication [67], W.A. Veech has studied almost automorphic functions on groups. We like to mention the contribution by M. Zaki [70] which provides a clear presentation of the study of almost automorphic functions with values in a Banach space. Zaki's work has been done under the supervision of Professor S. Zaidman of the University of Montreal, Canada, and has since strongly stimulated investigations in relation to the following problem:

What is the structure of bounded functions of the differential equation $x' = Ax + f$ where f is an almost automorphic function?

This equation was originally raised and solved by Bohr and Neugebauer for an almost periodic function f in a finite dimensional space. The generalization of this result to the larger class of almost automorphic functions in infinite dimensional spaces is not a trivial one. Indeed, it sometimes uses sophisticated techniques and strong tools from functional analysis and operator theory.

In this monograph we present several recent results from authors who contributed to solve the above problem and consider some nonlinear cases. We deal with classical solutions as well as the so-called mild solutions.

The concept of weak almost automorphy as presented by M. Zaki [70] is also discussed (Chap. 2, Sect. 2).

Also, continuous solutions on the non-negative semi-axis that approach almost automorphic functions at infinity are studied in Sections 5 and 6 of Chap. 2. In particular semi-groups of linear operators are considered as an independent subject in section of Chap. 3 and discussed in the context of the so-called Nemytskii and Stepanov theory of dynamical systems.

A wide range of situations is presented in Chaps. 4 through 6.

In Chap. 3, we present some results of the theory of almost periodic functions taking values in a locally convex space. We use a definition introduced in the literature by C. Corduneanu and developed by the author for the first time in [54]. Applications to abstract differential equations are given in Chaps. 7 and 8. At the end of each chapter, we have included a Notes section that gives some comments the main references used.

It is our hope that this monograph will constitute a useful reference textbook for post-graduate students and researchers in analysis, ordinary differential equations, partial differential equations, and dynamical systems.

May it stimulate new developments of the theory of almost automotrphic and almost periodic functions and enrich its applications to other fields.

It is a great pleasure to record our very sincere thanks to Professor Jerome A. Goldstein, a friend and mentor for over two decades and Professor Georges Anastassiou, who strongly encouraged us to complete this project.

We express my warm gratitude to Professor Constantin Corduneanu and Professor Joseph Auslander for their valuable comments and suggestions. Our thanks to our friend Professor Thomas Seidman who corrected some errors and Stephanie Smith for her extraordinary skill and patience in setting this text.

We also express our appreciation to the editorial assistance of Kluwer Academic Publishers, especially from Ana Bozicevic and Chris Curcio.

Finally, we owe a great deal to Professor Samuel Zaidman, who introduced us to the exciting world of mathematical research. His experience and outstanding contributions to mathematics have been a great source of inspiration to several young mathematicians.

Preface to the Second Edition

Since the publication of our book [55] in 2001, there has been a real rebirth of the theory of almost automorphic functions and applications to evolution equations as we expected. An incredible number of researchers have been attracted by this topic. This leads to a fast-growing number of publications.

We have received many helpful comments from colleagues and students, some pointing out typographical errors, others asking for clarification and improvement on some materials. In particular, Zheng, Ding, and N'Guérékata were able to answer the long-time open problem: **what is the "amount" of almost automorphic functions which are not almost periodic in the sense of Bohr?** The answer is that the space of almost periodic functions is a set of first category in the space of almost automorphic functions (cf. Chap. 1). Many other problems remain open, for instance the study of almost periodic functions taking values in non-locally convex spaces (cf. [30]).

Several generalizations were introduced in the literature including the study of almost automorphic sequences. The interplay between almost automorphy and almost periodicity is better known.

Researchers in the field overwhelmingly encouraged us to write a second edition including some of the fresh and most relevant contributions and references.

As in the first edition, we present the materials in a simplified and rigorous way. Each chapter is concluded with bibliographical notes showing the original sources of the results and further reading.

We are most grateful to our numerous co-authors and colleagues who made such great contributions to the theory of almost automorphy. We will not exhibit a list, which would be any way incomplete, but we hope our friends will be satisfied with our thanks and gratitude.

Finally, we thank our students Fatemeh Norouzi and Romario Gildas Foko Tiomela and our friend and colleague Alexander Pankov for their careful proof-reading and suggestions.

Baltimore, MD, USA
October 2020

Gaston M. N'Guérékata

Contents

Chapter 1
Introduction and Preliminaries

This monograph presents several recent developments on the theory of almost automorphic and almost periodic functions (in the sense of Bohr) with values in an abstract space and its application to abstract differential equations. We suppose that the reader is familiar with the fundamentals of Functional Analysis. However, to facilitate the understanding of the exposition, we give in the beginning, without proofs, some facts of the theory of topological vector spaces and operators which will be used later in the text.

1 Banach Spaces

We denote by $\mathbb{R}$ and $\mathbb{C}$ the fields of real and complex numbers, respectively. We will consider a (real or complex) normed space $\mathbb{X}$, that is a vector space over the field $\Phi = \mathbb{R}$ or $\mathbb{C}$ (respectively) with norm $\|\cdot\|$.

Definition 1.1 A sequence of vectors (x_n) in $\mathbb{X}$ is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists a natural number N such that $\|x_n - x_m\| < \epsilon$ for all $n, m > N$.

Proposition 1.2 *The following are equivalent:*

(i) (x_n) *is a Cauchy sequence.*
(ii) $\|x_{n_{k+1}} - x_{n_k}\| \to 0$ *as* $k \to \infty$, *for every increasing subsequence of positive integers* (n_k).

Proposition 1.3 *If* (x_n) *is a Cauchy sequence in a normed space* $\mathbb{X}$, *the sequence of reals* $(\|x_n\|)$ *is convergent.*

Definition 1.4 A Banach space $\mathbb{X}$ is a complete normed space, that is, a normed space $\mathbb{X}$ in which every Cauchy sequence is convergent to an element of $\mathbb{X}$.

G. M. N'Guérékata, *Almost Periodic and Almost Automorphic Functions in Abstract Spaces*, https://doi.org/10.1007/978-3-030-73718-4_1

Definition 1.5 A Banach space $\mathbb{X}$ is said to be uniformly convex if for every α, $0 < \alpha < 2$, there exists a number $\delta = \delta(\alpha) > 0$ such that for every $x, y \in \mathbb{X}$ with $\|x\| < 1$, $\|y\| < 1$, $\|x - y\| > \alpha$, we have $\|x + y\| \leq 2(1 - \delta)$.

Now if $x, y \in \mathbb{X}$ (not necessarily in the open unit ball), the conditions become

$$\left\| \frac{x+y}{2} \right\| \leq (1 - \delta) \cdot \max\{\|x\|, \|y\|\}$$

if

$$\|x - y\| \geq \alpha \cdot \max\{\|x\|, \|y\|\}.$$

We observe that Hilbert spaces are examples of uniformly convex Banach spaces.

Definition 1.6 A subset S of a normed space $\mathbb{X}$ is said to be open if for every $x \in S$, there exists $\epsilon > 0$ such that the open ball

$$B(x, \epsilon) := \{y \in \mathbb{X} \ : \ \|x - y\| < \epsilon\}$$

is included in S. S is said to be closed if its complement in $\mathbb{X}$ is open.

Proposition 1.7 *A subset S of a normed space $\mathbb{X}$ is closed if and only if every sequence of elements of S which converges in $\mathbb{X}$, has its limit in S.*

Definition 1.8 The closure of a subset S in a normed space $\mathbb{X}$, denoted $\overline{S}$, is the intersection of all closed sets containing S.

It is easy to verify the following:

Proposition 1.9 *Let S be a subset of a normed space $\mathbb{X}$; then*

$$\overline{S} = \{x \in \mathbb{X} \ : \ \exists (x_n) \subset S, \ \lim_{n \to \infty} x_n = x\}.$$

Definition 1.10 A subset S of a normed space $\mathbb{X}$ is said to be

(i) Dense in $\mathbb{X}$ if $\overline{S} = \mathbb{X}$;
(ii) Bounded in $\mathbb{X}$ if it is either empty or included in a closed ball;
(iii) Relatively compact in $\mathbb{X}$ if $\overline{S}$ is compact. Equivalently S is relatively compact if and only if every sequence in S contains a convergent sequence. It is observed that every relatively compact set is bounded.

Definition 1.11 Let $\mathbb{X}$ be a Banach space over the field $\Phi = \mathbb{R}$ or $\mathbb{C}$. The (continuous) dual space of $\mathbb{X}$ is the normed space of all bounded linear functionals $\varphi : \mathbb{X} \to \Phi$ which we denote $\mathbb{X}^*$.

We can rewrite Definition 1.10 (ii) as follows:

Definition 1.12 A subset S of a Banach space $\mathbb{X}$ is said to be bounded if $\varphi(S)$ is bounded in Φ for every $\varphi \in \mathbb{X}^*$.

Proposition 1.13 ([54]) *Weakly bounded sets are bounded in any Banach space* $\mathbb{X}$. *In particular every weakly convergent sequence is bounded in* $\mathbb{X}$.

We refer to $(\mathbb{X}^*)^* = \mathbb{X}^{**}$, *the bidual of* $\mathbb{X}$. $\mathbb{X}$ *can be considered as embedded in* $\mathbb{X}^{**}$ *as follows:*

For $x \in \mathbb{X}$, *let*

$$J(x) : \mathbb{X}^* \to \Phi (= \mathbb{R} \text{ or } \mathbb{C})$$

be defined by

$$J(x)[\varphi] = \varphi(x), \ \ \varphi \in \mathbb{X}^*.$$

Then $J(x)$ *is a linear form. It is continuous since*

$$|J(x)[\varphi]| = |\varphi(x)| \leq \|\varphi\| \|x\||, \ \forall \varphi \in \mathbb{X}^*.$$

Hence $J(x) \in \mathbb{X}^{**}$ *for all* $x \in \mathbb{X}$. *The map* $J : \mathbb{X} \to \mathbb{X}^{**}$ *defined this way is also linear and isometric. It is called the canonical embedding of* $\mathbb{X}$ *into its bidual* $\mathbb{X}^{**}$.

Definition 1.14 If the canonical embedding $J : \mathbb{X} \to \mathbb{X}^{**}$ is surjective, i.e. $\mathbb{X} = \mathbb{X}^{**}$, we say that $\mathbb{X}$ is reflexive.

Proposition 1.15 *If* $\mathbb{X}$ *is a reflexive Banach space and* (x_n) *is a bounded sequence, then we can extract a subsequence* (x'_n) *which will converge weakly to an element of* $\mathbb{X}$.

2 L^p Spaces

Let I be an open interval of $\mathbb{R}$ and denote by $C_c(I, \mathbb{X})$ the Banach space of all continuous functions $I \to \mathbb{X}$ with compact support.

Definition 1.16 A function $f : I \to \mathbb{X}$ is said to be measurable if there exists a set $S \subset I$ of measure 0 and a sequence $(f_n) \subset C_c(I, \mathbb{X})$ such that $f_n(t) \to f(t)$ for all $t \in I \setminus S$.

It is clear that if $f : I \to \mathbb{X}$ is measurable, then $\|f\| : I \to \mathbb{R}$ is measurable too.

Theorem 1.17 *Let* $f_n : I \to \mathbb{X}, \quad n = 1, 2, \ldots$ *be a sequence of measurable functions and suppose that* $f : I \to \mathbb{X}$ *and* $f_n(t) \to f(t)$, *as* $n \to \infty$, *for almost all* $t \in I$. *Then* f *is measurable.*

Proof We have $f_n \to f$ on $I \setminus S$, where S is a set of measure 0. Let $(f_{n.k})$ be a sequence of functions in $C_c(I, \mathbb{X})$ such that $f_{n.k} \to f$ almost everywhere on I as

$k \to \infty$. By Egorov's Theorem (cf. [69, p. 16]) applied to the sequence of functions $\|f_{n,k} - f_n\|$, there exists a set $S_n \subset I$ of measure less that $\frac{1}{2^n}$ such that $f_{n,k} - f \to f_n$ uniformly on $I \setminus S_n$, as $k \to \infty$.

Now let $k(n)$ be such that $\|f_{n,k(n)}\| < \frac{1}{n}$ on $I \setminus S_n$ and $F_n := f_{n,k(n)}$. Also let $B := S \cup (\bigcap_{m \geq 1} \bigcup_{n>m} S_n)$. Then it is clear that B is a subset of I of measure 0. Take $t \in I \setminus B$. So we get $f_n(t) \to f(t)$, as $n \to \infty$. On the other hand if n is large enough, $t \in I \setminus S_n$. It follows that $\|F_n - f\| < \frac{1}{n}$, which means that $F_n(t) \to f(t)$, as $n \to \infty$, and consequently, f is measurable. □

Remark 1.18 It is easy to observe that if $\phi : I \to \mathbb{R}$ and $f : I \to \mathbb{X}$ are measurable, then the product $\phi f : I \to \mathbb{X}$ is measurable too.

Theorem 1.19 (Pettis Theorem) *A function $f : I \to \mathbb{X}$ is measurable if and only if the following conditions hold:*

(a) f is weakly measurable (i.e. for every $x^* \in X^*$, the dual space of X, the function $x^* f : I \to \mathbb{X}$ is measurable).
(b) There exists a set $S \subset I$ of measure 0 such that $f(I \setminus S)$ is separable.

Proof See [69, p. 131]. □

We also have the following:

Theorem 1.20 *If $f : I \to \mathbb{X}$ is weakly continuous, then it is measurable.*

Theorem 1.21 (Bochner's Theorem) *Assume that $f : I \to \mathbb{X}$ is measurable. Then f is integrable if and only if $\|f\|$ is integrable. Moreover, we have*

$$\left\| \int_I f \right\| \leq \int_I \|f\|.$$

Proof Let $f : I \to \mathbb{X}$ be integrable. Then there exists a sequence of functions $f_n \in C_c(I, \mathbb{X})$, $n = 1, 2, \ldots$ such that $\int_I \|f_n(t) - f(t)\| dt \to 0$, as $n \to \infty$. Using the inequality $\|f\| \leq \|f - f_n\| + \|f_n\|$, for all n, we see that $\|f\|$ is integrable.

Conversely assume that $\|f\|$ is integrable. Let $F_n \in C_c(I, \mathbb{R})$, $n = 1, 2, \ldots$ be a sequence of continuous functions such that $\int_I |F_n - \|f\|| \to 0$ as $n \to \infty$ and $|F_n| \leq F$ almost everywhere for some $F : I \to \mathbb{R}$ with $\int_I |F| < \infty$.

Since f is measurable, there exists $f_n \in C_c(I, \mathbb{X})$, $n = 1, 2, \ldots$ such that $f_n \to f$ almost everywhere.

We now let

$$u_n := \frac{|F_n|}{\|f_n\| + \frac{1}{n}}, \quad n = 1, 2, \ldots$$

Then it is obvious that $u_n \leq F$, $n = 1, 2, \ldots$ and $u_n \to f$ almost everywhere on I. Therefore $\int_I \|u_n - f\| \to 0$ as $n \to \infty$ and consequently f is integrable.

Using the Lebesgue–Fatou Lemma (cf. [69]), we get

$$\left\| \int_I f \right\| \leq \lim_{n\to\infty} \left\| \int_I u_n \right\| \leq \int_I \|f\|.$$

This completes the proof. □

Theorem 1.22 (Lebesgue's Dominated Convergence Theorem) *Let $f_n : I \to \mathbb{X}$, $n = 1, 2, \ldots$ be a sequence of integrable functions and $g : I \to \mathbb{R}^+$ be an integrable function. Let also $f : I \to \mathbb{X}$ and assume that:*

(i) for all $n = 1, 2, \ldots$, $\|f_n\| \leq g$, almost everywhere on I.
(ii) $f_n(t) \to f(t)$, as $n \to \infty$ for all $t \in I$.

Then f is integrable on I and

$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$

Definition 1.23 Let $1 \leq p \leq \infty$. We will denote by $L^p(I, \mathbb{X})$ the space of all classes of equivalence (with respect to the equality on I) of measurable functions $f : I \to \mathbb{X}$ such that $\|f\|^p$ is integrable. If we equip $L^p(I, \mathbb{X})$ with the norm

$$\|f\|_p := \left(\int_I \|f(t)\|^p dt \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

and

$$\|f\|_\infty := ess \sup_I \|f(t)\|, \quad p = \infty,$$

then $L^p(I, \mathbb{X})$ turns out to be a Banach space.

We shall denote by $L^p_{loc}(I, \mathbb{X})$ the space of all (equivalence classes of) measurable functions $f : I \to \mathbb{X}$ such that the restriction of f to every bounded subinterval of I is in $L^p(I, \mathbb{X})$.

3 Linear Operators

Let us consider a normed space $\mathbb{X}$ and a linear operator $A : \mathbb{X} \to \mathbb{X}$. We define the norm of A by

$$|||A||| := \sup_{\|x\|=1} \|Ax\|.$$

Definition 1.24 A linear operator $A : \mathbb{X} \to \mathbb{X}$ is said to be continuous at $x \in \mathbb{X}$ if for any sequence $(x_n) \subset \mathbb{X}$ such that $x_n \to x$, we have $Ax_n \to Ax$, that is, $\|Ax_n - Ax\| \to 0$ as $\|x_n - x\| \to 0$.

If A is continuous at each $x \in Y \subset \mathbb{X}$, we say that A is continuous on Y.

Proposition 1.25 *A linear operator $A : \mathbb{X} \to \mathbb{X}$ is continuous (on $\mathbb{X}$) if and only if it is continuous at a point of $\mathbb{X}$.*

Based on the above Proposition, we generally prove continuity of a linear operator by checking its continuity at the zero vector.

Definition 1.26 A linear operator $A : \mathbb{X} \to \mathbb{X}$ is said to be bounded if there exists $M > 0$ such that $\|Ax\| \leq M\|x\|$ for all $x \in \mathbb{X}$.

We observe that a linear operator $A : \mathbb{X} \to \mathbb{X}$ is continuous if and only if it is bounded.

Proposition 1.27 (The Uniform Boundedness Principle) *Let $\mathcal{F}$ be a nonempty family of bounded linear operators over a Banach space $\mathbb{X}$. If $\sup_{A\in\mathcal{F}} \|Ax\| < \infty$ for each $x \in \mathbb{X}$, then $\sup_{A\in\mathcal{F}} |||A||| < \infty$.*

Definition 1.28 A linear operator A in a normed space $\mathbb{X}$ is said to be compact if AU is relatively compact, where U is the closed unit ball

$$U := \{x \in \mathbb{X} \ : \ \|x\| \leq 1\}.$$

Proposition 1.29 *If $\mathbb{X}$ is a Banach space, the linear operator $A : \mathbb{X} \to \mathbb{X}$ is compact if and only if for every bounded sequence $(x_n) \subset \mathbb{X}$, the sequence $(Ax_n) \subset \mathbb{X}$ has a convergent subsequence; in other words, AS is relatively compact for every bounded subset S of $\mathbb{X}$.*

4 Functions with Values in a Banach Space

We shall consider functions $x : I \to \mathbb{X}$ where I is an interval of the real number set $\mathbb{R}$ and $\mathbb{X}$ a Banach space.

Definition 1.30 A function $x(t)$ is said to be (strongly) continuous at a point $t_0 \in I$ if $\|x(t) - x(t_0)\| \to 0$ as $t \to t_0$ and strongly continuous on I if it is (strongly) continuous at each point of I. If t_0 is an end point of I, $t \to t_0$ (from the right or from the left), accordingly.

$x(t)$ is said to be weakly continuous on I if for any $\varphi \in \mathbb{X}^*$, the dual space of $\mathbb{X}$, the numerical function $(\varphi x)(t) : I \to \mathbb{R}$ is continuous. It is obvious that the strong continuity of x implies its weak continuity. The converse is not true in general.

In fact we have

Proposition 1.31 *If $x(t) : I \to \mathbb{X}$ is weakly continuous and has a range with a compact closure in $\mathbb{X}$, then $x(t)$ is strongly continuous on I.*

In this monograph, continuity will always denote strong continuity, unless otherwise explicitly specified.

Proposition 1.32 *Let $I = [a, b]$. Then the set $C(I, \mathbb{X})$ of all continuous functions $x(t) : I \to \mathbb{X}$ is a Banach space when equipped with the norm*

$$\|x\|_{C(I,\mathbb{X})} := \sup_{t \in I} \|x(t)\|.$$

Definition 1.33 A function $x(t) : I \to \mathbb{X}$ is said to be differentiable at an interior point t_0 of I if there exists some $y \in \mathbb{X}$ such that $\|\frac{x(t_0+\Delta t)-x(t_0)}{\Delta t} - y\| \to 0$ as $\Delta t \to 0$ and differentiable on an open subinterval of I if it is differentiable at each point of I. Such $y \in \mathbb{X}$, when it exists at t_0 is denoted $x'(t_0)$ and called the derivative of $x(t)$ at t_0.

Definition 1.34 If the function $x(t) : I \to \mathbb{X}$ is continuous on $I = [a, b]$, we define its integral on I (in the sense of Riemann) as the following limit:

$$\lim_{n \to \infty} \sum_{k=1}^{n} x(t_k) \Delta t_k,$$

where the diameter of the partition $a = t_0 < t_1 < \ldots < t_n = b$ of I tends to zero. When the limit exists we denote it by $\int_a^b x(t)dt$.

One can easily establish the estimate

$$\left\| \int_a^b x(t)dt \right\| \leq \int_a^b \|x(t)\|dt.$$

Improper integrals are defined as in the case of classical calculus. For instance, if the function is continuous on the interval $[a, \infty)$, then we define its integral on $[a, \infty)$ as follows:

$$\int_a^\infty x(t)dt = \lim_{b \to \infty} \int_a^b x(t)dt$$

if the limit exists in $\mathbb{X}$. This integral is said to be absolutely convergent if $\displaystyle\int_a^\infty \|x(t)\|dt < \infty$.

5 Semigroups of Linear Operators

Definition 1.35 Let $A : \mathbb{X} \to \mathbb{X}$ be a linear operator with domain $D(A) \subset \mathbb{X}$, a Banach space. The family $T = (T(t))_{t\geq 0}$ of bounded linear operators on $\mathbb{X}$ is said to be a C_0-semigroup if

(i) For all $x \in \mathbb{X}$, the mapping $T(t)x : \mathbb{R}^+ \to \mathbb{X}$ is continuous.
(ii) $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbb{R}^+$ (semigroup property).
(iii) $T(0) = I$, the identity operator.

The operator A is called the infinitesimal generator (or generator in short) of the C_0-semigroup T if

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t}$$

and

$$D(A) := \left\{ x \in \mathbb{X} \,/\, \lim_{t\to 0^+} \frac{T(t)x - x}{t} \;\; exists \right\}.$$

It is observed that S commutes with $T(t)$ on $D(A)$. We define a C_0-group in a similar way, by replacing $\mathbb{R}^+$ by $\mathbb{R}$.

For a bounded operator A, we have

$$T(t) := e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}.$$

Theorem 1.36 *Let $T = (T(t))_{t\geq 0}$ be a C_0-semigroup. Then there exists $K \geq 1$ and $\alpha \in \mathbb{R}$ such that*

$$\|T(t)\| \leq K e^{\alpha t}, \quad \forall t \geq 0.$$

If $\alpha < 0$, we say that T is exponentially stable.

Proposition 1.37

(a) The function $t \to \|T(t)\|$ from $\mathbb{R}^+ \to \mathbb{R}^+$ is measurable and bounded on any compact interval of $\mathbb{R}^+$.
(b) The domain $D(A)$ of its generator A is dense in $\mathbb{X}$.
(c) The generator A is a closed operator.

For more details, cf. [35] and [69].

6 Topological Vector Spaces

Let E be a vector space over the field Φ ($\Phi = \mathbb{R}$ or $\mathbb{C}$). We say that E is a topological vector space, which we denote $E = E(\tau)$, if E is equipped with a topology τ which is compatible to the algebraic structure of E.

It is easy to check that for all $a \in E$, the translation $f : E \to E$ defined by $f(x) = x + a$ is a homeomorphism. Thus if $\mho$ is a base of neighborhoods of the origin, $\mho + a$ is a base of neighborhoods of a. Consequently the whole topological structure of E will be determined by a base of neighborhoods of the origin.

In this book, we will mainly use neighborhoods of the origin, which we sometimes call neighborhoods in short.

Another interesting fact is that for every $\lambda \in \Phi$, $\lambda \neq 0$, the mapping $f : E \to E$ defined by $f(x) = \lambda x$ is a homeomorphism, so that λU will be a neighborhood (of the origin) if U is a neighborhood (of the origin), $\lambda \neq 0$.

Let us also recall the following:

Proposition 1.38 *If $\mho$ is a base of neighborhoods, then for each $U \in \mho$, we have:*

(i) U is absorbing, that is for each $x \in U$, there exists $\lambda > 0$ such that $x \in \alpha U$ for all α with $|\alpha| \geq \lambda$;
(ii) There exists $W \in \mho$ such that $W + W \subset U$;
(iii) There exists a balanced neighborhood V such that $V \subset U$ (A balanced or symmetric set is a set V such that $\alpha V = V$ if $|\alpha| = 1$).

A consequence of the above proposition is that every topological space E possesses a base of balanced neighborhood.

We will call a locally convex topological vector space (or shortly a locally convex space), every topological vector space which has a base of convex neighborhoods. It follows that in a locally convex space, any open set contains a convex, balanced, and absorbing open set.

A locally convex space whose topology is induced by an invariant complete metric is called a Fréchet space.

Proposition 1.39 *Let E be a vector space over the field Φ ($\Phi = \mathbb{R}$ or $\mathbb{C}$). A function $p : E \to \mathbb{R}^+$ is called a seminorm if*

(i) $p(x) \geq 0$ for every $x \in E$;
(ii) $p(\lambda x) = |\lambda| p(x)$, for every $x \in E$ and $\lambda \in \Phi$;
(iii) $p(x + y) \leq p(x) + p(y)$, for every $x, y \in E$.

It is noted that if p is a seminorm on E, then the sets $\{x \;:\; p(x) < \lambda\}$ and $\{x \;/\; p(x) \leq \lambda\}$, where $\lambda > 0$, are absorbing. They are also absolutely convex. We recall that a set $B \subset E$ is said to be absolutely convex if for every $x, y \in E$ and $\lambda, \mu \in \Phi$, with $|\lambda| + |\mu| \leq 1$, we have $\lambda x + \mu y \in B$.

Theorem 1.40 *For every set Q of seminorms on a vector space E, there exists a coarsest topology on E compatible with its algebraic structure and in which each* *seminorm in Q is continuous. Under this topology, E is a locally convex space and a base of neighborhoods is formed by the closed sets*

$$\{x \in E \ : \ \sup_{1 \le i \le n} p_i(x) \le \epsilon\},$$

where $\epsilon > 0$ and $p_i \in Q$, $i = 1, 2, \dots n$.

Also E will be separated if and only if for each $x \in E$, $x \neq 0$, there exists a seminorm $p \in Q$ such that $p(x) > 0$.

An important fact that will be used is the following consequence of the Hahn–Banach Extension Theorem:

Proposition 1.41 ([69, page 107]) *For each non-zero a in a locally convex space E, there exists a linear functional $\varphi \in E^*$, the dual space of E, such that $\varphi(a) \neq 0$.*

A subset S of a locally convex space is called totally bounded if, for every neighborhood U, there are $a_i \in S$, $i = 1, 2, \dots n$, such that

$$S \subset \cup_{i=1}^{n} (a_i + U).$$

It is clear that every totally bounded set is bounded. Also, the closure of a totally bounded set is totally bounded.

We observe [69, page 13] that in a complete metric space, total boundedness and relatively compactness are equivalent notions.

Now for functions of the real variable with values in a locally convex space E, we define continuity, differentiability, and integration as in [54, 56, 69].

We finally revisit Proposition 1.27 in the context of locally convex spaces as follows (cf. [45, page 199]):

Proposition 1.42 (Uniform Boundedness Principle) *Let $\varphi = \{A_\alpha \ : \ \alpha \in \Gamma\}$ where each $A_\alpha : E \to F$ is a bounded linear operator and E, F are Fréchet spaces. Suppose that $\{A_\alpha x \ : \ \alpha \in \Gamma\}$ is bounded for each $x \in E$. Then φ is uniformly bounded.*

Notes Details on this topic can be found in [66].

7 The Exponential of a Bounded Linear Operator

Let E be a complete, Hausdorff locally convex space.

Definition 1.43 A family of continuous linear operators $B_\alpha : E \to E$, $\alpha \in \Gamma$ is said to be equicontinuous if for any seminorm p, there exists a seminorm q such that

$$p(B_\alpha x) \le q(x), \quad \text{for any } x \in E, \quad \text{any } \alpha \in \Gamma.$$

Theorem 1.44 *Let $A : E \to E$ be a continuous linear operator such that the family $\{A^k : k = 1, 2, \ldots\}$ is equicontinuous. Then for each $x \in E$, $t \geq 0$, the series*

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k x$$

(where $A^0 = I$, the identity operator on E) is convergent.

Proof Let p be a seminorm on E. By equicontinuity of $\{A^k : k = 1, 2, \ldots\}$, there exists a seminorm q on E such that

$$p(A^k x) \leq q(x), \quad \text{for all } k, \text{ and } x \in E.$$

Therefore we have

$$p\left(\sum_{k=n}^{m} \frac{t^k}{k!} A^k x\right) \leq \sum_{k=n}^{n} \frac{t^k}{k!} p(A^k x) \leq q(x) \sum_{k=n}^{m} \frac{t^k}{k!},$$

which proves that the sequence $\sum_{k=0}^{n} \frac{t^k}{k!} A^k x$ is a Cauchy sequence in E. It is then convergent and we denote the limit by

$$e^{tA} x := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k x.$$

□

Theorem 1.45 *The mapping $x \to e^{tA} x$, $t \geq 0$, defines a continuous linear operator $E \to E$.*

Proof Consider the linear operators $A_n := \sum_{k=0}^{n} \frac{t^k}{k!} A^k$, $n = 0, 1, 2, \ldots$ The family $\{A_n : \ n = 0, 1, 2, \ldots\}$ is equicontinuous on any compact interval of $\mathbb{R}^+$.

Indeed, by equicontinuity of $\{A^k : \ k = 1, 2, \ldots\}$, if p is a given seminorm, then there exists a seminorm q such that

$$p(A_n x) \leq \sum_{k=0}^{n} \frac{t^k}{k!} p(A^k x) \leq q(x) \sum_{k=0}^{n} \frac{t^k}{k!} \leq q(x) e^t$$

for every $n = 0, 1, 2, \ldots$. It follows that

$$p(e^{tA} x) \leq q(x) e^t,$$

for every $t \geq 0$ and $x \in E$. This completes the proof. □

Theorem 1.46 *Let A and B be two continuous linear operators $E \to E$ such that $\{A^n; n = 1, 2, \ldots\}$ and $\{B^n; n = 1, 2, \ldots\}$ are equicontinuous. Assume that A and B commute, that is $AB = BA$; then*

$$e^{tA} \cdot e^{tB} = e^{t(A+B)}, \quad t \geq 0.$$

Proof The proof is similar to the numerical case, that is for any real numbers a and b, we have

$$\sum_{n=0}^{\infty} \frac{(ta)^n}{n!} . \sum_{n=0}^{\infty} \frac{(tb)^n}{n!} = \sum_{n=0}^{\infty} \frac{(t(a+b))^n}{n!}.$$

Indeed for any integer k and $x \in E$, we have

$$(A+B)^k x = \sum_{j=0}^{k} \binom{k}{j} A^j B^{k-j} x = \sum_{j=0}^{k} \binom{k}{j} B^{k-j} A^j x,$$

where $\binom{k}{j} = \frac{k!}{j!(k-j)!}$.

In the last equality, we used the fact that $AB = BA$. Let p be a given seminorm on E. Then there exists a seminorm q such that

$$\begin{aligned} p((A+B)^k x) &\leq \sum_{j=0}^{k} \binom{k}{j} p(B^{k-j} A^j x) \\ &\leq \sum_{j=0}^{k} \binom{k}{j} q(A^j x) \\ &\leq 2^k \sup_{j \geq 0} q(A^j x) \end{aligned}$$

since $\sum_{j=0}^{k} \binom{k}{j} = 2^k$.

This last inequality shows that the family $\left\{ \frac{(A+B)^k}{2^k} : k = 1, 2, \ldots \right\}$ is equicontinuous, so by Theorem (1.44), we can define $e^{t(A+B)}$ by

$$e^{t(A+B)} x := \sum_{n=0}^{\infty} \frac{(t(A+B))^n x}{n!}.$$

Now using the Cauchy product formula, we obtain

$$e^{tA} \cdot e^{tB} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \cdot \sum_{n=0}^{\infty} \frac{(tB)^n}{n!} = \sum_{n=0}^{\infty} C_n,$$

where

$$\begin{aligned} C_n &= \sum_{k=0}^{n} \frac{(tA)^k}{k!} \cdot \frac{(tB)^{n-k}}{(n-k)!} \\ &= \sum_{k=0}^{n} \frac{t^n}{k!(n-k)!} A^k B^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{t^n}{n!} A^k B^{n-k} \\ &= \sum_{k=0}^{n} \frac{(t(A+B))^n}{n!}. \end{aligned}$$

That means $\sum_{n=0}^{\infty} C_n = e^{t(A+B)}$. The proof is complete. □

Theorem 1.47 *Suppose that A is a continuous linear operator $E \to E$ such that $\{A^n; n = 1, 2, \ldots\}$ is equicontinuous. Then for every $x \in E$, we have*

$$\lim_{h \to 0^+} \left(\frac{e^{hA} - I}{h} \right) x = Ax.$$

Proof Let p be a seminorm. Then there exists a seminorm q such that

$$\begin{aligned} p((\tfrac{e^{hA}-I}{h})x - Ax) &= p(\tfrac{1}{h}(\sum_{n=0}^{\infty} \frac{h^n}{n!} A^n - I)x - Ax) \\ &\le p(\tfrac{1}{h}(\sum_{n=2}^{\infty} \frac{h^n}{n!} A^n x) \\ &\le \sum_{n=2}^{\infty} \frac{h^{n-1}}{n!} p(A^n x) \\ &\le q(x) \sum_{n=2}^{\infty} \frac{h^{n-1}}{n!} \\ &= q(x) \left(\tfrac{e^h - 1}{h} - 1 \right). \end{aligned}$$

And since $\lim_{h \to +} \frac{e^h - 1}{h} = 1$, we get the result. □

From the above, we can deduce that

$$\frac{d}{dt}e^{tA}x = e^{tA} \cdot Ax = Ae^{tA}x,$$

Using the semigroup property above, we get also

$$e^{(t+s)A} = e^{tA} \cdot e^{sA}.$$

We can use the same technique to prove similar results if $t \leq 0$ and establish e^{tA} for $t \in \mathbb{R}$.

We are now ready to prove the following:

Theorem 1.48 *The function* $e^{tA}x_0 : \mathbb{R} \to E$ *is the unique solution of the differential equation*

$$x'(t) = Ax(t), \quad t \in \mathbb{R}$$

satisfying $x(0) = x_0$.

Proof Suppose there were another solution $y(t)$ with $y(0) = x_0$. Consider the function $v(s) = e^{(t-s)A}y(s)$, with t fixed in $\mathbb{R}$; then we have

$$\begin{aligned} v'(s) &= -Ae^{(t-s)A}y(s) + e^{(t-s)A}y'(s) \\ &= -Ae^{(t-s)A}y(s) + e^{(t-s)A}Ay(s) \\ &= 0, \end{aligned}$$

for every $s \in \mathbb{R}$. Therefore, $v'(s) = 0$ on $\mathbb{R}$, so that

$$v(t) = v(0), \quad t \in \mathbb{R}$$

or

$$y(t) = e^{tA}y(0) = e^{tA}x_0, \quad t \in \mathbb{R}.$$

Since t is arbitrary, this completes the proof. □

Let us recall the following fixed point theorem from [15]:

Theorem 1.49 *Let D be a closed and convex subset of a Hausdorff locally convex space such that* $0 \in D$, *and let G be a continuous mapping of D into itself. If the implication*

$$(V = conv G(V), or\ V = G(V) \cup \{0\}) \Rightarrow V is\ relatively\ compact$$

holds for every subset V of D, then G has a fixed point.

8 Non-locally Convex Spaces

It is well known that an F-space $(X, +, \cdot, ||\cdot||)$ is a linear space (over the field $\Phi = \mathbb{R}$ or $K = \mathbb{C}$) such that $||x+y|| \leq ||x|| + ||y||$ for all $x, y \in X$, $||x|| = 0$ if and only if $x = 0$, $||\lambda x|| \leq ||x||$, for all scalars λ with $|\lambda| \leq 1$, $x \in X$, and with respect to the metric $D(x, y) = ||x - y||$, X is a complete metric space (see e.g. [25, p. 52], or [37]). Obviously D is invariant to translations.

In addition, if there exists $0 < p < 1$ with $||\lambda x|| = |\lambda|^p ||x||$, for all $\lambda \in K$, $x \in X$, then $||\cdot||$ will be called a p-norm and X will be called p-Fréchet space. (This is only a slight abuse of terminology. Note that in e.g. [10] these spaces are called p-Banach spaces). In this case, it is immediate that $D(\lambda x, \lambda y) = |\lambda|^p D(x, y)$, for all $x, y \in X$ and $\lambda \in \Phi$.

It is known that the F-spaces are not necessarily locally convex spaces. Three classical examples of p-Fréchet spaces, non-locally convex, are the Hardy space H^p with $0 < p < 1$ that consists in the class of all analytic functions $f : \mathbb{D} \to \mathbb{C}$, $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ with the property

$$||f|| = \frac{1}{2\pi} \sup \left\{ \int_0^{2\pi} |f(re^{it})|^p dt; r \in [0, 1) \right\} < +\infty,$$

the sequences space

$$l^p = \left\{ x = (x_n)_n; ||x|| = \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

for $0 < p < 1$, and the $L^p[0, 1]$ space, $0 < p < 1$, given by

$$L^p = L^p[0, 1] = \left\{ f : [0, 1] \to \mathbb{R}; ||f|| = \int_0^1 |f(t)|^p dt < \infty \right\}.$$

More generally, we may consider $L^p(\Omega, \Sigma, \mu)$, $0 < p < 1$, based on a general measure space (Ω, Σ, μ), with the p-norm given by $||f|| = \int_\Omega |f|^p d\mu$.

Some important characteristics of the F-spaces are given by the following remarks:

Remark 1.50

(1) Three of the basic results in Functional Analysis hold in F-spaces too : the Principle of Uniform Boundedness (see e.g. [25, p. 52]), the Open Mapping Theorem, and the Closed Graph Theorem (see e.g. [37, p. 9–10]).

But on the other hand, the Hahn–Banach Theorem fails in non-locally convex F-spaces. More exactly, if in an F-space the Hahn–Banach theorem holds, then that space is necessarily locally convex space (see e.g. [37, Chapter 4]).

(2) If $(X, +, \cdot, ||\cdot||)$ is a p-Fréchet space over the field Φ, $0 < p < 1$, then its dual X^* is defined as the class of all linear functionals $h : X \to \Phi$ which satisfy $|h(x)| \le |||h||| \cdot ||x||^{1/p}$, for all $x \in X$, where $|||h||| = \sup\{|h(x)|; ||x|| \le 1\}$ (see e.g. [10, pp. 4–5]). Note that $|||\cdot|||$ in fact is a norm on X^*.

For $0 < p < 1$, while $(L^p)^* = 0$, we have that $(l^p)^*$ is isometric to l^∞—the Banach space of all bounded sequences (see e.g. [37, p. 20–21]), therefore $(l^p)^*$ becomes a Banach space. Also, if $\phi \in (H^p)^*$ then there exists a unique g, analytic on $\mathbb{D}$ and continuous on the closure of $\mathbb{D}$, such that

$$\phi(f) = \frac{1}{2\pi} \lim_{r\to 1} \int_0^{2\pi} f(re^{it})g(e^{-it})dt,$$

for all $f \in H^p$ (see e.g. [26, p. 115, Theorem 7.5]). Moreover, $(H^p)^*$ becomes a Banach space with respect to the usual norm $|||\phi||| = \sup\{|\phi(f)|; ||f|| \le 1\}$ (see the same paper [4]).

In both cases of l^p and H^p, $0 < p < 1$, their dual spaces separate the points of corresponding spaces.

(3) The spaces l^p and H^p, $0 < p < 1$, have Schauder bases (see e.g. [37, p. 20], for l^p and [37, 64] for H^p). It is also worth to note that according to e.g. [28], every linear isometry T of H^p onto itself has the form

$$T(f)(z) = \alpha[\phi'(z)]^{1/p} f(\phi(z)), \tag{8.1}$$

where α is some complex number of modulus one and ϕ is some conformal mapping of the unit disc onto itself.

Chapter 2
Almost Automorphic Functions

1 Almost Automorphic Functions in a Banach Space

Definition 2.1 (S. Bochner [11–14]) Let $\mathbb{X}$ be a (real or complex) Banach space and $f \in C(\mathbb{R}, \mathbb{X})$. We say that f is almost automorphic if for every sequence of real numbers (s'_n) there exists a subsequence (s_n) such that

$$\lim_{m\to\infty}\lim_{n\to\infty} f(t + s_m - s_n) = f(t)$$

for each $t \in \mathbb{R}$.

This is equivalent to the following:

Definition 2.2 $f \in BC(\mathbb{R}, \mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers (s'_n) there exists a subsequence (s_n) such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

exists for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$.

Remark 2.3

- The function g in Definition 2.2 is measurable, but not necessarily continuous.
- If the convergence in Definition 2.1 is uniform on compact subsets of $\mathbb{R}$, then we say that f is compact almost automorphic.
- If the convergence in Definitions 2.1 and 2.2 is uniform in $t \in \mathbb{R}$, then f is almost periodic. This shows that the class of almost automorphic functions is larger than

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the class of almost periodic functions. We will show later that the inclusion is strict.

Theorem 2.4 ([27, 44, 58]) *If the function g in the Definition 2.2 is continuous, then f is uniformly continuous.*

Proof Suppose f is not uniformly continuous on $\mathbb{R}$. Then there exists a number $\epsilon > 0$ and two sequences (t_n') and (s_n') such that for each n, we have

$$\|f(s_n' - t_n') - f(s_n')\| > \epsilon$$

and

$$\lim_{n\to\infty} t_n' = 0.$$

In view of the almost automorphy of f, one can extract subsequences $(s_n) \subset (s_n')$ and $(s_n + t_n) \subset (s_n' + t_n')$ such that

$$g_1(t) := \lim_{n\to\infty} f(t + s_n)$$

exists for each $t \in \mathbb{R}$ and

$$g_2(t) := \lim_{n\to\infty} f(t + s_n + t_n)$$

for each $t \in \mathbb{R}$.

Here g_1 and g_2 are continuous by assumption. Therefore we have

$$\|g_1(0) - g_2(0)\| > \epsilon.$$

Define the set

$$\Omega := \left\{ t \in \mathbb{R} : \|g_1(t) - g_2(t)\| > \frac{3}{4}\epsilon \right\}.$$

Then Ω is an open set in view of the continuity of g_1 and g_2. It is also nonempty since $0 \in \Omega$.

Now define for each n the set

$$A_n := \left\{ t \in \mathbb{R} : \|f(t + s_m) - g_1(t)\| \le \frac{\epsilon}{4},\ \|f(t + s_m + t_m) - g_2(t)\| \le \frac{\epsilon}{4},\ \ m \ge n \right\}.$$

Each A_n is nonempty, because of the convergence to g_1 and g_2 above. Now let

$$B_n := A_n \cap \Omega, \quad n = 1, 2, \ldots$$

It is obvious that each B_n is nonempty (since $0 \in B_n$), and

$$\cup_{n=1}^{\infty} B_n = \Omega.$$

Let n_0 be large enough and take $t \in B_{n_0}$. Then $t \in \Omega$ that means

$$(\star) \quad \|g_1(t) - g_2(t)\| > \frac{3}{4}\epsilon.$$

But since Ω is open and g_1 is continuous, we may choose m large enough, $m \geq n_0$ such that

$$t + t_m \in \Omega$$

and

$$\|g_1(t + t_m) - g_1(t)\| < \frac{\epsilon}{4}.$$

Since $t \in B_{n_0}$, we have

$$\|f(t + t_m + s_m) - g_2(t)\| \leq \frac{\epsilon}{4}.$$

Also since $\Omega = \cup_{n=1}^{\infty} B_n$, there exists n_1 such that $t+t_m \in B_{n_1}$ implies $t+t_m \in A_{n_1}$. So

$$\|f(t + t_m + s_m) - g_1(t + t_m)\| \leq \frac{\epsilon}{4}.$$

Finally, we obtain

$$\begin{aligned}\|g_1(t) - g_2(t)\| &\leq \|g_1(t) - g_1(t + t_m)\| + \|g_1(t + t_m) - f(t + t_m + s_m)\| \\ &+ \|f(t + t_m + s_m) - g_2(t)\| \\ &\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{3}{4}\epsilon,\end{aligned}$$

which contradicts $(\star)$ and establishes the result. □

Theorem 2.5 *If f, f_1, f_2 are almost automorphic functions $\mathbb{R} \to X$ and λ is a scalar, then the following are true:*

(i) λf *and* $f_1 + f_2$ *are almost automorphic.*
(ii) $f_a(\cdot) := f(a + \cdot)$ *is almost automorphic for every* $a \in \mathbb{R}$.
(iii) $\sup_{t\in\mathbb{R}} \|f(t)\| < \infty$.
(iv) *The range* $R_f := \{f(t) : t \in \mathbb{R}\}$ *is relatively compact in* $\mathbb{X}$.

Proof Statements (i) and (ii) are obvious.

Let us prove (iii). Suppose by contradiction that $\sup_{t\in\mathbb{R}} \|f(t)\| = \infty$. Then there exists a sequence (s'_n) of real numbers such that

$$\lim_{n\to\infty} \|f(s'_n)\| = \infty.$$

Since f is almost automorphic, we can extract a subsequence $(s_n) \subset (s'_n)$ such that

$$\lim_{n\to\infty} f(s_n) = \alpha$$

for some $\alpha \in \mathbb{R}$, that is

$$\lim_{n\to\infty} \|f(s_n)\| = \|\alpha\| < \infty$$

which is a contradiction and establishes (iii).

(iv) Consider an arbitrary sequence $(f(s'_n))$ in R_f. Since f is almost automorphic, we can extract a subsequence $(s_n) \subset (s'_n)$ such that

$$\lim_{n\to\infty} f(s_n) = g(0),$$

where g is the function in Definition 2.2. This proves that R_f is relatively compact in X. □

Remark 2.6 It is easy to observe that $\sup_{t\in\mathbb{R}} \|g(t)\| = \sup_{t\in\mathbb{R}} \|f(t)\|$, which implies that $\overline{R_g} = \overline{R_f}$.

Theorem 2.7 *Let (f_n) be a sequence of almost automorphic functions in a Banach space $\mathbb{X}$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ uniformly in $t \in \mathbb{R}$.*

Then $f(t)$ is also almost automorphic.

Proof Let (s'_n) be a sequence of real numbers. By the diagonal procedure, we can extract a subsequence (s_n) of (s'_n) such that

$$\lim_{n\to\infty} f_i(t+s_n) = g_i(t) \tag{1.1}$$

for each $i = 1, 2, \ldots$ and each $t \in \mathbb{R}$.

We claim that the sequence of functions $(g_i(t))$ is a Cauchy sequence. Indeed if we write

$$\begin{aligned} g_i(t) - g_j(t) = g_i(t) - f_i(t+s_n) + f_i(t+s_n) - f_j(t+s_n) \\ + f_j(t+s_n) - g_j(t), \end{aligned}$$

and use the triangle inequality, we get

$$\|g_i(t) - g_j(t)\| \leq \|g_i(t) - f_i(t + s_n)\| + \|f_i(t + s_n) - f_j(t + s_n)\| \\ + \|f_j(t + s_n) - g_j(t)\|.$$

Let $\epsilon > 0$ be given. By uniform convergence of the sequence (f_n), we can find a natural number N such that for all $i, j > N$,

$$\|f_i(t + s_n) - f_j(t + s_n)\| < \epsilon,$$

for all $t \in \mathbb{R}$ and all $n = 1, 2, \ldots$. Using Eq. (1.1) and the completeness of the space $\mathbb{X}$, we can deduce the pointwise convergence of the sequence $(g_i(t))$, say to a function $g(t)$.

Let us prove that

$$\lim_{n\to\infty} f(t + s_n) = g(t)$$

and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

pointwise on $\mathbb{R}$.

Indeed, for each $i = 1, 2, \ldots,$ we get

$$\|f(t + s_n) - g(t)\| \leq \|f(t + s_n) - f_i(t + s_n)\| \\ + \|f_i(t + s_n) - g_i(t)\| + \|g_i(t) - g(t)\|.$$

Given $\epsilon > 0$, we can find some natural number M such that

$$\|f(t + s_n) - f_M(t + s_n)\| \leq \epsilon$$

for every $t \in \mathbb{R}$, $n = 1, 2, \ldots$ and $\|g_M(t) - g(t)\| < \epsilon$ for every $t \in \mathbb{R}$, so that

$$\|f(t + s_n) - g(t)\| \leq 2\epsilon + \|f_M(t + s_n) - g_M(t)\|$$

for every $t \in \mathbb{R}$, $n = 1, 2, \ldots$.

Now for every $t \in \mathbb{R}$, we can find some natural number K depending on ϵ and M such that

$$\|f_M(t + s_n) - g_M(t)\| < \epsilon$$

for every $n > K$.

Finally, we get

$$\|f(t+s_n)-g(t)\| < 3\epsilon$$

for $n \geq N_0$ where N_0 is some natural number depending on t and ϵ.

We can similarly prove that

$$\lim_{n\to\infty} g(t-s_n) = f(t).$$

□

Let us denote by $AA(\mathbb{X})$ (resp.$AA_c(\mathbb{X})$) the space of all almost automorphic functions (resp. compact almost automorphic) $f : \mathbb{R} \to \mathbb{X}$. It turns out from the above that $AA(\mathbb{X})$ and $AA_c(\mathbb{X})$ are closed subspaces of $BC(\mathbb{R}, \mathbb{X})$. Thus they are themselves Banach spaces under the supnorm

$$\|f\|_{AA(\mathbb{X})} := \sup_{t\in\mathbb{R}} \|f(t),$$

resp.

$$\|f\|_{AA_c(\mathbb{X})} := \sup_{t\in\mathbb{R}} \|f(t).$$

If we denote by $AP(\mathbb{X})$ the space of all almost periodic functions $f : \mathbb{R} \to \mathbb{X}$ (in the sense of Bohr, cf. [22], or Chapter 4 below), then it is obvious that

$$AP(\mathbb{X}) \subset AA_c(\mathbb{X}) \subset AA(\mathbb{X}) \tag{1.2}$$

and the inclusions are strict.

Let us state the following composition theorem:

Theorem 2.8 *Let* $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ *be Banach spaces over the same field* Φ, $f \in AA(\mathbb{X})$ *and* $\mathcal{R}_f := \{f(t) : t \in \mathbb{R}\}$ *is the range of* f. *If* $\phi : \mathcal{R}_f \to \mathbb{Y}$ *be a continuous and bounded application, then the composite function* $\phi \circ f : \mathcal{R}_f \to \mathbb{Y}$ *is also almost automorphic.*

Proof Let (s'_n) be sequence of real numbers. Since $f \in AA(\mathbb{X})$, there exists a subsequence (s_n) such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

exists for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t-s_n) = f(t)$$

for each $t \in \mathbb{R}$.

Since ϕ is continuous and bounded, we have

$$\lim_{n\to\infty}(\phi \circ f)(t+s_n) = \phi \circ \lim_{n\to\infty} f(t+s_n) = (\phi \circ g)(t)$$

exists for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty}(\phi \circ g)(t-s_n) = \phi \circ \lim_{n\to\infty} g(t-s_n) = (\phi \circ f)(t)$$

for each $t \in \mathbb{R}$. This shows that $\phi \circ f \in AA(\mathbb{X})$. □

The following corollary follows immediately:

Corollary 2.9 *If A is a bounded linear operator on $\mathbb{X}$ and $f \in AA(\mathbb{X})$, then $(Af)(\cdot) \in AA(\mathbb{X})$.*

Let us now give some examples of almost automorphic functions which are not almost periodic.

Example 2.10 (Levitan) Let $f \in AP(\mathbb{R})$ and $\phi : \mathcal{R}_f \to \mathbb{Y}$ be continuous and bounded. Then $\phi \circ f : \mathcal{R}_f \to \mathbb{Y}$ may not be almost periodic. For example, let

$$x(t) = \cos t + \cos\sqrt{2}t + 2$$

and

$$\phi(s) = \sin\frac{1}{s}.$$

Clearly $\phi(t) = \sin\left(\frac{1}{\cos t + \cos\sqrt{2}t + 2}\right)$ is almost automorphic. But since $\phi(s)$ is not uniformly continuous on $\overline{\mathcal{R}_x}$, then $\phi(t)$ is not almost periodic.

Example 2.11 (Veech) Consider the function $x : \mathbb{R} \to \mathbb{C}$ defined by

$$x(t) = e^{it} + e^{i\sqrt{2}t} + 2.$$

Let $\phi : \mathbb{C} \setminus \{0\} \to \Gamma$ where Γ is the unit circle in $\mathbb{C}$ be defined by

$$\phi(x) = \frac{x}{|x|}.$$

Thus $\phi(t) = \dfrac{e^{it} + e^{i\sqrt{2}t} + 2}{|e^{it} + e^{i\sqrt{2}t} + 2|}$ is almost automorphic but not almost periodic.

Remark 2.12 If $f \in AA(\mathbb{X})$ and $\phi : \mathcal{R}_f \subset \mathbb{R} \to \mathbb{Y}$ is not bounded, then $\phi \circ f : \mathcal{R}_f$ in the proof of Theorem 2.8 is not well-defined for all $t \in \mathbb{R}$, therefore $\phi \circ f : \mathcal{R}_f$

is not almost automorphic. For example, replace $\phi(x) = \sin\frac{1}{x}$ in the example above by $\phi(x) = \frac{1}{x}$.

Theorem 2.13 ([72]) *$AP(\mathbb{X})$ is a set of first category in $AA(\mathbb{X})$.*

Proof It suffices to observe that $AP(\mathbb{X})$ is a closed subset of $AA(\mathbb{X})$ equipped with the supnorm. Thus its interior is empty. □

Theorem 2.14 *Let $T = (T(t))_{t\in\mathbb{R}}$ be a one parameter group of strongly continuous linear operators uniformly bounded, i.e. there exists $M > 0$ such that $\sup_{t\in\mathbb{R}} \|T(t)\| \leq M$. Let $f \in AA(\mathbb{X})$ and $S = f(\mathbb{Q})$, where $\mathbb{Q}$ denotes the set of rational numbers, with the property that the function $T(\cdot)x \in AA(X)$ for each $x \in S$.*

Then $T(\cdot)f(\cdot) \in AA(\mathbb{X})$.

Proof Let $B = \{f(t) : t \in \mathbb{R}\}$ be the range of f. Then S is a countable and dense subset of B.

Let $S = (x_n)$; then $T(\cdot)x_n \in AA(\mathbb{X})$ for each $n = 1, 2, \ldots$. Consider an arbitrary sequence of real numbers (s'_n). Using the diagonal procedure, we can show that there exists a subsequence (s_n) of (s'_n) such that

$$\lim_{n\to\infty} T(s_n)x$$

exists for each $x \in S$. Pick $\overline{x} \in \overline{B}$. For any n, m, k we have

$$\begin{aligned}\|T(s_n)\overline{x} - T(s_m)\overline{x}\| \leq{} & \|T(s_n)\overline{x} - T(s_n)x_k\| \\ & + \|T(s_n)x_k - T(s_m)x_k\| \\ & + \|T(s_m)x_k - T(s_m)\overline{x}\|.\end{aligned}$$

Therefore $\|T(s_n)x_n - T(s_m)x_m\| \to 0$ since $x_n \to S$ and we have

$$\lim_{n,m\to\infty} \|T(s_n)\overline{x} - T(s_m)\overline{x}\| \leq 2M\|\overline{x} - x_k\|.$$

Consequently, in view of the density of S in $\overline{B}$, we can say that

$$\lim_{n\to\infty} T(s_n)\overline{x}$$

exists for every $\overline{x} \in \overline{B}$.

Now we observe that $\lim_{n\to\infty} T(s_n)x = y$ defines a mapping F from the linear space spanned by $\overline{B}$ into $\mathbb{X}$, namely

$$Fx = y \quad \text{if} \quad \lim_{n\to\infty} T(s_n)x = y. \tag{1.3}$$

The map F has the following properties:

(i) F is linear.
(ii) $\|Fx\| = \|y\| \leq \lim_{n\to\infty} \|T(s_n)x\| \leq M\|x\|$ for every x in the subspace spanned by $\overline{B}$.
(iii) F is one-to-one.
(iv) If (x_n) is a given sequence in $\overline{B}$ such that strong-$\lim_{n\to\infty} x_n = x$ exists, then strong-$\lim_{n\to\infty} T(s_n)x_n = Fx$ and strong-$\lim_{n\to\infty} Fx_n = Fx$.

Let $R_F := \{Fx : x \in \overline{B}\}$ be the range of F. Then we observe that

$$\lim_{n\to\infty} T(-s_n)y$$

exists for every $y \in R_F$.

It suffices to prove that

$$\lim_{n\to\infty} T(-s_n)y_m$$

exists for every $y_m \in F(S)$, where $y_m = F(x_m)$, $m = 1, 2, \ldots$.

Since $T(t)x_m \in AA(\mathbb{X})$ for each $m = 1, 2, \ldots,$ we have

$$\begin{aligned}
\lim_{n\to\infty} T(t+s_n)x_m &= \lim_{n\to\infty} T(t)T(s_n)x_m \\
&= T(t)\lim_{n\to\infty} T(s_n)x_m \\
&= T(t)Fx_m \\
&= T(t)y_m
\end{aligned}$$

pointwise on $\mathbb{R}$. Also we have

$$\begin{aligned}
\lim_{n\to\infty} T(t-s_n)y_m &= T(t)x_m \\
&= T(t)\lim_{n\to\infty} T(-s_n)y_m.
\end{aligned}$$

Now, for $t = 0$, we get

$$\lim_{n\to\infty} T(-s_n)y_m$$

exists for $m = 1, 2, \ldots$ and $T(0)x_m = x_m$. Hence, we get

$$\lim_{n\to\infty} T(-s_n)y$$

exists for every $y \in R_F$. This defines a linear map G on the linear subspace spanned by R_F where

$$Gy = \lim_{n\to\infty} T(-s_n)y.$$

It is easy to verify that G has the same properties as F and we have

$$GFx = x$$

for every $x \in \overline{B}$.

If (s'_n) is an arbitrary sequence of real numbers, we can extract a subsequence (s_n) such that

$$\lim_{n\to\infty} f(t+s_n) = g(t)$$

$$\lim_{n\to\infty} g(t-s_n) = f(t)$$

pointwise on $\mathbb{R}$ and

$$\lim_{n\to\infty} T(-s_n)x = y$$

exists for each $x \in \overline{B}$.

Now let us observe that for every $t \in \mathbb{R}$ and $n = 1, 2, \ldots,$ we have

$$f(t+s_n),\ g(t) \in \overline{B}.$$

Let $t \in \mathbb{R}$ be arbitrary. Then for every $n = 1, 2, \ldots$

$$T(t+s_n)f(t+s_n) = T(t)T(s_n)f(t+s_n)$$

so that

$$\lim_{n\to\infty} T(t+s_n)f(t+s_n) = T(t)Fg(t)$$

and

$$\lim_{n\to\infty} T(t-s_n)Fg(t-s_n) = T(t)\lim_{n\to\infty} T(-s_n)Fg(t-s_n)$$
$$= T(t)GFf(t).$$

The theorem is proved. □

Theorem 2.15 *Let $f \in AA(\mathbb{X})$. If $f(t) = 0$ for all $t > \alpha$ for some real number α, then $f(t) \equiv 0$ for all $t \in \mathbb{R}$.*

Proof It suffices to prove that $f(t) = 0$ for $t \le \alpha$. Consider the sequence of natural numbers $\mathbb{N} = (n)$. By assumption there exists a subsequence $(n_k) \subset (n)$ such that

$$\lim_{k\to\infty} f(t+n_k) = g(t)$$

exists for each $t \in \mathbb{R}$ and

$$\lim_{k\to\infty} g(t-n_k) = f(t)$$

for each $t \in \mathbb{R}$.

Obviously, for any $t \le \alpha$, we can find $(n_{k_j}) \subset (n_k)$ with $t + n_{k_j} > \alpha$ for all $j = 1, 2, \ldots$, so that $f(t + n_{k_j}) = 0$ for all $j = 1, 2, \ldots$. And since $\lim_{j\to\infty} f(t+n_{k_j}) = g(t)$, it yields $g(t) = 0$. Then we deduce that $f(t) = 0$.

The proof is complete. □

Theorem 2.16 *Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group and suppose that $x(t) := T(t)x_0 \in AA(\mathbb{X})$ for some $x_0 \in D(A)$, the domain of its infinitesimal generator A. Then*

$$\inf_{t\in\mathbb{R}} \|x(t)\| > 0, \quad or \quad x(t) \equiv 0 \quad for\ every\ t \in \mathbb{R}. \tag{1.4}$$

Proof Assume that $\inf_{t\in\mathbb{R}} \|T(t)x_0\| = 0$ and let (s'_n) be a sequence of real numbers such that $\lim_{n\to\infty} \|x(s'_n)\| = 0$. We can extract a subsequence (s_n) of (s'_n) such that

$$\lim_{n\to\infty} x(t+s_n) = y(t)$$

exists for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} y(t-s_n) = x(t)$$

for each $t \in \mathbb{R}$. We have in fact

$$y(t) = \lim_{n\to\infty} T(t+s_n)x_0 = T(t) \lim_{n\to\infty} T(s_n)x_0 = T(t) \lim_{n\to\infty} x(s_n) = 0$$

for each $t \in \mathbb{R}$. We deduce that $x(t) \equiv 0$ on $\mathbb{R}$, and the proof is complete. □

2 Weak Almost Automorphy

Definition 2.17 A weakly continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be weakly almost automorphic (in short w-almost automorphic) if for every sequence of real numbers (s'_n) there exists a subsequence (s_n) such that

$$\text{weak} - \lim_{n\to\infty} f(t+s_n) = g(t)$$

exists for each $t \in \mathbb{R}$ and

$$\text{weak} - \lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$.

Remark 2.18

(i) Every almost automorphic function is w-almost automorphic.
(ii) If $f : \mathbb{R} \to \mathbb{X}$ is w-almost automorphic, then the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(t) := (\varphi f)(t)$ with $\varphi \in \mathbb{X}^*$ the dual space of $\mathbb{X}$ is almost automorphic.

The following results are obvious and we omit the proof:

Theorem 2.19 *If f, f_1, f_2 are w-almost automorphic, then the following also are w-almost automorphic:*

(i) $f_1 + f_2$.
(ii) cf *for an arbitrary scalar* c.
(iii) $f_a(t) := f(t + a)$, *for any fixed real number* a.

We denote by $WAA(\mathbb{X})$ the vector space of all w-almost automorphic functions $f : \mathbb{R} \to \mathbb{X}$.

Theorem 2.20 *If $f \in WAA(\mathbb{X})$, then* $\sup_{t\in\mathbb{R}} \|f(t)\| < \infty$.

Proof Suppose by contradiction that $\sup_{t\in\mathbb{R}} \|f(t)\| = \infty$. Then there exists a sequence of real numbers (s_n') such that $\lim_{n\to\infty} \|f(s_n')\| = \infty$. Since f is w-almost automorphic, then we can find a subsequence (s_n) such that

$$weak - \lim_{n\to\infty} f(s_n) = \alpha \;\; exists.$$

$(f(s_n))$ is then a weakly convergent sequence, hence it is weakly bounded and therefore bounded by Proposition 1.41. This is a contradiction, and consequently, the theorem holds. □

Theorem 2.21 *If $f \in WAA(\mathbb{X})$, then*

$$\sup_{t\in\mathbb{R}} \|f(t)\| = \sup_{t\in\mathbb{R}} \|g(t)\|,$$

where g is the function defined in Definition 2.17.

Proof Since every weakly convergent sequence is bounded in norm (Proposition 1.41), and in particular if

$$weak - \lim_{n\to\infty} x_n = \alpha,$$

then

$$\|\alpha\| \leq \liminf_{n\to\infty} \|x_n\|$$

(cf. [69, Theorem 1, page 120]). Thus, for each $t \in \mathbb{R}$, we get

$$\|g(t)\| \leq \liminf_{n\to\infty} \|f(t+s_n)\| \leq \sup_{t\in\mathbb{R}} \|f(t)\| < \infty$$

and

$$\|f(t)\| \leq \liminf_{n\to\infty} \|g(t-s_n)\| \leq \sup_{t\in\mathbb{R}} \|g(t)\| < \infty.$$

The equality is now proved. □

The following result is easy to prove:

Theorem 2.22 *Let $f \in WAA(\mathbb{X})$ and $A \in B(\mathbb{X})$. Then $Af : \mathbb{R} \to \mathbb{X}$ is also w-almost automorphic.*

Theorem 2.23 *Let $f \in WAA(\mathbb{X})$ and suppose that its range $\mathcal{R}_f$ is relatively compact in $\mathbb{X}$. Then $f \in AA(\mathbb{X})$.*

Proof Let (s_n') be a sequence of real numbers. We can extract a subsequence $(s_n) \subset (s_n')$ such that

$$weak - \lim_{n\to\infty} f(t+s_n) = g(t)$$

exists for each $t \in \mathbb{R}$ and

$$weak - \lim_{n\to\infty} g(t-s_n) = f(t)$$

for each $t \in \mathbb{R}$. Now fix $t_0 \in \mathbb{R}$. Then we have

$$\lim_{n\to\infty} (\varphi f)(t_0+s_n) = (\varphi g)(t_0)$$

and

$$\lim_{n\to\infty} (\varphi g)(t_0-s_n) = (\varphi f)(t_0)$$

for every $\varphi \in \mathbb{X}^*$.

Observe that the range $\mathcal{R}_g$ of g is also relatively compact in $\mathbb{X}$.

Indeed, for every $\bar{t} \in \mathbb{R}$, $g(\bar{t})$ is the strong limit of the sequence $(f(\bar{t}+s_n))$ which is contained in the closure of $\mathcal{R}_f$; whence $g(\bar{t})$ is in the closure of $\mathcal{R}_f$, a compact set in $\mathbb{X}$.

Also from the weak convergence of the sequence $(g(\bar{t} - s_n))$ toward $f(\bar{t})$, for every $\bar{t} \in \mathbb{R}$, we have the strong convergence, so $f \in AA(\mathbb{X})$. □

3 Almost Automorphic Sequences

Similarly as for functions, we define below the almost automorphy of sequences. From now on, we will use the notation $l^\infty(\mathbb{X})$ to indicate the space of all bounded (two-sided) sequences in a Banach space $\mathbb{X}$ with supnorm, that is, if $x = (x_n)_{n\in\mathbb{Z}} \in l^\infty(\mathbb{X})$, then

$$\|x\| := \sup_{n\in\mathbb{Z}} \|x_n\|.$$

Definition 2.24 A sequence $x \in l^\infty(\mathbb{X})$ is said to be *almost automorphic* if for any sequence of integers (k_n'), there exists a subsequence (k_n) such that

$$\lim_{m\to\infty} \lim_{n\to\infty} x_{p+k_n-k_m} = x_p \tag{3.1}$$

for any $p \in \mathbb{Z}$.

The set of all almost automorphic sequences in $\mathbb{X}$ forms a closed subspace of $l^\infty(\mathbb{X})$, that is denoted by $aa(\mathbb{X})$. We can show that the range of an almost automorphic sequence is precompact. For each bounded sequence $g := (g_n)_{n\in\mathbb{Z}}$ in $\mathbb{X}$, we will denote by $S(k)g$ the k-translation of g in $l^\infty(\mathbb{X})$, i.e., $(S(k)g)_n = g_{n+k}, \forall n \in \mathbb{Z}$. And S stands for $S(1)$.

3.1 *Kadets Theorem*

Let c_0 be the Banach space of all numerical sequences $(a_n)_{n=1}^\infty$ such that $\lim_{n\to\infty} a_n = 0$, equipped with supnorm. In the simplest case, the problem we are considering becomes the following:

when is the integral of an almost automorphic function also almost automorphic?

We can take the same counterexample as in [38] to show that additional conditions should be imposed on the space $\mathbb{X}$.

Example 2.25 Consider the function $f(t)$ with values in c_0 defined by

$$f(t) = ((1/n)\cos(t/n))_{n=1}^\infty, \forall t \in \mathbb{R}.$$

The integral $F(t) = \int_0^t f(\xi)d\xi$ of $f(t)$ is $F(t) = (\sin(t/n))_{n=1}^{\infty}$. Obviously, f is almost periodic (so it is almost automorphic), and F is bounded. However, the range of F, as shown in [38, p. 81–82], is not precompact, so F cannot be almost automorphic.

The Kadets Theorem (see e.g. [38, Theorem 2, p. 86]) says that if f is almost periodic and F is bounded, then F is almost periodic if and only if $\mathbb{X}$ does not contain any subspace isomorphic to c_0. An extension of the Kadets Theorem to almost automorphic functions was given in [9].

The following extension of the Kadets Theorem to sequences will be used later on:

Lemma 2.26 *Assume that $x = (x_n)_{n\in\mathbb{Z}}$ is a sequence in a Banach space $\mathbb{X}$ that does not contain any subspace isomorphic to c_0, and the difference*

$$x - Sx = y \tag{3.2}$$

is almost automorphic. Then, the sequence x itself is almost automorphic.

Proof This lemma is a special case of [9, Theorem 1]. □

As it is well known (see e.g. [38]), a convex Banach space does not contain any subspace isomorphic to c_0. In particular, every finite dimensional space does not contain any subspace isomorphic to c_0.

Example 2.27 ([67]) Let $\theta \in \mathbb{Q}$. For $n \in \mathbb{Z}$, $\cos(2\pi n\theta) \neq 0$. We define

$$l_n = \text{sgn}\cos(2\pi n\theta) = 1, \quad if \;\; \cos(2\pi n\theta) > 0$$

$$l_n = \text{sgn}\cos(2\pi n\theta) = -1, \quad if \;\; \cos(2\pi n\theta) < 0.$$

Let f be defined by

$$f(t) = l_n + (t-n)(l_{n+1} - _n), \quad t \in [n, n+1].$$

It is obvious that f is compact almost automorphic but it is not almost periodic.

4 Asymptotically Almost Automorphic Functions

This section is devoted to the study of continuous functions $\mathbb{R} \to \mathbb{X}$ which approach almost automorphic functions as t tends to ∞. This theory was introduced in [52, 53]. The results obtained will be used to investigate the asymptotic behavior of solutions of differential equations and dynamical systems.

Definition 2.28 Let $\mathbb{X}$ be a (real or complex) Banach space. A function $f \in C(\mathbb{R}^+, \mathbb{X})$ is said to be asymptotically almost automorphic if it admits a decomposition

$$f(t) = g(t) + h(t), \ t \in \mathbb{R}^+,$$

where $g \in AA(\mathbb{X})$ and $h \in C_0(\mathbb{R}^+, \mathbb{X})$ (i.e. $h : \mathbb{R}^+ \to \mathbb{X}$ is continuous and $\lim_{t\to\infty} \|h(t)\| = 0$). The function $g(t)$ (resp. $h(t)$) is called the principal term (resp. corrective term) of $f(t)$.

The set of all asymptotically almost automorphic functions $f : \mathbb{R}^+ \to \mathbb{X}$ will be denoted $AAA(\mathbb{X})$.

Remark 2.29 It is obvious that if $f \in AA(\mathbb{X})$, then its restriction to $\mathbb{R}^+$ will belong to $AAA(\mathbb{X})$. It suffices to take $g = f$ on $\mathbb{R}$ and $h = 0$ on $\mathbb{R}^+$.

The following is quite obvious:

Theorem 2.30 *$AAA(\mathbb{X})$ is a linear vector space.*

We also have the following properties:

Theorem 2.31 *Let $\mathbb{X}$ be a Banach space over the field $\Phi = \mathbb{R}$ or $\mathbb{C}$ and $f : \mathbb{R}^+ \to \mathbb{X}$ and $\nu : \mathbb{R}^+ \to \Phi$ be asymptotically almost automorphic. Then the following functions are also asymptotically almost automorphic:*

(i) $f_a : \mathbb{R}^+ \to \mathbb{X}$ defined by $f_a(t) = f(t + a)$, $a \in \Phi$;
(ii) $\nu f : \mathbb{R}^+ \to \mathbb{X}$ defined by $(\nu f)(t) = \nu(t) f(t)$.

Proof Obvious. □

Proposition 2.32 ([52, 59]) *Let $f \in AAA(\mathbb{X})$ such that $f = g+h$ with $g \in AA(\mathbb{X})$ and $h \in C_0(\mathbb{R}^+, \mathbb{X})$. Then*

(i) The decomposition of f is unique.
(ii) $\overline{\{g(t) : \ t \in \mathbb{R}\}} \subset \overline{\{f(t) : \ t \in \mathbb{R}^+\}}$.
(iii) $AAA(\mathbb{X})$ is a Banach space under the norm

$$\|f\|_\star := \sup_{t\in\mathbb{R}^+} \|f(t)\|.$$

(iv) The range $\mathcal{R}_f := \{f(t); t \in \mathbb{R}^+\}$ of a function $f \in AAA(\mathbb{X})$ is relatively compact.
(v) $AAA(\mathbb{X}) = AA(\mathbb{X}) \oplus C_0(\mathbb{R}^+; \mathbb{X})$.

Proof

(i) Let $f \in AAA(\mathbb{X})$ such that

$$f(t) = g_i(t) + h_i(t), \ t \in \mathbb{R}^+,$$

where $g_i \in AA(\mathbb{X}),\ h_i \in C_0(\mathbb{R}^+, \mathbb{X})$ for $i = 1, 2$. Then for $t \in \mathbb{R}^+$ we have

$$g_1(t) - g_2(t) + h_1(t) - h_2(t) = 0.$$

Consequently,

$$\lim_{t\to\infty} g_1(t) - g_2(t) = 0.$$

Consider the sequence of natural numbers (n). Since $g_1 - g_2 \in AA(\mathbb{X})$, there exists a subsequence $(n_k) \subset (n)$ such that

$$\lim_{k\to\infty} g_1(t + n_k) - g_2(t + n_k) = F(t)$$

exists for each $t \in \mathbb{R}$ and

$$\lim_{k\to\infty} F(t - n_k) = g_1(t) - g_2(t)$$

for each $t \in \mathbb{R}$. This proves that $F(t) \equiv 0$ on $\mathbb{R}$ and consequently $g_1 - g_2 \equiv 0$ as well. It follows that $h_1(t) - h_2(t) = 0$, for every $t \in \mathbb{R}^+$. This completes the proof.

(ii) Let $(s'_n)_n$ be a sequence of reals such that $\lim_{n\to\infty} s'_n = +\infty$. Then consider the subsequence $(s_n)_n$ such that

$$k(t) := \lim_{n\to\infty} g(t + s_n)$$

is well-defined for each $t \in \mathbb{R}$ and

$$\lim_{t\to\infty} k(t - s_n) = g(t)$$

pointwise on $\mathbb{R}$.

Fix $t_0 \in \mathbb{R}$ and consider

$$f(t_0 + s_n) = g(t_0 + s_n) + h(t_0 + s_n).$$

Since $t_0 + s_n \to \infty$ as $n \to \infty$, we get

$$\lim_{n\to\infty} f(t_0 + s_n) = k(t_0).$$

Hence $k(t_0) \in \overline{\{f(t) : t \in \mathbb{R}^+\}}$; consequently $\overline{\{k(t) : t \in \mathbb{R}\} \subset \{f(t) : t \in \mathbb{R}^+\}}$.

But from the definition and property of k it is obvious that $\overline{\{k(t) : t \in \mathbb{R}\}} = \overline{\{g(t) : t \in \mathbb{R}\}}$. Thus

$$\overline{\{g(t):\ t\in\mathbb{R}\}}\subset\overline{\{f(t):\ t\in\mathbb{R}^+\}}.$$

This ends the proof.

(iii) From (ii), it is clear that

$$\sup_{t\in\mathbb{R}}\|g(t)\|\leq\sup_{t\in\mathbb{R}^+}\|f(t)\|.$$

Therefore we get

$$\|f\|_\star\leq|f|=\sup_{t\in\mathbb{R}}\|g(t)\|+\sup_{t\in\mathbb{R}^+}\|f(t)-g(t)\|\leq3\sup_{t\in\mathbb{R}^+}\|f(t)\|=3\|f\|_\star.$$

This shows that the norms $\|\cdot\|_\star$ and $|\cdot|$ are equivalent. But $(AAA(\mathbb{X}),|\cdot|)$ is a Banach space (cf. [57]), thus $(AAA(\mathbb{X}),\|\cdot\|_\star)$ is a Banach space too.

(iv) Note that the range of g is relatively compact since $g\in AA(\mathbb{X})$. It suffices to prove that range $\{h(t):\ t\in\mathbb{R}^+\}$ of h is relatively compact.

Now let $(s_n')_n$ be a sequence in $\mathbb{R}^+$.

Case I: $(s_n')_n$ is bounded. Then there exists a subsequence $(s_n)_n$ such that $\lim_{n\to\infty}s_n=t_0$ for some $t_0\in\mathbb{R}^+$. Since h is continuous, $\lim_{n\to\infty}h(s_n)=h(t_0)$.

Case II. $(s_n')_n$ is unbounded. Then there exists a subsequence $(s_k)_k$ such that $s_k\to+\infty$. Therefore $\lim_{n\to\infty}h(s_n)=0$.

We deduce from Case I and Case II that the range $\{h(t):\ t\in\mathbb{R}^+\}$ of h is relatively compact in $\mathbb{X}$. The proof is complete. □

Theorem 2.33 *Let* $(\mathbb{X},\|\cdot\|_\mathbb{X})$, $(\mathbb{Y},\|\cdot\|_\mathbb{Y})$ *be two Banach spaces over the field* Φ *and* $f\in AAA(\mathbb{X})$. *Let* $\phi:\mathbb{X}\to\mathbb{Y}$ *be a strongly continuous application. Assume that there exists a compact set* B *which contains* $\overline{\{f(t)\ :\ t\in\mathbb{R}^+\}}$ *and* $\overline{\{g(t)\ :\ t\in\mathbb{R}\}}$.

Then we have $\phi(f(t))\in AAA(\mathbb{Y})$.

Proof Let $f=g+h$ where g is the principal term of f and h its corrective term. Then $\phi(g(t))\in AA(\mathbb{Y})$ by Theorem 2.8. Since $\phi(f(t))$ and $\phi(g(t))$ are continuous, the function $\Gamma:\mathbb{R}^+\to\mathbb{Y}$ defined by

$$\Gamma(t)=\phi(f(t))-\phi(g(t)),\ t\in\mathbb{R}^+$$

is also continuous.

Let $\epsilon>0$ be given. By uniform continuity on the compact set B, we can choose $\delta>0$ such that

$$\|\phi(x)-\phi(y)\|_\mathbb{Y}<\epsilon\ \text{ if }\ \|x-y\|_\mathbb{X}<\delta,$$

with $x,y\in B$.

On the other hand, since $\lim_{t\to\infty} h(t) = 0$, there exists $t_0 = t_0(\delta) > 0$ such that

$$\|h(t\| = \|f(t) - g(t)\| < \delta \;\; \forall t > t_0.$$

Now if $t > t_0$, we obtain

$$\|\Gamma(t)\|_{\mathbb{Y}} = \|\phi(f(t)) - \phi(g(t))\|_{\mathbb{Y}} < \epsilon$$

which proves that $\lim_{t\to\infty} \Gamma(t) = 0$.

The function $\phi(f(t))$ is then asymptotically almost automorphic; its principal term is $\phi(g(t))$ and its corrective term is $\Gamma(t)$. This completes the proof. □

Bibliographical Notes Materials in this chapter are work of N'Guérékata [52, 53, 58, 59] and Zaki [70].

In [31, 34], the author developed the theory of almost automorphic functions in Fréchet spaces. A Fuzzy version of the theory was also developed in [29, 33].

In [32], and following Veech [67], the authors presented interesting results on almost automorphic groups and semigroups in Fréchet spaces.

Chapter 3
Almost Automorphy of the Function $f(t,x)$

1 The Nemytskii's Operator

Definition 3.1 A continuous function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is said to be almost automorphic if $f(t,x)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $x \in K$, where K is any bounded subset of $\mathbb{X}$. In other words for every sequence of real numbers (s'_n) there exists a subsequence (s_n) such that

$$g(t,x) = \lim_{n\to\infty} f(t+s_n,x)$$

is well-defined in $t \in \mathbb{R}$ for all K and

$$\lim_{n\to\infty} g(t-s_n,x) = f(t,x)$$

for all $t \in \mathbb{R}$ and $x \in K$.

We denote by $AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ the set of all such functions.

Theorem 3.2 *If $f, f_1, f_2 \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, then we have*

(i) $f_1 + f_2 \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$.
(ii) $\lambda f \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, *for any scalar* λ.

Proof Obvious. □

Theorem 3.3 *If $f \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$,then*

$$\sup_{t\in\mathbb{R}} \|f(t,x)\| = \sup_{t\in\mathbb{R}} \|g(t,x)\| = C_x < \infty$$

for x in any bounded set $K \subset \mathbb{X}$ where g is the function in Definition 3.1.

G. M. N'Guérékata, *Almost Periodic and Almost Automorphic Functions in Abstract Spaces*, https://doi.org/10.1007/978-3-030-73718-4_3

Proof It is analogous to the proof of Remark 2.6. □

Theorem 3.4 *If $f \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ is lipschitzian in x uniformly in $t \in \mathbb{R}$, then the function g as in Definition 3.1 is also lipschitzian with the same Lipschitz constant.*

Proof Let L be a Lipschitz constant for the function f, i.e.

$$\|f(t, x) - f(t, y)\| < L\|x - y\|$$

for x, y in any bounded subset K of $\mathbb{X}$ uniformly in $t \in \mathbb{R}$.

Let $t \in \mathbb{R}$ be arbitrary and $\varepsilon > 0$ and K a bounded set in $\mathbb{X}$ be given. Then for any sequence of real numbers (s'_n), there exists a subsequence (s_n) such that

$$\|f(t + s_n, x) - g(t, x)\| < \frac{\varepsilon}{2}$$

and

$$\|g(t - s_n, x) - f(t, x)\| < \frac{\varepsilon}{2}$$

for n sufficiently large and uniformly in $x \in K$.

Let us write for $x, y \in K$

$$\begin{aligned} g(t, x) - g(t, y) = g(t, x) - f(t + s_n, x) + f(t + s_n, x) - f(t + s_n, y) \\ + f(t + s_n, y) - g(t, y). \end{aligned}$$

For n sufficiently large we get

$$\|g(t, x) - g(t, y)\| < \varepsilon + L\|x - u\|.$$

And since ε is arbitrary we obtain

$$\|g(t, x) - g(t, y)\| \leq \varepsilon$$

uniformly for $x, y \in K$, which completes the proof. □

Theorem 3.5 ([39]) *Let $f \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and assume that $f(t, \cdot)$ is uniformly continuous on each bounded set $K \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$; in other words, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in K$ with $\|x - y\| < \delta$, then $\|f(t, x) - f(t, y)\| < \varepsilon$ for all $t \in \mathbb{R}$. Let $\varphi \in AA(\mathbb{X})$.*

Then the Nemytskii operator $\mathcal{N} : \mathbb{R} \to \mathbb{X}$ defined by $\mathcal{N}(\cdot) := f(\cdot, \varphi(\cdot))$ is in $AA(\mathbb{X})$.

Proof Let (s'_n) be a sequence of real numbers. Then there exists a subsequence $(s_n) \subset (s'_n)$ such that

(i) $\lim_{n\to\infty} f(t+s_n, x) = g(t, x)$, for each $t \in \mathbb{R}$ and $x \in \mathbb{X}$,
(ii) $\lim_{n\to\infty} g(t-s_n, x) = f(t, x)$, for each $t \in \mathbb{R}$ and $x \in \mathbb{X}$,
(iii) $\lim_{n\to\infty} \varphi(t+s_n) = \gamma(t)$ for each $t \in \mathbb{R}$,
(iv) $\lim_{n\to\infty} \gamma(t+s_n) = \varphi(t)$ for each $t \in \mathbb{R}$.

Let us define $G : \mathbb{R} \to \mathbb{X}$ by $G(t) = g(t, \gamma(t))$. Then we obtain

$$\lim_{n\to\infty} \mathcal{N}(t+s_n) = G(t)$$

and

$$\lim_{n\to\infty} G(t-s_n) = \mathcal{N}(t)$$

for each $t \in \mathbb{R}$.

Consider the inequality

$$\begin{aligned}\|\mathcal{N}(t+s_n) - G(t)\| \le \|f(t+s_n, \varphi(t+s_n)) - f(t+s_n, \gamma(t))\| \\ + \|f(t+s_n, \gamma(t)) - g(t, \gamma(t))\|.\end{aligned}$$

Since $\varphi \in AA(\mathbb{X})$, then φ and γ are bounded. Let us choose $K \in \mathbb{X}$ such that $\varphi(t), \gamma(t) \in K$ for all $t \in \mathbb{R}$. In view of (iii) and the uniform continuity of $f(t, x)$ in $x \in K$, we will have

$$\lim_{n\to\infty} \|f(t+s_n, \varphi(t+s_n)) - f(t+s_n, \gamma(t))\| = 0.$$

Now by (i), we get

$$\lim_{n\to\infty} \|f(t+s_n, \gamma(t)) - g(t, \gamma(t))\| = 0,$$

which proves that for each $t \in \mathbb{R}$

$$\lim_{n\to\infty} \mathcal{N}(t+s_n) = G(t).$$

Similarly, we can prove that

$$\lim_{n\to\infty} G(t-s_n) = \mathcal{N}(t)$$

for each $t \in \mathbb{R}$. The proof is now complete. □

Theorem 3.6 ([39]) *Let $f \in AAA(\mathbb{R}^+ \times \mathbb{X}, \mathbb{X})$ with principal term $g(t, x)$ and corrective term $h(t, x)$. Assume that $g(t, x)$ is uniformly continuous on any bounded set $K \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$. Assume also that $\varphi \in AAA(\mathbb{X})$. Then the Nemytskii operator $\mathcal{N} : \mathbb{R} \to \mathbb{X}$ defined by $\mathcal{N}(\cdot) := f(\cdot, \varphi(\cdot))$ is in $AAA(\mathbb{X})$.*

Proof Let $\alpha(t)$ and $\beta(t)$ be the principal and corrective terms of $\varphi(t)$, respectively. Let us write

$$f(t,\varphi(t)) = g(t,\alpha(t)) + f(t,\varphi(t)) - g(t,\alpha(t)) = g(t,\alpha(t)) + g(t,\varphi(t)) - g(t,\alpha(t)) + h(t,\varphi(t)).$$

In view of Theorem 3.5, $g(t,\alpha(t)) \in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$.

On the other hand, the uniform continuity of $g(t,\varphi(t))$ implies that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|g(t,\varphi(t)) - g(t,\alpha(t))\| < \varepsilon$$

if $\varphi(t), \alpha(t) \in K$ for any $t \in \mathbb{R}^+$ and a given bounded set $K \subset \mathbb{X}$ and $\|\varphi(t) - \alpha(t)\| < \delta$. Moreover since $\beta(t) \in C_0(\mathbb{R},\mathbb{X})$, there exists $T > 0$ such that

$$\|\varphi(t) - \alpha(t)\| = \|\beta(t)\| < \delta,$$

for $t > T$. Consequently, we get

$$\lim_{t\to\infty} \|g(t,\varphi(t)) - g(t,\alpha(t))\| = 0.$$

We know also that

$$\lim_{t\to\infty} \|h(t,\varphi(t))\| = 0.$$

This proves that

$$g(t,\varphi(t)) - g(t,\alpha(t)) + h(t,\varphi(t)) \in C_0(\mathbb{R}^+,\mathbb{X}),$$

and consequently

$$\mathcal{N}(\cdot) := f(\cdot,\varphi(\cdot)) \in AAA(\mathbb{X})$$

□

Bibliographical Notes Most of this chapter are contained in the first edition of this book. It is noted that C. Lizama and J.G Mesquita [41–43] and Milcé et al. [46, 47, 50, 62, 63] studied almost automorphy on time scales and its application to dynamic equations on time scales. This is another growing field which needs further investigation.

Chapter 4
Differentiation and Integration

1 Differentiation in $AA(\mathbb{X})$

Theorem 4.1 *Let $f \in AA(\mathbb{X})$ and suppose that its derivative f' exists and is uniformly continuous on $\mathbb{R}$. Then $f' \in AA(\mathbb{X})$.*

Proof Let $\varepsilon > 0$ be given; then using the uniform continuity of f' we can choose $\delta > 0$ such that for every pair of real numbers t_1, t_2 such that $|t_1 - t_2| < \delta$, we have $\|f'(t_1) - f'(t_2)\| < \epsilon$.

Now for arbitrary $t \in \mathbb{R}$ and $\delta \geq \frac{1}{n}$, we get

$$n(f(t + \frac{1}{n}) - f(t)) - f'(t) = n \int_0^{\frac{1}{n}} (f'(t+s) - f'(t))ds.$$

This equality shows that the sequence of almost automorphic functions $n(f(t + \frac{1}{n}))$ converges uniformly to $f'(t)$ on $\mathbb{R}$. We deduce that $f' \in AA(\mathbb{X})$. □

2 Integration in $AA(\mathbb{X})$

Let us introduce some useful notations due to S. Bochner in order to facilitate the exposition of the proofs. In what follows if $f : \mathbb{R} \to \mathbb{X}$ is a function and a sequence of real numbers $s = (s_n)$ is such that we have

$$\lim_{n\to\infty} f(t + s_n) = g(t)$$

pointwise in $\mathbb{R}$, we will write

$$T_s f = g.$$

G. M. N'Guérékata, *Almost Periodic and Almost Automorphic Functions in Abstract Spaces*, https://doi.org/10.1007/978-3-030-73718-4_4

Remark 4.2

- T_s is a linear operator.
  Indeed, given a fixed sequence $s = (s_n) \subset \mathbb{R}$, the domain of T_s is $D(T_s) = \{f : \mathbb{R} \to \mathbb{X} \,/\, T_s f \text{ exists}\}$. $D(T_s)$ is a linear set. Indeed if $f, f_1, f_2 \in D(T_s)$, then $f_1 + f_2 \in D(T_s)$ and $\lambda f \in D(T_s)$ for any scalar λ. And obviously $T_s(f_1 + f_2) = T_s f_1 + T_s f_2$ and $T_s(\lambda f) = \lambda T_s f$.
- Let us write $-s := (-s_n)$ and suppose $f \in D(T_s)$ and $T_s f \in D(T_{-s})$. Then the product $A_s = T_{-s}T_s$ is well-defined. It is easy to verify that A_s is also a linear operator.
- A_s maps bounded functions into bounded functions, and if $f \in AA(\mathbb{X})$, we get $A_s f = f$.

We are now ready to prove the following:

Theorem 4.3 *Let $f \in AA(\mathbb{X})$ and consider the function $F : \mathbb{R} \to \mathbb{X}$ defined by $F(t) = \int_0^t f(s)ds$. Then $F \in AA(\mathbb{X})$ if and only if its range $\mathcal{R}_F = \{F(t) \,/\, t \in \mathbb{R}\}$ is relatively compact in $\mathbb{X}$.*

Proof It suffices to prove that $F(t) \in AA(\mathbb{X})$ if $\mathcal{R}_F$ is relatively compact. Let (s_n'') be an arbitrary sequence of real numbers. Then there exists a subsequence $(s_n') \subset (s_n'')$ such that

$$\lim_{n\to\infty} f(t + s_n') = g(t)$$

exists for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t + s_n') = f(t)$$

for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} F(s_n') = \alpha_1$$

for some $\alpha_1 \in \mathbb{X}$.

For every $t \in \mathbb{R}$, we get

$$\begin{aligned} F(t + s_n') = \int_0^{t+s_n'} f(r)dr &= \int_0^{s_n'} f(r)dr + \int_{s_n'}^{t+s_n'} f(r)dr \\ &= F(s_n') + \int_{s_n'}^{t+s_n'} f(r)dr. \end{aligned}$$

Using the substitution $\sigma = r - s_n'$, we obtain

$$F(t + s_n') = F(s_n') + \int_0^t f(\sigma + s_n')d\sigma.$$

If we apply the Lebesgue Dominated Convergence Theorem, we then get

$$\lim_{n\to\infty} F(t+s'_n) = \alpha_1 + \int_0^t g(\sigma)d\sigma$$

for each $t \in \mathbb{R}$.

Let us observe that the range of the function $G(t) = \alpha_1 + \int_0^t g(r)dr$ is also relatively compact in $\mathbb{X}$ and

$$\sup_{t\in\mathbb{R}} \|G(t)\| \leq \sup_{t\in\mathbb{R}} \|f(t)\|$$

so that we can extract a subsequence $(s_n) \subset (s'_n)$ such that

$$\lim_{n\to\infty} G(-s_n) = \alpha_2$$

for some $\alpha_2 \in \mathbb{X}$.

Now we can write

$$G(t-s_n) = G(-s_n) + \int_0^t g(r-s_n)dr$$

so that

$$\lim_{n\to\infty} G(t-s_n) = \alpha_2 + \int_0^t f(r)dr = \alpha_2 + F(t).$$

Let us prove now that $\alpha_2 = \theta$, the null vector in $\mathbb{X}$.

Using notation in Remark 4.2 above, we get

$$A_s F = \alpha_2 + F,$$

where $s = (s_n)$.

Now it is easy to observe that F as well as α_2 belongs to the domain of A_s. Therefore $A_s F$ also is in the domain of A_s, and we deduce the equation

$$A_s^2 F = A_s\alpha_2 + A_s F = \alpha_2 + \alpha_2 + A_s F = 2\alpha_2 + F.$$

We can continue indefinitely the process to get

$$A_s^n F = n\alpha_2 + F, \quad \forall n = 1, 2, \ldots$$

But we have

$$\sup_{t\in\mathbb{R}} \|A_s^n F(t)\| \leq \sup_{t\in\mathbb{R}} \|F(t)\|$$

and $F(t)$ is a bounded function. This leads to a contradiction if $\alpha_2 \neq \theta$. Hence $\alpha_2 = \theta$ and $A_s F = F$ which proves that $F(t) \in AA(\mathbb{X})$. The proof is complete. □

Theorem 4.4 *Let $\mathbb{X}$ be a reflexive Banach space and $f \in WAA(\mathbb{X})$. Then the function $F : \mathbb{R} \to \mathbb{X}$ defined by $F(t) = \int_0^t f(s)ds$ is weakly almost automorphic if and only if it is bounded in norm.*

Proof This is similar to the proof of Theorem 4.3 above. It suffices to observe that in a reflexive Banach space, bounded sets are weakly relatively compact (Proposition 1.15) □

Theorem 4.5 *Let $\mathbb{X}$ be a uniformly convex Banach space and let $f \in AA(\mathbb{X})$. Then the function $F : \mathbb{R} \to \mathbb{X}$ defined by $F(t) = \int_0^t f(s)ds$ is almost automorphic if and only if it is bounded.*

Proof If $F \in AA(\mathbb{X})$, then it is bounded.

Conversely let us assume that $F(t)$ is bounded. Since $f \in AA(\mathbb{X})$, then $f \in WAA(\mathbb{X})$. On the other hand $\mathbb{X}$ is reflexive since it is uniformly convex. Using Theorem 4.3, we know that $F \in WAA(\mathbb{X})$.

Assume now that

$$weak - \lim_{n\to\infty} F(t + s_n) = G(t)$$

pointwise on $\mathbb{R}$. Observe that

$$\sup_{t\in\mathbb{R}} \|F(t)\| = \sup_{t\in\mathbb{R}} \|G(t)\|.$$

Let us show that the image of $F(t)$ is relatively compact in $\mathbb{X}$. Suppose it is not. Then there exists $\sigma > 0$ and a sequence (s''_n) of real numbers such that

$$\|F(s''_n) - F(s''_m)\| \geq 2\sigma, \quad n \neq m.$$

Since $f \in AA(\mathbb{X})$, we can extract a subsequence (s'_n) of (s''_n) such that

$$\lim_{n\to\infty} f(t + s'_n) = g(t)$$

pointwise on $\mathbb{R}$.

Observe that the sequence $(F(s'_n))$ is bounded in virtue of the assumption on f. Since $\mathbb{X}$ is also reflexive, we can extract a subsequence (s_n) of (s'_n) such that

$$weak - \lim_{n\to\infty} F(s_n) = \alpha$$

exists in $\mathbb{X}$. Let $t \in \mathbb{R}$ be arbitrary. Then we can write

$$F(t+s_n) = F(s_n) + \int_{s_n}^{t+s_n} f(s)ds = F(s_n) + \int_0^t f(s+s_n)ds$$

and deduce that

$$weak - \lim_{n\to\infty} F(t+s_n) = \alpha + \int_0^t g(s)ds$$

pointwise on any finite interval of $\mathbb{R}$. Let

$$H(t) = \alpha + \int_0^t g(s)ds, \quad t \in \mathbb{R}.$$

We get

$$\sup_{t\in\mathbb{R}} \|F(t)\| = \sup_{t\in\mathbb{R}} \|H(t)\| = M < \infty. \tag{2.1}$$

Let us fix $t \in \mathbb{R}$. Then

$$\begin{aligned}
\|F(t+s_n) - F(t+s_m)\| &= \|F(s_n) - F(s_m) + \int_0^t f(s+s_n) - f(s+s_m)ds\| \\
&\geq \|F(s_n) - F(s_m)\| - \|\int_0^t f(s+s_n) - f(s+s_m)ds\| \\
&\geq 2\sigma - \|\int_0^t f(s+s_n) - f(s+s_m)ds\| \\
&\geq \sigma \\
&\geq \frac{\sigma}{M}\max\{\|F(s+s_n)\|, \|F(s+s_m)\|\},
\end{aligned}$$

for $n, m > N = N(t)$. Since $\mathbb{X}$ is a uniformly convex Banach space, it follows that

$$\begin{aligned}
\left\|\frac{F(t+s_n) - F(t+s_m)}{2}\right\| &\leq \left(1 - \frac{\delta\sigma}{M}\right)[\max\{\|F(s+s_n)\|, \|F(s+s_m)\|\}] \\
&\leq \left(1 - \frac{\delta\sigma}{M}\right) M
\end{aligned}$$

for $n, m > N = N(t)$.

Now let $\varphi \in \mathbb{X}^*$ be arbitrary with $\|\varphi\| = 1$. We have

$$\left|\varphi(\frac{F(t+s_n) - F(t+s_m)}{2})\right| \leq \|\varphi\| \left\|\frac{F(t+s_n) - F(t+s_m)}{2}\right\|$$

$$\leq \left(1 - \frac{\delta\sigma}{M}\right) M.$$

Take the limit as $n, m \to \infty$ to both sides of the inequality and obtain

$$|\varphi(H(t))| \leq \left(1 - \frac{\delta\sigma}{M}\right) M.$$

Hence

$$\|H(t)\| \leq \left(1 - \frac{\delta\sigma}{M}\right) M$$

and

$$\begin{aligned}\sup_{t\in\mathbb{R}} \|H(t)\| &\leq \left(1 - \frac{\delta\sigma}{M}\right) M \\ &< M = \sup_{t\in\mathbb{R}} \|F(T)\|\end{aligned}$$

which contradicts (2.3) and prove that $F(t)$ is relatively compact. Thus, since $F \in WAA(\mathbb{X})$, then $F \in AA(\mathbb{X})$.

The proof is complete. □

Proposition 4.6 *Let $\mathbb{X}$ be a Banach space and $x \in C(\mathbb{R}^+, \mathbb{X})$, $f \in C(\mathbb{R}, \mathbb{X})$. Let $T = (T(t))_{t\in\mathbb{R}^+}$ be a C_0-semigroup of linear operators. Suppose that*

$$x(t) = T(t)x(0) + \int_0^t T(t-s)f(s)ds, \quad t \in \mathbb{R}^+.$$

Then for t given in $\mathbb{R}$ and $b > a > 0$, $a + t > 0$, we have

$$x(t+b) = T(t+a)x(b-a) + \int_{-a}^t T(t-s)f(s)ds.$$

Proof Since $t + b > t + a > 0$, we get

$$\begin{aligned}x(t+b) &= T(t+b)x(0) + \textstyle\int_0^{t+b} T(t+b-s)f(s)ds \\ &= T(t+a)T(b-a)x(0) + \textstyle\int_0^{t+b} T(t+b-s)f(s)ds.\end{aligned}$$

We also have

$$x(b-a) = T(b-a)x(0) + \int_0^{b-a} T(b-a-s)f(s)ds,$$

which gives

$$T(b-a)x(0) = x(b-a) - \int_0^{b-a} T(b-a-s)f(s)ds.$$

Substituting this into the expression for $x(t+b)$ gives

$$\begin{aligned} x(t+b) &= T(t+a)(x(b-a) - \int_0^{b-a} T(b-a-s)f(s)ds \\ &\quad + \int_0^{t+b} T(t+b-s)f(s)ds \\ &= T(t+a)x(b-a) + \int_{b-a}^{t+b} T(t+b-s)f(s)ds. \end{aligned}$$

And putting $s = r + b$ will give

$$\int_{b-a}^{t+b} T(t+b-s)f(s)ds = \int_{-a}^{t} T(t-r)f(r+b)dr.$$

The proof is now complete. □

Theorem 4.7 *Let $T = (T(t))_{t\in\mathbb{R}^+}$ be a C_0-semigroup of linear operators on the reflexive Banach space $\mathbb{X}$. Let $f \in AA(\mathbb{X})$ and $x \in BC(\mathbb{R}^+, \mathbb{X})$ with the integral representation*

$$x(t) = T(t)x(0) + \int_0^t T(t-s)f(s)ds,\ t \in \mathbb{R}^+.$$

Then there exists $y \in BC(\mathbb{R} \to \mathbb{X})$ such that

$$y(t) = T(t-t_0)y(t_0) + \int_{t_0}^t T(t-s)f(s)ds$$

for all $t_0 \in \mathbb{R}$ and all $t \geq t_0$.

Proof Let $x_n(t) := x(t+n),\ n = 1, 2, \ldots$. Since $\mathbb{X}$ is reflexive, every bounded subset of $\mathbb{X}$ is weakly sequentially compact. Also $\mathbb{X}$ is weakly sequentially complete, so there exists a subsequence $(n_{k,0})_{k=1}^{\infty} \in \mathbb{N}$ such that

$$weak - \lim_{k\to\infty} x_{n_{k,0}}(0) \equiv weak - \lim_{k\to\infty} x(n_{k,0}) = x_0$$

exists in $\mathbb{X}$.

Now we extract a subsequence $(n_{k,1})_{k=1}^{\infty} \subset (n_{k,0})_{k=1}^{\infty}$ such that

$$weak - \lim_{k\to\infty} x_{n_{k,1}}(-1) \equiv weak - \lim_{k\to\infty} x(-1+n_{k,1}) = x_1$$

exists in $\mathbb{X}$. We continue the process and take the diagonal sequence $(n_j)_{j=1}^{\infty}$ to obtain

$$weak - \lim_{j\to\infty} x_{n_j}(-N) = x_N$$

for each $N = 0, 1, 2, \ldots$.

Since $f \in AA(\mathbb{X})$, we can extract a subsequence $(n_i)_{i=1}^{\infty} \subset (n_j)_{j=1}^{\infty}$ such that

$$\lim_{i\to\infty} f(t+n_i) = g(t)$$

exists pointwise in $\mathbb{R}$ and

$$\lim_{i\to\infty} g(t-n_i) = f(t)$$

pointwise on $\mathbb{R}$ and

$$weak - \lim_{i\to\infty} x_{n_i}(-N) = x_N$$

for $N = 0, 1, 2, \ldots$

Now let us prove the following three properties:

(i) $weak - \lim_{i\to\infty} x_{n_i}(t) = z(t)$ pointwise on $\mathbb{R}$.
(ii) $\sup_{t\in\mathbb{R}} \|z(t)\|| < \infty$.
(iii) $z(t) = T(t-t_0)z(t_0) + \int_{t_0}^{t} T(t-s)g(s)ds$ for every $t \geq t_0$.

To prove (i), let us fix $t \in \mathbb{R}$ and choose N such that $t + N > 0$ and consider i such that $n_i > N$.

Letting $a = N$ and $n_i = b$ in Proposition 4.6 we get

$$x(t+n_i) = T(t+N)x(n_i - N) + \int_{-N}^{t} T(t-s)f(s+n_i)ds.$$

On the other hand, we observe that for t_0, t fixed in $\mathbb{R}$ with $t \geq t_0$ and for any sequence of real numbers (s_n), the sequence of functions $F_n : [t_0, t] \to \mathbb{X}$ defined by

$$F_n(s) := T(t-s)f(s+s_n), \quad n = 1, 2, \ldots$$

is a uniformly bounded sequence of strongly measurable functions since $f \in AA(\mathbb{X})$.

As $T = (T(t))_{t\in\mathbb{R}^+}$ is a C_0-semigroup, then we have

$$\|T(t)\| \leq Me^{\beta t}, \; t \in \mathbb{R}^+$$

for some constant $M \geq 1$ and $\beta \in \mathbb{R}$. Therefore $T(t-s)$ is uniformly bounded for $s \in [t_0, t]$. Observe also that the function $F : [t_0, t] \to \mathbb{R}$ defined by

$$F(s) = T(t-s)f(s)$$

is continuous.

We can now prove that $(T(t-s)f(s-n_i))$ is a sequence of uniformly bounded and strongly measurable functions which converge to $T(t-s)g(s)$ everywhere on the interval $[-N, t]$. Therefore

$$\lim_{i\to\infty} \int_{-N}^{t} T(t-s)f(s+n_i)ds = \int_{-N}^{t} T(t-s)g(s)ds$$

by the Lebesgue Dominated Convergence Theorem. Let us call

$$weak - \lim_{i\to\infty} x_{n_i}(t) = weak - \lim_{i\to\infty} x(t+n_i) = z(t)$$

so that

$$z(t) = T(t+N)x_N + \int_{-N}^{t} T(t-s)g(s)ds$$

for all $t \in \mathbb{R}$ and all N such that $t + N > 0$. Also

$$\|z(t)\| \leq \liminf_{i\to\infty} \|x(t+n_i)\| \leq M$$

for all $t \in \mathbb{R}$. Therefore

$$\sup_{t\in\mathbb{R}} \|z(t)\| \leq N < \infty,$$

which proves (i).

Now choose $t_0, t \in \mathbb{R}$ with $t \geq t_0$ and choose $N \in \mathbb{N}$ with $t_0 + N > 0$. Then we obtain

$$z(t+a) = T(t-t_0)x(t_0+a) + \int_{t_0}^{t} T(t-s)g(s)ds.$$

Since $z(t)$ is bounded in norm on $\mathbb{R}$, we may assume that the sequence $(z(t_0 - n_i))$ is weakly convergent in the reflexive Banach space $\mathbb{X}$.

Let us write

$$z(t-n_i) = T(t-t_0)z(t_0-n_i) + \int_{t_0}^{t} T(t-s)g(s-n_i)ds.$$

Since $f \in AA(\mathbb{X})$, then g is bounded in norm and $T(t-s)f(s)$ is strongly measurable. Therefore the sequence $(T(t-s)g(s-n_i))$ will be uniformly bounded on $[t_0, t]$ and strongly measurable, and

$$\lim_{i\to\infty} T(t-s)g(s-n_i) = T(t-s)f(s)$$

for every $s \in [t_0, t]$.

Hence

$$\lim_{i\to\infty} \int_{t_0}^{t} T(t-s)g(s-n_i)ds = \int_{t_0}^{t} T(t-s)f(s)ds$$

and

$$weak - \lim_{i\to\infty} z(t-n_i) = y(t)$$

exists in $\mathbb{X}$ for every $t \in \mathbb{R}$.

Then

$$y(t) = T(t-t_0)y(t_0) + \int_{t_0}^{t} T(t-s)f(s)ds$$

for every $t \geq t_0$, so $y(t)$ is defined on $\mathbb{R}$ and

$$\|y(t)\| \leq \liminf_{i\to\infty} \|z(t-n_i)\| \leq M$$

for all $t \in \mathbb{R}$.

Since we have

$$\sup_{t\in\mathbb{R}} \|z(t)\| \leq M,$$

then we also get

$$\sup_{t\in\mathbb{R}} \|y(t)\| \leq M.$$

The proof is now complete. □

3 Differentiation in $WAA(\mathbb{X})$

Definition 4.8 A Banach space $\mathbb{X}$ is said to be perfect if every bounded function $f : \mathbb{R} \to \mathbb{X}$ with an almost automorphic derivative f' is necessarily almost automorphic.

Remark 4.9 Every uniformly convex Banach space is a perfect Banach space (cf. Theorem 4.5)

Theorem 4.10 *Let $\mathbb{X}$ be a perfect Banach space and $f \in AAA(\mathbb{X})$ with principal term g and corrective term h. Assume that $f'(t)$ exists for every $t \in \mathbb{R}^+$ and $g'(t)$ exists for every $t \in \mathbb{R}$. If in addition $f' \in AAA(\mathbb{X})$, then $g'(t)$ and $h'(t)$ are respectively the principal and the corrective terms of f'.*

Proof We first note that $h'(t)$ exists on $\mathbb{R}^+$. Since $f' \in AAA(\mathbb{X})$, let us write

$$f'(t) = G(t) + H(t), \;\; t \in \mathbb{R}^+,$$

where $G(t)$ and $H(t)$ are respectively the principal and the corrective terms of $f'(t)$. We need to show that $G(t) = g'(t)$ for every $t \in \mathbb{R}$ and $H(t) = h'(t)$ for every $t \in \mathbb{R}^+$.

Let us consider the functions $\alpha : \mathbb{R} \to \mathbb{X}$ and $\beta : \mathbb{R}^+ \to \mathbb{X}$ defined by

$$\alpha(t) = \int_t^{t+\eta} G(s)ds, \;\; t \in \mathbb{R}$$

$$\beta(t) = \int_t^{t+\eta} H(s)ds, \;\; t \in \mathbb{R}^+$$

for a fixed real number η. Note that β is continuous on $\mathbb{R}^+$ and we have the inequality

$$\|\beta(t)\| \leq |\eta| \cdot \sup_{s \in I_\eta} \|H(s)\|,$$

where $I_\eta = [t + \eta, t]$ or $I_\eta = [t, t + \eta]$ according to the sign of η. Since $\lim_{t\to\infty} \|H(t)\| = 0$, we deduce that $\lim_{t\to\infty} \|\beta(t)\| = 0$.

Also, α is defined and continuous on $\mathbb{R}$; t is bounded on $\mathbb{R}$ since G is bounded on $\mathbb{R}$. Since $\mathbb{X}$ is a perfect Banach space, we deduce that $\alpha \in AA(\mathbb{X})$.

Now consider the equalities

$$f(t + \eta) - f(t) = \alpha(t) + \beta(t)$$

$$f(t + \eta) - f(t) = [g(t + \eta) - g(t)] + [h(t + \eta) - h(t)],$$

where $t \in \mathbb{R}^+$ and η is chosen so that $t + \eta \geq 0$. By the uniqueness of the decomposition of the asymptotically almost automorphic function $f(t+\eta) - f(t)$, we get

$$\alpha(t) = g(t+\eta) - g(t), \ t \in \mathbb{R}$$

$$\beta(t) = h(t+\eta) - h(t), \ t \in \mathbb{R}^+,$$

whence

$$G(t) = \lim_{\eta \to 0} \frac{\alpha(t)}{\eta} = \lim_{\eta \to 0} \frac{g(t+\eta) - g(t)}{\eta} = g'(t), \ t \in \mathbb{R}$$

$$H(t) = \lim_{\beta \to 0} \frac{\alpha(t)}{\eta} = \lim_{\eta \to 0} \frac{h(t+\eta) - h(t)}{\eta} = h'(t), \ t \in \mathbb{R}^+.$$

The proof is achieved. □

4 Integration in $AAA(\mathbb{X})$

Theorem 4.11 *Let $\mathbb{X}$ be a Banach space and $f \in AAA(\mathbb{X})$. Consider the function $F : \mathbb{R}^+ \to \mathbb{X}$ defined by $F(t) = \int_0^t f(s)ds$ and $G : \mathbb{R} \to \mathbb{X}$ defined by $G(t) = \int_0^t g(s)ds$ where g is the principal term of f. Assume G has a relatively compact range in $\mathbb{X}$ and $\int_0^\infty \|h(s)\|ds < \infty$ where h is the corrective term of f.*

Then $F \in AAA(\mathbb{X})$; its principal term will be $G(t) + \int_0^\infty h(s)ds$, and its corrective term $H(t) = -\int_t^\infty h(s)ds$.

Proof We observe that $G \in AA(\mathbb{X})$ by Theorem 4.3, so is $G(t) + \int_0^\infty h(s)ds$ since the improper integral $\int_0^\infty h(s)ds$ exists in $\mathbb{X}$.

Now the continuous function $H(t) = -\int_t^\infty h(s)ds, \ t \in \mathbb{R}^+$ satisfies the property $\lim_{t\to\infty} \|H(t)\| = 0$.

Finally, observe that we can write

$$F(t) = G(t) + \int_t^\infty h(s)ds + H(t), \ t \in \mathbb{R}^+.$$

The theorem is proved. □

Corollary 4.12 *Let $\mathbb{X}$ be a uniformly convex Banach space and $f \in AAA(\mathbb{X})$. Consider the function $F : \mathbb{R}^+ \to \mathbb{X}$ defined by $F(t) = \int_0^t f(s)ds$ and $G : \mathbb{R} \to \mathbb{X}$ defined by $G(t) \int_0^t g(s)ds$ where g is the principal term of f. Assume G has a bounded range in $\mathbb{X}$ and $\int_0^\infty \|h(s)\|ds < \infty$ where h is the corrective term of f.*

Then $F \in AAA(\mathbb{X})$*; its principal term will be* $G(t) + \int_0^\infty h(s)ds$ *and its corrective term* $H(t) = -\int_t^\infty h(s)ds$.

Proof Observe that $G \in AA(\mathbb{X})$ in view of Theorem 4.5. □

Bibliographical Notes The results of this chapter are the work of N'Guérékata and were presented in [52, 53] and the first edition of this book.

Chapter 5
Pseudo Almost Automorphy

1 Pseudo Almost Automorphic Functions

Let $AA_0(\mathbb{X})$, $\phi \in BC(\mathbb{R}, \mathbb{X})$ such that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(t)\| dt = 0,$$

(resp. $AA_0(\mathbb{R} \times \mathbb{X})$ the set of all functions $\phi \in BC(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ such that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(t, x)\| dt = 0$$

uniformly for x in any bounded set of $\mathbb{X}$.)

Definition 5.1 A function $f \in BC(\mathbb{R}, \mathbb{X})$ is said to be pseudo almost automorphic (and we write $f \in PAA(\mathbb{X})$) if it admits the decomposition

$$f = g + \phi,$$

where $g \in AA(\mathbb{X})$ and $\phi \in AA_0(\mathbb{X})$. Analogously a function $f \in BC(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ is said to be pseudo almost automorphic (and we write $f \in PAA(\mathbb{R} \times \mathbb{X})$) if it admits the decomposition

$$f = g + \phi,$$

where $g \in AA(\mathbb{R} \times \mathbb{X})$ and $\phi \in AA_0(\mathbb{R} \times \mathbb{X})$ uniformly for x in any bounded set of $\mathbb{X}$.

In either cases, g is called the principal term of f and ϕ its ergodic term.

G. M. N'Guérékata, *Almost Periodic and Almost Automorphic Functions in Abstract Spaces*, https://doi.org/10.1007/978-3-030-73718-4_5

Remark 5.2 The following are obvious:

(i) $PAA(\mathbb{X})$ and $PAA(\mathbb{R} \times \mathbb{X})$ are vector spaces.
(ii) $AAA(\mathbb{X}) \subset PAA(\mathbb{X})$.

Lemma 5.3 *Assume $f \in AA(\mathbb{X})$, write*

$$B_\varepsilon := \{\tau \in \mathbb{R} \ : \ \|f(t_0 + \tau) - f(t_0)\| < \varepsilon\},$$

where $\varepsilon > 0$ and $t_0 \in \mathbb{R}$ is fixed. Then there exists $s_1, s_2, \ldots, s_m \in \mathbb{R}$ such that

$$\cup_{i=1}^{m}(s_i + B_\varepsilon) = \mathbb{R}.$$

Theorem 5.4 *Let $f \in PAA(\mathbb{X})$ where $g \in AA(\mathbb{X})$ and $h \in AA_0(\mathbb{X})$ such that*

$$f = g + h.$$

Then

$$\{g(t) \ : \ t \in \mathbb{R}\} \subset \overline{\{f(t) \ : \ t \in \mathbb{R}\}}. \tag{1.1}$$

Proof Suppose that (1.1) is not true, then there exists $t_0 \in \mathbb{R}, \ \varepsilon > 0$ such that

$$\|g(t_0) - f(t)\| \geq 2\varepsilon, \quad t \in \mathbb{R}. \tag{1.2}$$

Let $s_1, s_2, \ldots, s_m$ be as in Lemma 5.3 and write

$$\tau_i = s_i - t_0, \quad i = 1, 2, \ldots, m, \qquad \eta = \max_{1 \leq i \leq m} |\tau_i|.$$

For $T \in \mathbb{R}$ with $|T| > \eta$, we let

$$B_{\varepsilon,T}^{(i)} := [-T + \eta - \tau_i, T - \eta - \tau_i] \cap (t_0 + B_\varepsilon), \quad i = 1, 2, \ldots, m,$$

where B_ε is as in Lemma 5.3. It is clear that

$$\cup_{i=1}^{m}(\tau_i + B_{\varepsilon,T}^{(i)}) = [-T + \eta, T - \eta].$$

Thus we obtain

$$\begin{aligned} 2(T - \eta) &= \text{mes}[-T + \eta, T - \eta] \\ &\leq \sum_{i=1}^{m} \text{mes}(\tau_i + B_{\varepsilon,T}^{(i)}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \text{mes}(\tau_i + B^{(i)}_{\varepsilon,T}) \\
&\le m \cdot \max_{1\le i\le m} \{\text{mes}(B^{(i)}_{\varepsilon,T})\} \\
&\le m \cdot \text{mes}([-T,T] \cap (t_0 + B_\varepsilon)), \qquad (1.3)
\end{aligned}
$$

if we observe that

$$B^{(i)}_{\varepsilon,T} \subset [-T,T] \cap (t_0 + B_\varepsilon),$$

for each $i = 1, 2, \dots, m$. Using Inequality (1.2) we have

$$\|h(t)\| = \|f(t) - g(t)\| \ge \|g(t_0) - f(t)\| - \|g(t) - g(t_0)\| > \varepsilon$$

for any $t \in t_0 + B_\varepsilon$.

This and Inequality (1.3) together give

$$\frac{1}{2T}\int_{-T}^{T} \|h(t)\|dt \ge \varepsilon \frac{T-\eta}{mT} \to \frac{\varepsilon}{m}, \quad as \quad T \to \infty.$$

This is a contradiction since $h \in AA_0(\mathbb{X})$ and establishes our claim (1.1). □

Theorem 5.5 *$PAA(\mathbb{X})$ is a Banach space if equipped with the supnorm $\|\cdot\|_0$.*

Proof Let $(f_n) \subset PAA(\mathbb{X})$ be a Cauchy sequence in $BC(\mathbb{R},\mathbb{X})$ with the supnorm. Then we have

$$f_n = g_n + h_n, \quad n = 1, 2, \dots,$$

where $g_n \in AA(\mathbb{X})$ and $h_n \in AA_0(\mathbb{X})$ for each $n = 1, 2, \dots$. From (1.1) we can say that (g_n) is also a Cauchy sequence in $BC(\mathbb{R},\mathbb{X})$, and consequently (h_n) also is a Cauchy sequence in $BC(\mathbb{R},\mathbb{X})$. Since this latter is a Banach space and both $AA(\mathbb{X})$ and $AA_0(\mathbb{X})$ are closed subspaces of $BC(\mathbb{R},\mathbb{X})$, there exists $g \in AA(\mathbb{R},\mathbb{X})$ and $h \in AA_0(\mathbb{X})$ such that

$$\|g_n - g\|_0 \to 0 \text{ and } \|h_n - h\|_0 \to 0, \quad \text{as} \quad n \to \infty.$$

This implies that

$$f_n \to g + h, \quad \text{as} \quad n \to \infty$$

in the supnorm. Since $g + h \in PAA(\mathbb{X})$, the conclusion follows: □

2 μ-Pseudo Almost Automorphic Functions

Let $\mathcal{B}$ be the Lebesgue σ-field of $\mathbb{R}$ and $\mathfrak{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu([a,b]) < \infty$ for all $a, b \in \mathbb{R}$ $(a < b)$.

Definition 5.6 Let $\mu \in \mathfrak{M}$. A function $f \in BC(\mathbb{R}, \mathbb{X})$ is said to be μ-ergodic if

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t)\| d\mu(t) = 0.$$

We denote the space of all such functions by $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$.

Definition 5.7 Let $\mu \in \mathfrak{M}$. A function $f \in C(\mathbb{R} \times \mathbb{Y} \to \mathbb{X})$ is said to be μ-ergodic if $f(\cdot, y)$ is bounded for each $y \in \mathbb{Y}$ and

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t, y)\| d\mu(t) = 0,$$

uniformly in $y \in \mathbb{Y}$. We denote the space of all such functions by $\varepsilon(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

For $h > 0$, consider the following spaces:

$$\varepsilon(\mathbb{X}, \mu, h) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \|f(\theta)\| \right) d\mu(t) = 0 \right\};$$

$$\varepsilon(\mathbb{Y}, \mathbb{X}, \mu, h) := \Big\{ f \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X}) : f(\cdot, y) \text{ is bounded for each } y \in \mathbb{Y}$$
$$\text{and } \lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \|f(\theta, y)\| \right) d\mu(t) = 0 \text{ uniformly in } y \in \mathbb{Y} \Big\}.$$

Definition 5.8 Let $\mu \in \mathfrak{M}$. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic if f is written in the form:

$$f = g + \phi,$$

where $g \in AA(\mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$.

We denote the space of all such functions by $PAA(\mathbb{R}, \mathbb{X}, \mu)$. Then we have

$$AA(\mathbb{R}, \mathbb{X}) \subset PAA(\mathbb{R}, \mathbb{X}, \mu) \subset BC(\mathbb{R}, \mathbb{X}).$$

Definition 5.9 Let $\mu \in \mathfrak{M}$. A continuous function $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic if f is written in the form:

$$f = g + \phi,$$

where $g \in AA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$.

Definition 5.10 Let $\mu \in \mathfrak{M}$. A function $f \in BC(\mathbb{R}, \mathbb{X})$ (respectively, $f \in BC(\mathbb{R}\times \mathbb{Y}, \mathbb{X})$) is called μ-pseudo almost automorphic of class h if it can be expressed as $f = g + \phi$, where $g \in AA(\mathbb{X})$ (respectively, $AA(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) and $\phi \in \varepsilon(\mathbb{X}, \mu, h)$ (respectively, $\varepsilon(\mathbb{Y}, \mathbb{X}, \mu, h)$). We denote by $PAA(\mathbb{X}, \mu, h)$ (respectively, $PAA(\mathbb{R}\times \mathbb{Y}, \mathbb{X}, \mu, h)$) the set of all such functions.

Lemma 5.11 *Let $\mu \in \mathfrak{M}$. Then $(\varepsilon(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

Proof It suffices to prove that $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is a closed subset of $BC(\mathbb{R}, \mathbb{X})$. To this end, let (f_n) be a sequence in $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ uniformly in $t \in \mathbb{R}$.

Take an arbitrary $r > 0$. Then we get

$$\int_{[-r,r]} \|f(t)\| d\mu(t) \le \int_{[-r,r]} \|f(t) - f_n(t)\| d\mu(t) + \int_{[-r,r]} \|f_n(t)\| d\mu(t),$$

which yields

$$\frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t)\| d\mu(t) \le \|f - f_n\|_\infty + \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f_n(t)\| d\mu(t).$$

Thus we obtain

$$\limsup_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t)\| d\mu(t) \le \|f - f_n\|_\infty, \ \forall n \in \mathbb{N}.$$

Now observing that $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$, we obtain that

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t)\| d\mu(t) = 0$$

which proves the result. □

Lemma 5.12 *Let $\mu \in \mathfrak{M}$. Then $(\varepsilon(\mathbb{X}, \mu, h), \|\cdot\|_\infty)$ is a Banach space.*

Proof Like in the Lemma above, it suffices to prove that $\varepsilon(\mathbb{X}, \mu, h)$ is a closed subset of $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. Let (f_n) be a sequence in $\varepsilon(\mathbb{X}, \mu, h)$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ uniformly in $t \in \mathbb{R}$. Let $r > 0$. Then we have

$$\int_{[-r,r]} \left(\sup_{\theta\in[t-h,t]} \|f(\theta)\| \right) d\mu(t)$$

$$\leq \int_{[-r,r]} \left(\sup_{\theta \in [t-h,t]} \|f(\theta) - f_n(\theta)\| \right) d\mu(t) + \int_{[-r,r]} \left(\sup_{\theta \in [t-h,t]} \|f_n(\theta)\| \right) d\mu(t),$$

we deduce that

$$\frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta \in [t-h,t]} \|f(\theta)\| \right) d\mu(t)$$

$$\leq \|f - f_n\|_\infty + \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta \in [t-h,t]} \|f_n(\theta)\| \right) d\mu(t);$$

it follows that

$$\limsup_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta \in [t-h,t]} \|f(\theta)\| \right) d\mu(t) \leq \|f - f_n\|_\infty \quad for\ all\ n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$, we deduce that

$$\lim_{n\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sup_{\theta \in [t-h,t]} \|f(\theta)\| \right) d\mu(t) = 0.$$

We also obtain that

$$\lim_{n\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t)\| d\mu(t) = 0.$$

i.e. $\varepsilon(\mathbb{X}, \mu, h)$ is closed in $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. □

In view of the definitions of $\varepsilon(\mathbb{X}, \mu, h)$ and $\varepsilon(\mathbb{Y}, \mathbb{X}, \mu, h)$ and the previous proof, it is clear that $\varepsilon(\mathbb{X}, \mu, h)$ and $\varepsilon(\mathbb{Y}, \mathbb{X}, \mu, h)$ are continuously embedded into $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ and $\varepsilon(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$, respectively. Furthermore, it is not hard to see that $\varepsilon(\mathbb{X}, \mu, h)$ and $\varepsilon(\mathbb{Y}, \mathbb{X}, \mu, h)$ are closed in $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ and $\varepsilon(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, \mu)$, respectively.

For $\mu \in \mathfrak{M}$ and $\tau \in \mathbb{R}$, we denote μ_τ the positive measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu_\tau(B) = \mu(\{a + \tau : a \in B\}) \quad for \ \ B \in \mathcal{B}. \tag{2.1}$$

From $\mu \in \mathfrak{M}$, we formulate the following hypothesis:
(A0) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu_\tau(B) \leq \beta\mu(B),$$

when $B \in \mathcal{B}$ satisfies $B \cap I = \emptyset$.

Lemma 5.13 *Let $\mu \in \mathcal{M}$. Then it satisfies (A0) if and only if μ and μ_τ are equivalent for all $\tau \in \mathbb{R}$.*

Proof It is obvious that (A0) holds if μ and μ_τ are equivalent. Conversely let $\tau \in \mathbb{R}$. By (A0), we can say that there exists $c = c(\tau) > 0$ and a bounded interval $I = I(\tau)$ such that for all $B \in \mathcal{B}$ satisfying $B \cap I = \emptyset$ we have

$$\mu(\{a + \tau \ : \ a \in B\}) \leq c\mu(B) \tag{2.2}$$

and

$$\mu(\{a - \tau \ : \ a \in B\}) \leq c\mu(B). \tag{2.3}$$

We can deduce that there exists a bounded interval J such that for all $B \in \mathcal{B}$, we have

$$B \cap J = \emptyset \Rightarrow B \cap I = \emptyset \tag{2.4}$$

and

$$\{a + \tau \ : \ a \in B\} \cap I = \emptyset. \tag{2.5}$$

Now from the above (2.2), (2.3), (2.4), and (2.5), we deduce that for all $B \in \mathcal{B}$ such that $B \cap J = \emptyset$ we have

$\mu(\{a + \tau \ : \ a \in B\}) \leq c\mu(B)$ and $\mu(B) \leq c\mu(\{a + \tau \ : \ a \in B\})$. Finally, combining the above we obtain

$$\frac{1}{c}\mu(B) \leq \mu_\tau(B) \leq c\mu(B)$$

for all $B \in \mathcal{B}$ such that $B \cap I = \emptyset$. This shows that $\mu \sim \mu_\tau$ and ends the proof. □

Theorem 5.14 *Let $\mu \in \mathfrak{M}$ satisfy (A0). Then $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant; therefore $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is also translation invariant.*

Proof We know that $AA(\mathbb{X})$ is translation invariant. It remains to prove that $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is also translation invariant, that is if $f \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$, then $f_\tau \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ as well for any $\tau \in \mathbb{R}$.

So let $f \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ and $\tau > 0$ is given. Since $\mu(\mathbb{R}) = +\infty$, then we can find $r_0 > 0$ such that $\mu([-r - |\tau|, r + |\tau|]) > 0$ for all $r \geq r_0$. For such r and given $\tau \in \mathbb{R}$ let us denote by M_τ the following mean:

$$M_\tau(r) := \frac{1}{\mu_\tau([-r, r])} \int_{[-r,r]} \|f(t)\| d\mu_\tau(t),$$

where μ_τ is defined as in (2.1). By the lemma above, μ and μ_τ are equivalent, thus $\varepsilon(\mathbb{R}, \mathbb{X}, \mu_\tau)$=$\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. Consequently $f \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu_\tau)$, which implies

$$\lim_{r\to\infty} M_\tau(r) = 0, \quad \forall \tau \in \mathbb{R}.$$

Take an arbitrary $B \in \mathcal{B}$ and consider its characteristic function χ_B. Then we can get

$$\int_{[-r,r]} \chi_B d\mu_\tau(t) = \int_{[-r+\tau,r+\tau]} \chi_B(t-\tau)d\mu(t)$$

for all $B \in \mathcal{B}$. Now since $t \to \|f(t)\|$ can be viewed as the pointwise limit of an increasing sequence of linear combination of characteristic functions, we deduce that

$$\int_{[-r,r]} \|f(t)\|d\mu_\tau(t) = \int_{[-r+\tau,r+\tau]} \|f(t-\tau)\|d\mu(t).$$

From the above, we obtain

$$M_\tau(r) = \frac{1}{\mu([-r+\tau, r+\tau])} \int_{[-r+\tau,r+\tau]} \|f(t-\tau)\|d\mu(t).$$

Let us denote $\tau^+ := max(\tau, 0)$ and $\tau^- := max(-\tau, 0)$. Then we have $|\tau| + \tau = 2\tau^+$ and $|\tau| - \tau = 2\tau^-$; thus

$$[-r+\tau-|\tau|, r+\tau+|\tau|] = [-r-2r^-, r+2r^+].$$

Consequently, we obtain

$$M_\tau(r+|\tau|) = \frac{1}{\mu([-r-2\tau^-, r+2\tau^+])} \int_{[-r-2\tau^-,r+2\tau^+]} \|f(t-\tau)\|d\mu(t).$$

Combining this equality with the following:

$$\frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t-\tau)\|d\mu(t) \le \frac{1}{\mu([-r,r])} \int_{[-r-2\tau^-,r+2\tau^+]} \|f(t-\tau)\|d\mu(t),$$

we get

$$\frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t-\tau)\|d\mu(t) \le \frac{\mu([-r-2\tau^-, r+2\tau^+])}{\mu([-r,r])} M_\tau(r+|\tau|),$$

which implies

$$\frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t-\tau)\|d\mu(t) \le \frac{\mu([-r-2|\tau|, r+2|\tau|])}{\mu([-r,r])} M_\tau(r+|\tau|).$$

Now it is clear from all of the above that

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t-\tau)\| d\mu(t) = 0,$$

which means the shift function $f_{-\tau} \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ for all $\tau \in \mathbb{R}$. The proof is complete. □

Theorem 5.15 *Let $\mu \in \mathfrak{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then the decomposition of a μ-pseudo almost automorphic function in the form $f = g + \phi$ where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$, is unique.*

Proof Let $f \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ and suppose that $f = g_i + h_i$ where $g_i \in AA(\mathbb{X})$, and $h_i \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$, for $i = 1, 2$. Then we have $0 = (g_1 - g_2) + (h_1 - h_2) \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ since $g_1 - g_2 \in AA(\mathbb{X})$ and $h_1 - h_2 \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. Now $(g_1 - g_2)(\mathbb{R}) \subset \{0\}$, therefore $g_1 - g_2 \in AA(\mathbb{X})$ and $h_1 - h_2 \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$, which implies $g_1 = g_2$ and $h_1 = h_2$. The proof is complete. □

Theorem 5.16 *Let $\mu \in \mathfrak{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then $(PAA(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

Proof Consider an arbitrary Cauchy sequence $(f_n) \subset PAA(\mathbb{R}, \mathbb{X}, \mu)$, and let $f_n = g_n + h_n$ where g_n and h_n are respectively its principal term and μ-ergodic perturbation of f_n. From a previous result we can say that $\|g_n - g_m\| \le \|f_n - f_m\|$ which shows that (g_n) is a Cauchy sequence in $AA(\mathbb{X})$. It implies that (h_n) is also a Cauchy sequence in $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. Therefore $\lim_{n\to\infty} g_n = g \in AA(\mathbb{X})$ and $\lim_{n\to\infty} h_n = h \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. Finally, $\lim_{n\to\infty} f_n = g + h \in PAA(\mathbb{R}, \mathbb{X}, \mu)$. This completes the proof. □

Bibliographical Notes The concept in Sect. 1 was suggested by N'Guérékata in [57, page 40] and developed by Liang et al. in [68]. Section 2 is devoted to a new approach to this study via measure theory introduced by Blot et al. and leading to a series of interesting papers [18–21] and many others.

Chapter 6
Stepanov-like Almost Automorphic Functions

1 Definitions and Properties

Definition 6.1 The Bochner transform $f^b := f^b(t,s),\ t \in \mathbb{R},\ s \in [0,1]$ of a function $f : \mathbb{R} \to \mathbb{X}$ is defined by

$$f^b(t,s) = f(t+s).$$

Remark 6.2 A function $\varphi(t,s),\ t \in \mathbb{R},\ s \in [0,1]$, is the Bochner transform of a certain function $f(t)$

$$\varphi(t,s) = f^b(t,s)$$

if and only if

$$\varphi(t+\tau, s-\tau) = \varphi(t,s)$$

for all $t \in \mathbb{R},\ s \in [0,1]$ and $\tau \in [s-1, s]$.

Definition 6.3 ([65]) Let $p \in [1,\infty)$. The space $BS^p(\mathbb{X})$ of Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}, L^p(0,1;\mathbb{X}))$.

Remark 6.4 $BS^p(\mathbb{X})$ turns out to be a Banach space if equipped with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} := \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(s)\|_{\mathbb{X}}^p ds \right)^{\frac{1}{p}}.$$

Definition 6.5 ([60]) The space $AS^p(\mathbb{X})$ of all S^p-almost automorphic functions consists of all $f \in BS^p(\mathbb{X})$ such that $f^b \in AA(L^p(0,1;\mathbb{X}))$.

G. M. N'Guérékata, *Almost Periodic and Almost Automorphic Functions in Abstract Spaces*, https://doi.org/10.1007/978-3-030-73718-4_6

Equivalently, a function $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0, 1; \mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers (s'_n) there exists a subsequence (s_n) and a function $g \in L^p_{loc}(\mathbb{R}, \mathbb{X}))$ such that

$$\left(\int_0^1 \|f(t + s_n + s) - g(t + s)\|^p ds\right)^{\frac{1}{p}} \to 0$$

and

$$\left(\int_0^1 \|g(t - s_n + s) - f(t + s)\|^p ds\right)^{\frac{1}{p}} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}$.

Remark 6.6

(i) If $1 \le p < q < \infty$ and $f \in L^q_{loc}(\mathbb{R}, \mathbb{X})$ is S^q-almost automorphic, then f is S^p-almost automorphic.
(ii) It is clear that $f \in AA_c(\mathbb{X})$ if and only if $f^b \in AA(L^\infty(0, 1; \mathbb{X}))$. Consequently $AA_c(\mathbb{X})$ can be considered as $AS^\infty(\mathbb{X})$.

Theorem 6.7 *The following are equivalent:*

(i) $f \in AS^p(\mathbb{X})$.
(ii) $f^b \in AA_c(L^p(0, 1; \mathbb{X}))$.
(iii) *For every sequence of real numbers (s'_n) there exists a subsequence (s_n) such that*

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

exists in the space $L^p_{loc}(\mathbb{R}, \mathbb{X})$ and

$$f(t) = \lim_{n\to\infty} g(t - s_n)$$

exists in the sense of $L^p_{loc}(\mathbb{R}, \mathbb{X})$.

Proof

$(ii) \Rightarrow (i)$ is straightforward.
$(iii) \Rightarrow (ii)$ Let us prove

$$\lim_{n\to\infty} f^b(t + s_n) = g^b(t)$$

in $C(\mathbb{R}; L^p(0, 1))$. Indeed we have

$$\sup_{t\in[-a,a]} \|f^b(t+s_n) - g^b(t)\|_{L^p(0,1;\mathbb{X})}$$
$$\leq \left(\int_{-a}^{a} \|f(t+s_n) - g(t)\|^p dt\right)^{\frac{1}{p}} \to 0.$$

Similarly, we can prove that

$$\lim_{n\to\infty} g^b(t-s_n) = f^b(t)$$

in $C(\mathbb{R}; L^p(0,1))$.

Now we prove the statement $(i) \Rightarrow (iii)$. Indeed, suppose

$$f^b(t+s_n) \to \nu(t)$$

pointwise on $\mathbb{R}$. By definition ν is a measurable function with values in $L^p(0,1;\mathbb{X})$. By Remark 6.2, $\nu(t) = g^b(t)$, where $g \in L^p_{loc}(\mathbb{R},\mathbb{X})$. Let

$$\epsilon_n := \int_{-a}^{a} \|f(t+s_n) - g(t)\|^p dt.$$

We prove that $\epsilon_n \to 0$ as $n \to \infty$. Indeed without loss of generality, we may assume that a is an integer. Then we have

$$\epsilon_n := \sum_{k=-a}^{a-1} \int_k^{k+1} \|f(t+s_n)-g(t)\|^p dt = \sum_{k=-a}^{a-1} \|f^b(k+s_n)-g^b(k)\|^p_{L^p(0,1;\mathbb{X})} \to 0,$$

which proves that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

in the space $L^p_{loc}(\mathbb{R},\mathbb{X})$. Similarly, we can prove that

$$f(t) = \lim_{n\to\infty} g(t-s_n)$$

in the sense of $L^p_{loc}(\mathbb{R},\mathbb{X})$. □

Now we prove the following:

Theorem 6.8 *$AS^p(\mathbb{X})$ is a closed linear subspace of $BS^p(\mathbb{X})$.*

Proof Let $f \in AS^p(\mathbb{X})$ and λ a scalar. Then it is obvious that $\lambda f \in AS^p(\mathbb{X})$.

Now let $f_1, f_2 \in AS^p(\mathbb{X})$. Then by definition, $f_1^b, f_2^b \in AA(L^p(0,1;\mathbb{X}))$. Now using the Minkowski's theorem, we have

$$\begin{aligned}\|f_1^b + f_2^b\|_{L^\infty(\mathbb{R},L^p)} &\leq \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f_1(s) + f_2(s)\|_{\mathbb{X}}^p ds\right)^{\frac{1}{p}} \\ &\leq \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f_1(s)\|_{\mathbb{X}}^p ds\right)^{\frac{1}{p}} + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f_2(s)\|_{\mathbb{X}}^p ds\right)^{\frac{1}{p}} \\ &= \|f_1^b\|_{L^\infty(\mathbb{R},L^p)} + \|f_2^b\|_{L^\infty(\mathbb{R},L^p)},\end{aligned}$$

which proves that $f_1 + f_2 \in AS^p(\mathbb{X})$.

Finally, it is easy, using again the Minkowski's theorem, to prove that if (f_n) is a sequence in $AS^p(\mathbb{X})$ which converges to f in the S^p norm, then $f \in AS^p(\mathbb{X})$.

The proof of the theorem is complete. □

Example 6.9 Let $f_1 \in AS^p(\mathbb{X})$ and $f_2 \in AA_c(\mathbb{X})$. Then $f_1 + f_2 \in AS^p(\mathbb{X})$ and $f_1 + f_2 \notin AA(\mathbb{X})$.

Example 6.10 Let (x_n) be an almost automorphic sequence with values in $\mathbb{X}$ and $\epsilon_0 \in (0, \frac{1}{2})$. Let the function $f : \mathbb{R} \to \mathbb{X}$ be defined by
$f(t) = x_n, \quad t \in (n - \epsilon_0, n + \epsilon_0)$ and $f(t) = 0$ otherwise. Then $f \in AS^p(\mathbb{X})$ for all p with $1 \leq p < \infty$. But $f \notin AA(\mathbb{X})$.

The following result is easy to prove:

Theorem 6.11 *Let $f \in AS^p(\mathbb{X})$ and $A \in L(\mathbb{X})$. Then $Af \in AS^p(\mathbb{X})$.*

Definition 6.12 A function f is said to be weakly S^p-almost automorphic if $\varphi f \in AS^p(\mathbb{X})$ for every $\varphi \in \mathbb{X}^*$.

Let us denote by $AS^p(\mathbb{X})$ the space of all S^p-almost automorphic functions with values in $\mathbb{X}$. Then we have the following:

$$AA(\mathbb{X}) \subset WAA(\mathbb{X})$$

$$AA_c(\mathbb{X}) \subset WAA_c(\mathbb{X})$$

$$AS^p(\mathbb{X}) \subset WAS^p(\mathbb{X}).$$

Proposition 6.13

(a) Suppose $f \in WAA(\mathbb{X})$. Then f is bounded and its range is separable. As a consequence $f \in L^\infty(\mathbb{R}, \mathbb{X})$.
(b) If $f \in WAS^p(\mathbb{X})$, then $f \in BS^p(\mathbb{X})$.

Proof

(a) This is Theorem 2.20 in Chap. 2.
(b) Follows from (a).

□

Proposition 6.14 *Let $\mathbb{X}_0 \subset \mathbb{X}_1$ be a continuous and dense embedding of Banach spaces.*

(a) Let $f \in L^\infty(\mathbb{R}, \mathbb{X}_0)$ be a weakly continuous function. If $f \in WAA(\mathbb{X}_1)$, then $f \in WAA(\mathbb{X}_0)$.
(b) If $f \in BS^p(\mathbb{X}_0)$ and $f \in WAS^p(\mathbb{X}_1)$, then $f \in WAS^p(\mathbb{X}_0)$.

Proof

(a) It is clear that the dual embedding $\mathbb{X}_1^* \subset \mathbb{X}_0^*$ is also continuous and dense. Therefore for $\varphi \in \mathbb{X}_0^*$, there exists a sequence $\varphi_n \in \mathbb{X}_1^*$ such that $\lim_{n\to\infty} \varphi_n = \varphi$ in $\mathbb{X}_0^*$. Obviously, we have

$$|(\varphi f)(t) - (\varphi_n f)(t)| \leq \|\varphi - \varphi_n\| \|f\|_{L^\infty(\mathbb{R}, \mathbb{X}_0)}.$$

Since all $(\varphi_n f)(t)$ are almost automorphic, the result follows.
(b) The proof of this case is similar.

□

Bibliographical Notes The results in this chapter are due to N'Guérékata and Pankov [60].

Chapter 7
Dynamical Systems and C_0-Semigroups

In this section, we are concerned with the behavior of asymptotically almost automorphic semigroups of linear operators $T = (T(t))_{t \geq 0}$ at $t \to \infty$. We present some topological and asymptotic properties based on the Nemytskii and Stepanov theory of dynamical systems.

First of all, we present a connection between abstract dynamical systems and C_0-semigroups of linear operators. $\mathbb{X}$ is a (real or complex) Banach space.

1 Abstract Dynamical Systems

Definition 7.1 A mapping $u : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ is called an (abstract) dynamical system if

(i) $u(0, x) = x$, for every $x \in \mathbb{X}$;
(ii) $u(\cdot, x) : \mathbb{R}^+ \to \mathbb{X}$ is continuous for any $t > 0$ and right-continuous at $t = 0$, for each $x \in \mathbb{X}$;
(iii) $u(t, \cdot) : \mathbb{X} \to \mathbb{X}$ is continuous for each $t \in \mathbb{R}^+$;
(iv) $u(t + s, x) = u(t, u(s, x))$ for all $t, s \in \mathbb{R}^+$ and $x \in \mathbb{X}$.

If $u : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ is a dynamical system, the mapping $u(\cdot, x) : \mathbb{R}^+ \to \mathbb{X}$ will be called a motion originating at $x \in \mathbb{X}$.

Now, we state and prove the following.

Theorem 7.2 *Every C_0-semigroup $(T(t))_{t \in \mathbb{R}^+}$ determines a dynamical system and conversely by defining $u(t, x) := T(t)x,\ t \in \mathbb{R}^+,\ x \in \mathbb{X}$.*

Proof Let $u(t, x)$ be a dynamical system in the sense of Definition 7.1 and consider

$$T(t)x = u(t, x),\ t \in \mathbb{R}^+,\ x \in \mathbb{X}.$$

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Then, obviously $T(0) = I$, the identity operator on $\mathbb{X}$, since for every $x \in \mathbb{X}$, $T(0)x = u(0, x) = x$.

Let $t, s \in \mathbb{R}^+$ and $x \in \mathbb{X}$; then, we have

$$T(t+s)x = u(t+s, x) = u(t, u(s, x))$$

by property (iv) of Definition 7.1. But we have also

$$T(t)T(s)x = T(t)u(s, x) = T(t, u(s, x))$$

using the definition of $T(t)x$. Therefore,

$$T(t+s)x = T(t)T(s)x,$$

for every $t, s \in \mathbb{R}^+$ and $x \in \mathbb{X}$, which proves the semigroup property

$$T(t+s)x = T(t)T(s)x,$$

for all $t, s \in \mathbb{R}^+$, $x \in \mathbb{X}$.

The continuity of $T(t)x : \mathbb{X} \to \mathbb{X}$ follows readily from property (iii) of Definition 7.1 for every $t \in \mathbb{R}^+$.

Now, we have

$$\lim_{t \to 0^+} T(t)x = \lim_{t \to o^+} u(t, x) = u(0, x) = x$$

using property (ii) and then property (i) in Definition 7.1. We have proved that $(T(t))_{t \in \mathbb{R}^+}$ is a C_0-semigroup.

Conversely, suppose we have a C_0-semigroup $(T(t))_{t \in \mathbb{R}^+}$ and define

$$u(t, x) = T(t)x, \ t \in \mathbb{R}^+, \ x \in \mathbb{X}.$$

Then, all the properties (i)–(iv) in Definition 7.1 are obviously true. Therefore, the mapping u is a dynamical system. □

Remark 7.3 The above result tells us that the notions of abstract dynamical systems and C_0-semigroups are equivalent. This fact provides a solid ground to study C_0-semigroups of linear operators as an independent topic.

2 Complete Trajectories

In this section, we will consider a C_0-semigroup of linear operators $(T(t))_{t \in \mathbb{R}+}$ such that the motion $T(t)x_0 : \mathbb{R}^+ \to \mathbb{X}$ is an asymptotically almost automorphic function with principal term $f(t)$.

Let us now introduce some notations and definitions. We recall that x_0 is some fixed element in $\mathbb{X}$.

Definition 7.4 A function $\varphi : \mathbb{R} \to \mathbb{X}$ is said to be a complete trajectory of T if it satisfies the functional equation $\varphi(t) = T(t-a)\varphi(a)$ for all $a \in \mathbb{R}$ and all $t \geq a$.

We have the following properties.

Theorem 7.5 *The principal term of $T(t)x_0$ is a complete trajectory for T.*

Proof We have $T(t)x_0 = f(t) + g(t),\ t \in \mathbb{R}^+$. Since $f \in AA(\mathbb{X})$, there exists a subsequence $(n_k) \subset (n) = \mathbb{N}$ such that

$$\lim_{k\to\infty} f(t+n_k) = g(t)$$

and

$$\lim_{k\to\infty} g(t-n_k) = f(t)$$

pointwise on $\mathbb{R}$.

Put $\varphi(t) = T(t)x_0$. Then, $\varphi(0) = x_0$. Let us fix $a \in \mathbb{R}$ and choose k large enough so that $a + n_k \geq 0$. If $s \geq 0$, then

$$\begin{aligned}\varphi(a+s+n_k) &= T(a+s+n_k)\varphi(0)\\ &= T(s)T(a+n_k)\varphi(0)\\ &= T(s)\varphi(a+n_k).\end{aligned}$$

Consequently,

$$f(a+s+n_k) + h(a+s+n_k) = T(s)\varphi(a+n_k),$$

where $s \geq 0$ and $a + n_k \geq 0$. But we have

$$\lim_{k\to\infty} f(a+s+n_k) = g(a+s)$$

and

$$\lim_{k\to\infty} h(a+s+n_k) = 0,$$

so

$$\lim_{k\to\infty} \varphi(a+s+n_k) = \lim_{k\to\infty} T(s)\varphi(a+n_k) = g(a+s).$$

We also have

$$\lim_{k\to\infty} \varphi(a+n_k) = g(a).$$

Using the continuity of $T(t)$, we get

$$\lim_{k\to\infty} T(s)\varphi(a+n_k) = T(s)g(a).$$

We can establish the following equality:

$$T(s)g(a) = g(a+s),\ \forall a \in \mathbb{R},\ \forall s \geq 0.$$

But we have

$$\lim_{k\to\infty} g(t-n_k) = f(t),\ t \in \mathbb{R}$$

and

$$g(a-n_k+s) = T(s)g(a-n_k),\ \forall a \in \mathbb{R},\ \forall s \geq 0.$$

Therefore,

$$\lim_{k\to\infty} g(a-n_k+s) = T(s)f(a),\ \forall a \in \mathbb{R},\ \forall s \geq 0,$$

so that

$$f(a+s) = T(s)f(a),\ \forall a \in \mathbb{R},\ \forall s \geq 0.$$

Finally, let us put $s = t - a$ with $t \geq 0$. Then,

$$f(t) = T(t-a)f(a),\ \forall a \in \mathbb{R},\ \forall t \geq a.$$

The proof is complete. □

Definition 7.6 The set

$$\omega^+(x_0) = \{y \in \mathbb{X}\ :\ \exists\, 0 \leq t_n \to \infty\ s.t.\ \lim_{n\to\infty} T(t_n)x_0 = y\}$$

is called the ω-limit of $f(t)$, the principal term of $T(t)x_0$.

$$\gamma^+(x_0) = \{T(t)x_0\ /\ t \in \mathbb{R}^+\}$$

is called the trajectory of $T(t)x_0$.

Theorem 7.7 $\omega^+(x_0) \neq \emptyset$.

Proof We let $t_n = n,\ n = 1, 2, \ldots$ Since $f \in AA(\mathbb{X})$, there exists a subsequence $(t_{n_k}) \subset (t_n)$ such that

$$\lim_{k\to\infty} f(t_{n_k}) = g(0).$$

But

$$\lim_{k\to\infty} T(t_{n_k})x_0 = \lim_{k\to\infty} f(t_{n_k}).$$

Thus, we get

$$\lim_{k\to\infty} T(t_{n_k})x_0 = g(0).$$

Consequently, $g(0) \in \omega^+(x_0)$, since $t_{n_k} \to \infty$ as $k \to \infty$. So, $\omega^+(x_0) \neq \emptyset$.
This completes the proof. □

Theorem 7.8 $\omega^+(x_0) = \omega_f^+(x_0)$.

Proof To see that $T(t)x_0$ and its principal term have the same ω-limit set, it suffices to observe that

$$\lim_{t\to\infty} T(t)x_0 = \lim_{t\to\infty} f(t).$$

□

Definition 7.9 A set $B \subset \mathbb{X}$ is said to be invariant under the semigroup $T = (T(t))_{t\in\mathbb{R}^+}$ if $T(t)y \in B$ for every $y \in B$ and $t \in \mathbb{R}^+$.

Theorem 7.10 *$\omega^+(x_0)$ is invariant under T.*

Proof Let $y \in \omega^+(x_0)$, so there exists $0 \le t_n \to \infty$ such that $\lim\limits_{t\to\infty} T(t_n)x_0 = y$. Consider the sequence (s_n) where $s_n = t + t_n,\ n = 1, 2, \ldots$, for a given $t \in \mathbb{R}^+$. Then, $s_n \to \infty$ as $n \to \infty$. We have

$$T(s_n)x_0 = T(t)T(t_n)x_0,\ n = 1, 2, \ldots$$

and $\lim\limits_{n\to\infty} T(s_n)x_0 = T(t)y$, using the continuity of $T(t)$. Therefore, $T(t)y \in \omega^+(x_0)$.
The proof is complete. □

Theorem 7.11 *$\omega^+(x_0)$ is closed.*

Proof Let $y \in \overline{\omega^+(x_0)}$, the closure of $\omega^+(x_0)$. Then, there exists a sequence of elements $y_m \in \omega^+(x_0),\ m = 1, 2, \ldots$, with $y_m \to y$. For each y_m, there exists $0 \le t_{m,n} \to \infty$ such that $\lim\limits_{n\to\infty} T(t_{m,n})x_0 = y_m$.

Recursively choose

$$
\begin{aligned}
t_{1,n_1} &> 1 && \text{such that } \|y_1 - T(t_{1,n_1})x_0\| < \frac{1}{2} \\
t_{2,n_2} &> max(2, t_{1,n_1}) && \text{such that } \|y_2 - T(t_{2,n_2})x_0\| < \frac{1}{2^2} \\
t_{3,n_3} &> max(3, t_{2,n_2}) && \text{such that } \|y_3 - T(t_{3,n_3})x_0\| < \frac{1}{2^3} \\
t_{k,n_k} &> max(k, t_{k-1,n_{k-1}}) && \text{such that } \|y_k - T(t_{k,n_k})x_0\| < \frac{1}{2^k}.
\end{aligned}
$$

Let $s_k = t_{k,n_k},\ k = 1, 2, \ldots$ Clearly, $0 < s_k \to \infty$ as $k \to \infty$, and we have

$$
\begin{aligned}
\|T(s_k)x_0 - y\| &\le \|T(s_k)x_0 - y_k\| + \|y_k - y\| \\
&< \frac{1}{2^k} + \|y_k - y\|.
\end{aligned}
$$

Since $\lim\limits_{k\to\infty} y_k = y$, we have $y \in \omega^+(x_0)$.

This achieves the proof. □

Theorem 7.12 *$\omega^+(x_0)$ is compact if $\gamma^+(x_0)$ is relatively compact.*

Proof It is obvious that $\omega^+(x_0) \subset \overline{\gamma^+(x_0)}$. But $\overline{\gamma^+(x_0)}$ is compact by assumption and $\omega^+(x_0)$ is a closed subset (cf. Theorem 7.11). Therefore, $\omega^+(x_0)$ is itself compact. □

Theorem 7.13 *$\gamma_f(x_0) := \{f(t)\ /\ t \in \mathbb{R}\}$ is invariant under the semigroup $T = (T(t))_{t\in\mathbb{R}^+}$.*

Proof We recall that $\gamma_f(x_0)$ is relatively compact since $f \in AA(\mathbb{X})$. Let $y \in \gamma_f(x_0)$. So, there exists $\sigma \in \mathbb{R}$ such that $y = f(\sigma)$. For arbitrary $a \in \mathbb{R}$ such that $\sigma \ge a$, we can write

$$y = f(\sigma) = T(\sigma - a)f(a),$$

since f is a complete trajectory (cf. Theorem 7.5). Now, let $t \ge 0$. Then,

$$
\begin{aligned}
T(t)y &= T(t + \sigma - a)f(a) \\
&= f(t + \sigma),
\end{aligned}
$$

i.e. $T(t)y \in \gamma_f(x_0)$, for every $t \ge 0$, which shows that $\gamma_f(x_0)$ is indeed invariant under the semigroup T. □

Theorem 7.14 *Let* $\nu(t) := \inf_{y \in \omega^+(x_0)} \|T(t)x_0 - y\|$. *Then,*

$$\lim_{t\to\infty} \nu(t) = 0.$$

Proof Suppose $\lim_{t\to\infty} \nu(t) \neq 0$. Then, there exists $\varepsilon > 0$ such that for every $n = 1, 2, \ldots$, there exists $t'_n \geq n$ such that $\nu(t'_n) \geq \varepsilon$, i.e.

$$\exists t'_n \geq n, \quad \|T(t'_n)x_0 - y\| \geq \varepsilon \quad \forall y \in \omega^+(x_0), \quad \forall n = 1, 2, \ldots$$

Let (t_n) be a subsequence of (t'_n) such that $(f(t_n))$ converges, say, to $\overline{y}$, as guaranteed by the relative compactness of $\gamma_f(x_0)$.

Now, since $t_n \to \infty$ as $n \to \infty$, we get

$$\lim_{n\to\infty} T(t_n)x_0 = \lim_{n\to\infty} f(t_n) = \overline{y}.$$

Therefore, $\overline{y} \in \omega^+(x_0)$, which is a contradiction. □

Remark 7.15 The minimality property above shows that the ω-limit set $\omega^+(x_0)$ is the smallest closed set toward which the asymptotically almost automorphic function $T(t)x_0$ tends to as $t \to \infty$.

Definition 7.16 $e \in \mathbb{X}$ is called a rest point for the semigroup $T = (T(t))_{t\in\mathbb{R}^+}$ if $T(t)e = e, \ \forall t \in \mathbb{R}^+$.

Theorem 7.17 *If* x_0 *is a rest point for the semigroup* $T = (T(t))_{t\in\mathbb{R}^+}$, *then* $\omega^+(x_0) = \{x_0\}$.

Proof Since $T(t)x_0 = x_0$, for every $t \geq 0$, then for every sequence of real numbers (t_n) such that $0 \leq t_n \to \infty$, we get $\lim_{n\to\infty} T(t_n)x_0 = x_0$, i.e. $x_0 \in \omega^+(x_0)$.

Now, let $y \in \omega^+(x_0)$. There exists $0 \leq s_n \to \infty$ such that $\lim_{n\to\infty} T(s_n)x_0 = y$. But $T(s_n)x_0 = x_0$ for all $n = 1, 2, \ldots$ Therefore, $x_0 = y$.

The proof is complete. □

Bibliographical Notes The results in this chapter are due to N'Guérékata and published for the first time in the first edition of this book.

Chapter 8
Almost Periodic Functions with Values in a Locally Convex Space

1 Almost Periodic Functions

Definition 8.1 Let $E = E(\tau)$ be a complete Hausdorff locally convex space. A function $f : \mathbb{R} \to E$ is said to be almost periodic if for every neighborhood (of the origin) U, there exists a real number $l > 0$ such that every interval $[a, a + l]$ contains at least one point s such that

$$f(t - s) - f(t) \in U, \ \forall t \in \mathbb{R}.$$

The numbers s depend on U and are called U-translation numbers, or U-almost periods of the function f.

Remark 8.2 In the case where E is a Banach space $\mathbb{X}$ with norm $\|\cdot\|$, Definition 8.1 can be rewritten as:
$f : \mathbb{R} \to X$ is said to be almost periodic if for every $\varepsilon > 0$, there exists a real number $l > 0$ such that every interval $[a, a + l]$ contains at least one point s such that

$$\sup_{t\in\mathbb{R}} \|f(t - s) - f(t)\| < \varepsilon.$$

The numbers s are called the ε-almost periods of f.

Remark 8.3

(i) From Definition 8.1, we observe that for each neighborhood U, the set of all U-translation numbers is relatively dense in $\mathbb{R}$.
(ii) It is obvious that every continuous periodic function $f : \mathbb{R} \to E$ is almost periodic.

G. M. N'Guérékata, *Almost Periodic and Almost Automorphic Functions in Abstract Spaces*, https://doi.org/10.1007/978-3-030-73718-4_8

We now present some elementary properties of almost periodic functions taking values in locally convex spaces.

Theorem 8.4 *If $f, f_1, f_2 : \mathbb{R} \to E$ are almost periodic and λ is a scalar, then the following functions are also almost periodic:*

(i) $f_1 + f_2$;
(ii) λf;
(iii) $\check{f}$ *defined by* $\check{f}(t) = f(-t)$ *for every* $t \in \mathbb{R}$.

Proof (i) and (ii) are obvious.

Let us prove (iii). Take U an arbitrary neighborhood of the origin. By almost periodicity of f, there exists $l > 0$ such that every interval $[a, a+l]$ contains at least a point s such that

$$f(t-s) - f(t) \in U, \ \forall t \in \mathbb{R}.$$

If we put $r = -t$, we get

$$\check{f}(r-s) - \check{f}(r) = f(-r+s) - f(-r) = f(t+s) - f(t).$$

Therefore $\check{f}(r-s) - \check{f}(r) \in U$ for every $r \in \mathbb{R}$, which proves almost periodicity of $\check{f}$ with $-s$ as U-translation numbers. □

We will denote by $AP(E)$ the space of all almost periodic functions $f : \mathbb{R} \to E$.

The following two results are easy to prove (cf. [51, 54]):

Theorem 8.5 *Let $f \in AP(E)$. Then f is uniformly continuous on $\mathbb{R}$.*

Theorem 8.6 *Let $f_n \in AP(E)$, $n = 1, 2, \ldots$ and suppose that $f_n \to f$ uniformly in $t \in \mathbb{R}$. Then $f \in AP(E)$.*

Theorem 8.7 *If $f \in AP(E)$, then its range $\{f(t) \ / \ t \in \mathbb{R}\}$ is totally bounded in E.*

Proof Let U be a neighborhood and V a symmetric neighborhood such that $V + V \subset U$; let $l = l(V)$ as in Definition 8.1. By the continuity of f, the set $\{f(t) \ / \ t \in [0, l]\}$ is compact in E. But in a locally convex space, every compact set is totally bounded; therefore there exists $x_1, x_2, \ldots, x_n \in E$ such that for every $t \in [0, l]$, we have

$$f(t) \in \cup_{j=1}^{n} (x_j + V).$$

Take now an arbitrary $t \in \mathbb{R}$ and consider $s \in [-t, -t+l]$ a V-translation number of the function f. Then we have

$$f(t+s) - f(t) \in V.$$

Choose x_k among $x_1, \ldots, x_n$ such that

$$f(t+s) \in x_k + V.$$

Let us write $f(t) - x_k = (f(t) - f(t+s)) + (f(t+s) - x_k)$. Then we have $f(t) - x_k \in V + V$, and therefore $f(t) - x_k \in U$, or $f(t) \in x_k + U$. Since t is an arbitrary real number, we conclude that

$$\{f(t) \ / \ t \in \mathbb{R}\} \subset \cup_{j=1}^{n} (x_j + U).$$

The proof is complete. □

Remark 8.8 If $f \in AP(E)$ with E a Fréchet space, then its range is relatively compact in E, since in every complete metric space, relative compactness and totally boundedness are equivalent notions. We conclude in this case that every sequence $(f(t_n))$ contains a convergent subsequence $(f(t_{n_k}))$.

Theorem 8.9 *Let E be a Fréchet space and $f \in AP(E)$. Then for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that $(f(t+s_n))$ is uniformly convergent in $t \in \mathbb{R}$.*

Proof Let (s_n) be a sequence of real numbers and consider the sequence of functions $f_{s_n} : \mathbb{R} \to E$ defined by $f_{s_n}(t) = f(t+s_n)$, $n = 1, 2, \ldots$ Let $S = (\eta_n)$ be a countable dense set in $\mathbb{R}$. By Remark 8.8, we can extract from $(f(\eta_1 + s_n))$ a convergent subsequence, since the set $\{f(t) \ / \ t \in \mathbb{R}\}$ is relatively compact in E.

Let $(f_{s_{1,n}})$ be the subsequence of (f_n) which converges at η_1.We apply the same argument to the sequence $(f_{s_{1,n}})$ to choose a subsequence $(f_{s_{2,n}})$ which converges at η_2. We continue the process and consider the diagonal sequence $(f_{s_{n,n}})$ which converges at η_n in S.

Call this last sequence (f_{r_n}). Now let us show that it is uniformly convergent on $\mathbb{R}$: that is, for every neighborhood U, there exists $N = N(U)$ such that

$$f(t+r_n) - f(t+r_m) \in U$$

for every $t \in \mathbb{R}$, if $n, m > N$.

Consider now an arbitrary neighborhood U and a symmetric neighborhood V such that $V + V + V + V + V \subset U$. Let $l = l(V) > 0$ as in Definition 8.1. Since f is uniformly continuous on $\mathbb{R}$ (Theorem 8.5), we can find $\delta = \delta(V) > 0$ such that

$$f(t) - f(t') \in V$$

for every $t, t' \in \mathbb{R}$ with $|t - t'| < \delta$.

Let us divide the interval $[0, l]$ into ν subintervals of lengths smaller than δ and choose in each interval a point of S, obtaining $S_0 = \{\xi_1, \ldots, \xi_\nu\}$. Since S_0 is a finite set, (f_{r_n}) is uniformly convergent over S_0; therefore there exists a natural number $N = N(V)$ such that

$$f(\xi_i + r_n) - f(\xi_i + r_m) \in V$$

for every $i = 1, \dots, \nu$, and for $n, m > N$.

Let $t \in \mathbb{R}$ be arbitrary and $s \in [-t, -t + l]$ such that $f(t + s) - f(t) \in V$. Let us choose ξ_i such that $|t + s - \xi_i| < \delta$; then

$$f(t + s + r_n) - f(\xi_i + r_n) \in V$$

for every n.

Let us write

$$\begin{aligned} f(t + r_n) - f(t + r_m) = (f(t + r_n) - f(r + r_n + s)) + (f(r + r_n + s) \\ - f(\xi_i + r_n)) + (f(\xi_i + r_n) - f(\xi_i + r_m)) \\ + (f(\xi_i + r_m) - f(t + r_m + s)) + (f(t + r_m + s) \\ - f(t + r_m)). \end{aligned}$$

Then it appears

$$f(t + r_n) - f(t + r_m) \in V + V + V + V + V \subset U$$

if $n, m > N$, which proves the uniform convergence of $(f(t + r_n))$. □

We are now ready to establish the following important result called also the **Bochner's criterion**:

Theorem 8.10 *Let E be a Fréchet space. Then $f \in AP(E)$ if and only if for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that $(f(t + s_n))$ converges uniformly in $t \in \mathbb{R}$.*

Proof The condition is necessary by Theorem 8.9.

Now we need to prove that it is sufficient. Suppose by contradiction that $f \notin AP(E)$. Then there exists a neighborhood U such that for every real number $l > 0$, there exists an interval of length l which contains no U-translation number of f, or there exists an interval $[-a, -a + l]$ such that for every $s \in [-a, -a + l]$, there exists $t = t_s$ such that $f(t + s) - f(t) \notin U$.

Let us consider $s_1 \in \mathbb{R}$ and an interval (a_1, b_1) with $b_1 - a_1 > 2|s_1|$ which contains no U-translation number of f. Now let $s_2 = \frac{(a_1 - b_1)}{2}$; then $s_2 - s_1 \in (a_1, b_1)$ and therefore $s_2 - s_1$ cannot be a U-translation number of f.

Let us consider another interval (a_2, b_2) with $b_2 - a_2 > 2(|s_1| + |s_2|)$, which contains no U-translation number of f. Let $s_3 = \frac{(a_2 - b_2)}{2}$; then $s_3 - s_1, s_3 - s_2 \in (a_2, b_2)$ and therefore $s_3 - s_1$ and $s_3 - s_2$ cannot be U-translation numbers of f.

We proceed and obtain a sequence (s_n) of real numbers such that no $s_m - s_n$ is a U-translation number of f, that is

$$f(t + s_m - s_n) - f(t) \notin U.$$

Putting $\sigma = t - s_n$, we get

$$f(\sigma + s_m) - f(\sigma + s_n) \notin U. \tag{1.1}$$

Suppose there exists a subsequence (s'_n) of (s_n) such that $(f(t + s'_n))$ converges uniformly in $t \in \mathbb{R}$. Then for every neighborhood V, there exists a natural number $N = N(V)$ such that, if $n, m > N$ (we may take $m > n$), we have

$$f(t + s'_m) - f(t + s'_n) \in V$$

for every $t \in \mathbb{R}$. This contradicts (2.1) and so establishes the sufficiency of the condition.

The proof is complete. □

Theorem 8.11 *Let $f \in AP(E)$. Then the following hold true:*

(i) $Af(t) \in AP(E)$ for every linear bounded operator A on E.
(ii) $\nu f \in AP(E)$ where $\nu : \mathbb{R} \to \Phi$ is almost automorphic.

Proof Trivial, cf for instance [51, 54]. □

Using the Bochner's criterion, one can easily prove the following:

Theorem 8.12 *Let E be a Fréchet space and $f_1, f_2 \in AP(E)$. Then the function $F : \mathbb{R} \to E \times E$ defined by $F(t) = (f_1(t), f_2(t))$ is also almost periodic.*

Corollary 8.13 *Let $f_1, f_2 \in AP(E)$ where E is a Fréchet space. Then for every neighborhood U, f_1 and f_2 have common U-translation numbers.*

Proof Let U be a neighborhood in E. Then by Theorem 8.12 the function $f(t) = (f_1(t), f_2(t)) \in AP(E \times E)$. Consider now s a U-translation number of f; then $f(t+s) - f(t) \in U \times U$ for every $t \in \mathbb{R}$, and therefore $f_i(t+s) - f_i(t) \in U, \quad i = 1, 2$ for every $t \in \mathbb{R}$. s is a U-translation number for f_1 and f_2. □

Theorem 8.14 *Let E be a Fréchet space. Then $AP(E)$ is also a Fréchet space.*

Proof Consider $BC(\mathbb{R}, E)$ the linear space of all bounded and continuous functions $\mathbb{R} \to E$ and denote by (p_n), $n \in \mathbb{N}$, the family of seminorms which generates the topology of E. Without loss of generality we may assume that $p_n \le p_{n+1}$, pointwise for $n \in \mathbb{N}$. Define

$$q_n(f) = \sup_{t \in \mathbb{R}} p_n(f(t)), \;\; n \in \mathbb{N}.$$

Obviously (q_n) forms a family of seminorms on $BC(\mathbb{R}, E)$. Moreover, it is clear that $q_n \le q_{n+1}$ for $n \in \mathbb{N}$. Define the pseudonorm

$$|f| := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{q_n(f)}{1 + q_n(f)}, \quad f \in BC(\mathbb{R}, E).$$

Obviously $BC(\mathbb{R}, E)$ with the above defined pseudonorm is a Fréchet space. It is also a closed linear subspace of $BC(\mathbb{R}, E)$. This completes the proof. □

2 Weakly Almost Periodic Functions

Definition 8.15 Let E be a complete Hausdorff locally convex space. A weakly continuous function $f : \mathbb{R} \to E$ is said to be weakly almost periodic if the numerical function $F(t) = (x^* f)(t)$ is almost periodic for every $x^* \in E^*$ the dual space of E.

We will denote by $WAP(E)$ the set of all weakly almost periodic functions $\mathbb{R} \to E$.

Remark 8.16

(i) Every weakly almost periodic function is weakly bounded.
(ii) Every almost periodic function is weakly almost periodic.

Theorem 8.17 *Let $f \in WAP(E) \cap C(\mathbb{R}, E)$. Assume that the set $\{F(t) \ / \ t \in \mathbb{R}\}$ be weakly bounded where the function $F : \mathbb{R} \to E$ is defined by $F(t) = \int_0^t f(s)ds$. Then $F \in WAP(E)$.*

Proof We first observe that the integral exists in E since f is (strongly) continuous on $\mathbb{R}$. Take $x^* \in E^*$, so $x^* f \in AP(\mathbb{R})$. By the continuity of x^*, $(x^* F)(t) = x^* \int_0^t f(s)ds = \int_0^t (x^* f)(s)ds$ which is bounded by assumption and so is almost periodic. The Theorem is proved. □

Theorem 8.18 *Let E be a Fréchet space and $f : \mathbb{R} \to E$. Then $f \in AP(E)$ if and only if $f \in WAP(E)$ and its range is relatively compact.*

Proof The condition is necessary by Remarks 8.8 and 8.16. Let us show by contradiction that it is sufficient.

Suppose there exists $t_0 \in \mathbb{R}$ such that f is discontinuous at t_0, so we can find a neighborhood U and two sequences of real numbers (s'_n) and (s''_n) such that

$$\lim_{n\to\infty} s'_n = 0 = \lim_{n\to\infty} s''_n$$

and

$$f(t_0 + s'_n) - f(t_0 + s''_n) \notin U, \ \forall n \in \mathbb{N}. \tag{2.1}$$

By the relative compactness of $\{f(t) \ / \ t \in \mathbb{R}\}$, we can extract (r'_n) and (r''_n) from (s'_n) and (s''_n) respectively, such that

$$\lim_{n\to\infty} f(t_0 + r'_n) = a_1 \in E$$

and

$$\lim_{n\to\infty} f(t_0 + r''_n) = a_2 \in E.$$

Consequently, $a_1 - a_2 \notin U$ by (2.1), and using the Hahn–Banach Theorem (Proposition 1.41 Chap. 1), we can find $x^* \in E^*$ such that $x^*(a_1 - a_2) \neq 0$, hence $x^*(a_1) \neq x^*(a_2)$. By the continuity of x^*, we have

$$x^*(a_1) = \lim_{n\to\infty} (x^* f)(t_0 + r'_n) = \lim_{n\to\infty} (x^* f)(t_0 + r''_n) = x^*(a_2)$$

which is a contradiction. So we conclude that f is continuous on $\mathbb{R}$. □

To prove the almost periodicity of f we need the following:

Lemma 8.19 *Let E be a Fréchet space and $\phi \in AP(E)$. Let (s_n) be a sequence of real numbers such that $\lim_{n\to\infty} \phi(s_n + \eta_k)$ exists for each $k = 1, 2, \ldots$ where the set (η_k) is dense in $\mathbb{R}$. Then the sequence $(\phi(t + s_n))$ is uniformly convergent in $t \in \mathbb{R}$.*

Proof (of Lemma 8.19) Suppose by contradiction that $(\phi(t + s_n))$ is not uniformly convergent in $t \in \mathbb{R}$. Then there exists a neighborhood U such that for every $N = 1, 2, \ldots$ there exists $n_N, m_N > N$ and $t_N \in \mathbb{R}$ such that

$$\phi(t_N + s_{n_N}) - \phi(t_N + s_{m_N}) \notin U.$$

By the Bochner's criterion (Theorem 8.10), we can extract two sequences $(s'_{n_N}) \subset (s_{n_N})$ and $(s'_{m_N}) \subset (s_{m_N})$ such that

$$\lim_{N\to\infty} \phi(t + s'_{n_N}) = g_1(t) \text{ uniformly in } t \in \mathbb{R}$$

and

$$\lim_{N\to\infty} \phi(t + s'_{m_N}) = g_2(t) \text{ uniformly in } t \in \mathbb{R}.$$

Let V be a symmetric neighborhood such that $V + V + V \subset U$. Then there exists $N_0 = N_0(V)$ such that if $N > n_0$, we have

$$\phi\left(t_N + s'_{n_N}\right) - g_1(t_N) \in V$$

and

$$\phi\left(t_N + s'_{m_N}\right) - g_2(t_N) \in V.$$

We deduce that $g_1(t_N) - g_2(t_N) \notin V$, otherwise we should have

$$\phi\left(t_N + s'_{n_N}\right) - \phi\left(t_N + s'_{m_N}\right) \in U$$

which contradicts (2.1).

Indeed if $g_1(t_N) - g_2(t_N) \in V$, then by writing

$$\begin{aligned}\phi\left(t_N + s'_{n_N}\right) - \phi\left(t_N + s'_{m_N}\right) &= \phi\left(t_N + s'_{n_N}\right) - g_1(t_N)\\ &\quad + g_1(t_N) - g_2(t_N)\\ &\quad + g_2(t_N) - \phi\left(t_N + s'_{m_N}\right)\end{aligned}$$

we obtain

$$\phi(t_N + s'_{n_N}) - \phi(t_N + s'_{m_N}) \in V + V + V \subset U.$$

Thus we have found a symmetric neighborhood V with the property that if N is large enough, there exists $t_N \in \mathbb{R}$ such that

$$g_1(t_N) - g_2(t_N) \notin V.$$

But this is impossible, because if we take a subsequence (ξ_k) of (η_k) with $\xi_k \to t_N$, then we would obtain

$$\lim_{N\to\infty} \phi\left(\xi_k + s'_{n_N}\right) = \lim_{N\to\infty} \phi\left(\xi_k + s'_{m_N}\right)$$

for every k.

Therefore $g_1(\xi_k) = g_2(\xi_k)$ for every k. By the continuity of g_1 and g_2, $g_1(t_N) = g_2(t_N)$, thus $g_1(t_N) - g_2(t_N)$ belongs to every neighborhood.

The lemma is proved. □

Proof (of Theorem 8.18 (continued)) Consider a sequence of real numbers (h_n) and a sequence of rational numbers (η_r). By the relative compactness of $\{f(t)\ /\ t \in \mathbb{R}\}$, we can extract a subsequence (h_n) (we do not change the notation) such that for each $r = 1, 2, \ldots$

$$\lim_{n\to\infty} f(\eta_r + h_n) = x_r$$

exists in E. Now the sequence $(f(\eta_r + h_n))$ is uniformly convergent in η_r, or we could find a neighborhood U and three subsequences $(\xi_r) \subset (\eta_r)$, $(h'_r) \subset (h_r)$, and $(h''_r) \subset (h_r)$ with

$$f(\xi_r + h'_r) - f(\xi_r + h''_r) \notin U. \tag{2.2}$$

By the relative compactness of $\{f(t)\ /\ t \in \mathbb{R}\}$, we may say that

$$\lim_{r\to\infty} f\left(\xi_r + h'_r\right) = b' \in E$$

$$\lim_{r\to\infty} f\left(\xi_r + h''_r\right) = b'' \in E.$$

Then, using (2.2), we get

$$b' - b'' \notin U.$$

By the Hahn–Banach Theorem, there exists $x^* \in E^*$ such that

$$x^*(b') \neq x^*(b'').$$

Now $x^* f \in AP(\mathbb{R})$, therefore it is uniformly continuous over $\mathbb{R}$.

Let us consider the functions (φ_n) defined on $\mathbb{R}$ by

$$\varphi_n(t) := (x^* f)(t + h_n), \quad n = 1, 2, \ldots$$

The equality

$$\varphi_n(t + s) - \varphi_n(t) = (x^* f)(t + s + h_n) - (x^* f)(t + h_n)$$

shows the almost periodicity of each φ_n, $n = 1, 2, \ldots$, if s is seen as a U-translation number of $(x^* f)(t)$. Also the sequence of functions (φ_n) is equicontinuous over $\mathbb{R}$ because $(x^* f)(t)$ is uniformly continuous on $\mathbb{R}$.

Since

$$\lim_{n\to\infty} f(\eta_r + h_n) = x_r,$$

we get

$$\lim_{n\to\infty} (x^* f)(n_r + h_n) = x^* x_r$$

for every $r = 1, 2, \ldots$ Therefore by Lemma 8.19, $((x^* f)(\eta_r + h_n))$ is uniformly convergent in t.

Consider now the sequences $(\xi_r + h'_r)$ and $(\xi_r + h''_r)$. By the Bochner's criterion, we can extract a subsequence from each sequence, respectively, such that, using the same notations, $((x^* f)(t + \xi_r + h'_r))$ and $((x^* f)(t + \xi_r + h''_r))$ are uniformly convergent in $t \in \mathbb{R}$.

Let us now prove that

$$\lim_{r\to\infty} (x^* f)(t + \xi_r + h'_r) = \lim_{r\to\infty} (x^* f)(t + \xi_r + h''_r).$$

Write $(x^* f)(t+\xi_r+h'_r)-(x^* f)(t+\xi_r+h''_r)$ as follows:

$$\begin{aligned}&(x^* f)(t+\xi_r+h'_r)-(x^* f)(t+\xi_r+h''_r)\\&\quad=(x^* f)(t+\xi_r+h'_r)-(x^* f)(t+\xi_r+h_r)\\&\qquad+(x^* f)(t+\xi_r+h_r)-(x^* f)(t+\xi_r+h''_r)\end{aligned}$$

and consider the following inequality (IN):

$$\begin{aligned}&|(x^* f)(t+\xi_r+h'_r)-(x^* f)(t+\xi_r+h''_r)|\\&\quad\leq|(x^* f)(t+\xi_r+h'_r)-(x^* f)(t+\xi_r+h_r)|\\&\qquad+|(x^* f)(t+\xi_r+h_r)-(x^* f)(t+\xi_r+h''_r)|\end{aligned}$$

which holds true for $r=1,2,\ldots$

Let $\varepsilon>0$ be given. Since $((x^* f)(t+h_r))$ is uniformly convergent in t, we can choose η_ε such that for $r,s>\eta_\varepsilon$, we obtain

$$|(x^* f)(t+h_s)-(x^* f)(t+h_r)|<\frac{\varepsilon}{2},\quad\forall t\in\mathbb{R}.$$

So, replacing t by $t+\xi_r$ gives,

$$|(x^* f)(t+\xi_r+h_s)-(x^* f)(t+\xi_r+h_r)|<\frac{\varepsilon}{2},$$

and consequently

$$|(x^* f)(t+\xi_r+h'_r)-(x^* f)(t+\xi_r+h_r)|<\frac{\varepsilon}{2},$$

$$|(x^* f)(t+\xi_r+h''_r)-(x^* f)(t+\xi_r+h_r)|<\frac{\varepsilon}{2}.$$

The inequality (IN) above gives

$$|(x^* f)(t+\xi_r+h'_r)-(x^* f)(t+\xi_r+h''_r)|<\varepsilon,\quad\forall t$$

which proves that

$$\lim_{r\to\infty}(x^* f)(t+\xi_r+h'_r)=\lim_{r\to\infty}(x^* f)(t+\xi_r+h''_r)$$

which contradicts $x^*(b')\neq x^*(b'')$ obtained earlier and uniform continuity of $(f(\eta_r+h_n))$ as well.

If $i,j>N$, we have

$$f(\eta_r+h_i)-f(\eta_r+j)\in U.$$

This proves that $f \in AP(E)$ by the Bochner's criterion. □

Theorem 8.20 *Let E be a Fréchet space. If $f \in AP(E)$ and $\{F(t) \ / \ t \in \mathbb{R}\}$ is relatively compact in E where $F(t) = \int_0^t f(s)ds$, then $F \in AP(E)$.*

Proof This is immediate by Theorems 8.17 and 8.18. □

Theorem 8.21 *Let E be a complete locally convex space and $f \in AP(E)$. If the derivative f' exists and is uniformly continuous on $\mathbb{R}$, then $f' \in AP(E)$.*

Proof This is similar to the proof of the almost automorphic case (Theorem 4.1). We consider the sequence of almost periodic functions $(n(f(t+\frac{1}{n}) - f(t)))$ and let $U = U(\varepsilon;\ p_i,\ 1 \leq i \leq k)$ be a neighborhood. Since $f'(t)$ is uniformly continuous on $\mathbb{R}$, we can choose $\delta = \delta(U) > 0$ such that

$$f'(t_1) - f(t_2) \in U$$

for every t_1, t_2 such that $|t_1 - t_2| < \delta$. Let us write

$$f'(t) - n\left(f\left(t + \frac{1}{n}\right) - f(t)\right) = n\int_0^{\frac{1}{n}} (f'(t) - f'(t+s))ds.$$

Then if $N = N(U) > \frac{1}{n}$ and $n > N$, we would obtain

$$p_i\left[f'(t) - n\left(f\left(t + \frac{1}{n}\right) - f(t)\right)\right] \leq n\int_0^{\frac{1}{n}} p_i\left[(f'(t) - f'(t+s))\right]ds < \varepsilon$$

for every seminorm p_i and every $t \in \mathbb{R}$. That shows that the sequence of almost periodic functions $\left(n\left(f(t+\frac{1}{n}) - f(t)\right)\right)$ converges uniformly to $f'(t)$ on $\mathbb{R}$. By Theorem 8.6, it follows that $f' \in AP(E)$. □

Theorem 8.22 *If $f : \mathbb{R} \to E$ (E a Fréchet space) is weakly bounded, then it is bounded.*

Proof For f to be weakly bounded means $\sup_{t\in\mathbb{R}} |x^* f(t)| < \infty$ for every $x^* \in E^*$. Suppose $f(\mathbb{R})$ is not bounded. Then there would exist a seminorm p such that $p(f(t_n)) \to \infty$ as $n \to \infty$ for some sequence of real numbers (t_n).

Let E_p be the completion of the normed space $E/kerp$ in the norm p. So E_p is a Banach space and $\tilde{f}(t_n) = f(t_n)/kerp$ is unbounded in E_p. Consequently there exists $\varphi \in E_p^*$ such that $|\varphi(\tilde{f}(t_n))| \to \infty$ as $n \to \infty$.

The natural map $J : E \to E_p$ is continuous, so its adjoint $J^* : E_p^* \to E^*$ is continuous. Finally setting $\psi = J^*(\varphi) \in E^*$, we have

$$|\psi(f(t_n))| = |J^*(\varphi)(f(t_n))| = |\varphi(\tilde{f}(t_n))| \to \infty$$

as $n \to \infty$. This completes the proof. □

Theorem 8.23 *Let E be a Fréchet space, $f \in WAP(E)$ and $A \in L(E)$ a bounded linear operator on E. Then $Af \in WAP(E)$.*

Proof Obvious. We leave it to the reader. □

Proposition 8.24 *Let E be a complete locally convex space and $f \in AP(E)$. Then for every sequence of real numbers (s_n), there exists a subsequence (s'_n) such that for every neighborhood U,*

$$f(t + s'_n) - f(t + s'_m) \in U$$

for every $t \in \mathbb{R}$ and every n, m.

Proof Let $U = U(\varepsilon;\ p_i,\ 1 \le i \le n)$ and $V = V(\frac{\varepsilon}{3};\ p_i,\ 1 \le i \le n)$ be a symmetric neighborhood such that $V + V + V \subset U$. By the definition of almost periodicity, there exists a number $l = l(V) > 0$ (depending also on U) such that every compact interval of length l contains a number τ such that

$$f(t + \tau) - f(t) \in V$$

for every $t \in \mathbb{R}$.

Consider now a given sequence of real numbers (s_n). For each s_n, we can find τ_n and σ_n such that $s_n = \tau_n + \sigma_n$ with τ_n a V-translation number of f and $\sigma_n \in [0, l]$. In fact it suffices to take $\tau_n \in [s_n - l, s_n]$ and then $\sigma_n = s_n - \tau_n$.

Since f is uniformly continuous, there exists $\delta = \delta(\varphi)$ such that

$$f(t') - f(t'') \in V$$

for all t', t'' with $|t' - t''| < 2\delta$.

Note that $\sigma_n \in [0, l]$ for all n. Hence by the Bolzano–Weierstrass Theorem, the sequence (σ_n) has a convergent subsequence, say (σ_{n_k}). Let $\sigma = \lim_{k\to\infty} \sigma_{n_k}$, which shows that $\sigma \in [0, l]$.

Now consider the subsequence of (σ_{n_k}) (we use the same notation) with

$$\sigma - \delta \le \sigma_{n_k} \le \sigma + \delta, \quad k = 1, 2, \ldots$$

and let (s_{n_k}) be the corresponding subsequence of (s_n) with

$$s_{n_k} = \tau_{n_k} + \sigma_{n_k}, \quad k = 1, 2, \ldots$$

Let us prove that

$$f(t + s_{n_k}) - f(t + s_{n_j}) \in U$$

for all t and all k, j. For this, let us write

$$f(t+s_{n_k}) - f(t+s_{n_j}) = f(t+\tau_{n_k}+\sigma_{n_k}) - f(t+\sigma_{n_k}) + f(t+\sigma_{n_k}) \\ - f(t+\sigma_{n_j}) + f(t+\sigma_{n_j}) - f(t+\tau_{n_j}+\sigma_{n_j}).$$

Because τ_{n_k} and τ_{n_j} are V-translation numbers of f, we have

$$f(t+\tau_{n_k}+\sigma_{n_k}) - f(t+\sigma_{n_k}) \in V$$

and

$$f(t+\sigma_{n_j}) - f(t+\tau_{n_j}+\sigma_{n_j}) \in V$$

for every t and every k, j. Also

$$f(t+\sigma_{n_k}) - f(t+\sigma_{n_j}) \in V$$

for every t and every k, j, since

$$|(t+\sigma_{n_k}) - (t+\sigma_{n_j})| = |\sigma_{n_k} - \sigma_{n_j}| \le |\sigma_{n_k} - \sigma| + |\sigma - \sigma_{n_j}| \le 2\delta.$$

The result is complete if we set $s'_k = s_{n_k}, \quad k = 1, 2, \ldots$ □

Theorem 8.25 *Let E be a Fréchet space and $(T(t))_{t\in\mathbb{R}}$ be an equicontinuous C_0-group of linear operators with $\{T(t)x;\ t \in \mathbb{R}\}$ relatively compact in E for every $x \in E$. Assume also that $f : \mathbb{R} \to E$ is a function with a relatively compact range in E. Then $\{T(t)f(t) :\ t \in \mathbb{R}\}$ is relatively compact in E.*

Proof Let (t''_n) be a sequence of real numbers. Since the range of $f(t)$ is relatively compact in E, we can extract a subsequence $(t'_n) \subset (t''_n)$ such that

$$\lim_{n\to\infty} f(t'_n) = x, \text{ exists in } E.$$

Further, by the assumption on $T(t)$, we can find a subsequence $(t_n) \subset (t'_n)$ such that $(T(t_n)x)$ is convergent, thus a Cauchy sequence in E.

Let us write

$$T(t_n)f(t_n) - T(t_m)f(t_m) = [T(t_n) - T(t_m)][f(t_n) - x] + [(T(t_n) - T(t_m))x] \\ + T(t_m)[f(t_n) - f(t_m)].$$

For an arbitrary seminorm p we have

$$p(T(t_n)f(t_n) - T(t_m)f(t_m)) \le p([T(t_n) - T(t_m)][f(t_n) - x]) \\ + p([(T(t_n) - T(t_m))x]) \\ + p(T(t_m)[f(t_n) - f(t_m)]).$$

Using the equicontinuity of $T(t)$, we can find a seminorm q such that

$$p(T(t_m)[f(t_n) - f(t_m)]) \leq q(f(t_n) - f(t_m))$$

and

$$p([T(t_n) - T(t_m)][f(t_n) - x]) \leq 2q(f(t_n) - x).$$

Now choose n large enough so that

$$q(f(t_n) - f(t_m)) < \frac{\varepsilon}{3} q(f(t_n) - x) < \frac{\varepsilon}{3}$$

and

$$q([T(t_n) - T(t_m)]x) < \frac{\varepsilon}{3}$$

$$p(T(t_n)f(t_n) - T(t_m)) < \varepsilon,$$

which shows that $(T(t_n)f(t_n))$ is a Cauchy sequence, thus convergent. The theorem is proved. □

Theorem 8.26 *Let E be a Fréchet space and consider an equicontinuous C_0-group of linear operators $(T(t))_{t\in\mathbb{R}}$ such that $T(t)x : \mathbb{R} \to E$ is almost periodic for every $x \in E$. Suppose also that $f \in AP(E)$. Then $T(t)f(t) \in AP(E)$.*

Proof Consider $U = U(\varepsilon;\ p_i,\ 1 \leq i \leq n)$ be a given neighborhood of the origin. Because of the equicontinuity of $T(t)$, one can find, for each semi-norm p_i, a seminorm q_i such that

$$p_i(T(t)x) \leq q_i(x)$$

for every $t \in \mathbb{R}$ and $x \in E$. Consider also the symmetric neighborhood

$$V = V\left(\frac{\varepsilon}{4};\ p_i, q_i,\ 1 \leq i \leq n\right).$$

Then $V + V + V + V \subset U$. Since $\{f(t) :\ t \in \mathbb{R}\}$ is totally bounded, there exists $t_1, \ldots, t_\nu$ such that

$$f(t) \in \bigcup_{k=1}^{\nu} (f(t_k) + V)$$

for every $t \in \mathbb{R}$.

Consider now the almost periodic functions

$$f(t),\ T(t)(f(t_k)),\quad k = 1, 2, \ldots, \nu.$$

These are the same V-translation numbers by Corollary 8.13; therefore we can find a number $l = l(V) > 0$ such that any interval $[a, a+l]$ contains at least one number s such that

$$f(t+s) - f(s) \in V \quad \text{for every} \quad t \in \mathbb{R}, \tag{2.3}$$

$$T(t+s)f(t_k) - T(t)f(t_k) \in V \quad \text{for every } t \in \mathbb{R} \tag{2.4}$$

and for every $k = 1, 2, \ldots, \nu$.

Take now an arbitrary $t \in \mathbb{R}$. Then there exists $(1 \le j \le \nu)$ such that

$$f(t) \in f(t_j) + V. \tag{2.5}$$

Write

$$\begin{aligned} T(t+s)f(t+s) - T(t)f(t) = T(t+s)(f(t+s) - f(t)) \\ +T(t+s)(f(t) - f(t_j)) \\ +T(t+s)f(t_j) - T(t)f(t_j) \\ +T(t)(f(t_j) - f(t)). \end{aligned}$$

For every seminorm p_i, we can find a seminorm q_i such that

$$\begin{aligned} p_i[T(t+s)f(t+s) - T(t)f(t)] &\le q_i(f(t+s) - f(t)) \\ &\quad +q_i(f(t) - f(t_j)) + p_i(T(t+s)f(t_j) \\ &\quad -T(t)f(t_j)) + q_i(f(t_j) - f(t)) \\ &\quad < \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} \\ &= \varepsilon \end{aligned}$$

using (2.3), (2.4), and (2.5) above. Thus we have

$$T(t+s)f(t+s) - T(t)f(t) \in U$$

for every $t \in \mathbb{R}$, which establishes the almost periodicity of $T(t)f(t)$. □

Definition 8.27 A Fréchet space E is said to be perfect if every bounded function $f : \mathbb{R} \to E$ with an almost periodic derivative f' is necessarily almost periodic.

Example 8.28 Denote by s the linear space of all real sequences

$$s := \{s = (x_n) \;/\; x_n \in \mathbb{R}, \;\; n = 1, 2, \ldots\}.$$

For each $n \in \mathbb{N}$, define $p_n(x) := |x_n|, \;\; x \in s$. Obviously p_n is a seminorm defined on s. Now define $q_n := p_1 \vee p_2 \vee \ldots \vee p_n$ for $n \in \mathbb{N}$. We have $q_n \le q_{n+1}$ for $n \in \mathbb{N}$. The space s considered with the family of seminorms (q_n) is a Fréchet space. Moreover, it can be proved (cf. [1] 17.7 p. 210) that each closed and bounded subset of s is compact. Thus, in particular, s is not a Banach space. Moreover in view of Theorem 8.20, s is perfect.

Definition 8.29 A function $f \in C(\mathbb{R}, \mathbb{X})$ is called periodic if there exists $l > 0$ such that

$$f(t+l) = f(t), \quad \forall t \in \mathbb{R}.$$

Here, l is called a period of f. We denote the collection of all such functions by $P(\mathbb{X})$. For $f \in P(\mathbb{X})$, we call l_0 the fundamental period if l_0 is the smallest period of f.

Remark 8.30 Similar to the proof in [22, p. 1], it is not difficult to show that if $f \in P(\mathbb{X})$ is not constant, then f has the fundamental period.

Theorem 8.31 ([72]) *Let X be a Banach space with norm $\|\cdot\|$, then $P(\mathbb{X})$ is a set of first category in $AP(\mathbb{X})$.*

Proof For $n = 1, 2, \ldots$, we denote

$$P_n = \{f \in C(\mathbb{R}, \mathbb{X}) \ : \ \exists l \in [n, n+1] \text{ such that } f(t+1) = f(t), \ \forall t \in \mathbb{R}\}.$$

Then, it is easy to see that

$$P(\mathbb{X}) = \bigcup_{n=1}^{\infty} P_n.$$

We divide the remaining proof into two steps.

Step 1 Every P_n is a closed subset of $AP(\mathbb{X})$.
Let $f \in AP(\mathbb{X}) \backslash P_n$. Then, for every $l \in [n, n+1]$, there exists $t_l \in \mathbb{R}$ such that $f(t_l + l) \neq f(t_l)$. Denote

$$\varepsilon_l := \frac{1}{4}\|f(t_l + l) - f(t_l)\| > 0, \quad l \in [n, n+1].$$

In addition, due to the continuity of f, for every $l \in [n, n+1]$, there exists $\delta_l > 0$ such that

$$\|f(t_l + s) - f(t_l)\| \geq 3\varepsilon_l, \quad \forall s \in (l - \delta_l, l + \delta_l). \tag{2.6}$$

Obviously, we have

$$[n, n+1] \subset \bigcup_{l \in [n, n+1]} (l - \delta_l, l + \delta_l).$$

Then, by the Heine-Borel theorem, there exists $l_1, \ldots l_k \in [n, n+1]$ such that

$$[n, n+1] \subset \bigcup_{i=1}^{k} (l_i - \delta_{l_i}, l_i + \delta_{l_i}),$$

where k is a fixed positive integer. Letting $\varepsilon = \min\limits_{1 \le i \le k} \{\varepsilon_{l_i}\}$, and

$$N(f, \varepsilon) := \{g \in AP(\mathbb{X}) : \|g - f\|_{AP(X)} < \varepsilon\},$$

for every $g \in N(f, \varepsilon)$, we claim that $g \notin P_n$. In fact, for every $l \in [n, n+1]$, there exists $i \in \{1, \ldots, k\}$ such that

$$l \in (l_i - \delta_{l_i}, l_i + \delta_{l_i}).$$

Then, by (4.3), we have

$$\|f(t_{l_i} + l) - f(t_{l_i})\| \ge 3\varepsilon_{l_i} \ge 3\varepsilon,$$

which yields that

$$\begin{aligned} \|g(t_{l_i} + l) - g(t_{l_i})\| &\ge \|f(t_{l_i} + l) - f(t_{l_i})\| - \|f(t_{l_i} + l) - g(t_{l_i} + l)\| \\ &\quad - \|f(t_{l_i}) - g(t_{l_i})\| \ge 3\varepsilon - \varepsilon - \varepsilon = \varepsilon > 0, \end{aligned}$$

where $\|g - f\|_{AP(\mathbb{R})} < \varepsilon$ was used. So, we know that $N(f, \varepsilon) \subset AP(\mathbb{X}) \backslash P_n$, which means that P_n is a closed subset of $AP(\mathbb{X})$.

Step 2 Every P_n has an empty interior.

It suffices to prove that for every $f \in P_n$ and $\delta > 0$, $N(f, \delta) \bigcap (AP(\mathbb{X}) \backslash P_n) \neq \emptyset$. Now let $f \in P_n$ and $\delta > 0$. In the following, we discuss by two cases:

Case I f is constant.

We denote

$$f_\delta(t) = \frac{\cos t + \cos(\sqrt{2}t)}{3} \cdot \delta \cdot x_0 + f(t), \quad t \in \mathbb{R}$$

where $x_0 \in \mathbb{X}$ is some constant with $\|x_0\| = 1$. Then $f_\delta \in N(f, \delta)$, and $f_\delta \notin P_n$ since f_δ is not periodic.

Case II f is not constant.

Let f be a fundamental period l_0. We denote

$$f_\delta(t) = f(t) + f\left(\frac{t}{\pi}\right) \cdot \frac{\delta}{M_f}, \quad t \in \mathbb{R},$$

where $M_f = \sup\limits_{t \in \mathbb{R}} \|f(t)\|$. Obviously, $f_\delta \in N(f, \delta)$. Also, we claim that $f_\delta \notin P_n$. In fact, if this is not true, then there exists $T \in [n, n+1]$ such that

$$f_\delta(t+T) = f_\delta(t), \quad t \in \mathbb{R},$$

i.e.

$$f(t+T) + f\left(\frac{t+T}{\pi}\right) \cdot \frac{\delta}{M_f} = f(t) + f\left(\frac{t}{\pi}\right) \cdot \frac{\delta}{M_f}, \quad t \in \mathbb{R}.$$

Let

$$F_1(t) = f(t+T) - f(t), \quad F_2(t) = \frac{\delta}{M_f}\left[f\left(\frac{t}{\pi}\right) - f\left(\frac{t+T}{\pi}\right)\right], \quad t \in \mathbb{R}.$$

Then $F_1(t) \equiv F_2(t)$. If $F_1(t) \equiv F_2(t) \equiv C$, where C is a fixed constant, then

$$f(t+T) = f(t) + C, \quad t \in \mathbb{R},$$

which yields

$$C = \frac{f(kT) - f(0)}{k} \to 0, \ k \to \infty,$$

since f is bounded. Thus, we have

$$f(t+T) = f(t), \quad f\left(\frac{t}{\pi}\right) = f\left(\frac{t+T}{\pi}\right), \quad t \in \mathbb{R}.$$

Noting that l_0 is the fundamental period of f and πl_0 is the fundamental period of $f\left(\frac{\cdot}{\pi}\right)$, there exist two positive integers p, q such that

$$pl_0 = T = q\pi l_0,$$

i.e. $\pi = \frac{p}{q}$, which is a contradiction. If $F_1 = F_2$ is not constant, then by Remark 8.30, we can assume that T_0 is the fundamental period of F_1 and F_2. Noting that l_0 is a period of F_1 and πl_0 is a period of F_2, similar to the above proof, we can also show that π is a rational number, which is a contradiction.

In conclusion, $P(\mathbb{X})$ is countable unions of closed subsets with empty interior. So $P(\mathbb{X})$ is a set of first category. □

Theorem 8.32 ([72]) *Let $\mathbb{X}$ be a Banach space. Then $AP(\mathbb{X})$ is a set of first category in $AA(\mathbb{X})$.*

Proof It suffices to note that $AP(\mathbb{X})$ is a proper closed subspace of $AA(\mathbb{X})$ equipped with the supnorm. Therefore it is of first category in $AA(\mathbb{X})$. □

3 Almost Periodicity of the Function $f(t,x)$

Definition 8.33 Let E be a Fréchet space. A function $f \in C(\mathbb{R} \times E, E)$ is said to be almost periodic in $t \in \mathbb{R}$ for each $x \in E$ if for each neighborhood of the origin U, there exists a real number $l > 0$ such that every interval $[a, a+l]$ contains at least a point τ such that

$$f(t+\tau, x) - f(t,x) \in U, \quad \text{for each } t \in \mathbb{R} \text{ and each } x \in E.$$

In view of the Bochner's criterion, this definition is equivalent to the following: $f \in C(\mathbb{R} \times E, E)$ is almost periodic in $t \in \mathbb{R}$ for each $x \in E$ if and only if for every sequence of real numbers (s_n') there exists a subsequence $(s_n) \subset (s_n')$ such that $(f(t+s_n, x))$ converges uniformly in $t \in \mathbb{R}$ and $x \in E$.

Theorem 8.34 *Let $f : \mathbb{R} \times E \to E$ be almost periodic in $t \in \mathbb{R}$ for each $x \in E$, and assume that f satisfies a Lipschitz condition in x uniformly in t, that is $d(f(t,x) - f(t,y)) \leq Ld(x,y)$ for all $t \in \mathbb{R}$ and $x, y \in E$, where d is a metric on E. Let $\phi : \mathbb{R} \to E$ be almost periodic. Then the Nemytskii's operator $\mathcal{N}$ defined by $\mathcal{N}(\cdot) := f(\cdot, \phi(\cdot))$ is almost periodic.*

Proof Trivial. We leave it to the reader. □

4 Equi-Asymptotically Almost Periodic Functions

In this section, we introduce the notion of equi-asymptotically almost periodicity (cf. [24]), and present some basic and interesting properties for equi-asymptotically almost periodic functions.

Definition 8.35 Let $\mathbb{X}$ be a Banach space. A set $F \subset C(\mathbb{R}, \mathbb{X})$ is called equi-asymptotically almost periodic if for every $\varepsilon > 0$, there exist a constant $M(\varepsilon) > 0$ and a relatively dense set $T(F, \varepsilon) \subset \mathbb{R}$ such that

$$\|f(t+\tau) - f(t)\| < \varepsilon,$$

for all $f \in F$, $t \in \mathbb{R}$ with $|t| > M(\varepsilon)$ and $\tau \in T(F, \varepsilon)$ with $|t+\tau| > M(\varepsilon)$.

Theorem 8.36 *Let $F \subset AAP(\mathbb{R}, \mathbb{X})$. Then the following assertions are equivalent:*

(i) *F is precompact in $AAP(\mathbb{R}, \mathbb{X})$.*
(ii) *F satisfy the following three conditions:*

  (a) *for every $t \in \mathbb{R}$, $\{f(t) : f \in F\}$ is precompact in $\mathbb{X}$.*
  (b) *F is equi-uniformly continuous.*
  (c) *F is equi-asymptotically almost periodic.*

(iii) *G is precompact in $AP(\mathbb{R}, X)$ (in short AP) and H is precompact in $C_0(\mathbb{R}, X)$ (in short C_0), where*

$$G = \{f_{AP} : f \in F\} \text{ and } H = \{f_{C_0} : f \in F\}.$$

Proof

(i) ⇒ (ii) Let F be precompact in $AAP(\mathbb{R}, \mathbb{X})$. Then, obviously, for every $t \in \mathbb{R}$, $\{f(t) : f \in F\}$ is precompact in X. In addition, for every $\varepsilon > 0$, there exist $f_1, f_2, \ldots, f_k \in F$ such that for every $f \in F$,

$$\min_{1 \le i \le k} \|f - f_i\| < \varepsilon,$$

where k is a positive integer dependent on ε. Combining this with the fact that $(f_i)_{i=1}^k$ is equi-uniformly continuous and equi-asymptotically almost periodic, we know that (b) and (c) hold.

(ii) ⇒ (iii) Let $(g_n) \subset G$. For every n, there exist $f_n \in F$ and $h_n \in H$ such that $f_n = g_n + h_n$. By (a) and (b), applying Arzela–Ascoli theorem and choosing diagonal sequence, we can get a subsequence of (f_n), which we still denote by (f_n) for convenience, such that $(f_n(t))$ is uniformly convergent on every compact subsets of $\mathbb{R}$.

Since (f_n) is equi-asymptotically almost periodic, for every $\varepsilon > 0$, there exists $l(\varepsilon), M(\varepsilon) > 0$ such that for every $t \in \mathbb{R}$ with $|t| > M(\varepsilon)$, there is a

$$\tau_t \in [M(\varepsilon) + 1 - t, M(\varepsilon) + 1 - t + l(\varepsilon)]$$

satisfying

$$\|f_n(t + \tau_t) - f_n(t)\| < \frac{\varepsilon}{3} \tag{4.1}$$

for all $n \in \mathbb{N}$. Noting that $(f_n(t))$ is uniformly convergent on

$$[-M(\varepsilon) - l(\varepsilon) - 1, M(\varepsilon) + l(\varepsilon) + 1],$$

for the above $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m \ge n \ge N$ and $t \in [-M(\varepsilon) - l(\varepsilon) - 1, M(\varepsilon) + l(\varepsilon) + 1]$,

$$\|f_m(t) - f_n(t)\| < \frac{\varepsilon}{3}. \tag{4.2}$$

Combining (4.1) and (4.2), for all $m \ge n \ge N$ and $t \in \mathbb{R}$ with $|t| > M(\varepsilon)$, we have

$$\|f_m(t) - f_n(t)\| \le \|f_m(t) - f_m(t + \tau_t)\| + \|f_m(t + \tau_t) - f_n(t + \tau_t)\|$$

$$+\|f_n(t+\tau_t)-f_n(t)\| \le \varepsilon,$$

which and (4.2) yield that $(f_n(t))$ is uniformly convergent on $\mathbb{R}$. In view of

$$\{g_m(t)-g_n(t) : t \in \mathbb{R}\} \subset \overline{\{f_m(t)-f_n(t) : t \in \mathbb{R}\}}$$

for all $m, n \in \mathbb{N}$, we conclude that $(g_n(t))$ is also uniformly convergent on $\mathbb{R}$, i.e. (g_n) is convergent in $AP(\mathbb{R}, X)$. So G is precompact in $AP(\mathbb{R}, X)$. In addition, it follows from the above proof that F is precompact, and thus H is also precompact.

(iii) $\Rightarrow$ (i) The proof is straightforward.

□

Remark 8.37 Theorem 8.36 can be seen as an extension of the corresponding compactness criteria for the subsets of $AP(\mathbb{R}, \mathbb{X})$ (cf. e.g., [22]).

Definition 8.38 $F \subset C_0(\mathbb{R}, X)$ is called equi-C_0 if

$$\lim_{|t|\to\infty} \sup_{f\in F} \|f(t)\| = 0.$$

Theorem 8.39 *The following two assertions are equivalent:*

(I) *F is equi-asymptotically almost periodic;*
(II) *G is equi-almost periodic and H is equi-C_0, where*

$$G = \{f_{AP} : f \in F\} \text{ and } H = \{f_{C_0} : f \in F\}.$$

Proof The proof from (II) to (I) is straightforward. We will only give the proof from (I) to (II) by using the idea in the proof of [71, p. 24, Theorem 2.5].

Since F is equi-asymptotically almost periodic, for every $k \in \mathbb{N}$, there exist a constant $M_k > 0$ and a relatively dense set $T(F, k) \subset \mathbb{R}$ such that

$$\|f(t+\tau)-f(t)\| < \frac{1}{k}, \tag{4.3}$$

for all $f \in F, t \in \mathbb{R}$ with $|t| > M_k$ and $\tau \in T(F, k)$ with $|t+\tau| > M_k$. Moreover, for every $f \in F \subset AAP(\mathbb{R}, \mathbb{X})$, noting that f is uniformly continuous, for the above $k \in \mathbb{N}$, there exists $\delta_k^f > 0$ such that

$$\|f(t_1)-f(t_2)\| < \frac{1}{k} \tag{4.4}$$

for all $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \delta_k^f$.

Now, for every $t \in \mathbb{R}$ and $k \in \mathbb{N}$, we choose $\tau_k^t \in T(F, k)$ with $t + \tau_k^t > M_k$. Also, we denote

$$g_k^f(t) = f(t + \tau_k^t), \quad t \in \mathbb{R},\ k \in \mathbb{N},\ f \in F.$$

Next, we divide the remaining proof into eight steps.

Step 1 For every $f \in F$, there holds

$$\left\| g_k^f(t_1) - g_k^f(t_2) \right\| < \frac{5}{k} \tag{4.5}$$

for all $k \in \mathbb{N}$, and $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \delta_k^f$.
In fact, by (4.3) and (4.4), we have

$$\begin{aligned}
\|g_k^f(t_1) - g_k^f(t_2)\| &= \|f(t_1 + \tau_k^{t_1}) - f(t_2 + \tau_k^{t_2})\| \\
&\le \|f(t_1 + \tau_k^{t_1}) - f(t_1 + \tau_k^{t_1} + \tau)\| \\
&\quad + \|f(t_1 + \tau_k^{t_1} + \tau) - f(t_2 + \tau_k^{t_1} + \tau)\| \\
&\quad + \|f(t_2 + \tau_k^{t_1} + \tau) - f(t_2 + \tau)\| \\
&\quad + \|f(t_2 + \tau) - f(t_2 + \tau + \tau_k^{t_2})\| \\
&\quad + \|f(t_2 + \tau + \tau_k^{t_2}) - f(t_2 + \tau_k^{t_2})\| < \frac{5}{k},
\end{aligned}$$

where $\tau \in T(F, k)$ satisfying

$$\min\left\{t_1 + \tau_k^{t_1} + \tau, t_2 + \tau_k^{t_1} + \tau, t_2 + \tau, t_2 + \tau + \tau_k^{t_2}\right\} > M_k.$$

Step 2 For every $k \in \mathbb{N}$, there holds

$$\|g_k^f(t + \tau) - g_k^f(t)\| < \frac{5}{k} \tag{4.6}$$

for all $f \in F$, $\tau \in T(F, k)$, and $t \in \mathbb{R}$.
In fact, by using (4.3), we have

$$\begin{aligned}
\|g_k^f(t + \tau) - g_k^f(t)\| &= \|f(t + \tau + \tau_k^{t+\tau}) - f(t + \tau_k^t)\| \\
&\le \|f(t + \tau + \tau_k^{t+\tau}) - f(t + \tau + \tau_k^{t+\tau} + \tau')\| \\
&\quad + \|f(t + \tau + \tau_k^{t+\tau} + \tau') - f(t + \tau_k^{t+\tau} + \tau')\| \\
&\quad + \|f(t + \tau_k^{t+\tau} + \tau') - f(t + \tau')\| \\
&\quad + \|f(t + \tau') - f(t + \tau' + \tau_k^t)\| \\
&\quad + \|f(t + \tau' + \tau_k^t) - f(t + \tau_k^t)\| < \frac{5}{k},
\end{aligned}$$

where $\tau' \in T(F, k)$ satisfying

$$\min\left\{t + \tau + \tau_k^{t+\tau} + \tau', t + \tau_k^{t+\tau} + \tau', t + \tau', t + \tau' + \tau_k^t\right\} > M_k.$$

Step 3 For every $n \in \mathbb{N}$, there holds

$$\left\| g_m^f(t) - g_n^f(t) \right\| < \frac{4}{n} \tag{4.7}$$

for all $f \in F$, $t \in \mathbb{R}$, and $m, n \in \mathbb{N}$ with $m \geq n$.
In fact, without loss of generality, we can assume that $M_{k+1} \geq M_k$ for all $k \in \mathbb{N}$. Then, by using (4.3), we have

$$\begin{aligned}
&\left\| g_m^f(t) - g_n^f(t) \right\| \\
&\quad = \| f(t + \tau_m^t) - f(t + \tau_n^t) \| \\
&\quad \leq \| f(t + \tau_m^t) - f(t + \tau_m^t + \tau) \| + \| f(t + \tau_m^t + \tau) - f(t + \tau_m^t + \tau + \tau_n^t) \| \\
&\qquad + \| f(t + \tau_m^t + \tau + \tau_n^t) - f(t + \tau + \tau_n^t) \| + \| f(t + \tau + \tau_n^t) - f(t + \tau_n^t) \| \\
&\qquad < \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \frac{1}{n} \leq \frac{4}{n},
\end{aligned}$$

where $\tau \in T(F, n)$ satisfying

$$\min\left\{t + \tau_m^t + \tau, t + \tau_m^t + \tau + \tau_n^t, t + \tau + \tau_n^t\right\} > M_m.$$

Step 4 Let

$$g^f(t) = \lim_{n \to \infty} g_n^f(t), \quad t \in \mathbb{R}, \ f \in F.$$

By Step 3, we know that for every $f \in F$, g^f is well-defined. Moreover, it follows from Step 3 that for every $n \in \mathbb{N}$, there holds

$$\| g^f(t) - g_n^f(t) \| \leq \frac{4}{n} \tag{4.8}$$

for all $f \in F$, R, and $n \in \mathbb{N}$.
Step 5 For every $f \in F$, g^f is uniformly continuous on $\mathbb{R}$.
In fact, by (4.5) and (4.8), we have

$$\begin{aligned}
\| g^f(t_1) - g^f(t_2) \| &\leq \| g^f(t_1) - g_n^f(t_1) \| + \| g_n^f(t_1) - g_n^f(t_2) \| \\
&\quad + \| g_n^f(t_2) - g^f(t_2) \| \leq \frac{4}{n} + \frac{5}{n} + \frac{4}{n} = \frac{13}{n},
\end{aligned}$$

for all $n \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \delta_n^f$.

Step 6 $\{g^f\}_{f \in F}$ is equi-almost periodic.

By (4.6) and (4.8), for every $n \in \mathbb{N}$, we get

$$\|g^f(t+\tau) - g^f(t)\| \leq \|g^f(t+\tau) - g_n^f(t+\tau)\| + \|g_n^f(t+\tau) - g_n^f(t)\| + \|g_n^f(t) - g^f(t)\| \leq \frac{4}{n} + \frac{5}{n} + \frac{4}{n} = \frac{13}{n},$$

for all $f \in F$, $\tau \in T(F, n)$, and $t \in \mathbb{R}$. Then, it follows that $\{g^f\}_{f \in F}$ is equi-almost periodic.

Step 7 $\{h^f\}_{f \in F}$ is equi-C_0, where $h^f(t) = f(t) - g^f(t)$ for all $f \in F$ and $t \in \mathbb{R}$. In fact, firstly, by Step 5, $h^f \in C(\mathbb{R}, X)$ for every $f \in F$; secondly, for every $n \in \mathbb{N}$, by (4.8) and the definition of τ_n^t, we have

$$\begin{aligned}\|h^f(t)\| &= \|f(t) - g^f(t)\| \\ &\leq \|f(t) - g_n^f(t)\| + \|g_n^f(t) - g^f(t)\| \\ &\leq \|f(t) - f(t+\tau_n^t)\| + \frac{4}{n} \\ &\leq \frac{1}{n} + \frac{4}{n} = \frac{5}{n},\end{aligned}$$

for all $f \in F$, $t \in \mathbb{R}$ with $|t| > M_n$. Thus, $\{h^f\}_{f \in F}$ is equi-C_0.

Step 8 It follows from the above proof that $G = \{g^f\}_{f \in F}$ and $H = \{h^f\}_{f \in F}$. This completes the proof.

□

Bibliographical Notes Section 1 in this chapter is in [51, 54]. Theorem 8.32 is in [72] with a different proof. Section 3 is a work by Ding et al. [24].

Chapter 9
Almost Periodic Functions with Values in a Non-locally Convex Space

In this section, $(X, +, \cdot, ||\cdot||)$ will be a p-Fréchet space with $0 < p < 1$ (over the field $\Phi = \mathbb{R}$ or $\mathbb{C}$). Also, denote $D(x, y) = ||x - y||$.

Similarly to [22], p. 137, a trigonometric polynomial of degree $\leq n$ with coefficients (and values) in the p-Fréchet space X, is defined as a finite sum of the form $T_n(t) = \sum_{k=1}^{n} c_k e^{i\lambda_n t}$, where $c_k \in X$, $k = 1, \ldots, n$.

Also, recall that $f : \mathbb{R} \to X$ is said to be continuous on $x_0 \in \mathbb{R}$ if $\forall \epsilon > 0$, there exists $\delta > 0$ such that $||f(x) - f(x_0)|| < \epsilon$, whenever $x \in \mathbb{R}$, $|x - x_0| < \delta$. From the triangle inequality satisfied by the p-norm $||\cdot||$, it easily follows the inequality $|\ \ ||x|| - ||y||\ \ | \leq ||x - y||$, which immediately implies that if f is continuous in x_0 as above, then the real-valued function $||f(t)||$ is also continuous at x_0.

1 Definitions and Properties

In this section, starting from a Bohr-kind definition for the almost periodicity, we develop a theory of almost periodic functions with values in a p-Fréchet space, $0 < p < 1$, similar to that for functions with values in a Banach space.

The following three points in Definition 3.1 represent the basic concepts in the theory of almost periodic functions with values in the p-Fréchet space X.

Definition 9.1 Let $f : \mathbb{R} \to X$ be continuous on $\mathbb{R}$.

(i) We say that **f is almost periodic** if $\forall \epsilon > 0$, there exists $l(\epsilon) > 0$ such that any interval of length $l(\epsilon)$ of the real line contains at least one point ξ with

$$||f(t + \xi) - f(t)|| < \epsilon, \ \forall t \in \mathbb{R}.$$

G. M. N'Guérékata, *Almost Periodic and Almost Automorphic Functions in Abstract Spaces*, https://doi.org/10.1007/978-3-030-73718-4_9

(ii) We say that **f is normal** if for any sequence $F_n : \mathbb{R} \to X$ of the form $F_n(x) = f(x + h_n)$, $n \in \mathbb{N}$, where $(h_n)_n$ is a sequence of real numbers, one can extract a subsequence of $(F_n)_n$, converging uniformly on $\mathbb{R}$ (i.e. $\forall (h_n)_n, \exists \left(F_{n_k}\right), \exists F : \mathbb{R} \to X$ (which may depend on $(h_n)_n$), such that $\lim_{k\to\infty} ||F_{n_k}(x) - F(x)|| = 0$, uniformly with respect to $x \in \mathbb{R}$).

(iii) We say that **f has the approximation property**, if $\forall \epsilon > 0$, there exists some trigonometric polynomial T with coefficients in X, such that $||f(x) - T(x)|| < \epsilon, \forall x \in \mathbb{R}$.

Let us denote $AP(X) = \{f : \mathbb{R} \to X;\ f$ is almost periodic$\}$. The next two theorems show that $AP(X)$ is a subclass of uniformly continuous bounded functions.

Remark 9.2 We can reformulate 9.1(i), as follows: $f : \mathbb{R} \to X$ is called almost periodic if for every $\varepsilon > 0$, there exists a relatively dense set $\{\tau\}_\varepsilon$, such that

$$\sup_{t\in\mathbb{R}} ||f(t+\tau) - f(t)|| \le \varepsilon, \text{ for all } \tau \in \{\tau\}_\varepsilon.$$

Also, each $\tau \in \{\tau\}_\varepsilon$ is called ε-almost period of f.

Remark 9.3 Theorems 10.5, 8.5, 8.6, and 8.10 and Remark 8.8 remain valid in p-Fréchet spaces, $0 < p < 1$.

Theorem 9.4 *If f has the approximation property, then it is almost periodic.*

Proof A function $f : \mathbb{R} \to \mathbb{X}$ is said to be τ-periodic if $f(t+\tau) = f(t)$ for all $t \in \mathbb{R}$. Obviously, any trigonometric polynomial with values in $\mathbb{X}$ is almost periodic. This with Theorem 8.6 completes the proof. □

Remark 9.5 Let us denote $AP(X) = \{f : \mathbb{R} \to X;\ f$ is B-almost periodic$\}$, and for $f \in AP(X)$, let us define $\|f\|_b = \sup\{\|f(t)\|;\ t \in \mathbb{R}\}$. By Theorem 3.2, we get $\|f\|_b < +\infty$. It easily follows that $||\cdot||_b$ is also a p-norm on the space

$$BC(\mathbb{R}, X) = \{f : \mathbb{R} \to X \text{ is continuous and bounded on } \mathbb{R}\}.$$

In addition, since (X, D), where D is defined by $D(x, y) = \|x - y\|$, is a complete metric space, by standard reasonings, it follows that $BC(\mathbb{R}, X)$ becomes complete metric space with respect to the metric $D_b(f, g) = ||f - g||_b$, that is, $(BC(\mathbb{R}, X), ||\cdot||_b)$ becomes a p-Fréchet space.

Then, Theorems 3.2 and 3.5 show that $AP(X)$ is a closed subset of $BC(\mathbb{R}, X)$, i.e. $(AP(X), D_b)$ is a complete metric space, and therefore $(AP(X), ||\cdot||_b)$ becomes a p-Fréchet space. By similar reasonings with those in the proofs of Theorems 6.9 and 6.10 in [22], pp. 142–143 (where we define on X^m the p-norm $||x||_m = \sum_{k=1}^{m} ||x_k||$ and the metric $D_m(x, y) = \sum_{i=1}^{m} D(x_i, y_i), \forall x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in X^m$), we can state the following compactness criterion.

Theorem 9.6 *The necessary and sufficient condition that a family $\mathcal{A} \subset AP(\mathbb{X})$ be relatively compact is that the following properties hold true:*

(i) *$\mathcal{A}$ is equicontinuous;*
(ii) *$\mathcal{A}$ is equi-almost periodic;*
(iii) *for any $t \in \mathbb{R}$, the set of values of functions from $\mathcal{A}$ be relatively compact in $\mathbb{X}$.*

In what follows, we consider the concept of Bochner's transform. Thus, Bochner's transform of f in $BC(\mathbb{R}, X)$ is denoted by $B(f) := \tilde{f}$ and is defined by $\tilde{f} : \mathbb{R} \to BC(\mathbb{R}, X)$, $\tilde{f}(s) \in BC(\mathbb{R}, X)$, $\tilde{f}(s)(t) = f(t+s)$, for all $t \in \mathbb{R}$.

The properties of the Bochner transform can be summarized in the following theorem:

Theorem 9.7 *(i)* $||\tilde{f}(s)||_b = ||f(\cdot + s)||_b = ||\tilde{f}(0)||_b$, *for all* $s \in \mathbb{R}$;
(ii) $||\tilde{f}(s + \tau) - \tilde{f}(s)||_b = sup\{||f(t + \tau) - f(t)||; t \in \mathbb{R}\} = ||\tilde{f}(\tau) - \tilde{f}(0)||_b$, *for all* $s, \tau \in \mathbb{R}$;
(iii) *f is B-almost periodic if and only if $\tilde{f}$ is B-almost periodic, with the same set of ε-almost periods $\{\tau\}_\varepsilon$;*
(iv) *$\tilde{f}$ is B-almost periodic, if and only if there exists a relatively dense sequence in $\mathbb{R}$, denoted by $\{s_n; n \in \mathbb{N}\}$, such that the set of functions $\{\tilde{f}(s_n); n \in \mathbb{N}\}$ is relatively compact in the complete metric space $(BC(\mathbb{R}, \mathbb{X}), D_b)$;*
(v) *$\tilde{f}$ is B-almost periodic, if and only $\tilde{f}(\mathbb{R})$ is relatively compact in the complete metric space $(BC(\mathbb{R}, \mathbb{X}), D_b)$;*
(vi) *(Bochner's criterion) f is B-almost periodic if and only if $\tilde{f}(\mathbb{R})$ is relatively compact in the complete metric space $(BC(\mathbb{R}, \mathbb{X}), D_b)$.*

Proof It is absolutely similar to the proof for Banach space valued functions, see e.g. [3, pp. 7–9].

Now, we are in position to prove the following sufficient condition for B-almost periodicity in p-Fréchet spaces, $0 < p < 1$. □

Theorem 9.8 *Let $f \in BC(\mathbb{R}, \mathbb{X})$. Let us suppose that there exists a relatively dense set of real numbers (s_n), such that*

(i) *the set $\{f(s_n); n \in \mathbb{N}\}$ is relatively compact in the metric space (X, D) and*
(ii) *for any $n, m \in \mathbb{N}$, the relation*

$$||f(s_n) - f(s_m)|| \geq c||f(\cdot + s_n) - f(\cdot + s_m)||_b$$

holds with $c > 0$ independent of n, m.

Then, f is almost periodic.

Proof The inequality in statement together with Theorem 9.7, obviously implies

$$D[f(s_n), f(s_m)] = ||f(s_n) - f(s_m)|| \geq c||\tilde{f}(s_n) - \tilde{f}(s_m)||_b = cD_b[\tilde{f}(s_n), \tilde{f}(s_m)].$$

Since by hypothesis, the set $\{f(s_n); n \in \mathbb{N}\}$ is relatively compact in the metric space (X, D), it has a convergent subsequence $(f(s'_n))_n$, which therefore is a

Cauchy sequence in the complete metric space (X, D), so it is convergent. The above inequality implies that $(\tilde{f}(s'_n))_n$ is also a Cauchy sequence in the complete metric space $(C(\mathbb{R}, \mathbb{X}), D_b)$, so it is convergent. Combined with Theorem 3.11(iv), it follows that $\tilde{f}$ is almost periodic, which combined with Theorem 3.11(iii), implies that f is almost periodic. The theorem is proved. □

2 Weakly Almost Periodic Functions

In what follows, we will consider the concept of weakly almost periodicity, at least in the cases of l^p and H^p spaces, $0 < p < 1$. Indeed, according to Remark 9.13(1) in Sect. 2, the dual spaces $(l^p)^*$ and $(H^p)^*$ are non-null. In addition, since $\{e_i, i \in \mathbb{N}\}$, with $e_i = (\delta_{i,n})_n \in l^p$, $\delta_{i,n}$, Kronecker's symbol, is a basis in l^p (see e.g. [5], p. 20), and since any $e_i^* : l^p \to \mathbb{R}$ is linear and continuous (see e.g. [5], p. 12, Theorem 1.8), it easily follows that $\{p_\varphi; \varphi \in (l^p)^*\}$, with $p_\varphi(x) = |\varphi(x)|$, for all $x \in l^p$, defines a sufficient family of seminorms on l^p, which evidently induces a weak topology on l^p, namely a locally convex Hausdorff topology on l^p.

Also, since according to e.g. [37, p. 35], the point evaluations $\varphi_z(f) = f(z)$, $z \in \mathbb{D}$, satisfy $\varphi_z \in (H^p)^*$, for all $z \in \mathbb{D}$, again it easily follows that $\{p_\varphi(x); \varphi \in (H^p)^*\}$ with $p_\varphi(x) = |\varphi(x)|$, for all $x \in H^p$, defines a sufficient family of seminorms on H^p, which evidently induces a locally convex Hausdorff (weak) topology on H^p.

Definition 9.9 Let $X = l^p$ or $X = H^p$ with $0 < p < 1$. A function $f : \mathbb{R} \to X$ is called weakly almost periodic, if $f : \mathbb{R} \to X$ is continuous and almost periodic, considering X endowed with the (weak) locally convex topology as above (see e.g. [2, pp. 159–160], or [7, 8]).

Remark 9.10 Obviously that Definition 9.9 has no sense for the p-Fréchet space $L^p[0, 1]$, $0 < p < 1$, whose dual is $\{0\}$.

Theorem 9.11 *Let $X = l^p$ or $X = H^p$, $0 < p < 1$. The necessary and sufficient condition that the function $f : \mathbb{R} \to X$ be weakly almost periodic is that for any $\varphi \in X^*$, the numerical function $h : \mathbb{R} \to \mathbb{R}$, defined by $h(t) = \varphi[f(t)]$, be almost periodic.*

Proof It is similar to the proof for Banach space-valued functions (see Theorem 6.1.7, p. 160 in [2]). □

Theorem 9.12 *Let $X = l^p$ or $X = H^p$, $0 < p < 1$. The necessary and sufficient condition that the function $f : \mathbb{R} \to X$ be almost periodic is that f be weakly almost periodic and that $f(\mathbb{R})$ be relatively compact.*

Proof Since for any $\varphi \in X^*$ and all $t, \tau \in \mathbb{R}$, we have

$$|\varphi[f(t+\tau)] - \varphi[f(r)]| \leq |||\varphi||| \cdot ||f(t+\tau) - f(t)||^{1/p},$$

the usual (strong) almost periodicity and continuity of f immediately imply that f is weakly almost periodic and weakly continuous. This together with Theorem 3.6 immediately proves the necessity of theorem.

To prove the sufficiency, we would need the analogue for p-Fréchet space of the following result of Philips for Banach space (see the proof of Theorem 6.18, pp. 160–161 in [2]): from any bounded sequence $(\varphi_n)_n$ in X^* defined on a relatively compact subset $A \subset X$, one can extract a convergent subsequence on A. □

Remark 9.13

(1) In the case when X is a p-Fréchet space endowed with the p-norm $||\cdot||$, $0 < p < 1$, in [3, p. 102] (see also [1, p. 158]), the integral was introduced as follows. First, for $a = a_0 < a_1 < \ldots < a_n = b$, a partition of $[a, b]$, a step function on $[a, b]$ is of the form $s(x) = \sum_{k=0}^{n-1} y_k \cdot \chi_{[a_k, a_{k+1})}(x)$ (where $\chi_{[a_k,a_{k+1})}$ is the characteristic function of $[a_k, a_{k+1})$ and $y_k \in X$, $k = 0, 1, \ldots, n-1$), and its integral on $[a, b]$ is defined by $\int_a^b s(x)dx = \sum_{k=0}^{n-1} y_k(a_{k+1} - a_k)$. Then, since any continuous function $f : [a, b] \to X$ is uniformly continuous on $[a, b]$, it is easy to show that it is the uniform limit on $[a, b]$ of the sequences of step functions $s_n(x)$, $n \in \mathbb{N}$ defined by

$$s_n(x) = \sum_{k=0}^{n-1} f(a_k) \cdot \chi_{[a_k,a_{k+1})}(x),\ a_k = a + k\frac{b-a}{n},\ k = 0, 1, \ldots, n-1,$$

and the integral of f will be defined by $\int_a^b f(x)dx \in X$, where

$$\lim_{n\to\infty} \left\| \int_a^b f(x)dx - \int_a^b s_n(x)dx \right\| = 0.$$

(It is easy to see that the above $\int_a^b f(x)dx$ does not depend on the sequence of step functions uniformly convergent to f.) Unfortunately, the fundamental theorem of calculus stated in [3, Theorem 2, pp. 104–105] (see also [10], pp. 161–162) seems to be not valid in general, since for a continuous function $f : [a, b] \to X$, for the integral $F(t) = \int_a^t f(x)dx$, we have

$$\left\| \frac{F(t+h) - F(t)}{h} - f(t) \right\| = \left\| \frac{\int_t^{t+h} [f(x) - f(t)]dx}{h} \right\|,$$

but we do not get, in general, the estimate

$$\left\| \int_t^{t+h} [f(x) - f(t)]dx \right\| \leq |h|^p ||f(u) - f(t)||,$$

with u between t and $t+h$, as claimed in [10], p. 162 (which would imply that $\lim_{h \to 0} \left\| \frac{F(t+h) - F(t)}{h} - f(t) \right\| = 0$). As a first consequence, it follows that the implication "f' uniformly continuous and f B-almost periodic imply f' B-almost periodic" does not hold, although in the case of Banach space-valued functions, it is valid (see e.g. [3, Theorem VI, p. 6]).

(2) One could also adopt the more particular definition (of Riemann-type) for the integral on $[a, b]$ of a function $f : [a, b] \to X$, as unique limit of all the Riemann sums $\sum_{k=0}^{n-1} f(\xi_k)(a_{k+1} - a_k)$, with $\xi_k \in [a_k, a_{k+1}]$. Unfortunately, for this kind of integral too, the property $||\lambda x|| = |\lambda|^p ||x||$, where $0 < p < 1$, produces a bad estimate for the difference between the Riemann sums attached to two functions $f, g : [a, b] \to X$, namely

$$\begin{aligned} &\left\| \sum_{k=0}^{n-1} f(\xi_k)(a_{k+1} - a_k) - \sum_{k=0}^{n-1} g(\xi_k)(a_{k+1} - a_k) \right\| \\ &\leq \sum_{k=0}^{n-1} (a_{k+1} - a_k)^p \, \| f(\xi_k) - g(\xi_k) \| \\ &\not\leq \sum_{k=0}^{n-1} (a_{k+1} - a_k) ||f(\xi_k) - g(\xi_k)|| \end{aligned}$$

(in fact, since $0 < p < 1$, for $a_{k+1} - a_k \leq 1$, we have $(a_{k+1} - a_k)^p \geq (a_{k+1} - a_k)$, which is the case for n sufficiently large). This fact that has the similar effect as for the first integral, namely the fundamental theorem of calculus for this second integral also does not hold.

(3) From Remarks 9.13(1) and (2), it is evident that for a continuous $f : [a, b] \to X$, the inequality

$$\left\| \int_a^b f(t)dt \right\| \leq \int_a^b ||f(t)||dt$$

does not hold.

Now, if we introduce (as in the case of Banach space-valued functions) the mean value

$$M(f) = \lim_{T \to +\infty} \frac{1}{T} \cdot \int_0^T f(t)\, dt \in X\,,$$

where the limit is considered in the metric space (X, D) (i.e. there exists $M(f) \in X$ with $\lim_{T \to +\infty} D\left(M(f), \frac{1}{T} \cdot \int_0^T f(t)\, dt\right) = 0$), then because of Remark 9.13, it seems that $M(f)$ does not exist for any $f \in AP(X)$, since in the proof for the case of Banach space-valued functions, the inequality mentioned in Remark 9.13(3) is essential. This has as an effect the fact that, in general, one cannot attach Fourier series to a function $f \in AP(X)$ and the fact that the almost periodicity of f does not imply the approximation property mentioned in Definition 3.1(iii).

(4) In [6], a theory of the semigroups of linear and continuous operators is developed. As one of the applications, it was obtained that the initial value problem in the p-Fréchet space X, $0 < p < 1$,

$$\frac{du(t)}{dt} = A[u(t)],\ t \geq 0, u(0) = x \in X$$

(where $A : X \to X$ is linear and continuous) has as the unique solution $u(t) = T(t)(x)$, with $T(t)(x) = \lim_{n \to \infty} \left(I - \frac{t}{n} A\right)^{-n} (x)$, the limit being in the p-norm in X. On the other hand, taking into account Remarks 9.13(1) and 9.13(2), it follows that the inhomogeneous Cauchy problem

$$\frac{du(t)}{dt} = A[u(t)] + f(t), t \geq 0, u(0) = x \in X,$$

in general, seems to not have mild solution, in the sense that, even if we can define it as usual, it does not satisfy the differential equation.

(5) It is easy to construct almost periodic functions $f : \mathbb{R} \to X$ for which there exists $M(f)$ and the fundamental theorem of calculus holds. Indeed, any f of the form $c \cdot g$, where $c \in X$ and $g : \mathbb{R} \to \mathbb{R}$ is almost periodic, satisfies these two requirements.

Remark 9.14 The results in this section are contributions from Gal and N'Guérékata [30].

3 Applications

Firstly, we illustrate the idea of propagation of almost periodicity from the input data to the solutions of a simple differential equation in a p-Fréchet space $(X, ||\cdot||)$. In this sense, we present the following.

Theorem 9.15 *Let $f : \mathbb{R} \to \mathbb{R}$ be a usual almost periodic function and $c \in X$. Then, the function $y : \mathbb{R} \to X$ given by*

$$y(t) = c \cdot \int_{-\infty}^{t} e^{u-t} f(u)du, \ t \in \mathbb{R},$$

is B-almost periodic and satisfies the differential equation

$$y'(t) + y(t) = c \cdot f(t),$$

for all $t \in \mathbb{R}$.

Here, $y'(t)$ *is defined as usual, that is, the limit in the metric* $D(x, y) = ||x - y||$, *given by*

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}.$$

Proof Let us denote $F(t) = \int_{-\infty}^{t} e^{u-t} f(u)du$, $t \in \mathbb{R}$. By the classical theory, F is a usual almost periodic function. Then, by

$$||c \cdot F(t) - c \cdot F(t+\tau)|| = |F(t) - F(t+\xi)|^p \cdot ||c||,$$

it is immediate that $y(t) = c \cdot F(t)$ is B-almost periodic in the sense of Definition 3.1(i).

Since

$$F'(t) = f(t) - \int_{-\infty}^{t} e^{u-t} f(u), \forall t \in \mathbb{R},$$

it easily follows that $y'(t) = c \cdot \left[f(t) - \int_{-\infty}^{t} e^{u-t} f(u)du \right]$ and that $y(t)$ satisfies the differential equation, which proves the theorem. □

Bibliographical Notes The materials in this chapter are due to Gal and N'Guérékata [30]. As one can see, there are some open problems that need further investigations.

Chapter 10
The Equation x'(t)=A(t)x(t)+f(t)

1 The Equation x'(t)=A(t)x(t)+f(t)

Case I Let us first assume that $\mathbb{X}$ is of **finite dimension**, say $\mathbb{X} = \mathbb{C}^n$.

Let us consider the inhomogeneous linear evolution equations of the form

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t), \quad t \in \mathbb{R},\ x(t) \in \mathbb{X}, \tag{1.1}$$

where $A(\cdot)$ is a τ-periodic (unbounded) linear operator-valued function and $f \in AA(\mathbb{X})$.

Theorem 10.1 ([40, 55]) *Every bounded solution on the whole real line of Eq. (1.1) is in* $AA(\mathbb{X})$.

Proof First, we note that by Floquet theory of periodic ordinary differential equations and by Proposition 2.11 [40], without loss of generality, we may assume that A is independent of t.

Next, we will show that the problem can be reduced to the one-dimensional case. In fact, if A is independent of t, by a change of variable if necessary, we may assume that A is of Jordan normal form. In this direction, we can go further with the assumption that A has only one Jordan box; that is, we have to prove the theorem for equations of the form

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

Let us consider the last equation involving $x_n(t)$. We have

G. M. N'Guérékata, *Almost Periodic and Almost Automorphic Functions in Abstract Spaces*, https://doi.org/10.1007/978-3-030-73718-4_10

$$\dot{x}_n(t) = \lambda x_n(t) + f_n(t), \quad t \in \mathbb{R},\ x(t) \in \mathbb{C}^n.$$

If $\Re\lambda \neq 0$, then we can easily check that either

$$y(t) = \int_{-\infty}^{t} e^{\lambda(t-\xi)} f(\xi) d\xi \quad (\Re\lambda < 0)$$

or

$$z(t) = \int_{t}^{\infty} e^{\lambda(t-\xi)} f(\xi) d\xi \quad (\Re\lambda > 0)$$

is a unique bounded solution of Eq. (1.1). Moreover, by Proposition 2.11 [40] in both cases, $y(t)$ and $z(t)$ are in $AA(\mathbb{X})$. Hence, x_n is in $AA(\mathbb{X})$.

If $\Re\lambda = 0$, then $\lambda = i\eta$ for $\eta \in \mathbb{R}$. By assumption, there is a constant c such that the function

$$x_n(t) := ce^{i\eta t} + \int_0^t e^{i\eta(t-\xi)} f(\xi) d\xi$$

is bounded on $\mathbb{R}$. This yields the boundedness of $\int_0^t e^{-i\eta\xi} f(\xi) d\xi$ on $\mathbb{R}$. Hence, $\int_0^t e^{-i\eta\xi} f(\xi) d\xi$ is in $AA(\mathbb{X})$. Finally, this yields that x_n is in $AA(\mathbb{X})$.

Let us consider next the equation involving x_{n-1} and x_n. Since x_n is in $AA(\mathbb{X})$, by repeating the above argument, we can show that x_{n-1} is also in $AA(\mathbb{X})$. Continuing this process, we can show that all $x_k(\cdot)$ are in $AA(\mathbb{X})$. The proof is complete. □

Case II Let us now consider Eq. (1.1) in an **infinite dimensional Banach space** $\mathbb{X}$ where $f \in AA(\mathbb{X})$, and $A(t)$ generates a 1-periodic evolutionary process $(U(t,s))_{t \geq s}$ in $\mathbb{X}$, that is, a two-parameter family of bounded linear operators that satisfies the following conditions:

(i) $U(t,t) = I$ for all $t \in \mathbb{R}$,
(ii) $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r$,
(iii) The map $(t,s) \mapsto U(t,s)x$ is continuous for every fixed $x \in \mathbb{X}$,
(iv) $U(t+1, s+1) = U(t,s)$ for all $t \geq s$ (1-periodicity),
(v) $\|U(t,s)\| \leq Ne^{\omega(t-s)}$ for some positive N, ω independent of $t \geq s$.

We emphasize that the above choice of the period of the equations is merely for the simplification of the notation but does not mean a restriction. We refer the reader to [6, 17, 36] for more information on the applications of this concept of evolutionary processes to partial differential equations.

Definition 10.2 An $\mathbb{X}$-valued continuous function u on $\mathbb{R}$ is said to be a mild solution of Eq. (1.1) if

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi) f(\xi) d\xi, \quad \forall t \geq s; t, s \in \mathbb{R}. \tag{1.2}$$

Lemma 10.3 ([40]) *Let u be a bounded mild solution of Eq. (1.1) on $\mathbb{R}$ and f be almost automorphic. Then, u is almost automorphic if and only if the sequence $(u(n))_{n\in\mathbb{Z}}$ is almost automorphic.*

Proof

Necessity: Obviously, if u is almost automorphic, the sequence $(u(n))_{n\in\mathbb{Z}}$ is almost automorphic.

Sufficiency: Let the sequence $(u(n))_{n\in\mathbb{Z}}$ be almost automorphic. We now prove that u is almost automorphic. The proof is divided into several steps:

Step 1: We first suppose that (n'_k) is a given sequence of integers. Then, there exist a subsequence (n_k) and a sequence $(v(n))$ such that

$$\lim_{k\to\infty} u(n+n_k) = v(n); \qquad \lim_{k\to\infty} v(n-n_k) = u(n), \quad \forall n \in \mathbb{Z} \tag{1.3}$$

$$\lim_{k\to\infty} f(t+n_k) = g(t); \qquad \lim_{k\to\infty} g(t-n_k) = f(t), \quad \forall t \in \mathbb{R}. \tag{1.4}$$

For every fixed $t \in \mathbb{R}$, let us denote by $[t]$ the integer part of t. Then, define

$$v(\eta) := U(\eta, [t])v([t]) + \int_{[t]}^{\eta} U(\eta,\xi)g(\xi)d\xi, \quad \eta \in [[t],[t]+1).$$

In this way, we can define v on the whole line $\mathbb{R}$. Now, we show that

$$\lim_{k\to\infty} u(t+n_k) = v(t).$$

In fact,

$$\begin{aligned}\lim_{k\to\infty}\|u(t+n_k)-v(t)\| &\le \lim_{k\to\infty}\|U(t+n_k,[t]+n_k)u([t]+n_k)-U(t,[t])v([t])\|\\ &\quad+\lim_{k\to\infty}\int_{[t]}^{t}\|U(t,\eta)\|\|f(\eta+n_k)-g(\eta)\|d\eta\\ &=\lim_{k\to\infty}\|U(t,[t])u([t]+n_k)-U(t,[t])v([t])\|\\ &\quad+\lim_{k\to\infty}\int_{[t]}^{t}\|U(t,\eta)\|\|f(\eta+n_k)-g(\eta)\|d\eta=0.\end{aligned}$$

Similarly, we can show that

$$\lim_{k\to\infty}\|v(t-n_k)-u(t)\| = 0.$$

Step 2: Now, we consider the general case where $(s'_k)_{k\in\mathbb{Z}}$ may not be an integer sequence. The main lines are similar to those in Step 1 combined with the strong continuity of the process and the precompactness of the range of the function f.

Set $n'_k = [s'_k]$ for every k. Since $(t_k)_{k\in\mathbb{Z}}$, where $t_k := s'_k - [s'_k]$, is a sequence in $[0, 1)$, we can choose a subsequence (n_k) from $\{n'_k\}$ such that $\lim_{k\to\infty} t_k = t_0 \in [0, 1]$ and (1.3) holds for a function v, as shown in Step 1.

Let us first consider the case $0 < t_0 + t - [t_0 + t]$. We show that

$$\lim_{k\to\infty} u(t_k + t + n_k) = \lim_{k\to\infty} u(t_0 + t + n_k) = v(t_0 + t). \tag{1.5}$$

In fact, for sufficiently large k, from the above assumption, we have $[t_0+t] = [t_k+t]$. Using the 1-periodicity of the process $(U(t,s))_{t\geq s}$, we have

$$\|u(t_k + t + n_k) - u(t_0 + t + n_k)\| \leq A(k) + B(k), \tag{1.6}$$

where $A(k)$ and $B(k)$ are defined and estimated as below. By the 1-periodicity of the process $(U(t,s))_{t\geq s}$, we have

$$\begin{aligned} A(k) := &\|U(t_k + t + n_k, [t_k + t] + n_k)u([t_k + t] + n_k) \\ &-U(t_0 + t + n_k, [t_0 + t] + n_k)u([t_0 + t] + n_k)\| \\ = &\|U(t_k + t, [t_0 + t])u([t_0 + t] + n_k) - U(t_0 + t, [t_0 + t])u([t_0 + t] + n_k)\|. \end{aligned}$$

Using the strong continuity of the process $(U(t,s))_{t\geq s}$ and the precompactness of the range of the sequence $(u(n))_{n\in\mathbb{Z}}$, we have $\lim_{k\to\infty} A(k) = 0$. Next, we define

$$B(k) := \|\int_{[t_k+t]+n_k}^{t_k+t+n_k} U(t_k+t+n_k, \eta)f(\eta)d\eta - \int_{[t_0+t]+n_k}^{t_0+t+n_k} U(t_0+t+n_k, \eta)f(\eta)d\eta\|.$$

By the 1-periodicity of the process $(U(t,s))_{t\geq s}$ and $[t_0 + t] = [t_k + t]$, we have

$$\begin{aligned} B(k) = &\left\| \int_0^{t_k+t-[t_k+t]} U(t_k + t + n_k, [t_0 + t] + n_k + \theta)f([t_0 + t] + n_k + \theta)d\theta \right. \\ &\left. - \int_0^{t_0+t-[t_0+t]} U(t_0 + t + n_k, [t_0 + t] + n_k + \theta)f([t_0 + t] + n_k + \theta)d\theta \right\| \\ = &\left\| \int_0^{t_k+t-[t_0+t]} U(t_k + t - [t_0 + t], \theta)f([t_0 + t] + n_k + \theta)d\theta \right. \\ &\left. - \int_0^{t_0+t-[t_0+t]} U(t_0 + t - [t_0 + t], \theta)f([t_0 + t] + n_k + \theta)d\theta \right\|. \end{aligned}$$

From the strong continuity of the process $(U(t,s))_{t\geq s}$ and the precompactness of the range of f, it follows that $\lim_{k\to\infty} B(k) = 0$. So, in view of Step 1, we see that (1.5) holds.

Next, we consider the case when $t_0 + t - [t_0 + t] = 0$, that is, $t_0 + t$ is an integer. If $t_k + t \geq t_0 + t$, we can repeat the above argument. So, we omit the details. Now, suppose that $t_k + t < t_0 + t$. Then,

$$\|u(t_k + t + n_k) - u(t_0 + t + n_k)\| \leq C(k) + D(k), \tag{1.7}$$

where $C(k)$ and $D(k)$ are defined and estimated as below:

$$\begin{aligned} C(k) &:= \|U(t_k + t + n_k, [t_k + t] + n_k)u([t_k + t] + n_k) \\ &\quad - U(t_0 + t + n_k, t_0 + t - 1 + n_k)u(t_0 + t - 1 + n_k)\| \\ &= \|U(t_k + t, t_0 + t - 1)u(t_0 + t - 1 + n_k) \\ &\quad - U(t_0 + t, t_0 + t - 1)u(t_0 + t - 1 + n_k)\|. \end{aligned}$$

Now, using the strong continuity of the process $(U(t,s))_{t \geq s}$ and the precompactness of the range of the sequence $(u(n))_{n \in \mathbb{Z}}$, we obtain $\lim_{k \to \infty} C(k) = 0$.

As for $D(k)$, we have

$$\begin{aligned} D(k) &:= \left\| \int_{[t_k+t]+n_k}^{t_k+t+n_k} U(t_k + t + n_k, \eta) f(\eta) d\eta \right. \\ &\quad \left. - \int_{[t_0+t]+n_k-1}^{t_0+t+n_k} U(t_0 + t + n_k, \eta) f(\eta) d\eta \right\| \\ &= \left\| \int_{[t_0+t]+n_k-1}^{t_k+t+n_k} U(t_k + t + n_k, \eta) f(\eta) d\eta \right. \\ &\quad \left. - \int_{[t_0+t]+n_k-1}^{t_0+t+n_k} U(t_0 + t + n_k, \eta) f(\eta) d\eta \right\| \\ &= \left\| \int_0^{t_k+1-t_0} U(t_k + t, t_0 + t - 1 + \theta) f(t_0 + t + n_k - 1 + \theta) d\theta \right. \\ &\quad \left. - \int_0^1 U(t_0 + t, t_0 + t - 1 + \theta) f(t_0 + t + n_k - 1 + \theta) d\theta \right\|. \end{aligned}$$

From the strong continuity of the process $(U(t,s))_{t \geq s}$ and the precompactness of the range of f, it follows that $\lim_{k \to \infty} D(k) = 0$. This finishes the proof of the lemma. □

Theorem 10.4 ([40]) *Let $A(t)$ in Eq. (1.1) generate a 1-periodic strongly continuous evolutionary process, and let f be almost automorphic. Assume further that the space $\mathbb{X}$ does not contain any subspace isomorphic to c_0, and the part of spectrum of the monodromy operator $U(1,0)$ on the unit circle is countable. Then, every bounded mild solution of Eq. (1.1) on the real line is almost automorphic.*

Proof The theorem is an immediate consequence of the results above. In fact, we need only to prove the sufficiency. Let us consider the discrete equation

$$u(n+1) = U(n+1,n)u(n) + \int_n^{n+1} U(n+1,\xi)f(\xi)d\xi, \quad n \in \mathbb{Z}.$$

From the 1-periodicity of the process $(U(t,s))_{t\geq s}$, this equation can be rewritten in the form

$$u(n+1) = Bu(n) + y_n, \quad n \in \mathbb{Z}, \tag{1.8}$$

where

$$B := U(1,0);\ y_n := \int_n^{n+1} U(n+1,\xi)f(\xi)d\xi, \quad n \in \mathbb{Z}.$$

We are going to show that the sequence $(y_n)_{n\in\mathbb{Z}}$ defined as above is almost automorphic. In fact, since f is automorphic, for any given sequence (n'_k), there are a subsequence (n_k) and a measurable function g such that $\lim_{k\to\infty} f(t+n_k) = g(t)$ and $\lim_{m\to\infty} g(t-n_m) = f(t)$ for every $t \in \mathbb{R}$. Therefore, if we set

$$w_n = \lim_{k\to\infty} \int_{n+n_k}^{n+n_k+1} U(n+n_k,\xi)f(\xi)d\xi, \quad n \in \mathbb{Z},$$

then, by the 1-periodicity of $(U(t,s))_{t\geq s}$ and the Lebesgue Dominated Convergence Theorem, we have

$$w_n = \lim_{k\to\infty} \int_n^{n+1} U(n,\eta)f(n_k+\eta)d\eta = \int_n^{n+1} U(n,\eta)g(\eta)d\eta.$$

Therefore, $\lim_{k\to\infty} y_{n+n_k} = w_n$ for every $n \in \mathbb{Z}$. Similarly, we can show that $\lim_{k\to\infty} w_{n-n_k} = y_n$.

By Lemma 2.11 [48], since $(u(n))$ is a bounded solution of (1.8), $\mathbb{X}$ does not contain any subspace isomorphic to c_0, and the part of spectrum of $U(1,0)$ on the unit circle is countable, $(u(n))$ is almost automorphic. By Lemma 10.3, this yields that the solution u itself is almost automorphic. □

Now, let us consider Eq. (1.1) where $A(t) = A$.

Theorem 10.5 ([70]) *Suppose that A generates an asymptotically stable C_0-semigroup $(T(t))_{t\geq 0}$, that is,*

$$\lim_{t\to\infty} T(t)x = 0, \quad \textit{for every}\ \ x \in \mathbb{X},$$

and $f \in AA(\mathbb{X})$*. If* $x(t)$ *is a mild solution of Eq.* (1.1) *with a relatively compact range in* $\mathbb{X}$*, then* $x \in AA(\mathbb{X})$*.*

Proof $x(t)$ will admit the representation

$$x(t) = T(t-a)x(a) + \int_a^t T(t-\xi)f(\xi)d\xi$$

for each $a \in \mathbb{R}$ and $t \geq a$.

Let (s'_n) be a sequence of real numbers. Since $f \in AA(\mathbb{X})$, we can find a subsequence $(s_n) \subset (s'_n)$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well-defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t-s_n) = f(t)$$

for each $t \in \mathbb{R}$.

Suppose now that the sequence $(x(t_0+s_n))$ is not convergent for some $t_0 \in \mathbb{R}$. Then, there exist some $\alpha > 0$ and two subsequences (σ'_n) and (σ''_n) of (s_n) such that

$$\|x(t_0+\sigma'_n) - x(t_0+\sigma''_n)\| > \alpha \tag{1.9}$$

for $n = 1, 2, \ldots$.

We have, for $a \leq t_0$

$$\begin{aligned} x(t_0+\sigma'_n) - x(t_0+\sigma''_n) = {} & T(t_0-a)[x(t_0+\sigma'_n) - x(t_0+\sigma''_n)] \\ & + \textstyle\int_a^{t_0} T(t_0-\xi)[f(\xi+\sigma'_n) - f(\xi-\sigma''_n)]d\xi. \end{aligned}$$

Let $K = \overline{\{x(t) \,/\, t \in \mathbb{R}\}}$ be the closure of the range of $x(t)$; by assumption K is compact in $\mathbb{X}$.

Since $\lim_{t\to\infty} T(t)x = 0$ for every $x \in \mathbb{X}$, it is easy to observe that $\lim_{t\to\infty} T(t)x = 0$ uniformly in any compact subset of $\mathbb{X}$. Thus we can choose some $a < 0$ such that

$$\|T(t_0-a)x(a+\sigma'_n)\| < \frac{\varepsilon}{3}$$

and

$$\|T(t_0-a)x(a+\sigma''_n)\| < \frac{\varepsilon}{3}$$

for all $n = 1, 2, \ldots$ Now, fix a and put

$$F_n(\xi) := T(t-\xi)[f(\xi+s_n') - f(\xi+s_n'')]$$

with $a \leq \xi \leq t_0$. Since $\sup_{t\in\mathbb{R}} \|f(t)\| < \infty$, and $\|T(t_0)\| \leq M$ for some $M > 0$, we get

$$\|F_n(\xi)\| \leq \|T(t_0-\xi)\|(\|f(\xi+s_n')\| + \|f(\xi+s_n'')\|) \leq L$$

for some $L < \infty$.

Also, we observe that $\lim_{n\to\infty} F_n(\xi) = 0$ in the strong sense for every ξ since $\lim_{n\to\infty} f(t+s_n)$ exists for every t, and (σ_n') and (σ_n'') both are subsequences of (s_n). Finally, $F_n(\xi)$ is measurable for each $n = 1, 2, \ldots$

Using Lebesgue's Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_a^t T(t-\xi)[f(\xi+\sigma_n') - f(\xi+\sigma_n'')]d\xi = 0.$$

This implies the existence of some positive integer N such that

$$\|x(t_0+\sigma_n') - x(t_0+\sigma_n'')\| < \varepsilon \text{ if } n > N,$$

which contradicts (1.9).

Consequently, we deduce that the sequence $(x(t+s_n))$ is convergent in $\mathbb{X}$ for $t \in \mathbb{R}$.

Let $y(t) := \lim_{n\to\infty} x(t+s_n), \quad t \in \mathbb{R}$. It follows that

$$y(t) = T(t-a)y(a) + \int_a^t T(t-\xi)g(\xi)d\xi$$

for every $a \in \mathbb{R}$ and $t \geq a$. Moreover, $\Gamma := \{y(t) \,/\, t \in \mathbb{R}\} \subset K$. And consequently, Γ is relatively compact in $\mathbb{X}$. We may assume that

$$\lim_{n\to\infty} y(t-s_n) = u(t)$$

pointwise on $\mathbb{R}$.

Using the same argument as above, we can get

$$u(t) = T(t-a)u(a) + \int_a^t T(t-\xi)g(\xi)d\xi$$

for every $a \in \mathbb{R}$ and $t \geq a$. We also have

$$\{u(t) \ / \ t \in \mathbb{R}\} \subset \overline{\{y(t) \ / \ t \in \mathbb{R}\}} \subset K.$$

It remains to prove that $u(t) = x(t), t \in \mathbb{R}$.

Let us write $y(t - s_n) - x(t) = T(t-a)y(a - s_n) - T(t-a)x(a) + \int_a^t T(t-\xi)(g(\xi - s_n) - f(s))d\xi$.

Fix $t \in \mathbb{R}$, and let $\varepsilon > 0$. Since K is compact, one may choose $a < 0$ large enough such that

$$\|T(t-a)y(a-s_n)\| < \frac{\varepsilon}{3}, \quad \forall n$$

and

$$\|T(t-a)x(a)\| < \frac{\varepsilon}{3}.$$

The second inequality is based on the assumption that $\lim_{t \to \infty} T(t)x = 0$. Now, fix a, and let

$$H_n(\xi) := T(t-\xi)(g(\xi - s_n) - f(\xi)).$$

It is clear that each $H_n(\xi)$, $n = 1, 2, \ldots,$ is bounded in norm since $\sup_{t \geq 0} \|T(t)\| < \infty$ and $\sup_{t \in \mathbb{R}} \|f(t)\| < M < \infty$. By Lebesgue's Dominated Convergence Theorem, we get

$$\lim_{n \to \infty} \int_a^t T(t-\xi)(g(\xi - s_n) - f(\xi))d\xi = 0.$$

We then obtain

$$\|y(t - s_n) - x(t)\| < \varepsilon,$$

if $n > N$ for some given positive integer N. This implies that $x(t) = u(t)$ for each $t \in \mathbb{R}$. The proof is complete. □

Theorem 10.6 (N'Guérékata [55]) *Assume that A is bounded and $f \in AA(\mathbb{X})$. Let $x(t)$ be a (strong) solution of Eq. (1.1) with a relatively compact range in $\mathbb{X}$. Assume also that there exists a finite dimensional subspace $\mathbb{X}_1$ of $\mathbb{X}$ with the properties:*

(i) $e^{tA}u \in \mathbb{X}_1, \quad \forall u \in \mathbb{X}_1$;
(ii) $Ax(0) \in \mathbb{X}_1$;
(iii) $(e^{tA} - I)f(s) \in \mathbb{X}_1 \quad \forall t, s \in \mathbb{R}$.

Then, $x \in AA(\mathbb{X})$.

Proof Consider the projection $P : \mathbb{X} \to \mathbb{X}_1$. Then, we have $\mathbb{X} = \mathbb{X}_1 \oplus N(P)$, where $N(P)$ is the null space of P. Note that $Q = I - P$ is the projection on $N(P)$. Both P and Q are bounded linear operators.

Let $x(t)$ be a solution of Eq. (1.1). Then, we can write

$$x(t) = x_1(t) + y(t), \quad t \in \mathbb{R},$$

where $x_1(t) = Px(t) \in \mathbb{X}_1$ and $y(t) = Qx(t) \in N(P)$.

Since the range of $x(t)$ is relatively compact in $\mathbb{X}$, so are the ranges of $x_1(t)$ and $y(t)$ as we can easily observe. Also,

$$x'(t) = x_1'(t) + y(t) = Ax_1(t) + Ay(t) + Pf(t) + Qf(t), \quad t \in \mathbb{R}. \tag{1.10}$$

$x(t)$ has the integral representation

$$\begin{aligned} x(t) &= e^{tA}x(0) + \textstyle\int_0^t e^{(t-\xi)A} f(\xi)d\xi \\ &= e^{tA}x(0) + \textstyle\int_0^t f(\xi)d\xi + \int_0^t (e^{(t-\xi)A} - I) f(\xi)d\xi. \end{aligned}$$

Using assumption (iii), we can deduce that $\int_0^t (e^{(t-\xi)A} - I) f(\xi)d\xi \in \mathbb{X}_1$; then, applying Q to both sides of the last equation above, we get

$$y(t) = Qe^{tA}x(0) + Q\int_0^t f(\xi)d\xi = Qe^{tA}x(0) + \int_0^t Qf(\xi)d\xi.$$

Thus,

$$y'(t) = QAe^{tA}x(0) + Qf(t) = Qf(t)$$

since $Ax(0) \in \mathbb{X}_1$, so $e^{tA}Ax(0) \in \mathbb{X}_1$ by (ii).

Now, $Qf(t) : \mathbb{R} \to \mathbb{X}$ is an almost automorphic function since Q is a bounded linear operator. Hence, $y'(t) \in AA(\mathbb{X})$. Thus, $y(t) \in AA(\mathbb{X})$ since its range is relatively compact in $\mathbb{X}$ in view of Theorem 4.3.

Now, if we apply P to both sides of Eq. (1.10), we get in $\mathbb{X}_1$ the following equation:

$$x_1'(t) = PAx_1(t) + PAy(t) + P^3 f(t) + PAf(t), \quad t \in \mathbb{R}.$$

We observe that the function $g(t) := PAy(t) + P^3 f(t) + PAf(t)$ is almost automorphic.

Now, the operator PA restricted to the subspace $\mathbb{X}_1$ is a matrix and the function $x_1(t)$ is bounded since its range is relatively compact. So, we deduce that it is almost automorphic in view of Theorem 4.5.

Finally, $x(t) \in AA(\mathbb{X})$ as the sum of two almost automorphic functions.

The proof is complete. □

Now we consider in a general Banach space $\mathbb{X}$, the equation:

$$x'(t) = (A + B)x(t), \quad t \in \mathbb{R} \tag{1.11}$$

and the associated inhomogeneous one

$$x'(t) = (A + B)x(t) + f(t), \quad t \in \mathbb{R}. \tag{1.12}$$

.

We make the following assumptions:

(i) A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ such that $T(t)x : \mathbb{R} \to \mathbb{X}$ is almost automorphic for each $x \in \mathbb{X}$
(ii) There exists a finite dimensional subspace $\mathbb{X}_1$ of $\mathbb{X}$ such that $D(A) \cap \mathbb{X}_1$ is dense in $\mathbb{X}$.
(iii) The projection $P : \mathbb{X} \to \mathbb{X}_1$ commutes with A.
(iv) B is a continuous linear operator such that $B(\mathbb{X}) = \mathbb{X}_1$.

Theorem 10.7 (N'Guérékata [55]) *Under assumptions (i)–(iv), every bounded solution of Eq. (1.11) is almost automorphic.*

Proof We recall that P is a bounded linear operator and has the property

$$\mathbb{X} = \mathbb{X}_1 \oplus N(P),$$

where $N(P)$ is the kernel of P. We set $Q = I - P$.

Now, if $x(t)$ is a bounded solution of Eq. (1.11), then it can be decomposed as

$$x(t) = x_1(t) + x_2(t) \quad t \in \mathbb{R},$$

where $x_1(t) = Px(t) \in \mathbb{X}_1$ and $x_2(t) = Qx(t) \in N(P)$ are also bounded.

First, let us show that $x_2(t) \in AA(\mathbb{X})$. We have

$$\begin{aligned} x_2'(t) = \tfrac{d}{dt}Qx(t) = Q\tfrac{d}{dt}x(t) &= Q(A + B)x(t) \\ &= QAx(t), \text{ since } QBx(t) = 0 \\ &= AQx(t), \text{ since } A \text{ and } Q \text{ commute} \\ &= Ax_2(t). \end{aligned}$$

Thus, we can write $x_2(t) = T(t)x_2(0), \quad t \in \mathbb{R}$, which shows that $x_2(t) \in AA(\mathbb{X})$.

Now, if we apply P to Eq. (1.11) and use the commutativity of A and P, we obtain

$$x_1'(t) = (A + PB)x_1(t) + PBx_2(t),$$

where $g(t) := PBx_2(t) \in AA(\mathbb{X})$.

It is clear that $A + PB = A + B$ is a linear operator restricted to $D(A) \cap \mathbb{X}_1 \equiv \mathbb{X}_1$ because of assumption (ii). Since $x_1(t)$ is bounded, it is almost automorphic (Theorem 4.5). Finally, $x(t) \in AA(\mathbb{X})$ as the sum of two almost automorphic functions. The proof is complete. □

Theorem 10.8 *Assume that assumptions (i)–(iv) above are satisfied and* $f \in AA(\mathbb{X})$*. Then, every solution of Eq. (1.12) with a relatively compact range is almost automorphic.*

Proof We start the proof as in Theorem 10.7 with the same notations. Consider a solution $x(t)$ of Eq. (1.12) with a relatively compact range in $\mathbb{X}$, and let

$$x(t) = x_1(t) + x_2(t), \quad t \in \mathbb{R},$$

as above. Observe that the range of $x_2(t)$ is also relatively compact in $\mathbb{X}$. It is easy to check that it satisfies the following equation in $N(P)$:

$$x_2'(t) = Ax_2(t) + Qf(t), \quad t \in \mathbb{R}.$$

The function $Qf(t) : \mathbb{R} \to N(P)$ is almost automorphic since Q is bounded. We deduce that $x_2(t) \in AA(\mathbb{X})$ in view of Theorem (4.3).

In applying P to Eq. (1.12), we obtain in the finite dimensional space $\mathbb{X}_1$ the equation

$$x_1'(t) = (A + PB)x_1(t) + g(t),$$

where $g(t) := PBx_2(t) + Pf(t)$ is an almost automorphic function $\mathbb{R} \to \mathbb{X}_1$. As in Theorem 10.7, $A + PB = A + B$ on $D(A) \cap X_1$. Now, since $x_1(t)$ has a relatively compact range and thus it is bounded in the finite dimensional space $\mathbb{X}_1$, we conclude in view of Theorem 4.5 that it is almost automorphic. Finally, $x(t)$ is almost automorphic as the sum of two almost automorphic functions. The proof is now complete. □

Bibliographical Notes This section is essentially based on the contributions by Zaki, Zaidman, and N'Guérékata. Theorem 10.5 is a slight generalization of a result by Zaki [70]. The results by N'Guérékata and Pankov in [61] provide a nontrivial interplay between Bohr almost periodicity, Besicovich almost periodicity, and almost automorphy of solutions of some almost periodic elliptic equations.

The interaction between the spectral theory of functions and mild solutions of evolution equations with almost automorphic forcing terms was initiated by

Diagana, N'Guérékata, and Nguyen Van Minh in [23] using the concept of uniform spectrum; cf. also [48, 49]. That interaction brings more operator theory, harmonic analysis' ideas, and complex functions to these differential equations. The reader can obtain a complete presentation of results in this direction in the recent book by N'Guérékata [59].

Chapter 11
Almost Periodic Solutions of the Differential Equation in Locally Convex Spaces

As an application of results obtained in Chap. 3, we will study conditions for almost periodicity of solutions of the linear differential equation $x'(t) = Ax(t) + f(t), \ t \in \mathbb{R}$ and the associated homogeneous equation in locally convex spaces. We will start with the case of a bounded linear operator A and then study the general case of a (eventually unbounded) linear operator A which generates an equicontinuous C_0-semigroup of linear operators.

1 Linear Equations

Definition 11.1 A Fréchet space E is said to be perfect if it satisfies the following property:

(F) Every function $f : \mathbb{R} \to E$ with a bounded range $\{f(t) : \ t \in \mathbb{R}\}$ and $f'(t)$ almost periodic is necessarily almost periodic.

Example 11.2 Denote by s the linear space of all real numbers $s = \{x = (x_n) : x_n \in \mathbb{R}, \ \ n = 1, 2, \ldots\}$. For each n, define

$$p_n(x) := |x_n|, \quad x \in s.$$

Obviously p_n is a seminorm defined on s. Define

$$q_n := p_1 \vee p_2 \vee \ldots \vee p_n, \quad n = 1, 2, \ldots$$

We have $q_n \leq q_{n+1}, \quad n = 1, 2, \ldots$ The space s considered with the family of seminorms (q_n) is a Fréchet space. Moreover, it can be proved (see [1], 17.7, p. 210) that each closed and bounded subset of s is compact. Thus, in particular, s is a Banach space. Moreover, in view of Theorem 8.12, s is perfect.

G. M. N'Guérékata, *Almost Periodic and Almost Automorphic Functions in Abstract Spaces*, https://doi.org/10.1007/978-3-030-73718-4_11

1.1 The Homogeneous Equation $x' = Ax$

Consider in a complete Hausdorff locally convex space E the equation

$$x'(t) = Ax(t), \ t \in \mathbb{R}. \tag{1.1}$$

Theorem 11.3 ([51]) *Let E be a perfect Fréchet space. Assume that*

(i) A is a compact linear operator on E;
(ii) $\{A^k : \ k = 1, 2, \ldots\}$ is equicontinuous;
(iii) for every seminorm p, there exists a seminorm q such that

$$p(e^{tA}x) \le q(x), \ \ \textit{for every } t \in \mathbb{R}, \ x \in E.$$

Then, the unique solution of Eq. (1.48) is almost periodic in E.

Proof Let $x(t) = e^{tA}x_0$ be the unique solution of Eq. (1.1). Then, by (iii), it is bounded. Since E is a perfect Fréchet space, it suffices to prove that $x'(t)$ is almost periodic (cf. Property (F) above).

Now, assumption (i) implies that the set $\{x'(t) : \in \mathbb{R}\}$ is also relatively compact in E.

Let (s'_n) be an arbitrary sequence of real numbers; then, we can extract a subsequence (s_n) such that $(x'(s_n))$ is convergent and thus is a Cauchy sequence in E.

But we have

$$\begin{aligned} x'(t + s_n) &= Ax(t + s_n) \\ &= Ae^{(t+s_n)A}x_0 \\ &= Ae^{tA}e^{s_nA}x_0 \\ &= Ae^{tA}x(s_n) \\ &= e^{tA}Ax(s_n) \\ &= e^{tA}x'(s_n) \end{aligned}$$

for every $n = 1, 2, \ldots$ and every $t \in \mathbb{R}$.

If p is a seminorm on E, then there exists a seminorm q on E such that

$$\begin{aligned} p(x'(t + s_n) - x'(t + s_m)) &= p(e^{tA}(x'(s_n) - x'(s_m))) \\ &\le q(x'(s_n) - x'(s_m)) \end{aligned}$$

for every n, m and every $t \in \mathbb{R}$.

It follows that $(x'(t + s_n))$ is uniformly Cauchy in t; therefore, it is uniformly convergent in t.

We conclude by Bochner's criterion that $x'(t)$ is almost periodic. This completes the proof. □

1.2 The Inhomogeneous Case

Now, let us consider in a perfect Fréchet space E the inhomogeneous equation:

$$x'(t) = Ax(t) + f(t), \quad t \in \mathbb{R}, \tag{1.2}$$

with the following assumptions:

(H1) A is a compact linear operator on E;
(H2) $\{A^k;\ k = 1, 2, \ldots\}$ is equicontinuous;
(H3) for every seminorm p_n, there exists a seminorm q_n such that

$$p_n(e^{tA}x) \leq q_n(x), \quad \text{for every } t \in \mathbb{R},\ x \in E;$$

(H4) $f \in AP(E)$ and for each $p_n \in \mathcal{P}$, there exists a function $\psi_{p_n} : \mathbb{R} \to \mathbb{R}^+$ with

$$p_n(f(s)) \leq \psi_{p_n}(s) \ \text{ and } \ \int_{-\infty}^{\infty} \psi_{p_n}(s)ds < \infty.$$

Now, let us state and prove the following.

Theorem 11.4 ([16]) *Under assumptions (H1)–(H4), every solution of Eq. (1.2) is almost periodic in E.*

Proof By Theorem 11.3, the function $e^{tA}x(0) \in AP(E)$.

Now, let $F(t) := \int_0^t e^{(t-s)A} f(s)ds$. It is also immediate that $s \to e^{-sA} f(s)$ is in $AP(E)$ based on Theorem 8.26. In view of assumptions (H5) and (H8), $F(t)$ is bounded over $\mathbb{R}$. We then deduce that $\int_0^t e^{-sA} f(s)ds \in AP(E)$ since E is a perfect Fréchet space. But this last integral is equal to $e^{-tA}F(t)$. Applying again Theorem 8.26, we obtain $e^{tA}(e^{-tA}F(t)) = F(t)$ is almost periodic. The proof is now complete. □

Let us now consider a Fréchet space in which property (F) may not hold.

Theorem 11.5 ([51, 55]) *Let E be a Fréchet space (not necessarily perfect), and assume that assumptions (i)–(iii) of Theorem 11.3 are satisfied. Assume also that the range $\mathcal{R}(A)$ of the operator A is dense in E.*

Then, every solution of Eq. (1.1) is almost periodic.

Proof Let us observe that the first part of the proof of Theorem 11.3 tells us that if $x(t) = e^{tA}x_0$ is a solution of Eq. (1.48) with $x_0 \in D(A)$ the domain of A, then $x'(t)$ will be almost periodic. □

Lemma 11.6 *Every solution of Eq. (1.48) with initial data in $\mathcal{R}(A)$, the range of A, is almost periodic.*

Proof Let $a \in \mathcal{R}(A)$ and consider the unique solution $y(t)$ with $y(0) = a$. There exists $x_0 \in D(A)$ such that $Ax_0 = a$. We have

$$y(t) = e^{tA}a = e^{tA}Ax_0 = Ae^{tA}x_0 = Ax(t) = x'(t),$$

where $x(t) = e^{tA}x_0$. Therefore, $x'(t)$, and consequently $y(t)$, is almost periodic.

□

Proof (of Theorem 11.5 (continued)) Consider the solution $x(t)$ of Eq. (1.48) with $x(0) \in E$. Since $\mathcal{R}(A)$ is dense in E, there exists a sequence (a_n) in $\mathcal{R}(A)$ such that

$$\lim_{n\to\infty} a_n = x(0).$$

Consider a sequence of solutions $(y_n(t))$ with $y_n(0) = a_n, \quad n = 1, 2, \ldots$ By the above, each $y_n(t)$ is almost periodic. □

Now, to prove almost periodicity of $x(t)$, it suffices to prove that $(y_n(t))$ converges to $x(t)$ uniformly in $t \in \mathbb{R}$. We have

$$x(t) = e^{tA}x(0), \quad y_n(t) = e^{tA}a_n.$$

So, given a seminorm p, there exists a seminorm q by (iii) such that

$$p(y_n(t) - x(t)) = p(e^{tA}(a_n - x(0))) \leq q(a_n - x(0))$$

for every $n = 1, 2, \ldots$ and every $t \in \mathbb{R}$. The conclusion follows immediately.

Bibliographical Notes The contributions in this chapter are due to D. Bugajewski and G.M. N'Guérékata [16, 55].

Appendix

The function spaces can be arranged in the following chart:

	Type		Asymptotic		Pseudo
Almost Automorphic	AA	$\subset$	AAA	$\subset$	PAA
	$\cup$		$\cup$		$\cup$
Compact Almost Automorhic	AA_c	$\subset$	AAA_c	$\subset$	PAA_c
	$\cup$		$\cup$		$\cup$
Almost Periodic	AP	$\subset$	AAP	$\subset$	PAP
	$\cup$		$\cup$		$\cup$
ω-Periodic	P_ω	$\subset$	AP_ω	$\subset$	PP_ω
			$\cap$		
			SAP_ω		

G. M. N'Guérékata, *Almost Periodic and Almost Automorphic Functions in Abstract Spaces*, https://doi.org/10.1007/978-3-030-73718-4

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