A Quick Criterion for Calculating Waiting Phenomena at Intersections

Raffaele Mauro, Marco Guerrieri, and Andrea Pompigna

Abstract Calculations of queues length and waiting times at intersections are essential to evaluate the quality of circulation at road junctions (Level of Service, LOS). These calculations are carried out with the theory of waiting phenomena (probabilistic and/or deterministic queue theory) and different models are adopted, depending on whether the operating conditions of the traffic are stationary or not. In technical practice for some time, both for sub-saturation and over-saturation situations for the intersection arms, the formulations of the time-dependent queues obtained with the so-called criterion of coordinates have been used. Depending on the degree of saturation of an input arm (traffic intensity), this criterion allows the transition from probabilistic solutions to deterministic ones. In the paper, after a brief review of time-dependent solutions, a quick criterion is provided for calculating the length of queues and waiting times in the event of peak traffic—as well as the duration of the effects of the latter—obtained under specific characteristics of the arrival processes at the intersection; a demonstration is given of how this criterion leads to solutions conforming to the deterministic type; estimates of the errors, which arise from the criterion developed in this paper to replace a time-dependent formulation, are provided in terms of confidence intervals with varying the degree of saturation.

Keywords Queuing theory · Non-stationary queues · Time-dependent queues · Renewal processes · Unsignalized intersections

1 Introduction

In Fig. 1 we consider the basic case of a road intersection in which only two traffic streams interact with each other. The major flow Q on the major street crosses the intersection, while the minor flow q on the minor street turns right.

Q has priority (priority flow) over q (non-priority flow). For this rule of priority of Q over q, the vehicles of flow q can form a queue. The interaction between q and Q can be modeled with the simplest of the queue models [1]. The waiting system of

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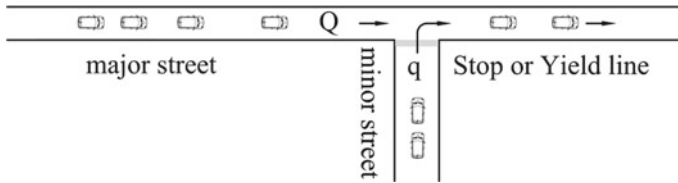
Fig. 1 Example of traffic flows with different priority at an intersection

Fig. 1 has a single service point (at Stop or Yield line) and a single serving channel (minor street). The discipline of the system service is First In–First Out (FIFO). The intersection of Fig. 1 can be studied with the mathematical queue theory [2, 3]. The theory of queues includes deterministic and probabilistic models to calculate waiting system states. Generally, the state variables $E[L_q]$ and $E[W_q]$ are used. $E[L_q]$ is the average length of the queue. $E[W_q]$ is the average waiting time in the queue. Other state variables are also used, which are deduced directly from $E[L_q]$ and $E[W_q]$. In a waiting system, inputs and outputs are sequences of events over time. The incoming flows are the arrivals in the system. The outgoing flows are the departures from the system (from the queue). Deterministic solutions are used when the count $x_t = A(t)$ of arrivals and the count $y_t = D(t)$ of departures are regular ($VAR[x_t] = VAR[y_t] = 0$) [4]. The cumulated functions $A(t)$ and $D(t)$ are step-functions of time t, with inter-arrival and departure times of constant amplitude over certain time intervals. So if we interpolate these step-functions with continuous functions of time, the rates $dA(t)/dt$ and $dD(t)/dt$ do not have random fluctuations. In deterministic solutions $E[L_q]$ and $E[W_q]$ are functions of time, like the other state variables. Thus, deterministic solutions are called time-dependent solutions. The probabilistic solutions of queue theory are used for random arrivals and departures [2, 3]. In this case, the queue length values L_q in the time form the random processes of queue. Every value L_q has its own probability of occurring. If the probability law of L_q does not change over time, evidently also $E[L_q]$ and $E[W_q]$ do not change over time. In this case, the waiting system is stationary. If the probability law of the queue length L_q varies instant by instant, waiting system is not in a steady state, but in a transitory state. Probabilistic solutions for non-stationary states are time-dependent solutions because they are obtained by probability law of L_q as time functions. In general, probabilistic solutions for stationary conditions are very simple mathematical expressions and $E[L_q]$ and $E[W_q]$ can be calculated easily. On the contrary, the solutions for non-stationary conditions are not simple mathematical expressions and the calculation of $E[L_q]$ and $E[W_q]$ is also not immediate. Thus, probabilistic time-dependent solutions are of little use in practical applications and heuristic solutions have been obtained for queues in transient states.

2 Operating Conditions of Intersections

In Highway engineering, the system shown in Fig. 1 is of significant interest. All the methods for calculating the performances for the unsignalized intersections are derived from the simple waiting system of Fig. 1. The operating conditions of the intersections depend on the traffic demand that must be served (Q and q of Fig. 1). It is intuitive that this intersection is in a steady state if Q and q are constant during an infinite time interval. If c is the capacity of the turn right maneuver by the vehicles of q, for the steady state it must be $q < c$. In all the other conditions of Q, q and c, the intersection is not in a steady-state condition. In general, c depends on Q, on the geometric layout of the intersection G and on human factor parameters θ of the drivers at intersections, i.e. $c = f(Q, G, \theta)$. Steady-state conditions are not realistic in actual traffic operating conditions because traffic demand always varies over time. Therefore, also Q and q always fluctuate during the different time periods of the day. Moreover, for $\rho > 0.8 - 0.6$, the queue length and the waiting times by stationary solutions tend to become infinite [5]. ρ is the traffic intensity, i.e. is the ratio between the average service time $1/c$ and the average interval between two arrivals $1/q$. This result is clearly unrealistic, because the time intervals with $\rho > 0.8 - 0.6$ are of finite length, and so the queue cannot grow indefinitely. In technical practice, in steady-state conditions but with $\rho > 0.8 - 0.6$ and in no steady-state conditions, heuristic solutions [4] are usefully used to calculate the state variables of the waiting system of Fig. 1. As mentioned, the non-stationary probabilistic solutions are too complicated to be easily implemented in the calculations. Heuristic solutions are also time dependent, since they relate to non-stationary conditions. These heuristic solutions are obtained with the method of the coordinates transformation [6] from the stationary solutions tending asymptotically to the deterministic solutions. The heuristic solutions $E[\bullet] = f(\rho)$ are continuous functions of ρ. ρ continuously assumes values in the interval $[0, +\infty]$, so all the possible operating conditions of the intersection in Fig. 1 can be analyzed in a unitary way from a low traffic intensity ($\rho \ll 1$) to a congested traffic situation ($\rho \geq 1$) (Fig. 2), as in the case of traffic peak (Fig. 3).

Heuristic solutions also allow the study of the effects of traffic peaks. However, if traffic peaks involve high levels of congestion, arrivals in the queue and departures from the queue are less random and more regular. Thus, with increasing congestion ($\rho \to +\infty$) the heuristic solutions tend to asymptotically coincide with the deterministic solutions. In operational terms the intersection of Fig. 1 is congested if $q > c$ ($\rho = q/c > 1$), or if $q \leq c$ ($\rho = q/c \leq 1$) but the queue evolves in the time from an already long extension.

To study the effects of traffic peaks at intersections, or in other traffic waiting systems, other solutions for peak traffic (also asymptotic formulations) can be used. These formulations have simpler mathematics than heuristic formulations. Furthermore, there are interesting mathematical relationships for asymptotic and deterministic formulations for congested traffic. These issues are covered in the following points of this paper.

Fig. 2 Stationary, deterministic and heuristic solutions for continuous functions $E[\bullet] = f(\rho)$
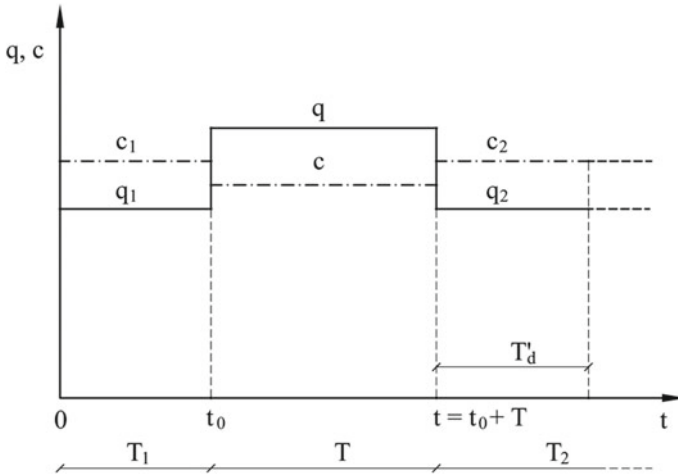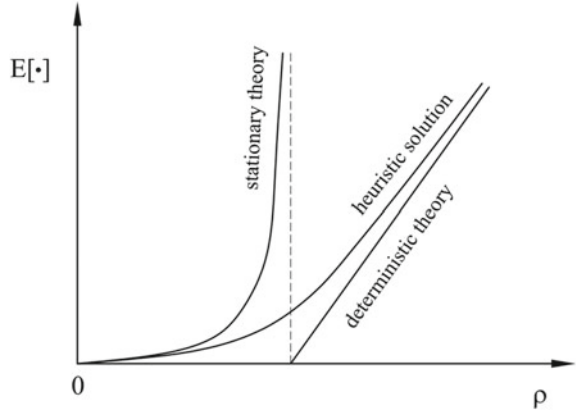
Fig. 3 Example case of a traffic peak

3 Approximate Solutions for Traffic Peaks

To analyze the intersection in Fig. 1 the state variables are:

- L number of vehicles in the system, i.e. waiting on the minor road. If the system is not empty, L is the number of vehicles in the queue L_q plus the vehicle in first position (i.e. Yield or Stop line):

$$L = L_q + 1 \tag{1}$$

- $\tau_y = s$ service time. s is the time spent by the vehicle in first position waiting to perform the maneuver;

- W_q time spent in queue. W_q is the time that passes between the instant in which a vehicle joins the queue and the instant in which it occupies the first position;
- W waiting time in the system. W is the total waiting time spent in the waiting line:

$$W = W_q + s \tag{2}$$

L_q, L, W_q, W and s are random variables. The mean values $E[L]$ and $E[W]$ are used in applications as state variables for the system. $E[L]$ and $E[W]$ are preferred to the mean values $E[L_q]$ and $E[W_q]$ for L_q and W_q because L and W contain more information than for L_q and W_q. Let be L at $t = 0$, L_0, $L_0 = 0$ (no vehicles initially observed in the system for a generic time interval $[0, t]$). To obtain the L equation (as function of time t), a balance equation can be written between the number x_t of vehicles arriving in the queue and the number y_t of vehicles leaving the queue at the end of a generic time interval $[0, t]$:

$$L_t = x_t - y_t \tag{3}$$

Now, let be the system characterized by a continuous and long queue with respect to the single vehicle. Also let be waiting times long if compared with the average service times. In this case, it may be appropriate and useful to replace these discontinuous trends with continuous functions that allow a "smoothing" over time. If x_t and y_t assumes sufficiently large values with respect to the unit, therefore, only small relative variations with respect to the average value can be expected. In these conditions, the negligible variations with respect to the average value allow to use a so-called first order approximation for x_t and y_t. Thus the continuous time and discrete valued random processes x_t and y_t are replaced with $A(t)$ and $D(t)$. $A(t)$ and $D(t)$ are continuous-time and continuous valued deterministic processes. Therefore, within this approximation and with $L_0 = 0$ we obtain:

$$L_t^* = A(t) - D(t) \tag{4}$$

where also L_t^* is a continuous and deterministic function of time. If $1/\tau_x(t) = q_t = dA(t)/dt$ and $1/\tau_y(t) = c_t = dD(t)/dt$, we have that $A_t = \int_0^t q(u)du$ and $Dt = \int_0^t c(u)du$. Assuming that $q_t = q$ and $c_t = c$ have constant values in the interval $[0, t]$, in the more general case of non-zero initial queue L_0 we obtain:

$$L_t^* = L_0 + t(q-c) = t(\rho - 1)c \tag{5}$$

L_t^* represents the first order (or deterministic) fluid approximation for L_t [2, 3]. This approximation (i.e. $L_t \approx L_t^*$) only considers the accumulation that occurs due to the saturation of the system. In compliance with the Law of Large Numbers, its degree of approximation increases with increasing ρ over 1 and up to $+\infty$, as the basic hypotheses relating to the negligibility of the variations with respect to the average values are more sustainable. Still operating in the stochastic field, the

problem of the peak of traffic can be tackled using the Renewal Theory [7]. Taking into account the balance Eq. (3) with a non-zero initial queue and under the same hypotheses of the fluid approximation, in terms of expected value we can write:

$$E[L_t] = E[L_0] + E[x_t] - E[y_t] \tag{6}$$

If the intersection is congested in [0, t] the queue is always present on the minor street. Under such conditions it is very likely that in the interval [0, t] the system is always busy. In this case, the times of departure of the vehicles that leave the system can be considered with the same probability distribution as the service times. The departure times are therefore independent from the arrival times so:

$$VAR[L_t] = VAR[L_0] + VAR[x_t] + VAR[y_t] \tag{7}$$

The renewal processes theory allows to express the values $E[\bullet]$ and $VAR[\bullet]$ for x_t and y_t, and finally for L_t. By a renewal processes, if vehicles arrive in the system and leave the system continuously (i denotes a generic vehicle), this processes can be interpreted as a sequence of replacements/substitutions with random times of replacements $\tau_{y,i}$ and random time of substitutions $\tau_{x,i}$, Let the random variables $\tau_{y,i}$ and $\tau_{x,i}$ be independent and identically distributed respectively with mean μ_y and μ_x and variance σ^2_y and σ^2_x. It has been widely demonstrated that, whatever the distribution of $\tau_{y,i}$ and $\tau_{x,i}$, the realizations over time of the counting processes of departures y_t and arrivals x_t asymptotically follows (i.e. $t \rightarrow +\infty$) a normal distribution respectively with mean t/μ_y and t/μ_x and variance $t(\sigma^2_y/\mu^3_y)$ and $t(\sigma^2_x/\mu^3_x)$. With $C_y = \sigma_y/\mu_y$ ($C_x = \sigma_x/\mu_x$) we can write that for $t \rightarrow +\infty$:

$$y_t \sim N\left(t/\mu_y; \ C_y^2 \ t/\mu_y\right) \tag{8}$$

$$x_t \sim N(t/\mu_x; \ C_x^2 \ t/\mu_x) \tag{9}$$

Assuming that the parameters of the distributions do not vary over time, with $1/\mu_x = q$ and $\mu_y = E[s]$ we have:

$$y_t \sim N\left(t/s; \ C_y^2 \ t/E[s]\right) \tag{10}$$

$$x_t \sim N\left(t \ q; \ C_x^2 \ t \ q\right) \tag{11}$$

In view of this, Eqs. (6) and (7) approximate asymptotically with the following equations:

$$E[L_t] \sim E[L_0] + qt - t/E[s] = E[L_0] + t/E[s](\rho - 1) \tag{12}$$

$$VAR[L_t] \sim VAR[L_0] + C_x^2 t \ q + C_y^2 t/E[s] \tag{13}$$

The renewal processes theory proves that this asymptotic approximation for $t \rightarrow +\infty$ holds with good approximation if $t > \mu_y/C^2_y$ and $t > \mu_x/C^2_x$ [8]. It should be noted that the same result as in Eqs. (12) and (13) for the approximation with $\rho > 1$ and $t \rightarrow +\infty$ arises from a diffusive or second-order approach [2, 3], which proposes to approximate x_t and y_t (considered independent under the usual hypotheses of extended queue) as normally distributed, in accordance with the Central Limit Theorem [3]. An important result is proven for exponential distributions of arrival and departure inter-times (i.e. x_t and y_t realization of counting processes with Poissonian arrivals and departures). With reference to the system in Fig. 1, this is an M/M/1/∞/FIFO type queue system. In this case $C_y = C_x = 1$ and the approximations for $\rho > 1$ (and asymptotically $\rightarrow +\infty$) are exact for each value of t [8, 9]. In this case, in fact, y_t and x_t follow two Poisson distributions with mean $t\mu_y = t/E$ [s] and $t\mu_x = t\, q$. It turns out that:

$$E[L_t] = E[L_0] + qt - t/E[s] = E[L_0] + t(\rho - 1)/E[s] \qquad (14)$$

$$VAR[L_t] = VAR[L_0] + t\, q + t/E[s] = VAR[L_0] + t(\rho + 1)/E[s] \qquad (15)$$

4 Traffic Peaks Solution for Random Arrivals and Exponential Service Times

4.1 Queue Growth in Saturation Conditions

In the following, the simple traffic situation of Fig. 3 is considered for the intersection of Fig. 1. In Fig. 3 the minor flow $q_1 = q_2$ are constant and less than the capacities $c_1 = c_2$ before and after the peak period T. During the same peak period the flow q is greater than the flows before and after this period. Furthermore q during the peak period T exceeds capacity c. So in the interval T, i.e. between instants t_0 and $t_0 + T$, the intersection of Fig. 1 is subject to a traffic peak.

In this way the waiting system of Fig. 1 can be considered in a steady state at the instant t_0 starting from which the peak demand q occurs. The effects of the traffic peak on the intersection are affected beyond T, for a time interval T'_d starting from $t = t_0 + T$ and included in T_2. For the situation in Fig. 3, if L_0 is the steady state queue at the instant $t = 0$ before the start of the traffic peak, for Poissonian arrivals with rate q and service times distributed exponentially with mean E [s] ($\rho_0 = q_1/c_1$), we have that:

$$E[L_t] = \rho_0/(1 - \rho_0) + t(\rho - 1)/E[s] \qquad (16)$$

$$VAR[L_t] = \rho_0/(1 - \rho_0)^2 + t(\rho + 1)/E[s] \qquad (17)$$

Equation (16) allows to obtain the expected value of the waiting time in the system W_t of a vehicle arriving at the instant t. This vehicle will see L_t vehicles waiting in queue, so neglecting that the vehicle currently in service at time t may be partially served, its waiting time S_{Lt} is equal to the sum of L_t exponential service times with mean $E[s]$. As is demonstrated for renewal processes [7], S_{Lt} is distributed according to an Erlang variable of parameters L_t and $1/E[s]$, with expected value $L_t E[s]$ and variance $L_t (E[s])^2$. For the hypotheses considered above, the expected value and the variance of the waiting time W_t of the vehicle arriving at the instant t conditioned with respect to the presence of total L_t vehicles in the system can be approximated as follows [9]:

$$E[W_t | L_t] \approx E[L_t] E[s] \tag{17}$$

$$VAR[W_t | L_t] \approx E[L_t](E[s])^2 \tag{18}$$

In this way the expected value for waiting time W_t, considering a vehicle arriving at the instant t (i.e., at the end of the peak period) is:

$$E[W_t] = E[s]\rho_0/(1 - \rho_0) + t(\rho - 1) \tag{19}$$

Furthermore, considering that [9]:

$$VAR[W_t] = E[L_t] VAR[W_t | L_t] + VAR[L_t] E[W_t | L_t] \tag{20}$$

using the previous equations we also obtain the approximation for the variance of W_t.

4.2 Saturation Queue Discharge

With reference to Fig. 3, we want to find the duration of the time interval T'_d starting from $t = t_0 + T$ and included in T_2. In this interval, the effects of peak traffic that arose during the T interval are exhausted. If $L_e = \rho_2/(1 - \rho_2)$ is the steady state number of vehicle in the system with traffic demand q_2 and capacity c_2 ($\rho_2 = q_2/c_2$), using Eqs. (16) and (17) we can approximate $E[T'_d]$ and $VAR[T'_d]$. After some calculations, we have that [9]:

$$E[T'_d] = (E[L_T] - L_e)/(c_2(1 - \rho_2)) \tag{21}$$

$$VAR[T'_d] = 2\rho_2(E[L_T] - L_e)/(c_2^2(1 - \rho_2)^3) \\ + (E[L_T] + VAR[L_T])/(c_2^2(1 - \rho_2)^2) \tag{22}$$

5 Comparisons Between Approximate and Heuristic Solutions

Firstly we propose some comparison tests for the queue growth between Approximate solutions (cf. 3.1) with Heuristic solutions (cf. 2). We refer to Kimber and Hollis [6] Heuristic formulation as proposed by [4, 10] with:

$$L_t = \frac{1}{2}\left[\sqrt{A^2 + B} - A\right] \tag{23}$$

where

$$A = (1 - \rho)ct + 1 - L_0 \tag{24}$$

$$B = 4(L_0 + \rho ct) \tag{25}$$

For the intersection of Fig. 1 we consider the traffic condition of Fig. 3, but in this case with $\rho_1 = 0$, therefore $L_0 = 0$. L_t is calculated with Eq. (16) and with Eqs. (23)–(25) for ρ ranging between 1.0 and 2.0, and $T = 10$ min. It is considered $1/E[s]$ $= c = 0.278$ veh/s (1000 veh/h). Also, Fig. 4 shows the upper limit of Čebyšëv (*CSL*) interval at 85% for Eq. (16), considering the variance value obtained using Eq. (17).

The comparison shows the deviations between Heuristic and Approximate solutions which are more and more reduced with increasing the traffic congestion. The deviation values appear to be extremely low compared to CSL at 85%. For the analyzed case study, Fig. 5 shows the percentage deviation (plotted in logarithmic scale) between Heuristic and Approximate solution related to T in the range from 1 to 60 min. It is worth pointing out that in the time intervals usually used for the performance analysis of road intersections, the deviations between the two time-dependent solution types, already for $\rho = 1.2$ are less than 8 and 2.5% in case of T $= 15$ min and $T = 60$ min respectively.
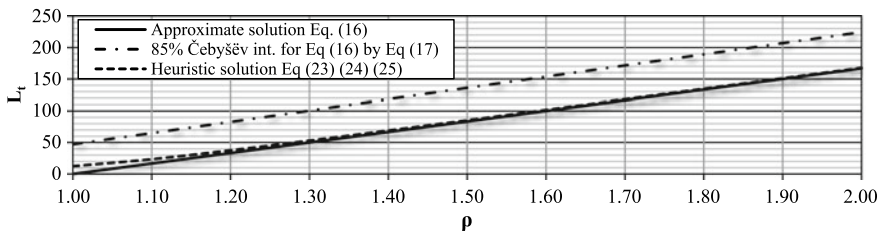

Fig. 4 Comparison between heuristic and approximate solutions

Figure 6 shows L_t values (plotted in logarithmic scale) obtained with the two time-dependent solution and the CSL at 85% in function of ρ, in the case of non-zero initial queue $L_0 = 9$ veh ($c_1 = 0.278$ veh/s $q_1 = 0.25$ veh/s, with $\rho_1 = 0.9$), under the condition $T = 15$ min and $1/E[s] = c = c_1$.

In accordance with L_t values of Fig. 6, the percentage deviation of Heuristic and Approximate solutions have been calculated in function of ρ ($\rho = 1.0–2.0$) and T ($T = 1–60$ min) as shown in Fig. 7a. Figure 7b reports similar assessments for the case $L_0 = 19$ veh ($c_1 = 0.556$ veh/s $= 2000$ veh/h and $q_1 = 0.528$ veh/s $= 1900$ veh/h, with $\rho_1 = 0.95$) and capacity $1/E[s] = c = c_1$. Thus, obviously, for fixed T and ρ values, the deviation between the two time-dependent solution types decreases as L_0 increases.

Finally, Fig. 8 shows the evolution of queue length in a traffic saturation state, for a time interval of 60 min, under the following conditions: initial queue $L_0 = 9$ veh, $c_1 = 0.278$ veh/s, $q_1 = 0.25$ veh/s ($\rho_1 = 0.9$), $1/E[s] = c = c_1$ and $q = 0.333$ veh/s $= 1200$ veh/h. Let be note here that for every solutions, the trend of L_k in Fig. 8 has been obtained considering subsequent regular intervals $k_i = 1, 2, … 20$, each of 3 min. Once again, for the Approximate solution (Eq. 16) the 85% Čebyšëv interval values over the time (namely in function of k_i) are given. According to Eqs. (23), (24) and (25), in Fig. 8 for the Heuristic solution two trends are reported:

a)   the first one with re-initialization of the queue at the beginning of each 3 min time interval respect to the initial equilibrium one ($L_0 = 9$ vehicles at the beginning of each interval);
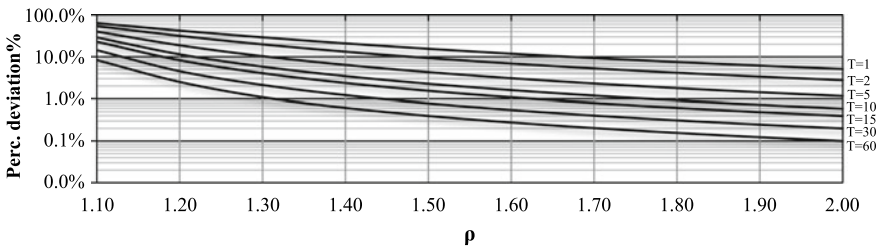

Fig. 5 Percentage deviation between Heuristic and Approximate solutions in function of ρ
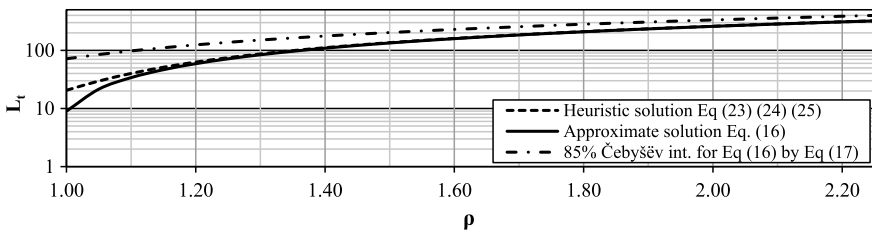

Fig. 6 L_t values in function of ρ (case of non-zero initial queue: $L_0 = 9$ veh)

b)   the second one with the evolution of the queue at the beginning of each 3 min
time interval considering the queue at the end of the previous interval [10].

As expected, the curves of Fig. 8 highlight a substantially convergence of the
results obtained with the two different time-dependent solutions.

It is interesting to test the Eqs. (21) and (22) in order to evaluate the time interval
T'_d obtained with the Approximate solution in comparison with the Heuristic one
(queue discharge, cf. 3.2). In this regard, and in compliance with Fig. 3, the following
traffic condition was analyzed: $q_1 = 300$ veh/h, $c_1 = 450$ veh/h ($\rho_1 = 0.667$); $q =$
600 veh/h, $c = 450$ veh/h ($\rho = 1.334$), $T = 10$ min and $q_2 = q_1 = 300$ veh/h, $c_2 =$
$c_1 = 450$ veh/h ($\rho_1 = 0.667$). By means of Eqs. (23)–(25), a queue length $L_T =$
29 has been obtained. Using Eqs. (16) and (17), it results $E[L_T] = 27$ and $VAR[L_T]$
$= 6$. These values demonstrate that the Approximate solution is very close to the
Heuristic one.

Now we want to evaluate the time interval T'_d that, once the traffic peak period
T is over, the system employs reducing the queue L_T to the stationary value for ρ_1
$= 0.667$, which is $L_e = 2$ veh. Using Eq. (21), it results $E[T'_d] = 600$ s and (VAR
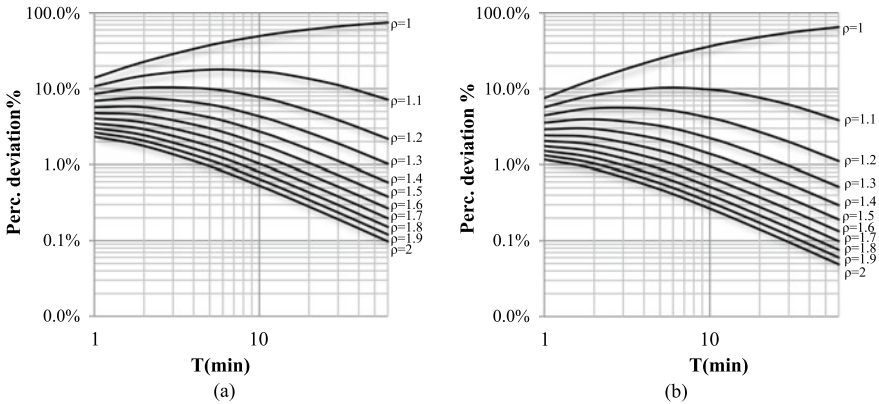$[T'_d])^{1/2} = 277$ s.

Fig. 7   Percentage deviation between the two time-dependent solutions (**a**: $L_0 = 9$ veh; **b**: $L_0 = 19$
veh)

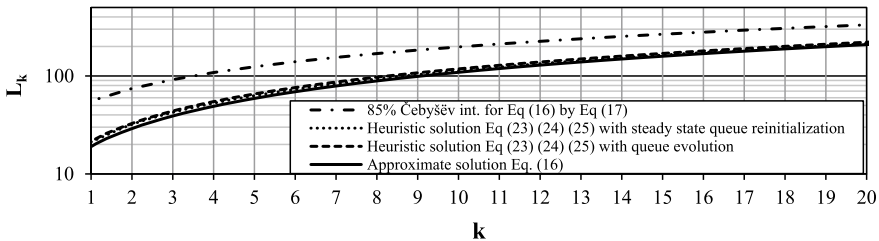**Fig. 8**   Evolution of queue considering subsequent time intervals $k_i = 1, 2, … 20$, each of 3 min

To estimate a realistic duration T'_d by Heuristic solutions, some corrections to increase accuracy of calculation of queue discharge are required [6]. In our case (e.g. $L_T > 2L_e$) we consider the following relationships [6]:

$$L_t = L_T + (\rho_2 - L_T/(L_T + 1))c_2 t \quad if \ \ 0 \le t \le t_C \tag{26}$$

$$L_t = 2L_e - \frac{1}{2}\left[\sqrt{A^2 + B} - A\right] \ \text{if } t > t_C \tag{27}$$
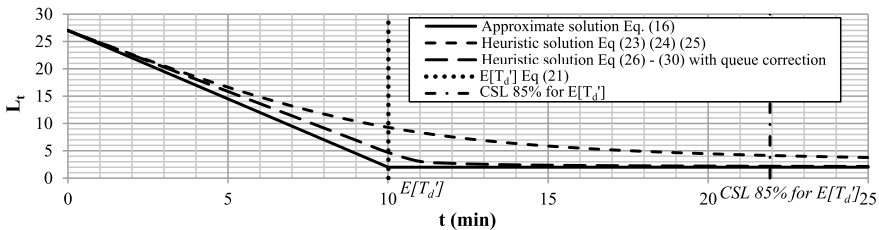
with

$$A = (1 - \rho_2)(t - t_C)c_2 + 1 \tag{28}$$

$$B = 4\rho_2 c_2(t - t_C) \tag{29}$$

$$t_C = (2L_e - L_T)/c_2(\rho_2 - L_T/(L_T + 1)) \tag{30}$$

For determining T'_d, it is essential to find the value of t for which Eq. (26) is satisfied with $L_t = L_e = 27$ veh for every $0 \le t \le t_C$, or else Eq. (27) with values according to Eqs. (28) and (29) if $t > t_C$. In both cases, t_C is calculated with Eq. (30). In the example under study, it results $t_C = 619$ s. Fig. 9 shows the values of L_t in function of t, calculated with Eq. (16) (Approximate solution) and using the set of relationships (26)–(30) (Heuristic equation with correction for queue discharge). Moreover, in Fig. 9 are plotted: the L_t values estimated with Eqs. (23)–(25); the value $E[T'_d] = 600$ s (from Eq. 21); the CSL 85%, obtained considering $(VAR [T'_d])^{1/2} = 277$ s (from Eq. 22). For t $= E[T'_d] = 600$ s, evaluated with Eq. (21)—for which it results $E[L_t] = L_e = 2$ veh with Eq. (16)—the queue length estimated with the corrected Heuristic solution is slightly below 5 veh, decreasing to 3 veh after just 2 min. Therefore, the value $E[T'_d]$ can be considered a good approximation of the time taken for the queue discharge due to the traffic peak. Based on these considerations, the proposed method (Approximate solution) turns out to be very accurate and even simpler than the Heuristic solution.

Fig. 9 Comparison among Heuristic solution with—without queue discharge correction and approx. solution

6 Conclusions

This article demonstrates that in congested traffic conditions, the approximate deterministic solutions for the calculation of the state variables of a road intersection can be deduced as an asymptotic result of the renewal theory. In congested traffic conditions, these approximate solutions have proven to be substantially coincident with the heuristic solutions by coordinates transformation method. The differences obtained by applying the two time-dependent solutions have been estimate as function of traffic intensity. These differences that we get, even if small, are to be considered in the context of the underlying uncertainty in road intersections calculation, primarily for traffic demand and driving behavior parameters. The results have shown that the proposed approximate solutions, marked by a reliable and rapid mathematical approach, could be widely and profitably used in traffic analysis. It should be noted that the solutions obtained were tested by taking comparison with theoretical queue models, on the basis of the most widespread and shared assumptions in the literature for the analysis of unsignalized intersections for arrival and service time probability distributions. The possibility of further testing the acceptability of the same assumptions through comparisons against real traffic scenarios in situations of congestion at unsignalized intersections represents an important topic that is interesting to deepen. This in the continuous search for calculation criteria that, as well as being quick and easy to formulate mathematically, are effectively able to represent real world occurrences.

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