Asymptotic Behavior of Solutions for a Fourth Order Parabolic Equation with Gradient Nonlinearity via the Galerkin Method

Nobuhito Miyake and Shinya Okabe

Abstract In this paper we consider the initial-boundary value problem for a fourth order parabolic equation with gradient nonlinearity. The problem is regarded as the L^2-gradient flow for an energy functional which is unbounded from below. We first prove the existence and the uniqueness of solutions to the problem via the Galerkin method. Moreover, combining the potential well method with the Galerkin method, we study the asymptotic behavior of global-in-time solutions to the problem.

Keywords Fourth order parabolic equation · Gradient nonlinearity · Epitaxial growth · Galerkin method

1 Introduction

We consider the following initial-boundary value problem for a fourth order parabolic equation with gradient nonlinearity:

$$\begin{cases} \partial_t u + (-\Delta)^2 u = -\nabla \cdot (|\nabla u|^{p-2} \nabla u) & \text{in} \quad \Omega \times (0, T), \\ \partial_\nu u = \partial_\nu \Delta u = 0 & \text{on} \quad \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0(\cdot) & \text{in} \quad \Omega. \end{cases} \tag{P}$$

Here, $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a smooth bounded domain, $u_0 \in L^2(\Omega)$, $p > 2$, $T > 0$, $\partial_t := \partial/\partial t$ and ∂_ν denotes the outer normal derivative to $\partial\Omega$. In this paper we show the existence and the uniqueness of local-in-time solutions to problem (P) and consider the asymptotic behavior of global-in-time solutions to problem (P) via the Galerkin method.

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Fourth order parabolic equations with gradient nonlinearity appear in a model of thin film growth. King–Stein–Winkler [8] studied the following continuum model for epitaxial thin film growth proposed by Ortiz–Repetto–Si [11], based on phenomenological considerations by Zangwill [15]:

$$\begin{cases} \partial_t u + (-\Delta)^2 u = \nabla \cdot f(\nabla u) + g & \text{in} \quad \Omega \times (0, T), \\ \partial_\nu u = \partial_\nu \Delta u = 0 & \text{on} \quad \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0(\cdot) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

In the paper [8], they assumed that (1) has a gradient structure and the corresponding energy is bounded from below (for example, $f(z) = |z|^{p-2}z - z$ and $g \equiv 0$). Under these conditions, they studied the existence of global-in-time solutions and large time behavior of solutions to (1). Recently problem (1) was studied in the mathematical literature (e.g., see [4, 17, 18]). However, the approaches which were used in these papers cannot be applied directly to problem (P). Indeed, problem (P) is regarded as the L^2-gradient flow for the energy functional

$$E(u) := \frac{1}{2} \int_\Omega |\Delta u|^2 \, dx - \frac{1}{p} \int_\Omega |\nabla u|^p \, dx$$

and the functional E is unbounded from below due to $p > 2$.

On the other hand, problem (P) was studied by Sandjo–Moutari–Gningue [13] via the semigroup approach. They showed the existence of local-in-time solutions to problem (P) under the condition $3 < p < 4$. The assumption for p was required for the Lipschitz continuity of the nonlinear term and hence this approach cannot be adapted for the case $2 < p < 3$ in problem (P). However, as in the result [7] which is the whole space case for problem (P), the restriction $p > 3$ should be eliminated.

In this paper we prove the existence of local-in-time solutions to problem (P) in the case

(a) $u_0 \in H^2_{\mathcal{N}}(\Omega)$ and $2 < p < p_S$, or (b) $u_0 \in L^2_{\mathcal{N}}(\Omega)$ and $2 < p < p_*$,

where

$$L^2_{\mathcal{N}}(\Omega) := \left\{ v \in L^2(\Omega) \,\middle|\, \int_\Omega v \, dx = 0 \right\} \subset L^2(\Omega),$$
$$H^2_{\mathcal{N}}(\Omega) := \left\{ v \in H^2(\Omega) \cap L^2_{\mathcal{N}}(\Omega) \mid \partial_\nu v = 0 \text{ on } \partial\Omega \right\} \subset H^2(\Omega),$$

and

$$p_* := 2 + \frac{4}{N+2}, \qquad p_S := \begin{cases} \dfrac{2N}{N-2} & \text{if} \quad N \geq 3, \\ \infty & \text{if} \quad N = 2. \end{cases}$$

Moreover, we study the asymptotic behavior of global-in-time solutions to problem (P) in the case (a). In order to formulate a definition of the solution to problem (P), we set

$$\mathscr{V} := \left\{\varphi \in H^1(0,T; L^2_{\mathscr{N}}(\Omega)) \cap L^2(0,T; H^2_{\mathscr{N}}(\Omega)) \mid \nabla\varphi \in (L^p(0,T; L^p(\Omega)))^N\right\}.$$

Definition 1.1 Let $u_0 \in L^2_{\mathscr{N}}(\Omega)$ and $T > 0$. We say that a function

$$u \in C([0,T]; L^2_{\mathscr{N}}(\Omega)) \cap L^2(0,T; H^2_{\mathscr{N}}(\Omega)) \quad \text{with} \quad \nabla u \in (L^p(0,T; L^p(\Omega)))^N$$

is a solution to problem (P) in $\Omega \times [0,T]$ if u satisfies

$$\begin{aligned}&\int_\Omega [u(T)\varphi(T) - u_0\varphi(0)]\,dx \\ &\quad - \int_0^T \int_\Omega u\partial_t\varphi\,dx\,dt + \int_0^T \int_\Omega \left[\Delta u\Delta\varphi - |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\right] dx\,dt = 0\end{aligned} \tag{2}$$

for $\varphi \in \mathscr{V}$. Moreover, we say that u is a global-in-time solution to problem (P) if u is a solution to problem (P) in $\Omega \times [0,T']$ for all $T' > 0$.

The first and the second main results of this paper are the existence and the uniqueness of local-in-time solutions to problem (P) in the case (a) and (b) respectively:

Theorem 1.1 *Let $u_0 \in H^2_{\mathscr{N}}(\Omega)$ and assume that $2 < p < p_S$. Then the following hold*:

(i) *There exist $T > 0$ and a solution u to problem* (P) *in $\Omega \times [0,T]$. Moreover, the solution u satisfies*

$$u \in H^1(0,T; L^2_{\mathscr{N}}(\Omega)) \cap C_{\mathrm{w}}([0,T]; H^2_{\mathscr{N}}(\Omega)) \quad \textit{with} \quad \nabla u \in (C([0,T]; L^p(\Omega)))^N.$$

(ii) *If u_1 and u_2 are solutions to problem* (P) *in $\Omega \times [0,T]$ for some $T > 0$ and satisfy*

$$\nabla u_1, \nabla u_2 \in (L^\infty(0,T; L^p(\Omega)))^N,$$

then it holds that $u_1 \equiv u_2$ in $\Omega \times [0,T]$.

Theorem 1.2 *Let $u_0 \in L^2_{\mathscr{N}}(\Omega)$ and assume that $2 < p < p_*$. Then there exist $T > 0$ and a unique solution u to problem* (P) *in $\Omega \times [0,T]$.*

The precise definition of the space $C_{\mathrm{w}}([0,T]; H^2_{\mathscr{N}}(\Omega))$, see Sect. 2.

The other purpose of this paper is to study the asymptotic behavior of global-in-time solutions to problem (P) in the case (a). In order to state our main result on this topic, we introduce several notations. Let $0 < \mu_1 < \mu_2 < \cdots$ be the strictly monotone increasing divergent sequence of all eigenvalues of the boundary value problem

$$\begin{cases} -\Delta\psi = \mu\psi & \text{in} \quad \Omega, \\ \partial_\nu \psi = 0 & \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega \psi \, dx = 0. \end{cases} \tag{3}$$

Let P_0 be the zero map. For each $k \in \mathbb{N}$, we define P_k as the projection from $L^2_{\mathscr{N}}(\Omega)$ to the subspace spanned by the eigenfunctions corresponding to μ_1, μ_2, $\cdots$, μ_k. Let $k_* \in \mathbb{N}$ be the number such that

$$\mu_{k_*}^2 < (p-1)\mu_1^2 \le \mu_{k_*+1}^2. \tag{4}$$

We note that the condition $p > 2$ implies the existence of the number k_*. As a class of initial data, we set

$$W := \left\{ v \in H^2_{\mathscr{N}}(\Omega) \mid E(v) < d, \ I(v) > 0 \right\},$$

where I denotes the Nehari functional given by

$$I(v) := \int_\Omega |\Delta v|^2 \, dx - \int_\Omega |\nabla v|^p \, dx, \quad v \in H^2_{\mathscr{N}}(\Omega),$$

and

$$d := \left(\frac{1}{2} - \frac{1}{p}\right) S_p^{\frac{p}{p-2}} > 0, \quad S_p := \inf_{v \in H^2_{\mathscr{N}}(\Omega), v \neq 0} \frac{\|\Delta v\|^2_{L^2(\Omega)}}{\|\nabla v\|^2_{L^p(\Omega)}} > 0. \tag{5}$$

Then the third main result of this paper is stated as follows:

Theorem 1.3 *Let $u_0 \in W$ and assume that $2 < p < p_S$. Then problem (P) possesses the unique global-in-time solution u such that*

$$\|\Delta u(t)\|_{L^2(\Omega)} = O(e^{-\mu_1^2 t}) \quad as \quad t \to \infty. \tag{6}$$

Moreover, it holds that

$$\|u(t) - P_{k-1}u(t)\|_{L^2(\Omega)} = O(e^{-\mu_k^2 t}) \qquad as \quad t \to \infty, \quad 1 \le k \le k_*, \tag{7}$$

$$\|u(t) - P_{k_*}u(t)\|_{L^2(\Omega)} = O(e^{-(1-\varepsilon)(p-1)\mu_1^2 t}) \quad as \quad t \to \infty, \quad 0 < \varepsilon < 1, \tag{8}$$

where k_* is the positive integer satisfying (4).

One of the main ingredients of the strategy in the proof of Theorem 1.3 is the potential well method introduced by Sattinger [14] and Payne–Sattinger [12]. The assumption $u_0 \in W$ implies that E is bounded from below along the orbit of the solution to problem (P) starting from u_0. Then, combining the Galerkin method, one can prove that problem (P) has a global-in-time solution. Indeed, [5] proved the existence of global-in-time solutions to a related problem by the same strategy. However, the strategy does not show the existence of local-in-time solutions to problem (P) with more general initial data. Moreover, it is not clear how to derive the asymptotic behavior of global-in-time solutions to problem (P), because useful mathematical tools such as the comparison principle do not hold for fourth order parabolic problems. In Theorems 1.1 and 1.2, making use of the Galerkin method and the Aubin–Lions–Simon compactness theorem, we prove the existence of local-in-time solutions to problem (P) without using the potential well method. To the best of our knowledge, this is the first paper to prove the solvability of problem (P) for $u_0 \in L^2(\Omega)$ and $2 < p < 3$. Moreover, our argument based on the Galerkin method and the potential well method enable to derive the precise asymptotic behavior of the global-in-time solutions as in Theorem 1.3.

This paper is organized as follows. We introduce some notations and collect several useful propositions in Sect. 2. We construct a solution to problem (P) with the Galerkin method in Sect. 3. In Sect. 4, we give the characterization of stable sets and study the asymptotic behavior of global-in-time solutions which converge to 0. In appendix, we prove the uniqueness of solutions to problem (P) under the assumption in Theorems 1.1 and 1.2.

2 Preliminaries

In this section we collect several notations and propositions which are used in this paper. In what follows, we rewrite the norm $\|\cdot\|_{L^q(\Omega)}$ as $\|\cdot\|_q$ for $q \in [1, \infty]$ and the L^2-inner product $(\cdot, \cdot)_{L^2(\Omega)}$ as $(\cdot, \cdot)_2$.

We mention several remarks on $L^2_{\mathscr{N}}(\Omega)$ and $H^2_{\mathscr{N}}(\Omega)$. As stated in [8], the map $\Delta \colon H^2_{\mathscr{N}}(\Omega) \to L^2_{\mathscr{N}}(\Omega)$ is a homeomorphism and hence there exists a constant $c_1 = c_1(N) > 0$ such that

$$c_1^{-1} \sum_{k=0}^{2} \|\nabla^k v\|_2^2 \le \|v\|_{H^2_{\mathscr{N}}}^2 := \|\Delta v\|_2^2, \quad v \in H^2_{\mathscr{N}}(\Omega). \tag{9}$$

Moreover, $H^2_{\mathscr{N}}(\Omega)$ is a Hilbert space with the inner product

$$(v, w)_{H^2_{\mathscr{N}}} := \int_\Omega \Delta v \Delta w \, dx, \quad v, w \in H^2_{\mathscr{N}}(\Omega).$$

Let X be a real Banach space and X^* denote the dual space of X. We denote $C_{\mathrm{w}}([0,T];X)$ by the set of all X-valued weak continuous functions. Here, we say that u is an X-valued weak continuous function if for all $F \in X^*$ the function

$$[0,T] \ni s \mapsto {}_{X^*}\langle F, u(s)\rangle_X \in \mathbb{R}$$

is continuous.

By (5) we see that S_p is the best constant for the following inequality:

$$S_p \|\nabla v\|_p^2 \le \|\Delta v\|_2^2, \quad v \in H^2_{\mathcal{N}}(\Omega). \tag{10}$$

We collect several useful propositions. The Gagliard–Nirenberg inequality (cf. [1, Theorems 5.2 and 5.8]) and (9) lead the following interpolation inequality:

Proposition 2.1 *Let $2 < p < p_S$. Then there exists a positive constant c_2 depending only on N, Ω and p such that*

$$\|\nabla v\|_p \le c_2 \|\Delta v\|_2^{\theta} \|v\|_2^{1-\theta}, \quad v \in H^2_{\mathcal{N}}(\Omega),$$

where

$$\theta := \frac{N(p-2)}{4p} + \frac{1}{2} \in \left(\frac{1}{2}, 1\right). \tag{11}$$

Next we mention the property of the space of weak continuous functions (cf. [2, Lemma II.5.9]).

Proposition 2.2 *Let $T > 0$, X be a separable and reflexive real Banach space and Y be a real Banach space such that the embedding $X \subset Y$ is continuous. Then*

$$L^\infty(0,T;X) \cap C_{\mathrm{w}}([0,T];Y) = C_{\mathrm{w}}([0,T];X).$$

We introduce the Aubin–Lions–Simon compactness theorem (cf. [2, Theorem II.5.16]).

Proposition 2.3 *Let X_0, X_1, X_2 be Banach spaces with the following properties*:

- *The embedding $X_0 \subset X_1$ is compact.*
- *The embedding $X_1 \subset X_2$ is continuous.*

Let $T > 0$, q, $r \in [1, \infty]$ and set

$$E_{q,r} := \left\{ v \in L^q(0,T;X_0) \;\middle|\; \partial_t v \in L^r(0,T;X_2) \right\}.$$

Then the following hold:

- *If $q \neq \infty$, then the embedding $E_{q,r} \subset L^q(0,T;X_1)$ is compact.*
- *If $q = \infty$ and $r > 1$, then the embedding $E_{q,r} \subset C([0,T];X_1)$ is compact.*

We close Sect. 2 with a remark on the assumption on initial data.

Remark 2.1 Without loss of generality, we may assume that u_0 and the solution u to problem (P) satisfy

$$\int_\Omega u\,dx = \int_\Omega u_0\,dx = 0, \quad t \in (0,T).$$

Indeed, setting

$$\tilde{u}(x,t) := u(x,t) - \frac{1}{|\Omega|}\int_\Omega u_0\,dx \quad \text{and} \quad \tilde{u}_0(x) := u_0(x) - \frac{1}{|\Omega|}\int_\Omega u_0\,dx,$$

we see that $\tilde{u}$ is a solution to problem (P) with $\tilde{u}_0$. Formally, integrating the both side of the equation in problem (P), we have

$$\frac{d}{dt}\int_\Omega u(x,t)\,dx = \frac{d}{dt}\int_\Omega \tilde{u}(x,t)\,dx = 0$$

for $t \in (0,T)$. Hence it holds that

u is a solution to problem (P)

$\iff$ $\tilde{u}$ is a solution to problem (P) replaced u_0 by $\tilde{u}_0$.

3 Proof of Theorems 1.1 and 1.2

In this section we prove Theorems 1.1 and 1.2. We first construct approximate solutions we use in the both of proofs. We consider the eigenvalue problem for the Laplace equation under Neumann boundary condition:

$$\begin{cases} -\Delta\psi = \lambda\psi & \text{in} \quad \Omega, \\ \partial_\nu\psi = 0 & \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega \psi\,dx = 0. \end{cases} \tag{12}$$

Let $\{(\lambda_k, \psi_k)\}_{k=1}^\infty$ be a family of pairs with the following properties:

$$\begin{cases} \text{For each } k \in \mathbb{N},\ (\lambda,\psi) = (\lambda_k,\psi_k) \text{ satisfies (12)}, \\ 0 < \lambda_1 \le \lambda_2 \le \cdots, \\ \{\psi_k\}_{k=1}^\infty : \text{orthonormal basis of } L^2_{\mathcal{N}}(\Omega), \\ \{\lambda_k^{-1}\psi_k\}_{k=1}^\infty : \text{orthonormal basis of } H^2_{\mathcal{N}}(\Omega). \end{cases} \tag{13}$$

For each $m \in \mathbb{N}$, we consider the solution $a^m = (a_1^m, \cdots, a_m^m)$ to the following ODE system:

$$\begin{cases} \dfrac{da_k^m}{dt}(t) + \lambda_k^2 a_k^m(t) = \displaystyle\int_\Omega \Big| \sum_{i=1}^m a_i^m(t) \nabla \psi_i \Big|^{p-2} \sum_{j=1}^m a_j^m(t) \nabla \psi_j \cdot \nabla \psi_k \, dx \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad k \in \{1, \cdots, m\}, \ t \in (0, T), \\ a_k^m(0) = (u_0, \psi_k)_2 \qquad\qquad\qquad\quad \text{for} \quad k \in \{1, \cdots, m\}. \end{cases} \tag{14}$$

We remark that there exists a unique solution $a^m \in C^1([0, T]; \mathbb{R}^m)$ to (14) for some $T > 0$ in the classical sense. Moreover, the solution can be uniquely extended if $|a_k^m|$ is bounded for each $k \in \{1, \cdots, m\}$. Define $T_m > 0$ as the maximal existence time of the solution to (14), that is,

$$T_m := \sup \left\{ \tau > 0 \;\middle|\; \exists a^m \in C^1([0, \tau]; \mathbb{R}^m) \colon \text{unique classical solution to (14)} \right\}.$$

Define u^m as

$$u^m(x, t) := \sum_{k=1}^m a_k^m(t) \psi_k(x), \quad (x, t) \in \Omega \times [0, T_m).$$

Since $\{\psi_k\}_{k=1}^\infty$ satisfies (12) and (13), u^m satisfies

$$\frac{d}{dt}(u^m(t), \psi_k)_2 + (\Delta u^m(t), \Delta \psi_k)_2 = \int_\Omega |\nabla u^m(t)|^{p-2} \nabla u^m(t) \cdot \nabla \psi_k \, dx \tag{15}$$

for $k \in \{1, \cdots, m\}$ and $t \in [0, T_m)$. Multiplying $\partial_t a_k^m$ by (15), summing k from 1 to m and integrating it on (t', t), we have

$$E(u^m(t)) + \int_{t'}^t \|\partial_t u^m(\tau)\|_2^2 \, d\tau = E(u^m(t')), \quad 0 \le t' \le t < T_m. \tag{16}$$

Similarly, multiplying a_k^m by (15), we obtain

$$\|u^m(t)\|_2^2 + \int_{t'}^t I(u^m(\tau)) \, d\tau = \|u^m(t')\|_2^2, \quad 0 \le t' \le t < T_m. \tag{17}$$

In particular, (16) implies that $E(u^m(t))$ is non-increasing with respect to t.

We prove Theorems 1.1 and 1.2 in Sects. 3.1 and 3.2, respectively.

3.1 Proof of Theorem 1.1

By a standard argument in [10] we can prove the uniqueness of solutions to problem (P). Thus we postpone the proof of the uniqueness to Sect. 5.1.

We divide the proof into three steps.

Step 1: *We derive a priori estimate of* $\{u^m\}_{m=1}^{\infty}$. Since $u_0 \in H^2_{\mathcal{N}}(\Omega)$, (10) and (13) imply that

$$K = K(u_0) := \sup_{m\in\mathbb{N}} E(u^m(0)) < \infty, \quad E(u^m(0)) \to E(u_0) \quad \text{as} \quad m \to \infty.$$

Moreover, it follows from (10), (12) and (13) that

$$\begin{aligned}\|\nabla u^m(0)\|_p^2 &\le S_p^{-1}\|\Delta u^m(0)\|_2^2 \\ &= S_p^{-1}\sum_{k=1}^{m}\lambda_k^2(u_0,\psi_k)_2^2 = S_p^{-1}\sum_{k=1}^{m}(u_0,\lambda_k^{-1}\psi_k)_{H^2_{\mathcal{N}}}^2 \le S_p^{-1}\|\Delta u_0\|_2^2.\end{aligned} \tag{18}$$

Let $L > S_p^{-p/2}\|\Delta u_0\|_2^p$ and set

$$T_{L,m} := \sup\left\{\tau > 0 \,\middle|\, \sup_{0\le t\le\tau}\|\nabla u^m(t)\|_p^p \le L\right\} \le T_m.$$

Since $\nabla u^m \in (C([0,T_m); L^p(\Omega)))^N$, we deduce from (18) that $T_{L,m} > 0$. By (16) and the definition of E we have

$$\|\Delta u^m(t)\|_2^2 = 2E(u^m(t)) + \frac{2}{p}\|\nabla u^m(t)\|_p^p \le 2K + \frac{2}{p}L, \tag{19}$$

$$E(u^m(t)) = \frac{1}{2}\|\Delta u^m(t)\|_2^2 - \frac{1}{p}\|\nabla u^m(t)\|_p^p \ge -\frac{1}{p}L, \tag{20}$$

$$\int_0^t \|\partial_t u^m(\tau)\|_2^2\,d\tau = E(u^m(0)) - E(u^m(t)) \le K + \frac{1}{p}L, \tag{21}$$

for $t \in [0, T_{L,m}]$, where we used (20) in (21).

Step 2: *We show a lower estimate for* $T_{L,m}$. It follows from (21) that

$$\begin{aligned}\|u^m(t) - u^m(t')\|_2^2 &= \int_\Omega\left(\int_{t'}^t \partial_t u^m(\tau)\,d\tau\right)^2 dx \\ &\le (t-t')\int_{t'}^t \|\partial_t u^m(\tau)\|_2^2\,d\tau \le \left(K + \frac{1}{p}L\right)(t-t')\end{aligned} \tag{22}$$

for $0 \le t' \le t \le T_{L,m}$. Moreover, we observe from Proposition 2.1, (19) and (22) that

$$\begin{aligned}\|\nabla u^m(t) - \nabla u^m(t')\|_p &\le c_2 \|\Delta u^m(t) - \Delta u^m(t')\|_2^{\theta} \|u^m(t) - u^m(t')\|_2^{1-\theta} \\ &\le 2^{3\theta/2} c_2 \left(K + \frac{L}{p}\right)^{1/2} (t - t')^{(1-\theta)/2}\end{aligned} \tag{23}$$

for $0 \le t' \le t \le T_{L,m}$, where c_2 is obtained in Proposition 2.1 and θ is as in (11). Hence by (18) and (23) we see that

$$\begin{aligned}\|\nabla u^m(t)\|_p &\le \|\nabla u^m(0)\|_p + 2^{3\theta/2} c_2 \left(K + \frac{L}{p}\right)^{1/2} t^{(1-\theta)/2} \\ &\le S_p^{-1/2} \|\Delta u_0\|_2 + 2^{3\theta/2} c_2 \left(K + \frac{L}{p}\right)^{1/2} T_{L,m}^{(1-\theta)/2}\end{aligned}$$

for $t \in [0, T_{L,m}]$. Combining this estimate with the definition of $T_{L,m}$, we have

$$S_p^{-1/2} \|\Delta u_0\|_2 + 2^{3\theta/2} c_2 \left(K + \frac{L}{p}\right)^{1/2} T_{L,m}^{(1-\theta)/2} \ge L^{1/p},$$

that is,

$$T_{L,m} \ge T_* := \left[\frac{p}{2^{3\theta} c_2^2 (Kp + L)} \left(L^{1/p} - S_p^{-1/2} \|\Delta u_0\|_2 \right)^2 \right]^{1/(1-\theta)} \tag{24}$$

for $m \in \mathbb{N}$.

Step 3: *We construct a solution to problem* (P). We observe from (19), (21) and (24) that

$$\sup_{t \in [0, T_*]} \|\Delta u^m(t)\|_2^2 \le 2K + \frac{2}{p} L, \qquad \int_0^{T_*} \|\partial_t u^m(\tau)\|_2^2 \, d\tau \le K + \frac{1}{p} L.$$

Then, up to subsequence, there exists $u \in L^\infty(0, T_*; H^2_{\mathscr{N}}(\Omega)) \cap H^1(0, T_*; L^2_{\mathscr{N}}(\Omega))$ such that

$$u^m \overset{*}{\rightharpoonup} u \quad \text{weakly-* in} \quad L^\infty(0, T_*; H^2_{\mathscr{N}}(\Omega)) \quad \text{as} \quad m \to \infty, \tag{25}$$

$$u^m \rightharpoonup u \quad \text{weakly} \quad \text{in} \quad H^1(0, T_*; L^2_{\mathscr{N}}(\Omega)) \quad \text{as} \quad m \to \infty. \tag{26}$$

Moreover, by Proposition 2.3 we have

$$\nabla u^m \to \nabla u \quad \text{in} \quad (C([0, T_*]; L^p(\Omega)))^N \quad \text{as} \quad m \to \infty.$$

By Proposition 2.2 we see that u belongs to $C_{\mathrm{w}}([0, T_*]; H^2_{\mathcal{N}}(\Omega))$.

Recalling that u^m satisfies

$$\int_0^{T_*} \int_\Omega \left[\partial_t u^m \psi_k \zeta + \Delta u^m \Delta \psi_k \zeta - |\nabla u^m|^{p-2} \nabla u^m \cdot \nabla \psi_k \zeta \right] dt\, dx = 0$$

for $m \in \mathbb{N}$, $k \in \mathbb{N}$ and $\zeta \in C^\infty([0, T_*])$, we deduce from (25) and (26) that

$$\begin{aligned} &\int_\Omega [u(T_*)\zeta(T_*) - u_0 \zeta(0)]\, \eta\, dx \\ &\quad + \int_0^{T_*} \int_\Omega \left[-u\eta \partial_t \zeta + \Delta u \Delta \eta \zeta - |\nabla u|^{p-2} \nabla u \cdot \nabla \eta \zeta \right] dx\, dt = 0 \end{aligned}$$

for $\eta \in H^2_{\mathcal{N}}(\Omega)$ and $\zeta \in C^\infty([0, T_*])$. By a density argument (cf. [16, Proposition 23.23 (iii)]), we see that u satisfies (2) for $\varphi \in \mathcal{V}$. Hence we complete the proof of Theorem 1.1.

3.2 *Proof of Theorem 1.2*

Along the same line as in Sect. 3.1, we postpone the proof of the uniqueness of solutions to Sect. 5.2. We divide the proof of Theorem 1.2 into three steps.

Step 1: *We derive a priori estimate of* $\{u^m\}_{m=1}^\infty$. Since $u_0 \in L^2_{\mathcal{N}}(\Omega)$, relation (13) implies that $\|u^m(0)\|_2^2 \le \|u_0\|_2^2$ for $m \in \mathbb{N}$. Let $L > 0$ and set

$$\tilde{T}_{L,m} := \sup \left\{ \tau > 0 \;\middle|\; \int_0^\tau \|\nabla u^m(t)\|_p^p\, dt \le L \right\} \in (0, T_m].$$

By (17) and the definition of I we have

$$\|u^m(t)\|_2^2 + \int_0^t \|\Delta u^m(\tau)\|_2^2\, d\tau = \|u^m(0)\|_2^2 + \int_0^t \|\nabla u^m(\tau)\|_p^p\, d\tau \le \|u_0\|_2^2 + L \tag{27}$$

for $t \in [0, \tilde{T}_{L,m}]$.

Step 2: *We show a lower estimate for* $\tilde{T}_{L,m}$. It follows from Proposition 2.1 that

$$\begin{aligned}
&\int_0^t \|\nabla u^m(\tau)\|_p^p \, d\tau \\
&\le c_2^p \int_0^t \|\Delta u^m(\tau)\|_2^{\theta p} \|u^m(\tau)\|_2^{(1-\theta)p} \, d\tau \\
&\le c_2^p \left(\sup_{\tau\in[0,t]} \|u^m(\tau)\|_2^2 \right)^{(1-\theta)p/2} \int_0^t \|\Delta u^m(\tau)\|_2^{\theta p} \, d\tau \\
&\le c_2^p \left(\sup_{\tau\in[0,t]} \|u^m(\tau)\|_2^2 \right)^{(1-\theta)p/2} \left(\int_0^t \|\Delta u^m(\tau)\|_2^2 \, d\tau \right)^{\theta p/2} t^{1-\theta p/2}
\end{aligned} \tag{28}$$

for $t \in [0, \tilde{T}_{L,m}]$, where θ is as in (11). Here, we note that the condition $2 < p < p_*$ is equivalent to $0 < \theta p < 2$. If $\tilde{T}_{L,m} < \infty$, it follows from (27) and (28) that

$$L = \int_0^{\tilde{T}_{L,m}} \|\nabla u^m(\tau)\|_p^p \, d\tau \le c_2^p (\|u_0\|_2^2 + L)^{p/2} \tilde{T}_{L,m}^{1-\theta p/2},$$

that is,

$$\tilde{T}_{L,m} \ge \tilde{T}_* := \left(\frac{L}{c_2^p (\|u_0\|_2^2 + L)^{p/2}} \right)^{-(1-\theta p)/2}.$$

Step 3: *We construct a solution to problem* (P). We observe from (27) that

$$\sup_{t\in[0,T_*]} \|u^m(t)\|_2^2 \le \|u_0\|_2^2 + L, \qquad \int_0^{T_*} \|\Delta u^m(\tau)\|_2^2 \, d\tau \le \|u_0\|_2^2 + L. \tag{29}$$

These clearly imply that

$$\sup_{t\in[0,T_*]} \sum_{l=1}^{m} |(u^m(t), \psi_l)_2|^2 \le \|u_0\|_2^2 + L. \tag{30}$$

Moreover, by (15) and (29) we have

$$\begin{aligned}
&|(u^m(t), \psi_k)_2 - (u^m(t'), \psi_k)| \\
&\quad \le \int_{t'}^t \|\Delta u^m(\tau)\|_2 \|\Delta \psi_k\|_2 \, d\tau + \int_{t'}^t \int_\Omega \|\nabla u^m(\tau)\|_p^{p-1} \|\nabla \psi_k\|_p \, d\tau \\
&\quad \le (\|u_0\|_2^2 + L)^{1/2} \|\Delta \psi_k\|_2 (t - t')^{1/2} + L^{(p-1)/p} \|\nabla \psi_k\|_p (t - t')^{1/p}
\end{aligned}$$

for $1 \le k \le m$ and $0 \le t' \le t \le \tilde{T}_*$. Then, extracting a subsequence, we find $c_k \in C([0, \tilde{T}_*])$ such that

$$(u^m(t), \psi_k)_2 \to c_k(t) \quad \text{uniformly in} \quad [0, \tilde{T}_*] \quad \text{as} \quad m \to \infty$$

for $k \in \mathbb{N}$. Set

$$u(t) := \sum_{k=1}^{\infty} c_k(t)\psi_k, \quad t \in [0, \tilde{T}_*].$$

Then it follows from (13), (29) and (30) that $\|u(t)\|_2^2 \le \|u_0\|_2^2 + L$ and

$$\begin{aligned} &|(u^m(t) - u(t), \eta)_2| \\ &\le \sum_{k=1}^{M} |(\eta, \psi_k)_2| \left|(u^m(t) - u(t), \psi_k)_2\right| + \left|\left(u^m(t) - u(t), \sum_{k=M+1}^{\infty} (\eta, \psi_k)_2\psi_k\right)_2\right| \\ &\le \sum_{k=1}^{M} |(\eta, \psi_k)_2| \left|(u^m(t) - u(t), \psi_k)_2\right| + 2^{1/2}(\|u_0\|_2^2 + L)^{1/2} \left(\sum_{k=M+1}^{\infty} (\eta, \psi_k)_2^2\right)^{1/2} \end{aligned}$$

for $\eta \in L^2_{\mathcal{N}}(\Omega)$. Hence we see that $(u^m(t), \eta)_2$ converges to $(u(t), \eta)_2$ uniformly in $[0, \tilde{T}_*]$ for $\eta \in L^2_{\mathcal{N}}(\Omega)$ and $u \in C_{\mathrm{w}}([0, \tilde{T}_*]; L^2_{\mathcal{N}}(\Omega))$. Along the same argument as in step 3 in Sect. 3.1, we see that

$$\begin{aligned} &u^m \rightharpoonup u && \text{weakly in} \quad L^2(0, T_*; H^2_{\mathcal{N}}(\Omega)) && \text{as} \quad m \to \infty, \\ &\frac{\partial u^m}{\partial x_j} \rightharpoonup \frac{\partial u}{\partial x_j} && \text{weakly in} \quad L^p(0, T_*; L^p(\Omega)) && \text{as} \quad m \to \infty \quad \text{for} \quad 1 \le j \le N \end{aligned}$$

and u satisfies (2) for $\varphi \in \mathcal{V}$. Moreover, we observe from the same argument as in [10, Chapter III, Section 4] that $u \in C([0, \tilde{T}_*]; L^2_{\mathcal{N}}(\Omega))$. Hence we complete the proof of Theorem 1.2.

4 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. We first show two lemmas.

Lemma 4.1 *Let $v \in H^2_{\mathcal{N}}(\Omega) \setminus \{0\}$ and assume that $2 < p < p_S$. Then*

$$\lambda_* := \left(\frac{\|\Delta v\|_2^2}{\|\nabla v\|_p^p}\right)^{\frac{1}{p-2}} > 0$$

is the unique maximum point of the function

$$(0, \infty) \ni \lambda \mapsto E(\lambda v) \in \mathbb{R}$$

and λ_* satisfies

$$E(\lambda_* v) = \left(\frac{1}{2} - \frac{1}{p}\right) \left(\frac{\|\Delta v\|_2^2}{\|\nabla v\|_p^2}\right)^{\frac{p}{p-2}}, \qquad I(\lambda v) \begin{cases} > 0 & \text{if} \quad \lambda \in (0, \lambda_*), \\ = 0 & \text{if} \quad \lambda = \lambda_*, \\ < 0 & \text{if} \quad \lambda \in (\lambda_*, \infty). \end{cases}$$

Moreover, it holds that

$$d = \inf_{v \in H^2_{\mathcal{N}}(\Omega), v \neq 0} E(\lambda_* v),$$

where d is defined in (5)*.*

Lemma 4.1 follows from a direct calculation. Thus we omit the proof.

Lemma 4.2 *Let $u_0 \in W$ and assume that $2 < p < p_S$. Then there exists $m_* \in \mathbb{N}$ such that $T_m = \infty$ and $u^m(t) \in W$ for $t \in (0, \infty)$ if $m \geq m_*$, where $\{u^m(t)\}_{m=1}^{\infty}$ denotes the family of functions constructed in the proof of Theorem* 1.1.

Proof Since $u_0 \in W$, we find $m_* \in \mathbb{N}$ such that

$$u^m(0) \in W \quad \text{if} \quad m \geq m_*.$$

This together with (16) implies that $E(u^m(t)) < d$ for $m \geq m_*$ and $t \in (0, T_m)$. From now on, we let $m \geq m_*$. Assume that $I(u^m(t_*)) = 0$ holds for some $t_* \in (0, T_m)$. Then by Lemma 4.1 we have

$$E(u^m(t_*)) = \left(\frac{1}{2} - \frac{1}{p}\right) \left(\frac{\|\Delta u^m(t_*)\|_2^2}{\|\nabla u^m(t_*)\|_p^2}\right)^{\frac{p}{p-2}} \geq d$$

and it contradicts to $E(u^m(t_*)) < d$. Hence $u^m(t) \in W$ for $t \in (0, T_m)$. Moreover, this together with (17) implies that

$$\sum_{k=1}^{m} (a_k^m(t))^2 = \|u^m(t)\|_2^2 = \|u^m(0)\|_2^2 - \int_0^t I(u^m(\tau))\, d\tau \leq \|u_0\|_2^2$$

for $t \in (0, T_m)$. Thus we have $T_m = \infty$.

We are in a position to prove Theorem 1.3.

Proof (Theorem** 1.3**) Let $\{u^m(t)\}_{m=1}^{\infty}$ be the family of functions constructed in the proof of Theorem 1.1 and $\{\mu_k\}_{k=1}^{\infty}$ be as in (3). By the variational characterization of eigenvalues, we have

$$\mu_k^2 \|v - P_{k-1}v\|_2^2 \le \|\Delta(v - P_{k-1}v)\|_2^2, \quad k \in \mathbb{N},\ v \in H^2_{\mathcal{N}}(\Omega), \tag{31}$$

where P_k is as in Sect. 1.

Let $m_* \in \mathbb{N}$ be the number obtained in Lemma 4.2. From now on, we let $m \ge m_*$. By Lemma 4.2 we have

$$E(u^m(t)) < d, \quad t > 0. \tag{32}$$

Moreover, since

$$E(u^m(t)) = \left(\frac{1}{2} - \frac{1}{p}\right) \|\Delta u^m(t)\|_2^2 + \frac{1}{p} I(u^m(t)), \quad t > 0, \tag{33}$$

it follows from Lemma 4.2 that

$$\|\Delta u^m(t)\|_2^2 < \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} E(u^m(t)), \quad t > 0. \tag{34}$$

Hence, combining (32) with (34), we have

$$\|\Delta u^m(t)\|_2^2 < S_p^{p/(p-2)}, \quad t > 0, \tag{35}$$

where we use (5) in (35).

In the following, we consider the decaying estimate for u^m. We divide the proof into five steps.

Step 1: *We show that* $\|\Delta u^m(t)\|_2 \to 0$ *as* $t \to \infty$. The argument in Step 1 is based on [6] (see also [9]). Combining with (10), (16) and (34), we see that

$$\begin{aligned}
\|\nabla u^m(t)\|_p^p &\le S_p^{-\frac{p}{2}} \|\Delta u^m(t)\|_2^p \\
&< S_p^{-\frac{p}{2}} \left(\frac{1}{2} - \frac{1}{p}\right)^{-\frac{p-2}{2}} E(u^m(t))^{\frac{p-2}{2}} \|\Delta u^m(t)\|_2^2 \\
&\le S_p^{-\frac{p}{2}} \left(\frac{1}{2} - \frac{1}{p}\right)^{-\frac{p-2}{2}} E(u^m(0))^{\frac{p-2}{2}} \|\Delta u^m(t)\|_2^2 \\
&= \left(\frac{E(u^m(0))}{d}\right)^{\frac{p-2}{2}} \|\Delta u^m(t)\|_2^2
\end{aligned}$$

for $t > 0$. This implies that

$$I(u^m(t)) > \gamma \|\Delta u^m(t)\|_2^2, \quad t > 0, \tag{36}$$

where

$$\gamma := 1 - \sup_{m \geq m_*} \left(\frac{E(u^m(0))}{d} \right)^{\frac{p-2}{2}} \in (0, 1).$$

Hence by (33) and (36) we obtain

$$E(u^m(t)) < \left[\frac{1}{\gamma} \left(\frac{1}{2} - \frac{1}{p} \right) + \frac{1}{p} \right] I(u^m(t)), \quad t > 0. \tag{37}$$

On the other hand, we deduce from (17), (31) and (34) that

$$\int_t^\infty I(u^m(\tau))\, d\tau \leq \|u(t)\|_2^2 \leq \mu_1^{-2} \|\Delta u^m(t)\|_2^2 < \mu_1^{-2} \left(\frac{1}{2} - \frac{1}{p} \right)^{-1} E(u^m(t)) \tag{38}$$

for $t > 0$. Thus it follows from (37) and (38) that

$$\int_t^\infty E(u^m(\tau))\, d\tau < A E(u^m(t)), \quad \text{i.e.,} \quad \frac{d}{dt} \left[e^{\frac{t}{A}} \int_t^\infty E(u^m(\tau))\, d\tau \right] < 0, \tag{39}$$

for $t > 0$, where

$$A := \frac{1}{\mu_1^2} \left(\frac{1}{\gamma} + \frac{2}{p-2} \right).$$

Therefore, combining (32) with (39), we have

$$\int_t^\infty E(u^m(\tau))\, d\tau < e^{1-\frac{t}{A}} \int_A^\infty E(u^m(\tau))\, d\tau < A e^{1-\frac{t}{A}} E(u^m(A)) < A d e^{1-\frac{t}{A}} \tag{40}$$

for $t > A$. Since it follows from $u^m(t) \in W$ and (33) that $E(u^m(t)) > 0$, by (16) and (34) we see that

$$\begin{aligned} \int_t^\infty E(u^m(\tau))\, d\tau &\geq \int_t^{A+t} E(u^m(\tau))\, d\tau \\ &\geq A E(u^m(A+t)) > A \left(\frac{1}{2} - \frac{1}{p} \right) \|\Delta u^m(A+t)\|_2^2 \end{aligned} \tag{41}$$

for $t > 0$. This together with (40) implies that

$$\|\Delta u^m(t)\|_2^2 \le e^2 S_p^{\frac{p}{p-2}} e^{-\frac{t}{A}}, \quad t > 2A.$$

Therefore, recalling (35), we obtain

$$\|\Delta u^m(t)\|_2^2 \le c_A e^{-\frac{t}{A}}, \quad t > 0, \tag{42}$$

where c_A is a constant which is independent of m and t.

Step 2: *We derive a modified decay rate.* Fix $\varepsilon \in (0, 1)$ arbitrarily. We first prove the following: there exists $c_\varepsilon > 0$ such that

$$\|u^m(t)\|_2 \le c_\varepsilon e^{-(1-\varepsilon)\mu_1^2 t}, \quad t > 0. \tag{43}$$

Multiplying a_k^m by (15) and summing k from 1 to m, we see that

$$\frac{1}{2}\frac{d}{dt}\|u^m(t)\|_2^2 + \|\Delta u^m(t)\|_2^2 = \|\nabla u^m(t)\|_p^p, \quad t > 0. \tag{44}$$

This together with (10) implies that

$$\frac{1}{2}\frac{d}{dt}\|u^m(t)\|_2^2 + \|\Delta u^m(t)\|_2^2 \le S_p^{-p/2}\|\Delta u^m(t)\|_2^p, \quad t > 0. \tag{45}$$

Thanks to (42), we find $T_\varepsilon > 0$ such that

$$S_p^{-p/2}\|\Delta u^m(t)\|_2^{p-2} < \varepsilon, \quad t > T_\varepsilon. \tag{46}$$

Thus we observe from (45) and (46) that

$$\frac{d}{dt}\|u^m(t)\|_2^2 + 2(1-\varepsilon)\|\Delta u^m(t)\|_2^2 \le 0, \quad t > T_\varepsilon.$$

Hence by (31), we have

$$\frac{d}{dt}\left(e^{2(1-\varepsilon)\mu_1^2 t}\|u^m(t)\|_2^2\right) \le 0, \quad \text{i.e.,} \quad e^{2(1-\varepsilon)\mu_1^2 t}\|u^m(t)\|_2^2 \le e^{2(1-\varepsilon)\mu_1^2 T_\varepsilon}\|u^m(T_\varepsilon)\|_2^2$$

for $t > T_\varepsilon$. This clearly implies (43), because it follows from (9) and (35) that $\|u^m(t)\|_2^2 < c_1 S_p^{p/(p-2)}$ for $t > 0$.

Moreover, it follows from (17), (37) and (43) that for $\varepsilon \in (0, 1)$

$$\int_t^\infty E(u^m(\tau))\, d\tau < c_\varepsilon \left[\frac{1}{\gamma}\left(\frac{1}{2} - \frac{1}{p}\right) + \frac{1}{p}\right] e^{-2(1-\varepsilon)\mu_1^2 t}, \quad t > 0.$$

Along the same argument as in (41), we have

$$\int_t^\infty E(u^m(\tau))\,d\tau \ge \left(\frac{1}{2}-\frac{1}{p}\right)\|\Delta u^m(t+1)\|_2^2, \quad t>0.$$

Thus we find that $\tilde{c}_\varepsilon > 0$ such that

$$\|\Delta u^m(t)\|_2 \le \tilde{c}_\varepsilon e^{-(1-\varepsilon)\mu_1^2 t}, \quad t>0. \tag{47}$$

Step 3: *We derive the precise decay rate*: *there exist* $c_*, \tilde{c}_* > 0$ *such that*

$$\|u^m(t)\|_2 \le c_* e^{-\mu_1^2 t}, \quad t>0, \tag{48}$$

$$\|\Delta u^m(t)\|_2 \le \tilde{c}_* e^{-\mu_1^2 t}, \quad t>0. \tag{49}$$

Fix $\varepsilon \in (0,1)$ arbitrarily. By (31), (44), (47) and Proposition 2.1 we have

$$\frac{d}{dt}\|u^m(t)\|_2^2 + 2\mu_1^2\|u^m(t)\|_2^2 \le 2c_2^p\tilde{c}_\varepsilon^{\theta p}e^{-\theta p(1-\varepsilon)\mu_1^2 t}\|u^m(t)\|_2^{(1-\theta)p}, \quad t>0,$$

that is,

$$\frac{d}{dt}\left(e^{2\mu_1^2 t}\|u^m(t)\|_2^2\right) \le 2c_2^p\tilde{c}_\varepsilon^{\theta p}e^{(\theta p\varepsilon-(p-2))\mu_1^2 t}\left(e^{2\mu_1^2 t}\|u^m(t)\|_2^2\right)^{(1-\theta)p/2}, \quad t>0,$$

where θ is as in (11). Since $p>2$ implies that

$$(1-\theta)p = \frac{2N-(N-2)p}{4} \in (0,1),$$

we have

$$\left(e^{2\mu_1^2 t}\|u^m(t)\|_2^2\right)^{1-(1-\theta)p/2} \le \|u_0\|_2^{2-(1-\theta)p} + \frac{4c_2^p\tilde{c}_\varepsilon^{\theta p}}{2-(1-\theta)p}\int_0^t e^{(\theta p\varepsilon-(p-2))\mu_1^2 s}\,ds$$

for $t>0$. Taking $\varepsilon>0$ sufficiently small, we see that (48) holds for some positive constant $c_*>0$. Similarly to (47), we obtain (49).

Step 4: *We show the asymptotic behavior of* u^m. Fix $\varepsilon \in (0,1)$ and $1 \le k \le k_*$ arbitrarily. We first prove the following: there exists $c_{\varepsilon,k} > 0$ such that

$$\|u^m(t) - P_{k-1}u^m(t)\|_2 \le c_{\varepsilon,k}e^{-(1-\varepsilon)\mu_k^2 t}, \quad t>0, \tag{50}$$

where μ_k, P_k are as in Sect. 1. Similarly to (44), we see that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u^m(t) - P_{k-1}u^m(t)\|_2^2 + \|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2^2 \\ &\quad = \int_\Omega |\nabla u^m(t)|^{p-2}\nabla u^m(t)\cdot\nabla(u^m(t) - P_{k-1}u^m(t))\,dx, \quad t > 0.\end{aligned} \tag{51}$$

Combining (51) with (10) and (49) we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u^m(t) - P_{k-1}u^m(t)\|_2^2 + \|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2^2 \\ &\quad \le \|\nabla u^m(t)\|_p^{p-1}\|\nabla(u^m(t) - P_{k-1}u^m(t))\|_p \\ &\quad \le S_p^{-p/2}\tilde{c}_*^{p-1}e^{-(p-1)\mu_1^2 t}\|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2 \\ &\quad \le \varepsilon\|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2^2 + \frac{S_p^{-p}\tilde{c}_*^{2(p-1)}}{4\varepsilon}e^{-2(p-1)\mu_1^2 t}\end{aligned}$$

for $t > 0$. Hence by (31) we obtain

$$\frac{d}{dt}\left(e^{2(1-\varepsilon)\mu_k^2 t}\|u^m(t) - P_{k-1}u^m(t)\|_2^2\right) \le \frac{S_p^{-p}\tilde{c}_*^{2(p-1)}}{2\varepsilon}e^{-2((p-1)\mu_1^2-\mu_k^2)t}$$

and we can find the constant $c_{\varepsilon,k} > 0$ satisfying (50). Along the same line as the above argument, we find $c_{\varepsilon,k_*+1} > 0$ such that

$$\|u^m(t) - P_{k_*}u^m(t)\|_2 \le c_{\varepsilon,k_*+1}e^{-(1-\varepsilon)(p-1)\mu_1^2 t}, \quad t > 0. \tag{52}$$

On the other hand, we observe from (10), (49), (50) and (51) that

$$\begin{aligned}&\int_t^\infty \|\Delta(u^m(\tau) - P_{k-1}u^m(\tau))\|_2^2\,dt \\ &\le \frac{1}{2}\|u^m(t) - P_{k-1}u^m(t)\|_2^2 + \int_t^\infty \|\nabla u^m(\tau)\|_p^{p-1}\|\nabla(u^m(t) - P_{k-1}u^m(t))\|_p\,d\tau \\ &\le \frac{1}{2}c_{\varepsilon,k}e^{-2(1-\varepsilon)\mu_k^2 t} + \tilde{c}_*^{p-1}S_p^{-p/2}\int_t^\infty e^{-(p-1)\mu_1^2\tau}\|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2\,d\tau \\ &\le \frac{1}{2}c_{\varepsilon,k}e^{-2(1-\varepsilon)\mu_k^2 t} + \frac{\tilde{c}_*^{2(p-1)}S_p^{-p}}{2}\int_t^\infty e^{-2(p-1)\mu_1^2\tau}\,d\tau \\ &\quad + \frac{1}{2}\int_t^\infty \|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2^2\,d\tau\end{aligned}$$

and hence there exists $\tilde{c}_{\varepsilon,k} > 0$ such that

$$\int_t^\infty \|\Delta(u^m(\tau) - P_{k-1}u^m(\tau))\|_2^2 \, dt \le \tilde{c}_{\varepsilon,k} e^{-2(1-\varepsilon)\mu_k^2 t}, \quad t > 0. \tag{53}$$

We improve the decay rate in (50). By (10), (31), (48) and (51) we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u^m(t) - P_{k-1}u^m(t)\|_2^2 + \mu_k^2\|u^m(t) - P_{k-1}u^m(t)\|_2^2 \\ &\quad \le S_p^{-p/2}\tilde{c}_*^{p-1} e^{-(p-1)\mu_1^2 t}\|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2 \end{aligned}$$

and hence

$$\begin{aligned} &\frac{d}{dt}\left(e^{2\mu_k^2 t}\|u^m(t) - P_{k-1}u^m(t)\|_2^2\right) \\ &\le 2S_p^{-p/2}\tilde{c}_*^{p-1} e^{(2\mu_k^2-(p-1)\mu_1^2)t}\|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2 \\ &\le S_p^{-p}\tilde{c}_*^{2(p-1)} e^{2((1+\delta)\mu_k^2-(p-1)\mu_1^2)t} + e^{2(1-\delta)\mu_k^2 t}\|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2^2 \end{aligned}$$

for $\delta \in (0, 1)$. Integrating this inequality, we see that

$$\begin{aligned} &e^{2\mu_k^2 t}\|u^m(t) - P_{k-1}u^m(t)\|_2^2 \\ &\le \|u_0\|_2^2 + S_p^{-p}\tilde{c}_*^{2(p-1)} \int_0^t e^{2((1+\delta)\mu_k^2-(p-1)\mu_1^2)\tau}\, d\tau \\ &+ \int_0^t e^{2(1-\delta)\mu_k^2\tau}\|\Delta(u^m(\tau) - P_{k-1}u^m(\tau))\|_2^2\, d\tau \\ &\le \|u_0\|_2^2 + S_p^{-p}\tilde{c}_*^{2(p-1)} \int_0^t e^{2((1+\delta)\mu_k^2-(p-1)\mu_1^2)\tau}\, d\tau \\ &+ \int_0^\infty \|\Delta(u^m(\tau) - P_{k-1}u^m(\tau))\|_2^2\, d\tau \\ &+ 2(1-\delta)\mu_k^2 \int_0^t e^{2(1-\delta)\mu_k^2\tau} \int_\tau^\infty \|\Delta(u^m(\eta) - P_{k-1}u^m(\eta))\|_2^2\, d\eta\, d\tau. \end{aligned}$$

Taking $\varepsilon > 0$, $\delta > 0$ sufficiently small, by (53) we find $\tilde{c}_k > 0$ such that

$$\|u^m(t) - P_{k-1}u^m(t)\|_2 \le \tilde{c}_k e^{-\mu_k^2 t}, \quad t > 0. \tag{54}$$

Step 5: *Letting $m \to \infty$, we obtain the conclusion.* It follows from the proof of Theorem 1.1 and Lemma 4.2 that u^m satisfies

$$u^m \overset{*}{\rightharpoonup} u \quad \text{weakly-* in} \quad L^\infty(0,T;H^2_{\mathcal{N}}(\Omega)) \quad \text{as} \quad m \to \infty, \tag{55}$$

$$u^m \to u \quad \text{in} \quad C([0,T];L^2_{\mathcal{N}}(\Omega)) \quad \text{as} \quad m \to \infty, \tag{56}$$

for $T \in (0,\infty)$. Combining (56) with (52) and (54), we obtain (7) and (8).

We prove (6). Fix $t > 0$ arbitrarily. Then (49) and (55) imply that

$$\operatorname*{ess\,sup}_{\tau\in(t,t+1)} \|\Delta u(\tau)\|_2 \le \liminf_{m\to\infty} \operatorname*{ess\,sup}_{\tau\in(t,t+1)} \|\Delta u^m(\tau)\|_2 \le \tilde{c}_* e^{-\mu_1^2 t}$$

and hence there exists $N = N_t \subset (t, t+1)$ of measure zero such that

$$\|\Delta u(\tau)\|_2 \le \tilde{c}_* e^{-\mu_1^2 t}, \quad \tau \in (t,t+1) \setminus N_t.$$

Let $\{\tau_k\}_{k=1}^\infty \subset (t,\infty) \setminus N_t$ such that $\tau_k \to t$ as $k \to \infty$. Since $u \in C_{\mathrm{w}}([0,\infty); W^{2,2}_{\mathcal{N}}(\Omega))$, we have

$$\|\Delta u(t)\|_2 \le \liminf_{k\to\infty} \|\Delta u(\tau_k)\|_2 \le \tilde{c}_* e^{-\mu_1^2 t}.$$

Since $t > 0$ is arbitrary, we obtain (6). Therefore, Theorem 1.3 follows.

5 Appendix

In this section, we prove the uniqueness of solutions to problem (P). Let u_1 and u_2 be solutions to problem (P). In what follows, by the letter C we denote generic positive constants (which may depend on u_1 and u_2) and they may have different values also within the same line.

Let $t_0, t_1 \in (0,T)$ with $t_0 < t_1$, $h \in (0, \min\{t_0, T - t_1, t_1 - t_0\}/2)$. Define

$$W_h(x,t) := \frac{1}{h}\int_t^{t+h} \chi_{[t_0,t_1]}(\tau) w(x,\tau)\,d\tau, \quad \tilde{W}_h(x,t) := \frac{1}{h}\int_{t-h}^{t} W_h(x,\tau)\,d\tau,$$

for $(x,t) \in \Omega \times [0,T]$, where $w := u_1 - u_2$ and

$$\chi_{[t_0,t_1]}(t) := \begin{cases} 1 & \text{if} \quad t \in [t_0,t_1], \\ 0 & \text{if} \quad t \in \mathbb{R} \setminus [t_0,t_1]. \end{cases}$$

We remark that $\tilde{W}_h$ belongs to $\mathscr{V}$ and satisfies the following properties (cf. [3]):

$$\tilde{W}_h \to \chi_{[t_0,t_1]} w \quad \text{as} \quad h \searrow 0 \quad \text{in} \quad L^2(0,T; H^2_{\mathscr{N}}(\Omega)), \tag{57}$$

$$\nabla \tilde{W}_h \to \chi_{[t_0,t_1]} \nabla w \quad \text{as} \quad h \searrow 0 \quad \text{in} \quad (L^p(0,T; L^p(\Omega)))^N. \tag{58}$$

Taking $\varphi = \tilde{W}_h$ in (2), we see that

$$\begin{aligned}
&-\int_0^T \int_\Omega w \partial_t \tilde{W}_h \, dx \, dt + \int_0^T \int_\Omega \Delta w \Delta \tilde{W}_h \, dx \, dt \\
&= \int_0^T \int_\Omega (|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2) \cdot \nabla \tilde{W}_h \, dx \, dt \\
&\le C \int_0^T (\|\nabla u_1(t)\|_p^{p-2} + \|\nabla u_2(t)\|_p^{p-2}) \|\nabla w(t)\|_p \|\nabla \tilde{W}_h(t)\|_p \, dt.
\end{aligned} \tag{59}$$

On the other hand, setting

$$w_h(x,t) := \frac{1}{h} \int_t^{t+h} w(x,\tau) \, d\tau,$$

we see that

$$\begin{aligned}
&-\int_0^T \int_\Omega w \partial_t \tilde{W}_h \, dx \, dt \\
&= \int_0^T \int_\Omega \partial_t w_h(x,t) W_h(x,t) \, dx \, dt \\
&= \int_{t_0}^{t_1-h} \int_\Omega \partial_t w_h(x,t) w_h(x,t) \, dx \, dt + \left(\int_{t_0-h}^{t_0} + \int_{t_1-h}^{t_1} \right) \int_\Omega \partial_t w_h(x,t) W_h(x,t) \, dx \, dt \\
&= \frac{1}{2} (\|w_h(t_1-h)\|_2^2 - \|w_h(t_0)\|_2^2) + \left(\int_{t_0-h}^{t_0} + \int_{t_1-h}^{t_1} \right) \int_\Omega \partial_t w_h(x,t) W_h(x,t) \, dx \, dt \\
&\to \frac{1}{2} (\|w(t_1)\|_2^2 - \|w(t_0)\|_2^2) \quad \text{as} \quad h \searrow 0.
\end{aligned}$$

This together with (57), (58) and (59) implies that

$$\begin{aligned}
&\frac{1}{2} (\|w(t_1)\|_2^2 - \|w(t_0)\|_2^2) + \int_{t_0}^{t_1} \|\Delta w(t)\|_2^2 \, dt \\
&\quad \le C \int_{t_0}^{t_1} (\|\nabla u_1(t)\|_p^{p-2} + \|\nabla u_2(t)\|_p^{p-2}) \|\nabla w(t)\|_p^2 \, dt.
\end{aligned} \tag{60}$$

Since $t_0, t_1 \in (0,T)$ are arbitrary, (60) holds for $t_0, t_1 \in [0,T]$.

We prove the uniqueness of solutions to problem (P) in Theorems 1.1 and 1.2 in Sect. 5.1 and Sect. 5.2, respectively.

5.1 Uniqueness of Solutions in Theorem 1.1

Since $\nabla u_1, \nabla u_2 \in (L^\infty(0,T;L^p(\Omega)))^N$ and $w(0)=0$, we observe from (60) that

$$\|w(t)\|_2^2 + 2\int_0^t \|\Delta w(\tau)\|_2^2\,d\tau \le C\int_0^t \|\nabla w(\tau)\|_p^2\,d\tau$$

for $t \in [0,T]$. By Proposition 2.1 and the Young inequality we have

$$\|w(t)\|_2^2 \le C\int_0^t \|w(\tau)\|_2^2\,d\tau, \quad \text{i.e.,} \quad \frac{d}{dt}\left(e^{-Ct}\int_0^t \|w(\tau)\|_2^2\,d\tau\right) \le 0.$$

This implies that $w \equiv 0$ and we complete the proof of the uniqueness in Theorem 1.1.

5.2 Uniqueness of Solutions in Theorem 1.2

Fix $t_0, t_1 \in [0,T]$ arbitrarily. By Proposition 2.1 and $u_j \in C([0,T];L^2_{\mathcal{N}}(\Omega))$ we have

$$\begin{aligned}
&\int_{t_0}^{t_1} \|\nabla u_j(t)\|_p^{p-2}\|\nabla w(t)\|_p^2\,dt \\
&\quad \le C\left(\sup_{t\in[t_0,t_1]}\|w(t)\|_2^2\right)^{1-\theta}\int_{t_0}^{t_1}\|\Delta u_j(t)\|_2^{(p-2)\theta}\|\Delta w(t)\|_2^{2\theta}\,dt
\end{aligned} \tag{61}$$

for $j=1,2$. Moreover, since $2<p<p_*$ is equivalent to $0<\theta p<2$, it holds that

$$\begin{aligned}
&\int_{t_0}^{t_1}\|\Delta u_j(t)\|_2^{(p-2)\theta}\|\Delta w(t)\|_2^{2\theta}\,dt \\
&\quad \le C(t_1-t_0)^{1-\theta p/2}\left(\int_{t_0}^{t_1}\|\Delta u_j(t)\|_2^2\,dt\right)^{(p-2)\theta/2}\left(\int_{t_0}^{t_1}\|\Delta w(t)\|_2^2\,dt\right)^{\theta} \\
&\quad \le C(t_1-t_0)^{1-\theta p/2}\left(\int_{t_0}^{t_1}\|\Delta w(t)\|_2^2\,dt\right)^{\theta}
\end{aligned}$$

for $j = 1, 2$. This together with (60) and (61) we have

$$\begin{aligned}
&\frac{1}{2}(\|w(t_1)\|_2^2 - \|w(t_0)\|_2^2) + \int_{t_0}^{t_1} \|\Delta w\|_2^2\,dt \\
&\quad\le C(t_1 - t_0)^{1-\theta p/2} \left(\int_{t_0}^{t_1} \|\Delta w(t)\|_2^2\,dt\right)^{\theta} \left(\sup_{t\in[t_0,t_1]} \|w(t)\|_2^2\right)^{1-\theta} \\
&\quad\le \frac{1}{2}\int_{t_0}^{t_1} \|\Delta w(t)\|_2^2\,dt + C(t_1 - t_0)^{(2-\theta p)/2(1-\theta)} \sup_{t\in[t_0,t_1]} \|w(t)\|_2^2,
\end{aligned}$$

that is,

$$\|w(t_1)\|_2^2 \le \|w(t_0)\|_2^2 + c_0(t_1 - t_0)^{(2-\theta p)/2(1-\theta)} \sup_{t\in[t_0,t_1]} \|w(t)\|_2^2, \tag{62}$$

where $c_0 > 0$ is a constant depending on N, p, Ω, u_1 and u_2. Let $\delta > 0$ be such that

$$c_0\delta^{(2-\theta p)/2(1-\theta)} < \frac{1}{2}.$$

Since $w(0) = 0$, we observe from (62) and $0 < \theta p < 2$ that $w(t) = 0$ for $t \in [0, \delta]$. Iterating this argument, we obtain $w \equiv 0$ in $[0, T]$ and we complete the proof of the uniqueness in Theorem 1.2.

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