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Geometric Properties for Parabolic and Elliptic PDE's

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Vincenzo Ferone • Tatsuki Kawakami •
Paolo Salani • Futoshi Takahashi
Editors

Geometric Properties for Parabolic and Elliptic PDEs

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Editors
Vincenzo Ferone
Dipartimento di Matematica e Applicazioni
"R. Caccioppoli"
Università degli Studi di Napoli Federico II
Napoli, Italy

Tatsuki Kawakami
Applied Mathematics and Informatics
Course, Faculty of Advanced Science
and Technology
Ryukoku University
Otsu, Shiga, Japan

Paolo Salani
DIMAI Dipartimento di Matematica e
Informatica "U. Dini"
Università degli Studi di Firenze
Firenze, Italy

Futoshi Takahashi
Department of Mathematics
Osaka City University
Osaka, Japan

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Preface

This monograph contains contributions from the speakers at the 6th Italian-Japanese Workshop on Geometric Properties for Parabolic and Elliptic PDEs, which was held in Cortona (Italy) during May 20–24, 2019. The first five workshops were held in Sendai (Japan, 2009), Cortona (Italy, 2011), Tokyo (Japan, 2013), Palinuro (Italy, 2015), and Osaka (Japan, 2017), and in all the occasions the proceedings were subsequently published: see, respectively, Discrete Contin. Dyn. Syst. Ser. S 4 (2011), Springer INdAM Series 2 (2013), Kodai Math. J. 37 (2014), Springer Proceedings in Mathematics & Statistics 176 (2016) and Applicable Analysis 98, Issue 10 (2020). Based on the success of the previous workshops and the associated publications, we believe that this monograph will be of great interest for the mathematical community and in particular for researchers studying parabolic and elliptic PDEs.

As would be expected from such a wide topic, the contributions are very diverse. They cover many different fields of current research as follows: optimization problems, Sobolev inequalities, Hardy-type inequalities on trees, quasilinear chemotaxis system, linear and nonlinear parabolic and elliptic equations, neutral inclusion, Lane-Emden equation, fourth-order parabolic equation, and semilinear damped wave equation. In order to guarantee quality, all the papers have been submitted to two referees, chosen among the experts on related topics.

Naples, Italy — Vincenzo Ferone
Otsu, Japan — Tatsuki Kawakami
Firenze, Italy — Paolo Salani
Osaka, Japan — Futoshi Takahashi

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About the Editors

Vincenzo Ferone is a full professor of mathematical analysis at Dipartimento di Matematica e Applicazioni "R. Caccioppoli" of Università di Napoli Federico II. His main research interests concern various topics in the calculus of variations and in the theory of partial differential equations, such as optimization problems on classes of equimeasurable functions, comparison results for solutions to PDEs, optimization, existence and regularity for solutions to PDEs, and isoperimetric inequalities.

Tatsuki Kawakami has been a professor in the Faculty of Advanced Science and Technology at Ryukoku University since 2019. He is interested in research fields such as elliptic and parabolic PDEs, the existence of solutions and their asymptotic behavior, dynamic boundary problems, and related topics.

Paolo Salani is a full professor of mathematical analysis in the Dipartimento di Matematica e Informatica "U. Dini" of Università degli Studi di Firenze. His main research interests concern the study of geometric properties (such as convexity and starshape) of solutions to elliptic and parabolic PDEs, overdetermined problems, geometric and analytic inequalities of isoperimetric flavor, and convex geometry and analysis.

Futoshi Takahashi has been a professor in the Department of Mathematics at Osaka City University since 2009. He has been interested in research fields such as variational methods, functional inearities, and related nonlinear PDEs.

Poincaré and Hardy Inequalities on Homogeneous Trees

Elvise Berchio, Federico Santagati, and Maria Vallarino

Abstract We study Hardy-type inequalities on infinite homogeneous trees. More precisely, we derive optimal Hardy weights for the combinatorial Laplacian in this setting and we obtain, as a consequence, optimal improvements for the Poincaré inequality.

Keywords Graphs · Poincaré–Hardy inequalities · Homogeneous trees

1 Introduction

Given a linear, elliptic, second-order, symmetric nonnegative operator P on Ω, where Ω is a (e.g. Euclidean) domain, a *Hardy weight* is a nonnegative function W such that the following inequality holds

$$q(u) \geq \int_\Omega W u^2 \,\mathrm{d}x \quad \forall u \in C_c^\infty(\Omega), \tag{1.1}$$

where $q(u) = \langle u, Pu \rangle$ is the quadratic form associated to P. Clearly, the final (and most ambitious) goal is to get weights W such that inequality (1.1) is not valid for $V \geq W$, $V \neq W$, i.e. the operator $P - W$ is *critical* in the sense of [18, Definition 2.1]. When $P = -\Delta$ is the Laplace–Beltrami operator on a Riemannian manifold, the problem of the existence of Hardy weights has been widely studied in the literature, either in the Euclidean setting, see e.g. [4, 9, 10, 20, 27–29] or on general manifolds, see e.g. [11, 17, 18, 24, 26, 30, 35]. Recently, the attention has also been devoted to the discrete setting, see e.g. [8, 21–23] and references therein.

The present paper is motivated by some recent results obtained in [5], see also [1] and [6], within the context of Cartan–Hadamard manifolds M. In particular, when

E. Berchio (✉) · F. Santagati · M. Vallarino
Dipartimento di Scienze Matematiche "Giuseppe Luigi Lagrange", Dipartimento di Eccellenza 2018–2022, Politecnico di Torino, Torino, Italy
e-mail: elvise.berchio@polito.it; federico.santagati@polito.it; maria.vallarino@polito.it

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M is the hyperbolic space $\mathbb{H}^N$, i.e. the simplest example of manifold with negative sectional curvature, the following Hardy weight has been determined when P being the Laplace–Beltrami operator $-\Delta_{\mathbb{H}^N}$ on $\mathbb{H}^N$ with $N \geq 3$:

$$W(r) = \frac{(N-1)^2}{4} + \frac{1}{4\,r^2} + \frac{(N-1)(N-3)}{4}\frac{1}{\sinh^2 r},$$

where $r = d(o, x) > 0$ denotes the geodesic distance of x from a fixed pole $o \in \mathbb{H}^N$. Besides, it is proved that the operator $-\Delta_{\mathbb{H}^N} - W$ is *critical* in $\mathbb{H}^N \setminus \{o\}$. It is worth noticing that the number $\frac{(N-1)^2}{4}$ in $W(r)$ coincides with the bottom of the L^2-spectrum of $-\Delta_{\mathbb{H}^N}$. Hence, the existence of the above weight yields the following improved Poincaré inequality:

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} - \frac{(N-1)^2}{4}\int_{\mathbb{H}^N} u^2 \,\mathrm{d}v_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} R\, u^2 \,\mathrm{d}v_{\mathbb{H}^N} \quad \forall u \in C_c^\infty(\mathbb{H}^N),$$

where the remainder term is

$$R(r) = \frac{1}{4\,r^2} + \frac{(N-1)(N-3)}{4}\frac{1}{\sinh^2 r} \sim \frac{1}{4\,r^2} \quad \text{as } r \to +\infty, \tag{1.2}$$

and, as a consequence of the criticality issue, all constants in (1.2) turn out to be sharp.

Let $\Gamma = (V, E)$ denote a locally finite graph, where V and E denote a countably infinite set of vertices and the set of edges respectively. We recall that the combinatorial Laplacian Δ of a function f in the set $C(V)$ of real valued functions defined on V is defined by

$$\Delta f(x) := \sum_{y\sim x}\Big(f(x) - f(y)\Big) = m(x) f(x) - \sum_{y\sim x} f(y) \qquad \forall x \in V\,,$$

where $m(x)$ is the degree of x, i.e. the number of neighbors of x. The existence of Hardy weights for the combinatorial Laplacian or for more general operators on graphs has been recently studied in literature (see again [8, 21–23]).

We set our analysis on the case where the graph Γ is the homogeneous tree $\mathbb{T}_{q+1}$, i.e. a connected graph with no loops such that every vertex has $q+1$ neighbours, and we focus on the transient case, namely we always assume $q \geq 2$. $\mathbb{T}_{q+1}$ has been the object of investigation of many papers either in the field of harmonic analysis or of PDEs, see e.g. to [2, 3, 7, 12–16, 19, 31]. In particular, the homogeneous tree is in many respects a discrete analogue of the hyperbolic plane; we refer the reader to [7] for a discussion on this point. Therefore, since $\mathbb{T}_{q+1}$ is the basic example of graph of exponential growth, as $\mathbb{H}^N$ is the basic example of Riemannian manifold with exponential growth, it is natural to investigate whether the above mentioned results in $\mathbb{H}^N$ have a counterpart in $\mathbb{T}_{q+1}$: this will be the main goal of the paper.

In $\mathbb{T}_{q+1}$ the operator Δ is bounded on ℓ^2 and its ℓ^2-spectrum is given by $[(q^{1/2}-1)^2, (q^{1/2}+1)^2]$ (see [16]). Hence the following Poincaré inequality holds

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2 \geq \Lambda_q \sum_{x\in\mathbb{T}_{q+1}}\varphi^2(x) \qquad \forall\varphi\in C_0(\mathbb{T}_{q+1}),$$

with $\Lambda_q := (q^{1/2}-1)^2$.

By Keller et al. [22, Theorem 0.2] a Hardy weight for Δ on a transient graph Γ, is given by $W_{opt} = \frac{\Delta G_o^{1/2}}{G_o^{1/2}}$, where $G_o(x) := G(x,o)$ is the positive minimal Green function and o is a fixed point. Furthermore, W_{opt} is optimal in the sense of Definition 2.2 below and this implies, in particular, that the operator $\Delta - W_{opt}$ is *critical*. If $\Gamma = \mathbb{T}_{q+1}$, then the function G_o can be written explicitly, see Proposition 2.3 below, and W_{opt} reads as follows:

$$W_{opt}(x) = \begin{cases} \Lambda_q + q^{1/2} - q^{-1/2} & \text{if } |x| = 0, \\ \Lambda_q & \text{if } |x| \geq 1\,, \end{cases} \tag{1.3}$$

where, for each vertex $x \in \Gamma$, $|x| = d(x,o)$ and d is the usual discrete metric. By exploiting the super-solutions technique, in the present paper we provide the following new family of Hardy weights for Δ on $\mathbb{T}_{q+1}$:

$$W_{\beta,\gamma}(x) = \begin{cases} q+1-q^{1/2}(\frac{1}{\gamma}+\frac{1}{\gamma q}) & \text{if } |x| = 0, \\ q+1-q^{1/2}(2^\beta+\gamma) & \text{if } |x| = 1, \\ q+1-q^{1/2}[(1+\frac{1}{|x|})^\beta + (1-\frac{1}{|x|})^\beta] & \text{if } |x| \geq 2, \end{cases}$$

where $0 \leq \beta \leq \log_2 q^{1/2}$ and $q^{-1/2} \leq \gamma \leq q^{-1/2} + q^{1/2} - 2^\beta$. Moreover, if $\beta = 1/2$ we prove that the weight $W_{1/2,\gamma}$ is optimal (see again Definition 2.2), hence the operator $\Delta - W_{1/2,\gamma}$ is *critical*. We notice that

$$W_{\beta,\gamma}(x) = \Lambda_q + q^{1/2}\frac{\beta(1-\beta)}{|x|^2} + o\Big(\frac{1}{|x|^2}\Big) \qquad \text{as } |x| \to \infty\,,$$

hence the slowest decay at infinity occurs exactly for $\beta = 1/2$.

It is readily seen that the quadratic form inequality associated to $\Delta - W_{opt}$ in (1.3) can be read as an (optimal) *local* improvement of the Poincaré inequality on $\mathbb{T}_{q+1}$ at o. A direct inspection reveals that the weights $W_{\beta,\gamma}$ satisfy $W_{\beta,\gamma} > \Lambda_q$ on $\mathbb{T}_{q+1}$ for all $0 \leq \beta \leq \log_2\big(\frac{3}{2} - \frac{1}{2q}\big)$ and $\frac{1}{2} + \frac{1}{2q} \leq \gamma \leq 2 - 2^\beta$. Hence, for such values of β and γ, we derive the following family of *global* improved Poincaré inequalities:

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2 \geq \sum_{x\in\mathbb{T}_{q+1}}\Lambda_q\varphi^2(x)+\sum_{x\in\mathbb{T}_{q+1}}R_{\beta,\gamma}(x)\varphi^2(x) \qquad \forall\varphi\in C_0(\mathbb{T}_{q+1}), \tag{1.4}$$

where

$$0\leq R_{\beta,\gamma}(x)=\begin{cases} q^{1/2}(2-\frac{1}{\gamma}-\frac{1}{\gamma q}) & \text{if } |x|=0,\\ q^{1/2}(2-2^{\beta}-\gamma) & \text{if } |x|=1,\\ q^{1/2}\Big(2-(1+\frac{1}{|x|})^{\beta}-(1-\frac{1}{|x|})^{\beta}\Big) & \text{if } |x|\geq 2.\end{cases}$$

It is worth noticing that the maximum of $R_{\beta,\gamma}$ at o is reached by choosing γ as large as possible, namely by taking $\gamma = 2-2^{\beta}$. Since such value is maximum for $\beta = 0$, we conclude that, among the weights $W_{\beta,\gamma}$ improving the Poincaré inequality, the largest at o is $W_{0,1}\equiv W_{opt}$.

Even if (1.4) improves globally the Poincaré inequality, we do not know whether this improvement is sharp on the whole $\mathbb{T}_{q+1}$. Nevertheless, a *sharp* improvement is provided by the critical weight $W_{1/2,\gamma}$ outside the ball $B_2(o)$. More precisely, there holds

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2 \geq \sum_{x\in\mathbb{T}_{q+1}}\Lambda_q\varphi^2(x)+\sum_{x\in\mathbb{T}_{q+1}}\overline{R}(x)\varphi^2(x), \qquad \forall\varphi\in C_0(\mathbb{T}_{q+1}\setminus B_2(o)),$$

where

$$\overline{R}(x)=q^{1/2}\left[2-\left(1+\frac{1}{|x|}\right)^{1/2}-\left(1-\frac{1}{|x|}\right)^{1/2}\right] \qquad \text{if } |x|\geq 2$$

and the constant $q^{1/2}$ is sharp. Notice that

$$\overline{R}(x)\sim q^{1/2}\frac{1}{4|x|^2} \quad \text{as } |x|\to+\infty,$$

namely the decay of the remainder term is of the same order of that provided by (1.2) in $\mathbb{H}^N$, thereby confirming the analogy between $\mathbb{T}_{q+1}$ and $\mathbb{H}^N$.

Following the arguments used in the particular case of a homogeneous tree, in the last part of the paper we find a class of Hardy weights for the combinatorial Laplacian on rapidly growing *radial* trees, i.e. trees where the number of neighbours of a vertex x only depends on the distance of x from a fixed vertex o. This is a first result which might shed light on future related investigations on more general graphs.

The paper is organized as follows. In Sect. 2 we introduce the notation and we state our main results, namely Theorem 2.7, where we provide a family of optimal weights for Δ on $\mathbb{T}_{q+1}$, and Theorem 2.11 where we state the related improved Poincaré inequality. Section 3 is devoted to the proof of the statements of Sect. 2. Finally, in Sect. 4 we present a generalization of our results in the context of radial trees.

2 Notation and Main Results

We consider a graph $\Gamma = (V, E)$, where V and E denote a countably infinite set of vertices and the set of edges respectively, with the usual discrete metric d. If $(x, y) \in E$ we say that x and y are neighbors and we write $x \sim y$. We assume that Γ is a connected graph, that is, for every $x, y \in V$ there exists a finite sequence of vertices $x_1, \dots, x_n$ such that $x_0 = x$, $x_n = y$ and $x_j \sim x_{j+1}$ for $j = 0, \dots, n-1$. We also require that $(x, y) \in E$ if and only if $(y, x) \in E$. We use the notation $m(x)$ to indicate the degree of x, that is the number of edges that are attached to x and we assume that Γ is locally finite, i.e. $m(x) < \infty$ for all $x \in V$. When a vertex $o \in V$ is fixed let $x \mapsto |x|$ be the function which associates to each vertex x the distance $d(x, o)$ and define $B_r(o) = \{x \text{ s.t. } |x| < r\}$. We denote by $C(V)$ the set of real valued function defined on V and by $C_0(V)$ the subspace consisting on finitely supported functions. Finally, we introduce the space of square summable functions

$$\ell^2(V) = \{f \in C(V) \text{ s.t. } \sum_{x \in V} f^2(x) < +\infty\}.$$

This is a Hilbert space with the inner product

$$\langle f, g \rangle = \sum_{x \in V} f(x) g(x),$$

and the induced norm $\|f\| = \sqrt{\langle f, f \rangle}$. As shown in [33, 34]

$$\langle \Delta\varphi, \varphi \rangle_{\ell^2} = \frac{1}{2} \sum_{\substack{x, y \in V \\ x \sim y}} \Big(\varphi(x) - \varphi(y) \Big)^2 \qquad \forall \varphi \in C_0(V).$$

More generally, we consider Schrödinger operators $H = \Delta + Q$ where Q is any potential. A function f is called H-(super)harmonic in V if

$$Hf(x) = 0 \quad (Hf(x) \geq 0) \qquad \forall x \in V.$$

By Hardy-type inequality for a positive Schrödinger operator H we mean an inequality of the form

$$\langle H\varphi, \varphi\rangle \geq \langle W\varphi, \varphi\rangle \qquad \forall \varphi \in C_0(V),$$

where $W \not\equiv 0$ is a nonnegative function in $C(V)$. We write $h(\varphi)$ and $W(\varphi)$ in place of $\langle H\varphi, \varphi\rangle$ and $\langle W\varphi, \varphi\rangle$, respectively. In particular we denote $h_\Delta(\varphi) = \langle \Delta\varphi, \varphi\rangle$.

In [22] the authors introduce the notion of optimal weight for a Hardy-type inequality; we recall some fundamental definitions that we need in the later discussion.

Definition 2.1 Let h be a quadratic form associated with a Schrödinger operator H, such that $h \geq 0$ on $C_0(V)$. The form h is called **subcritical** in V if there is a nonnegative $W \in C_0(V)$, $W \not\equiv 0$, such that $h - W \geq 0$ on $C_0(V)$. A positive form h which is not subcritical is called **critical** in V.

In [23, Theorem 5.3] it is shown that the criticality of h is equivalent to the existence of a unique positive function which is H-harmonic. Such a function is called the ground state of h and we have the further definition:

Definition 2.2 Let h be a quadratic form associated with a Schrödinger operator H. We say that a positive function $W : V \to [0, \infty)$ is an **optimal** Hardy weight for h in V if

- $h - W$ is critical in V (**criticality**);
- $h - W \geq \lambda W$ fails to hold on $C_0(V \setminus K)$ for all $\lambda > 0$ and all finite $K \subset V$ (**optimality near infinity**);
- the ground state of $h - W$, $\Psi \notin \ell^2_W$ (**null-criticality**), namely

$$\sum_{x\in V} \Psi^2(x)W(x) = +\infty.$$

In the following, for shortness, we will say that the operator H is critical if and only if its associated quadratic form h is critical.

Finally, we recall that a function $u : V \to \mathbb{R}$ is named proper on V if $u^{-1}(K)$ is finite for all compact sets $K \subset u(V)$.

2.1 Hardy-Type Inequalities on $\mathbb{T}_{q+1}$

In this subsection we shall state various Hardy-type inequalities on the homogeneous tree $\mathbb{T}_{q+1}$ with $q \geq 2$. We start with an optimal inequality for Δ obtained by combining the explicit formula of the Green function and [22, Theorem 0.2].

Proposition 2.3 *For all $\varphi \in C_0(\mathbb{T}_{q+1})$ the following inequality holds:*

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2 \geq \sum_{x\in\mathbb{T}_{q+1}} W_{opt}(x)\varphi^2(x),$$

where

$$W_{opt}(x) = \begin{cases}\Lambda_q + q^{1/2} - q^{-1/2} & \text{if } |x| = 0,\\ \Lambda_q & \text{if } |x| \geq 1.\end{cases} \tag{2.1}$$

Furthermore, the weight W_{opt} is optimal for Δ.

Remark 2.4 As a consequence of the results of [22, Theorem 0.2] it follows that $G^{1/2}$ is the ground state of $h_\Delta - W_{opt}$. Furthermore, it is readily checked that

$$\sum_{x\in\mathbb{T}_{q+1}} G(x)W_{opt}(x) = +\infty\,,$$

namely $G^{1/2} \notin \ell^2_{W_{opt}}$.

In the next theorem we state a family of Hardy-type inequalities depending on two parameters β, γ. The weights $W_{\beta,\gamma}$ provided can be seen as a generalization of W_{opt}. Indeed, if we fix $\beta = 0$ and $\gamma = 1$ in the statement below, we obtain W_{opt}.

Theorem 2.5 *For all $0 \leq \beta \leq \log_2 q^{1/2}$ and $q^{-1/2} \leq \gamma \leq q^{1/2} + q^{-1/2} - 2^\beta$ the following inequality holds*

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2 \geq \sum_{x\in\mathbb{T}_{q+1}} W_{\beta,\gamma}(x)\varphi^2(x) \qquad \forall\varphi \in C_0(\mathbb{T}_{q+1})\,,$$

where $W_{\beta,\gamma} \geq 0$ is defined as follows:

$$W_{\beta,\gamma}(x) = \begin{cases} q+1-q^{1/2}(\frac{1}{\gamma}+\frac{1}{q\gamma}) & \text{if } |x| = 0,\\ q+1-q^{1/2}(2^\beta+\gamma) & \text{if } |x| = 1,\\ q+1-q^{1/2}[(1+\frac{1}{|x|})^\beta + (1-\frac{1}{|x|})^\beta] & \text{if } |x| \geq 2.\end{cases}$$

Remark 2.6 Notice that

$$W_{\beta,\gamma}(x) = \Lambda_q + q^{1/2}\frac{\beta(1-\beta)}{|x|^2} + o\Big(\frac{1}{|x|^2}\Big) \qquad \text{as } |x| \to \infty.$$

Since

$$\max_{\beta} \beta(1-\beta) = 1/4,$$

which is reached for $\beta = 1/2$, $W_{1/2,\gamma}$ is the largest among the $W_{\beta,\gamma}$ at infinity.

On the other hand, in order to maximize the value of $W_{\beta,\gamma}$ at o, γ has to be taken as large as possible, namely $\gamma = q^{-1/2} + q^{1/2} - 2^{\beta}$. Since this quantity is maximum for $\beta = 0$, the largest weight at o is $W_{0,\overline{\gamma}}$ with $\overline{\gamma} = q^{-1/2} + q^{1/2} - 1$. Notice that: $W_{0,\overline{\gamma}} \equiv W_{opt}$ for $|x| \geq 2$, while $W_{0,\overline{\gamma}}(o) > W_{opt}(o)$ and $W_{opt}(|x| = 1) > W_{0,\overline{\gamma}}(|x| = 1)$, hence the two weights are not globally comparable.

The previous remark suggests that, in order to have the largest weight at infinity, one has to fix $\beta = 1/2$ in Theorem 2.5. This intuition is somehow confirmed by the statement below.

Theorem 2.7 *For all $q^{-1/2} \leq \gamma \leq q^{-1/2} + q^{1/2} - 2^{1/2}$ the following inequality holds*

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \geq \sum_{x\in\mathbb{T}_{q+1}} W_{1/2,\gamma}(x)\varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1}),$$

where

$$W_{1/2,\gamma}(x) = \begin{cases} q+1-q^{1/2}(\frac{1}{\gamma}+\frac{1}{q\gamma}) & \text{if } |x| = 0,\\ q+1-q^{1/2}(2^{1/2}+\gamma) & \text{if } |x| = 1,\\ q+1-q^{1/2}[(1+\frac{1}{|x|})^{1/2}+(1-\frac{1}{|x|})^{1/2}] & \text{if } |x| \geq 2. \end{cases}$$

Furthermore, the weights $W_{1/2,\gamma}$ are optimal Hardy weights for Δ in the sense of Definition 2.2.

Using the same argument it is also possible to show that the weights we obtained in Theorem 2.5 are optimal near infinity, i.e. the constant is sharp in $\mathbb{T}_{q+1} \setminus K$ for every compact set K.

Corollary 2.8 *For all $0 \leq \beta < \min\{\log_2 q^{1/2}, 1\}$ and $q^{-1/2} \leq \gamma \leq q^{-1/2} + q^{1/2} - 2^{\beta}$ the following inequality holds*

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \geq \sum_{x\in\mathbb{T}_{q+1}} W_{\beta,\gamma}(x)\varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1}). \tag{2.2}$$

Moreover, the constant 1 *in front of the r.h.s. term is sharp at infinity, in the sense that inequality* (2.2) *fails on* $C_0(\mathbb{T}_{q+1} \setminus K)$ *if we replace* $W_{\beta,\gamma}$ *with* $CW_{\beta,\gamma}$*, for all* $C > 1$ *and all compact set* K*.*

2.2 Improved Poincaré Inequalities

We shall provide three examples of improved Poincaré inequalities derived by the Hardy-type inequalities stated in the previous subsection. We recall that the Poincaré inequality on $\mathbb{T}_{q+1}$ reads

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2 \geq \Lambda_q \sum_{x\in\mathbb{T}_{q+1}} \varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1}), \tag{2.3}$$

and the constant Λ_q is sharp in the sense that the above inequality cannot hold with a constant $\Lambda > \Lambda_q$.

The following improved Poincaré inequality is an immediate consequence of Theorem 2.3.

Proposition 2.9 *The following inequality holds*

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2$$
$$\geq \Lambda_q \sum_{x\in\mathbb{T}_{q+1}} \varphi^2(x) + \sum_{x\in\mathbb{T}_{q+1}} R_q(x)\varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1}), \tag{2.4}$$

where

$$R_q(x) = \begin{cases} q^{1/2} - q^{-1/2} & \text{if } |x| = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the operator $\Delta - \Lambda_q - R_q$ *is critical, hence the inequality does not hold with any* $R > R_q$*.*

Notice that (2.4) improves (2.3) only locally, namely at o. The next statement provides a global improvement of (2.3).

Theorem 2.10 *For all* $0 \le \beta \le \log_2\left(\frac{3}{2} - \frac{1}{2q}\right)$ *and* $\frac{1}{2} + \frac{1}{2q} \le \gamma \le 2 - 2^{\beta}$, *it holds*

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \ge \Lambda_q \sum_{x \in \mathbb{T}_{q+1}} \varphi^2(x) + \sum_{x \in \mathbb{T}_{q+1}} R_{\beta,\gamma} \varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1}), \tag{2.5}$$

where

$$0 \le R_{\beta,\gamma}(x) = \begin{cases} q^{1/2}(2 - \frac{1}{\gamma} - \frac{1}{q\gamma}) & \text{if } |x| = 0, \\ q^{1/2}(2 - 2^{\beta} - \gamma) & \text{if } |x| = 1, \\ q^{1/2}\Big(2 - (1 + \frac{1}{|x|})^{\beta} - (1 - \frac{1}{|x|})^{\beta}\Big) & \text{if } |x| \ge 2. \end{cases}$$

Notice that (2.5) improves globally (2.3) but it gives no information about the sharpness of $R_{\beta,\gamma}$. A sharp improvement is instead provided by the next theorem which holds for functions supported outside the ball $B_2(o)$.

Theorem 2.11 *The following inequality holds*

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \ge \sum_{x \in \mathbb{T}_{q+1}} \Lambda_q \varphi^2(x) + \sum_{x \in \mathbb{T}_{q+1}} \overline{R}(x) \varphi^2(x) \qquad \forall \varphi \in C_0(\mathbb{T}_{q+1} \setminus B_2(o)), \tag{2.6}$$

where

$$\overline{R}(x) = q^{1/2}\left[2 - \left(1 + \frac{1}{|x|}\right)^{1/2} - \left(1 - \frac{1}{|x|}\right)^{1/2}\right] \qquad \text{if } |x| \ge 2.$$

Moreover, the constant $q^{1/2}$ *is sharp in the sense that inequality* (2.6) *cannot hold if we replace the remainder term* $\overline{R}$ *with* $C\Big[2 - (1 + \frac{1}{|x|})^{1/2} - (1 - \frac{1}{|x|})^{1/2}\Big]$ *and* $C > q^{1/2}$.

3 Proofs of the Results

We collect here the proofs of the results stated in Sect. 2.

3.1 Proofs of Hardy-Type Inequalities

Proof of Proposition 2.3 Consider the function $\tilde{u}(x) = \sqrt{G(x,o)}$, where G is the Green function on $\mathbb{T}_{q+1}$. By Keller et al. [22, Theorem 0.2] we only need to show that

$$\frac{\Delta\tilde{u}(x)}{\tilde{u}(x)} = W_{opt}(x).$$

By the explicit formula for the Green function on $\mathbb{T}_{q+1}$ given in [32, Lemma 1.24] we have

$$\tilde{u}(x) = \sqrt{\frac{q}{q-1}\left(\frac{1}{q}\right)^{|x|}}.$$

For $x \neq o$, we obtain that

$$\frac{\Delta\tilde{u}(x)}{\tilde{u}(x)} = \left(q + 1 - \left(\frac{1}{q}\right)^{1/2} - q\left(\frac{1}{q}\right)^{-1/2}\right) = \left(q^{1/2} - 1\right)^2 = \Lambda_q.$$

For $x = o$ we get

$$\begin{aligned}\frac{\Delta\tilde{u}(o)}{\tilde{u}(o)} &= \left[(q+1)\left(\frac{q}{q-1}\right)^{1/2} - (q+1)\left(\frac{1}{(q-1)^{1/2}}\right)\right]\left(\frac{q-1}{q}\right)^{1/2} \\ &= q + 1 - \frac{q+1}{q^{1/2}} = \Lambda_q + q^{1/2} - q^{-1/2} > \Lambda_q.\end{aligned}$$

□

Proof of Theorem 2.5 The statement follows from [8, Proposition 3.1] by providing a suitable positive super-solution to the equation $\Delta u = W_{\beta,\gamma} u$ in $\mathbb{T}_{q+1}$. To this aim, we define the function:

$$u_{\beta,\gamma}(x) = \begin{cases} q^{-|x|/2}|x|^{\beta} & \text{if } |x| \geq 1, \\ \gamma & \text{if } |x| = 0. \end{cases} \tag{3.1}$$

Now, by writing $u = u_{\beta,\gamma}$, we have

$$\frac{\Delta u(o)}{u(o)} = q + 1 - (q+1)\frac{q^{-1/2}}{\gamma} = q + 1 - q^{1/2}\left(\frac{1}{\gamma} + \frac{1}{q\gamma}\right),$$

which is nonnegative if $\gamma \geq q^{-1/2}$.

Next, for every x such that $|x| = 1$, we have

$$\frac{\Delta u(x)}{u(x)} = q + 1 - q\frac{q^{-1}2^\beta}{q^{-1/2}} - \frac{\gamma}{q^{-1/2}} = q + 1 - q^{1/2}(2^\beta + \gamma),$$

which is nonnegative if $\gamma \leq q^{1/2} + q^{-1/2} - 2^\beta$. The restriction $\beta \leq 1/2 \log_2 q$ comes out to make the inequality consistent $q^{-1/2} \leq \gamma \leq q^{-1/2} + q^{1/2} - 2^\beta$.

Finally, for every x such that $|x| \geq 2$, we have

$$\begin{aligned}\frac{\Delta u(x)}{u(x)} &= q + 1 - q\frac{q^{-(|x|+1)/2}(|x|+1)^\beta}{q^{-|x|/2}|x|^\beta} - \frac{q^{-(|x|-1)/2}(|x|-1)^\beta}{q^{-|x|/2}|x|^\beta} \\ &= q + 1 - q^{1/2}\left[\left(1 + \frac{1}{|x|}\right)^\beta + \left(1 - \frac{1}{|x|}\right)^\beta\right] \geq 0. \end{aligned} \tag{3.2}$$

If $\beta \leq 1$, then the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = x^\beta$ is concave. It follows that

$$f\left(\frac{1}{2}\left(1 + \frac{1}{|x|}\right) + \frac{1}{2}\left(1 - \frac{1}{|x|}\right)\right) = f(1) \geq \frac{1}{2}f\left(1 + \frac{1}{|x|}\right) + \frac{1}{2}f\left(1 - \frac{1}{|x|}\right),$$

that is equivalent to

$$2 \geq \left(1 + \frac{1}{|x|}\right)^\beta + \left(1 - \frac{1}{|x|}\right)^\beta.$$

Then,

$$\begin{aligned}\frac{\Delta u(x)}{u(x)} &= q + 1 - q^{1/2}\left[\left(1 + \frac{1}{|x|}\right)^\beta + \left(1 - \frac{1}{|x|}\right)^\beta\right] \geq q + 1 - 2q^{1/2} \\ &= \Lambda_q > 0 \qquad \forall |x| \geq 2, \end{aligned}$$

which proves (3.2).

If $\log_2 q^{1/2} \geq \beta > 1$, notice that the function $h : [2, +\infty) \to \mathbb{R}$ defined by $h(x) = (1 + \frac{1}{x})^\beta + (1 - \frac{1}{x})^\beta$ is decreasing. Then h reaches its maximum at 2. Thus to show the validity of (3.2) it suffices to prove that

$$h(x) \leq h(2) = \left(\frac{3}{2}\right)^\beta + \left(\frac{1}{2}\right)^\beta \leq q^{1/2} + q^{-1/2}. \tag{3.3}$$

Notice that for every $\beta \geq 1$

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\left[\left(\frac{3}{2}\right)^\beta + \left(\frac{1}{2}\right)^\beta\right] = 2^{-\beta}(3^\beta \log(3/2) - \log(2)) \geq 0.$$

Hence

$$\left(\frac{3}{2}\right)^{\beta}+\left(\frac{1}{2}\right)^{\beta}\le\left(\frac{3}{2}\right)^{\log_2 q^{1/2}}+\left(\frac{1}{2}\right)^{\log_2 q^{1/2}}$$
$$\le 2^{\log_2 q^{1/2}}+2^{-\log_2 q^{1/2}}=q^{1/2}+q^{-1/2},$$

so that (3.3) holds and the proof is concluded. □

Remark 3.1 Note that the statement of Theorem 2.5 can be enriched by considering the family of radial functions

$$u_{\alpha,\beta,\gamma}(x)=\begin{cases}q^{\alpha|x|}|x|^{\beta} & \text{if } |x|\ge 1,\\ \gamma & \text{if } |x|=0,\end{cases}$$

with $\alpha\in\mathbb{R}$ and β and γ as in Theorem 2.5. Indeed, a straightforward computation shows that for $|x|\ge 2$

$$W_{\alpha,\beta,\gamma}(x)=\frac{\Delta u_{\alpha,\beta,\gamma}(x)}{u_{\alpha,\beta,\gamma}(x)}=q+1-q^{\alpha+1}\left(1+\frac{1}{|x|}\right)^{\beta}-q^{-\alpha}\left(1-\frac{1}{|x|}\right)^{\beta}.$$

Nevertheless,

$$W_{\alpha,\beta,\gamma}(x)=q+1-q^{1+\alpha}-q^{-\alpha}+o(1)\qquad\text{as } |x|\to+\infty,$$

which is maximum for $\alpha=-1/2$. Therefore, the choice $\alpha=-1/2$ turns out to be the best to get a weight as large as possible at ∞.

We shall now prove our main result, i.e. Theorem 2.7.

Proof of Theorem 2.7 Consider the Schrödinger operator $H:=\Delta+Q$, with

$$Q(x)=\begin{cases}0 & \text{if } |x|=0,\\ q^{1/2} & \text{if } |x|=1,\\ -\Lambda_q & \text{if } |x|\ge 2.\end{cases}$$

Step 1. We construct an optimal Hardy weight for H. To this aim, we exploit [22, Theorem 1.1] that provides an optimal Hardy weight for a Schrödinger operator H by using H-harmonic functions.

For the sake of completeness we start by briefly recalling the statement of [22, Theorem 1.1]: given two positive H-superharmonic functions u, v which are H-harmonic outside a finite set, if the function $u_0:=u/v$ is proper and $\sup_{x\sim y}u_0(x)/u_0(y)<+\infty$, then $\widetilde{W}:=\frac{H[(uv)^{1/2}]}{(uv)^{1/2}}$ is an optimal weight for H.

Next we define

$$u(x) := \begin{cases} \gamma & \text{if } |x| = 0, \\ q^{-|x|/2} & \text{if } |x| \geq 1, \end{cases}$$

$$v(x) := \begin{cases} \gamma & \text{if } |x| = 0, \\ |x| q^{-|x|/2} & \text{if } |x| \geq 1. \end{cases}$$

Now we show that u, v satisfy the hypothesis of the above-mentioned theorem. Indeed,

$$Hu(o) = (q+1)(\gamma - q^{-1/2}) + Q(o)\gamma \geq 0,$$
$$Hv(o) = (q+1)(\gamma - q^{-1/2}) + Q(o)\gamma \geq 0.$$

If $|x| = 1$, then

$$\begin{aligned} Hu(x) &= (q+1)q^{-1/2} - qq^{-1} - \gamma + q^{-1/2}q^{1/2} = q^{1/2} + q^{-1/2} - \gamma \geq 2^{1/2}, \\ Hv(x) &= (q+1)q^{-1/2} - 2q^{-1}q - \gamma + Q(x)q^{-1/2} \\ &\geq q^{1/2} + q^{-1/2} - 2 - q^{-1/2} - q^{1/2} + 2^{1/2} + 1 = 2^{1/2} + 1 - 2 > 0. \end{aligned}$$

If $|x| \geq 2$, then

$$\begin{aligned} Hu(x) &= (q+1)q^{-|x|/2} - qq^{-(|x|+1)/2} - q^{-(|x|-1)/2} - \Lambda_q q^{-|x|/2} \\ &= q^{-|x|/2}(q + 1 - 2q^{1/2} - \Lambda_q) = 0; \\ Hv(x) &= (q+1)|x|q^{-|x|/2} - (|x|+1)qq^{-(|x|+1)/2} \\ &\quad - (|x|-1)q^{-(|x|-1)/2} - \Lambda_q q^{-|x|/2} \\ &= |x|q^{-|x|/2}(q + 1 - 2q^{1/2} - \Lambda_q) = 0. \end{aligned}$$

Define now

$$u_0(x) := \frac{u(x)}{v(x)} = \begin{cases} 1 & \text{if } |x| = 0, \\ \frac{1}{|x|} & \text{otherwise.} \end{cases}$$

The function u_0 is proper because $\lim_{|x|\to\infty} u_0(|x|) = 0$ and $u_0(|x|) > u_0(|x|+1) > 0$ for all $|x| \geq 1$, thus $u_0^{-1}(K)$ is finite for all compact set $K \subset (0, \infty)$. Now consider $x \sim y$ and compute

$$\frac{u_0(x)}{u_0(y)} = \begin{cases} 1 & \text{if } |x| = 0, \\ 1/\gamma & \text{if } |y| = 0 \text{ and } |x| = 1, \\ 1 + \frac{1}{|x|} & \text{if } |y| = |x| + 1 \text{ and } |x| \geq 1, \\ 1 - \frac{1}{|x|} & \text{if } |y| = |x| - 1 \text{ and } |x| \geq 2. \end{cases}$$

Thus $\sup_{\substack{x,y\in\mathbb{T}_{q+1} \\ x\sim y}} \frac{u_0(x)}{u_0(y)} < +\infty$. Hence, from [22, Theorem 1.1] we conclude that the weight

$$\begin{aligned} \widetilde{W}(x) :&= \frac{H[(uv)^{1/2}](x)}{(uv)^{1/2}(x)} = \frac{\Delta(uv)^{1/2}(x)}{(uv)^{1/2}(x)} + Q(x) \\ &= \begin{cases} (q+1)(1 - \frac{q^{-1/2}}{\gamma}) & \text{if } |x| = 0, \\ (q+1) - q^{1/2}(2^{1/2} + \gamma) + q^{1/2} & \text{if } |x| = 1, \\ (q+1) - q^{1/2}[(1 + \frac{1}{|x|})^{1/2} + (1 - \frac{1}{|x|})^{1/2}] - \Lambda_q & \text{if } |x| \geq 2 \end{cases} \end{aligned}$$

is an optimal weight for H.

Step 2. We derive an optimal Hardy weight for Δ. To this aim we prove that the three conditions of Definition 2.2 are satisfied by the operator $\Delta - W_{1/2,\gamma}$, where $W_{1/2,\gamma} := \widetilde{W} - Q$.

- *Criticality:* the optimal Hardy inequality, obtained considering the quadratic form h associated with H, namely

$$\begin{aligned} &\frac{1}{2} \sum_{\substack{x,y\in\mathbb{T}_{q+1} \\ x\sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 + \sum_{x\in\mathbb{T}_{q+1}} Q(x)\varphi^2(x) \\ &\geq \sum_{x\in\mathbb{T}_{q+1}} \left(\frac{\Delta(uv)^{1/2}(x)}{(uv)^{1/2}(x)} + Q(x)\right)\varphi^2(x) \end{aligned}$$

  is equivalent to the Hardy inequality associated to Δ

$$\frac{1}{2} \sum_{\substack{x,y\in\mathbb{T}_{q+1} \\ x\sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 \geq \sum_{x\in\mathbb{T}_{q+1}} \frac{\Delta(uv)^{1/2}(x)}{(uv)^{1/2}(x)}\varphi^2(x) \qquad \forall\varphi \in C_0(\mathbb{T}_{q+1}).$$

Moreover,

$$W_{1/2,\gamma}(x) := \frac{\Delta(uv)^{1/2}(x)}{(uv)^{1/2}(x)}$$
$$= \begin{cases} q+1-q^{1/2}(\frac{1}{\gamma}+\frac{1}{q\gamma}) & \text{if } |x|=0, \\ q+1-q^{1/2}(2^{1/2}+\gamma) & \text{if } |x|=1, \\ q+1-q^{1/2}[(1+\frac{1}{|x|})^{1/2}+(1-\frac{1}{|x|})^{1/2}] & \text{if } |x|\geq 2, \end{cases}$$

is nonnegative. The optimality of $\widetilde{W}$ for H implies that it does not exist a nonnegative function $f \not\equiv 0$ such that

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2-\sum_{x\in\mathbb{T}_{q+1}}W_{1/2,\gamma}(x)\varphi^2(x)\geq\sum_{x\in\mathbb{T}_{q+1}}f(x)\varphi^2(x),$$

or, equivalently, $\Delta - W_{1/2,\gamma}$ is critical.

- *Null-criticality of* $\Delta - W_{1/2,\gamma}$ *with respect to* $W_{1/2,\gamma}$: the function $z = (uv)^{1/2}$ is the ground state of $h_\Delta - W_{1/2,\gamma}$. Notice that

$$W_{1/2,\gamma}(x) > W_{opt}(x) \qquad \text{if } |x|\geq 2,$$
$$z(x) > G^{1/2}(x) \qquad \text{if } |x|\geq 2,$$

where W_{opt} is defined by (2.1) and G is the Green function. Then by Remark 2.4

$$\sum_{x\in\mathbb{T}_{q+1}} z^2(x)W_{1/2,\gamma}(x) = +\infty.$$

- *Optimality near infinity* : suppose by contradiction that there exist $\overline{\lambda} > 0$ and a compact set $K \subset \mathbb{T}_{q+1}$ such that

$$\frac{1}{2}\sum_{\substack{x,y\in\mathbb{T}_{q+1}\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2-\sum_{x\in\mathbb{T}_{q+1}}W_{1/2,\gamma}(x)\varphi^2(x)$$
$$\geq \overline{\lambda}\sum_{x\in\mathbb{T}_{q+1}}W_{1/2,\gamma}(x)\varphi^2(x), \tag{3.4}$$

for all $\varphi \in C_0(\mathbb{T}_{q+1} \setminus K)$. Then, (3.4) holds true on $C_0(\mathbb{T}_{q+1} \setminus (K \cup B_2(o)))$. Notice that $W_{opt}\varphi^2 \leq W_{1/2,\gamma}\varphi^2$ for all $\varphi \in C_0(\mathbb{T}_{q+1} \setminus (K \cup B_2(o)))$. It follows that

$$\frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 - \sum_{x \in \mathbb{T}_{q+1}} W_{opt}(x)\varphi^2(x)$$

$$\geq \frac{1}{2} \sum_{\substack{x,y \in \mathbb{T}_{q+1} \\ x \sim y}} \Big(\varphi(x) - \varphi(y)\Big)^2 - \sum_{x \in \mathbb{T}_{q+1}} W_{1/2,\gamma}(x)\varphi^2(x)$$

$$\geq \overline{\lambda} \sum_{x \in \mathbb{T}_{q+1}} W_{1/2,\gamma}(x)\varphi^2(x) \geq \overline{\lambda} \sum_{x \in \mathbb{T}_{q+1}} W_{opt}(x)\varphi^2(x),$$

for all $\varphi \in C_0(\mathbb{T}_{q+1} \setminus (K \cup B_2(o)))$. This is a contradiction because W_{opt} is optimal for Δ. We checked the three conditions given in Definition 2.2. Hence $W_{1/2,\gamma}$ is optimal for Δ.

□

Proof of Corollary 2.8 For $\beta < \min\{1/2 \log q, 1\}$ we have that $W_{\beta,\gamma} > W_{opt}$ on $B_2(o)^c$. Then, the thesis follows by repeating the same argument used for proving (3.4). □

3.2 *Proof of Improved Poincaré Inequalities*

Proof of Theorem 2.10 Given $W_{\beta,\gamma} = \frac{\Delta u_{\beta,\gamma}}{u_{\beta,\gamma}}$, where $u_{\beta,\gamma}$ is defined by (3.1), it is easy to check that $W_{\beta,\gamma}$ is larger than Λ_q on $B_2(o)$ choosing the parameters $0 \leq \beta \leq \log_2\left(\frac{3}{2} - \frac{1}{2q}\right)$ and $\frac{1}{2} + \frac{1}{2q} \leq \gamma \leq 2 - 2^\beta$.

Indeed,

$$q + 1 - (q+1)q^{-1/2}/\gamma \geq q + 1 - 2q^{1/2}$$

is equivalent to $\frac{1}{2} + \frac{1}{2q} \leq \gamma$, and

$$q + 1 - q^{1/2}(2^\beta + \gamma) \geq q + 1 - 2q^{1/2}$$

is equivalent to $\gamma \leq 2 - 2^\beta$. Notice that for this choice of γ and β it follows that $\beta \leq \log_2(\frac{3}{2}) < 1$, and we already proved in Theorem 2.5 that $W_{\beta,\gamma} \geq \Lambda_q$ on $B_2(o)^c$ for all $0 \leq \beta < 1$. □

Proof of Theorem 2.11 We know from Theorem 2.7 that the optimal weight $W_{1/2,\gamma}$ is larger than Λ_q for $|x| \geq 2$. Then we can define

$$\overline{R}(x) = W_{1/2,\gamma}(x) - \Lambda_q \qquad \forall x \in \mathbb{T}_{q+1} \setminus B_2(o),$$

and (2.6) follows. The sharpness of $q^{1/2}$ is consequence of the optimality of $\widetilde{W}$ for H where $\widetilde{W}$ and H are chosen such as in the proof of Theorem 2.7. □

4 Hardy-Type Inequalities on Rapidly Growing Radial Trees

In view of the results obtained on the homogeneous tree, here we attempt to generalise the family of Hardy inequalities given in Theorem 2.7 on a more general context, namely on radial trees. This could pave the way to future investigations on more general nonradial trees, by means of suitable comparison theorems, as in the Riemannian setting.

Let $T = (V, E)$ be an infinite tree. We call T a **radial** tree if the degree m depends only on $|x|$ (see e.g. [8, 33]). In the following we set $\overline{m} = m - 1$ to lighten the notation. For future purposes, we also note that the volume of the ball $B_n(o)$ is given by

$$\begin{aligned}
&\#B_1(o) = 1, \\
&\#B_2(o) = 2 + \overline{m}(0), \\
&\#B_3(o) = 2 + \overline{m}(0) + (\overline{m}(0) + 1)\overline{m}(1), \\
&\qquad \vdots \\
&\#B_n(o) = 1 + (\overline{m}(0) + 1)[1 + \overline{m}(1) + \overline{m}(1)\overline{m}(2) + \ldots + \overline{m}(1)\overline{m}(2)\overline{m}(3)\ldots\overline{m}(n-2)].
\end{aligned}$$

If particular, if $T = \mathbb{T}_{q+1}$, then $\overline{m} \equiv q$ and we have that $\#B_n(o) \sim q^{n-1}$ as $n \to +\infty$.

Next, recalling that the proof of Theorem 2.5 relies on the exploitation of the superharmonic functions $u_{\alpha,\beta}$ and that $u_{\alpha,\beta}(x) = |x|^\beta q^{\alpha|x|}$ for all $|x| \geq 1$, by analogy, we consider on T the family of positive and radial functions:

$$u_{\alpha,\beta}(x) := |x|^\beta \Psi^\alpha(|x|) \qquad \text{if } |x| \geq 1\,. \tag{4.1}$$

Regarding the choice of the function Ψ, since in $\mathbb{T}_{q+1}$ the function $q^{|x|}$ is related to $\#B_{|x|+1}(o)$ and since $\frac{\#B_{|x|+1}(o)}{\#B_{|x|}(o)} \sim q = \overline{m}$ as $|x| \to +\infty$, we assume that it satisfies the following condition

$$\Psi(|x| + 1) = \overline{m}(|x|)\Psi(|x|) \quad \text{for all } |x| \geq 1. \tag{4.2}$$

Clearly, if $T = \mathbb{T}_{q+1}$, then (4.2) holds by taking $\Psi(|x|) = q^{|x|}$. We note that, conversely, for a given positive Ψ, condition (4.2) characterizes the tree we are dealing with through its degree, see Remark 4.2 below.

By showing that the function $u_{-1/2,\beta}$ is superharmonic on T, we obtain the following result.

Proposition 4.1 *Let* $\Psi : (0, +\infty) \to \mathbb{R}$ *be a positive function such that the map* $(0, +\infty) \ni s \mapsto \frac{\Psi(s+1)}{\Psi(s)}$ *is nondecreasing and let* T *be a radial tree with degree* $\overline{m} + 1$ *satisfying condition* (4.2)*. Then, for all* $\beta < 1$ *and* $\frac{1}{\Psi^{1/2}(1)} \le \gamma \le \frac{1}{\Psi^{1/2}(1)}\Big(\overline{m}(1) + 1 - \overline{m}^{1/2}(1)2^{\beta}\Big)$ *the following inequality holds*

$$\frac{1}{2}\sum_{\substack{x,y\in T\\ x\sim y}}\Big(\varphi(x)-\varphi(y)\Big)^2 \ge \sum_{x\in T} W_{\beta,\gamma}\varphi^2(x) \qquad \forall \varphi \in C_0(T)\,,$$

where $W_{\beta,\gamma}$ *is the positive weight*

$$W_{\beta,\gamma}(x) := \begin{cases} \overline{m}(0) + 1 - \frac{\overline{m}(0)+1}{\gamma\,\Psi^{1/2}(1)} & \text{if } |x| = 0,\\ \overline{m}(1) + 1 - \overline{m}^{1/2}(1)2^{\beta} - \Psi^{1/2}(1)\gamma & \text{if } |x| = 1,\\ \overline{m}(|x|) + 1 - \overline{m}^{1/2}(|x|)\Big(1+\frac{1}{|x|}\Big)^{\beta} - \overline{m}^{1/2}(|x|-1)\Big(1-\frac{1}{|x|}\Big)^{\beta} & \text{if } |x| \ge 2. \end{cases}$$

Remark 4.2 It is readily seen that, by taking $\Psi(s) = q^s$ in Proposition 4.1, we get $T = \mathbb{T}_{q+1}$ and we re-obtain Theorem 2.7; however, Proposition 4.1 gives no information about the criticality of the operator $\Delta - W_{\beta,\gamma}$ on T. We also note that condition (4.2) yields rapidly growing trees, such as those generated, for instance, by the maps $\Psi_a(s) = e^{s^a}$ with $a > 1$.

Proof The proof follows the same lines of the proof of Theorem 2.5, namely we show that the function $u_{\alpha,\beta}$ in (4.1), with $\alpha = -1/2$ and $\beta < 1$, is superharmonic in $T \setminus B_2(0)$ and that it can be properly extended to o in order to get a superharmonic function on the whole T. Hence the statement follows by invoking [8, Proposition 3.1].

If $\beta < 1$ and $|x| \ge 2$ we have

$$\begin{aligned}\Delta u_{-1/2,\beta}(x) &= \Big(\overline{m}(|x|)+1\Big)|x|^{\beta}\Psi^{-1/2}(|x|) - \overline{m}^{1/2}(|x|)(|x|+1)^{\beta}\Psi^{-1/2}(|x|) + \\ &\quad - (|x|-1)^{\beta}\overline{m}^{1/2}(|x|-1)\Psi^{-1/2}(|x|)\\ &= u_{-1/2,\beta}(x)\\ &\qquad \times \Big(\overline{m}(|x|) + 1 - \overline{m}^{1/2}(|x|)\Big(1+\frac{1}{|x|}\Big)^{\beta} - \overline{m}^{1/2}(|x|-1)\Big(1-\frac{1}{|x|}\Big)^{\beta}\Big).\end{aligned}$$

Since by hypothesis the function $\overline{m}$ is nondecreasing, we get

$$\begin{aligned}
&\Delta u_{-1/2,\beta}(x) \\
&= u_{-1/2,\beta}(x)\Bigg(\left(\overline{m}^{1/2}(|x|)\right) - 1\Big)^2 + \overline{m}^{1/2}(|x|)\Big(2 - \Big(1 + \frac{1}{|x|}\Big)^\beta - \Big(1 - \frac{1}{|x|}\Big)^\beta\Big) \\
&\quad + \Big(\overline{m}^{1/2}(|x|) - \overline{m}^{1/2}(|x|-1)\Big)\Big(1 - \frac{1}{|x|}\Big)^\beta\Bigg) > 0,
\end{aligned}$$

for all $|x| \geq 2$.

Then we choose $\gamma := u_{-1/2,\beta}(o)$ such that $\Delta u_{-1/2,\beta}$ is nonnegative in $B_2(o)$. By a direct computation we have

$$\Delta u_{-1/2,\beta}(o) = (\overline{m}(0) + 1)(\gamma - \Psi^{-1/2}(1)) \geq 0,$$

for $\gamma \geq \Psi^{-1/2}(1)$. Furthermore, for $|x| = 1$ we get

$$\Delta u_{-1/2,\beta}(x) = (\overline{m}(1) + 1)\Psi^{-1/2}(1) - \overline{m}(1)2^\beta\Psi^{-1/2}(2) - \gamma \geq 0,$$

for $\gamma \leq \Psi^{-1/2}(1)\left(\overline{m}(1) + 1 - \overline{m}^{1/2}(1)2^\beta\right)$. This concludes the proof. □

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References

1. Akutagawa, K., Kumura, H.: Geometric relative Hardy inequalities and the discrete spectrum of Schrödinger operators on manifolds. Calc. Var. Part. Diff. Eq. **48**, 67–88 (2013)
2. Anker, J.-Ph., Martinot, P., Pedon, E., Setti, A.G.: The Shifted Wave Equation on Damek-Ricci Spaces and on Homogeneous Trees. Trends in Harmonic Analysis, Springer INdAM Ser., vol. 3, pp. 1–25. Springer, Milan (2013)
3. Astengo, F.: An uncertainty principle on homogeneous trees. Proc. Am. Math. Soc. **131**(10), 3155–3161 (2003)
4. Barbatis, G., Filippas, S., Tertikas, A.: A unified approach to improved L^p Hardy inequalities with best constants. Trans. Am. Soc. **356**, 2169–2196 (2004)
5. Berchio, E., Ganguly, D., Grillo, G.: Sharp Poincaré-Hardy and Poincaré-Rellich inequalities on the hyperbolic space. J. Funct. Anal. **272**(4), 1661–1703 (2017)

6. Berchio, E., Ganguly, D., Grillo, G., Pinchover, Y.: An optimal improvement for the Hardy inequality on the hyperbolic space and related manifolds. Proc. R. Soc. Edinb. Sect. A **150**(4), 1699–1736 (2020)
7. Boiko, T., Woess, W.: Moments of Riesz measures on Poincaré disk and homogeneous tree-a comparative study. Expo. Math. **33**(3), 353–374 (2015)
8. Bonnefont, M., Golenia, S.: Essential spectrum and Weyl asymptotics for discrete Laplacians. Ann. Fac. Sci. Toulouse Math. **24**(3), 563–624 (2015)
9. Brezis, H., Marcus, M.: Hardy's inequalities revisited. Ann. Scuola Norm. Sup. Cl. Sci. (4) **25**, 217–237 (1997)
10. Brezis, H., Vazquez, J.L.: Blow-up solutions of some nonlinear elliptic problems. Rev. Mat. Univ. Complut. Madrid **10**(2), 443–469 (1997)
11. Carron, G.: Inégalités de Hardy sur les variétés Riemanniennes non-compactes. J. Math. Pures Appl. (9) **76**(10), 883–891 (1997)
12. Celotto, D., Meda, S., Wróbel, B.: L^p spherical multipliers on homogeneous trees. Studia Math. **247**(2), 175–190 (2019)
13. Cohen, J.M., Colonna, F., Singman, D.: Biharmonic Green functions on homogeneous trees. Mediterr. J. Math. **6**(3), 249–271 (2009)
14. Cohen, J.M., Colonna, F., Picardello, M.A., Singman, D.: Bergman spaces and Carleson measures on homogeneous isotropic trees. Potential Anal. **44**(4), 745–766 (2016)
15. Cowling, M., Meda, S., Setti, A.G.: An overview of harmonic analysis on the group of isometries of a homogeneous tree. Exposition. Math. **16**(5), 385–423 (1998)
16. Cowling, M., Meda, S., Setti, A.G.: Estimates for functions of the Laplace operator on homogeneous trees. Trans. Am. Math. Soc. **352**(9), 4271–4293 (2000)
17. D'Ambrosio, L., Dipierro, S.: Hardy inequalities on Riemannian manifolds and applications. Ann. Inst. H. Poincaré Anal. Non Linéaire **31**(3), 449–475 (2014)
18. Devyver, B., Fraas, M., Pinchover, Y.: Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon. J. Funct. Anal. **266**(7), 4422–4489 (2014)
19. Figà-Talamanca, A., Nebbia, C.: Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees. London Math. Soc. Lecture Notes Series, vol. 162. Cambridge Univ. Press, Cambridge (1991)
20. Gazzola, F., Grunau, H., Mitidieri, E.: Hardy inequalities with optimal constants and remainder terms. Trans. Am. Math. Soc. **356**(6), 2149–2168 (2004)
21. Keller, M., Pinchover, Y., Pogorzelski, F.: An improved discrete hardy inequality. Am. Math. Mon. **125**(4), 347–350 (2018)
22. Keller, M., Pinchover, Y., Pogorzelski, F.: Optimal Hardy inequalities for Schrödinger operators on graphs. Commun. Math. Phys. **358**(2), 767–790 (2018)
23. Keller, M., Pinchover, Y., Pogorzelski, F.: Criticality theory for Schrödinger operators on graphs. J. Spectr. Theory **10**, 73–114 (2020)
24. Kombe, I., Ozaydin, M.: Hardy-Poincaré, Rellich and uncertainty principle inequalities on Riemannian manifolds. Trans. Am. Math. Soc. **365**(10), 5035–5050 (2013)
25. Kovarik, H., Pinchover, Y.: On minimal decay at infinity of Hardy-weights. Commun. Contemp. Math. **22**, 1950046 (2020)
26. Kristaly, A.: Sharp uncertainty principles on Riemannian manifolds: the influence of curvature. J. Math. Pures Appl. (9) **119**, 326–346 (2018)
27. Kufner, A., Opic, B.: Hardy-Type Inequalities, Pitman Research Notes in Math., vol. 219. Longman, Harlow (1990)
28. Marcus, M., Mizel, V.J., Pinchover, Y.: On the best constant for Hardy's inequality in $\mathbb{R}^n$. Trans. Am. Math. Soc. **350**(8), 3237–3255 (1998)
29. Mitidieri, E.: A simple approach to Hardy inequalities. Mat. Zametki **67**(4), 563–572 (2000)
30. Ngô, Q.A., Nguyen, V.H.: Sharp constant for Poincaré-type inequalities in the hyperbolic space. Acta Math. Vietnam **44**(3), 781–795 (2019)
31. Pagliacci, M., Picardello, M.A.: Heat diffusion on homogeneous trees. Adv. Math. **110**(2), 175–190 (1995)

32. Woess, W.: Random Walks on Infinite Graphs and Groups. Cambridge University Press, Cambridge (2000)
33. Wojciechowski, R.: Stochastic completeness of graphs. PhD Thesis (2007)
34. Wojciechowski, R.: Heat kernel and essential spectrum of infinite graphs. Indiana Univ. Math. J. **58**, 1419–1442 (2009)
35. Yang, Q., Su, D., Kong, Y.: Hardy inequalities on Riemannian manifolds with negative curvature. Commun. Contemp. Math. **16**(2), 1350043, 24 pp. (2014)

Ground State Solutions for the Nonlinear Choquard Equation with Prescribed Mass

Silvia Cingolani and Kazunaga Tanaka

Abstract We study existence of radially symmetric solutions for the nonlocal problem:

$$\begin{cases} -\Delta u + \mu u = (I_\alpha * F(u))f(u) & \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = c \end{cases} \tag{1}$$

where $N \geq 3$, $\alpha \in (0, N)$, $c > 0$, $I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}$ is the Riesz potential, $F \in C^1(\mathbb{R}, \mathbb{R})$, $F'(s) = f(s)$, μ is a unknown Lagrange multiplier. Using a Lagrange formulation of the problem (1), we develop new deformation arguments under a version of the Palais-Smale condition introduced in the recent papers (Hirata and Tanaka, Adv Nonlinear Stud 19:263–290, 2019; Ikoma and Tanaka, Adv Differ Equ 24:609–646, 2019) and we prove the existence of a ground state solution for the nonlinear Choquard equation with prescribed mass, when F satisfies Berestycki-Lions type conditions.

Keywords Nonlinear Choquard equation · Normalized solutions · Nonlocal nonlinearities · Positive solutions · Lagrange formulation

S. Cingolani (✉)
Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, Bari, Italy
e-mail: silvia.cingolani@uniba.it

K. Tanaka
Department of Mathematics, School of Science and Engineering, Waseda University, Shinjuku-ku, Tokyo, Japan
e-mail: kazunaga@waseda.jp

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1 Introduction

In this paper we study existence of radially symmetric solutions of the nonlocal equation

$$\begin{cases} -\Delta u + \mu u = (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = c \\ v \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 3$, $\alpha \in (0, N)$, $c > 0$, $I_\alpha : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha |x|^{N-\alpha}}$$

and $F \in C^1(\mathbb{R}, \mathbb{R})$ and $f(s) = F'(s)$, μ is a Lagrange multiplier.

In 1954 the equation in (1.1) with $N = 3$, $\alpha = 2$ and $F(s) = \frac{1}{2}|s|^2$ was introduced by Pekar in [31] to describe the quantum theory of a polaron at rest and in 1976 it was arisen in the work of Choquard on the modeling of an electron trapped in its own hole, in a certain approximation to Hartree–Fock theory of one-component plasma [21] (see also [12, 13]). In particular it corresponds to the stationary nonlinear Hartree equation. Indeed if v is a solution of (1.1), then the wave function $\psi(x,t) = e^{i\mu t}v(x)$ is a solitary wave of the time-dependent Hartree equation

$$i\psi_t = -\Delta\psi - (\frac{1}{4\pi|x|} * |\psi|^2)\psi \quad \text{in} \mathbb{R} \times \mathbb{R}^3. \tag{1.2}$$

In literature the nonlocal equation in (1.1) is usually called the *nonlinear Choquard equation*.

Finally, we recall that the three-dimensional nonlocal equation was also proposed by Penrose [26, 32–34] in his discussion on the self-gravitational collapse of a quantum mechanical wave-function and in that context it is known as the Schrödinger-Newton equation (see also [36]).

In 1977 Lieb [21] proved the existence and uniqueness up to translations of the unique ground state solution of the equation

$$-\Delta u + \mu u = (\frac{1}{4\pi|x|} * |u|^2)u, \quad x \in \mathbb{R}^3$$

where μ is a fixed positive constant.

Due to its physical relevance, the existence of infinitely many standing wave solutions to (1.2) with prescribed L^2- norm has been faced by P.L. Lions in [23].

Variational methods were also employed to derive existence and multiplicity results of standing wave solutions for the nonlinear Choquard equation without prescribed mass [5, 11, 19, 25, 27, 28, 30] and to study concentration phenomena in the semiclassical limit [6, 9, 10, 20, 29, 35, 37]. Recently the existence of L^2- normalized solutions for the nonlinear Choquard equations has been investigated when $F(s) = |s|^p$ or it is monotone and satisfies Ambrosetti-Rabinowitz assumptions in [2, 38]. We also mention existence results of solutions for the Schrödinger-Poisson equations with prescribed mass [3, 8, 18, 24].

In this work by means of a new deformation approach, we study the nonlinear Choquard equation (1.1), where $\alpha \in (0, N)$ and $F(s)$ is a general nonlinearity which does not satisfy a monotonicity assumption nor Ambrosetti-Rabinowitz type condition [1].

To give our main result, we assume that

(f1) $f \in C(\mathbb{R}, \mathbb{R})$;
(f2) there exists $C > 0$ such that for every $s \in \mathbb{R}$,

$$|sf(s)| \le C(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha+2}{N}});$$

(f3) $F(s) = \int_0^s f(t)dt$ satisfies

$$\lim_{s\to 0} \frac{F(s)}{|s|^{\frac{N+\alpha}{N}}} = 0, \qquad \lim_{s\to +\infty} \frac{F(s)}{|s|^{\frac{N+\alpha+2}{N}}} = 0;$$

(f4) there exists $s_0 \in \mathbb{R}$, $s_0 \neq 0$ such that $F(s_0) > 0$;
(f5) f is odd and f is positive on $(0, +\infty)$.

We remark that the exponent $\frac{N+\alpha+2}{N}$ appears as a L^2-critical exponent for the Choquard equations and the conditions (f1)–(f4) correspond to L^2-subcritical growths.

For this general class of nonlinearities, related to [4, 28], we introduce a Lagrangian formulation of the nonlocal problem (1.1) and we extend a new approach introduced by Hirata and the second author [15] for the local case. One advantage of this method is that it can be also useful to derive multiplicity results of normalized solutions in several different frameworks. Recently, existence and multiplicity of L^2-normalized solutions for fractional scalar field equations are obtained in [7], detecting mini-max structures in a product space, by means of a *Pohozaev's mountain*. A similar approach to [7] has been performed in the present paper. Precisely a radially symmetric solution (μ, u) of (1.1) corresponds to a critical point of the functional $\mathcal{T} : (0, \infty) \times H_r^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{T}(\mu, u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\, dx + \frac{\mu}{2}\big(\|u\|_2^2 - c\big). \qquad (1.3)$$

Using a new variant of the Palais-Smale condition [15, 16], which takes into account Pohozaev's identity, we will prove a deformation theorem, which enables us to apply minimax arguments in the space $\mathbb{R} \times H_r^1(\mathbb{R}^N)$. As shown in [7], our deformation arguments show that solutions without Pohozaev identity are deformable with a suitable deformation flow. Therefore critical points with Pohozaev identity are just *essential* since they give a topological contribution.

We state our main results.

Theorem 1.1 *Suppose $N \geq 3$, (f1)–(f4). Then there exists c_0 such that for any $c > c_0$, the problem (1.1) has a radially symmetric solution. In addition if (f5) holds, the solution is positive.*

Theorem 1.2 *Suppose $N \geq 3$, (f1)–(f4) and*

$$\lim_{s\to 0} \frac{F(s)}{|s|^{\frac{N+\alpha+2}{N}}} = +\infty. \tag{1.4}$$

Then for any $c > 0$, the problem (1.1) has a radially symmetric solution. In addition if (f5) holds, the solution is positive.

We remark that solutions obtained in above theorems are ground state solutions, that is, they have least energy among all solutions. See Remarks 5.3 and 5.4.

2 Functional Settings

In what follows we use the notation:

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, dx \right)^{1/2},$$

$$\|u\|_r = \left(\int_{\mathbb{R}^N} |u|^r \, dx \right)^{1/r} \quad \text{for } r \in [1, \infty),$$

$$B(p, R) = \{x \in \mathbb{R}^N;\ |x - p| < R\}.$$

We recall the following generalized Hardy-Littlewood-Sobolev inequality [22].

Proposition 2.1 *Let $p, r > 1$ and $\alpha \in (0, N)$ with $\frac{1}{p} + \frac{1}{r} = \frac{N+\alpha}{N}$. Then there exists a constant $C = C(N, \alpha, p, r) > 0$ such that*

$$\left| \int_{\mathbb{R}^N} (I_\alpha * f) g \, dx \right| \leq C \|f\|_p \|g\|_r \tag{2.1}$$

for all $f \in L^p(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$.

In what follows we denote by p the L^2 critical exponent, i.e.

$$p = \frac{N+\alpha+2}{N}.$$

We consider the functional $\mathcal{T} : (0,\infty) \times H^1_r(\mathbb{R}^N) \to \mathbb{R}$ defined by (1.3). Using Proposition 2.1, it is easy to see that $\mathcal{T}(\mu, u) \in C^1((0,\infty)\times H^1_r(\mathbb{R}^N), \mathbb{R})$. Moreover (μ, u) solves problem (1.1) if and only if $\partial_u \mathcal{T}(\mu, u) = 0$ and $\partial_\mu \mathcal{T}(\mu, u) = 0$.

Moreover we define the functional $\mathcal{J} : (0,\infty) \times H^1_r(\mathbb{R}^N) \to \mathbb{R}$ by setting

$$\mathcal{J}(\mu, u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\, dx + \frac{\mu}{2}\|u\|_2^2. \tag{2.2}$$

For a fixed $\mu > 0$, u is critical point of $\mathcal{J}(\mu, \cdot)$ means that u solves

$$\begin{cases} -\Delta u + \mu u = (I_\alpha * F(u))f(u) & \text{in } \mathbb{R}^N, \\ \quad u \in H^1_r(\mathbb{R}^N). \end{cases} \tag{2.3}$$

It is immediate that

$$\mathcal{T}(\mu, u) = \mathcal{J}(\mu, u) - \frac{\mu}{2}c.$$

Finally by Proposition 3.1 in [28] each solution u of (2.3) belongs to $W^{2,2}_{loc}(\mathbb{R}^N)$ and it satisfies Pohozaev's identity (see [28, Proposition 3.5])

$$\frac{N-2}{2}\|\nabla u\|_2^2 + \frac{N}{2}\mu\|u\|_2^2 = \frac{N+\alpha}{2}\mathcal{D}(u), \tag{2.4}$$

where we set

$$\mathcal{D}(u) = \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\, dx.$$

Inspired by Pohozaev's identity, we also introduce the functional $\mathcal{P} : (0,\infty) \times H^1_r(\mathbb{R}^N) \to \mathbb{R}$ by setting

$$\mathcal{P}(\mu, u) = \frac{N-2}{2}\|\nabla u\|_2^2 - \frac{N+\alpha}{2}\mathcal{D}(u) + \frac{N}{2}\mu\|u\|_2^2. \tag{2.5}$$

3 Geometry of $\mathcal{T}(\mu, u)$

In [28] Moroz and Van Schaftingen showed that, under the assumption (f1)–(f5), for any $\mu > 0$ the functional

$$u \in H_r^1(\mathbb{R}^N) \mapsto \mathcal{J}(\mu, u) \in \mathbb{R}$$

has the Mountain Pass geometry. Precisely, set

$$\Gamma_\mu = \{\gamma(t) \in C([0, 1], H_r^1(\mathbb{R}^N));\ \gamma(0) = 0,\ \mathcal{J}(\mu, \gamma(1)) < 0\}$$

the following MP value

$$a(\mu) = \inf_{\gamma\in\Gamma_\mu} \max_{t\in[0,1]} \mathcal{J}(\mu, \gamma(t))$$

is well defined and attained at a positive radially symmetric ground state solution u_μ. Now set

$$\Sigma = \{(\mu, u) \in (0, \infty) \times H_r^1(\mathbb{R}^N);\ \mathcal{P}(\mu, u) > 0\} \cup \{(\mu, 0);\ \mu > 0\},$$

we denote

$$\partial\Sigma = \{(\mu, u) \in (0, \infty) \times H_r^1(\mathbb{R}^N);\ \mathcal{P}(\mu, u) = 0,\ u \neq 0\}$$

the boundary of Σ with respect to the relative topology to the set $(0, \infty) \times H_r^1(\mathbb{R}^N)$. We observe that

$$\{(\mu, 0) \mid \mu > 0\} \subset int(\Sigma). \tag{3.1}$$

We prove the following proposition.

Proposition 3.1

(i) $\mathcal{J}(\mu, u) \geq 0$ *for all* $(\mu, u) \in \Sigma$.
(ii) $\mathcal{J}(\mu, u) \geq a(\mu) > 0$ *for all* $(\mu, u) \in \partial\Sigma$.

Proof We notice that for all $(\mu, u) \in \Sigma$

$$\begin{aligned}\mathcal{J}(\mu, u) &\geq \mathcal{J}(\mu, u) - \frac{\mathcal{P}(\mu, u)}{N+\alpha} \\ &= \frac{\alpha+2}{2(N+\alpha)}\|\nabla u\|_2^2 + \frac{\alpha\mu}{2(N+\alpha)}\|u\|_2^2\end{aligned}$$

and thus (i) follows.

The proposition (ii) follows from the fact that the mountain pass level $a(\mu)$ coincides with the ground state energy level c (see Theorem 1 in [28]). ■

To see that $\mathcal{T}(\mu, u) = \mathcal{J}(\mu, u) - \frac{\mu}{2}c$ has a MP geometry in $(0, \infty) \times H_r^1(\mathbb{R}^N)$, it is crucial to analyze the behavior of $a(\mu)$ as $\mu \to \infty$.

Lemma 3.2

$$\lim_{\mu\to\infty} \frac{a(\mu)}{\mu} = \infty.$$

Proof We write $p = \frac{N+\alpha+2}{N}$ and $q = \frac{N+\alpha}{N}$. By (f3), for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$|F(s)| \le \delta|s|^p + C_\delta|s|^q \quad \text{for all } s \in \mathbb{R}.$$

For $v(x) \in H_r^1(\mathbb{R}^N)$, setting $u_s(x) = s^{N/2}v(sx)$, we have

$$\begin{aligned}
\mathcal{D}(u_s) &= \mathcal{D}(s^{N/2}v(sx)) = s^{-N-\alpha}\mathcal{D}(s^{N/2}v(x)) \\
&\le s^{-N-\alpha}\int_{\mathbb{R}^N}(I_\alpha * (\delta s^{\frac{N}{2}p}|v|^p + C_\delta s^{\frac{N}{2}q}|v|^q))(\delta s^{\frac{N}{2}p}|v|^p + C_\delta s^{\frac{N}{2}q}|v|^q) \\
&= s^2\int_{\mathbb{R}^N}(I_\alpha * (\delta|v|^p + C_\delta s^{-1}|v|^q))(\delta|v|^p + C_\delta s^{-1}|v|^q) \\
&\equiv s^2 D_{\delta, C_\delta s^{-1}}(v).
\end{aligned} \tag{3.2}$$

Here we write for $\delta > 0$ and $A \ge 0$,

$$D_{\delta,A}(v) = \int_{\mathbb{R}^N}(I_\alpha * (\delta|v|^p + A|v|^q))(\delta|v|^p + A|v|^q)\,dx,$$

$$\mathcal{J}_{\delta,A}(v) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 - \frac{1}{2}D_{\delta,A}(u).$$

We also denote by $b(\delta, A)$ the MP value of $\mathcal{J}_{\delta,A}(v)$. Taking into account the continuity and monotonicity property of $b(\delta, A)$ with respect of each variable δ and A and noting that $\mathcal{J}_{\delta,A}(v)$ satisfies (PS) condition, we have

$$b(\delta, A) \to b(\delta, 0) \quad \text{as } A \to 0,$$

$$b(\delta, 0) \to \infty \quad \text{as } \delta \to 0^+.$$

Thus, we have from (3.2) that

$$\mathcal{J}(\mu, u_s) \ge s^2\left(\frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\mu s^{-2}\|v\|_2^2 - \frac{1}{2}D_{\delta, C_\delta s^{-1}}(v)\right).$$

Setting $s = \sqrt{\mu}$, we have

$$\mathcal{J}(\mu, u_{\sqrt{\mu}}) \geq \mu \mathcal{J}_{\delta, C_\delta \mu^{-1/2}}(v)$$

and thus

$$\frac{a(\mu)}{\mu} \geq b(\delta, C_\delta \mu^{-1/2}).$$

Therefore we have

$$\liminf_{\mu \to \infty} \frac{a(\mu)}{\mu} \geq \lim_{A \to 0} b(\delta, A) = b(\delta, 0).$$

Since $\delta > 0$ is arbitrary, we have

$$\lim_{\mu \to \infty} \frac{a(\mu)}{\mu} = \infty.$$

■

Corollary 3.3 *For any $c > 0$, there exists $B_c \in \mathbb{R}$ such that*

$$\mathcal{T}(\mu, u) \geq B_c \quad \textit{for all } (\mu, u) \in \partial\Sigma.$$

Next we show

Proposition 3.4 *Assume* (1.4) *in addition to* (f1)–(f5). *Then*

$$\lim_{\mu \to 0} \frac{a(\mu)}{\mu} = 0. \tag{3.3}$$

Proof We fix $u \in H_r^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\|u\|_\infty = 1$. We note that there exists $L_s > 0$ such that

$$F(su(x)) \geq \sqrt{L_s} s^p |u(x)|^p \quad \text{for all } s \in (0, 1] \text{ and } x \in \mathbb{R}^N,$$

$$L_s \to \infty \quad \text{as } s \to 0.$$

Recalling $D_{1,0}(u) = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p$, we have for $t > 0$

$$\begin{aligned}
\mathcal{J}(\mu, su(x/t)) &\leq \frac{1}{2} s^2 t^{N-2} \|\nabla u\|_2^2 + \frac{\mu}{2} s^2 t^N \|u\|_2^2 - \frac{1}{2} L_s s^{2p} t^{N+\alpha} D_{1,0}(u) \\
&= \mu^{-\frac{N-2}{2}} \left(\frac{1}{2} s^2 \tau^{N-2} \|\nabla u\|_2^2 + \frac{1}{2} s^2 \tau^N \|u\|_2^2 - \frac{1}{2} L_s \mu^{\frac{N-2}{2}} \mu^{-\frac{N+\alpha}{2}} s^{2p} \tau^{N+\alpha} D_{1,0}(u) \right) \\
&= \mu^{-\frac{N-2}{2}} s^2 \left(\frac{1}{2} \|\nabla u\|_2^2 \tau^{N-2} + \frac{1}{2} \|u\|_2^2 \tau^N - \frac{1}{2} L_s \mu^{-\frac{2+\alpha}{2}} s^{2p-2} D_{1,0}(u) \tau^{N+\alpha} \right).
\end{aligned}$$

after setting $t = \mu^{-1/2}\tau$. Moreover setting $s = \mu^{N/4}$, we have

$$\mathcal{J}(\mu, \mu^{N/4}u(x/(\mu^{-1/2}\tau)) \le \mu\left(\frac{1}{2}\|\nabla u\|_2^2\tau^{N-2} + \frac{1}{2}\|u\|_2^2\tau^N - \frac{1}{2}L_{\mu^{N/4}}D_{1,0}(u)\tau^{N+\alpha}\right).$$

For $\mu \in (0, 1)$,

$$\tau \mapsto \mu^{N/4}u(x/\mu^{-1/2}\tau);\ (0, \infty) \to H_r^1(\mathbb{R}^N)$$

can be regarded as a path in Γ_μ. Thus

$$\frac{a(\mu)}{\mu} \le \max_{\tau\in(0,\infty)}\left(\frac{1}{2}\|\nabla u\|_2^2\tau^{N-2} + \frac{1}{2}\|u\|_2^2\tau^N - \frac{1}{2}L_{\mu^{N/4}}D_{1,0}(u)\tau^{N+\alpha}\right).$$

Since $L_{\mu^{N/4}} \to \infty$ as $\mu \to 0$, we have

$$R.H.S. \to 0 \quad \text{as } \mu \to 0^+.$$

Thus we have the conclusion. ■

4 Palais-Smale-Pohozaev Condition

Under the conditions (f1)–(f5), it seems hard to verify the standard Palais-Smale condition for the functional $\mathcal{T}(\lambda, u)$. As in [15] we introduce a compactness condition which is weaker than the standard Palais-Smale condition (see also [14, 16, 17]). Precisely, we give the following definition. In this section we set $\mu = e^\lambda$, with $\lambda \in \mathbb{R}$ and we write

$$\mathcal{T}(\lambda, u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\mathcal{D}(u) + \frac{1}{2}e^\lambda(\|u\|_2^2 - c) : \mathbb{R} \times H_r^1(\mathbb{R}^N) \to \mathbb{R}.$$

We also write

$$\mathcal{P}(\lambda, u) = \frac{N-2}{2}\|\nabla u\|_2^2 - \frac{N+\alpha}{2}\mathcal{D}(u) + \frac{N}{2}e^\lambda\|u\|_2^2,$$
$$\Sigma = \{(\lambda, u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N);\ \mathcal{P}(\lambda, u) > 0\} \cup \{(\lambda, 0);\ \lambda \in \mathbb{R}\}.$$

Definition 4.1 For $b \in \mathbb{R}$, we say that $\mathcal{T}(\lambda, u)$ satisfies the Palais-Smale-Pohozaev condition at level b (shortly the $(PSP)_b$ condition), if for any sequence $(\lambda_n, u_n) \subset$

$\mathbb{R} \times H_r^1(\mathbb{R}^N)$ such that

$$\mathcal{T}(\lambda_n, u_n) \to b,$$

$$\partial_\lambda \mathcal{T}(\lambda_n, u_n) \to 0.$$

$$\partial_u \mathcal{T}(\lambda_n, u_n) \to 0 \quad \textit{strongly in } (H_r^1(\mathbb{R}^N))^*,$$

$$\mathcal{P}(\lambda_n, u_n) \to 0,$$

it happens that (λ_n, u_n) has a strongly convergent subsequence in $\mathbb{R} \times H_r^1(\mathbb{R}^N)$.

We remark that this compactness condition takes into consideration of the scaling properties of $\mathcal{T}(\lambda, u)$, through Pohozaev functional $\mathcal{P}(\lambda, u)$.

We will show the following crucial result.

Theorem 4.2 *Assume* (f1)–(f5). *Let* $b \in \mathbb{R}$, $b < 0$. *Then* $\mathcal{T}(\lambda, u)$ *satisfies the* $(PSP)_b$ *condition on* $\mathbb{R} \times H_r^1(\mathbb{R}^N)$.

Proof Let $b < 0$ and $(\lambda_j, u_j) \subset \mathbb{R} \times H_r^1(\mathbb{R}^N)$ such that

$$\mathcal{T}(\lambda_j, u_j) \to b, \tag{4.1}$$

$$\partial_\lambda \mathcal{T}(\lambda_j, u_j) \to 0, \tag{4.2}$$

$$\partial_u \mathcal{T}(\lambda_j, u_j) \to 0 \quad \textit{strongly in } (H_r^1(\mathbb{R}^N))^*, \tag{4.3}$$

$$\mathcal{P}(\lambda_j, u_j) \to 0. \tag{4.4}$$

First we note that by (4.2)

$$e^{\lambda_j}(\|u_j\|_2^2 - c) \to 0. \tag{4.5}$$

Step 1: λ_j is bounded from below as $j \to +\infty$.

We have

$$\begin{aligned}
o(1) = \mathcal{P}(\lambda_j, u_j) &= \frac{N-2}{2}\|\nabla u_j\|_2^2 \\
&\quad + (N+\alpha)\Big\{\mathcal{T}(\lambda_j, u_j) - \frac{1}{2}\|\nabla u_j\|_2^2 - \frac{e^{\lambda_j}}{2}\big(\|u_j\|_2^2 - c\big)\Big\} + \frac{N}{2}e^{\lambda_j}\|u_j\|_2^2 \\
&= -\frac{\alpha+2}{2}\|\nabla u_j\|_2^2 + (N+\alpha)\Big\{\mathcal{T}(\lambda_j, u_j) - \frac{e^{\lambda_j}}{2}\big(\|u_j\|_2^2 - c\big)\Big\} + \frac{N}{2}e^{\lambda_j}\|u_j\|_2^2 \\
&= -\frac{\alpha+2}{2}\|\nabla u_j\|_2^2 + (N+\alpha)(b + o(1)) + \frac{N}{2}e^{\lambda_j}c + o(1).
\end{aligned}$$

Here we used (4.5).

If λ_j is not bounded below, there exists a subsequence, still denoted by λ_j such that $\lambda_j \to -\infty$. From the above identity, we derive a contradiction, since $b < 0$.

Step 2: $\|u_j\|_2^2 \to c$ as $j \to +\infty$.

It follows from (4.5).

Step 3: $\|\nabla u_j\|_2^2$, λ_j are bounded.

Since $\varepsilon_j \equiv \|\partial_u \mathcal{T}(\lambda_j, u_j)\|_{(H_r^1(\mathbb{R}^N))^*} \to 0$, we have

$$\|\nabla u_j\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * F(u_j)) f(u_j) u_j dx + e^{\lambda_j} \|u_j\|_2^2 \le \varepsilon_j \|u_j\|_{H^1}. \tag{4.6}$$

We observe that by (f3) for $\delta > 0$ fixed, there exists $C_\delta > 0$ such that

$$|F(s)| \le \delta |s|^p + C_\delta |s|^{\frac{N+\alpha}{N}}$$

where $p = \frac{N+\alpha+2}{N}$ and thus

$$\|F(u_j)\|_{\frac{2N}{N+\alpha}} \le \delta \||u_j|^p\|_{\frac{2N}{N+\alpha}} + C_\delta \||u_j|^{\frac{N+\alpha}{N}}\|_{\frac{2N}{N+\alpha}} = \delta \|u_j\|^p_{\frac{2Np}{N+\alpha}} + C_\delta \|u_j\|_2^{\frac{N+\alpha}{N}}.$$

Therefore by (f2) we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} (I_\alpha * |F(u_j)|)|f(u_j)u_j|\, dx \\
&\le C \|F(u_j)\|_{\frac{2N}{N+\alpha}} \|f(u_j)u_j\|_{\frac{2N}{N+\alpha}} \\
&\le C(\delta \|u_j\|^p_{\frac{2Np}{N+\alpha}} + C_\delta \|u_j\|_2^{\frac{N+\alpha}{N}}) \cdot C'(\|u_j\|^p_{\frac{2Np}{N+\alpha}} + \|u_j\|_2^{\frac{N+\alpha}{N}}) \\
&= CC'\delta \|u_j\|^{2p}_{\frac{2Np}{N+\alpha}} + CC'(\delta + C_\delta)\|u_j\|^p_{\frac{2Np}{N+\alpha}} \|u_j\|_2^{\frac{N+\alpha}{N}} + CC'C_\delta \|u_j\|_2^{\frac{2(N+\alpha)}{N}} \\
&= CC'\delta \|u_j\|^{2p}_{\frac{2Np}{N+\alpha}} + CC'(\delta + C_\delta)(\frac{\delta}{2} \|u_j\|^{2p}_{\frac{2Np}{N+\alpha}} + \frac{1}{2\delta} \|u_j\|_2^{\frac{2(N+\alpha)}{N}}) + CC'C_\delta \|u_j\|_2^{\frac{2(N+\alpha)}{N}} \\
&\le C''\delta \|u_j\|^{2p}_{\frac{2Np}{N+\alpha}} + C''_\delta \|u_j\|_2^{\frac{2(N+\alpha)}{N}}
\end{aligned}$$

and thus

$$\begin{aligned}
\|\nabla u_j\|_2^2 + e^{\lambda_j} \|u_j\|_2^2 &\le \int_{\mathbb{R}^N} (I_\alpha * |F(u_j)|)|f(u_j)u_j| dx + \varepsilon_j \|u_j\|_{H^1} \\
&\le C''\delta \|\nabla u_j\|_2^2 \|u_j\|_2^{2(p-1)} + C''_\delta \|u_j\|_2^{\frac{2(N+\alpha)}{N}} + \varepsilon_j \|u_j\|_{H^1}.
\end{aligned}$$

Since $\|u_j\|_2^2 = c + o(1)$, we have

$$(1 - C''\delta(c + o(1))^{p-1})\|\nabla u_j\|_2^2 + e^{\lambda_j}(c + o(1))$$
$$\leq C''_\delta(c + o(1))^{\frac{N+\alpha}{N}} + \varepsilon_j(\|\nabla u_j\|_2^2 + c + o(1))^{1/2}.$$

For δ small enough, we have boundedness of $\|\nabla u_j\|_2$ and e^{λ_j}. Thus λ_j cannot go to $+\infty$ and thus by Step 1 we infer that λ_j is bounded.

Step 4: By the steps 1–2, the sequence (λ_j, u_j) is bounded in $\mathbb{R} \times H_r^1(\mathbb{R}^N)$ and thus after extracting a subsequence, denoted in the same way, we may assume that $\lambda_j \to \lambda_0$ and $u_j \to u_0$ weakly in $H_r^1(\mathbb{R}^N)$ for some $(\lambda_0, u_0) \in \mathbb{R} \times H_r^1(\mathbb{R}^N)$. Taking into account the assumptions (f1)–(f4), we have

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_j)) f(u_j) u_0 \, dx \to \int_{\mathbb{R}^N} (I_\alpha * F(u_0)) f(u_0) u_0 \, dx$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_j)) f(u_j) u_j \, dx \to \int_{\mathbb{R}^N} (I_\alpha * F(u_0)) f(u_0) u_0 \, dx.$$

Thus we derive that $\langle \partial_u \mathcal{T}(\lambda_j, u_j), u_j \rangle \to 0$ and $\langle \partial_u \mathcal{T}(\lambda_j, u_j), u_0 \rangle \to 0$, and thus

$$\|\nabla u_j\|_2^2 + e^{\lambda_0}\|u_j\|_2^2 \to \|\nabla u_0\|_2^2 + e^{\lambda_0}\|u_0\|_2^2$$

which implies $u_j \to u_0$ strongly in $H_r^1(\mathbb{R}^N)$.

■

Remark 4.3 We emphasize that the (PSP) condition does not hold at level $b = 0$. Indeed we can consider the unbounded sequence $(\lambda_j, 0)$ with $\lambda_j \to -\infty$, for which we have

$$\mathcal{T}(\lambda_j, 0) = \partial_\lambda \mathcal{T}(\lambda_j, 0) = -\frac{e^{\lambda_j}}{2} c \to 0$$

and

$$\partial_u \mathcal{T}(\lambda_j, 0) = 0, \quad \mathcal{P}(\lambda_j, 0) = 0.$$

Now we denote

$$K_b = \{(\lambda, u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N) \mid \mathcal{T}(\lambda, u) = b, \partial_\lambda \mathcal{T}(\lambda, u) = 0, \partial_u \mathcal{T}(\lambda, u) = 0\}.$$

Clearly, $\partial_u \mathcal{T}(\lambda, u) = 0$ implies $\mathcal{P}(\lambda, u) = 0$.

Now for each $c \in \mathbb{R}$ we introduce the following notation

$$[\mathcal{T} \le c] = \{(\lambda, u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N) \mid \mathcal{T}(\lambda, u) \le c\}.$$

Following arguments strictly related to [15, Proposition 3.1, Corollary 4.3], we can establish the following deformation theorem for the functional $\mathcal{T}(\lambda, u)$.

Theorem 4.4 *Assume* (f1)–(f4). *Assume* $b < 0$. *Then* K_b *is compact in* $\mathbb{R} \times H_r^1(\mathbb{R}^N)$ *and* $K_b \cap (\mathbb{R} \times \{0\}) = \emptyset$.

Moreover for any open neighborhood U *of* K_b *and* $\bar{\varepsilon} > 0$ *there exists* $\varepsilon \in (0, \bar{\varepsilon})$ *and a continuous map*

$$\eta(t, \lambda, u) : [0, 1] \times \mathbb{R} \times H_r^1(\mathbb{R}^N) \to \mathbb{R} \times H_r^1(\mathbb{R}^N)$$

such that

(1^o) $\eta(0, \lambda, u) = (\lambda, u) \quad \forall(\theta, u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N)$;
(2^o) $\eta(t, \lambda, u) = (\lambda, u) \quad \forall(\theta, u) \in [\mathcal{T} \le b - \bar{\varepsilon}]$;
(3^o) $\mathcal{T}(\eta(t, \lambda, u)) \le \mathcal{T}(\lambda, u) \quad \forall(t, \lambda, u) \in [0, 1] \times \mathbb{R} \times H_r^1(\mathbb{R}^N)$;
(4^o) $\eta(1, [\mathcal{T} \le b + \varepsilon] \setminus U) \subset [\mathcal{T} \le b - \varepsilon]$;
(5^o) $\eta(1, [\mathcal{T} \le b + \varepsilon]) \subset [\mathcal{T} \le b - \varepsilon] \cup U$;
(6^o) *If* $K_b = \emptyset$, *we have* $\eta(1, [\mathcal{T} \le b + \varepsilon]) \subset [\mathcal{T} \le b - \varepsilon]$.

5 A Minimax Theorem

We go back to notation (2.2), i.e., we use μ instead of λ.

For any $c > 0$, let B_c be the constant defined in Corollary 3.3.

As a minimax class for $\mathcal{T}$, we define

$$\begin{aligned}\Lambda_c = \{\xi(t) \in C([0, 1], (0, \infty) \times H_r^1(\mathbb{R}^N));\ \xi(0) \in (0, \infty) \times \{0\},\\ \mathcal{T}(\xi(0)) \le B_c - 1,\ \xi(1) \notin \Sigma \text{ and } \mathcal{T}(\xi(1)) \le B_c - 1\}.\end{aligned}$$

From the observation in Sect. 4, we can see $\Lambda_c \neq \emptyset$. See also proof of (i) of the following Proposition 5.1. By (3.1) clearly $\xi([0, 1]) \cap \partial\Sigma \neq \emptyset$ for each $\xi \in \Lambda_c$.

Then from Corollary 3.3, a minimax value

$$\beta_c = \inf_{\xi \in \Lambda_c} \max_{t \in [0,1]} \mathcal{T}(\xi(t)) \tag{5.1}$$

is well-defined and finite. Since Palais-Smale-Pohozaev condition holds just for $b \in (-\infty, 0)$, it is important to estimate β_c. We have the following proposition.

Proposition 5.1

(i) *Assume* (f1)–(f5). *Then for sufficiently large* $c > 0$ *there exists* $\xi(t) \in \Lambda_c$ *such that*

$$\max_{t\in[0,1]} \mathcal{T}(\xi(t)) < 0. \tag{5.2}$$

(ii) *Assume (1.4) in addition to* (f1)–(f5). *Then for any* $c > 0$ *there exists* $\xi(t) \in \Lambda_c$ *with the property (5.2).*

(iii) $\lim_{c\to\infty} \frac{\beta_c}{c} = -\infty$.

Proof Let $\mu > 0$. Since u_μ is a MP solution, we have $\mathcal{J}(\mu, u_\mu(x/t)) \to -\infty$ as $t \to +\infty$. Therefore there exists an optimal path γ_μ such that

$$a(\mu) = \max_{t\in[0,1]} \mathcal{J}(\mu, \gamma_\mu(t))$$

and

$$\mathcal{T}(\mu, \gamma_\mu(1)) \le B_c - 1. \tag{5.3}$$

We also note that $\mathcal{T}(t, 0) \to -\infty$ as $t \to \infty$. Therefore joining the $\gamma_\mu(t)$ with the path $(\mu + Lt, 0)$ with L large enough, we can find a path $\xi(t)$ such that for μ sufficiently large

$$\max_{t\in[0,1]} \mathcal{T}(\xi(t)) < 0 \quad \text{and} \quad \xi(t) \in \Lambda_c$$

and thus we have (i).

(ii) If (1.4) holds, we can apply Proposition 3.4 and so we have

$$\lim_{\mu\to 0} \frac{a(\mu) - \frac{c}{2}\mu}{\mu} = -\frac{c}{2} < 0. \tag{5.4}$$

By (5.3) and (5.4) we infer that for $\mu > 0$ sufficiently small there exists $\xi_\mu(t) = (\mu, \gamma_\mu(t))$ such that

$$\max_{t\in[0,1]} \mathcal{T}(\xi(t)) < 0$$

and thus (ii).

(iii) Finally we have for any $\mu > 0$

$$\beta_c \le \max_{t\in[0,1]} \mathcal{T}(\xi(t)) = a(\mu) - \frac{\mu}{2}c \tag{5.5}$$

and thus

$$\limsup_{c\to\infty} \frac{\beta_c}{c} \le \lim_{c\to\infty} \frac{a(\mu)}{c} - \frac{\mu}{2} = -\frac{\mu}{2}.$$

Since μ is arbitrary, we have (iii). ■

We also have

Proposition 5.2

(i) *For sufficiently large* $c > 0$, $\beta_c < 0$.
(ii) *If (1.4) holds, then* $\beta_c < 0$ *for all* $c > 0$.

Proof By Proposition 5.1, we infer (i).
(ii) From (1.4) and Proposition 3.4, we derive for any $c > 0$

$$\beta_c = \inf_{\mu>0} \left(a(\mu) - \frac{1}{2}\mu c \right) < 0.$$

■

Finally using Theorem 4.4, we derive that the level β_c, defined in (5.1), is critical and thus Theorems 1.1 and 1.2 hold. For the positivity of the solutions, we remind to Remark 5.3.

Remark 5.3 We remark that the Mountain Pass solutions found in Theorems 1.1 and 1.2 are minimizers of the functional $\mathcal{L} : S_c \to \mathbb{R}$ defined by

$$\mathcal{L}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\, dx$$

on the sphere

$$S_c = \{\, u \in H_r^1(\mathbb{R}^N) \mid \|u\|_2^2 = c\},$$

namely,

$$\beta_c = \kappa_c,$$

where β_c is given in (5.1) and $\kappa_c = \inf_{u\in S_c} \mathcal{L}(u)$.
We just give an outline of the proof. Firstly we notice that we already proved that

$$\kappa_c > -\infty$$

and clearly the solution u_* obtained in Theorems 1.1 and 1.2 satisfies

$$0 > \beta_c = \mathcal{L}(u_*) \ge \kappa_c. \tag{5.6}$$

On the other hand, for a minimizer u_0 of $\mathcal{L}$ on S_c, that is, $\mathcal{L}(u_0) = \kappa_c$ (for the existence of a minimizer, see Remark 5.4 below), there exists a Lagrange multiplier $\mu_0 \in \mathbb{R}$ such that

$$-\Delta u_0 + \mu_0 u_0 = (I_\alpha * F(u_0)) f(u_0),$$

In particular, we have

$$\|\nabla u_0\|_2^2 + \mu_0 \|u_0\|_2^2 = \int_{\mathbb{R}^N} (I_\alpha * F(u_0)) f(u_0) u_0 \, dx. \tag{5.7}$$

We note that $\mu_0 > 0$ and u_0 satisfies Pohozaev identity. In fact, we consider a $\mathbb{R}$-action $\Phi : \mathbb{R} \times S_c \to S_c$ defined by

$$(\Phi_\theta v)(x) = e^{\frac{N}{2}\theta} v(e^\theta x). \tag{5.8}$$

Then we have $\|\Phi_\theta u\|_2^2 = \|u\|_2^2$ and

$$\mathcal{L}(\Phi_\theta u_0) = \frac{1}{2} e^{2\theta} \|\nabla u_0\|_2^2 - \frac{1}{2} e^{-(N+\alpha)\theta} \int_{\mathbb{R}^N} (I_\alpha * F(e^{\frac{N}{2}\theta} u_0(x))) F(e^{\frac{N}{2}\theta} u_0(x)) \, dx.$$

Since u_0 is a minimizer, we have $\frac{d}{d\theta}\big|_{\theta=0} \mathcal{L}(\Phi_\theta u_0) = 0$, that is,

$$\|\nabla u_0\|_2^2 + \frac{N+\alpha}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u_0)) F(u_0) \, dx - \frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u_0)) f(u_0) u_0 \, dx = 0. \tag{5.9}$$

From $\mathcal{L}(u_0) = \kappa_c < 0$, we have

$$\frac{1}{2} \|\nabla u_0\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u_0)) F(u_0) \, dx = \kappa_c < 0. \tag{5.10}$$

It follows form (5.7), (5.9), (5.10) that $\mu_0 > 0$ and Pohozaev identity:

$$\frac{N-2}{2} \|\nabla u_0\|_2^2 + \frac{N}{2} \mu_0 \|u_0\|_2^2 = \frac{N+\alpha}{2} \mathcal{D}(u_0). \tag{5.11}$$

Thus from the argument for Theorems 1.1 and 1.2, we can find a path $\xi_0(t) \in \Lambda_c$ such that

$$\max_{t \in [0,1]} \mathcal{T}(\xi_0(t)) = \mathcal{L}(u_0) = \kappa_c,$$

which implies $\beta_c \le \kappa_c$. Together with (5.6) we have $\beta_c = \kappa_c$.

If (f5) holds, the Mountain Pass solutions found in Theorems 1.1 and 1.2 are positive (see [28, Proposition 5.2]).

Remark 5.4 (Existence of a Minimizer of $\mathcal{L}$ on S_c) In order to show the existence of the minimizer u_0 of $\mathcal{L}$ on S_c, we use the action Φ_θ on S_c defined in (5.8). We introduce

$$\widehat{\mathcal{L}}(\theta, u) = \mathcal{L}(\Phi_\theta u) : \mathbb{R} \times S_c \to \mathbb{R}.$$

We note that

$$\inf_{(\theta,u)\in\mathbb{R}\times S_c} \widehat{\mathcal{L}}(\theta, u) = \kappa_c.$$

Applying Ekeland's Principle to $\widehat{\mathcal{L}} : \mathbb{R} \times S_c \to \mathbb{R}$, we find a sequence $(\theta_j, u_j) \subset \mathbb{R} \times S_c$ such that

$$\widehat{\mathcal{L}}(\theta_j, u_j) \to \kappa_c,\ \partial_\theta \widehat{\mathcal{L}}(\theta_j, u_j) \to 0,\ d_u \widehat{\mathcal{L}}(\theta_j, u_j) \to 0$$

Setting $\hat{u}_j(x) = e^{\frac{N}{2}\theta_j} u_j(e^{\theta_j} x)$, we observe for a suitable $\lambda_j \in \mathbb{R}$, $(\lambda_j, \hat{u}_j)$ is a $(PSP)_{\kappa_c}$-sequence for $\mathcal{T}(\lambda, u)$ introduced in Sect. 4. Thus, thanks to Theorem 4.2, after extracting a subsequence, $(\hat{u}_j)$ converges to a minimizer of $\mathcal{L}$ on S_c, provided $\kappa_c < 0$.

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References

1. Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**, 349–381 (1973)
2. Bartsch, T., Liu, Y., Liu, Z.: Normalized solutions for a class of nonlinear Choquard equations. SN Partial Differ. Equ. Appl. **1**, 34 (2020)
3. Bellazzini, J., Jeanjean, L., Luo, T.: Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations. Proc. Lond. Math.. Soc. **107**, 303–339 (2013)
4. Berestycki, H., Lions, P.-L.: Nonlinear scalar field equations. I. Existence of a ground state. Arch. Rational Mech. Anal. **82**, 313–345 (1983)
5. Cingolani, S., Clapp, M., Secchi, S.: Multiple solutions to a magnetic nonlinear Choquard equation. Z. Angew. Math. Phys. **63**, 233–248 (2012)
6. Cingolani, S., Clapp, M., Secchi, S.: Intertwining semiclassical solutions to a Schrödinger-Newton system. Discrete Contin. Dynam. Syst. Series S **6**, 891–908 (2013)

7. Cingolani, S., Gallo, M., Tanaka, K.: Normalized solutions for fractional nonlinear scalar field equation via Lagrangian formulation (submitted), arXiv:2103.10747
8. Cingolani, S., Jeanjean, L.: Stationary solutions with prescribed L^2-norm for the planar Schrödinger-Poisson system. SIAM J. Math. Anal. **51**, 3533–3568 (2019)
9. Cingolani, S., Secchi, S., Squassina, M.: Semiclassical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities. Proc. Roy. Soc. Edinburgh **140 A**, 973–1009 (2010)
10. Cingolani, S., Tanaka, K.: Semi-classical states for the nonlinear Choquard equations: existence, multiplicity and concentration at a potential well. Rev. Math. Iberoamericana **35**, 1885–1924 (2019)
11. Clapp, M., Salazar, D.: Positive and sign changing solutions to a nonlinear Choquard equation. J. Math. Anal. Appl. **407**, 1–15 (2013)
12. Fröhlich, J., Lenzmann, E.: Mean-field limit of quantum Bose gases and nonlinear Hartree equation. In: Séminaire: Équations aux Dérivées Partielles 2003–2004, Exp. No. XIX, 26 pp., Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau (2004)
13. Fröhlich, J., Tsai, T.-P., Yau, H.-T.: On the point-particle (Newtonian) limit of the non-linear Hartree equation. Comm. Math. Phys. **225**, 223–274 (2002)
14. Hirata, J., Ikoma, N., Tanaka, K.: Nonlinear scalar field equations in $\mathbb{R}^N$: mountain pass and symmetric mountain pass approaches. Topol. Methods Nonlinear Anal. **35**, 253–276 (2010)
15. Hirata, J., Tanaka, K.: Nonlinear scalar field equations with L^2 constraint: Mountain Pass and Symmetric Mountain Pass Approaches. Adv. Nonlinear Stud. **19**, 263–290 (2019)
16. Ikoma, N., Tanaka, K.: A note on deformation argument for L^2 constraint problems. Adv. Differ. Equ. **24**, 609–646 (2019)
17. Jeanjean, L.: Existence of solutions with prescribed norm for semilinear elliptic equations. Nonlinear Anal. **28**, 1633–1659 (1997)
18. Jeanjean, L., Luo, T.: Sharp nonexistence results of prescribed L^2-norm solutions for some class of Schrödinger-Poisson and quasi-linear equations. Z. Angew. Math. Phys. **64**, 937–954 (2013)
19. Lenzmann, E.: Uniqueness of ground states for pseudorelativistic Hartree equations. Anal. PDE **2**, 1–27 (2009)
20. Li, Y., Zhao, D., Wang, Q.: Concentration behavior of nonlinear Hartree-type equation with almost mass critical exponent. Z. Angew. Math. Phys. **70**, paper 28, 17 pp (2019)
21. Lieb, E.H.: Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. Stud. Appl. Math. **57**, 93–105 (1977)
22. Lieb, E.H.: Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. Ann. Math. **118**, 349–374 (1983)
23. Lions, P.-L.: The Choquard equation and related questions. Nonlinear Anal. Theory Methods Appl. **4**, 1063–1073 (1980)
24. Luo, T.: Multiplicity of normalized solutions for a class of nonlinear Schrödinger-Poisson-Slater equations. J. Math. Anal. Appl. **416**, 195–204 (2014)
25. Ma, L., Zhao, L.: Classification of positive solitary solutions of the nonlinear Choquard equation. Arch. Rational Mech. Anal. **195**, 455–467 (2010)
26. Moroz, I.M., Penrose, R., Tod, P.: Spherically-symmetric solutions of the Schrödinger-Newton equations, Topology of the Universe Conference (Cleveland, OH, 1997). Class. Quantum Grav. **15**, 2733–2742 (1998)
27. Moroz, V., Van Schaftingen, J.: Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics. J. Funct. Anal. **265**, 153–184 (2013)
28. Moroz, V., Van Schaftingen, J.: Existence of groundstates for a class of nonlinear Choquard equations. Trans. Am. Math. Soc. **367**, 6557–6579 (2015)
29. Moroz, V., Van Schaftingen, J.: Semi-classical states for the Choquard equations. Calc. Var. Partial Differ. Equ. **52**, 199–235 (2015)
30. Moroz, V., Van Schaftingen, J.: A guide to the Choquard equation. J. Fixed Point Theory Appl. **19**, 773–813 (2017)
31. Pekar, S.: Untersuchung über die Elektronentheorie der Kristalle. Akademie, Berlin (1954)

32. Penrose, R.: On gravity's role in quantum state reduction. Gen. Rel. Grav. **28**, 581–600 (1996)
33. Penrose, R.: Quantum computation, entanglement and state reduction. R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. **356**, 1927–1939 (1998)
34. Penrose, R.: The Road to Reality. A Complete Guide to the Laws of the Universe. Alfred A. Knopf, New York (2005)
35. Secchi, S.: A note on Schrödinger–Newton systems with decaying electric potential. Nonlinear Analysis **72**, 3842–3856 (2010)
36. Tod, P.: The ground state energy of the Schrödinger-Newton equation. Phys. Lett. A **280**, 173–176 (2001)
37. Wei, J., Winter, M.: Strongly interacting bumps for the Schrödinger–Newton equation. J. Math. Phys. **50**, 012905 (2009)
38. Ye, H.: Mass minimizers and concentration for nonlinear Choquard equations in $\mathbb{R}^N$. Topol. Methods Nonlinear Anal. **48**, 393–417 (2016)

Optimization of the Structural Performance of Non-homogeneous Partially Hinged Rectangular Plates

Alessio Falocchi

Abstract We consider a non-homogeneous partially hinged rectangular plate having structural engineering applications. In order to study possible remedies for torsional instability phenomena we consider the gap function as a measure of the torsional performances of the plate. We treat different configurations of load and we study which density function is optimal for our aims. The analysis is in accordance with some results obtained studying the corresponding eigenvalue problem in terms of maximization of the ratio of specific eigenvalues. Some numerical experiments complete the analysis.

Keywords Gap function · Torsional instability · Mass density

1 Introduction

We study a long narrow rectangular thin plate $\Omega \subset \mathbb{R}^2$, hinged at the short edges and free on the remaining two, see [12]. This plate may model the deck of a bridge; since this kind of structure exhibits problems of flutter instability, e.g. see [13, 15, 18], we optimize its design in order to reduce the phenomenon. To this aim one may vary the shape of the plate, see [6], or modify the materials composing it, see [3, 7, 8].

Here we fix the geometry of the plate, assuming that it has length π and width 2ℓ with $2\ell \ll \pi$ so that

$$\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2 ;$$

we assume that the plate is not homogeneous, i.e. it features variable density function $p = p(x, y)$; our aim is to find the optimal density configuration in order to improve the structural performance of the plate.

A. Falocchi (✉)
Dipartimento di Scienze Matematiche, Politecnico di Torino, Torino, Italy
e-mail: alessio.falocchi@polito.it

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In a rectangular plate it is possible to distinguish vertical and torsional oscillations; the most problematic are the second ones, that may cause the collapse of the structure, see [13]. Then we consider a functional, named *gap function*, able to measure the torsional performance of the plate, see also [5]. In particular, this functional measures the gap between the displacements of the two free edges of the structure; the higher is the gap the higher is the torsional motion of the plate. More precisely, we maximize the maximum of the absolute value of the gap function in a class of external forcing term; then we consider its minimization in a class of density functions. Hence, our final goal is to find the worst force and the best density in order to reduce the torsional oscillation of the plate.

Since the explicit solution of this *minimaxmax* problem is currently out of reach, we proceed testing the plate with some motivated external forces. Then we consider different densities $p(x, y)$ in order to understand how the gap function varies; the choice of $p(x, y)$ is driven by some results proposed in [3]. Here the authors present a study on the correspondent weighted eigenvalue problem and they compare different density functions in order to find the optimal, maximizing the ratio between the first torsional eigenvalue and the previous longitudinal; they tested some density functions proposing theoretical and numerical justifications. We point out that the study of a ratio of eigenvalues has some limits; first of all it requires to consider two specific eigenvalues, moreover the direct optimization of the ratio is very involved. As a consequence, the question is often dealt with in terms of minimization or maximization of a single eigenvalue, see [3] for details. Here we compare the density functions proposed in [3] and we observe that $p(x, y)$ optimal for [3] are optimal also with respect to the reduction of the gap function. This result confirms that the gap function is a reliable measure for the torsional performances of rectangular plates; furthermore, it is a useful tool to get information on optimal reinforces in order to reduce torsional instability phenomena.

The paper is organized as follows. In Sect. 2 we introduce some preliminaries and notations and we define longitudinal and torsional modes of vibration. In Sect. 3 we define the gap function, we write the *minimaxmax* problem we are interested in and we state the existence results, proved in Sect. 6. In Sect. 4 we describe the density functions that are meaningful for our aims. In Sect. 5 we study the problem considering external forces in $L^2(\Omega)$ and providing some numerical experiments to support the theoretical results.

2 Preliminaries and Variational Setting

2.1 Definition of the Problem

We derive the stationary equation which we are interested in from the energy of the system; we denote by $u = u(x, y)$ the vertical displacement of the plate Ω having mass surface density $p = p(x, y)$. In general, since we are dealing with a non-homogeneous plate, we may consider the modulus of Young $E = E(x, y)$ and the Poisson ratio $\sigma = \sigma(x, y)$ of the materials forming the plate not constant. We suppose that an external force for the unit mass $f = f(x, y)$ acts on the plate in the vertical direction. Thanks to the Kirchhoff-Love theory [14, 16], the energy of the plate is given by

$$\mathbb{E}(u) = \frac{h^3}{12}\int_\Omega \frac{E}{1-\sigma^2}\left(\frac{(\Delta u)^2}{2} + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy})\right)dxdy - \int_\Omega pfu\,dxdy,$$

where h is its constant thickness, see also [12].

To proceed with the classical minimization of the functional, we need some information on the regularity of the functions representing the materials composing the plate, i.e. $p(x, y)$, $E(x, y)$, $\sigma(x, y)$. We consider the possibility that the plate is composed by different materials, hence we cannot assume the continuity of the previous functions. In general discontinuous Young modulus and Poisson ratio generate some mathematical troubles in finding the minimization problem in strong form. For the civil engineering applications, which we are interested in, we point out that the Poisson ratio does not vary so much with respect to the possible choice of the materials; therefore, as a first approach, we suppose E and σ constant in space, while the density of the plate is in general variable and possibly discontinuous. Hence we have

$$\mathbb{E}(u) = \frac{Eh^3}{12(1-\sigma^2)}\int_\Omega \left(\frac{(\Delta u)^2}{2} + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy})\right)dxdy - \int_\Omega pfu\,dxdy;$$

in this framework we minimize the energy functional, we divide the differential equation for the flexural rigidity $\frac{Eh^3}{12(1-\sigma^2)}$ and, including it in the density function, we obtain

$$\begin{cases} \Delta^2 u = p(x, y)f(x, y) & \text{in } \Omega \\ u(0, y) = u_{xx}(0, y) = u(\pi, y) = u_{xx}(\pi, y) = 0 & \text{for } y \in (-\ell, \ell) \\ u_{yy}(x, \pm\ell) + \sigma u_{xx}(x, \pm\ell) = u_{yyy}(x, \pm\ell) + (2-\sigma)u_{xxy}(x, \pm\ell) = 0 & \text{for } x \in (0, \pi)\,. \end{cases} \tag{2.1}$$

The boundary conditions on the short edges are of Navier type, see [17], and model the situation in which the plate is hinged on $\{0, \pi\} \times (-\ell, \ell)$. Instead, the boundary conditions on the large edges are of Neumann type, modeling the fact that the deck is free to move vertically; for the Poisson ratio we shall assume

$$\sigma \in \left(0, \frac{1}{2}\right), \tag{2.2}$$

since most of the materials have values in this range.

In the sequel we denote by $\|\cdot\|_q$ the norm related to the Lebesgue spaces $L^q(\Omega)$ with $1 \le q \le \infty$ and we refer to q' as the conjugate of q, i.e. $1/q + 1/q' = 1$ with the usual conventions; moreover, given a functional space $V(\Omega)$, in the notation of the correspondent norm and scalar product we shall omit the set Ω, e.g. $\|\cdot\|_V := \|\cdot\|_{V(\Omega)}$.

In the next sections we study the behaviour of the plate with respect to different weight functions p and external forcing terms f.

2.2 Families of Forcing Terms and Weight Functions

We introduce

$$\mathcal{F}_V := \{f \in V(\Omega) : \|f\|_V = 1\}$$

the set of admissible forcing terms, fixed a certain functional space V. We introduce a family of weights to which p belongs

$$\mathcal{P}^{\alpha,\beta}_{L^\infty} := \left\{p \in L^\infty(\Omega) : \alpha \le p \le \beta\,,\ p(x,y) = p(x,-y) \text{ a.e. in } \Omega,\ \int_\Omega p\,dxdy = |\Omega|\right\} \tag{2.3}$$

where $\alpha, \beta \in \mathbb{R}^+$ with $\alpha < \beta$ fixed. When f belongs to certain functional spaces, we need further regularity on the weight functions; therefore we introduce a second family

$$\mathcal{P}^{\alpha,\beta}_{H^2} := \left\{p \in H^2(\Omega) :\ p \in \mathcal{P}^{\alpha,\beta}_{L^\infty} \quad \text{and} \quad \exists\,\kappa > 1 \ :\ \|p\|_{H^2} \leqslant \kappa\sqrt{|\Omega|}\right\},$$

with $\alpha, \beta \in \mathbb{R}^+$ and $\alpha < \beta$ fixed. The integral condition in (2.3) represents the preservation of the total mass of the plate; this is our fixed parameter, useful to compare the results between different weights. The bound on $\|p\|_{H^2}$ in $\mathcal{P}^{\alpha,\beta}_{H^2}$ is merely a technical condition to gain compactness; by Hölder inequality the preservation of the total mass condition yields $\|p\|_{H^2} \ge \sqrt{|\Omega|}$. Therefore, we choose $\kappa > 1$ to exclude the trivial case $p \equiv 1$ in Ω. Indeed, we will always assume

$$0 < \alpha < 1 < \beta\,,$$

studying the effect of a non-constant weight on the solution of (2.1). The assumption $\alpha < 1 < \beta$ is not restrictive; if we assume $\beta = 1$, it must be $p \equiv 1$ a.e. in Ω, since otherwise we would have $\int_\Omega p\,dx\,dy < |\Omega|$; similarly, if we consider $\alpha = 1$.

Moreover, we are interested in designs which are symmetric with respect to the mid-line of the roadway, being ℓ very small with respect to π. From a mathematical point of view, this assures two classes of eigenfunctions for the correspondent eigenvalue problem, respectively, even or odd in the y-variable; we shall clarify this question in Sect. 2.4.

2.3 *Existence and Uniqueness Result*

We introduce the space

$$H^2_*(\Omega) = \left\{u \in H^2(\Omega) : u = 0 \text{ on } \{0, \pi\} \times (-\ell, \ell)\right\},$$

where we study the weak solution of (2.1). Let us observe that the condition $u = 0$ has to be meant in a classical sense because $\Omega \subset \mathbb{R}^2$ and the energy space $H^2_*(\Omega)$ embeds into continuous functions. Furthermore, $H^2_*(\Omega)$ is a Hilbert space when endowed with the scalar product

$$(u, v)_{H^2_*} := \int_\Omega \left[\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})\right] dx\, dy$$

and associated norm

$$\|u\|^2_{H^2_*} = (u, u)_{H^2_*},$$

which is equivalent to the usual norm in $H^2(\Omega)$, see [12, Lemma 4.1]. We denote by $H^{-2}_*(\Omega)$ the dual space of $H^2_*(\Omega)$ and $\langle \cdot, \cdot \rangle$ its dual product. We write the problem (2.1) in weak sense

$$(u, v)_{H^2_*} = \langle pf, v\rangle \qquad \forall v \in H^2_*(\Omega). \tag{2.4}$$

Let us clarify what we mean for the dual product in (2.4) with respect to the choice of f and p.

If $f \in \mathcal{F}_{L^q}$ with $q \in (1, \infty]$ and $p \in \mathcal{P}^{\alpha,\beta}_{L^\infty}$, we write $\int_\Omega pfv\, dxdy$ instead of $\langle pf, v\rangle$.

If $f \in H^{-2}_*(\Omega)$ we need further regularity on p, e.g. $p \in \mathcal{P}^{\alpha,\beta}_{H^2}$. We introduce the linear functional $T_f : H^2_*(\Omega) \to \mathbb{R}$ such that $T_f(v) := \langle f, v\rangle$ for all $v \in H^2_*(\Omega)$ and we define

$$\langle pf, v\rangle := T_f(pv) \qquad \forall v \in H^2_*(\Omega). \tag{2.5}$$

Indeed, $H^2_*(\Omega)$ is a Banach algebra, being the $H^2_*(\Omega)$-norm equivalent to the $H^2(\Omega)$-norm, see [1, Theorem 5.23] applied to the Sobolev space $W^{m,p}(\Omega)$ with

$m = p = 2$ and $\Omega \subset \mathbb{R}^2$ convex with Lipschitz boundary. Therefore, if $p \in \mathcal{P}^{\alpha,\beta}_{H^2}$ we get $K > 0$ such that

$$pv \in H^2_*(\Omega) \qquad \|pv\|_{H^2_*} \leq K\|p\|_{H^2_*}\|v\|_{H^2_*} \qquad \forall v \in H^2_*(\Omega).$$

We state the following result.

Proposition 2.1 *Let $f \in \mathcal{F}_V$ and $0 < \alpha < 1 < \beta$. If*

(i) $V = L^q(\Omega)$ with $q \in (1, \infty]$ and $p \in \mathcal{P}^{\alpha,\beta}_{L^\infty}$,
(ii) $V = H^{-2}_*(\Omega)$ and $p \in \mathcal{P}^{\alpha,\beta}_{H^2}$,

then the problem (2.4) admits a unique weak solution $u \in H^2_*(\Omega) \subset C^0(\overline{\Omega})$.

Proof By [12] we have that the bilinear form $(u, v)_{H^2_*}$ is continuous and coercive, hence to apply Lax Milgram Theorem we consider the functional $\langle pf, v\rangle$.

(i) If $p \in \mathcal{P}^{\alpha,\beta}_{L^\infty}$ and $f \in \mathcal{F}_{L^q}$ with $q \in (1, \infty]$ then $pf \in L^q(\Omega)$; moreover we have $\Omega \subset \mathbb{R}^2$ so that $H^2_*(\Omega)$ is embedded in $C^0(\overline{\Omega})$. Therefore, applying Hölder inequality, we obtain $C_1 > 0$ such that

$$|\langle pf, v\rangle| = \left|\int_\Omega pf v\, dxdy\right| \leqslant \|pf\|_q\|v\|_{q'} \leq C_1\|v\|_{H^2_*} \qquad \forall v \in H^2_*(\Omega),$$

so that $\langle pf, v\rangle$ is a linear and continuous functional.

(ii) By (2.5) we observe that $T_f(pv)$ is linear and continuous, indeed we have $C_2 > 0$ such that

$$|T_f(pv)| = |\langle f, pv\rangle| \leq \|f\|_{H^{-2}_*}\|pv\|_{H^2_*} \leq C_2\|v\|_{H^2_*} \qquad \forall v \in H^2_*(\Omega),$$

being $H^2_*(\Omega)$ a Banach algebra.

The solution u is continuous since the space $H^2_*(\Omega)$ embeds into $C^0(\overline{\Omega})$. □

2.4 Definition of Longitudinal and Torsional Modes

To tackle (2.1) we need some preliminary information on the associated eigenvalue problem:

$$\begin{cases} \Delta^2 u = \lambda p(x, y)u & \text{in } \Omega \\ u(0, y) = u_{xx}(0, y) = u(\pi, y) = u_{xx}(\pi, y) = 0 & \text{for } y \in (-\ell, \ell) \\ u_{yy}(x, \pm\ell) + \sigma u_{xx}(x, \pm\ell) = u_{yyy}(x, \pm\ell) + (2 - \sigma)u_{xxy}(x, \pm\ell) = 0 & \text{for } x \in (0, \pi)\,. \end{cases} \tag{2.6}$$

As in [9], we introduce the subspaces of $H^2_*(\Omega)$:

$$H^2_{\mathcal{E}}(\Omega) := \{u \in H^2_*(\Omega) : u(x,-y) = u(x,y) \quad \forall (x,y) \in \Omega\},$$
$$H^2_{\mathcal{O}}(\Omega) := \{u \in H^2_*(\Omega) : u(x,-y) = -u(x,y) \quad \forall (x,y) \in \Omega\},$$

where

$$H^2_{\mathcal{E}}(\Omega) \perp H^2_{\mathcal{O}}(\Omega), \qquad H^2_*(\Omega) = H^2_{\mathcal{E}}(\Omega) \oplus H^2_{\mathcal{O}}(\Omega)\,. \tag{2.7}$$

We say that the eigenfunctions in $H^2_{\mathcal{E}}(\Omega)$ are *longitudinal* modes and those in $H^2_{\mathcal{O}}(\Omega)$ are *torsional* modes. For all $u \in H^2_*(\Omega)$ we denote by $u^e = \frac{u(x,y)+u(x,-y)}{2} \in H^2_{\mathcal{E}}(\Omega)$ and $u^o = \frac{u(x,y)-u(x,-y)}{2} \in H^2_{\mathcal{O}}(\Omega)$ respectively its even and odd components. Moreover, we set

$$H^{-2}_{\mathcal{E}}(\Omega) := \{f \in H^{-2}_*(\Omega) : \langle f, v\rangle = 0 \quad \forall v \in H^2_{\mathcal{O}}(\Omega)\},$$
$$H^{-2}_{\mathcal{O}}(\Omega) := \{f \in H^{-2}_*(\Omega) : \langle f, v\rangle = 0 \quad \forall v \in H^2_{\mathcal{E}}(\Omega)\}.$$

Since $H^2_*(\Omega) = H^{-2}_{\mathcal{E}}(\Omega) \oplus H^{-2}_{\mathcal{O}}(\Omega)$, there exists a unique couple $(f^e, f^o) \in H^{-2}_{\mathcal{E}}(\Omega) \times H^{-2}_{\mathcal{O}}(\Omega)$ such that $f = f^e + f^o$ for all $f \in H^{-2}_*(\Omega)$. We endow the space $H^{-2}_*(\Omega)$ with the norm $\|f\|_{H^{-2}_*} := \sup_{\|v\|_{H^2_*}=1}\langle f, v\rangle$, observing that

$$\|f\|_{H^{-2}_*} = \max\{\|f^o\|_{H^{-2}_*}, \|f^e\|_{H^{-2}_*}\} \qquad \forall f \in H^{-2}_*(\Omega). \tag{2.8}$$

When $p \equiv 1$ the whole spectrum of (2.6) is determined explicitly in [12] and gives two class of eigenfunctions belonging respectively to $H^2_{\mathcal{E}}(\Omega)$ or $H^2_{\mathcal{O}}(\Omega)$. Thanks to the symmetry assumption on p we obtain the same distinction for all the linearly independent eigenfunctions of the weighted eigenvalue problem (2.6).

We denote by $\mu_m(p)$ and $\nu_m(p)$ respectively the ordered weighted longitudinal and torsional eigenvalues of (2.6), repeated with their multiplicity; moreover, we denote respectively by $z^p_m(x,y) \in H^2_{\mathcal{E}}(\Omega)$ and $\theta^p_m(x,y) \in H^2_{\mathcal{O}}(\Omega)$, the corresponding (ordered) longitudinal and torsional linearly independent eigenfunctions of (2.6). We consider the eigenfunctions normalized in $L^2_p(\Omega)$ ($L^2(\Omega)$-weighted), i.e.

$$\|\sqrt{p}\, z^p_m\|_2^2 = \int_\Omega p\,(z^p_m)^2\,dxdy = 1 \qquad \|\sqrt{p}\,\theta^p_m\|_2^2 = \int_\Omega p\,(\theta^p_m)^2\,dxdy = 1. \tag{2.9}$$

3 Gap Function

In real structures the most problematic motions are related to the torsional oscillations, i.e. those in which prevail torsional modes. How can we measure the torsional behaviour? By Proposition 2.1, the solution of (2.1) is continuous; hence, we define the *gap function*, see also [5],

$$\mathcal{G}_{f,p}(x) := u(x,\ell) - u(x,-\ell) \qquad \forall x \in [0,\pi], \tag{3.1}$$

depending on the weight p and on the external load f. This function gives for every $x \in [0,\pi]$ the difference between the vertical displacements of the free edges, providing a measure of the torsional response. The maximal gap is given by

$$\mathcal{G}^{\infty}_{f,p} := \max_{x\in(0,\pi)} |\mathcal{G}_{f,p}(x)|. \tag{3.2}$$

In this way we introduce the map $\mathcal{G}^{\infty}_{f,p} : \mathcal{F}_V \times \mathcal{P}^{\alpha,\beta}_W \to [0,+\infty)$ with $(f,p) \mapsto \mathcal{G}^{\infty}_{f,p}$, that we study respectively in the cases

$$\begin{aligned} &(i) \quad (V,W) = \big(L^q(\Omega), L^\infty(\Omega)\big) \text{ with } q \in (1,\infty] \\ &(ii) \quad (V,W) = \big(H^{-2}_*(\Omega), H^2(\Omega)\big) \end{aligned} \tag{3.3}$$

for which Proposition 2.1 assures the uniqueness of a solution to (2.1).

Our aim is to find the worst $f \in \mathcal{F}_V$, i.e. the forcing term that maximizes $\mathcal{G}^{\infty}_{f,p}$, and the best weight $p \in \mathcal{P}^{\alpha,\beta}_W$ that minimizes $\mathcal{G}^{\infty}_{f,p}$. More precisely we want to solve the *minimaxmax* problem

$$\mathcal{G}^{\infty} := \min_{p\in\mathcal{P}^{\alpha,\beta}_W} \max_{f\in\mathcal{F}_V} \max_{x\in(0,\pi)} |\mathcal{G}_{f,p}(x)|,$$

in the cases (3.3).

In Sect. 6 we prove the existence results.

Theorem 3.1 *Given* $p \in \mathcal{P}^{\alpha,\beta}_W$ *with* $0 < \alpha < 1 < \beta$, *if*

(i) $W = L^\infty(\Omega)$ *and* $f \in \mathcal{F}_V$ *with* $V = L^q(\Omega)$ $q \in (1,\infty]$,
(ii) $W = H^2(\Omega)$ *and* $f \in \mathcal{F}_V$ *with* $V = H^{-2}_*(\Omega)$,

then the problem

$$\mathcal{G}^{\infty}_p := \max_{f\in\mathcal{F}_V} \mathcal{G}^{\infty}_{f,p} \tag{3.4}$$

admits solution.

Theorem 3.2 *Given $f \in \mathcal{F}_V$, if*

(i) $V = L^q(\Omega)$ *with* $q \in (1, \infty]$ *and* $p \in \mathcal{P}_W^{\alpha,\beta}$ *(*$0 < \alpha < 1 < \beta$*) with* $W = L^\infty(\Omega)$*,*
(ii) $V = H_*^{-2}(\Omega)$ *and* $p \in \mathcal{P}_W^{\alpha,\beta}$ *(*$0 < \alpha < 1 < \beta$*) with* $W = H^2(\Omega)$*,*

then the problem

$$\min_{p \in \mathcal{P}_W^{\alpha,\beta}} \mathcal{G}_p^\infty, \tag{3.5}$$

admits solution.

The next result shows that for $p \in \mathcal{P}_W^{\alpha,\beta}$ (y-even), the worst force $f \in \mathcal{F}_V$ in terms of torsional performance can be sought in the class of the y-odd distributions or functions.

Proposition 3.3

(i) *Let* (V, W) *as in* (3.3)-(i) *then problem (3.4) is equivalent to*

$$\max\{\mathcal{G}_{f,p}^\infty : f \in \mathcal{F}_{L^q}, \ \ f(x, -y) = -f(x, y) \ a.e. \ in \ \Omega\}.$$

Moreover, if $q \in (1, \infty)$ *any maximizer is necessarily odd with respect to* y*.*

(ii) *Let* (V, W) *as in* (3.3)-(ii)*, then problem (3.4) is equivalent to*

$$\max\{\mathcal{G}_{f,p}^\infty : f \in H_{\mathcal{O}}^{-2}, \ \ \|f\|_{H_*^{-2}} = 1\}.$$

This proposition and its proof are inspired by Berchio et al. [7, Theorem 4.1–4.2], where a similar problem is dealt with and further results are given. We underline that the uniqueness of a y-odd maximizer is not guaranteed; indeed, solely in the case (3.3)-(i) with $q \in (1, \infty)$ we obtain *only* odd maximizers. In the cases (3.3)-(i) with $q = \infty$ and (3.3)-(ii) it is possible that other f, not necessarily odd, attain the maximum, see also [7].

4 The Choice of the Weight Function $p \in \mathcal{P}_{L^\infty}^{\alpha,\beta}$

About the choice of the weight function $p \in \mathcal{P}_W^{\alpha,\beta}$ we are mainly interested in density functions not necessarily continuous, hence we consider $W = L^\infty(\Omega)$; therefore, in the rest of the paper we focus on (3.4)–(3.5) in the case $(V, W) = \big(L^q(\Omega), L^\infty(\Omega)\big)$ with $q \in (1, \infty]$.

We refer to some results obtained on the correspondent eigenvalue problem (2.6) presented in [3]. Here the authors find the best rearrangement of materials in Ω

which maximizes the ratio between two selected eigenvalues of (2.6), considering the optimization problem:

$$\mathcal{R} = \sup_{p \in \mathcal{P}^{\alpha,\beta}_{L^\infty}} \frac{\nu(p)}{\mu(p)}, \tag{4.1}$$

where $\nu(p)$ and $\mu(p)$ are respectively a torsional and a longitudinal eigenvalue. The direct study of (4.1) is very involved, then there are some theoretical results on the problem of maximization of the first torsional eigenvalue or minimization of the first longitudinal eigenvalue with respect to p; these results give suggestions on (4.1) and support some conjectures also thanks to numerical experiments. More precisely, in [3] the authors proved theoretically that optimal weights $p(x, y)$ in increasing or reducing the first torsional or longitudinal eigenvalue must be of *bang-bang* type, i.e.

$$p(x, y) = \alpha\chi_S(x, y) + \beta\chi_{\Omega\setminus S}(x, y) \quad \text{for a.e. } (x, y) \in \Omega\,,$$

for a suitable set $S \subset \Omega$, $0 < \alpha < 1 < \beta$ and χ_S is the characteristic function of S. In other words, the plate must be composed by two different materials properly located in Ω; this is useful in engineering terms, since the manufacturing of two materials with constant density is simpler than the assemblage of a material having variable density. On the other hand this produces some mathematical troubles, for instance when we consider as external forcing term $f \in H^{-2}_*(\Omega)$, see Proposition 2.1.

In the sequel we distinguish five meaningful *bang-bang* configurations for $p \in \mathcal{P}^{\alpha,\beta}_{L^\infty}$; we list the cases representing on the right in black the localization of the reinforcing material on the plate:

(i) $\boxed{p \equiv 1}$
This is a particular case when $\alpha = \beta = 1$ that corresponds to the homogeneous plate; we do not apply reinforcements, but we consider this case to compare it with the non-homogeneous ones.

(ii) $\boxed{p^*(x, y)}$ 
This choice comes out from the study of the problem

$$\nu_1^{\alpha,\beta} := \sup_{p \in \mathcal{P}^{\alpha,\beta}_{L^\infty}} \nu_1(p)\,. \tag{4.2}$$

We call *optimal pair* for (4.2) a couple $(\widehat{p}, \theta_1^{\widehat{p}})$ such that $\widehat{p}$ achieves the supremum in (4.2) and $\theta_1^{\widehat{p}}$ is an eigenfunction of $\nu_1(\widehat{p})$. In [3] the following result is proved.

Proposition 4.1 ([3]) *Problem* (4.2) *admits an optimal pair* $(\widehat{p}, \theta_1^{\widehat{p}}) \in \mathcal{P}_{L^\infty}^{\alpha,\beta} \times H^2_{\mathcal{O}}(\Omega)$. *Furthermore,* $\theta_1^{\widehat{p}}$ *and* $\widehat{p}$ *are related as follows*

$$\widehat{p}(x, y) = \beta\chi_{\widehat{S}}(x, y) + \alpha\chi_{\Omega\setminus\widehat{S}}(x, y) \quad \textit{for a.e. } (x, y) \in \Omega\,,$$

where $\widehat{S} = \{(x, y) \in \Omega\ :\ (\theta_1^{\widehat{p}})^2(x, y) \leqslant \widehat{t}\}$ *for some* $\widehat{t} > 0$ *such that* $|\widehat{S}| = \frac{1-\alpha}{\beta-\alpha}\,|\Omega|$.

Since we do not know explicitly $\theta_1^{\widehat{p}}$, the function $\theta_1^{\widehat{p}}$ is replaced by the torsional eigenfunction $\theta_1^1(x, y)$ of (2.6) with $p \equiv 1$, i.e. an eigenfunction corresponding to $\nu_1(1)$. This is explicitly known, see [12]; for details on this choice see [3]. Therefore we consider

$$p^*(x, y) := \beta\chi_{S^*}(x, y) + \alpha\chi_{\Omega\setminus S^*}(x, y) \quad \text{for a.e. } (x, y) \in \Omega\,,$$

where $S^* := \{(x, y) \in \Omega\ :\ (\theta_1^1)^2(x, y) \leq t^*\}$ for $t^* > 0$ such that $|S^*| = \frac{1-\alpha}{\beta-\alpha}|\Omega|$.

(iii) $\boxed{\breve{p}(y)}$ 

In order to find a reinforce more suitable for manufacturing, inspired by $p^*(x, y)$, we consider a weight depending only on y and concentrated around the mid-line $y = 0$, i.e.

$$\breve{p}(x, y) = \breve{p}(y) := \beta\chi_{\breve{I}}(y) + \alpha\chi_{(-\ell,\ell)\setminus\breve{I}}(y) \quad \text{for a.e. } (x, y) \in \Omega\,,$$

where $\breve{I} := \big(-\frac{\ell(\beta-1)}{\beta-\alpha}, \frac{\ell(\beta-1)}{\beta-\alpha}\big)$.

(iv) $\boxed{\overline{p}_i(x),\, i \in \mathbb{N}^+}$ 

The reasons of this choice are quite involved. We give here only the main idea and for details we refer to [3].

For $i \in \mathbb{N}^+$, we set the minimum problem

$$\mu_i^{\alpha,\beta} := \inf_{p\in\mathcal{P}_{L^\infty}^{\alpha,\beta}} \mu_i(p)\,, \tag{4.3}$$

where $\mu_i(p)$ is the i-th longitudinal eigenvalue of (2.6). We call *optimal pair* for (4.3) a couple $(\overline{p}_i, z_i^{\overline{p}_i})$ such that $\overline{p}_i$ achieves the infimum in (4.3) and $z_i^{\overline{p}_i}$ is an eigenfunction of $\mu_i(\overline{p}_i)$. In [8, Theorem 3.2] the following result is proved.

Proposition 4.2 ([8]) *Set* $i = 1$, *then problem* (4.3) *admits an optimal pair* $(\overline{p}_1, z_1^{\overline{p}_1}) \in \mathcal{P}_{L^\infty}^{\alpha,\beta} \times H^2_{\mathcal{E}}(\Omega)$. *Furthermore,* $z_1^{\overline{p}_1}$ *and* $\overline{p}_1$ *are related as follows*

$$\overline{p}_1(x, y) = \alpha\chi_{S_1}(x, y) + \beta\chi_{\Omega\setminus S_1}(x, y) \quad \textit{for a.e. } (x, y) \in \Omega\,,$$

where $S_1 = \{(x, y) \in \Omega : (z_1^{\overline{p}_1})^2(x, y) \le t_1\}$ *for some* $t_1 > 0$ *such that* $|S_1| = \frac{\beta-1}{\beta-\alpha}|\Omega|$.

Things become more involved for higher longitudinal eigenvalues and we do not find an analytical expression as for $i = 1$. Focusing on upper bounds for $\mu_i(p)$, see [3], we propose the following approximated optimal weight for $\mu_i^{\alpha,\beta}$:

$$\overline{p}_i(x, y) = \overline{p}_i(x) := \beta\chi_{I_i}(x) + \alpha\chi_{(0,\pi)\setminus I_i}(x), \quad \text{for a.e. } (x, y) \in \Omega\,,$$

where $I_i := \bigcup_{h=1}^{i}\left(\frac{\pi}{2i}(2h-1) - \frac{\pi}{i}\frac{(1-\alpha)}{2(\beta-\alpha)}, \frac{\pi}{2i}(2h-1) + \frac{\pi}{i}\frac{(1-\alpha)}{2(\beta-\alpha)}\right)$.

(v) $\boxed{\overline{\overline{p}}(x)}$

We consider a weight concentrated near the short edges of the plate:

$$\overline{\overline{p}}(x, y) = \overline{\overline{p}}(x) := \alpha\chi_I(x) + \beta\chi_{(0,\pi)\setminus I}(x) \quad \text{for a.e. } (x, y) \in \Omega\,,$$

where $I := \left(\frac{\pi}{2} - \frac{\pi(\beta-1)}{2(\beta-\alpha)}, \frac{\pi}{2} + \frac{\pi(\beta-1)}{2(\beta-\alpha)}\right)$. This weight seems to be simple for manufacturing and reasonable in order to increase $\mathcal{R}$.

We denote by

$$\widehat{P}_{\alpha,\beta} := \{p \in \mathcal{P}_{L^\infty}^{\alpha,\beta} : p(x, y) \text{ coincides with } 1 \text{ or } p^*(x, y) \text{ or } \breve{p}(y) \text{ or } \overline{p}_{10}(x) \text{ or } \overline{\overline{p}}(x) \quad \forall (x, y) \in \Omega\};$$

we shall explain in the next section why we are interested in $\overline{p}_{10}(x)$ in the fourth case.

5 $L^2(\Omega)$ External Forcing Terms

When $f \in \mathcal{F}_{L^2}$ it is possible to obtain more information on the solution of (2.4) and, in turn, on the gap function. In this case we expand u in Fourier series, adopting an orthonormal basis of L^2_p composed by the eigenfunctions of (2.6). In Sect. 6 we prove the following result.

Proposition 5.1 *For* $m \in \mathbb{N}^+$, *we denote by* $\nu_m(p)$ *and* $\mu_m(p)$ *the eigenvalues of (2.6) and, respectively,* $\theta_m^p(x, y)$ *and* $z_m^p(x, y)$ *the corresponding normalized eigenfunctions, see* (2.9)*.*

If $f \in \mathcal{F}_{L^2}$ *and* $p \in \mathcal{P}_{L^\infty}^{\alpha,\beta}$ *then the unique solution of (2.4) reads*

$$u(x, y) = \sum_{m=1}^{\infty}\left[\frac{a_m}{\nu_m(p)}\theta_m^p(x, y) + \frac{b_m}{\mu_m(p)}z_m^p(x, y)\right] \tag{5.1}$$

and

$$\mathcal{G}_{f,p}(x) = 2\sum_{m=1}^{\infty} \frac{a_m}{\nu_m(p)}\theta_m^p(x,\ell) \qquad \forall x \in [0,\pi], \tag{5.2}$$

where

$$a_m := \int_\Omega pf\,\theta_m^p\,dxdy \qquad b_m := \int_\Omega pf\,z_m^p\,dxdy.$$

If $f \in \mathcal{F}_{L^2}$ *and* $f(x,-y) = -f(x,y)$ *a.e. in* Ω *then* $u(x,y) = \sum_{m=1}^{\infty} \frac{a_m}{\nu_m(p)}\theta_m^p(x,y)$.

Driven by Proposition 3.3, we shall consider y-odd forcing terms; in [2] the authors conjectured as worst forcing term

$$f_0(x,y) = \begin{cases} 1 & y \in [0,\ell] \\ -1 & y \in [-\ell,0). \end{cases}$$

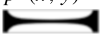
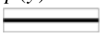
Since $\|f_0\|_2 = \sqrt{|\Omega|}$ and we are interested in $f \in \mathcal{F}_{L^2}$, we normalize f_0, i.e.

$$\overline{f}_0(x,y) = \begin{cases} \dfrac{1}{\sqrt{|\Omega|}} & y \in [0,\ell] \\ \dfrac{-1}{\sqrt{|\Omega|}} & y \in [-\ell,0). \end{cases}$$

We refer to Table 1 for numerical results about $\overline{f}_0$.

A physical interesting case is when f is in resonance with the structure, i.e. when f is a multiple of an eigenfunction of (2.6). The case in which f is proportional to

Table 1 The first torsional weighted eigenvalues $\nu_1(p)$, $\nu_2(p)$ and $\mathcal{G}^{\infty}_{f,p}$ defined in (3.2), assuming (5.3)–(5.4) and $N = 30$

	$p \equiv 1$	$p^*(x,y)$ 	$\breve{p}(y)$ 	$\overline{p}_{10}(x)$ 	$\overline{\overline{p}}(x)$ 
$\nu_1(p) \times 10^{-4}$	1.09	**1.98**	1.75	1.09	1.56
$\nu_2(p) \times 10^{-4}$	4.38	6.88	**7.01**	4.37	4.14
$\mathcal{G}^{\infty}_{\overline{f}_0,p} \times 10^4$	9.32	**6.09**	6.99	9.32	7.00
$\mathcal{G}^{\infty}_{\overline{f}_1,p} \times 10^4$	1.23×10	**6.74**	7.71	1.23×10	8.21
$\mathcal{G}^{\infty}_{\overline{f}_2,p} \times 10^4$	3.08	**1.93**	**1.93**	3.11	3.38

In bold we highlight the best values with respect to the weight tested a longitudinal mode is not interesting from our point of view since the gap function vanishes. Hence, we consider f proportional to the j-th torsional mode, i.e.

$$f_j(x, y) = \theta_j^p(x, y);$$

since $\|f_j\|_2 \neq 1$, we consider $\overline{f}_j(x, y) = \theta_j^p(x, y)/\|\theta_j^p\|_2$ so that $\overline{f}_j \in \mathcal{F}_{L^2}$ for all $j \in \mathbb{N}^+$. Trough Proposition 5.1, we readily obtain

$$a_m = \begin{cases} 1/\|\theta_j^p\|_2 & m = j \\ 0 & m \neq j \end{cases} \qquad u(x, y) = \frac{\theta_j^p(x, y)}{\nu_j(p)\|\theta_j^p\|_2} \qquad \mathcal{G}_{f_j,p}(x) = 2\frac{\theta_j^p(x, \ell)}{\nu_j(p)\|\theta_j^p\|_2}.$$

We provide now some numerical results considering a narrow plate, as it may be the deck of a suspension bridge, composed by typical materials adopted for these structures, i.e.

$$\ell = \frac{\pi}{150} \qquad \sigma = 0.2, \tag{5.3}$$

for details see [4, 10, 11]. We point out that with these parameters the eigenvalues of the homogeneous plate ($p \equiv 1$) are ordered in the following sequence

$$\mu_1(1) < \ldots < \mu_{10}(1) < \nu_1(1) < \mu_{11}(1) < \ldots$$

Hence, the longitudinal eigenvalue closest to the first torsional from below is $\mu_{10}(1)$; for this reason we consider $p \in \widehat{P}_{\alpha,\beta}$ fixing $i = 10$ for the fourth reinforce $\overline{p}_{10}$ proposed in Sect. 4. On the choice of the values $0 < \alpha < 1 < \beta$ related to the family $\mathcal{P}_{L^\infty}^{\alpha,\beta}$, for the applicative purpose we may strengthen the plate with steel and we may consider the other material composed by a mixture of steel and concrete; therefore, the denser material has approximately triple density with respect to the weaker. Thus, we assume

$$\alpha = 0.5 \qquad \beta = 1.5. \tag{5.4}$$

The numerical computation of the gap function in (5.2) is obtained truncating the Fourier series at a certain $N \geq 1$, integer; we compute the weighted eigenvalues and eigenfunctions of (2.6), exploiting the explicit information we have in the case $p \equiv 1$, see [12], and adopting the same numerical procedure described in [3].

In Table 1 we present the maximum values assumed by the gap function with respect to the choice of $f \in \mathcal{F}_{L^2}$ and $p \in \widehat{P}_{\alpha,\beta}$; as one can expect, for $f = \overline{f}_0$ the absolute maximum is always attained in $x = \pi/2$, while for $f = \overline{f}_j$ is assumed where $\sin(jx)$ has stationary points; indeed, $\theta_j^p(x, \pm\ell)$ is qualitatively similar to $\pm A\sin(jx)$ ($A \in \mathbb{R}^+$, $j \in \mathbb{N}^+$), see Fig. 1.

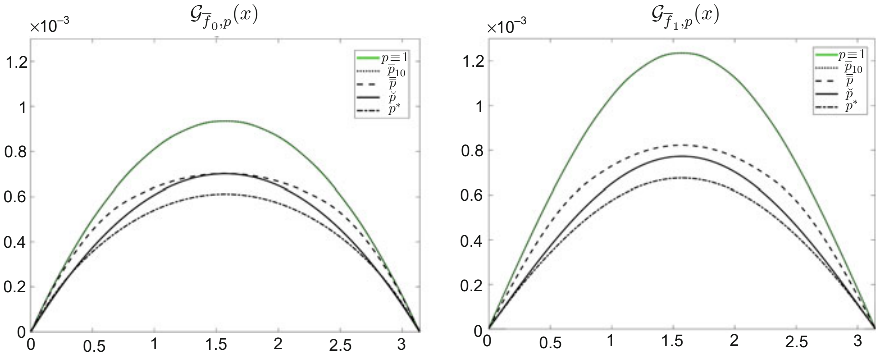


Fig. 1 Plots of the gap functions $\mathcal{G}_{\overline{f}_0,p}(x)$ and $\mathcal{G}_{\overline{f}_1,p}(x)$ for $x \in [0, \pi]$, varying p, assuming (5.3)–(5.4) and $N = 30$

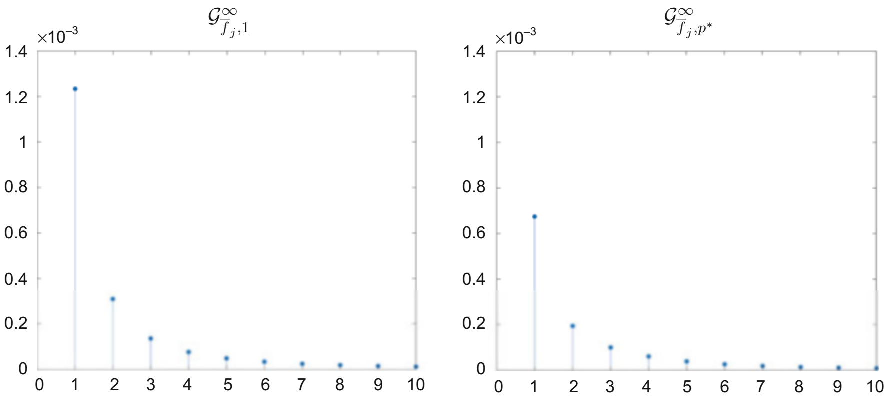


Fig. 2 Plots of $j \mapsto \mathcal{G}^{\infty}_{\overline{f}_j,1}$ and $j \mapsto \mathcal{G}^{\infty}_{\overline{f}_j,p^*}$, assuming (5.3)–(5.4) and $N = 30$

In Fig. 2 we plot $j \mapsto \mathcal{G}^{\infty}_{\overline{f}_j,p}$ when the plate is homogeneous and $p = p^*$; through this result we conjecture that the gap function reduces in amplitude when $\overline{f}_j$ is in resonance with higher torsional modes.

The choice to strengthen the plate with densities like $\overline{p}_i(x)$ $(i \in \mathbb{N}^+)$ needs some remarks. In this paper we considered only the case $\overline{p}_{10}(x)$, because it is emblematic for all $\overline{p}_i(x)$; indeed, the values of $\overline{p}_{10}(x)$ in Table 1 are very similar to those related to $\overline{p}_i(x)$ with $i = 4, \ldots, 15$, hence we do not show them. We point out that these reinforces are thought to reduce the i-th longitudinal eigenvalue, see [3]. From our analysis we observe that they are not so useful in modifying the torsional eigenvalues and in lowering the gap function; this is confirmed also by Fig. 1 where the gap function related to $\overline{p}_{10}(x)$ is very close to the gap function of the homogeneous plate. Numerically we observe that this trend is less and less remarked as we increase the size of ℓ with respect to (5.3). Hence, for $\ell \gg \frac{\pi}{150}$, e.g. $\ell = \frac{\pi}{15}$, it is possible that weights as $\overline{p}_i(x)$ ($i \in \mathbb{N}^+$) play a role in the torsional performance of the plate, but this overcomes our applicative purposes.

The worst situation among the tested external forces appears when $f = \overline{f}_1$ followed by $f = \overline{f}_0$; this suggests that the forces $f \in \mathcal{F}_{L^2}$ which maintain the same (and opposite) sign along the two free edges of the plate seem to be the candidate solutions of (3.4). Among the weight considered, the possible optimal reinforces of (3.5) are $p^*(x, y)$ or $\breve{p}(y)$, see Fig. 1. The weight $p^*(x, y)$ provides very good results for our aims, while $\breve{p}(y)$ is more suitable to maximize the second torsional eigenvalue; this is also confirmed by the value of $\mathcal{G}^{\infty}_{\overline{f}_2,p}$, i.e. the maximum of the gap function when f is in resonance with the second torsional weighted eigenfunction. In general, this agrees with the results obtained in [3], in which the problem is dealt with a different point of view, based on the maximization of the eigenvalues ratio $\mathcal{R}$ in (4.1).

6 Proofs

6.1 *Proof of Theorem 3.1*

Fixed $p \in \mathcal{P}^{\alpha,\beta}_W$ with $0 < \alpha < 1 < \beta$, we prove the continuity of the map $f \mapsto \mathcal{G}^{\infty}_{f,p}$ in the following lemma.

Lemma 6.1 *Let (V, W) the couple of functional spaces defined respectively in (3.3)-(i) or in (3.3)-(ii). The map $\mathcal{G}^{\infty}_{f,p} : V \to [0, +\infty)$ is continuous when V is endowed with the weak* topology.*

Proof Let $\{f_n\}_n \subset V$ be such that $f_n \overset{*}{\rightharpoonup} f$ in V for $n \to +\infty$. Denoting by u_n the solution of (2.4) corresponding to f_n, we have

$$(u_n, v)_{H^2_*} = \langle pf_n, v\rangle \qquad \forall v \in H^2_*(\Omega); \tag{6.1}$$

since $f_n \overset{*}{\rightharpoonup} f$ in V, its V norm is bounded, then the above equality with $v = u_n \in H^2_*(\Omega) \subset C^0(\overline{\Omega})$ gives respectively in the cases (3.3)-(i) and (3.3)-(ii)

$$\begin{aligned}
&(i) \qquad \|u_n\|^2_{H^2_*} = \left|\int_\Omega f_n\, pu_n\, dxdy\right| \leqslant \beta \int_\Omega |f_n u_n|\, dxdy \leqslant \beta\|f_n\|_q\|u_n\|_{q'} \le C_3\|u_n\|_{H^2_*},\\
&(ii) \qquad \|u_n\|^2_{H^2_*} = \left|\langle pf_n, u_n\rangle\right| = \left|\langle f_n, pu_n\rangle\right| \leqslant \|f_n\|_{H^{-2}_*}\|pu_n\|_{H^2_*} \le C_4\|u_n\|_{H^2_*},
\end{aligned} \tag{6.2}$$

in which in the last inequality we used that $H^2_*(\Omega)$ is a Banach algebra. Therefore $\|u_n\|_{H^2_*} \le C$ for some $C > 0$; thus we obtain, up to a subsequence, $u_n \rightharpoonup \overline{u}$ in

$H^2_*(\Omega)$. Denoting by V' the dual space of V, we get $pv \in V'$; hence we pass to the limit (6.1)

$$(\overline{u}, v)_{H^2_*} = \langle f, pv\rangle \qquad \forall v \in H^2_*(\Omega),$$

obtaining by the uniqueness that $\overline{u}$ is the weak solution of (2.4).

The embedding $H^2_*(\Omega) \subset C^0(\overline{\Omega})$ is compact, therefore $u_n \to \overline{u}$ in $C^0(\overline{\Omega})$, implying that the gap function $\mathcal{G}_{f_n,p}(x)$ converges uniformly to $\mathcal{G}_{f,p}(x)$ as $n \to +\infty$ for all $x \in [0, \pi]$. Therefore $\mathcal{G}^{\infty}_{f_n,p} \to \mathcal{G}^{\infty}_{f,p}$ as $n \to +\infty$. □

Proof of Theorem 3.1 Completed Let $p \in \mathcal{P}^{\alpha,\beta}_W$ fixed and $\{f_n\} \subset \mathcal{F}_V$ a maximizing sequence for (3.4); since $\|f_n\|_V = 1$, we have, up to a subsequence, $f_n \overset{*}{\rightharpoonup} \overline{f}$ in V. By the lower semi continuity of the norms we have $\|\overline{f}\|_V \leqslant \|f_n\|_V = 1$. Through Lemma 6.1 we obtain

$$\max_{f\in\mathcal{F}_V} \mathcal{G}^{\infty}_{f,p} = \mathcal{G}^{\infty}_{\overline{f},p};$$

we prove that $\|\overline{f}\|_V = 1$. For contradiction we suppose $\|\overline{f}\|_V < 1$; hence, we set $\widehat{f} = \overline{f}/\|\overline{f}\|_V$ and by linearity we obtain $\mathcal{G}^{\infty}_{\widehat{f},p} = \mathcal{G}^{\infty}_{\overline{f},p}/\|\overline{f}\|_V > \mathcal{G}^{\infty}_{\overline{f},p}$. This is absurd. □

6.2 *Proof of Theorem 3.2*

In the proof we shall use the compactness of the set $\mathcal{P}^{\alpha,\beta}_W$; if $W = L^{\infty}(\Omega)$ the set $\mathcal{P}^{\alpha,\beta}_{L^{\infty}}$ is compact for the L^{∞} weak* topology, see [3, Lemma 5.2]. If $W = H^2(\Omega)$ we prove the following result.

Lemma 6.2 *The set $\mathcal{P}^{\alpha,\beta}_{H^2}$ with $0 < \alpha < 1 < \beta$ is compact for the H^2 weak topology.*

Proof Let $\{p_n\}_n \subset \mathcal{P}^{\alpha,\beta}_{H^2}$, then by definition $\|p_n\|_{H^2} \leq \kappa\sqrt{|\Omega|}$, hence, up to a subsequence, we have $p_n \rightharpoonup \overline{p}$ in $H^2(\Omega)$ (as $n \to +\infty$) for some $\overline{p} \in H^2(\Omega)$ and

$$\|\overline{p}\|_{H^2} \leqslant \liminf_{n\to+\infty} \|p_n\|_{H^2} \leq \kappa\sqrt{|\Omega|};$$

due to the compact embedding $H^2(\Omega) \subset C^0(\overline{\Omega})$, we obtain $p_n \to \overline{p}$ uniformly as $n \to \infty$. This implies $\alpha \leqslant \overline{p} \leq \beta$ and $\overline{p}(x, -y) = \overline{p}(x, y)$ for all $(x, y) \in \Omega$; moreover, passing the limit under the integral, we obtain $|\Omega| = \int_{\Omega} p_n \, dx\, dy \to \int_{\Omega} \overline{p}\, dx\, dy$, implying $\int_{\Omega} \overline{p}\, dx\, dy = |\Omega|$.

Therefore the limit point $\overline{p} \in \mathcal{P}^{\alpha,\beta}_{H^2}$ and $\mathcal{P}^{\alpha,\beta}_{H^2}$ is compact for the H^2 weak topology. □

Fixed $f \in \mathcal{F}_V$, we endow the spaces

$$\begin{aligned} &(i) \qquad L^\infty(\Omega) \quad \text{with the weak* topology,} \\ &(ii) \qquad H^2(\Omega) \quad \text{with the weak topology} \end{aligned} \tag{6.3}$$

and we prove the continuity of the map $p \mapsto \mathcal{G}_p^\infty$ in the next lemma.

Lemma 6.3 *Let* (V, W) *the couple of functional spaces defined respectively in* (3.3)-*(i) or in* (3.3)-*(ii). The map* $\mathcal{G}_p^\infty : \mathcal{P}_W^{\alpha,\beta} \to [0, +\infty)$ *is continuous when* W *is endowed with the proper topology in* (6.3).

Proof Let $\{p_n\}_n \subset \mathcal{P}_W^{\alpha,\beta}$ be such that

$$\begin{aligned} &(i) \qquad \text{if } \ W = L^\infty(\Omega) \qquad p_n \overset{*}{\rightharpoonup} p \qquad \text{in} \qquad L^\infty(\Omega) \\ &(ii) \qquad \text{if } \ W = H^2(\Omega) \qquad p_n \rightharpoonup p \qquad \text{in} \qquad H^2(\Omega) \end{aligned}$$

for $n \to +\infty$; since $\mathcal{P}_W^{\alpha,\beta}$ is compact for the respective topology (6.3), then $p \in \mathcal{P}_W^{\alpha,\beta}$.

We denote by u_n the solution of (2.4) corresponding to p_n and we get

$$(u_n, v)_{H^2_*} = \langle p_n f, v \rangle \qquad \forall v \in H^2_*(\Omega); \tag{6.4}$$

the above equality with $v = u_n \in H^2_*(\Omega) \subset C^0(\overline{\Omega})$ gives respectively in the cases (3.3)-(i) and (3.3)-(ii)

$$\begin{aligned} &(i) \qquad \|u_n\|^2_{H^2_*} = \left| \int_\Omega p_n \, f u_n \, dxdy \right| \leqslant \|p_n\|_\infty \|f u_n\|_1 \leqslant \beta \|f\|_q \|u_n\|_{q'} \leq C_5 \|u_n\|_{H^2_*}, \\ &(ii) \qquad \|u_n\|^2_{H^2_*} = \left| \langle f, p_n u_n \rangle \right| \leqslant \|f\|_{H^{-2}_*} \|p_n u_n\|_{H^2_*} \leq C_6 \|u_n\|_{H^2_*}, \end{aligned} \tag{6.5}$$

in which, in the last inequality we use that $H^2_*(\Omega)$ is a Banach algebra, (2.5) and $p_n \rightharpoonup p$ in $H^2(\Omega)$. This implies $\|u_n\|_{H^2_*} \leqslant \overline{C}$ for some $\overline{C} > 0$; thus we get, up to a subsequence, $u_n \rightharpoonup \overline{u}$ in $H^2_*(\Omega)$ and we pass to the limit (6.4)

$$(\overline{u}, v)_{H^2_*} = \langle f, pv \rangle \qquad \forall v \in H^2_*(\Omega),$$

obtaining by the uniqueness that $\overline{u}$ is the weak solution of (2.4).

As in Lemma 6.1 we use the compact embedding $H^2_*(\Omega) \subset C^0(\overline{\Omega})$, implying that the gap function $\mathcal{G}_{p_n}(x)$ converges uniformly to $\mathcal{G}_p(x)$ as $n \to +\infty$ for all $x \in [0, \pi]$. □

Proof of Theorem 3.2 Completed By Lemma 6.3 we have that $p \mapsto \mathcal{G}_p^\infty$ is continuous on $\mathcal{P}_W^{\alpha,\beta}$ with respect to the proper topology associated to W in (6.3).

Moreover the set $\mathcal{P}_W^{\alpha,\beta}$ is compact for the correspondent topology, see [3, Lemma 5.2] and Lemma 6.2; this readily implies the existence of the minimum (3.5). □

6.3 *Proof of Proposition 3.3*

We follow the lines of [7, Section 9], beginning with the second statement.

(ii) Let $f \in \mathcal{F}_{H_*^{-2}}$ and $u_f \in H_*^2(\Omega)$ the solution of (2.4). Being $p(x, y)$ even with respect to y, we use the decomposition (2.7) and we rewrite (2.4) as

$$(u_f^o, v^o)_{H_*^2} + (u_f^e, v^e)_{H_*^2} = \langle pf^o, v^o\rangle + \langle pf^e, v^e\rangle \qquad \forall v \in H_*^2(\Omega). \tag{6.6}$$

By (3.1) we have $\mathcal{G}_{f,p}(x) = u^o(x, \ell) - u^o(x, -\ell)$; therefore, if $f^o = 0$ then $u^o = 0$ and $\mathcal{G}^\infty_{f,p} = 0$, implying that f cannot be a solution of (3.4). Through (2.8) we infer the existence of $\gamma \in (0, 1]$ such that $\gamma = \|f^o\|_{H_*^{-2}} \leqslant \|f\|_{H_*^{-2}} = 1$. By linearity and (6.6) we observe that the problem $(w, v)_{H_*^2} = \frac{1}{\gamma}\langle pf^o, v\rangle$ admits as solution $w = \frac{u^o}{\gamma}$ for all $v \in H_*^2(\Omega)$. Hence, by linearity, $\mathcal{G}^\infty_{\frac{f_0}{\gamma},p} = \frac{1}{\gamma}\mathcal{G}^\infty_{f,p} \geq \mathcal{G}^\infty_{f,p}$. Therefore for all $f \in \mathcal{F}_{H_*^{-2}}$ there exists $g \in H_{\mathcal{O}}^{-2}(\Omega)$ ($g = f^o/\gamma$) such that $\mathcal{G}^\infty_{g,p} \geq \mathcal{G}^\infty_{f,p}$, giving the thesis.

(i) In [7, Lemma 9.1] it is proved the following result: for $q \in [1, \infty]$, $a > 0$ and $\phi \in L^q(]-a, a[)$ it holds

$$\|\phi^o\|_{L^q(]-a,a[)} \leqslant \|\phi\|_{L^q(]-a,a[)}. \tag{6.7}$$

Hence for every $q \in (1, \infty]$, (6.7) combined with the arguments used in the proof of Proposition 3.3-(ii) yields that f odd with respect to y is a maximizer.

For $q \in (1, \infty)$ we suppose, by contradiction, that $f \in \mathcal{F}_{L^q}$ is a non-odd maximizer. We point out that the inequality (6.7) is strict for $q \in (1, \infty)$ if and only if ϕ is non-odd ($\phi \not\equiv \phi^o$), see again [7, Lemma 9.1] for a proof. Therefore, being $f \not\equiv f^o$, we get $\|f^o\|_q < \|f\|_q = 1$; we take $\overline{f} = f^o/\|f^o\|_q$, so that $\|\overline{f}\|_q = 1$. Since f^e does not play a role in the gap function, we have $\mathcal{G}^\infty_{\overline{f},p} = \frac{\mathcal{G}^\infty_{f,p}}{\|f^o\|_q} > \mathcal{G}^\infty_{f,p}$. This is absurd. □

6.4 *Proof of Proposition 5.1*

We choose $\{z_m^p, \theta_m^p\}_{m=1}^\infty$ as orthonormal basis of $L_p^2(\Omega)$ (and orthogonal basis of $H_*^2(\Omega)$). Since $f \in L^2(\Omega) \subset L_p^2(\Omega)$ we expand it in Fourier series

$$f(x, y) = \sum_{m=1}^{\infty} \left[a_m\theta_m^p(x, y) + b_m z_m^p(x, y)\right],$$

with $a_m, b_m \in \mathbb{R}$ defined as

$$a_m := \int_\Omega pf\,\theta_m^p\,dxdy \qquad b_m := \int_\Omega pf\,z_m^p\,dxdy.$$

We write

$$u(x,y) = \sum_{m=1}^{\infty} \left[\alpha_m \theta_m^p(x,y) + \beta_m z_m^p(x,y)\right] ,$$

where $\alpha_m, \beta_m \in \mathbb{R}$ are defined as

$$\alpha_m := \int_\Omega pu\,\theta_m^p\,dxdy \qquad \beta_m := \int_\Omega pu\,z_m^p\,dxdy.$$

For all $m \in \mathbb{N}^+$, z_m^p and θ_m^p solve:

$$\begin{aligned} (z_m^p, v)_{H_*^2} &= \mu_m(p)\,(p\,z_m^p, v)_{L^2} \quad \forall v \in H_*^2(\Omega) \\ (\theta_m^p, v)_{H_*^2} &= \nu_m(p)\,(p\,\theta_m^p, v)_{L^2} \quad \forall v \in H_*^2(\Omega)\,. \end{aligned} \tag{6.8}$$

Then considering (2.4) with $v = \theta_m^p$, $v = z_m^p$ and putting $v = u$ in (6.8) we have

$$\alpha_m = \frac{a_m}{\nu_m(p)} \qquad \beta_m = \frac{b_m}{\mu_m(p)}$$

and (5.1).

Now we verify that $u(x,y)$ written in Fourier series as (5.1) belongs to $H_*^2(\Omega)$. Through (6.8) we obtain that $\left\{\frac{\theta_m^p}{\sqrt{\nu_m(p)}}, \frac{z_m^p}{\sqrt{\mu_m(p)}}\right\}_{m=1}^{\infty}$ is an orthonormal basis in $H_*^2(\Omega)$; therefore, if $\left\{\frac{a_m}{\sqrt{\nu_m(p)}}, \frac{b_m}{\sqrt{\mu_m(p)}}\right\}_m \subset \ell^2(\mathbb{N}^+)$ we infer $u \in H_*^2(\Omega)$. We recall the variational representation of the eigenvalues of (2.6): for every $m \in \mathbb{N}^+$ it holds

$$\lambda_m(p) = \inf_{\substack{W_m \subset H_*^2(\Omega) \\ \dim W_m = m}} \sup_{u \in W_m \setminus \{0\}} \frac{\|u\|_{H_*^2}^2}{\|\sqrt{p}u\|_2^2},$$

implying the stability inequality

$$\frac{\lambda_m(1)}{\beta} \leqslant \lambda_m(p) \leqslant \frac{\lambda_m(1)}{\alpha}\,,$$

for every $m \in \mathbb{N}^+$. In [12, Theorem 7.6] the authors find explicit bounds for the eigenvalues when the plate is homogeneous ($p \equiv 1$); in general it holds $\lambda_m(1) > (1-\sigma)^2 m^4$, where σ is the Poisson ratio, see (2.2). Then we obtain

$$\lambda_m(p) \geqslant \frac{\lambda_m(1)}{\beta} > \frac{(1-\sigma)^2 m^4}{\beta}$$

so that, being $\|f\|_2 = \|\sqrt{p}\theta_m^p\|_2 = 1$,

$$\frac{|a_m|}{\sqrt{\nu_m(p)}} \le \frac{\sqrt{\beta}\|f\|_2\|p\theta_m^p\|_2}{(1-\sigma)m^2} \le \frac{\beta\|\sqrt{p}\theta_m^p\|_2}{(1-\sigma)m^2} = \frac{\beta}{(1-\sigma)m^2} \qquad \frac{|b_m|}{\sqrt{\mu_m(p)}} \le \frac{\beta}{(1-\sigma)m^2}$$

and

$$\sum_{m=1}^{\infty} \frac{|a_m|^2}{\nu_m(p)} + \frac{|b_m|^2}{\mu_m(p)} \leqslant \frac{2\beta^2}{(1-\sigma)^2} \sum_{m=1}^{\infty} \frac{1}{m^4} < \infty.$$

Through (5.1) we get

$$\mathcal{G}_{f,p}(x) = 2\sum_{m=1}^{\infty} \frac{a_m}{\nu_m(p)} \theta_m^p(x,\ell) \qquad \forall x \in [0,\pi],$$

since $z_m^p(x, y)$ is y-even.

If f is y-odd then $b_m = 0$. □

7 Conclusions

In this paper we consider a stationary forced problem for a non-homogeneous partially hinged rectangular plate, possibly modeling the deck of a bridge, on which a non constant density function $p(x, y)$, embodying the non-homogeneity, is given. The main aim is to optimize the torsional performance of the plate, measured through the so called gap function $\mathcal{G}_{f,p}(x)$, see (3.1), with respect to both the weight p and the external forcing term f; thus, we deal with the problem (3.2) where f and p belong to suitable classes of functions.

In Theorem 3.1 we prove the existence of an optimal force f solution of (3.4) fixed the weight p in proper functional spaces, while in the Theorem 3.2 we prove the converse, i.e. the existence of an optimal density p solution of (3.5) fixed f. Currently to find explicitly the solutions of (3.4) and (3.5) seems out of reach, therefore we propose some choices of f and p and we proceed numerically. In Proposition 3.3 we prove symmetry properties on the solutions of (3.4); motivated by this result, we focus on y-odd forces f as optimal candidates of (3.4). On the other hand about the possible optimal weight functions we study five meaningful density configurations; the latter are inspired by [3], where a similar problem in terms of weight optimization of the ratio between a torsional and a longitudinal eigenvalue is given, see (4.1).

We propose some numerical experiments when $f \in L^2(\Omega)$, because it is representative of the applications we have in mind; in this case, we state and prove Proposition 5.1 allowing to find a numerical scheme useful to determine the approximated solutions. Our analysis is performed imposing as parameters (5.3)–(5.4), having sense in terms of civil engineering applications. We summarize our main outcomes:

- The forces $f \in \mathcal{F}_{L^2}$ which maintain the same (and opposite) sign along the two free edges of the plate (e.g. $\overline{f}_0$, $\overline{f}_1$) seem to be the worst in terms of torsional performance of the plate for each density function.
- If we consider $f \propto \theta_j^p(x, y)$, i.e. proportional to the j-th weighted torsional eigenfunction, we get the corresponding maximum of the gap function decreasing with respect to j for every density function; this means that the worst case is recorded for $j = 1$, i.e. when f is in resonance with the first weighted torsional eigenfunction, see Fig. 2.
- To improve the torsional performance of the plate, we suggest to strengthen it with a density function like $p^*(x, y)$ or $\check{p}(y)$, see Sect. 4. These weights have a strong effect in increasing the first torsional eigenvalues and they reduce the maximum of the gap function more than the others.
- Weights as $\overline{p}_i(x)$ ($i \in \mathbb{N}^+$), useful to reduce the i-th longitudinal eigenvalue, generally do not affect the torsional response of the plate. We recorded the same behaviour as in the homogeneous plate, hence we do not suggest this kind of reinforce.

A future development in this field is the study of the corresponding evolutionary problem. We point out that the presence of a possibly discontinuous coefficient $p(x, y)$ in front of the time-derivative term may lead to some problems, even just in writing the equation in strong form.

Other researches may focus on other forces and density functions; is there a density function that maximizes the second torsional eigenvalue better than those in $\widehat{P}_{\alpha,\beta}$? How does the gap function vary in correspondence of such weight? In [3] it is pointed out that $p^*(x, y)$ may be the candidate maximizer of the first torsional eigenvalue, but nothing is said about the maximizer of the second torsional eigenvalue. It may be interesting to study this issue, since the deck of a suspension bridge seems to be more prone to develop torsional instability on the second torsional eigenvalue, see for instance [10, 13].

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References

1. Adams, R., Fournier, J.: Sobolev Spaces. Academic, London (2003)
2. Antunes, P., Gazzola, F.: Some solutions of minimaxmax problems for the torsional displacements of rectangular plates. Z. Angew. Math. Mech. **98**, 1974–1991 (2018)
3. Berchio, E., Falocchi, A.: Maximizing the ratio of eigenvalues of non-homogeneous partially hinged plates. J. Spectral Theory (to appear). arxiv: 1907.11097
4. Berchio, E., Ferrero, A., Gazzola, F.: Structural instability of nonlinear plates modelling suspension bridges: mathematical answers to some long-standing questions. Nonlinear Anal. Real World Appl. **28**, 91–125 (2016)
5. Berchio, E., Buoso, D., Gazzola, F.: A measure of the torsional performances of partially hinged rectangular plates. In: Constanda, C., Dalla Riva, M., Lamberti, P.D., Musolino, P. (eds.) Integral Methods in Science and Engineering, vol. 1, Theoretical Techniques, pp. 35–46. Birkhäuser, Basel (2017)
6. Berchio, E., Buoso, D., Gazzola, F.: On the variation of longitudinal and torsional frequencies in a partially hinged rectangular plate. ESAIM Control Optim. Calc. Var. **24**, 63–87 (2018)
7. Berchio, E., Buoso, D., Gazzola, F., Zucco, D.: A minimaxmax problem for improving the torsional stability of rectangular plates. J. Optim. Theory Appl. **177**, 64–92 (2018)
8. Berchio, E., Falocchi, A., Ferrero, A., Ganguly, D.: On the first frequency of reinforced partially hinged plates. Commun. Contemp. Math. **23**(3), 1950074, 37 pp. (2021)
9. Bonheure, D., Gazzola, F., Moreira dos Santos, E.: Periodic solutions and torsional instability in a nonlinear nonlocal plate equation. SIAM J. Math. Anal. **51**(4), 3052–3091 (2019)
10. Crasta, G., Falocchi, A., Gazzola, F.: A new model for suspension bridges involving the convexification of the cables. Z. Angew. Math. Phys. **71**, 93 (2020)
11. Falocchi, A.: Torsional instability in a nonlinear isolated model for suspension bridges with fixed cables and extensible hangers. IMA J. Appl. Math. **83**, 1007–1036 (2018)
12. Ferrero, A., Gazzola, F.: A partially hinged rectangular plate as a model for suspension bridges. Discr. Contin. Dyn. Syst. A. **35**, 5879–5908 (2015)
13. Gazzola, F.: Mathematical Models for Suspension Bridges, MS&A, vol. 15. Springer, Berlin (2015)
14. Kirchhoff, G.R.: Über das gleichgewicht und die bewegung einer elastischen scheibe. J. Reine Angew. Math. **40**, 51–88 (1850)
15. Larsen, A.: Aerodynamics of the Tacoma Narrows Bridge - 60 years later. Struct. Eng. Int. **4**, 243–248 (2000)
16. Love, A.E.H.: A Treatise on the Mathematical Theory of Elasticity, 4th edn. Cambridge Univ. Press, Cambridge (1927)
17. Navier, C.L.: Extraits des recherches sur la flexion des plans élastiques. Bulletin des Sciences de la Société Philomathique de Paris, pp. 92–102 (1823)
18. Rocard, Y.: Dynamic Instability: Automobiles, Aircraft, Suspension Bridges. Crosby Loockwood, London (1957)

Energy-Like Functional in a Quasilinear Parabolic Chemotaxis System

Kentaro Fujie

Abstract This note deals with a one-dimensional quasilinear chemotaxis system. The first part summarizes recent results, in which a new energy-like functional is introduced and plays a key role. In the latter half, the energy-like functional will be derived in a more general situation.

Keywords Chemotaxis · Global existence · Lyapunov functional

1 Summary of Recent Results

Consider the following one-dimensional quasilinear chemotaxis system

$$\begin{cases} \partial_t u = \partial_x \left(a(u)\partial_x u - u\partial_x v\right) & \text{in } (0,T)\times(0,1), \\ \partial_t v = \partial_x^2 v - v + u & \text{in } (0,T)\times(0,1), \\ \partial_x u = \partial_x v = 0 & \text{on } (0,T)\times\{0,1\}, \\ u(0,\cdot) = u_0, \quad v(0,\cdot) = v_0 & \text{in } (0,1), \end{cases} \tag{1.1}$$

where the nonlinearity $a \in C[0,\infty) \cap C^2(0,\infty)$ is positive and $(u_0, v_0) \in (W^{1,\infty}(0,1))^2$ is the pair of nonnegative initial data. Local existence and uniqueness of classical solutions are known, see [3, 8].

The typical choice of the nonlinearity is $a(u) = (1+u)^p$ with $p \in \mathbb{R}$. In the higher dimensional setting $n \geq 2$, the power $p = 1 - \frac{2}{n}$ is critical, that is, global existence for any initial data for $p > 1 - \frac{2}{n}$ (see [13]) and finite-time blowups when $p < 1 - \frac{2}{n}$ (see [9]) are known. In the critical case $p = 1 - \frac{2}{n}$, solutions exist

K. Fujie (✉)
Research Alliance Center for Mathematical Sciences, Tohuku University, Sendai, Japan
e-mail: fujie@tohoku.ac.jp

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globally for small data; whereas finite time blow-up solutions are constructed for large initial data, see [12] in dimension 2 and [11] in dimensions 3 and 4.

In view of the above, the particular choice $a(u) = 1/(1+u)$ is a natural candidate for a critical one in the one-dimensional setting. For the subcritical case $a(u) = (1+u)^{-p}$, $p < 1$, solutions emanating from any smooth data exist globally in time and remain bounded, see [3], while in the supercritical case ($p > 1$) finite time blow-up solutions are constructed under some additional restrictions in [8]. In the critical case $a(u) = 1/(1+u)$ global existence result for small data is known [3].

Recently, in [2, 5] the following result was proven.

Theorem 1.1 *Assume that the function* $a \in C[0,\infty) \cap C^2(0,\infty)$ *satisfies the following:*

(A1) there is $\alpha > 0$ *such that* $sa(s) \le \alpha$ *for any* $s \ge 0$*,*
(A2) $\int_1^\infty a(s)\,ds = \infty$*, i.e.* $a \notin L^1(1,\infty)$.

Then the problem (1.1) *has a unique classical positive solution, which exists globally in time. Moreover, the solution* (u, v) *of* (1.1) *is bounded.*

Remark 1.2 The function $a(s) = 1/(1+s)$ satisfies assumptions (A1) and (A2). Therefore no mass critical phenomenon occurs in the natural candidate for the critical case.

Let us recall the classical Lyapunov functional associated with (1.1) is

$$L(u,v) := \int_0^1 b(u)\,dx - \int_0^1 uv\,dx + \frac{1}{2}\|v\|_{H^1(0,1)}^2,$$

where $b \in C^2(0,\infty)$ satisfies $b''(r) = \frac{a(r)}{r}$ for $r > 0$ and $b(1) = b'(1) = 0$. It satisfies, see [8],

$$\frac{d}{dt}L(u,v) = -\int_0^1 |\partial_t v|^2\,dx - \int_0^1 u\left|\frac{a(u)}{u}\partial_x u - \partial_x v\right|^2 dx.$$

Since the solution (u, v) has the mass conservation law:

$$\int_0^1 u(t)\,dx = \int_0^1 u_0\,dx,$$

the embedding theorem allows us to have

$$\int_0^1 uv\,dx \le \|v\|_{L^\infty(0,1)}\int_0^1 u\,dx \le \frac{1}{4}\|v\|_{H^1(0,1)}^2 + C\|u_0\|_{L^1(0,1)}^2,$$

which yields that L is bounded from below:

$$|L(u(t), v(t))| \leq C \quad \text{for all } t > 0 \tag{1.2}$$

with some C.

In the study of the higher dimensional system, the Lyapunov functional plays a crucial role to decide behaviour of solutions, however the a priori estimate (1.2) seems poor in the case $a(u) = 1/(1+u)$. Actually, when $a(u) = 1/(1+u)$ the a priori estimate $\int_0^1 b(u) \simeq \int_0^1 u$ is not enough to guarantee global existence.

In [5, Lemma 3.3] the authors found the new Lyapunov-like functional which yields an estimate sufficient to prove global existence result.

Lemma 1.3 *Let* (u, v) *be a solution of* (1.1) *in* $(0, T) \times (0, 1)$ *and let* T *be the maximal existence time of the classical solution. Then the following identity holds:*

$$\frac{d}{dt}\mathcal{F}(u) + \mathcal{D}(u, v) = \int_0^1 \frac{ua(u)(v + \partial_t v)^2}{4}\, dx, \tag{1.3}$$

where

$$\mathcal{F}(u) := \frac{1}{2}\int_0^1 \frac{(a(u))^2}{u}|\partial_x u|^2\, dx - \int_0^1 u \int_1^u a(r)\, dr\, dx,$$

$$\mathcal{D}(u, v) := \int_0^1 ua(u)\left|\partial_x\left(\frac{a(u)}{u}\partial_x u\right) - \partial_x^2 v + \frac{(v + \partial_t v)}{2}\right|^2 dx.$$

Since the right-hand side of the above identity is not zero, the functional $\mathcal{F}(u)$ could increase. In [2, 5] it is proved that the growth rate can be controlled and the a priori estimate from the identity is sufficient to derive global existence and boundedness of solutions.

Remark 1.4 The natural question is what the functional $\mathcal{F}(u)$ represents. In [6] the following interpretation is given; if one consider the upper equation of (1.1) as a continuity equation of density u with a velocity field given by $V := \frac{a(u)}{u}\partial_x u - \partial_x v$, then the kinetic energy is

$$\int_0^1 uV^2\, dx = \int_0^1 u\left|\frac{a(u)}{u}\partial_x u - \partial_x v\right|^2 dx.$$

We remark that the principle term of the above is included in the functional $\mathcal{F}(u)$.

2 Nonlinear Sensitivity

Consider the following one-dimensional quasilinear chemotaxis system with nonlinear sensitivity:

$$\begin{cases} \partial_t u = \partial_x \left(D(u)\partial_x u - S(u)\partial_x v\right) & \text{in } (0,T)\times(0,1), \\ \partial_t v = \partial_x^2 v - v + u & \text{in } (0,T)\times(0,1), \\ \partial_x u = \partial_x v = 0 & \text{on } (0,T)\times\{0,1\}, \\ u(0,\cdot) = u_0, \quad v(0,\cdot) = v_0 & \text{in } (0,1), \end{cases} \tag{2.1}$$

where

$$D(u) = (1+u)^{-p}, \quad S(u) = u(1+u)^{-q} \qquad \text{with } p, q \in \mathbb{R}.$$

Furthermore we assume $(u_0, v_0) \in (W^{1,\infty}(0,1))^2$ is the pair of nonnegative initial data. Local existence and uniqueness of classical solutions are known, see [4, 10]. In this section we introduce a new energy-like identity in the system (2.1) according to the same spirit of the previous section.

Let us first recall the classical Lyapunov functional associated with (2.1), see [15]; the following identity holds:

$$\frac{d}{dt}\mathcal{L}(u,v) + \int_0^1 |\partial_t v|^2 + \int_0^1 S(u)\cdot\left|\frac{D(u)}{S(u)}\partial_x u - \partial_x v\right|^2 = 0,$$

where

$$\mathcal{L}(u,v) := \int_0^1 G(u) - \int_0^1 uv + \frac{1}{2}\|v\|^2_{H^1(0,1)},$$
$$G(s) := \int_1^s \int_1^\sigma \frac{D(\tau)}{S(\tau)}\,d\tau\,d\sigma.$$

The next lemma claims a crucial identity, which is the generalization of [7, Lemma 2.1].

Lemma 2.1 *Let $\phi \in C^3((0,1))$. Then the following identity holds:*

$$S(\phi)\partial_x\mathcal{M}(\phi) = \partial_x\left(S(\phi)D(\phi)\partial_x\left(\frac{D(\phi)}{S(\phi)}\partial_x\phi\right)\right) + \frac{(D(\phi))^2 S''(\phi)}{2S(\phi)}|\partial_x\phi|^2\partial_x\phi,$$

where

$$\mathcal{M}(\phi) := \frac{D(\phi)D'(\phi)}{S(\phi)}|\partial_x\phi|^2 - \frac{(D(\phi))^2 S'(\phi)}{2(S(\phi))^2}|\partial_x\phi|^2 + \frac{(D(\phi))^2}{S(\phi)}\partial_x^2\phi.$$

Proof The left-hand side of the identity is calculated as

$$\begin{aligned}
&S(\phi)\partial_x\mathcal{M}(\phi)\\
&= S(\phi)\Bigg(\frac{(D'(\phi))^2}{S(\phi)}|\partial_x\phi|^2\partial_x\phi + \frac{D(\phi)D''(\phi)}{S(\phi)}|\partial_x\phi|^2\partial_x\phi\\
&\qquad\qquad - \frac{D(\phi)D'(\phi)S'(\phi)}{(S(\phi))^2}|\partial_x\phi|^2\partial_x\phi + \frac{D(\phi)D'(\phi)}{S(\phi)}\partial_x(|\partial_x\phi|^2)\Bigg)\\
&\quad + S(\phi)\Bigg(-\frac{D(\phi)D'(\phi)S'(\phi)}{(S(\phi))^2}|\partial_x\phi|^2\partial_x\phi - \frac{(D(\phi))^2 S''(\phi)}{2(S(\phi))^2}|\partial_x\phi|^2\partial_x\phi\\
&\qquad\qquad + \frac{(D(\phi))^2(S'(\phi))^2}{(S(\phi))^3}|\partial_x\phi|^2\partial_x\phi - \frac{(D(\phi))^2 S'(\phi)}{2(S(\phi))^2}\partial_x(|\partial_x\phi|^2)\Bigg)\\
&\quad + S(\phi)\left(\frac{2D(\phi)D'(\phi)}{S(\phi)}\partial_x^2\phi\partial_x\phi - \frac{(D(\phi))^2 S'(\phi)}{(S(\phi))^2}\partial_x^2\phi\partial_x\phi + \frac{(D(\phi))^2}{S(\phi)}\partial_x^3\phi\right)\\
&= (D'(\phi))^2|\partial_x\phi|^2\partial_x\phi + D(\phi)D''(\phi)|\partial_x\phi|^2\partial_x\phi - \frac{2D(\phi)D'(\phi)S'(\phi)}{S(\phi)}|\partial_x\phi|^2\partial_x\phi\\
&\quad + D(\phi)D'(\phi)\partial_x(|\partial_x\phi|^2) - \frac{(D(\phi))^2 S''(\phi)}{2S(\phi)}|\partial_x\phi|^2\partial_x\phi\\
&\quad + \frac{(D(\phi))^2(S'(\phi))^2}{(S(\phi))^2}|\partial_x\phi|^2\partial_x\phi - \frac{(D(\phi))^2 S'(\phi)}{2S(\phi)}\partial_x(|\partial_x\phi|^2)\\
&\quad + 2D(\phi)D'(\phi)\partial_x^2\phi\partial_x\phi - \frac{(D(\phi))^2 S'(\phi)}{S(\phi)}\partial_x^2\phi\partial_x\phi + (D(\phi))^2\partial_x^3\phi.
\end{aligned}$$

Since the direct calculations yield

$$\begin{aligned}
\partial_x\left(D(\phi)D'(\phi)|\partial_x\phi|^2\right) &= (D'(\phi))^2|\partial_x\phi|^2\partial_x\phi + D(\phi)D''(\phi)|\partial_x\phi|^2\partial_x\phi\\
&\quad + D(\phi)D'(\phi)\partial_x(|\partial_x\phi|^2),\\
\partial_x\left((D(\phi))^2\partial_x^2\phi\right) &= 2D(\phi)D'(\phi)\partial_x^2\phi\partial_x\phi + (D(\phi))^2\partial_x^3\phi,
\end{aligned}$$

we compute that

$$S(\phi)\partial_x\mathcal{M}(\phi) = \partial_x\left(D(\phi)D'(\phi)|\partial_x\phi|^2\right) + \partial_x\left((D(\phi))^2\partial_x^2\phi\right) - \frac{2D(\phi)D'(\phi)S'(\phi)}{S(\phi)}|\partial_x\phi|^2\partial_x\phi$$
$$- \frac{(D(\phi))^2S''(\phi)}{2S(\phi)}|\partial_x\phi|^2\partial_x\phi + \frac{(D(\phi))^2(S'(\phi))^2}{(S(\phi))^2}|\partial_x\phi|^2\partial_x\phi$$
$$- \frac{(D(\phi))^2S'(\phi)}{2S(\phi)}\partial_x(|\partial_x\phi|^2) - \frac{(D(\phi))^2S'(\phi)}{S(\phi)}\partial_x^2\phi\partial_x\phi.$$

Due to the identity

$$\partial_x\phi\partial_x^2\phi = \frac{1}{2}\partial_x\left(|\partial_x\phi|^2\right),$$

we arrive at

$$\begin{aligned}S(\phi)\partial_x\mathcal{M}(\phi) &= \partial_x\left(D(\phi)D'(\phi)|\partial_x\phi|^2\right) + \partial_x\left((D(\phi))^2\partial_x^2\phi\right)\\
&\quad - \frac{2D(\phi)D'(\phi)S'(\phi)}{S(\phi)}|\partial_x\phi|^2\partial_x\phi - \frac{(D(\phi))^2S''(\phi)}{2S(\phi)}|\partial_x\phi|^2\partial_x\phi\\
&\quad + \frac{(D(\phi))^2(S'(\phi))^2}{(S(\phi))^2}|\partial_x\phi|^2\partial_x\phi - \frac{(D(\phi))^2S'(\phi)}{S(\phi)}\partial_x(|\partial_x\phi|^2)\\
&= \partial_x\left(D(\phi)D'(\phi)|\partial_x\phi|^2 + (D(\phi))^2\partial_x^2\phi - \frac{(D(\phi))^2S'(\phi)}{S(\phi)}|\partial_x\phi|^2\right)\\
&\quad + \frac{(D(\phi))^2S''(\phi)}{2S(\phi)}|\partial_x\phi|^2\partial_x\phi\\
&= \partial_x\left(S(\phi)D(\phi)\partial_x\left(\frac{D(\phi)}{S(\phi)}\partial_x\phi\right)\right) + \frac{(D(\phi))^2S''(\phi)}{2S(\phi)}|\partial_x\phi|^2\partial_x\phi.\end{aligned}$$

The proof is completed. □

In light of the above identity we have the following lemma, which is the generalization of [5, Lemma 3.1].

Lemma 2.2 *Let* (u, v) *be a solution of* (2.1) *in* $(0, T) \times (0, 1)$. *Then the following identity holds:*

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 \frac{(D(u))^2}{S(u)}|\partial_x u|^2\right) + \int_0^1 S(u)D(u)\left|\partial_x\left(\frac{D(u)}{S(u)}\partial_x u\right)\right|^2$$
$$= \int_0^1 S(u)D(u)\partial_x^2 v \cdot \partial_x\left(\frac{D(u)}{S(u)}\partial_x u\right) + \int_0^1\left(\frac{D(u)}{S(u)}\partial_x u - \partial_x v\right)\cdot\frac{(D(u))^2S''(u)}{2S(u)}|\partial_x u|^2\partial_x u.$$

Proof Multiplying the first equation of (2.1) by $\mathcal{M}(u)$ and integrating over (0, 1), we have that

$$\int_0^1 \partial_t u \mathcal{M}(u) = \int_0^1 \partial_x \left(D(u)\partial_x u - S(u)\partial_x v\right) \mathcal{M}(u) = \int_0^1 \partial_x \left(S(u)\left(\frac{D(u)}{S(u)}\partial_x u - \partial_x v\right)\right) \mathcal{M}(u).$$

By the integration by parts and Lemma 2.1, it follows that

$$\begin{aligned}
\int_0^1 \partial_t u \mathcal{M}(u) &= -\int_0^1 \left(\frac{D(u)}{S(u)}\partial_x u - \partial_x v\right) \cdot S(u)\partial_x \mathcal{M}(u) \\
&= -\int_0^1 \left(\frac{D(u)}{S(u)}\partial_x u - \partial_x v\right) \cdot \partial_x \left(S(u)D(u)\partial_x \left(\frac{D(u)}{S(u)}\partial_x u\right)\right) \\
&\quad - \int_0^1 \left(\frac{D(u)}{S(u)}\partial_x u - \partial_x v\right) \cdot \frac{(D(u))^2 S''(u)}{2S(u)} |\partial_x u|^2 \partial_x u \\
&= \int_0^1 \partial_x \left(\frac{D(u)}{S(u)}\partial_x u - \partial_x v\right) \cdot \left(S(u)D(u)\partial_x \left(\frac{D(u)}{S(u)}\partial_x u\right)\right) \\
&\quad - \int_0^1 \left(\frac{D(u)}{S(u)}\partial_x u - \partial_x v\right) \cdot \frac{(D(u))^2 S''(u)}{2S(u)} |\partial_x u|^2 \partial_x u \\
&= \int_0^1 S(u)D(u) \left|\partial_x \left(\frac{D(u)}{S(u)}\partial_x u\right)\right|^2 - \int_0^1 S(u)D(u)\partial_x^2 v \cdot \partial_x \left(\frac{D(u)}{S(u)}\partial_x u\right) \\
&\quad - \int_0^1 \left(\frac{D(u)}{S(u)}\partial_x u - \partial_x v\right) \cdot \frac{(D(u))^2 S''(u)}{2S(u)} |\partial_x u|^2 \partial_x u. \qquad (2.2)
\end{aligned}$$

On the other hand, we infer that

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_0^1 \frac{(D(u))^2}{S(u)}|\partial_x u|^2\right) \\
&= \int_0^1 \frac{2D(u)D'(u)S(u) - (D(u))^2 S'(u)}{2(S(u))^2} |\partial_x u|^2 \partial_t u + \int_0^1 \frac{(D(u))^2}{S(u)} \partial_x u \partial_x \partial_t u \\
&= \int_0^1 \frac{2D(u)D'(u)S(u) - (D(u))^2 S'(u)}{2(S(u))^2} |\partial_x u|^2 \partial_t u - \int_0^1 \partial_x \left(\frac{(D(u))^2}{S(u)}\partial_x u\right) \partial_t u.
\end{aligned}$$

Since

$$\begin{aligned}
&-\int_0^1 \partial_x \left(\frac{(D(u))^2}{S(u)}\partial_x u\right) \partial_t u \\
&= -\int_0^1 \frac{2D(u)D'(u)S(u) - (D(u))^2 S'(u)}{(S(u))^2} |\partial_x u|^2 \partial_t u - \int_0^1 \frac{(D(u))^2}{S(u)} \partial_x^2 u \partial_t u,
\end{aligned}$$

we deduce that

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 \frac{(D(u))^2}{S(u)}|\partial_x u|^2\right) = -\int_0^1 \partial_t u \mathcal{M}(u). \tag{2.3}$$

Combining (2.2) and (2.3), we complete the proof. □

Here we define the function Ψ as follows:

$$\Psi(\phi) := \int_1^\phi \left(\int_1^r \frac{\tau D(\tau)S'(\tau)}{S(\tau)}\,d\tau + rD(r)\right) dr.$$

Lemma 2.3 *Let* (u, v) *be a solution of* (2.1) *in* $(0, T) \times (0, 1)$. *Then the following identity holds:*

$$\frac{d}{dt}\int_0^1 \Psi(u) = \int_0^1 \partial_x \left(\frac{D(u)}{S(u)}\partial_x u\right) uS(u)D(u) + \int_0^1 \partial_x\,(uS(u)D(u)) \cdot \partial_x v.$$

Proof Testing the first equation of (2.1) by

$$\int_1^u \frac{rD(r)S'(r)}{S(r)}\,dr + uD(u),$$

and integrating over $(0, 1)$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \Psi(u) \\
&= \int_0^1 \partial_t u \left(\int_1^u \frac{rD(r)S'(r)}{S(r)}\,dr\right) + \int_0^1 uD(u)\partial_t u \\
&= -\int_0^1 (D(u)\partial_x u - S(u)\partial_x v) \cdot \frac{uD(u)S'(u)}{S(u)}\partial_x u - \int_0^1 \partial_x(uD(u))(D(u)\partial_x u - S(u)\partial_x v).
\end{aligned}$$

Since it follows from a straightforward computation that

$$-\int_0^1 D(u)\partial_x u \cdot \frac{uD(u)S'(u)}{S(u)}\partial_x u - \int_0^1 \partial_x(uD(u))D(u)\partial_x u = \int_0^1 \partial_x \left(\frac{D(u)}{S(u)}\partial_x u\right) uS(u)D(u),$$

$$\int_0^1 S(u)\partial_x v \cdot \frac{uD(u)S'(u)}{S(u)}\partial_x u + \int_0^1 \partial_x(uD(u))S(u)\partial_x v = \int_0^1 \partial_x\,(uS(u)D(u)) \cdot \partial_x v,$$

we conclude the proof. □

Now we are in the position to construct the announced functional.

Proposition 2.4 *Let* (u, v) *be a solution of* (2.1) *in* $(0, T) \times (0, 1)$. *The following identity is satisfied:*

$$\frac{d}{dt}\mathcal{F}(u(t)) + \mathcal{D}(u(t), v(t)) = \int_0^1 \frac{S(u)D(u)(v + \partial_t v)^2}{4} + \int_0^1 \left(\frac{D(u)}{S(u)}\partial_x u - \partial_x v\right) \cdot \frac{(D(u))^2 S''(u)}{2S(u)} |\partial_x u|^2 \partial_x u, \tag{2.4}$$

where

$$\mathcal{F}(u(t)) := \frac{1}{2}\int_0^1 \frac{(D(u))^2}{S(u)}|\partial_x u|^2 - \int_0^1 \Psi(u),$$

$$\mathcal{D}(u(t), v(t)) := \int_0^1 S(u)D(u)\left|\partial_x\left(\frac{D(u)}{S(u)}\partial_x u\right) - \partial_x^2 v + \frac{(v + \partial_t v)}{2}\right|^2.$$

Proof Multiplying the second equation of (2.1) by $S(u)D(u)\partial_x^2 v$ and integrating over (0, 1) we have that

$$\begin{aligned}\int_0^1 S(u)D(u)\partial_t v \partial_x^2 v &= \int_0^1 S(u)D(u)|\partial_x^2 v|^2 - \int_0^1 v \cdot S(u)D(u)\partial_x^2 v + \int_0^1 uS(u)D(u)\partial_x^2 v \\ &= \int_0^1 S(u)D(u)|\partial_x^2 v|^2 - \int_0^1 v \cdot S(u)D(u)\partial_x^2 v - \int_0^1 \partial_x (uS(u)D(u)) \cdot \partial_x v.\end{aligned} \tag{2.5}$$

Combining Lemma 2.3 and (2.5) we get

$$\frac{d}{dt}\left(-\int_0^1 \Psi(u)\right) + \int_0^1 S(u)D(u)|\partial_x^2 v|^2 - \int_0^1 v \cdot S(u)D(u)\partial_x^2 v - \int_0^1 S(u)D(u)\partial_t v \partial_x^2 v = -\int_0^1 \partial_x\left(\frac{D(u)}{S(u)}\partial_x u\right) uS(u)D(u),$$

and then using the second equation of (2.1) we see that

$$\frac{d}{dt}\left(-\int_0^1 \Psi(u)\right) + \int_0^1 S(u)D(u)|\partial_x^2 v|^2 - \int_0^1 v \cdot S(u)D(u)\partial_x^2 v - \int_0^1 S(u)D(u)\partial_t v \partial_x^2 v = -\int_0^1 \partial_x\left(\frac{D(u)}{S(u)}\partial_x u\right)(\partial_t v - \partial_x^2 v + v)S(u)D(u).$$

Thus it follows from Lemma 2.2 that

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 \frac{(D(u))^2}{S(u)}|\partial_x u|^2 - \int_0^1 \Psi(u)\right) + \int_0^1 S(u)D(u)\left|\partial_x\left(\frac{D(u)}{S(u)}\partial_x u\right) - \partial_x^2 v\right|^2$$
$$+ \int_0^1 S(u)D(u)(v+\partial_t v)\partial_x\left(\frac{D(u)}{S(u)}\partial_x u - \partial_x v\right)$$
$$= \int_0^1 \left(\frac{D(u)}{S(u)}\partial_x u - \partial_x v\right)\cdot\frac{(D(u))^2 S''(u)}{2S(u)}|\partial_x u|^2\partial_x u,$$

that is,

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 \frac{(D(u))^2}{S(u)}|\partial_x u|^2 - \int_0^1 \Psi(u)\right) + \int_0^1 S(u)D(u)\left|\partial_x\left(\frac{D(u)}{S(u)}\partial_x u\right) - \partial_x^2 v + \frac{(v+\partial_t v)}{2}\right|^2$$
$$= \int_0^1 \frac{S(u)D(u)(v+\partial_t v)^2}{4} + \int_0^1 \left(\frac{D(u)}{S(u)}\partial_x u - \partial_x v\right)\cdot\frac{(D(u))^2 S''(u)}{2S(u)}|\partial_x u|^2\partial_x u,$$

which is the desired inequality. □

By the direct calculations we can confirm that (1.3) is equal to (2.4) with $q = 0$. Moreover, the functional in (2.4) also represents the kinetic energy of the system as we noted in Remark 1.4. Unfortunately, for the case of general sensitivity $S(u)$, we have one involved term in the right hand side of (2.4). This term disturbs us to apply similar estimates in [2, 5].

Remark 2.5 In the earlier literature (see [14, 15], also the survey [1, Section 3.6]), $p - q = 1$ is the natural candidate for the critical condition.

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References

1. Bellomo, N., Bellouquid, A., Tao, Y., Winkler, M.: Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues. Math. Models Methods Appl. Sci. **25**, 1663–1763 (2015)
2. Bieganowski, B., Cieślak, T., Fujie, K., Senba, T.: Boundedness of solutions to the critical fully parabolic quasilinear one-dimensional Keller-Segel system. Math. Nachr. **292**, 724–732 (2019)
3. Burczak, J., Cieślak, T., Morales-Rodrigo, C.: Global existence vs. blowup in a fully parabolic quasilinear 1D Keller-Segel system. Nonlinear Anal. **75**, 5215–5228 (2012)
4. Cieślak, T.: Quasilinear nonuniformly parabolic system modelling chemotaxis. J. Math. Anal. Appl. **326**, 1410–1426 (2007)
5. Cieślak, T., Fujie, K.: No critical nonlinear diffusion in 1D quasilinear fully parabolic chemotaxis system. Proc. Am. Math. Soc. **146**, 2529–2540 (2018)
6. Cieślak, T., Fujie, K.: Some remarks on well-posedness of the higher-dimensional chemorepulsion system. Bull. Pol. Acad. Sci. Math. **67**, 165–178 (2019)

7. Cieślak, T., Fujie, K.: Global existence in the 1D quasilinear parabolic-elliptic chemotaxis system with critical nonlinearity. Discrete Contin. Dyn. Syst. Ser. S **13**, 165–176 (2020)
8. Cieślak, T., Laurençot, Ph.: Finite time blow-up for a one-dimensional quasilinear parabolic-parabolic chemotaxis system. Ann. Inst. H. Poincaré Anal. Non Linéaire **27**, 437–446 (2010)
9. Cieślak, T., Stinner, C.: Finite-time blowup and global-in-time unbounded solutions to a parabolic-parabolic quasilinear Keller-Segel system in higher dimensions. J. Differ. Equ. **252**, 5832–5851 (2012)
10. Horstmann, D., Winkler, M.: Boundedness vs. blow-up in a chemotaxis system. J. Differ. Equ. **215**, 52–107 (2005)
11. Laurençot, Ph., Mizoguchi, N.: Finite time blowup for the parabolic-parabolic Keller-Segel system with critical diffusion. Ann. Inst. H. Poincaré Anal. Non Linéaire **34**, 197–220 (2017)
12. Mizoguchi, N., Winkler, M.: Finite-time blow-up in the two-dimensional Keller-Segel system. Preprint
13. Senba, T., Suzuki, T.: A quasi-linear parabolic system of chemotaxis. Abstr. Appl. Anal. **2006**, Art. ID 23061, 21 pp. (2006)
14. Tao, Y., Winkler, M.: Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity. J. Differ. Equ. **252**, 692–715 (2012)
15. Winkler, M.: Does a 'volume-filling effect' always prevent chemotactic collapse? Math. Methods Appl. Sci. **33**, 12–24 (2010)

Solvability of a Semilinear Heat Equation via a Quasi Scale Invariance

Yohei Fujishima and Norisuke Ioku

Abstract Solvability of semilinear heat equations with general nonlinearity is investigated. Applying a quasi scale invariant transformation, we clarify the threshold singularity of initial data for existence and nonexistence results.

Keywords Nonlinear heat equation · Local in time solvability · Singular initial data · Quasi scale invariance

1 Introduction

In this paper we study the existence and nonexistence of solutions for a semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + f(u) & \text{in } \mathbb{R}^N \times (0, T), \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

where $\partial_t = \partial/\partial t$, $N \geq 1$, $T > 0$, u_0 is a nonnegative measurable initial function and $f \in C^1([0, \infty))$ is a positive monotonically increasing function in $(0, \infty)$, that is,

$$f(s) > 0, \quad f'(s) \geq 0 \quad \text{for all } s \in (0, \infty). \tag{2}$$

Y. Fujishima
Department of Mathematical and Systems Engineering, Faculty of Engineering, Shizuoka University, Hamamatsu, Japan
e-mail: fujishima@shizuoka.ac.jp

N. Ioku (✉)
Mathematical Institute, Tohoku University, Sendai, Japan
e-mail: ioku@tohoku.ac.jp

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It is well known that if u_0 is bounded there exists a unique classical solution of the problem (1) satisfying the initial condition in the sense that $\|u(t) - e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)} \to 0$ $(t \to 0)$ for general nonlinearities $f \in C^1([0,\infty))$. On the other hand, for unbounded initial data results on the existence or nonexistence of solutions heavily depend on the growth rate of the nonlinear term f and the singularity of the initial data u_0.

The model problem in this direction is the solvability of the Cauchy problem

$$\begin{cases} \partial_t u = \Delta u + u^p, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^N. \end{cases} \tag{3}$$

This problem has been studied extensively by many mathematicians since the pioneering work of Fujita [6]. Baras–Pierre [1] investigated the initial trace problem, that determined a necessary condition on the initial data such that the problem (3) possesses a local in time solution. The following can be obtained as a corollary of Baras–Pierre [1]: There exists $C > 0$ such that the problem (3) cannot possess nonnegative local in time solution if u_0 satisfies

$$u_0(x) \geq C|x|^{-\frac{2}{p-1}}, \qquad p > 1 + \frac{2}{N},$$

in a neighborhood of the origin. On the other hand, Ishige–Kawakami–Sierżęga [17, Corollary 3.2] obtained a sufficient condition for the existence of the problem (3). They determined that there exists $\tilde{C} > 0$ such that if u_0 satisfies

$$u_0(x) \leq \tilde{C}|x|^{-\frac{2}{p-1}}, \qquad p > 1 + \frac{2}{N},$$

then there exists a nonnegative global in time solution of the problem (3). It is remarkable that they proved a corresponding result for a system of parabolic equations with power type nonlinearities. Similar results on the solvability in a weak space can be found in [16, 19, 31].

The aim of this paper is to generalize these results to nonlinearities f satisfying (2) and determine a threshold singularity of initial data u_0 which separates the existence and nonexistence of a local in time nonnegative classical solution of (1). To this end, we apply the quasi scale invariance which was introduced in [4, 5]. In what follows we always assume

$$F(s) := \int_s^\infty \frac{du}{f(u)} < \infty \quad (s > 0). \tag{4}$$

The quasi scaling function is defined by

$$u_\lambda(x,t) := F^{-1}\Big[\frac{1}{\lambda^2}F(u(\lambda x, \lambda^2 t))\Big] \quad (\lambda > 0), \tag{5}$$

where F^{-1} is the inverse function of F. Then the function u_λ satisfies

$$\partial_t u_\lambda = \Delta u_\lambda + f(u_\lambda) + \frac{f(u_\lambda)|\nabla u|^2}{f(u)^2 F(u)}\Big[f'(u)F(u) - f'(u_\lambda)F(u_\lambda)\Big] \tag{6}$$

if and only if u satisfies $\partial_t u = \Delta u + f(u)$. It should be mentioned that the scaling (5) coincides with the well known scaling

$$u_\lambda(x,t) := \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t) \qquad (\lambda > 0), \tag{7}$$

if we choose $f(u) = u^p$. We focus on the limit $f'(s)F(s)$ as $s \to \infty$:

$$A := \lim_{s\to\infty} f'(s)F(s), \tag{8}$$

since the behavior of the function $f'F$ controls the remainder term in (6). Furthermore, we define $\tilde{p}$ as the Hölder conjugate of A, that is,

$$\tilde{p} := \frac{A}{A-1} \quad \text{if } 1 < A < \infty,$$

which can be considered as the growth rate of $f(u)$. Indeed, one can check easily that if $f(u) = u^p$ then $A = \frac{p}{p-1}$ and hence $\tilde{p} = p$.

It is remarkable that if $f \in C^2([0,\infty))$, the constants A and $\tilde{p}$ coincide with $\lim_{s\to\infty} \frac{f'(s)^2}{f(s)f''(s)}$ and $\lim_{s\to\infty} \frac{sf'(s)}{f(s)}$, respectively, where the limits of the same functions as $s \to 0$ are introduced by Dupaigne-Farina [3]. Indeed, by the de l'Hôpital rule we have

$$A = \lim_{s\to\infty} \frac{F(s)}{1/f'(s)} = \lim_{s\to\infty} \frac{(F(s))'}{\left(\frac{1}{f'(s)}\right)'} = \lim_{s\to\infty} \frac{f'(s)^2}{f(s)f''(s)},$$

$$\lim_{s\to\infty} \frac{sf'(s)}{f(s)} = \lim_{s\to\infty} \frac{s}{\frac{f(s)}{f'(s)}} = \lim_{s\to\infty} \frac{1}{\frac{f'(s)^2 - f(s)f''(s)}{f'(s)^2}} = \tilde{p}.$$

This makes it easier to calculate A and $\tilde{p}$ if $f \in C^2([0,\infty))$.

To state the results, we define the uniformly local Lebesgue space for $1 \le p < \infty$:

$$L^p_{ul,\rho}(\mathbb{R}^N) := \left\{ u \in L^1_{loc}(\mathbb{R}^N) : \|u\|_{L^p_{ul,\rho}(\mathbb{R}^N)} := \sup_{y\in\mathbb{R}^N} \|u\chi_{B_\rho(y)}\|_{L^p(\mathbb{R}^N)} < \infty \right\},$$

where $\chi_A(\cdot)$ is the characteristic function of $A \subset \mathbb{R}^N$.

We are ready to state our main results.

Theorem 1 *Let* $N \geq 1$, $\rho > 0$ *and* $f \in C^1([0,\infty))$ *satisfy* (2) *and* (4). *Assume that the limit* $A = \lim_{s\to\infty} f'(s)F(s)$ *exists and* $1 < A < 1 + \frac{N}{2}$, *equivalently* $1 + \frac{2}{N} < \tilde{p} < +\infty$. *Then, there exist* $0 < C_1 < C_2$ *such that the following hold.*

(i) (Existence) *Assume in addition that there exists a constant* $s_1 > 0$ *such that*

$$f'(s)F(s) \leq A \quad \text{for all } \; s \geq s_1. \tag{9}$$

If u_0 *satisfies* $0 \leq u_0(x) \leq \max\left\{0, F^{-1}\left(\frac{|x|^2}{C_1}\right)\right\}$ *for all* $x \in \mathbb{R}^N$ *then there exists* $T > 0$ *and a local in time classical solution* $u \in C^{2,1}(\mathbb{R}^N \times (0,T))$ *satisfying*

$$\lim_{t\to 0} \|u(t) - e^{t\Delta}u_0\|_{L^\alpha_{ul,\rho}(\mathbb{R}^N)} = 0 \tag{10}$$

for all $1 \leq \alpha < \frac{N}{2}(\tilde{p} - 1)$, *and*

$$u(t) \to u_0 \;\text{ in the sense of distributions.} \tag{11}$$

Furthermore, if f *satisfies* $f'(s)F(s) \leq A$ *for all* $s > 0$ *and* $f(0) = 0$ *then the solution exists globally in time.*

(ii) (Nonexistence) *Assume in place of (i) that* f *is convex in* $[0,\infty)$. *If* u_0 *satisfies* $u_0(x) \geq \max\left\{0, F^{-1}\left(\frac{|x|^2}{C_2}\right)\right\}$ *for all* $x \in \mathbb{R}^N$, *then there cannot exist a nonnegative classical solution of* (1) *which satisfies the initial condition in the sense of* (10).

It should be mentioned that Theorem 1 coincides with known results if $f(u) = u^p$ $(1 + \frac{2}{N} < p)$. Moreover, we have the following result with $f(u) = u^p[\log(e+u)]^q$ $(1 + \frac{2}{N} < p,\; 0 \leq q)$ as a direct application of Theorem 1.

Corollary 1 *Let* $N \geq 1$, $\rho > 0$, $1 + \frac{2}{N} < p$, $0 \leq q$ *and*

$$f(s) = s^p\left[\log(e+s)\right]^q, \quad s > 0. \tag{12}$$

Let $u_0(x) = C|x|^{-\frac{2}{p-1}}\left[\log\left(e + \frac{1}{|x|}\right)\right]^{-\frac{q}{p-1}}$. *Then there exists* $0 < C_1 < C_2$ *such that the following hold.*

(i) (Existence) *If* $0 < C \leq C_1$ *then there exists* $T > 0$ *and a local in time classical solution* $u \in C^{2,1}(\mathbb{R}^N \times (0,T))$ *of* (1) *with* (12) *satisfying* (10) *and* (11).

(ii) (Nonexistence) *If* $C \geq C_2$ *then there cannot exist a nonnegative classical solution of* (1) *with* (12) *which satisfies the initial condition in the sense of* (10).

Remark 1 In this paper we exclude the endpoint cases $A = 1 + \frac{N}{2}$ $(\tilde{p} = 1 + \frac{2}{N})$ and $A = 1$ $(\tilde{p} = \infty)$. For the model problem (2) with the Fujita exponent $p = 1 + \frac{2}{N}$, Baras–Pierre [1] proved that there cannot exist a nonnegative classical solution of (3) with $p = 1 + \frac{2}{N}$ for

$$u_0(x) = C|x|^{-N}\left[\log\left(e + \frac{1}{|x|}\right)\right]^{-\frac{N}{2}-1}$$

whenever C is sufficiently large. On the other hand, Hisa–Ishige [10] obtained conversely that there exists a nonnegative classical solution of (3) with $p = 1 + \frac{2}{N}$ for u_0 with sufficiently small $C > 0$. As for the superpower type nonlinearity $\tilde{p} = \infty$, exponential nonlinearities are considered in [7, 11–15, 32, 34]. These endpoint cases $A = 1 + \frac{N}{2}$ ($\tilde{p} = 1 + \frac{2}{N}$) and $A = 1$ ($\tilde{p} = \infty$) for general nonlinear term $f(u)$ will be discussed in a forthcoming paper.

It would be worthwhile to state some of related works for the problem (1) and its model case (3). Classification of the existence or nonexistence in Lebesgue spaces for the model problem (3) was studied by Weissler [35, 36], and uniqueness and nonuniqueness was studied by Brezis-Cazenave [2], Ni–Sacks [27], and Terraneo [33]. In [5], the authors investigated a generalization of Weissler's result to the problem (1) with general nonlinearity by introducing the invariant integral

$$\int_{\mathbb{R}^N} \frac{1}{F(u_\lambda(x,0))^{\frac{N}{2}}} dx = \int_{\mathbb{R}^N} \frac{1}{F(u_0(x))^{\frac{N}{2}}} dx \qquad (\lambda > 0)$$

under the scaling (5). The transformation (5) and the limit (8) were introduced in [5]. See also [2, 8, 18, 22–24, 30] for the existence and nonexistence of solutions for nonlinear parabolic equations and their qualitative properties. A singular stationary solution is considered in [26]. We also refer to [21] and [29], which include further numerous references on the topic.

At the end of this section, we summarize the outline of the proof of the existence part in Theorem 1, so that the importance of the limit A and its Hölder conjugate $\tilde{p}$ is clarified. The following generalized Cole–Hopf transformation, which is established in our previous paper [5], plays an essential role:

Proposition 1 (Proposition 3.1 in [5]) *Let f and g satisfy* (2) *and* (4)*, and define*

$$u(x,t) := F^{-1}\left(G(v(x,t))\right) \quad \textit{with} \quad G(v) := \int_v^\infty \frac{ds}{g(s)} \tag{13}$$

for $v \in C^{2,1}(\mathbb{R}^N \times (0,T))$ and some $T > 0$. Then the equation $\partial_t v = \Delta v + g(v)$ is equivalent to

$$\partial_t u - \Delta u - f(u) = \frac{|\nabla u|^2}{f(u)F(u)}\Big[g'(v)G(v) - f'(u)F(u)\Big]. \tag{14}$$

Proposition 1 suggests that a function g satisfying $g'(s)G(s) \equiv A$ is a typical nonlinearity having the property $A = \lim_{s\to+\infty} f'(s)F(s)$. Since $G'(s) = -\frac{1}{g(s)}$, the equation $g'(s)G(s) \equiv A$ can be regarded as an ordinary differential equation, and its simplest solution is $g(s) = (A-1)s^{\frac{A}{A-1}}$. The following corollary is a direct consequence of Proposition 1 with $g(s) = (A-1)s^{\frac{A}{A-1}} = \frac{1}{\tilde{p}-1}s^{\tilde{p}}$.

Corollary 2 *Let $A > 1$. Let f satisfy (2) and (4) and assume that the limit $A = \lim_{s\to\infty} f'(s)F(s)$ exists. Let $v \in C^{2,1}(\mathbb{R}^N \times (0,T))$ with $T > 0$ satisfies*

$$\partial_t v = \Delta v + (A-1)v^{\frac{A}{A-1}} = \Delta v + \frac{1}{\tilde{p}-1}v^{\tilde{p}}.$$

Then, the transformed function $\overline{u}(x,t) := F^{-1}\left(v(x,t)^{-(\tilde{p}-1)}\right)$ satisfies

$$\partial_t\overline{u} - \Delta\overline{u} - f(\overline{u}) = \frac{|\nabla\overline{u}|^2}{f(\overline{u})F(\overline{u})}\left(A - f'(\overline{u})F(\overline{u})\right).$$

Once we obtain a solution v to the initial value problem of the model equation,

$$\begin{cases} \partial_t v = \Delta v + |v|^p & \text{in } \mathbb{R}^N \times (0,T), \\ v(x,0) = v_0 & \text{in } \mathbb{R}^N, \end{cases}$$

which comes from the transformation in Corollary 2, then the function $\overline{u}$ defined in Corollary 2 is a supersolution of the original problem (1) provided that $f'(u)F(u) \le A$ and $v_0(x) = F(u_0(x))^{-\frac{1}{\tilde{p}-1}}$. Then applying the monotone method developed in [17] and [31], we obtain the existence of a solution of (1). The proof of Proposition 1 can be found in [5].

2 Preliminaries

Here we summarize several properties which are used in the proof. The following pointwise estimate is used to obtain the convergence to the initial data.

Proposition 2 (Lemma 3.1 in [5]) *Let $f \in C^1([0,\infty))$ satisfy (2) and (4). Define $h(\sigma) := f\left(F^{-1}(\sigma)\right)$. Let $A > 0$ and assume that $f'(s)F(s) \le A$ for all $s \ge s_1$ for some $s_1 > 0$. Then there exists a constant $C > 0$ such that $h(\sigma) \le C\sigma^{-A}$ for all sufficiently small $\sigma > 0$.*

Consider the following semilinear heat equation:

$$\begin{cases} \partial_t v = \Delta v + |v|^p & \text{in } \mathbb{R}^N \times (0, T), \\ v(x, 0) = v_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{15}$$

where $T > 0$. We first state an existence result for the problem (15) in uniformly local weak Lebesgue spaces. To show our main result, we recall the definition of the symmetric decreasing rearrangement (the Schwarz symmetrization). Let $A \subset \mathbb{R}^N$ be a measurable set and $A^\sharp$ be the ball centered at the origin with the same volume as A. Then for any measurable function f over $\mathbb{R}^N$ which satisfies $|\{x; |f(x)| > \lambda\}| < \infty$ for all $\lambda > 0$, the symmetric decreasing rearrangement of f denoted by $f^\sharp$ is defined by

$$f^\sharp(x) = \int_0^\infty \chi_{\{|f|>\lambda\}^\sharp}(x)d\lambda,$$

where χ_A is the characteristic function of $A \subset \mathbb{R}^N$. We denote $L^p(\mathbb{R}^N)$ the Lebesgue spaces endowed with the norm $\|u\|_{L^p(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |u(x)|^p dx\right)^{\frac{1}{p}}$. For $1 \le p < \infty$ and $1 \le q < \infty$, the weak Lebesgue space $L^{p,\infty}(\mathbb{R}^N)$ and the Lorentz space $L^{p,q}(\mathbb{R}^N)$ are defined by

$$L^{p,\infty}(\mathbb{R}^N) := \left\{ u \in L^1_{loc}(\mathbb{R}^N) : \|u\|_{L^{p,\infty}(\mathbb{R}^N)} := \sup_{x \in \mathbb{R}^N} |x|^{\frac{N}{p}} |u^\sharp(x)| < \infty \right\},$$

$$L^{p,q}(\mathbb{R}^N) := \left\{ u \in L^1_{loc}(\mathbb{R}^N) : \|u\|_{L^{p,q}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} \left(|x|^{\frac{N}{p}} |u^\sharp(x)| \right)^q \frac{dx}{|x|^N} \right)^q < \infty \right\}.$$

See [9, Section 1.4.2] and [20, Section 3] for more details on weak Lebesgue spaces and the Schwarz symmetrization. We define the uniformly local Lebesgue space for $1 \le p < \infty$ as follows:

$$L^p_{ul,\rho}(\mathbb{R}^N) := \left\{ u \in L^1_{loc}(\mathbb{R}^N) : \|u\|_{L^p_{ul,\rho}(\mathbb{R}^N)} := \sup_{y \in \mathbb{R}^N} \|u \chi_{B_\rho(y)}\|_{L^p(\mathbb{R}^N)} < \infty \right\}.$$

In the same manner, the uniformly local weak Lebesgue space $L^{p,\infty}_{ul,\rho}(\mathbb{R}^N)$ is defined by

$$L^{p,\infty}_{ul,\rho}(\mathbb{R}^N) := \left\{ u \in L^1_{loc}(\mathbb{R}^N) : \|u\|_{L^{p,\infty}_{ul,\rho}(\mathbb{R}^N)} := \sup_{y \in \mathbb{R}^N} \|u \chi_{B_\rho(y)}\|_{L^{p,\infty}(\mathbb{R}^N)} < \infty \right\}.$$

We denote by $\mathscr{L}^p_{ul,\rho}(\mathbb{R}^N)$ the closure of the space of bounded uniformly continuous functions $BUC(\mathbb{R}^N)$ in the space $L^p_{ul,\rho}(\mathbb{R}^N)$, that is,

$$\mathscr{L}^p_{ul,\rho}(\mathbb{R}^N) := \overline{BUC(\mathbb{R}^N)}^{\|\cdot\|_{L^p_{ul,\rho}(\mathbb{R}^N)}}.$$

Proposition 3 (Solvability in a Uniformly Local Weak Lebesgue Spaces) *Let $N \geq 1$, $1 + \frac{2}{N} < p < \infty$, and $r = \frac{N}{2}(p-1)$. There exists $\varepsilon > 0$ such that for any initial data $v_0 \in L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)$ satisfying*

$$\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \leq \varepsilon$$

there exists a classical solution $v \in C((0,\rho^2); L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)) \cap L^\infty_{loc}((0,\rho^2); L^\infty(\mathbb{R}^N))$ of (15) satisfying

$$\lim_{t \to 0} \|v(t) - e^{t\Delta} v_0\|_{L^\alpha_{ul,\rho}(\mathbb{R}^N)} = 0 \tag{16}$$

for $1 \leq \alpha < r$, and

$$v(t) \to v_0 \ \text{in the sense of distributions.} \tag{17}$$

Remark 2 If initial data v_0 is sufficiently small in $L^{r,\infty}(\mathbb{R}^N)$, the solution exists globally in time. This coincides with the case of $\rho = \infty$ in Proposition 3.

Remark 3 For the case $\alpha = r = \frac{N}{2}(p-1)$, the convergence

$$\lim_{t \to 0} \|v(t) - e^{t\Delta} v_0\|_{L^{r,\infty}(\mathbb{R}^N)} = 0 \tag{18}$$

does not hold in general. Indeed, if $v_0 = C|x|^{-\frac{2}{p-1}}$ and $C > 0$ is small enough there exists a self similar solution v in the form

$$v(x,t) = t^{-\frac{1}{p-1}} f(x/\sqrt{t}), \quad x \in \mathbb{R}^N,\ t > 0,$$

where $f \in L^\infty(\mathbb{R}^N)$. See [29, Theorem 20.19]. A simple calculation shows us that this solution satisfies

$$\begin{aligned} v(x,t) - e^{t\Delta} v_0(x) &= \int_0^t \int_{\mathbb{R}^N} (4\pi(t-s))^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} s^{-\frac{p}{p-1}} |f(y/\sqrt{s})|^p \, dy ds \\ &= t^{-\frac{1}{p-1}} \int_0^1 \int_{\mathbb{R}^N} (4\pi(1-s))^{-\frac{N}{2}} e^{-\frac{|x/\sqrt{t}-y|^2}{4(1-s)}} s^{-\frac{p}{p-1}} |f(y/\sqrt{s})|^p \, dy ds. \end{aligned}$$

Therefore,

$$\|v(\cdot,t)-e^{t\Delta}v_0\|_{L^{r,\infty}(\mathbb{R}^N)} = \sup_{x\in\mathbb{R}^N} |x|^{\frac{2}{p-1}} \left(v(\cdot,t)-e^{t\Delta}v_0(\cdot)\right)^{\sharp}(x)$$

$$= \sup_{x\in\mathbb{R}^N} \Bigg[\left(\frac{|x|}{\sqrt{t}}\right)^{\frac{2}{p-1}} \times$$

$$\left(\int_0^1 \int_{\mathbb{R}^N} (4\pi(1-s))^{-\frac{N}{2}} e^{-\frac{|\cdot-y|^2}{4(1-s)}} s^{-\frac{p}{p-1}} \left| f\left(\frac{y}{\sqrt{s}}\right)\right|^p dyds\right)^{\sharp} \left(\frac{x}{\sqrt{t}}\right)\Bigg]$$

$$= \tilde{C}$$

for some constant $\tilde{C} > 0$ which does not depend on $t > 0$. Here we used the fact that $\left(h(\lambda\cdot)\right)^{\sharp}(x) = h^{\sharp}(\lambda x)$ for $\lambda > 0$ and $h : \mathbb{R}^N \to \mathbb{R}$.

Proposition 3 is essentially obtained by Ishige-Kawakami-Sierżęga [17, Corollary 3.2]. We give an alternative proof of Proposition 3 by applying standard contraction mapping argument.

The following linear estimate plays an essential role in the proof of Proposition 3. In the following we consider the rescaled function $f_t(x) := t^{-\frac{N}{2}} f\left(\frac{x}{\sqrt{t}}\right)$ for $t > 0$.

Proposition 4 *Let $1 < q < p < \infty$ and $F(x)$, $H(x)$ be two real valued functions in $\mathbb{R}^N$ satisfying $|F(x)| \le H(x)$ for all $x \in \mathbb{R}^N$. Assume that H is bounded, integrable, radially symmetric, and decreasing in $\mathbb{R}^N$. Then, for any function $g \in L^{q,\infty}_{ul,\rho}(\mathbb{R}^N)$, $F_t * g(x)$ is well-defined for all $x \in \mathbb{R}^N$. Furthermore, the estimate*

$$\|F_t * g\|_{L^p_{ul,\rho}(\mathbb{R}^N)} \le \left(C_1\|H\|_{L^1}\rho^{-N\left(\frac{1}{q}-\frac{1}{p}\right)} + C_2\|H\|_{L^r}\, t^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right)\|g\|_{L^{q,\infty}_{ul,\rho}(\mathbb{R}^N)}$$

holds, where r satisfies $1+\frac{1}{p} = \frac{1}{q}+\frac{1}{r}$, and C_1, C_2 are positive constants depending on N, p, q.

Corollary 3 ($L^p_{ul,\rho}$–$L^{q,\infty}_{ul,\rho}$ Estimates) *If $1 \le q \le p \le \infty$, there exists $C > 0$ depending only N such that*

$$\|e^{t\Delta}v_0\|_{L^p_{ul,\rho}(\mathbb{R}^N)} \le C\left(\rho^{-N\left(\frac{1}{q}-\frac{1}{p}\right)} + t^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right)\|v_0\|_{L^q_{ul,\rho}(\mathbb{R}^N)}$$

for all $v_0 \in L^q_{ul,\rho}(\mathbb{R}^N)$.

If $1 < q < p < \infty$, there exists $C_{p,q} > 0$ depending only p, q, N such that

$$\|e^{t\Delta}v_0\|_{L^p_{ul,\rho}(\mathbb{R}^N)} \le C_{p,q}\left(\rho^{-N\left(\frac{1}{q}-\frac{1}{p}\right)} + t^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right)\|v_0\|_{L^{q,\infty}_{ul,\rho}(\mathbb{R}^N)}$$

for all $v_0 \in L^{q,\infty}_{ul,\rho}(\mathbb{R}^N)$.

The first assertion is nothing but Corollary 3.1 in Maekawa–Terasawa [25]. The second assertion is a direct consequence of Proposition 4 by taking $F(x) = H(x) = (4\pi)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4}}$.

Remark 4 By Chebyshev's inequality, there holds

$$\|e^{t\Delta} v_0\|_{L^{p,\infty}_{ul,\rho}(\mathbb{R}^N)} \le \|e^{t\Delta} v_0\|_{L^{p}_{ul,\rho}(\mathbb{R}^N)}$$

for all $v_0 \in L^{q,\infty}_{ul,\rho}(\mathbb{R}^N)$. This together with Corollary 3 yields $L^{p,\infty}_{ul,\rho}$–$L^{q,\infty}_{ul,\rho}$ estimates of the heat kernel.

Proof of Proposition 4 The proof is essentially the same as the proof of Theorem 3.1 in Maekawa–Terasawa [25]. Here we only explain the difference between the proof in [25] and ours. Our argument relies on the following weak Young inequality (cf. [9, Theorem 1.4.24]): for any $1 < q < p < \infty$ there exists $C > 0$ such that

$$\|f * g\|_{L^p(\mathbb{R}^N)} \le C\|f\|_{L^r(\mathbb{R}^N)}\|g\|_{L^{q,\infty}(\mathbb{R}^N)} \quad \text{for all } f \in L^r(\mathbb{R}^N),\ g \in L^{q,\infty}(\mathbb{R}^N), \tag{19}$$

where r satisfies $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Maekawa–Terasawa applied Young's inequality for (3.13) in [25]. Replacing it with the weak Young inequality (19) for (3.13) in [25], we obtain the desired estimate. □

We are now in position to prove Proposition 3.

Proof of Proposition 3 Let $1 + \frac{2}{N} < p < \infty$, $r = \frac{N(p-1)}{2}$, and $v_0 \in L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)$. Take q such that $\max\{p, r\} < q < pr$ and define $\sigma := \frac{N}{2}\left(\frac{1}{r} - \frac{1}{q}\right)$ and $C' := 2\max\{C_{r,r}, C_{q,r}\}$, where $C_{a,b}$ is a constant which is determined in Corollary 3. We consider the following set of functions:

$$X := \left\{ \begin{array}{r} v \in C\big((0,\rho^2); L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)\big) \cap L^\infty_{loc}\big((0,\rho^2); \mathscr{L}^q_{ul,\rho}(\mathbb{R}^N)\big) : \\ \sup_{0<t<\rho^2} \|v(t)\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \le C'\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)}, \\ \sup_{0<t<\rho^2} t^\sigma \|v(t)\|_{L^{q}_{ul,\rho}(\mathbb{R}^N)} \le C'\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \end{array} \right\},$$

equipped with the metric $d_X(u, v) := \sup\{t^\sigma\|u(t) - v(t)\|_{L^q_{ul,\rho}(\mathbb{R}^N)} : 0 < t < \rho^2\}$. Then (X, d_X) is a complete metric space. Remark that $\mathscr{L}^q_{ul,\rho}(\mathbb{R}^N) \subset L^q_{ul,\rho}(\mathbb{R}^N) \subset L^r_{ul,\rho}(\mathbb{R}^N)$ since $r < q$. For any $v \in X$, define

$$\Phi(v) := e^{t\Delta} v_0 + \int_0^t e^{(t-s)\Delta}|v(s)|^p\, ds.$$

We show that Φ is a contraction map from X to itself if the initial data u_0 is sufficiently small in $L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)$. Assume that $0 < t < \rho^2$. It follows from Corollary 3 that

$$\begin{aligned}
\|\Phi(v)\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} &\le C_{r,r}\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} + \int_0^t \|e^{(t-s)\Delta}|v(s)|^p\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)}\,ds \\
&\le C_{r,r}\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} + C_{r,\frac{q}{p}}\int_0^t \left(\rho^{-N\left(\frac{p}{q}-\frac{1}{r}\right)} + (t-s)^{-\frac{N}{2}\left(\frac{p}{q}-\frac{1}{r}\right)}\right)\|v(s)\|^p_{L^q_{ul,\rho}(\mathbb{R}^N)}\,ds.
\end{aligned}$$

This together with $v \in X$ yields that

$$\begin{aligned}
&\|\Phi(v)\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \\
&\le C_{r,r}\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} + C_{r,\frac{q}{p}}\left(t^{1-\sigma p}\rho^{-N\left(\frac{p}{q}-\frac{1}{r}\right)}\right)C'^p\|v_0\|^p_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \\
&\quad + C_{r,\frac{q}{p}}\left(t^{1-\sigma p-\frac{N}{2}\left(\frac{p}{q}-\frac{1}{r}\right)}\int_0^1 (1-s)^{-\frac{N}{2}\left(\frac{p}{q}-\frac{1}{r}\right)}s^{-\sigma p}\,ds\right)C'^p\|v_0\|^p_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)}.
\end{aligned} \tag{20}$$

Then, since $-\sigma p > -1$ and $1-\sigma p - \frac{N}{2}\left(\frac{p}{q}-\frac{1}{r}\right) = 0$, we have

$$\sup_{0<t<\rho^2}\|\Phi(v)\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \le C_{r,r}\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} + C_1\|v_0\|^p_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \tag{21}$$

for some constant $C_1 > 0$. Similarly, we have

$$\begin{aligned}
t^\sigma\|\Phi(v)\|_{L^q_{ul,\rho}(\mathbb{R}^N)} &\le C_{q,r}\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} + t^\sigma\int_0^t \|e^{(t-s)\Delta}|v(s)|^p\|_{L^q_{ul,\rho}(\mathbb{R}^N)}\,ds \\
&\le C_{q,r}\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \\
&\quad + t^\sigma C_{q,\frac{q}{p}}\int_0^t\left(\rho^{-N\left(\frac{p}{q}-\frac{1}{q}\right)} + (t-s)^{-\frac{N}{2}\left(\frac{p}{q}-\frac{1}{q}\right)}\right)\|v(s)\|^p_{L^q_{ul,\rho}(\mathbb{R}^N)}\,ds \\
&\le C_{q,r}\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} + C_{q,\frac{r}{p}}\left(t^{\sigma+1-\sigma p}\rho^{-N\left(\frac{p}{q}-\frac{1}{q}\right)}\right)C'^p\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \\
&\quad + C_{q,\frac{r}{p}}\left(t^{\sigma+1-\sigma p-\frac{N}{2}\left(\frac{p}{q}-\frac{1}{q}\right)}\int_0^1(1-s)^{-\frac{N}{2}\left(\frac{p}{q}-\frac{1}{q}\right)}s^{-\sigma p}\,ds\right)C'^p\|v_0\|^p_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)}.
\end{aligned} \tag{22}$$

Then, since $-\sigma p > -1$, $-\frac{N}{2}\left(\frac{p}{q}-\frac{1}{q}\right) > -1$ and $\sigma + 1 - \sigma p - \frac{N}{2}\left(\frac{p}{q}-\frac{1}{q}\right) = 0$, we obtain

$$\sup_{0<t<\rho^2} t^\sigma\|\Phi(v)\|_{L^q_{ul,\rho}(\mathbb{R}^N)} \le C_{q,r}\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} + C_2\|v_0\|^p_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \tag{23}$$

for some $C_2 > 0$. Therefore, if $\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)}$ is sufficiently small then the right hand sides of (21) and (23) are bounded by $C'\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)}$. Combining this, (21), and (23), we obtain

$$\sup_{0<t<\rho^2} \|\Phi(v)\|_{L^{r,\infty}_{ul,\rho}} \le C'\|v_0\|_{L^{r,\infty}_{ul,\rho}} \quad \text{and} \quad \sup_{0<t<\rho^2} t^{\sigma}\|\Phi(v)\|_{L^{q}_{ul,\rho}} \le C'\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)}.$$

Furthermore, it follows from the smoothing effect of the heat kernel that $\Phi(v) \in L^{\infty}_{loc}\big((0,\rho^2);\mathscr{L}^q_{ul,\rho}(\mathbb{R}^N)\big)$. It remains to prove that $\Phi(v) \in C\big((0,\rho^2); L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)\big)$ if $v \in X$. Take $t > 0$ and $h > 0$. Then,

$$\begin{aligned} &\|\Phi\big(v(t+h)\big) - \Phi\big(v(t)\big)\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \\ &= \left\| \int_0^{t+h} e^{(t+h-s)\Delta}|v(s)|^p ds - \int_0^t e^{(t-s)\Delta}|v(s)|^p ds \right\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \\ &\le \int_t^{t+h} \|e^{(t+h-s)\Delta}|v(s)|^p\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} ds \\ &\quad + \int_0^t \|e^{(t-s)\Delta}(e^{h\Delta}|v(s)|^p - |v(s)|^p)\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} ds. \end{aligned} \tag{24}$$

By the similar argument as in (20), there holds for some $C_3 > 0$ that

$$\begin{aligned} &\int_t^{t+h} \|e^{(t+h-s)\Delta}|v(s)|^p\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} ds \\ &\qquad \le C_3 \int_{\frac{t}{t+h}}^1 (1-s)^{-\frac{N}{2}\left(\frac{p}{q}-\frac{1}{r}\right)} s^{-\sigma p} ds \|v_0\|^p_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \to 0 \ (h \to 0). \end{aligned}$$

We estimate the second term of the right hand side in (24). By the similar argument as in (20), we have

$$\begin{aligned} &\int_0^t \|e^{(t-s)\Delta}(e^{h\Delta}|v(s)|^p - |v(s)|^p)\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} ds \\ &\qquad \le \int_0^t (t-s)^{-\frac{N}{2}\left(\frac{p}{q}-\frac{1}{r}\right)} \|(e^{h\Delta}-1)|v(s)^p|\|_{L^{\frac{q}{p},\infty}_{ul,\rho}(\mathbb{R}^N)}. \end{aligned}$$

Since there exists $C_4 > 0$ such that

$$\|(e^{h\Delta}-1)|v(s)^p|\|_{L^{\frac{q}{p},\infty}_{ul,\rho}(\mathbb{R}^N)} \le C_4\|v(s)\|_{L^q_{ul,\rho}(\mathbb{R}^N)} \le C_4 s^{-\sigma p}\|v_0\|_{L^q_{ul,\rho}(\mathbb{R}^N)}$$

for all $h > 0$ and

$$\lim_{h\to 0} \|(e^{h\Delta} - 1)|v(s)^p|\|_{L^{\frac{q}{p},\infty}_{ul,\rho}(\mathbb{R}^N)} = 0 \text{ a.e. } s > 0$$

by $|v(s)|^p \in \mathscr{L}^{\frac{q}{p}}_{ul,\rho}(\mathbb{R}^N)$, one can apply the dominated convergence theorem to obtain

$$\lim_{h\to 0} \int_0^t \|e^{(t-s)\Delta}(e^{h\Delta}|v(s)|^p - |v(s)|^p)\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} ds = 0.$$

Thus $\Phi(v) \in C\big((0,\rho^2); L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)\big)$ if $v \in X$. This completes the proof of that Φ is a map from X to itself.

By the similar argument, we have for $v, w \in X$ that

$$\sup_{0<t<\rho^2} t^\sigma \|\Phi(v) - \Phi(w)\|_{L^q_{ul,\rho}(\mathbb{R}^N)} \le C_5 \|v_0\|^{p-1}_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \sup_{0<t<\rho^2} t^\sigma \|v - w\|_{L^q_{ul,\rho}(\mathbb{R}^N)} \tag{25}$$

for some $C_5 > 0$. This proves that Φ is a contraction map from X to itself if $\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)}$ is sufficiently small. Thus, by the contraction mapping theorem we obtain a fixed point $v \in X$.

It remains to prove the convergence to the initial data (16) and (17). Let $1 \le \alpha < r$. Take β such that $\alpha < \beta < r < p\beta$. Then, it is easily obtained by a similar argument to (20) that

$$\begin{aligned} \|v(t) - e^{t\Delta}v_0\|_{L^\alpha_{ul,\rho}(\mathbb{R}^N)} &\le \int_0^t \|e^{(t-s)\Delta}|v(s)|^p\|_{L^\alpha_{ul,\rho}(\mathbb{R}^N)} \\ &\le \int_0^t \|e^{(t-s)\Delta}|v(s)|^p\|_{L^\beta_{ul,\rho}(\mathbb{R}^N)} \\ &\le t^{1-\frac{N}{2}\left(\frac{p}{r}-\frac{1}{\beta}\right)} \|v_0\|^p_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \to 0 \ (t \to 0), \end{aligned} \tag{26}$$

since $1 - \frac{N}{2}\left(\frac{p}{r} - \frac{1}{\beta}\right) > 0$ and $L^{q_2}_{ul,\rho}(\mathbb{R}^N) \subset L^{q_1}_{ul,\rho}(\mathbb{R}^N)$ if $q_1 \le q_2$. This proves (16). We next prove (17). Fix $\phi \in C_0^\infty(\mathbb{R}^N)$. Then there holds

$$\begin{aligned} &\int_{\mathbb{R}^N} (v(x,t) - v_0(x))\,\phi(x)dx \\ &= \int_{\mathbb{R}^N} \big(v(x,t) - e^{t\Delta}v_0(x)\big)\,\phi(x)dx + \int_{\mathbb{R}^N} \big(e^{t\Delta}v_0(x) - v_0(x)\big)\,\phi(x)dx. \end{aligned} \tag{27}$$

Since suppϕ is compact, there exists $k \in \mathbb{N}$ and $\{x_j\}_{j=1}^k \subset \mathbb{R}^N$ such that supp$\phi \subset \cup_{j=1}^k B_\rho(x_j)$. Therefore, for $1 < \alpha < r$ we have by (16) that

$$\begin{aligned}\left|\int_{\mathbb{R}^N} \left(v(x,t) - e^{t\Delta}v_0(x)\right)\phi(x)dx\right| &\le \sum_{j=1}^{k}\int_{B_\rho(x_j)} \left|(v(x,t) - e^{t\Delta}v_0(x))\phi(x)\right| dx \\ &\le \|v(t) - e^{t\Delta}v_0\|_{L^\alpha_{ul,\rho}(\mathbb{R}^N)} \times \|\phi\|_{L^{\alpha'}(\mathbb{R}^N)} \\ &\to 0\end{aligned} \tag{28}$$

as $t \to 0$. Moreover,

$$\begin{aligned}\left|\int_{\mathbb{R}^N} \left(e^{t\Delta}v_0(x) - v_0(x)\right)\phi(x)dx\right| &= \left|\int_{\mathbb{R}^N} v_0(x)\left(e^{t\Delta}\phi(x) - \phi(x)\right)dx\right| \\ &\le \|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)}\|e^{t\Delta}\phi - \phi\|_{L^{r',1}(\mathbb{R}^N)} \to 0\end{aligned} \tag{29}$$

as $t \to 0$. Here we applied the Hölder inequality in Lorentz spaces [9, Theorem 1.4.17] and the density of $C_0^\infty(\mathbb{R}^N) \subset L^{r',1}(\mathbb{R}^N)$. More general density result for smooth functions can be found in [28]. Combining (27), (28), and (29), we obtain (17).

Smoothness of the solution is proved by the standard bootstrap argument. See for instance [29, p. 81]. This completes the proof of Proposition 3. □

3 Proof of the Results

3.1 *Proof of the Existence Result (Theorem 1 (i))*

Let $r = \frac{N}{2}(\tilde{p} - 1)$. We consider the following initial value problem:

$$\begin{cases} \partial_t v = \Delta v + (A-1)v^{\frac{A}{A-1}} = \Delta v + \dfrac{1}{\tilde{p}-1}v^{\tilde{p}} & \text{in } \mathbb{R}^N \times (0,T), \\ v(x,0) = v_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{30}$$

where $T > 0$. Define

$$v_0(x) := \max\left\{F(u_0(x))^{-\frac{1}{\tilde{p}-1}}, F(s_1)^{-\frac{1}{\tilde{p}-1}}\right\}, \tag{31}$$

where s_1 is the constant appearing in (9). By the assumption on u_0, the initial data v_0 satisfies

$$v_0(x) \le \max\left\{C_1^{\frac{1}{\tilde{p}-1}}|x|^{-\frac{2}{\tilde{p}-1}}, F(s_1)^{-\frac{1}{\tilde{p}-1}}\right\}. \tag{32}$$

It follows from

$$\left(v_0\chi_{B_\rho(y)}\right)^\sharp(x) \le \max\left\{C_1^{\frac{1}{\tilde{p}-1}}|x|^{-\frac{2}{\tilde{p}-1}}, F(s_1)^{-\frac{1}{\tilde{p}-1}}\right\}\chi_{B_\rho(0)} \quad \text{for } y \in \mathbb{R}^N$$

that

$$\|v_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \le \max\left\{C_1^{\frac{1}{\tilde{p}-1}}, \rho^{\frac{2}{\tilde{p}-1}}F(s_1)^{-\frac{1}{\tilde{p}-1}}\right\}.$$

Hence one can apply Proposition 3 if C_1 and ρ are sufficiently small, and obtain that there exists a local in time classical solution v of (30) satisfying (17). It should be remarked that

$$v_0(x) \ge F(s_1)^{-\frac{1}{\tilde{p}-1}} > 0 \quad \text{in } \mathbb{R}^N \tag{33}$$

and hence

$$v(x,t) \ge (e^{t\Delta}v_0)(x) \ge F(s_1)^{-\frac{1}{\tilde{p}-1}} \tag{34}$$

in $\mathbb{R}^N \times (0,T)$.

Here we construct a super solution for the original problem (1). Define the function $\overline{u} \in C^{2,1}(\mathbb{R}^N \times (0,T))$ by

$$\overline{u}(x,t) := F^{-1}\left(v(x,t)^{-\frac{1}{A-1}}\right) = F^{-1}\left(v(x,t)^{-(\tilde{p}-1)}\right). \tag{35}$$

Applying Corollary 2, we have

$$\partial_t\overline{u} - \Delta\overline{u} - f(\overline{u}) = \frac{|\nabla\overline{u}|^2}{f(\overline{u})F(\overline{u})}\left(A - f'(\overline{u})F(\overline{u})\right).$$

It follows from (34) that

$$\overline{u}(x,t) \ge F^{-1}(F(s_1)) = s_1 \quad \text{in } \mathbb{R}^N \times (0,T). \tag{36}$$

Thus, since $f'(\overline{u})F(\overline{u}) \le A$ in $\mathbb{R}^N \times (0,T)$ by (9) and (36), we obtain

$$\partial_t\overline{u} \ge \Delta\overline{u} + f(\overline{u}) \quad \text{in } \mathbb{R}^N \times (0,T). \tag{37}$$

Furthermore, by (31) we see that

$$\overline{u}(x,0) = \max\{u_0(x), s_1\} \geq u_0(x) \quad \text{in } \mathbb{R}^N. \tag{38}$$

Therefore, we see that the function $\bar{u}$ is a supersolution of (1). Applying the monotone methods (see the proof of Theorem 1.1 [5] for more details), we conclude that there exists a local in time classical solution u for the problem (1).

We turn to the proof of the convergence to the initial data (10) and (11). Let $1 \leq \alpha < r$. It follows from $u(x,t) \leq \overline{u}(x,t)$ and Proposition 2 with $f(s) = s^{\frac{A}{A-1}} = s^{\tilde{p}}$ that

$$\begin{aligned} |u(t) - e^{t\Delta}u_0| &= \int_0^t e^{(t-s)\Delta} f(u(s))ds \\ &\leq \int_0^t e^{(t-s)\Delta} f(\overline{u}(s))\, ds \leq \tilde{C} \int_0^t e^{(t-s)\Delta} v(s)^{\tilde{p}}\, ds \end{aligned} \tag{39}$$

for some $\tilde{C} > 0$. We have by the same argument as in (26) that

$$\|u(t) - e^{t\Delta}u_0\|_{L^\alpha_{ul,\rho}(\mathbb{R}^N)} \leq \tilde{C} \int_0^t \|e^{(t-s)\Delta} v(s)^{\tilde{p}}\|_{L^\alpha_{ul,\rho}(\mathbb{R}^N)}\, ds \to 0$$

as $t \to 0$. This proves (10). It remains to prove (11). Fix $\phi \in C_0^\infty(\mathbb{R}^N)$. Then there holds

$$\begin{aligned} \int_{\mathbb{R}^N} (u(x,t) - u_0(x))\,\phi(x)dx &= \int_{\mathbb{R}^N} \left(u(x,t) - e^{t\Delta}u_0(x)\right)\phi(x)dx \\ &\quad + \int_{\mathbb{R}^N} \left(e^{t\Delta}u_0(x) - u_0(x)\right)\phi(x)dx. \end{aligned} \tag{40}$$

It follows from the compactness of suppϕ that there exists $k \in \mathbb{N}$ and $\{x_j\}_{j=1}^k \subset \mathbb{R}^N$ such that supp$\phi \subset \cup_{j=1}^k B_\rho(x_j)$. Hence, by (10) we have

$$\begin{aligned} \left|\int_{\mathbb{R}^N} \left(u(x,t) - e^{t\Delta}u_0(x)\right)\phi(x)dx\right| &\leq \sum_{j=1}^k \int_{B_\rho(x_j)} \left|(u(x,t) - e^{t\Delta}u_0(x))\phi(x)\right| dx \\ &\leq \|u(t) - e^{t\Delta}u_0\|_{L^\alpha_{ul,\rho}(\mathbb{R}^N)} \|\phi\|_{L^{\alpha'}(\mathbb{R}^N)} \to 0 \end{aligned} \tag{41}$$

as $t \to 0$. To estimate the second term of (40), we prove the following pointwise estimate:

$$F^{-1}\left([a + b(s - s_1)]^{-(\tilde{p}-1)}\right) \leq s \quad (s > s_1), \tag{42}$$

where

$$a = F(s_1)^{-(A-1)}, \quad b = (A-1)F(s_1)^{-A}(-F'(s_1)).$$

Observe that $a, b > 0$ since F is positive and strictly decreasing. Since $f'(s)F(s) \le A$ $(s > s_1)$ and $f'(s) = \frac{F''(s)}{F'(s)^2}$, it suffices to solve

$$F(s)F''(s) - AF'(s)^2 \le 0.$$

Multiplying $F(s)^{-A}$ and integrating on (s_1, s), we have

$$F(s)^{-A}F'(s) \le F(s_1)^{-A}F'(s_1).$$

Again, integrating on (s_1, s), we obtain

$$F(s)^{-(A-1)} \ge F(s_1)^{-(A-1)} - (A-1)F(s_1)^{-A}F'(s_1)(s - s_1).$$

This proves (42). Clearly the inequality (42) yields

$$F^{-1}\left(\frac{|x|^2}{C}\right) \le \frac{C^{1/(\tilde{p}-1)}|x|^{-\frac{2}{\tilde{p}-1}} - a}{b} + s_1 \quad \text{for} \quad |x| < \left(\frac{C}{a^{\tilde{p}-1}}\right)^{\frac{1}{2}},$$

hence

$$u_0(x) \le C_6|x|^{-\frac{2}{\tilde{p}-1}} + C_7 \quad (x \in \mathbb{R}^N) \tag{43}$$

for some $C_6, C_7 > 0$. Therefore, $u_0 \in L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)$. Combining this and the similar argument to (29), we have

$$\left| \int_{\mathbb{R}^N} \left(e^{t\Delta}u_0(x) - u_0(x)\right)\phi(x)dx \right| \le \|u_0\|_{L^{r,\infty}_{ul,\rho}(\mathbb{R}^N)} \|e^{t\Delta}\phi - \phi\|_{L^{r',1}(\mathbb{R}^N)} \to 0 \tag{44}$$

as $t \to 0$. The estimates (41) and (44) yield the desired convergence (11).

It remains to prove the global existence if f satisfies $f'(s)F(s) \le A$ for all $s > 0$ and $f(0) = 0$. Since $f(0) = 0$ and $f \in C^1([0, \infty))$, the mean value theorem implies that there exists $a > 0$ such that $f(s) = f(s) - f(0) \le as$ for small $s > 0$. Hence $\lim_{s \to 0} F(s) = \infty$, and thus $F^{-1}(s) > 0$ for all $s > 0$. Therefore u_0 satisfies

$$u_0(x) \le F^{-1}\left(\frac{|x|^2}{C}\right),$$

and $v_0(x) := F(u_0(x))^{-\frac{1}{\tilde{p}-1}}$ satisfies

$$\|v_0\|_{L^{r,\infty}(\mathbb{R}^N)} \le C_1^{\frac{1}{\tilde{p}-1}}.$$

Therefore, Proposition 3 and Remark 2 give us a global in time classical solution of (30), and the function $\bar{u}$ defined in (35) becomes a global supersolution in time of the original problem (1). The monotone methods prove the existence of a global in time classical solution. This completes the proof of Theorem 1 (i). □

3.2 Proof of the Nonexistence Result (Theorem 1 (ii))

Fix C_2 and u_0 such that

$$(4\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4}} |y|^2 dy < C_2 \quad \text{and} \quad u_0(x) \ge \max\left\{F^{-1}\left(\frac{|x|^2}{C_2}\right), 0\right\}.$$

The proof proceeds by contradiction. Assume that there exists a nonnegative solution. Then, by Fujishima and Ioku [5, Lemma 4.1], the initial data satisfies

$$\|e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)} \le F^{-1}(t) \tag{45}$$

for small $t > 0$. Here we need the convexity of f which is assumed in Theorem 1 (ii).

On the other hand, since F^{-1} is a convex function, it follows from Jensen's inequality that

$$\begin{aligned}
\|e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^N)} &\ge (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4t}} F^{-1}\left(\frac{|y|^2}{C_2}\right) dy \\
&\ge F^{-1}\left((4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4t}} \frac{|y|^2}{C_2} dy\right) \\
&= F^{-1}\left((4\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4}} \frac{|y|^2}{C_2} dy \times t\right)
\end{aligned}$$

for all $t > 0$. Remark that

$$\tilde{C} := (4\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4}} \frac{|y|^2}{C_2} dy < 1.$$

Applying the mean value theorem to F^{-1}, we have

$$F^{-1}(\tilde{C}t) = F^{-1}(t) + f\left(F^{-1}(C't)\right) \times (1-\tilde{C})t$$

for some C' ($\tilde{C} < C' < 1$). This together with (45) yields that

$$F^{-1}(t) \geq F^{-1}(t) + f\left(F^{-1}(C't)\right) \times (1-\tilde{C})t.$$

This contradicts the positivity of f. □

Remark 5 F^{-1} is a convex function under the assumption (2). Indeed, direct computations show

$$\frac{d^2}{ds^2}F^{-1}(s) = f'(F^{-1}(s)) \times f(F^{-1}(s)) \geq 0$$

for all $s > 0$.

Remark 6 We used positivity of the initial data and solutions in both proofs of the existence and the nonexistence in Theorem 1. Indeed, for the existence, our argument relies on the super-subsolution method. Thanks to positivity of the initial data and solutions, the zero function is clearly a subsolution. For the nonexistence, we also need positivity to derive the decay estimate of the heat kernel (45). For more details, see [5, Lemma 4.1].

3.3 Proof of Corollary 1

We apply Theorem 1 for $f(s) := s^p\left[\log(e+s)\right]^q$ $(1+\frac{2}{N} < p,\ 0 \leq q)$. We have by direct computations that

$$f'(s) = ps^{p-1}\left[\log(e+s)\right]^q + q\frac{s^p}{s+e}\left[\log(e+s)\right]^{q-1}.$$

Thus

$$0 \leq f'(s) \leq ps^{p-1}\left[\log(e+s)\right]^q + qs^{p-1}\left[\log(e+s)\right]^{q-1}$$

for all $s > 0$. Moreover,

$$f''(s) = p(p-1)s^{p-2}\left[\log(e+s)\right]^q + 2pq\frac{s^{p-1}}{s+e}\left[\log(e+s)\right]^{q-1}$$
$$- q\frac{s^p}{(s+e)^2}\left[\log(e+s)\right]^{q-1} + q(q-1)\frac{s^p}{(s+e)^2}\left[\log(e+s)\right]^{q-2}$$

$$= p(p-1)s^{p-2}\left[\log(e+s)\right]^q + q\frac{s^{p-1}}{s+e}\left[\log(e+s)\right]^{q-1}\left\{p - \frac{s}{s+e}\right\}$$
$$+ q\frac{s^p}{s+e}\left[\log(e+s)\right]^{q-2}\left\{p\log(e+s) + (q-1)\frac{s}{s+e}\right\} \geq 0$$

for sufficiently large $s > 0$. Integrating by parts, we have

$$\begin{aligned} F(s) &= \int_s^\infty \frac{d\eta}{\eta^p\left[\log(e+\eta)\right]^q} \\ &= \left[\frac{\eta^{1-p}}{1-p}\left[\log(e+\eta)\right]^{-q}\right]_s^\infty - \frac{q}{p-1}\int_s^\infty \frac{\eta^{1-p}}{e+\eta}\left[\log(e+\eta)\right]^{-q-1} d\eta \\ &= \frac{s^{1-p}}{p-1}\left[\log(e+s)\right]^{-q} - \frac{q}{p-1}\int_s^\infty \frac{\eta^{1-p}}{e+\eta}\left[\log(e+\eta)\right]^{-q-1} d\eta. \end{aligned}$$

Therefore,

$$f'(s)F(s) \leq \frac{p}{p-1} + \frac{q}{p-1}\left[\log(e+s)\right]^{-1} - \frac{pq}{p-1}\int_s^\infty \frac{\eta^{1-p}}{\eta+e}\left[\log(e+\eta)\right]^{-q-1} d\eta.$$

Define

$$h(s) = \frac{1}{p}\left[\log(e+s)\right]^{-1} - \int_s^\infty \frac{\eta^{1-p}}{\eta+e}\left[\log(e+\eta)\right]^{-q-1} d\eta.$$

Since $h(s) \to 0$ $(s \to \infty)$, it suffices to prove $h'(s) > 0$ for sufficiently large $s > 0$. Indeed, there holds

$$h'(s) = s^{-p}\left[\log(e+s)\right]^{-q-1}\left\{\frac{p-1/2}{p}\frac{s}{s+e} - \frac{p-1}{p}\right\}$$
$$+ \frac{s^{1-p}}{e+s}\left[\log(e+s)\right]^{-q-1}\left\{\frac{1}{2p} - \frac{q+1}{p}\left[\log(e+s)\right]^{-1}\right\} > 0$$

for sufficiently large $s > 0$. Hence, $f'(s)F(s) \leq \frac{p-1}{p}$ for sufficiently large $s > 0$. Therefore, f satisfies all assumptions in Theorem 1.

We next calculate the behavior of $F^{-1}\left(|x|^2\right)$. Since

$$\lim_{s\to 0}\frac{F(s)}{s^{1-p}\left[\log(e+s)\right]^{-q}} = \lim_{s\to\infty}\frac{F(s)}{s^{1-p}\left[\log(e+s)\right]^{-q}} = \frac{1}{p-1},$$

there exists $C' > 0$ such that

$$\frac{1}{C'} s^{1-p} \left[\log(e+s)\right]^{-q} \le F(s) \le C' s^{1-p} \left[\log(e+s)\right]^{-q} \tag{46}$$

for all $s > 0$. Hence, $F(s) = C_s s^{1-p} \left[\log(e+s)\right]^{-q}$ for some C_s $(1/C' \le C_s \le C')$. Define

$$\tilde{F}(s) = s^{-\frac{1}{p-1}} \left[\log\left(e + \frac{1}{s}\right)\right]^{-\frac{q}{p-1}}.$$

It follows from

$$\frac{\tilde{F}(F(s))}{s} = C_s^{-\frac{1}{p-1}} \left[\frac{\log(e+s)}{\log(e + \frac{1}{C_s} s^{p-1} [\log(e+s)]^q)}\right]^{\frac{q}{p-1}}$$

that

$$\frac{1}{\tilde{C}} \tilde{F}(s) \le F^{-1}(s) \le \tilde{C} \tilde{F}(s)$$

for some $\tilde{C} > 0$. This shows us that

$$\frac{1}{\tilde{C}} |x|^{-\frac{2}{p-1}} \left[\log\left(e + \frac{1}{|x|}\right)\right]^{-\frac{q}{p-1}} \le F^{-1}\left(|x|^2\right) \le \tilde{C} |x|^{-\frac{2}{p-1}} \left[\log\left(e + \frac{1}{|x|}\right)\right]^{-\frac{q}{p-1}}.$$

Therefore, applying Theorem 1 with $u_0 = C|x|^{-\frac{2}{p-1}} \left[\log\left(e + \frac{1}{|x|}\right)\right]^{-\frac{q}{p-1}}$, we obtain Corollary 1. □

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References

1. Baras, P., Pierre, M.: Critère d'existence de solutions positives pour des équations semi-linéaires non monotones. Ann. Inst. H. Poincaré Anal. Non Linéaire **2**, 185–212 (1985)
2. Brezis, H., Cazenave, T.: A nonlinear heat equation with singular initial data. J. Anal. Math. **68**, 277–304 (1996)
3. Dupaigne, L., Farina, A.: Stable solutions of $-\Delta = f(u)$ in $\mathbb{R}^N$. J. Eur. Math. Soc. **12**, 855–882 (2010)
4. Fujishima, Y.: Blow-up set for a superlinear heat equation and pointedness of the initial data. Discrete Contin. Dyn. Syst. A **34**, 4617–4645 (2014)

5. Fujishima, Y., Ioku, N.: Existence and nonexistence of solutions for the heat equation with a superlinear source term. J. Math. Pures Appl. (9) **118**, 128–158 (2018)
6. Fujita, H.: textitOn some nonexistence and nonuniqueness theorems for nonlinear parabolic equations. Nonlinear Funct. Anal. **18**, 105–113 (1970)
7. Furioli, G., Kawakami, T., Ruf, B., Terraneo, E.: Asymptotic behavior and decay estimates of the solutions for a nonlinear parabolic equation with exponential nonlinearity. J. Differ. Equ. **262**, 145–180 (2017)
8. Giga, Y.: Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system. J. Differ. Equ. **62**, 415–421 (1986)
9. Grafakos, L.: Classical Fourier Analysis. Graduate Texts in Mathematics, vol. 249. Springer, New York (2008)
10. Hisa, K., Ishige, K.: Existence of solutions for a fractional semilinear parabolic equation with singular initial data. Nonlinear Anal. **175**, 108–132 (2018)
11. Ibrahim, S., Jrad, R., Majdoub, M., Saanouni, T.: Local well posedness of a 2D semilinear heat equation. Bull. Belg. Math. Soc. Simon Stevin **21**, 535–551 (2014)
12. Ibrahim, S., Kikuchi, H., Nakanishi, K., Wei, J.: Non–uniqueness for an energy–critical heat equation on $\mathbb{R}^2$. arXiv:1903.06729
13. Ioku, N.: The Cauchy problem for heat equations with exponential nonlinearity. J. Differ. Equ. **251**, 1172–1194 (2011)
14. Ioku, N., Ruf, B., Terraneo, E.: Existence, non-existence, and uniqueness for a heat equation with exponential nonlinearity in $\mathbb{R}^N$. Math. Phys. Anal. Geom. **18**, 29 (2015)
15. Ioku, N., Ruf, B., Terraneo, E.: Non-uniqueness for a critical heat equation in two dimensions with singular data. Ann. Inst. H. Poincaré Anal. Non Linéaire **36**, 2027–2051 (2019)
16. Ishige, K., Kawakami, T., Kobayashi, K.: Global solutions for a nonlinear integral equation with a generalized heat kernel. Discrete Contin. Dyn. Syst. Ser. S **7**, 767–783 (2014)
17. Ishige, K., Kawakami, T., Sierżęga, M.: Supersolutions of parabolic systems with power nonlinearities. J. Differ. Equ. **260**, 6084–6107 (2016)
18. Ishige, K., Sato, R.: Heat equation with a nonlinear boundary condition and uniformly local L^r spaces. Discrete Contin. Dyn. Syst. A **36**, 2627–2652 (2016)
19. Kozono, H., Yamazaki, M.: Semilinear heat equations and the Navier–Stokes equation with distributions in new function spaces as initial data. Commun. Partial Differ. Equ. **19**, 959–1014 (1994)
20. Lieb, E., Loss, M.: Analysis, 2nd edn. American Math. Soc., Providence (1994)
21. Ladyženskaja, O.A., Solonnikov, N.S., Ural'ceva, N.N.: Linear and Quasilinear Equations of Parabolic Type. Amer. Math. Soc., Providence (1968)
22. Laister, R., Robinson, J.C., Sierżęga, M.: Non-existence of local solutions for semilinear heat equations of Osgood type. J. Differ. Equ. **255**, 3020–3028 (2013)
23. Laister, R., Robinson, J.C., Sierżęga, M., Vidal-López, A.: A complete characterization of local existence of semilinear heat equations in Lebesgue spaces. Ann. Inst. H. Poincaré Anal. Non Linéaire **33**, 1519–1538 (2016)
24. Laister, R., Robinson, J.C., Sierżęga, M.: A necessary and sufficient condition for uniqueness of the trivial solution in semilinear parabolic equations. J. Differ. Equ. **262**, 4979–4987 (2017)
25. Maekawa, Y., Terasawa, Y.: The Navier-Stokes equations with initial data in uniformly local L^p spaces. Differ. Integr. Equ. **19**, 369–400 (2006)
26. Miyamoto, Y.: A limit equation and bifurcation diagrams of semilinear elliptic equations with general supercritical growth. J. Differ. Equ. **264**, 2684–2707 (2018)
27. Ni, W.-M., Sacks, P.: Singular behavior in nonlinear parabolic equations. Trans. Am. Math. Soc. **287**, 657–671 (1985)
28. Nakai, E., Tomita, N., Yabuta, K.: Density of the set of all infinitely differentiable functions with compact support in weighted Sobolev spaces. Sci. Math. Jpn. **60**(1), 121–127 (2004)
29. Quittner, P., Souplet, P.: Superlinear Parabolic Problems, Blow-up, Global Existence and Steady States. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser, Basel (2007)
30. Robinson, J.C., Sierżęga, M.: A note on well-posedness of semilinear reaction-diffusion problem with singular initial data. J. Math. Anal. Appl. **385**, 105–110 (2012)

31. Robinson, J.C., Sierżęga, M.: Supersolutions for a class of semilinear heat equations. Rev. Mat. Complut. **26**, 341–360 (2013)
32. Ruf, B., Terraneo, E.: The Cauchy problem for a semilinear heat equation with singular initial data. Progr. Nonlinear Differ. Equ. Appl. **50**, 295–309 (2002)
33. Terraneo, E.: Non-uniqueness for a critical non-linear heat equation. Commun. Partial Differ. Equ. **27**, 185–218 (2002)
34. Vazquez, J.L.: Domain of existence and blowup for the exponential reaction-diffusion equation. Indiana Univ. Math. J. **48**, 677–709 (1999)
35. Weissler, F.B.: Local existence and nonexistence for semilinear parabolic equations in L^p. Indiana Univ. Math. J. **29**, 79–102 (1980)
36. Weissler, F.B.: Existence and nonexistence of global solutions for a semilinear heat equation. Israel Math. J. **38**, 29–40 (1981)

Bounds for Sobolev Embedding Constants in Non-simply Connected Planar Domains

Filippo Gazzola and Gianmarco Sperone

Abstract In a bounded non-simply connected planar domain Ω, with a boundary split in an interior part and an exterior part, we obtain bounds for the embedding constants of some subspaces of $H^1(\Omega)$ into $L^p(\Omega)$ for any $p > 1$, $p \neq 2$. The subspaces contain functions which vanish on the interior boundary and are constant (possibly zero) on the exterior boundary. We also evaluate the precision of the obtained bounds in the limit situation where the interior part tends to disappear and we show that it does not depend on p. Moreover, we emphasize the failure of symmetrization techniques in these functional spaces. In simple situations, a new phenomenon appears: the existence of a break even surface separating masses for which symmetrization increases/decreases the Dirichlet norm. The question whether a similar phenomenon occurs in more general situations is left open.

Keywords Embedding constants · Pyramidal functions · Symmetrization

1 Introduction

In the plane $\mathbb{R}^2$ we consider an open, bounded, connected, and simply connected domain K, with Lipschitz boundary ∂K. Then we remove K, seen as an obstacle, from a larger square Q such that $\partial K \cap \partial Q = \emptyset$, and we define the domain

$$Q = (-L, L)^2 , \qquad \Omega = Q \setminus \overline{K},$$

where $L > \text{diam}(K)$, as shown in Fig. 1.

F. Gazzola (✉)
Dipartimento di Matematica, Politecnico di Milano, Milan, Italy
e-mail: filippo.gazzola@polimi.it

G. Sperone
Department of Mathematical Analysis, Charles University in Prague, Prague, Czech Republic
e-mail: sperone@karlin.mff.cuni.cz

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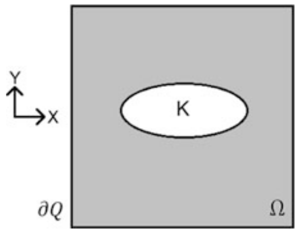

Fig. 1 The planar domain Ω with a smooth obstacle K

We focus our attention on the first order Hilbertian Sobolev space of functions vanishing on ∂K, which is a proper part of $\partial\Omega$ having positive 1D-measure:

$$H^1_*(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial K\}.$$

This space is rigorously defined as the closure of the space $\mathcal{C}^\infty_c(\overline{Q} \setminus \overline{K})$ with respect to the Dirichlet norm: this is legitimate since $|\partial K| > 0$ and the Poincaré inequality holds in $H^1_*(\Omega)$.

Motivated by the target of finding explicit thresholds for bifurcation from uniqueness in stationary Navier-Stokes equations modeling a flow around an obstacle, in a recent paper [7] we bounded some Sobolev embedding constants for $H^1_*(\Omega) \subset L^4(\Omega)$. We obtained a universal bound on the flow velocity for the appearance of a lift force on the obstacle K exerted by a fluid entering Q with constant velocity. In the present paper we drop this physical motivation and we focus our attention on the functional analytic aspect and on the possibility of obtaining similar inequalities in $L^p(\Omega)$ for any $p > 1$, $p \neq 2$. To the best of our knowledge, bounds in spaces of functions vanishing on a proper part of the boundary were obtained in the past only for the critical Sobolev embedding [1, 9] (thereby in space dimension $n \geq 3$), where one can exploit scaling methods since the optimal constant does not depend on the domain.

Given a subset $D \subset \mathbb{R}^2$ and $p > 1$, throughout the paper we denote by $\|\cdot\|_{p,D}$ the norm of the space $L^p(D)$. The relative capacity of K with respect to Q is defined by

$$\mathrm{Cap}_Q(K) = \min_{v \in H^1_0(Q)} \left\{ \int_Q |\nabla v|^2\, dx \;\middle|\; v = 1 \text{ in } K \right\} \tag{1.1}$$

and the relative capacity potential $\psi \in H^1_0(Q)$, which achieves the minimum in (1.1), satisfies

$$\Delta\psi = 0 \text{ in } \Omega, \qquad \psi = 0 \text{ on } \partial Q, \qquad \psi = 1 \text{ in } K, \qquad \mathrm{Cap}_Q(K) = \|\nabla\psi\|^2_{2,\Omega}.$$

Then we consider a proper subspace of $H^1_*(\Omega)$, namely

$$H^1_c(\Omega) = \{v \in H^1_*(\Omega) \mid v \text{ is constant on } \partial Q\}, \tag{1.2}$$

that can be rigorously characterized by using the relative capacity potential ψ. Indeed,

$$H_c^1(\Omega) = H_0^1(\Omega) \oplus \mathbb{R}(\psi - 1)\,, \quad H_0^1(\Omega) \perp \mathbb{R}(\psi - 1)\,,$$

so that $H_0^1(\Omega)$ has codimension 1 within $H_c^1(\Omega)$ and the "missing dimension" is spanned by the function $\psi - 1$, see [7] for the details. Since Ω is a planar domain, the embedding $H_*^1(\Omega) \subset L^p(\Omega)$ holds for any $1 < p < \infty$, and we define the Sobolev constants

$$\mathcal{S}_p = \min_{w \in H_*^1(\Omega)\setminus\{0\}} \frac{\|\nabla w\|_{2,\Omega}^2}{\|w\|_{p,\Omega}^2}, \qquad \mathcal{S}_p^0 = \min_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla w\|_{2,\Omega}^2}{\|w\|_{p,\Omega}^2},$$
$$\mathcal{S}_p^1 = \min_{w \in H_c^1(\Omega)\setminus\{0\}} \frac{\|\nabla w\|_{2,\Omega}^2}{\|w\|_{p,\Omega}^2}\,. \tag{1.3}$$

Due to the inclusions $H_0^1(\Omega) \subset H_c^1(\Omega) \subset H_*^1(\Omega)$, we have $\mathcal{S}_p \le \mathcal{S}_p^1 \le \mathcal{S}_p^0$, for every $p > 1$.

In Sect. 2 we obtain bounds for the constants $\mathcal{S}_p^0$ and $\mathcal{S}_p^1$, extending the results in [7] where only the case $p = 4$ was considered. To this end, we repeatedly use some sharp Gagliardo-Nirenberg inequalities due to del Pino-Dolbeault [5] and the behavior of pyramidal functions introduced in [7]. It turns out that the cases $p > 2$ and $p < 2$ require slightly different approaches. We obtain both lower and upper bounds for the constants $\mathcal{S}_p^0$ and $\mathcal{S}_p^1$ defined in (1.3) and we show that they are quite precise. In particular, we analyze the case where the obstacle tends to vanish ($|K| \to 0$) and we show that the ratio between these bounds converges to a *universal constant* $\pi/4 \approx 0.79$, independently of the value of $p > 1$ ($p \neq 2$), see Theorem 2.3. Our bounds do not depend on the position of the obstacle and it is therefore natural to expect that they might be improved, see Problem 2.1.

In Sect. 3 we address the question whether symmetrization techniques might be employed to obtain additional bounds. It turns out that, at least in its simplest forms, symmetrization is of no help in annuli, see Theorem 3.1. In its proof we exhibit examples where any of the possible inequalities may hold: in case of different (constant) conditions on the two connected components of the boundary

> there is no a priori monotonicity of the Dirichlet norm under decreasing rearrangement neither from an annulus into itself, nor from an annulus into a disk with the same measure.

Moreover, we determine explicitly a "break even surface" which separates the cases where the mass of the gradient increases or decreases after symmetrization. We believe that this phenomenon deserves further investigation, see Problem 3.1.

2 Bounds for the Sobolev Embedding Constants

In Sect. 2.1 we provide lower bounds for the Sobolev embedding constants (1.3), for a general Lipschitz obstacle K. Then, in Sect. 2.2 we derive upper bounds for these constants and quantify the accuracy of our estimates when K is a square.

2.1 Lower Bounds

As mentioned in the introduction, the cases $p \lesseqgtr 2$ are different and we consider first the case $p < 2$. Let

$$\mu_0 = \text{the first zero of the Bessel function of first kind of order zero} \approx 2.40483\,. \tag{2.1}$$

Then we have

Theorem 2.1 *For any* $1 < p < 2$ *and* $u \in H_0^1(\Omega)$ *one has*

$$\|u\|_{p,\Omega}^2 \leq \frac{|\Omega|^{\frac{2}{p}}}{\pi} \min\left\{\frac{1}{\mu_0^2}, \frac{1}{2\pi}\frac{|Q|}{|\Omega|}, \left(\frac{p}{2}\right)^{\frac{4-p}{2-p}}\right\} \|\nabla u\|_{2,\Omega}^2\,. \tag{2.2}$$

For any $1 < p < 2$ *and* $u \in H_c^1(\Omega)$ *one has*

$$\|u\|_{p,\Omega}^2 \leq \frac{|\Omega|^{\frac{2}{p}}}{\pi}\left(\frac{p}{2}\right)^{\frac{4-p}{2-p}}\left(1+\frac{1}{2}\sqrt{\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\log\left(\frac{|Q|}{|K|}\right)}\right)^{\frac{2(p-1)}{p}}$$
$$\times\left[1+\frac{1}{2}\sqrt{\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\log\left(\frac{|Q|}{|K|}\right)}\right.$$
$$\left.+\frac{p}{2-p}\frac{|K|}{|\Omega|}\left(\frac{1}{4}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\log\left(\frac{|Q|}{|K|}\right)\right)^{p-1}\right]^{\frac{2}{p}}\|\nabla u\|_{2,\Omega}^2. \tag{2.3}$$

Proof We begin by proving the following Poincaré inequality in Ω:

$$\|u\|_{2,\Omega} \leq \min\left\{\frac{1}{\mu_0}\sqrt{\frac{|\Omega|}{\pi}}, \frac{1}{\pi}\sqrt{\frac{|Q|}{2}}\right\}\|\nabla u\|_{2,\Omega} \qquad \forall u \in H_0^1(\Omega). \tag{2.4}$$

Through the Faber-Krahn inequality [6, 8] we first bound the $L^2(\Omega)$-norm of functions in terms of their Dirichlet norm by using the Poincaré inequality in Ω^*, namely a disk having the same measure as Ω. Since $|\Omega| = |Q| - |K|$, the radius of Ω^* is given by

$$R = \sqrt{\frac{|\Omega|}{\pi}} = \sqrt{\frac{|Q| - |K|}{\pi}}\,.$$

Since the Poincaré constant (least eigenvalue of $-\Delta$) in the unit disk is given by μ_0^2, see (2.1), the Poincaré constant in Ω^* is given by μ_0^2/R^2, which means that

$$\min_{w\in H_0^1(\Omega)} \frac{\|\nabla w\|_{2,\,\Omega}}{\|w\|_{2,\,\Omega}} \geq \min_{w\in H_0^1(\Omega^*)} \frac{\|\nabla w\|_{2,\,\Omega^*}}{\|w\|_{2,\,\Omega^*}} = \frac{\mu_0}{R}\,.$$

Therefore,

$$\|u\|_{2,\,\Omega} \leq \frac{R}{\mu_0}\|\nabla u\|_{2,\,\Omega} = \frac{1}{\mu_0}\sqrt{\frac{|\Omega|}{\pi}}\,\|\nabla u\|_{2,\,\Omega} \qquad \forall u \in H_0^1(\Omega),$$

which provides the first bound in (2.4). On the other hand, the least eigenvalue for the problem $-\Delta v = \lambda v$ in $H_0^1(Q)$ is given by $\lambda = \pi^2/2L^2$. Therefore, the Poincaré inequality in Q reads

$$\|u\|_{2,\,Q} \leq \frac{\sqrt{2}L}{\pi}\,\|\nabla u\|_{2,\,Q} = \frac{1}{\pi}\sqrt{\frac{|Q|}{2}}\,\|\nabla u\|_{2,\,Q} \qquad \forall u \in H_0^1(Q),$$

yielding the second bound in (2.4) since any function of $H_0^1(\Omega)$ can be extended by 0 in K, thereby becoming a function in $H_0^1(Q)$.

The first two bounds in (2.2) are obtained after applying both Hölder's inequality and (2.4)

$$\begin{aligned}\|u\|_{p,\,\Omega}^p &\leq |\Omega|^{\frac{2-p}{2}}\,\|u\|_{2,\,\Omega}^p \\ &\leq \min\left\{\frac{|\Omega|}{\left(\mu_0\sqrt{\pi}\right)^p},\ |\Omega|^{\frac{2-p}{2}}\left(\frac{1}{\pi}\sqrt{\frac{|Q|}{2}}\right)^p\right\}\|\nabla u\|_{2,\,\Omega}^p \qquad \forall u \in H_0^1(\Omega).\end{aligned}$$

To prove the third bound in (2.2), we recall the following (optimal) Gagliardo-Nirenberg inequality in $\mathbb{R}^2$ given by del Pino-Dolbeault [5, Theorem 2]:

$$\|u\|_{p,\,\Omega} \leq \pi^{\frac{p-2}{2p}}\left(\frac{p}{2}\right)^{\frac{4-p}{2p}}\|\nabla u\|_{2,\,\Omega}^{\frac{2-p}{p}}\,\|u\|_{2(p-1),\,\Omega}^{\frac{2(p-1)}{p}} \qquad \forall u \in H_0^1(\Omega) \qquad \forall p \in (1,2). \tag{2.5}$$

Since functions in $H^1_0(\Omega)$ may be extended by zero outside Ω, they can be seen as functions defined over the whole plane. An application of the Hölder inequality shows that

$$\|u\|_{2(p-1),\,\Omega} \leq |\Omega|^{\frac{2-p}{2p(p-1)}}\,\|u\|_{p,\,\Omega} \qquad \forall u \in H^1_0(\Omega)$$

which, combined with (2.5), yields the third bound in (2.2).

In order to prove (2.3) we restrict our attention to functions $u \in H^1_c(\Omega) \setminus H^1_0(\Omega)$: this restriction will be justified a posteriori because, if we manage proving (2.3) for these functions, then it will also hold for functions in $H^1_0(\Omega)$ since the constant in (2.2) is smaller, see also Fig. 3 below. For functions $u \in H^1_c(\Omega) \setminus H^1_0(\Omega)$, it suffices to analyze the case where $u \geq 0$ in Ω (by replacing u with $|u|$), $u = 1$ on ∂Q (by homogeneity), and we define a.e. in Q the function

$$v(x, y) = \begin{cases} 1 - u(x, y) & \text{if } (x, y) \in \Omega \\ 1 & \text{if } (x, y) \in K, \end{cases}$$

so that $v \in H^1_0(Q)$ and, after a zero extension outside Q, v satisfies (2.5). Let us put

$$A_p = A_p(u) \doteq \pi^{\frac{p-2}{2}} \left(\frac{p}{2}\right)^{\frac{4-p}{2}} \|\nabla v\|^{2-p}_{2,\,Q} = \pi^{\frac{p-2}{2}} \left(\frac{p}{2}\right)^{\frac{4-p}{2}} \|\nabla u\|^{2-p}_{2,\,\Omega}\,,$$

so that (2.5) reads

$$\int_Q |v|^p \leq A_p \int_Q |v|^{2(p-1)}$$
$$\Longrightarrow \int_\Omega \left[|1-u|^p + \frac{|K|}{|\Omega|} - A_p\left(|1-u|^{2(p-1)} + \frac{|K|}{|\Omega|}\right)\right] \leq 0. \tag{2.6}$$

The next step consists in finding $\alpha \in (0, 1)$ and $\beta > 0$ (possibly depending on p, but having ratio independent of u) for which

$$|1-s|^p - A_p|1-s|^{2(p-1)} + (1 - A_p)\frac{|K|}{|\Omega|} \geq \alpha s^p - \beta A_p^{\frac{p}{2-p}} \qquad \forall s \geq 0. \tag{2.7}$$

Given any $p \in (0, 1)$ and $\gamma \in \mathbb{R}$, the function $s \mapsto |1-s|^p - A_p|1-s|^{2(p-1)} + \gamma$ is symmetric with respect to $s = 1$, so it suffices to find $\alpha \in (0, 1)$ and $\beta > 0$ ensuring (2.7) for every $s \geq 1$. Thus, for all such α and β we define the function

$$\varphi_p(s) = (s-1)^p - A_p(s-1)^{2(p-1)} - \alpha s^p + (1 - A_p)\frac{|K|}{|\Omega|} + \beta A_p^{\frac{p}{2-p}} \qquad \forall s \geq 1,$$

and we seek $\alpha \in (0, 1)$ and $\beta > 0$ in such a way that φ_p has a non-negative minimum value at some $s > 1$. Equivalently, we seek $\gamma > 0$ such that φ_p attains its minimum at $s_0 = 1 + \gamma A_p$, that is,

$$\varphi'(s_0) = \gamma^{p-1} A_p^{p-1} \left[p - 2(p-1)\gamma^{p-2} A_p^{p-1} \right] - p\alpha(1 + \gamma A_p)^{p-1} = 0,$$

thus fixing α in dependence of u through the expression

$$\alpha = \frac{1}{p} \left(\frac{\gamma A_p}{1 + \gamma A_p} \right)^{p-1} \left[p - 2(p-1)\gamma^{p-2} A_p^{p-1} \right] \in (0, 1)$$
$$\iff \quad \gamma > \left(\frac{2p-2}{p} \right)^{\frac{1}{2-p}} A_p^{\frac{p-1}{2-p}} .$$

By imposing $\varphi(s_0) \geq 0$, we obtain the following lower bound for β:

$$\beta \geq A_p^{\frac{p}{p-2}} \left[\alpha \left(1 + \gamma A_p\right)^p + \gamma^{2p-2} A_p^{2p-1} \left(1 - \gamma^{2-p} A_p^{1-p}\right) \right] + \frac{A_p - 1}{A_p^{\frac{p}{2-p}}} \frac{|K|}{|\Omega|}.$$

This condition is certainly satisfied if we choose

$$\beta = A_p^{\frac{p}{p-2}} \left[\alpha \left(1 + \gamma A_p\right)^p + \gamma^{2p-2} A_p^{2p-1} \left(1 - \gamma^{2-p} A_p^{1-p}\right) \right] + A_p^{\frac{2p-2}{p-2}} \frac{|K|}{|\Omega|} \geq 0 \,,$$

where one should take $\gamma \leq A_p^{\frac{p-1}{2-p}}$ in order to ensure that $\beta \geq 0$. With the above choices of α and β we obtain the ratio

$$\begin{aligned} \frac{\beta}{\alpha} = & \frac{p A_p^{\frac{p}{p-2}} \left(\frac{\gamma A_p}{1 + \gamma A_p} \right)^{1-p}}{p - 2(p-1)\gamma^{p-2} A_p^{p-1}} \\ & \times \left\{ \frac{1 + \gamma A_p}{p} (\gamma A_p)^{p-1} \left(p - 2(p-1)\gamma^{p-2} A_p^{p-1} \right) \right. \\ & \left. + \gamma^{2p-2} A_p^{2p-1} \left(1 - \gamma^{2-p} A_p^{1-p}\right) + A_p \frac{|K|}{|\Omega|} \right\} , \end{aligned} \tag{2.8}$$

which depends on u and on $\gamma > 0$ such that

$$\left(\frac{2p-2}{p} \right)^{\frac{1}{2-p}} A_p^{\frac{p-1}{2-p}} < \gamma \leq A_p^{\frac{p-1}{2-p}} .$$

Hence, we still have the freedom of choosing γ. By taking $\gamma = A_p^{\frac{p-1}{2-p}}$ (which, numerically, appears to be close to the global minimum of the right-hand side of (2.8)), we obtain

$$\frac{\beta}{\alpha} = \left(1 + \frac{1}{A_p(u)^{\frac{1}{2-p}}}\right)^{p-1} \left(1 + \frac{1}{A_p(u)^{\frac{1}{2-p}}} + \frac{p}{2-p} \frac{1}{A_p(u)^{\frac{2p-2}{2-p}}} \frac{|K|}{|\Omega|}\right), \tag{2.9}$$

where we emphasized the dependence of A on u. In order to obtain an upper bound for the ratio β/α independent of u, we use [7, Remark 2.1] which states that

$$\|\nabla u\|_{2,\,\Omega}^2 \geq \frac{4\pi}{\log(|Q|) - \log(|K|)} \qquad \forall u \in H_c^1(\Omega) \text{ s.t. } u = 1 \text{ on } \partial Q,\ u \geq 0 \text{ in } \Omega,$$

thus yielding

$$A_p(u) \geq 2^{2-p} \left(\frac{p}{2}\right)^{\frac{4-p}{2}} \left(\log\left(\frac{|Q|}{|K|}\right)\right)^{\frac{p-2}{2}} \qquad \forall u \in H_c^1(\Omega) \text{ s.t. } u = 1 \text{ on } \partial Q,\ u \geq 0 \text{ in } \Omega.$$

Hence, from (2.9) we obtain the following uniform bound (independent of u)

$$\frac{\beta}{\alpha} \leq \left(1 + \frac{1}{2}\sqrt{\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \log\left(\frac{|Q|}{|K|}\right)}\right)^{p-1} \times \left[1 + \frac{1}{2}\sqrt{\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \log\left(\frac{|Q|}{|K|}\right)} + \frac{p}{2-p} \frac{|K|}{|\Omega|} \left(\frac{1}{4}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \log\left(\frac{|Q|}{|K|}\right)\right)^{p-1}\right].$$

In turn, from (2.6), by replacing s with $|u|$ in (2.7) and integrating over Ω, we obtain

$$\begin{aligned}\|u\|_{p,\Omega}^{p} &\le \frac{\beta}{\alpha}\, A_p(u)^{\frac{p}{2-p}}\, |\Omega| \\ &\le \pi^{-\frac{p}{2}} \left(\frac{p}{2}\right)^{\frac{p(4-p)}{2(2-p)}} |\Omega| \left(1+\frac{1}{2}\sqrt{\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \log\left(\frac{|Q|}{|K|}\right)}\right)^{p-1} \\ &\quad \times \left[1+\frac{1}{2}\sqrt{\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \log\left(\frac{|Q|}{|K|}\right)} \right. \\ &\quad \left. +\frac{p}{2-p}\,\frac{|K|}{|\Omega|}\left(\frac{1}{4}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \log\left(\frac{|Q|}{|K|}\right)\right)^{p-1}\right] \|\nabla u\|_{2,\Omega}^{p},\end{aligned}$$

for every $u \in H_c^1(\Omega)$ such that $u = 1$ on ∂Q and $u \ge 0$ in Ω. The bound in (2.3) follows by taking the p-roots in the last inequality. □

Remark 2.1 We point out that (2.4) provides an upper bound for the Poincaré constant in $H_0^1(\Omega)$ (for $p = 2$). On the other hand, a bound for the Poincaré constant in $H_c^1(\Omega)$ for $p = 2$ cannot be obtained by taking the limit in (2.3) when $p \to 2$, because the right-hand side of (2.3) blows up. This is the reason why the analysis of the case $p = 2$ has been excluded in the present article.

We now turn to the case $p > 2$.

Theorem 2.2 *For any $p > 2$ and $u \in H_0^1(\Omega)$ one has*

$$\|u\|_{p,\Omega}^{2} \le \frac{|\Omega|^{\frac{2}{p}}}{\pi} \left(\frac{p+2}{4}\right)^{\frac{p-6}{p}} \min\left\{\frac{1}{\mu_0^2},\ \frac{1}{2\pi}\frac{|Q|}{|\Omega|}\right\}^{\frac{2}{p}} \|\nabla u\|_{2,\Omega}^{2}. \tag{2.10}$$

For any $p>2$ *and* $u\in H^1_c(\Omega)$ *one has*

$$\|u\|^2_{p,\,\Omega}\le\frac{|\Omega|^{\frac{2}{p}}}{\pi}\left(\frac{p+2}{4}\right)^{\frac{p-6}{p-2}}\left(1+\frac{1}{2}\sqrt{\left(\frac{p+2}{4}\right)^{\frac{6-p}{p-2}}\log\left(\frac{|Q|}{|K|}\right)}\right)^{\frac{2(p-1)}{p}} \tag{2.11}$$

$$\times\left[1+\frac{1}{2}\sqrt{\left(\frac{p+2}{4}\right)^{\frac{6-p}{p-2}}\log\left(\frac{|Q|}{|K|}\right)}\right.$$

$$\left.+\frac{2p}{p-2}\frac{|K|}{|\Omega|}\left(\frac{1}{4}\left(\frac{p+2}{4}\right)^{\frac{6-p}{p-2}}\log\left(\frac{|Q|}{|K|}\right)\right)^{\frac{p+2}{4}}\right]^{\frac{2}{p}}\|\nabla u\|^2_{2,\,\Omega}.$$

Proof For $p>2$, del Pino-Dolbeault [5, Theorem 1] obtained the optimal constant for the following Gagliardo-Nirenberg inequality in $\mathbb{R}^2$:

$$\|u\|_{p,\,\Omega}\le\pi^{\frac{2-p}{4p}}\left(\frac{p+2}{4}\right)^{\frac{p-6}{4p}}\|\nabla u\|^{\frac{p-2}{2p}}_{2,\,\Omega}\,\|u\|^{\frac{p+2}{2p}}_{\frac{p}{2}+1,\,\Omega}\qquad\forall u\in H^1_0(\Omega). \tag{2.12}$$

As in Theorem 2.1, we notice that functions in $H^1_0(\Omega)$ may be extended by zero outside Q, so they can be seen as functions defined over the whole plane. For general exponents, the optimal constant in the Gagliardo-Nirenberg inequality is not known, this is why we introduce the $L^{\frac{p}{2}+1}$-norm. By combining (2.12) with the following form of the Hölder inequality

$$\|u\|^{\frac{p}{2}+1}_{\frac{p}{2}+1,\,\Omega}\le\|u\|_{2,\,\Omega}\,\|u\|^{p/2}_{p,\,\Omega}\qquad\forall u\in L^p(\Omega)\,,$$

we infer that

$$\|u\|^2_{p,\,\Omega}\le\pi^{\frac{2-p}{p}}\left(\frac{p+2}{4}\right)^{\frac{p-6}{p}}\|\nabla u\|^{\frac{2(p-2)}{p}}_{2,\,\Omega}\,\|u\|^{4/p}_{2,\,\Omega}\qquad\forall u\in H^1_0(\Omega)\,. \tag{2.13}$$

Then (2.10) is obtained after inserting (2.4) into (2.13).

The proof of (2.11) follows exactly the procedure employed in the proof of inequality (2.3) given in Theorem 2.1, and therefore is omitted here. □

Remark 2.2 Notice that the minimum in (2.2) and (2.10) is the consequence of the Poincaré inequality (2.4) that we only use in the space $H_0^1(\Omega)$. In particular, from (2.10) we deduce that

$$\|u\|_{p,\Omega}^2 \leq \frac{|\Omega|^{2/p}}{\pi \mu_0^{4/p}} \left(\frac{p+2}{4}\right)^{\frac{p-6}{p}} \|\nabla u\|_{2,\Omega}^2 \qquad \text{for all } u \in H_0^1(\Omega) \text{ and } p > 2. \tag{2.14}$$

On the other hand, by applying firstly (2.12) and then Hölder's inequality, we also have

$$\|u\|_{p,\Omega}^2 \leq \frac{|\Omega|^{2/p}}{\pi} \left(\frac{p+2}{4}\right)^{\frac{p-6}{p-2}} \|\nabla u\|_{2,\Omega}^2 \qquad \text{for all } u \in H_0^1(\Omega) \text{ and } p > 2. \tag{2.15}$$

The ratio between the constants appearing in the right-hand sides of (2.14) and (2.15) is plotted in Fig. 2 as a function of $p > 2$, showing that the smallest constant corresponds to (2.14).

Theorems 2.1 and 2.2 yield (unpleasant) lower bounds for the Sobolev constants in (1.3): it suffices to take the inverse of the constants appearing in (2.2), (2.3), (2.10) and (2.11). The lower bounds for $\mathcal{S}_p^1$ may be treated as functions of $|Q|/|K| \in [1, \infty)$: regardless of the value of $p > 1$, they vanish like $\left[\log(|Q|/|K|)\right]^{-1}$ as $|Q|/|K| \to \infty$, see the plots in Fig. 3 where we also compare them with the (larger) lower bound for $\mathcal{S}_p^0$. One should also compare this uniform asymptotic behavior with the result of Theorem 2.3.

In the case when K is a square, the explicit lower bounds for $\mathcal{S}_p^1$ are as follows:

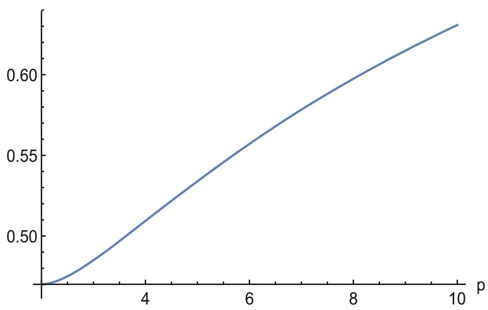

Fig. 2 Ratio between the embedding constants given in (2.14) and (2.15)

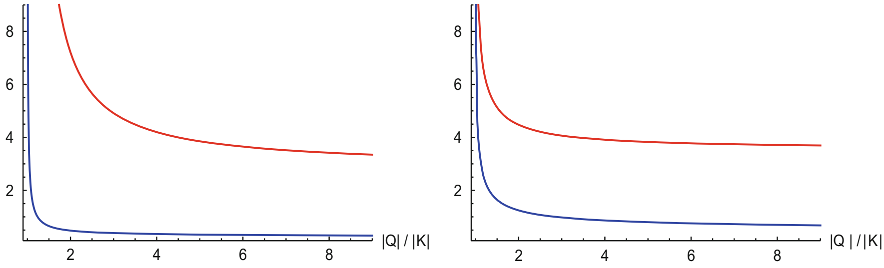

Fig. 3 Behavior of the lower bounds for $\mathcal{S}_0^p$ (red) and $\mathcal{S}_1^p$ (blue) as functions of $|Q|/|K|$, when $p = 3/2$ (left) and $p = 6$ (right)

Corollary 2.1 *For $0 < a < L$, suppose that $K = (-a, a)^2$. Then, for every $1 < p < 2$ we have*

$$\mathcal{S}_p^1 \geq L^{-4/p} \frac{\pi}{4^{2/p}} \left[1 - \left(\frac{L}{a}\right)^{-2}\right]^{-\frac{2}{p}} \left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \left(1 + \sqrt{\frac{1}{2}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \log\left(\frac{L}{a}\right)}\right)^{\frac{2(1-p)}{p}}$$

$$\times \left[1 + \sqrt{\frac{1}{2}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \log\left(\frac{L}{a}\right)} \right.$$

$$\left. + \frac{p}{2-p} \frac{1}{\left(\frac{L}{a}\right)^2 - 1} \left(\frac{1}{2}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \log\left(\frac{L}{a}\right)\right)^{p-1}\right]^{-\frac{2}{p}},$$

and for every $p > 2$ we have

$$\mathcal{S}_p^1 \geq L^{-4/p} \frac{\pi}{4^{2/p}} \left[1 - \left(\frac{L}{a}\right)^{-2}\right]^{-\frac{2}{p}}$$

$$\times \left(\frac{p+2}{4}\right)^{\frac{6-p}{p-2}} \left(1 + \sqrt{\frac{1}{2}\left(\frac{p+2}{4}\right)^{\frac{6-p}{p-2}} \log\left(\frac{L}{a}\right)}\right)^{\frac{2(1-p)}{p}}$$

$$\times \left[1 + \sqrt{\frac{1}{2}\left(\frac{p+2}{4}\right)^{\frac{6-p}{p-2}} \log\left(\frac{L}{a}\right)} \right.$$

$$\left. + \frac{2p}{p-2} \frac{1}{\left(\frac{L}{a}\right)^2 - 1} \left(\frac{1}{2}\left(\frac{p+2}{4}\right)^{\frac{6-p}{p-2}} \log\left(\frac{L}{a}\right)\right)^{\frac{p+2}{4}}\right]^{-\frac{2}{p}}.$$

Problem 2.1 The bounds obtained in Theorems 2.1 and 2.2 for the Sobolev constants merely depend on the measure of the obstacle K but they *do not depend on its position nor on its shape*. It is natural to conjecture that obstacles close to ∂Q might generate larger Sobolev constants. Moreover, it is well-known that Steiner symmetrization [10] preserves the L^p norms of functions and reduces their Dirichlet norm, see [2–4, 11] and references therein. In our 2D setting, the Steiner symmetrization produces rearrangements that gain symmetry about a line. We are here interested in a finite number of iterations by symmetrizing about the four lines $x = 0$, $y = 0$ and $y = \pm x$, namely the axes of symmetry of Q. Then, it appears interesting to find the shape and the position of the optimal obstacle minimizing the Sobolev constants among obstacles K of given measure.

2.2 Upper Bounds

It is natural to wonder whether the lower bounds for $\mathcal{S}_p^0$ and $\mathcal{S}_p^1$ so far obtained are accurate. This can be tested through suitable upper bounds. For $\mathcal{S}_p^0$ we take the function $w(x, y) = \cos(\frac{\pi x}{2L})\cos(\frac{\pi y}{2L})$, defined for $(x, y) \in \overline{Q}$, so that $w \in H_0^1(Q)$ and

$$\|w\|_{p,Q}^2 = \left[\frac{2L}{\sqrt{\pi}}\,\frac{\Gamma\left(\frac{1+p}{2}\right)}{\Gamma\left(1+\frac{p}{2}\right)}\right]^{4/p}, \quad \|\nabla w\|_{2,Q}^2 = \frac{\pi^2}{2}$$

$$\Longrightarrow \quad \mathcal{S}_p^0 \leq \frac{\pi^2}{2}\left[\frac{\sqrt{\pi}}{2L}\,\frac{\Gamma\left(1+\frac{p}{2}\right)}{\Gamma\left(\frac{1+p}{2}\right)}\right]^{4/p}. \tag{2.16}$$

Notice that the upper bound (2.16) holds for any obstacle K.

In order to derive an upper bound for $\mathcal{S}_p^1$, we recall the definition of *pyramidal function*, introduced in [7, Theorem 2.2]. For $0 < d \leq a < L$, suppose that $K = (-a, a) \times (-d, d)$ and divide the domain Ω into four trapezia T_1, T_2, T_3, T_4 as in the left picture in Fig. 4. By *pyramidal function* we mean any function having the level lines as in the right picture of Fig. 4, namely level lines parallel to ∂Q (and to the rectangle K) in each of the trapezia. In particular, pyramidal functions are constant on ∂K and constitute the following convex subset of $H_0^1(Q)$:

$$\mathcal{P}(Q) = \{u \in H_0^1(Q) \mid u = 1 \text{ in } K,\ u = u(y) \text{ in } T_1 \cup T_3,\ u = u(x) \text{ in } T_2 \cup T_4\}. \tag{2.17}$$

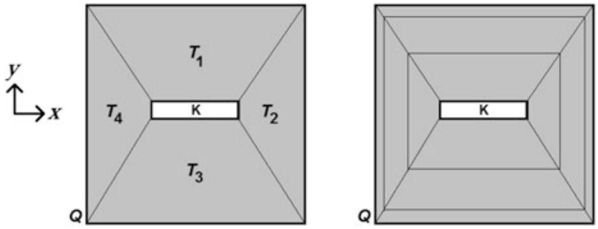

Fig. 4 The domain Ω (left) and the level lines of pyramidal functions (right)

Now, any $V^\phi \in \mathcal{P}(Q)$ is fully characterized by a (continuous) function

$$\phi \in H^1([0,1];\mathbb{R}) \quad \text{such that} \quad \phi(0) = 1\,, \quad \phi(1) = 0\,,$$

giving the values of V^ϕ on the oblique edges of the trapezia. For instance, consider the right trapezia $T_5, T_6 \subset Q$ being, respectively, half of the trapezia T_1 and T_2, defined by

$$T_5 = \left\{ (x,y) \in Q \;\middle|\; d < y < L,\ 0 < x < a + \frac{L-a}{L-d}(y-d) \right\},$$
$$T_6 = \left\{ (x,y) \in Q \;\middle|\; a < x < L,\ 0 < y < d + \frac{L-d}{L-a}(x-a) \right\}.$$

Since V^ϕ is a function of y in T_1 and a function of x in T_2, ϕ and V^ϕ are linked through the formulas

$$V^\phi(x,y) = \phi\left(\frac{y-d}{L-d}\right) \qquad \forall (x,y) \in T_5,$$
$$V^\phi(x,y) = \phi\left(\frac{x-a}{L-a}\right) \qquad \forall (x,y) \in T_6. \tag{2.18}$$

Whence,

$$\frac{\partial V^\phi}{\partial y}(x,y) = \frac{1}{L-d}\phi'\left(\frac{y-d}{L-d}\right) \quad \forall (x,y) \in T_5,$$
$$\frac{\partial V^\phi}{\partial x}(x,y) = \frac{1}{L-a}\phi'\left(\frac{x-a}{L-a}\right) \quad \forall (x,y) \in T_6. \tag{2.19}$$

To avoid tedious computations, we restrict again our attention to the case $d = a$ (squared obstacle). The next result gives an upper bound for the constants $\mathcal{S}_p^1$ and measures the precision of the bounds in the limit situation where K is a vanishing square. Interestingly, the ratio between our lower and upper bounds for $\mathcal{S}_p^1$ converges to a limit that is independent of p.

Theorem 2.3 *For $0 < a < L$, suppose that $K = (-a, a)^2$. Then, for every $p > 1$ ($p \neq 2$) we have*

$$\mathcal{S}_p^1 \leq \frac{8^{1-\frac{2}{p}}}{L^{4/p}} \left(\frac{L}{a}\right)^{4/p} \log\left(\frac{L}{a}\right) \left(\int_1^{L/a} t \log^p(t)\, dt\right)^{-2/p}. \tag{2.20}$$

Moreover, the ratio between the lower bounds in Corollary 2.1 and the upper bound (2.20) tends to $\pi/4 \approx 0.79$ as $L/a \to \infty$, ***independently of the value of*** *$p > 1$ ($p \neq 2$).*

Proof Let $\mathcal{P}(Q)$ be as in (2.17) and let $V^\phi \in \mathcal{P}(Q)$ be defined by (2.18) with

$$\phi(s) = \log\left(\frac{a + (L-a)s}{L}\right) \Big/ \log\left(\frac{a}{L}\right) \qquad \forall s \in [0, 1].$$

For symmetry reasons, the contribution of $|\nabla V^\phi|$ over $T_1 \cup T_3$ is four times the contribution over T_5, whereas the contribution of $|\nabla V^\phi|$ over $T_2 \cup T_4$ is four times the contribution over T_6. By taking into account all these facts, in particular (2.19), we infer that

$$\begin{aligned}
\|\nabla V^\phi\|_{2,\Omega}^2 &= 4 \int_a^L \int_0^y \left|\frac{\partial V^\phi}{\partial y}\right|^2 dx\, dy + 4 \int_a^L \int_0^x \left|\frac{\partial V^\phi}{\partial x}\right|^2 dy\, dx \\
&= 4 \int_a^L y \left|\frac{\partial V^\phi}{\partial y}\right|^2 dy + 4 \int_a^L x \left|\frac{\partial V^\phi}{\partial x}\right|^2 dx \\
&= \frac{8}{L-a} \int_0^1 [a + (L-a)s]\, \phi'(s)^2\, ds \\
&= 8 \left[\log\left(\frac{L}{a}\right)\right]^{-1}.
\end{aligned} \tag{2.21}$$

In a similar fashion, for every $p > 1$ we have

$$\begin{aligned}
\|1 - V^\phi\|_{p,\Omega}^p &= 4 \int_a^L \int_0^y \left|1 - V^\phi(y)\right|^p dx\, dy + 4 \int_a^L \int_0^x \left|1 - V^\phi(x)\right|^p dy\, dx \\
&= 4 \int_a^L y \left|1 - V^\phi(y)\right|^p dy + 4 \int_a^L x \left|1 - V^\phi(x)\right|^p dx \\
&= 8(L-a) \int_0^1 [a + (L-a)s]\, |1 - \phi(s)|^p\, ds.
\end{aligned}$$

Through the change of variable $t = a + (L - a)s$, for $s \in [0, 1]$, we then obtain

$$\|1 - V^\phi\|_{p,\Omega}^p = 8a^2 \left[\log\left(\frac{L}{a}\right)\right]^{-p} \int_1^{L/a} t \log^p(t)\, dt. \tag{2.22}$$

We finally notice that if $v \in \mathcal{P}(Q)$, then $1 - v \in H_c^1(\Omega)$ with $v = 1$ on ∂Q. Therefore,

$$\mathcal{S}_p^1 \le \min_{v\in\mathcal{P}(Q)} \frac{\|\nabla v\|_{2,\Omega}^2}{\|1 - v\|_{p,\Omega}^2} \le \frac{\|\nabla V^\phi\|_{2,\Omega}^2}{\|1 - V^\phi\|_{p,\Omega}^2} \qquad \forall p > 1,$$

which yields (2.20) in view of (2.21) and (2.22).

Next, for any $1 < p < 2$, denote by $\mathcal{R}(z)$ the ratio between the lower bound for $\mathcal{S}_p^1$ given in Corollary 2.1 and the just proved upper bound (2.20), as a function of $z = L/a$: for every $z > 1$ we have

$$\mathcal{R}(z) = \frac{\pi\, 2^{\frac{2}{p}-3} \left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \left(\frac{1}{z^2-1} \int\limits_1^z t \log^p(t)\, dt\right)^{\frac{2}{p}} \left(1 + \sqrt{\frac{1}{2}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \log(z)}\right)^{\frac{2(1-p)}{p}}}{\log(z) \left[1 + \sqrt{\frac{1}{2}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \log(z)} + \frac{p}{2-p} \frac{1}{z^2-1} \left(\frac{1}{2}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \log(z)\right)^{p-1}\right]^{\frac{2}{p}}},$$

so that

$$\mathcal{R}(z) \sim \pi\, 2^{\frac{2}{p}-2} \left(\frac{1}{z^2 \log^p(z)} \int\limits_1^z t \log^p(t)\, dt\right)^{\frac{2}{p}} \qquad \text{as } z \to \infty.$$

An application of L'Hôpital's rule yields

$$\lim_{z\to\infty} \frac{1}{z^2 \log^p(z)} \int\limits_1^z t \log^p(t)\, dt = \frac{1}{2},$$

which concludes the proof, since the limit in the case $p > 2$ can be treated exactly in the same way. □

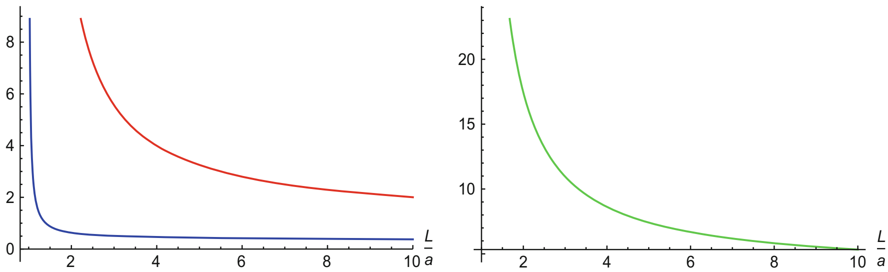

Fig. 5 On the left: behavior of the lower and upper bounds for $\mathcal{S}_3^1$ as a function of L/a. On the right: ratio between the upper and lower bounds for $\mathcal{S}_3^1$ as a function of L/a

Remark 2.3 If $p > 1$ is an integer we may explicitly compute

$$\int_1^{L/a} t \log^p(t)\, dt = p! \left[\left(\frac{L}{a}\right)^2 \sum_{k=0}^{p} \frac{(-1)^k}{2^{k+1}} \frac{1}{(p-k)!} \left(\log\left(\frac{L}{a}\right)\right)^{p-k} - \frac{(-1)^p}{2^{p+1}} \right].$$

By dropping the multiplicative term $L^{-4/p}$, the lower and upper bounds for $\mathcal{S}_p^1$ in Corollary 2.1 and in Theorem 2.3 can be treated as functions of $L/a \in (1, \infty)$. The plots in Fig. 5 describe the overall behavior for $p = 3$. Qualitatively, the same plots are found for any value of $p > 1$ ($p \neq 2$).

3 Failure of Elementary Symmetrization Methods

Theorems 2.1 and 2.2 may be extended to any space dimension $n \geq 3$ and any $1 < p < \frac{2n}{n-2}$ but the question whether they might be improved arises naturally. In particular, one wonders whether some symmetrization techniques [11] could be used. In this section we show that, at least in its simplest forms, symmetrization is of no help: we argue in any space dimension $n \geq 2$ because this creates no additional difficulties.

For any $R > 0$ we denote by $B_R \subset \mathbb{R}^n$ the n-dimensional ball of radius R centered at the origin. In the next statement we show that if we compare the Dirichlet norm of a (radial) function in this annulus with that of its decreasing rearrangement, nothing can be said a priori: both inequalities may occur.

Theorem 3.1 *There exist radial functions* $f_1, f_2, f_3, f_4 \in H_c^1(B_2 \setminus B_1)$ *such that*

$$\|\nabla f_1\|_{2,\, B_2\setminus B_1} < \|\nabla g_1\|_{2,\, B_2\setminus B_1}\,, \qquad \|\nabla f_2\|_{2,\, B_2\setminus B_1} > \|\nabla g_2\|_{2,\, B_2\setminus B_1} \tag{3.1}$$

and, for $R = \sqrt[n]{2^n - 1}$,

$$\|\nabla f_3\|_{2,\,B_2\setminus B_1} < \|\nabla g_3\|_{2,\,B_R}\,, \qquad \|\nabla f_4\|_{2,\,B_2\setminus B_1} > \|\nabla g_4\|_{2,\,B_R}\,, \tag{3.2}$$

where g_i *denotes the decreasing rearrangement of* f_i*, for* $i \in \{1, 2, 3, 4\}$.

Proof First we prove (3.1). In the annulus $B_2 \setminus B_1$, take any positive strictly increasing radial function $f = f(r)$ over the interval $[1, 2]$ such that $f(1) = 0$ and $f(2) = 1$. Its decreasing rearrangement within the annulus is given by

$$g(r) = f\Big(\sqrt[n]{2^n + 1 - r^n}\Big) \qquad \forall r \in (1, 2).$$

Hence, as expected, we have

$$\begin{aligned}\int_1^2 r^{n-1} g(r)^p\, dr &= \int_1^2 r^{n-1} f\Big(\sqrt[n]{2^n + 1 - r^n}\Big)^p\, dr \\ &= \int_1^2 t^{n-1} f(t)^p\, dt \qquad \forall p > 1,\end{aligned}$$

where we used the change of variables

$$t = \sqrt[n]{2^n + 1 - r^n} \quad \Longleftrightarrow \quad r = \sqrt[n]{2^n + 1 - t^n}. \tag{3.3}$$

On the other hand, we have

$$g'(r) = -r^{n-1}\Big(2^n + 1 - r^n\Big)^{\frac{1}{n}-1} f'\Big(\sqrt[n]{2^n + 1 - r^n}\Big) \qquad \forall r \in (1, 2),$$

so that, using again (3.3),

$$\begin{aligned}\int_1^2 r^{n-1} f'(r)^2\, dr &= \int_1^2 t^{n-1} f'\Big(\sqrt[n]{2^n + 1 - t^n}\Big)^2\, dt \\ &= \int_1^2 \frac{\Big(2^n + 1 - t^n\Big)^{2-\frac{2}{n}}}{t^{n-1}} g'(t)^2\, dt.\end{aligned}$$

The "break even" in the integral occurs whenever

$$\frac{\Big(2^n + 1 - t^n\Big)^{2-\frac{2}{n}}}{t^{n-1}} = t^{n-1} \iff t = r^* \doteq \left(2^{n-1} + \frac{1}{2}\right)^{\frac{1}{n}}.$$

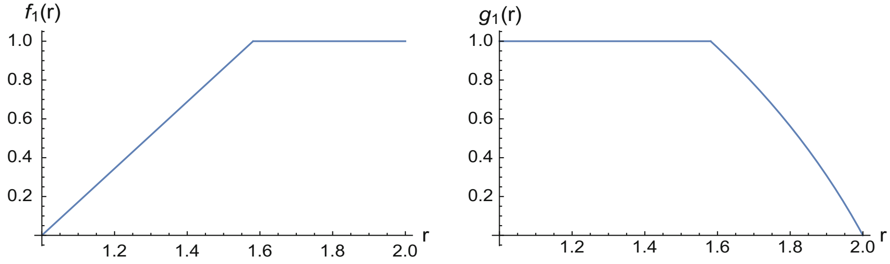

Fig. 6 Plot of f_1 and of its symmetric decreasing rearrangement g_1, with $n = 2$ and $r^* = \sqrt{5/2}$

Let us consider first the function (see Fig. 6 when $n = 2$)

$$f_1(r) = \begin{cases} \frac{r-1}{r^*-1} & \text{if } 1 < r \le r^* \\ 1 & \text{if } r^* \le r < 2 \end{cases} \qquad \Longrightarrow \qquad f_1'(r) = \begin{cases} \frac{1}{r^*-1} & \text{if } 1 < r < r^* \\ 0 & \text{if } r^* < r < 2 \end{cases},$$

so that

$$g_1(r) = \begin{cases} 1 & \text{if } 1 < r \le r^* \\ \frac{\sqrt[n]{2^n+1-r^n}-1}{r^*-1} & \text{if } r^* \le r < 2 \end{cases} \qquad \Longrightarrow$$

$$g_1'(r) = \begin{cases} 0 & \text{if } 1 < r < r^* \\ \frac{-r^{n-1}}{r^*-1}\,\frac{1}{(2^n+1-r^n)^{1-1/n}} & \text{if } r^* < r < 2\,. \end{cases}$$

Then we have

$$\int_1^2 r^{n-1} f_1'(r)^2\,dr < \int_1^2 t^{n-1} g_1'(t)^2\,dt\,,$$

which proves the first of (3.1).

Next, consider the function (see Fig. 7 when $n = 2$)

$$f_2(r) = \begin{cases} 0 & \text{if } 1 < r \le r^* \\ \frac{r-r^*}{2-r^*} & \text{if } r^* \le r < 2 \end{cases} \qquad \Longrightarrow \qquad f_2'(r) = \begin{cases} 0 & \text{if } 1 < r < r^* \\ \frac{1}{2-r^*} & \text{if } r^* < r < 2 \end{cases},$$

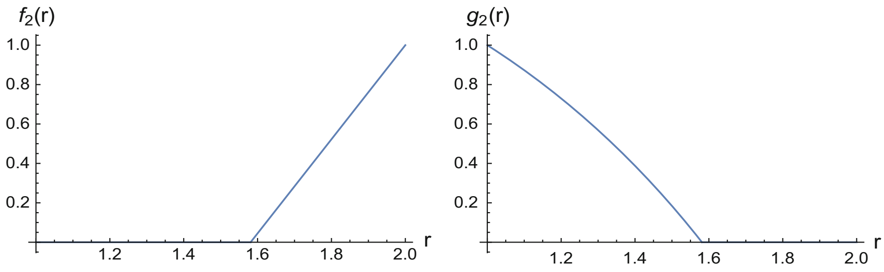

Fig. 7 Plot of f_2 and of its symmetric decreasing rearrangement g_2, with $n = 2$ and $r^* = \sqrt{5/2}$

so that

$$g_2(r) = \begin{cases} \frac{\sqrt[n]{2^n+1-r^n}-r^*}{2-r^*} & \text{if } 1 < r \le r^* \\ 0 & \text{if } r^* \le r < 2 \end{cases} \implies$$

$$g_2'(r) = \begin{cases} \frac{-r^{n-1}}{2-r^*} \frac{1}{(2^n+1-r^n)^{1-1/n}} & \text{if } 1 < r < r^* \\ 0 & \text{if } r^* < r < 2\,. \end{cases}$$

Then we have

$$\int_1^2 r^{n-1} f_2'(r)^2 \, dr > \int_1^2 t^{n-1} g_2'(t)^2 \, dt\,,$$

which proves the second inequality in (3.1).

Let us now prove (3.2). Notice that $|B_2 \setminus B_1| = \omega_n(2^n - 1)$, where ω_n is the measure of the unit ball B_1. Hence, the disk D of radius $R = \sqrt[n]{2^n - 1}$ has the same measure as $B_2 \setminus B_1$ so that $B_R = (B_2 \setminus B_1)^*$. Consider a positive strictly increasing radial function $f = f(r)$ over the interval $(1, 2)$, then its decreasing rearrangement within the disc B_R is given by

$$g(r) = f\left(\sqrt[n]{2^n - r^n}\right) \qquad \forall r \in \left(0, \sqrt[n]{2^n - 1}\right).$$

We have again

$$\int_0^{\sqrt[n]{2^n-1}} r^{n-1} g(r)^p \, dr = \int_1^2 t^{n-1} f(t)^p \, dt \qquad \forall p > 1.$$

where we used the change of variables

$$t = \sqrt[n]{2^n - r^n} \iff r = \sqrt[n]{2^n - t^n}. \tag{3.4}$$

On the other hand, we have

$$g'(r) = -r^{n-1}\Big(2^n - r^n\Big)^{\frac{1}{n}-1} f'\Big(\sqrt[n]{2^n - r^n}\Big) \qquad \forall r \in \Big(0, \sqrt[n]{2^n - 1}\Big).$$

so that, using again (3.4),

$$\int_1^2 r^{n-1} f'(r)^2\, dr = \int_0^{\sqrt[n]{2^n-1}} \frac{\Big(2^n - t^n\Big)^{2-\frac{2}{n}}}{t^{n-1}} g'(t)^2\, dt.$$

The "break even" in the integral occurs whenever

$$\frac{\Big(2^n - t^n\Big)^{2-\frac{2}{n}}}{t^{n-1}} = t^{n-1} \iff t = r^* \doteq 2^{1-\frac{1}{n}}.$$

Let us consider first the function (see Fig. 8 when $n = 2$)

$$f_3(r) = \begin{cases} \frac{r-1}{r^*-1} & \text{if } 1 < r \le r^* \\ 1 & \text{if } r^* \le r < 2 \end{cases} \qquad \Longrightarrow \qquad f_3'(r) = \begin{cases} \frac{1}{r^*-1} & \text{if } 1 < r < r^* \\ 0 & \text{if } r^* < r < 2 \end{cases},$$

so that

$$g_3(r) = \begin{cases} 1 & \text{if } 0 < r < r^* \\ \frac{\sqrt[n]{2^n - r^n} - 1}{r^* - 1} & \text{if } r^* < r < \sqrt[n]{2^n - 1} \end{cases}$$

and

$$g_3'(r) = \begin{cases} 0 & \text{if } 0 < r < r^* \\ \frac{-r^{n-1}}{r^*-1} \frac{1}{(2^n - r^n)^{1-1/n}} & \text{if } r^* < r < \sqrt[n]{2^n - 1}. \end{cases}$$

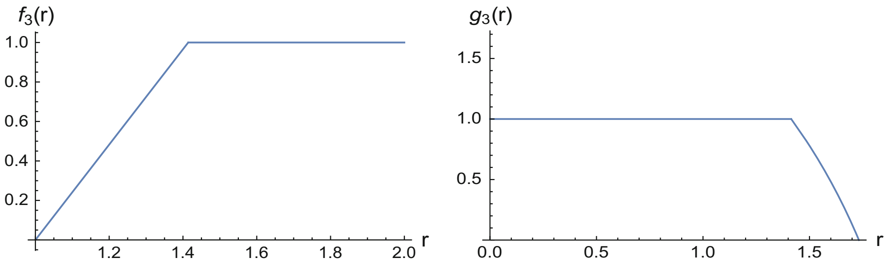

Fig. 8 Plot of f_3 and of its symmetric decreasing rearrangement g_3, with $n = 2$ and $r^* = \sqrt{2}$

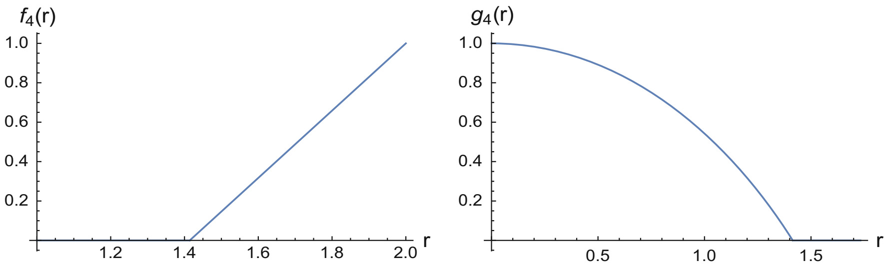

Fig. 9 Plot of f_4 and of its symmetric decreasing rearrangement g_4, with $n = 2$ and $r^* = \sqrt{2}$

Then

$$\int_1^2 r^{n-1} f_3'(r)^2\,dr < \int_0^{\sqrt[n]{2^n-1}} t^{n-1} g_3'(t)^2\,dt\,,$$

thereby proving the first inequality in (3.2).

Finally, consider the function (see Fig. 9 when $n = 2$)

$$f_4(r) = \begin{cases} 0 & \text{if } 1 < r \le r^* \\ \frac{r-r^*}{2-r^*} & \text{if } r^* \le r < 2 \end{cases} \qquad \Longrightarrow \qquad f_4'(r) = \begin{cases} 0 & \text{if } 1 < r < r^* \\ \frac{1}{2-r^*} & \text{if } r^* < r < 2 \end{cases},$$

so that

$$g_4(r) = \begin{cases} \frac{\sqrt[n]{2^n-r^n}-r^*}{2-r^*} & \text{if } 0 < r \le r^* \\ 0 & \text{if } r^* \le r < \sqrt[n]{2^n-1} \end{cases}$$

and

$$g_4'(r) = \begin{cases} \frac{-r^{n-1}}{2-r^*}\,\frac{1}{(2^n-r^n)^{1-1/n}} & \text{if } 0 < r < r^* \\ 0 & \text{if } r^* < r < \sqrt[n]{2^n-1}\,. \end{cases}$$

Then we have

$$\int_1^2 r^{n-1} f_4'(r)^2\,dr > \int_0^{\sqrt[n]{2^n-1}} t^{n-1} g_4'(t)^2\,dt\,,$$

proving also the second inequality in (3.2). □

One then naturally wonders if a result similar to Theorem 3.1 holds in *any* non-simply connected domain, that is

Problem 3.1 Let $\Omega \subset \mathbb{R}^n$ be the difference between two simply connected bounded convex domains Q and K such that $K \subset Q$ and $\partial K \cap \partial Q = \emptyset$. Define $H^1_c(\Omega)$ as in (1.2), and for any $f \in H^1_c(\Omega)$, let f^* be the symmetric decreasing rearrangement of f on Ω^*, the n-dimensional ball having the same measure as Ω. Does there exists a break even $(n-1)$-dimensional surface such that if $f \in H^1_c(\Omega)$ concentrates its mass inside (resp. outside) this surfaces, then the Dirichlet norm of f in Ω is strictly smaller (resp. larger) than the Dirichlet norm of f^* in Ω^*? The same question may be formulated by considering the symmetric decreasing rearrangement of f on the annulus whose inner ball has the same measure of K and whose outer ball has the same measure as Q.

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References

1. Abdellaoui, B., Colorado, E., Peral, I.: Some critical quasilinear elliptic problems with mixed Dirichlet-Neumann boundary conditions: relation with Sobolev and Hardy-Sobolev optimal constants. J. Math. Anal. Appl. **332**(2), 1165–1188 (2007)
2. Alvino, A., Trombetti, G., Lions, P.-L.: On optimization problems with prescribed rearrangements. Nonlinear Anal. **13**, 185–220 (1989)
3. Brothers, J., Ziemer, W.: Minimal rearrangements of Sobolev functions. J. Reine Angew. Math. **384**, 153–179 (1988)
4. Burchard, A.: Steiner symmetrization is continuous in $W^{1,p}$. Geom. Funct. Anal. **7**, 823–860 (1997)
5. del Pino, M., Dolbeault, J.: Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. J. Math. Pures Appl. **81**, 847–875 (2002)
6. Faber, G.: Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt, pp. 169–172. Verlag der Bayerischen Akademie der Wissenschaften, München (1923)
7. Gazzola, F., Sperone, G.: Steady Navier-Stokes equations in planar domains with obstacle and explicit bounds for unique solvability. Arch. Ration. Mech. Anal. **238**(3), 1283–1347 (2020)
8. Krahn, E.: Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises. Math. Ann. **94**(1), 97–100 (1925)
9. Lions, P.-L., Pacella, F., Tricarico, M.: Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions. Indiana Univ. Math. J. **37**(2), 301–324 (1988)
10. Steiner, J.: Einfacher Beweise der isoperimetrischen Hauptsätze. J. Reine Angew. Math. **18**, 281–296 (1838). Reprinted in Gesammelte Werke 2 (Berlin 1882), pp. 77–91 (1838)
11. Talenti, G.: The art of rearranging. Milan J. Math. **84**(1), 105–157 (2016)

Sharp Estimate of the Life Span of Solutions to the Heat Equation with a Nonlinear Boundary Condition

Kotaro Hisa

Abstract Consider the heat equation with a nonlinear boundary condition

$$
\text{(P)} \qquad \begin{cases} \partial_t u = \Delta u, & x \in \mathbf{R}^N_+, \quad t > 0, \\ -\dfrac{\partial u}{\partial x_N} u = u^p, & x \in \partial \mathbf{R}^N_+, \quad t > 0, \\ u(x, 0) = \kappa \psi(x), & x \in \overline{\mathbf{R}^N_+}, \end{cases}
$$

where $N \geq 1$, $p > 1$, $\kappa > 0$ and ψ is a nonnegative measurable function in $\mathbf{R}^N_+ := \{y \in \mathbf{R}^N : y_N > 0\}$. Let us denote by $T(\kappa\psi)$ the life span of solutions to problem (P). We investigate the relationship between the singularity of ψ at the origin and $T(\kappa\psi)$ for sufficiently large $\kappa > 0$ and the relationship between the behavior of ψ at the space infinity and $T(\kappa\psi)$ for sufficiently small $\kappa > 0$. Moreover, we obtain sharp estimates of $T(\kappa\psi)$, as $\kappa \to \infty$ or $\kappa \to +0$.

Keywords Life span · Heat equation · Nonlinear boundary condition · Blow-up

1 Introduction

Consider the heat equation with a nonlinear boundary condition

$$
\begin{cases} \partial_t u = \Delta u, & x \in \mathbf{R}^N_+, \quad t > 0, \\ -\dfrac{\partial u}{\partial x_N} = u^p, & x \in \partial \mathbf{R}^N_+, \quad t > 0, \end{cases} \tag{1.1}
$$

K. Hisa (✉)
Department of Mathematics, Tokyo Institute of Technology, Meguro-ku, Tokyo, Japan
e-mail: hisa.k.ab@m.titech.ac.jp

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with the initial condition

$$u(x,0) = \kappa\psi(x), \qquad x \in D := \overline{\mathbf{R}^N_+}, \tag{1.2}$$

where $N \geq 1$, $p > 1$, $\kappa > 0$ and ψ is a nonnegative measurable function in $\mathbf{R}^N_+ := \{y \in \mathbf{R}^N : y_N > 0\}$. Let $T(\kappa\psi)$ denote the maximal existence time of the minimal solution to problem (1.1) with (1.2). We call $T(\kappa\psi)$ the life span of solutions to problem (1.1) with (1.2) (see Definitions 1.1 and 1.2). The life spans depend on a lot of factors such as diffusion effect, nonlinearity of equations, boundary conditions and the singularity or the decay of initial functions and they have been studied, see e.g., [5, 10, 13, 14]. For related results on semilinear parabolic equations, see e.g., [4, 5, 8–11, 13–21] and references therein.

Problem (1.1) can be physically interpreted as a nonlinear radiation law and it has been studied in many papers (see e.g., [1–3, 5–7, 10, 12–14]). Among others, the author of this paper and Ishige [10] obtained necessary conditions and sufficient conditions for the solvability of problem (1.1) and identified the strongest singularity of the initial function for the existence of solutions to problem (1.1). In this paper, applying the results in [10], we obtain sharp estimates of the life span $T(\kappa\psi)$ as $\kappa \to \infty$ or $\kappa \to +0$ and show that the behavior of the life span $T(\kappa\psi)$ as $\kappa \to \infty$ and $\kappa \to +0$ depends on the singularity and the decay of ψ, respectively. The proofs of our results require careful treatments of parameters in the results in [10].

Before stating the main results of this paper, we have to define the life span $T(\kappa\psi)$ of solutions to (1.1) with (1.2) exactly. To do that, we formulate the definition of solutions to (1.1). Let $G = G(x, y, t)$ be the Green function for the heat equation on $\mathbf{R}^N_+$ with the homogeneous Neumann boundary condition. For $y = (y_1, \cdots, y_N) \in \mathbf{R}^N$, y' is given by $y' = (y_1, \cdots, y_{N-1})$.

Definition 1.1 Let u be a nonnegative and continuous function in $D \times (0, T)$, where $0 < T < \infty$.

- Let φ be a nonnegative measurable function in $\mathbf{R}^N_+$. We say that u is a solution to (1.1) in $[0, T)$ with $u(0) = \varphi$ if u satisfies

$$u(x,t) = \int_D G(x,y,t)\varphi(y)\,dy + \int_0^t \int_{\mathbf{R}^{N-1}} G(x,y',0,t-s)u(y',0,s)^p\,dy'ds$$

for $(x, t) \in D \times (0, T)$.
- We say that u is a minimal solution to (1.1) in $[0, T)$ with $u(0) = \varphi$ if u is a solution to (1.1) in $[0, T)$ with $u(0) = \varphi$ and satisfies

$$u(x,t) \leq w(x,t) \quad \text{in} \quad D \times (0,T)$$

for any solution w to (1.1) in $[0, T)$ with $w(0) = \varphi$.

Remark 1.1 Let u be a solution to problem (1.1) with $u(0) = \varphi$ in the sense of Definition 1.1. Then u satisfies the initial condition in the sense of distributions, that is,

$$\lim_{t\to+0}\int_D u(y,t)\eta(y)\,dy = \int_D \varphi(y)\eta(y)\,dy$$

for all $\eta \in C_0(\mathbf{R}^N)$.

Since the minimal solution is unique, we can define the life span $T(\kappa\psi)$ as follows:

Definition 1.2 The life span $T(\kappa\psi)$ of solutions to (1.1) with (1.2) is defined by the maximal existence time of the minimal solution to (1.1) with (1.2).

Next, we set up notation. For any $x \in \mathbf{R}^N$ and $r > 0$, set

$$B_+(x,r) := \{y \in \mathbf{R}^N : |x-y| < r\} \cap D.$$

For any set E, let χ_E be the characteristic function which has value 1 in E and value 0 outside E. For any two nonnegative functions f_1 and f_2 defined in $(0,\infty)$, we write $f_1(\tau) \sim f_2(\tau)$ as $\tau \to \infty$(resp. $+0$) if there exists a constant $C > 0$ such that $C^{-1} f_2(\tau) \le f_1(\tau) \le C f_1(\tau)$ for sufficiently large (resp. small) $\tau > 0$.

Now we are ready to state the main results of this paper. In Theorem 1.1 we obtain the relationship between the singularity of ψ and the life span $T(\kappa\psi)$ as $\kappa \to \infty$ and give sharp estimates to the life span as $\kappa \to \infty$. Appendix contains a brief summary of Theorem 1.1 (see Tables 1, 2 and 3 in Appendix). In what follows we set $p_* := 1 + 1/N$.

Theorem 1.1 *Assume that*

$$\psi(x) := |x|^{-A}\left[\log\left(e + \frac{1}{|x|}\right)\right]^{-B} \chi_{B_+(0,1)}(x) \in L^1(\mathbf{R}^N_+) \setminus L^\infty(\mathbf{R}^N_+),$$

where $0 \le A \le N$ *and*

$$B > 0 \quad \text{if} \quad A = 0, \qquad B \in \mathbf{R} \quad \text{if} \quad 0 < A < N, \qquad B > 1 \quad \text{if} \quad A = N. \tag{1.3}$$

Then $T(\kappa\psi) \to 0$ *as* $\kappa \to \infty$ *and the following holds:*

(i) $T(\kappa\psi)$ *satisfies*

$$T(\kappa\psi) \sim \begin{cases} \left[\kappa(\log\kappa)^{-B}\right]^{-\frac{2(p-1)}{-A(p-1)+1}} & \text{if} \quad A < \min\left\{N, \dfrac{1}{p-1}\right\}, \\ \left[\kappa(\log\kappa)^{-B+1}\right]^{-\frac{2(p-1)}{-A(p-1)+1}} & \text{if} \quad 1 < p < p_*, \ A = N, \ B > 1, \end{cases}$$

and

$$|\log T(\kappa\psi)| \sim \begin{cases} \kappa^{\frac{1}{B}} & \text{if} \quad p > p_*, \ A = \dfrac{1}{p-1}, \ B > 0, \\ \kappa^{\frac{1}{B-N-1}} & \text{if} \quad p = p_*, \ A = N, \ B > N+1, \end{cases}$$

as $\kappa \to \infty$*;*

(ii) *Let* $p > p_*$. *If, either*

$$A > 1/(p-1) \quad \text{and} \quad B \in \mathbf{R} \qquad \text{or} \qquad A = 1/(p-1) \quad \text{and} \quad B < 0,$$

then problem (1.1) *with* (1.2) *possesses no local-in-time solutions for all* $\kappa > 0$. *If*

$$A = 1/(p-1) \quad \text{and} \quad B = 0,$$

then problem (1.1) *with* (1.2) *possesses no local-in-time solutions for sufficiently large* $\kappa > 0$*;*

(iii) *Let* $p = p_*$. *If*

$$A = N \quad \text{and} \quad B < N+1,$$

then problem (1.1) *with* (1.2) *possesses no local-in-time solutions for all* $\kappa > 0$. *If*

$$A = N \quad \text{and} \quad B = N+1,$$

then problem (1.1) *with* (1.2) *possesses no local-in-time solutions for sufficiently large* $\kappa > 0$.

We remark that when ψ is as in Theorem 1.1, ψ satisfies (1.3) if and only if $\psi \in L^1_{loc}(\mathbf{R}^N_+)$. It is obvious that $T(\kappa\psi) = 0$ for all $\kappa > 0$ if (1.3) does not hold.

Remark 1.2 When $B = 0$, Ishige and Sato [13] have already obtained sharp estimates of the life span $T(\kappa\psi)$ as $\kappa \to \infty$ in the case when $\psi(x) = |x|^{-A}$ in a neighborhood of the origin, where

$$0 \le A < N \quad \text{if} \quad 1 < p < p_* \qquad \text{and} \qquad 0 \le A < \frac{1}{p-1} \quad \text{if} \quad p \ge p_*,$$

and proved that $T(\kappa\psi) \sim \kappa^{-\frac{2(p-1)}{-A(p-1)+1}}$ as $\kappa \to \infty$. This also follows from Theorem 1.1.

Remark 1.3 Let $N = 1$ and let ψ be a continuous, positive and bounded function in $\mathbf{R}$. Fernández Bonder and Rossi [5] obtained the precise asymptotic behavior of the life span $T(\kappa\psi)$ as $\kappa \to \infty$, that is, $\lim_{\kappa\to\infty} \kappa^{2(p-1)} T(\kappa\psi) = T(\psi(0))$.

Theorem 1.2 gives sharp estimates to the life span $T(\kappa\psi)$ as $\kappa \to +0$ with ψ behaving like $|x|^{-A} (A > 0)$ at the space infinity. Appendix contains a brief summary of Theorem 1.2 (see Tables 4 and 5 in Appendix).

Theorem 1.2 *Let $A > 0$ and $\psi(x) = (1 + |x|)^{-A}$. Then $T(\kappa\psi) \to \infty$ as $\kappa \to 0$ and the following holds:*

(i) *Let $1 < p < p_*$ or $0 < A < 1/(p-1)$. Then*

$$T(\kappa\psi) \sim \begin{cases} \kappa^{-\left(\frac{1}{2(p-1)} - \frac{1}{2}\min\{A,N\}\right)^{-1}} & \text{if} \quad A \neq N, \\ \left(\dfrac{\kappa^{-1}}{\log(\kappa^{-1})}\right)^{\left(\frac{1}{2(p-1)} - \frac{N}{2}\right)^{-1}} & \text{if} \quad A = N, \end{cases}$$

as $\kappa \to +0$;

(ii) *Let $p = p_*$ and $A \geq 1/(p-1)$. Then*

$$\log T(\kappa\psi) \sim \begin{cases} \kappa^{-(p-1)} & \text{if} \quad A > N, \\ \kappa^{-\frac{p-1}{p}} & \text{if} \quad A = N, \end{cases}$$

as $\kappa \to +0$;

(iii) *Let $p > p_*$ and $A \geq 1/(p-1)$. Then problem (1.1) with (1.2) possesses a global-in-time solution if $\kappa > 0$ is sufficiently small.*

Remark 1.4 Sharp estimates of the life span $T(\kappa\psi)$ as $\kappa \to +0$ have been already obtained in some cases. Specifically, if ψ satisfies

$$\psi(x) = (1 + |x|)^{-A} \quad (A > 0)$$

for all $x \in D$, then the following holds:

$$T(\kappa\psi) \sim \begin{cases} \kappa^{-\left(\frac{1}{2(p-1)} - \frac{A}{2}\right)^{-1}} & \text{if} \quad p \geq p_*, \quad 0 \leq A < 1/(p-1), \\ \kappa^{-\left(\frac{1}{2(p-1)} - \frac{1}{2}\min\{A,N\}\right)^{-1}} & \text{if} \quad p < p_*, \quad A \neq N, \\ \left(\dfrac{\kappa^{-1}}{\log(\kappa^{-1})}\right)^{\left(\frac{1}{2(p-1)} - \frac{N}{2}\right)^{-1}} & \text{if} \quad p < p_*, \quad A = N, \end{cases}$$

as $\kappa \to +0$ (see [13]).

Finally, we show that $\lim_{\kappa\to 0} T(\kappa\psi) = \infty$ does not necessarily hold for problem (1.1) if ψ has an exponential growth as $x_N \to \infty$.

Theorem 1.3 *Let $p > 1$, $\lambda > 0$ and $\psi(x) := \exp(\lambda x_N^2)$. Then*

$$\lim_{\kappa\to+0} T(\kappa\psi) = (4\lambda)^{-1}. \tag{1.4}$$

Remark 1.5 Let $\psi(x) = \exp(\lambda x_N^2)$. Set

$$v(x,t) := \int_D G(x,y,t)\psi(y)\,dy.$$

Then v is a solution to

$$\begin{cases} \partial_t v = \Delta v, & x \in \mathbf{R}^N_+, \quad t > 0, \\ -\dfrac{\partial v}{\partial x_N} = 0, & x \in \partial\mathbf{R}^N_+, \quad t > 0, \\ v(x,0) = \psi(x), & x \in \overline{\mathbf{R}^N_+}, \end{cases}$$

and

$$v(x,t) = (1 - 4\lambda t)^{-\frac{1}{2}} \exp\left(\frac{\lambda x_N^2}{1 - 4\lambda t}\right),$$

where $N \geq 1$. Moreover, v does not exist after $t = (4\lambda)^{-1}$.

The rest of this paper is organized as follows. In Sect. 2, we review some of the facts on the solvability of problem (1.1), which have been already obtained in [10]. In Sect. 3, we give upper estimates and lower estimates to the life span $T(\kappa\psi)$ as $\kappa \to \infty$ (see Propositions 3.1 and 3.2). By combining these estimates, we can prove Theorem 1.1. In Sect. 4, we prove Theorem 1.2 by the same method as in Sect. 3 (see Propositions 4.1 and 4.2) and prove Theorem 1.3. Appendix contains summaries of Theorems 1.1 and 1.2.

2 Necessary Conditions and Sufficient Conditions for the Solvability of Problem (1.1)

In what follows the letter C denotes a generic positive constant depending only on N and p. For any $L \geq 0$, we set

$$D_L := \{(x', x_N) : x' \in \mathbf{R}^{N-1}, x_N \geq L^{\frac{1}{2}}\},$$
$$D'_L := \{(x', x_N) : x' \in \mathbf{R}^{N-1}, 0 \leq x_N < L^{\frac{1}{2}}\}.$$

Now we review necessary conditions for the solvability of problem (1.1), which have been obtained in [10].

Theorem 2.1 *Let $p > 1$ and u be a solution to (1.1) in $[0, T)$ with $u(0) = \varphi$, where $0 < T < \infty$. Then for any $\delta > 0$, there exists $\gamma_1 = \gamma_1(N, p, \delta) > 0$ such that*

$$\sup_{x\in\mathbf{R}^N} \exp\left(- (1+\delta)\frac{x_N^2}{4\sigma^2}\right) \int_{B_+(x,\sigma)} \varphi(y)\, dy \le \gamma_1 \sigma^{N-\frac{1}{p-1}} \tag{2.1}$$

for $0 < \sigma \le T^{1/2}$. In particular, in the case of $p = p_*$, there exists $\gamma_1' = \gamma_1'(N, \delta) > 0$ such that

$$\sup_{x\in\mathbf{R}^N} \exp\left(- (1+\delta)\frac{x_N^2}{4\sigma^2}\right) \int_{B_+(x,\sigma)} \varphi(y)\, dy \le \gamma_1' \left[\log\left(e + \frac{T^{\frac{1}{2}}}{\sigma}\right)\right]^{-N} \tag{2.2}$$

for $0 < \sigma \le T^{1/2}$.

Remark 2.1 If $1 < p \le p_*$ and $\mu \not\equiv 0$ in D, then problem (1.1) possesses no nonnegative global-in-time solutions. See [3] and [7].

Next, we review sufficient conditions for the solvability of problem (1.1), which have been obtained also in [10]. For any measurable function ϕ in $\mathbf{R}^N$ and any bounded Borel set E, we set

$$\fint_E \phi(y)\, dy = \frac{1}{|E|}\int_E \phi(y)\, dy, \quad \phi_E(x) := \phi(x)\chi_E(x),$$

where $|E|$ is the Lebesgue measure of E.

Theorem 2.2 *Let $1 < p < p_*$, $T > 0$ and $\delta \in (0, 1)$. Set $\lambda := (1-\delta)/4T$. Then there exists $\gamma_2 = \gamma_2(N, p, \delta) > 0$ with the following property: If φ is a nonnegative measurable function in $\mathbf{R}^N_+$ satisfying*

$$\sup_{x\in D} \fint_{B_+(x,T^{1/2})} e^{-\lambda y_N^2} \varphi(y)\, dy \le \gamma_2 T^{-\frac{1}{2(p-1)}}, \tag{2.3}$$

then there exists a solution u to (1.1) in $[0, T)$ with $u(0) = \varphi$.

Theorem 2.3 *Let $p > 1$, $a \in (1, p)$, $T > 0$ and $\delta \in (0, 1)$. Let φ be a nonnegative measurable function in $\mathbf{R}^N_+$. Set $\varphi_1 := \varphi_{D_T}$, $\varphi_2 := \varphi_{D_T'}$ and $\lambda := (1-\delta)/4T$. Then there exists $\gamma_3 = \gamma_3(N, p, a, \delta) > 0$ with the following property: Assume that φ_1 satisfies*

$$\sup_{x\in D} \fint_{B_+(x,T^{1/2})} e^{-\lambda y_N^2} \varphi_1(y)\, dy \le \gamma_3 T^{-\frac{1}{2(p-1)}}. \tag{2.4}$$

Furthermore, assume that φ_2 *satisfies*

$$\sup_{x\in D'_T}\left[\fint_{B_+(x,\sigma)}\varphi_2(y)^a\,dy\right]^{\frac{1}{a}} \le \gamma_3\sigma^{-\frac{1}{p-1}} \quad \text{for } 0<\sigma\le T^{\frac{1}{2}}. \tag{2.5}$$

Then there exists a solution u *to* (1.1) *in* $[0,T)$ *with* $u(0)=\varphi$.

Theorem 2.4 *Let* $p = p_*$, $T > 0$ *and* $\delta \in (0,1)$. *Let* φ *be a nonnegative measurable function in* $\mathbf{R}^N_+$. *Set* $\varphi_1 := \varphi_{D_T}$, $\varphi_2 := \varphi_{D'_T}$, $\lambda := (1-\delta)/4T$ *and*

$$\Phi(s) := s[\log(e+s)]^N, \quad \rho(s) := s^{-N}\left[\log\left(e+\frac{1}{s}\right)\right]^{-N} \quad \text{for} \quad s>0. \tag{2.6}$$

Then there exists $\gamma_4 = \gamma_4(N,\delta) > 0$ *with the following property: Assume that* φ_1 *satisfies*

$$\sup_{x\in D_T}\fint_{B_+(x,T^{1/2})} e^{-\lambda y_N^2}\varphi_1(y)\,dy \le \gamma_4 T^{-\frac{1}{2(p-1)}}. \tag{2.7}$$

Furthermore, assume that φ_2 *satisfies*

$$\sup_{x\in D'_T}\Phi^{-1}\left[\fint_{B_+(x,\sigma)}\Phi(T^{\frac{1}{2(p-1)}}\varphi_2(y))\,dy\right] \le \gamma_4\rho(\sigma T^{-\frac{1}{2}}) \quad \text{for } 0<\sigma\le T^{\frac{1}{2}}. \tag{2.8}$$

Then there exists a solution u *to* (1.1) *in* $[0,T)$ *with* $u(0)=\varphi$.

3 Proof of Theorem 1.1

For simplicity of notation, we write T_κ instead of $T(\kappa\psi)$. In this section we study the behavior of T_κ as $\kappa\to\infty$ and prove Theorem 1.1. In order to prove Theorem 1.1, we obtain upper and lower estimates of T_κ as $\kappa\to\infty$. Proposition 3.1 gives upper estimates of T_κ as $\kappa\to\infty$. In the rest of this paper, for any two nonnegative functions f_1 and f_2 defined in a subset E of $[0,\infty)$, we write $f_1(t) \asymp f_2(t)$ for all $t\in E$ if $C^{-1}f_2(t)\le f_1(t)\le Cf_2(t)$ for all $t\in E$.

Proposition 3.1 *Let* ψ *be a nonnegative measurable function in* D *such that*

$$\psi(y) \ge |y|^{-A}\left[\log\left(e+\frac{1}{|y|}\right)\right]^{-B}, \qquad y\in B_+(0,1), \tag{3.1}$$

where $0 \le A \le N$ *and* B *are as in* (1.3)*. Then* $\lim_{\kappa\to\infty} T(\kappa\psi) = 0$*. Furthermore, the following holds:*

(i) *Let* $1 < p < p_*$*. Then there exists* $\gamma > 0$ *such that*

$$T(\kappa\psi) \le \gamma[\kappa(\log\kappa)^{-B}]^{-\frac{2(p-1)}{-A(p-1)+1}} \qquad \text{if} \quad A < N, \ B \in \mathbf{R}, \tag{3.2}$$

$$T(\kappa\psi) \le \gamma[\kappa(\log\kappa)^{-B+1}]^{-\frac{2(p-1)}{-A(p-1)+1}} \qquad \text{if} \quad A = N, \ B > 1, \tag{3.3}$$

for sufficiently large $\kappa > 0$*;*

(ii) *Let* $p > p_*$*. If, either*

$$A > 1/(p-1) \quad \text{and} \quad B \in \mathbf{R} \qquad \text{or} \qquad A = 1/(p-1) \quad \text{and} \quad B < 0, \tag{3.4}$$

then problem (1.1) *with* (1.2) *possesses no local-in-time solutions for all* $\kappa > 0$*. If*

$$A = 1/(p-1) \quad \text{and} \quad B = 0,$$

then problem (1.1) *with* (1.2) *possesses no local-in-time solutions for sufficiently large* $\kappa > 0$*. Furthermore,*

(a) *if* $A < 1/(p-1)$*, then* (3.2) *holds;*
(b) *if* $A = 1/(p-1)$ *and* $B > 0$*, then there exists* $\gamma' > 0$ *such that*

$$T(\kappa\psi) \le \exp(-\gamma'\kappa^{\frac{1}{B}})$$

for sufficiently large $\kappa > 0$*;*

(iii) *Let* $p = p_*$*. If*

$$A = N \quad \text{and} \quad B < N+1,$$

then problem (1.1) *with* (1.2) *possesses no local-in-time solutions for all* $\kappa > 0$*. If*

$$A = N \quad \text{and} \quad B = N+1,$$

then problem (1.1) *with* (1.2) *possesses no local-in-time solutions for sufficiently large* $\kappa > 0$*. Furthermore,*

(c) *if* $A < N$*, then* (3.2) *holds;*
(d) *if* $A = N$ *and* $B > N+1$*, then there exists* $\gamma'' > 0$ *such that*

$$T(\kappa\psi) \le \exp(-\gamma''\kappa^{\frac{1}{B-N-1}})$$

for sufficiently large $\kappa > 0$*.*

Proof We assume that (1.1) with (1.2) possesses a solution in $[0, T_\kappa)$. For any $p > 1$, by (2.1) and (3.1) we can find a constant $\gamma_1 > 0$ such that

$$\gamma_1 \sigma^{N-\frac{1}{p-1}} \geq \kappa \int_{B_+(0,\sigma)} \psi(y)\, dy \geq \kappa \int_{B_+(0,\sigma)} |y|^{-A} \left[\log \left(e + \frac{1}{|y|} \right) \right]^{-B} dy > 0 \tag{3.5}$$

for $0 < \sigma \leq T_\kappa^{1/2}$. Firstly, we show that $\lim_{\kappa \to \infty} T_\kappa = 0$ by contradiction. Assume that there exist $\{\kappa_j\}_{j=1}^\infty$ and $c_* > 0$ such that

$$\lim_{j \to \infty} \kappa_j = \infty, \quad T_{\kappa_j} > c_*^2 \quad \text{for all } j = 1, 2, \cdots .$$

By (3.5) with $\sigma = c_*$, we have

$$\gamma_1 c_*^{N-\frac{1}{p-1}} \geq \kappa_j \int_{B_+(0,c_*)} |y|^{-A} \left[\log \left(e + \frac{1}{|y|} \right) \right]^{-B} dy, \quad j = 1, 2, \cdots ,$$

where γ_1 is a constant independent of κ_j. Since $\lim_{j \to \infty} \kappa_j = \infty$, we have a contradiction. Since c_* is arbitrary, we have

$$\lim_{\kappa \to \infty} T_\kappa = 0.$$

Without loss of generality we can assume that $T_\kappa > 0$ is sufficiently small.

We prove assertion (i). Let $1 < p < p_*$. For any $p > 1$, by (3.5) we have

$$\gamma_1 \geq \begin{cases} C\kappa \sigma^{-A+\frac{1}{p-1}} \left[\log \left(e + \sigma^{-1} \right) \right]^{-B} & \text{if} \quad A < N, \ B \in \mathbf{R}, \\ C\kappa \sigma^{-N+\frac{1}{p-1}} \left[\log \left(e + \sigma^{-1} \right) \right]^{-B+1} & \text{if} \quad A = N, \ B > 1, \end{cases} \tag{3.6}$$

for $0 < \sigma \leq T_\kappa^{1/2}$ and sufficiently large $\kappa > 0$. We notice that for any $a_1 > 0$ and $a_2 \in \mathbf{R}$,

$$\Psi(\tau) := \tau^{a_1} [\log(e + \tau^{-1})]^{a_2} \text{ is increasing for sufficiently small } \tau > 0, \tag{3.7}$$

Ψ^{-1} satisfies

$$\Psi^{-1}(\tau) \asymp \tau^{\frac{1}{a_1}} [\log(e + \tau^{-1})]^{-\frac{a_2}{a_1}} \quad \text{for sufficiently small } \tau > 0 \tag{3.8}$$

and $\Psi^{-1}(\tau)$ is also increasing sufficiently small $\tau > 0$. We consider the case where $A < N$ and $B \in \mathbf{R}$. Set

$$a_1 := -A + \frac{1}{p-1} > 0 \quad \text{and} \quad a_2 := -B.$$

By (3.6)–(3.8) we have

$$\begin{aligned}\sigma &\leq C\Psi^{-1}(C\gamma_1\kappa^{-1})\\ &\leq C(C\gamma_1\kappa^{-1})^{(-A+\frac{1}{p-1})^{-1}}[\log(e+(C\gamma_1\kappa^{-1})^{-1})]^{B(-A+\frac{1}{p-1})^{-1}}\\ &\leq C[\kappa(\log\kappa)^{-B}]^{-\frac{p-1}{-A(p-1)+1}}\end{aligned}$$

for $0 < \sigma \leq T_\kappa^{1/2}$. Setting $\sigma = T_\kappa^{1/2}$, we obtain (3.2). Similarly, we can obtain (3.3). Thus assertion (i) follows.

We prove assertion (ii). Let $p > p_*$. We can assume that

$$A < N \quad \text{and} \quad B \in \mathbf{R} \qquad \text{or} \qquad A = N \quad \text{and} \quad B > 1 \tag{3.9}$$

because

$$\int_{B_+(0,\sigma)} |y|^{-A}\left[\log\left(e+\frac{1}{|y|}\right)\right]^{-B} dy = \infty$$

for all $\sigma > 0$ when A and B do not satisfy (3.9). By (3.5), this implies that $T_\kappa = 0$ for all $\kappa > 0$. By (3.9), we have (3.6). Since A and B satisfy (3.4), the right hand side of (3.6) goes to infinity as $\sigma \to +0$. This implies that $T_\kappa = 0$ for all $\kappa > 0$. In the case where $A = 1/(p-1)$ and $B = 0$ (this condition also satisfies (3.9)), it follows from (3.6) that

$$\gamma_1 \geq C\kappa. \tag{3.10}$$

Since (3.10) does not hold for sufficiently large $\kappa > 0$, this implies that $T_\kappa = 0$ for sufficiently large $\kappa > 0$. Furthermore, if $A < 1/(p-1)$, we obtain (3.2) by a similar argument to the proof of assertion (i). Then we obtain (a). It remains to consider the case where $A = 1/(p-1)$ and $B > 0$. Since $T_\kappa > 0$ is sufficiently small, by (3.6) we have

$$\gamma\kappa^{-1} \geq C[\log(e+T_\kappa^{-\frac{1}{2}})]^{-B} \geq C[\log(T_\kappa^{-\frac{1}{2}})]^{-B}.$$

Since $B > 0$, this implies that there exists a constant $\gamma' > 0$ such that

$$T_\kappa \le \exp(-\gamma' \kappa^{\frac{1}{B}})$$

for sufficiently large $\kappa > 0$ and (b) follows. Thus assertion (ii) is proved.

Finally, we prove assertion (iii). Let $A = N$. Since $p = p_*$ and $B > 1$, by (2.2) we have

$$\begin{aligned} \gamma_1' \left[\log\left(e + \frac{T_\kappa^{\frac{1}{2}}}{\sigma} \right) \right]^{-N} &\ge \kappa \int_{B_+(0,\sigma)} \psi(y)\, dy \\ &\ge \kappa \int_{B_+(0,\sigma)} |y|^{-A} \left[\log\left(e + \frac{1}{|y|} \right) \right]^{-B} dy \\ &\ge C\kappa [\log(e + \sigma^{-1})]^{-B+1} \end{aligned} \tag{3.11}$$

for $0 < \sigma \le T_\kappa^{1/2}$. In the case of $B < N + 1$, we see that (3.11) does not hold for sufficiently small $\sigma > 0$. This implies that $T_\kappa = 0$ for all $\kappa > 0$. In the case of $B = N + 1$, it follows from (3.11) with $\sigma = T_\kappa (< T_\kappa^{1/2})$ that

$$\gamma_1' [\log(e + T_\kappa^{-\frac{1}{2}})]^{-N} \ge C\kappa [\log(e + T_\kappa^{-1})]^{-N}.$$

This inequality implies that

$$\gamma_1' \ge C\kappa. \tag{3.12}$$

Since (3.12) does not hold for sufficiently large $\kappa > 0$, this implies that $T_\kappa = 0$ for sufficiently large $\kappa > 0$. In the case of $A < N$, since (3.6) holds, we obtain (3.2) by a similar argument to the proof of assertion (i). Then we obtain (c). In the case where $A = N$ and $B > N + 1$, since $T_\kappa > 0$ is sufficiently small, by (3.11) with $\sigma = T_\kappa (< T_\kappa^{1/2})$ we have

$$C\gamma_1' \kappa^{-1} \le [\log(e + T_\kappa^{-1})]^{-B+N+1} \le [\log(T_\kappa^{-1})]^{-B+N+1}.$$

Since $B - N - 1 > 0$, this implies that there exists a constant $\gamma'' > 0$ such that

$$T_\kappa \le \exp(-\gamma'' \kappa^{\frac{1}{B-N-1}})$$

for sufficiently large $\kappa > 0$. Thus assertion (iii) follows and the proof of Proposition 3.1 is complete. □

In Proposition 3.2 we obtain lower estimates of T_κ as $\kappa \to \infty$ and show the optimality of the estimates of T_κ in Proposition 3.1.

Proposition 3.2 *Let ψ be a nontrivial nonnegative measurable function in D such that supp $\psi \subset B(0,1)$ and*

$$\psi(y) \le |y|^{-A}\left[\log\left(e+\frac{1}{|y|}\right)\right]^{-B}, \qquad y \in B_+(0,1), \tag{3.13}$$

where $0 \le A \le N$ and B are as in (1.3).

(i) *Let $1 < p < p_*$. Then there exists $\gamma > 0$ such that*

$$T(\kappa\psi) \ge \gamma[\kappa(\log\kappa)^{-B}]^{-\frac{2(p-1)}{-A(p-1)+1}} \qquad \text{if} \quad A < N, \; B \in \mathbf{R}, \tag{3.14}$$
$$T(\kappa\psi) \ge \gamma[\kappa(\log\kappa)^{-B+1}]^{-\frac{2(p-1)}{-A(p-1)+1}} \qquad \text{if} \quad A = N, \; B > 1.$$

(ii) *Let $p > p_*$.*

(a) *If $A < 1/(p-1)$, then (3.14) holds;*
(b) *If $A = 1/(p-1)$ and $B > 0$, then there exists $\gamma' > 0$ such that*

$$T(\kappa\psi) \ge \exp(-\gamma'\kappa^{\frac{1}{B}})$$

for sufficiently large $\kappa > 0$;

(iii) *Let $p = p_*$.*

(c) *If $A < N$, then (3.14) holds;*
(d) *If $A = N$ and $B > N+1$, then there exists $\gamma'' > 0$ such that*

$$T(\kappa\psi) \ge \exp(-\gamma''\kappa^{\frac{1}{B-N-1}})$$

for sufficiently large $\kappa > 0$.

Proof We first consider the case where $p > p_*$ and $A < 1/(p-1)(< N)$. Let $a \in (1, p)$ be such that $aA < N$. By the Jensen inequality and (3.13), we have

$$\begin{aligned}
\sigma^{\frac{1}{p-1}} \sup_{x\in D} \fint_{B_+(x,\sigma)} \kappa\psi(y)\,dy &\le \sigma^{\frac{1}{p-1}} \sup_{x\in D}\left[\fint_{B_+(x,\sigma)} [\kappa\psi(y)]^a\,dy\right]^{\frac{1}{a}} \\
&\le C\kappa\sigma^{\frac{1}{p-1}}\left[\fint_{B_+(0,\sigma)} |y|^{-Aa}\left[\log\left(L+\frac{1}{|y|}\right)\right]^{-aB} dy\right]^{\frac{1}{a}} \\
&\le C\kappa\sigma^{\frac{1}{p-1}-A}\left[\log\left(e+\frac{1}{\sigma}\right)\right]^{-B}
\end{aligned} \tag{3.15}$$

for sufficiently small $\sigma > 0$. Let c be a sufficiently small positive constant and set

$$\tilde{T}_\kappa := c[\kappa(\log\kappa)^{-B}]^{-\frac{2(p-1)}{-A(p-1)+1}}.$$

Since $A < 1/(p-1)$, by (3.7) we have

$$\begin{aligned} C\kappa\sigma^{\frac{1}{p-1}-A}\left[\log\left(e+\frac{1}{\sigma}\right)\right]^{-B} &\le C\kappa\sigma^{\frac{1}{p-1}-A}\left[\log\left(e+\frac{1}{\sigma}\right)\right]^{-B}\Bigg|_{\sigma=\tilde{T}_\kappa^{1/2}} \\ &\le Cc^{\frac{1}{2(p-1)}-\frac{A}{2}} \end{aligned}$$

for $0 < \sigma \le \tilde{T}_\kappa^{1/2}$ and sufficiently large $\kappa > 0$. Taking a sufficiently small $c > 0$ if necessary, we obtain

$$C\kappa\sigma^{\frac{1}{p-1}-A}\left[\log\left(e+\frac{1}{\sigma}\right)\right]^{-B} \le \gamma_3, \tag{3.16}$$

for $0 < \sigma \le \tilde{T}_\kappa^{1/2}$ and sufficiently large $\kappa > 0$, where γ_3 is as in Theorem 2.3. Then (3.15) and (3.16) yield (2.4) and (2.5). Applying Theorem 2.3, we see that (1.1) with (1.2) possesses a solution in $[0, \tilde{T}_\kappa)$ and

$$T_\kappa \ge \tilde{T}_\kappa = c[\kappa(\log\kappa)^{-B}]^{-\frac{2(p-1)}{-A(p-1)+1}}$$

for sufficiently large $\kappa > 0$. So we have (a). Similarly, we have (b) and (c). Furthermore, we can prove (3.14) in the case of $1 < p < p_*$ by using the above argument with $a = 1$ and applying Theorem 2.2.

Next we consider the case where $1 < p < p_*$, $A = N$ and $B > 1$, let c be a sufficiently small positive constant and set

$$\tilde{T}'_\kappa := c[\kappa(\log\kappa)^{-B+1}]^{-\frac{2(p-1)}{-A(p-1)+1}}.$$

Taking a sufficiently small $c > 0$ if necessary, by (3.13) we have

$$\begin{aligned} &\tilde{T}'^{\frac{1}{2(p-1)}}_\kappa \sup_{x\in D} \fint_{B_+(x,\tilde{T}'^{1/2}_\kappa)} \kappa\psi(y)\,dy \\ &\le C\kappa\tilde{T}'^{\frac{1}{2(p-1)}}_\kappa \fint_{B_+(0,\tilde{T}'^{1/2}_\kappa)} |y|^{-N}\left[\log\left(\frac{1}{|y|}\right)\right]^{-B} dy \\ &\le C\kappa\tilde{T}'^{\frac{1}{2(p-1)}-\frac{N}{2}}_\kappa \left[\log\left(\frac{1}{\tilde{T}'^{1/2}_\kappa}\right)\right]^{-B+1} \le Cc^{\frac{1}{2(p-1)}-\frac{N}{2}} \le \gamma_2 \end{aligned} \tag{3.17}$$

for sufficiently large $\kappa > 0$, where γ_2 is as in Theorem 2.2. Then (3.17) yields (2.3). Applying Theorem 2.2, we see that (1.1) with (1.2) possesses a solution in $[0, \tilde{T}'_\kappa)$ and

$$T_\kappa \geq \tilde{T}'_\kappa = c[\kappa(\log\kappa)^{-B+1}]^{-\frac{2(p-1)}{-A(p-1)+1}}$$

for sufficiently large $\kappa > 0$. So we have assertion (i).

It remains to prove (d). Let $p = p_*$, $A = N$ and $B > N+1$. Let c be a sufficiently small positive constant and set

$$\hat{T}_\kappa := \exp(-c^{-1}\kappa^{\frac{1}{B-N-1}})$$

for sufficiently large $\kappa > 0$. We can assume that $\hat{T}_\kappa > 0$ is sufficiently small. Let Φ be as in (2.6). By (3.7) and (3.8) with $a_1 = 1$ and $a_2 = N$, we see that Φ^{-1} satisfies

$$\Phi^{-1}(\tau) \asymp \tau[\log(e+\tau^{-1})]^{-N} \quad \text{for sufficiently small } \tau > 0$$

and $\Phi^{-1}(\tau)$ is increasing for sufficiently small $\tau > 0$. Similarly to (3.15), we have

$$\begin{aligned} &\sup_{x\in D}\Phi^{-1}\left[\fint_{B_+(x,\sigma)}\Phi\left(\hat{T}_\kappa^{\frac{1}{2(p-1)}}\kappa\psi(y)\right)dy\right] \\ &\qquad \leq \Phi^{-1}\left[\fint_{B_+(0,\sigma)}\Phi\left(\hat{T}_\kappa^{\frac{N}{2}}\kappa|y|^{-N}\left[\log\left(e+\frac{1}{|y|}\right)\right]^{-B}\right)dy\right] \end{aligned} \tag{3.18}$$

for $0 < \sigma \leq \hat{T}_\kappa^{1/2}$. Since

$$\begin{aligned} &\log\left[e+\hat{T}_\kappa^{\frac{N}{2}}\kappa|y|^{-N}\left[\log\left(e+\frac{1}{|y|}\right)\right]^{-B}\right] \\ &\leq \log\left[\left(e+\hat{T}_\kappa^{\frac{N}{2}}\kappa|y|^{-N}\right)\left(e+\left[\log\left(e+\frac{1}{|y|}\right)\right]^{-B}\right)\right] \\ &\leq \log\left[C\hat{T}_\kappa^{\frac{N}{2}}\kappa|y|^{-N}\right] \leq C\log\left[\hat{T}_\kappa^{\frac{N}{2}}\kappa|y|^{-N}\right] \leq C\log\frac{1}{|y|} \end{aligned}$$

for $y \in B_+(0,\sigma)$, $0 < \sigma \leq \hat{T}_\kappa^{1/2}$ and sufficiently large κ, we have

$$\begin{aligned} &\fint_{B_+(0,\sigma)}\Phi\left(\hat{T}_\kappa^{\frac{N}{2}}\kappa|y|^{-N}\left[\log\left(e+\frac{1}{|y|}\right)\right]^{-B}\right)dy \\ &= \fint_{B_+(0,\sigma)}\hat{T}_\kappa^{\frac{N}{2}}\kappa|y|^{-N}\left[\log\left(e+\frac{1}{|y|}\right)\right]^{-B}\left[\log\left[e+\hat{T}_\kappa^{\frac{N}{2}}\kappa|y|^{-N}\left[\log\left(e+\frac{1}{|y|}\right)\right]^{-B}\right]\right]^N dy \\ &\leq C\hat{T}_\kappa^{\frac{N}{2}}\kappa\fint_{B_+(x,\sigma)}|y|^{-N}\left[\log\frac{1}{|y|}\right]^{-B+N}dy \leq C\kappa\sigma^{-N}\hat{T}_\kappa^{\frac{1}{2(p-1)}}\left[\log\frac{1}{\sigma}\right]^{-B+N+1} \end{aligned}$$

for $0 < \sigma \le \hat{T}_\kappa^{1/2}$ and sufficiently large $\kappa > 0$. This together with (3.18) implies that

$$\begin{aligned}
&\sup_{x\in D} \fint_{B_+(x,\sigma)} \hat{T}_\kappa^{\frac{1}{2(p-1)}} \kappa\psi(y)\,dy \le \sup_{x\in D} \Phi^{-1}\left[\fint_{B_+(x,\sigma)} \Phi\left(\hat{T}_\kappa^{\frac{1}{2(p-1)}}\kappa\psi(y)\right)dy\right] \\
&\le \Phi^{-1}\left(\fint_{B_+(0,\sigma)} \Phi\left(\hat{T}_\kappa^{\frac{N}{2}}\kappa|y|^{-N}\left[\log\left(e+\frac{1}{|y|}\right)\right]^{-B}\right)dy\right) \\
&\le \Phi^{-1}\left(C\kappa\sigma^{-N}\hat{T}_\kappa^{\frac{1}{2(p-1)}}\left[\log\frac{1}{\sigma}\right]^{-B+N+1}\right) \\
&\le C\kappa\sigma^{-N}\hat{T}_\kappa^{\frac{N}{2}}\left[\log\frac{1}{\sigma}\right]^{-B+N+1}\left(\log\left[e+C\hat{T}_\kappa^{\frac{N}{2}}\kappa\sigma^{-N}\left[\log\frac{1}{\sigma}\right]^{-B+N+1}\right]\right)^{-N} \\
&\le C\kappa\sigma^{-N}\hat{T}_\kappa^{\frac{N}{2}}\left[\log\frac{1}{\sigma}\right]^{-B+1}
\end{aligned} \tag{3.19}$$

for $0 < \sigma \le \hat{T}_\kappa^{1/2}$ and sufficiently large $\kappa > 0$. On the other hand, since $\hat{T}_\kappa > 0$ is sufficiently small, we have

$$\rho(\sigma\hat{T}_\kappa^{-\frac{1}{2}}) = \sigma^{-N}\hat{T}_\kappa^{\frac{N}{2}}\left[\log\left(e+\frac{\hat{T}_\kappa^{\frac{1}{2}}}{\sigma}\right)\right]^{-N} \ge \sigma^{-N}\hat{T}_\kappa^{\frac{N}{2}}\left[\log\frac{1}{\sigma}\right]^{-N} \tag{3.20}$$

for $0 < \sigma \le \hat{T}_\kappa^{1/2}$ and sufficiently large κ, where ρ is as in (2.6). Since $B > N+1$ and

$$\kappa\left[\log\frac{1}{\sigma}\right]^{-B+1+N} \le C\kappa\left[\log\frac{1}{\hat{T}_\kappa^{\frac{1}{2}}}\right]^{-B+1+N} = Cc^{B-N-1}, \tag{3.21}$$

taking a sufficiently small $c > 0$ if necessary, (3.19), (3.20) and (3.21) yield (2.7) and (2.8). Applying Theorem 2.4, we see that

$$T_\kappa \ge \hat{T}_\kappa = \exp(-c^{-1}\kappa^{\frac{1}{B-N-1}})$$

for sufficiently large $\kappa > 0$. This implies (d). The proof of Proposition 3.2 is complete. □

4 Proofs of Theorems 1.2 and 1.3

We state two results on the behavior of T_κ as $\kappa \to +0$. If ψ is a bounded function in $\mathbf{R}^N$, then $T_\kappa \to \infty$ as $\kappa \to +0$ and the behavior of T_κ depends on the decay of ψ at the space infinity. In order to prove Theorem 1.2, It suffice to prove the following propositions. In Proposition 4.1 we obtain upper estimates of T_κ as $\kappa \to +0$.

Proposition 4.1 *Let $N \ge 1$ and $p > 1$. Let $A > 0$ and ψ be a nonnegative $L^\infty(D)$-function such that $\psi(x) \ge (1+|x|)^{-A}$ for $x \in D$.*

(i) *Let $p = p_*$ and $A \ge 1/(p-1) = N$. Then there exists $\gamma > 0$ such that*

$$\log T(\kappa\psi) \le \begin{cases} \gamma\kappa^{-(p-1)} & \text{if} \quad A > N, \\ \gamma\kappa^{-\frac{p-1}{p}} & \text{if} \quad A = N, \end{cases}$$

for sufficiently small $\kappa > 0$.

(ii) *Let $1 < p < p_*$ or $A < 1/(p-1)$. Then there exists $\gamma' > 0$ such that*

$$T(\kappa\psi) \le \begin{cases} \gamma'\kappa^{-\left(\frac{1}{2(p-1)} - \frac{1}{2}\min\{A,N\}\right)^{-1}} & \text{if} \quad A \ne N, \\ \gamma'\left(\dfrac{\kappa^{-1}}{\log(\kappa^{-1})}\right)^{\left(\frac{1}{2(p-1)} - \frac{N}{2}\right)^{-1}} & \text{if} \quad A = N, \end{cases}$$

for sufficiently small $\kappa > 0$.

Proof Since $\psi \in L^\infty(D)$, by Theorem 2.3 we have

$$T_\kappa \ge C\kappa^{-(p-1)}$$

for sufficiently small $\kappa > 0$. This implies that $\lim_{\kappa\to 0} T_\kappa = \infty$. Without loss of generality, we can assume that $T_\kappa > 0$ is sufficiently large. For any $p > 1$, we see that

$$\begin{aligned} \int_{B_+(0,\sigma)} \kappa\psi(y)\,dy &\ge \kappa\int_{B_+(0,\sigma)} (1+|y|)^{-A}\,dy \\ &\ge \begin{cases} C\kappa & \text{if} \quad \sigma > 1,\ A > N, \\ C\kappa\log(e+\sigma) & \text{if} \quad \sigma > 1,\ A = N, \\ C\kappa\sigma^{N-A} & \text{if} \quad \sigma > 1,\ A < N, \end{cases} \end{aligned} \tag{4.1}$$

for $\sigma > 1$ and sufficiently small $\kappa > 0$. In the case of $p = p_*$, it follows from (2.2) that

$$\int_{B_+(0,\sigma)} \kappa\psi(y)\,dy \le \gamma_1'\left[\log\left(e + \frac{T_\kappa^{\frac{1}{2}}}{\sigma}\right)\right]^{-N}$$

for $0 < \sigma \le T_\kappa^{1/2}$ and sufficiently small $\kappa > 0$. This implies that

$$\int_{B_+(0,T_\kappa^{1/4})} \kappa\psi(y)\,dy \le C\gamma_1'[\log T_\kappa]^{-N}, \tag{4.2}$$

$$\int_{B_+(0,T_\kappa^{1/2})} \kappa\psi(y)\,dy \le C\gamma_1', \tag{4.3}$$

for sufficiently small $\kappa > 0$. By (4.1) and (4.2) with $\sigma = T_\kappa^{1/4}$ we obtain assertion (i). Furthermore, by (4.1) and (4.3) with $\sigma = T_\kappa^{1/2}$ we obtain assertion (ii) in the case where $p = p_*$ and $A < 1/(p-1)$.

We prove assertion (ii) in the case of $1 < p < p_*$. By (2.1) we see that

$$\int_{B_+(0,T_\kappa^{1/2})} \kappa\psi(y)\,dy \le \gamma_1 T_\kappa^{\frac{N}{2}-\frac{1}{2(p-1)}}. \tag{4.4}$$

By (4.1) and (4.4), we obtain assertion (ii) in the case of $1 < p < p_*$. Similarly, we obtain assertion (ii) in the case of $p > p_*$. Thus Proposition 4.1 follows. □

In Proposition 4.2 we obtain lower estimates of T_κ as $\kappa \to +0$ and show the optimality of the estimates of T_κ in Proposition 4.1.

Proposition 4.2 *Let $N \ge 1$ and $p > 1$. Let $A > 0$ and ψ be a nonnegative measurable function in D such that supp $\psi \subset D$ and $0 \le \psi(x) \le (1+|x|)^{-A}$ for $x \in D$.*

(i) *Let $p = p_*$ and $A \ge 1/(p-1) = N$. Then there exists $\gamma > 0$ such that*

$$\log T(\kappa\psi) \ge \begin{cases} \gamma\kappa^{-(p-1)} & \text{if} \quad A > N, \\ \gamma\kappa^{-\frac{p-1}{p}} & \text{if} \quad A = N, \end{cases}$$

for sufficiently small $\kappa > 0$.

(ii) *Let $1 < p < p_*$ or $A < 1/(p-1)$. Then there exists $\gamma' > 0$ such that*

$$T(\kappa\psi) \ge \begin{cases} \gamma'\kappa^{-\left(\frac{1}{2(p-1)}-\frac{1}{2}\min\{A,N\}\right)^{-1}} & \text{if} \quad A \ne N, \\ \gamma'\left(\dfrac{\kappa^{-1}}{\log(\kappa^{-1})}\right)^{\left(\frac{1}{2(p-1)}-\frac{N}{2}\right)^{-1}} & \text{if} \quad A = N, \end{cases}$$

for sufficiently small $\kappa > 0$.

Proof Let $p = p_*$ and $A > N$. Let c be a sufficiently small positive constant and set

$$\hat{T}_\kappa := \exp(c\kappa^{-(p-1)}) = \exp(c\kappa^{-\frac{1}{N}}).$$

Let $L \geq e$ be such that

$$\tau[\log(L+\tau)]^{-N} \text{is increasing in } [0,\infty).$$

Then we see that $\Phi(\tau) \asymp \tau[\log(L+\tau)]^N$ and $\Phi^{-1}(\tau) \asymp \tau[\log(e+\tau)]^{-N} \asymp \tau[\log(L+\tau)]^{-N}$ for all $\tau > 0$. Similarly to (3.19), we have

$$\begin{aligned} \sup_{x\in D} \fint_{B_+(x,\sigma)} \hat{T}_\kappa^{\frac{1}{2(p-1)}} \kappa\psi(y)\,dy &\leq \sup_{x\in D} \Phi^{-1}\left[\fint_{B_+(x,\sigma)} \Phi\left(\hat{T}_\kappa^{\frac{1}{2(p-1)}}\kappa\psi(y)\right)dy\right] \\ &\leq \Phi^{-1}\left[\fint_{B_+(0,\sigma)} \Phi\left(\hat{T}_\kappa^{\frac{N}{2}}\kappa(1+|y|)^{-A}\right)dy\right] \end{aligned} \tag{4.5}$$

for all $\sigma > 0$. Since

$$\log\left[L+\hat{T}_\kappa^{\frac{N}{2}}\kappa(1+|y|)^{-A}\right] \leq \log(C\hat{T}_\kappa^{\frac{N}{2}}) \leq Cc\kappa^{-\frac{1}{N}} \tag{4.6}$$

for sufficiently small $\kappa > 0$, we have

$$\fint_{B_+(0,\sigma)} \Phi\left(\hat{T}_\kappa^{\frac{N}{2}}\kappa(1+|y|)^{-A}\right)dy \leq Cc^N\hat{T}_\kappa^{\frac{N}{2}} \fint_{B_+(0,\sigma)} (1+|y|)^{-A}\,dy \leq Cc^N\hat{T}_\kappa^{\frac{N}{2}}\sigma^{-N} \tag{4.7}$$

for $0 < \sigma \leq \hat{T}_\kappa^{1/2}$ and sufficiently small $\kappa > 0$. This together with (4.5) implies that

$$\begin{aligned} &\sup_{x\in D} \fint_{B_+(x,\sigma)} \hat{T}_\kappa^{\frac{1}{2(p-1)}} \kappa\psi(y)\,dy \leq \sup_{x\in D} \Phi^{-1}\left[\fint_{B_+(x,\sigma)} \Phi\left(\hat{T}_\kappa^{\frac{1}{2(p-1)}}\kappa\psi(y)\right)dy\right] \\ &\leq Cc^N\sigma^{-N}\hat{T}_\kappa^{\frac{N}{2}}\left(\log\left[L+Cc^N\hat{T}_\kappa^{\frac{N}{2}}\sigma^{-N}\right]\right)^{-N} \leq Cc^N\sigma^{-N}\hat{T}_\kappa^{\frac{N}{2}}\left(\log\left[L+\frac{\hat{T}_\kappa^{\frac{1}{2}}}{\sigma}\right]\right)^{-N} \\ &= Cc^N\rho(\sigma\hat{T}_\kappa^{-\frac{1}{2}}) \end{aligned}$$

for $0 < \sigma \leq \hat{T}_\kappa^{1/2}$ and sufficiently small $\kappa > 0$. Therefore, taking a sufficiently small $c > 0$ if necessary, we apply Theorem 2.4 to see that (1.1) with (1.2) possesses a solution in $[0, \hat{T}_\kappa)$ and

$$T_\kappa \geq \hat{T}_\kappa = \exp(c\kappa^{-(p-1)})$$

for all sufficiently small $\kappa > 0$.

In the case of $A = N$, setting

$$\check{T}_\kappa := \exp(c\kappa^{-\frac{p-1}{p}}) = \exp(c\kappa^{-\frac{1}{N+1}}),$$

similarly to (4.6) and (4.7), we have

$$
\begin{aligned}
&\fint_{B_+(0,\sigma)} \Phi\left(\check{T}_\kappa^{\frac{N}{2}}\kappa(1+|y|)^{-A}\right)dy \\
&\le C\kappa\check{T}_\kappa^{\frac{N}{2}}(\log\check{T}_\kappa)^N \fint_{B_+(x,\sigma)}(1+|y|)^{-N}\,dy \le C\kappa\check{T}_\kappa^{\frac{N}{2}}\sigma^{-N}(\log\check{T}_\kappa)^{N+1} = Cc^{N+1}\check{T}_\kappa^{\frac{N}{2}}\sigma^{-N}
\end{aligned}
$$

for $0<\sigma\le\check{T}_\kappa^{1/2}$ and sufficiently small $\kappa>0$. Then we apply the same argument as in the case of $A>N$ to see that

$$
T_\kappa \ge \check{T}_\kappa = \exp(c\kappa^{-\frac{p-1}{p}})
$$

for sufficiently small $\kappa>0$. Thus assertion (i) follows.

We show assertion (ii). Let $1<p<p_*$ and $0<A<N$. Let c be a sufficiently small positive constant and set

$$
\tilde{T}_\kappa := c\kappa^{-\left(\frac{1}{2(p-1)}-\frac{A}{2}\right)^{-1}}.
$$

Then

$$
\sup_{x\in D}\fint_{B_+(x,\tilde{T}_\kappa^{\frac{1}{2}})}\kappa\psi(y)\,dy \le C\kappa\fint_{B_+(0,\tilde{T}_\kappa^{\frac{1}{2}})}(1+|y|)^{-A}\,dy \le C\kappa\tilde{T}_\kappa^{-\frac{A}{2}} = Cc^{\frac{1}{2(p-1)}-\frac{A}{2}}\tilde{T}_\kappa^{-\frac{1}{2(p-1)}}
$$

for sufficiently small $\kappa>0$. Then we have assertion (ii) in the case where $1<p<p_*$ and $0<A<N$. Similarly, we can prove assertion (ii) in the other cases and assertion (ii) follows. Thus the proof of Proposition 4.2 is complete. □

Finally, we show that $\lim_{\kappa\to 0}T_\kappa=\infty$ does not necessarily hold for problem (1.1) if ψ has an exponential growth as $x_N\to\infty$.

Proof of Theorem 1.3 Let $\kappa>0$ and $\delta>0$. It follows from Theorem 2.1 that

$$
\begin{aligned}
\gamma_1 T_\kappa^{\frac{N}{2}-\frac{1}{2(p-1)}} &> \exp\left(-(1+\delta)\frac{x_N^2}{4T_\kappa}\right)\int_{B_+(x,T_\kappa^{1/2})}\kappa\psi(y)\,dy \\
&\ge C\exp\left(-(1+\delta)\frac{x_N^2}{4T_\kappa}\right)\kappa T_\kappa^{\frac{N}{2}}\exp\left(\lambda(x_N-T_\kappa^{\frac{1}{2}})^2\right) \\
&\ge C\kappa T_\kappa^{\frac{N}{2}}\exp\left\{\left(\lambda-\frac{1+\delta}{4T_\kappa}\right)x_N^2\right\}\exp\left(-2\lambda T_\kappa^{\frac{1}{2}}x_N+\lambda T_\kappa\right)
\end{aligned}
$$

for all $x \in D_{T_\kappa}$, where γ_1 is as in Theorem 2.1. Letting $x_N \to \infty$, we see that $\lambda - (1+\delta)/4T_\kappa \le 0$. Since $\delta > 0$ is arbitrary, we obtain

$$\limsup_{\kappa \to +0} T_\kappa \le (4\lambda)^{-1}. \tag{4.8}$$

On the other hand, it follows that

$$\fint_{B_+(x,\tilde{T}_\delta^{1/2})} \exp\left(-(1-\delta)\frac{y_N^2}{4\tilde{T}_\delta}\right) \kappa \exp(\lambda y_N^2)\, dy = \kappa, \quad x \in D_{\tilde{T}_\delta},$$

where $\tilde{T}_\delta := (1-\delta)/4\lambda$. Then we deduce from Theorem 2.3 that $T_\kappa \ge \tilde{T}_\delta$ for sufficiently small $\kappa > 0$. Since $\delta > 0$ is arbitrary, we obtain $\liminf_{\kappa \to +0} T_\kappa \ge (4\lambda)^{-1}$. This together with (4.8) implies (1.4). Thus Theorem 1.3 follows. □

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Appendix

By Theorem 1.1, we obtain Tables 1, 2 and 3. These tables show the behavior of the life span $T(\kappa\psi)$ as $\kappa \to \infty$ when ψ is as in Theorem 1.1, that is,

$$\psi(x) := |x|^{-A}\left[\log\left(e + \frac{1}{|x|}\right)\right]^{-B} \chi_{B_+(0,1)}(x) \in L^1(\mathbf{R}_+^N) \setminus L^\infty(\mathbf{R}_+^N),$$

Table 1 The behavior of T_κ in the case of $1 < p < p_*$ (as $\kappa \to \infty$)

B \ A	$A < N$	$A = N$
$B > 1$	$T_\kappa \sim [\kappa(\log\kappa)^{-B}]^{-\frac{2(p-1)}{-A(p-1)+1}}$	$T_\kappa \sim [\kappa(\log\kappa)^{-B+1}]^{-\frac{2(p-1)}{-A(p-1)+1}}$
$B \le 1$	$T_\kappa \sim [\kappa(\log\kappa)^{-B}]^{-\frac{2(p-1)}{-A(p-1)+1}}$	0

Table 2 The behavior of T_κ in the case of $p > p_*$ (as $\kappa \to \infty$)

B \ A	$A < \frac{1}{p-1}$	$A = \frac{1}{p-1}$	$\frac{1}{p-1} < A \le N$
$B > 0$	$T_\kappa \sim [\kappa(\log\kappa)^{-B}]^{-\frac{2(p-1)}{-A(p-1)+1}}$	$\lvert\log T_\kappa\rvert \sim \kappa^{\frac{1}{B}}$	0
$B = 0$	$T_\kappa \sim [\kappa(\log\kappa)^{-B}]^{-\frac{2(p-1)}{-A(p-1)+1}}$	0	0
$B < 0$	$T_\kappa \sim [\kappa(\log\kappa)^{-B}]^{-\frac{2(p-1)}{-A(p-1)+1}}$	0	0

Table 3 The behavior of T_κ in the case of $p = p_*$ (as $\kappa \to \infty$)

B \ A	$A < N$	$A = N$
$B > N+1$	$T_\kappa \sim \left[\kappa(\log\kappa)^{-B}\right]^{-\frac{2(p-1)}{-A(p-1)+1}}$	$\lvert\log T_\kappa\rvert \sim \kappa^{\frac{1}{B-N-1}}$
$B = N+1$	$T_\kappa \sim \left[\kappa(\log\kappa)^{-B}\right]^{-\frac{2(p-1)}{-A(p-1)+1}}$	0
$B < N+1$	0	0

Table 4 The behavior of T_κ in the case of $A \neq N$ (as $\kappa \to +0$)

p \ A	$A < \frac{1}{p-1}$	$A = \frac{1}{p-1}$	$A > \frac{1}{p-1}$
$p < p_*$	$T_\kappa \sim \kappa^{-\left(\frac{1}{2(p-1)} - \frac{1}{2}\min\{A,N\}\right)^{-1}}$	$T_\kappa \sim \kappa^{-\left(\frac{1}{2(p-1)} - \frac{N}{2}\right)^{-1}}$	$T_\kappa \sim \kappa^{-\left(\frac{1}{2(p-1)} - \frac{N}{2}\right)^{-1}}$
$p = p_*$	$T_\kappa \sim \kappa^{-\left(\frac{1}{2(p-1)} - \frac{A}{2}\right)^{-1}}$	($A = N$, see Table 5)	$\log T_\kappa \sim \kappa^{-(p-1)}$
$p > p_*$	$T_\kappa \sim \kappa^{-\left(\frac{1}{2(p-1)} - \frac{A}{2}\right)^{-1}}$	∞	∞

Table 5 The behavior of T_κ in the case of $A = N$ (as $\kappa \to +0$)

p \ A	$A = N$
$p < p_*$	$T_\kappa \sim \left(\frac{\kappa^{-1}}{\log(\kappa^{-1})}\right)^{\left(\frac{1}{2(p-1)} - \frac{N}{2}\right)^{-1}}$
$p = p_*$	$\log T_\kappa \sim \kappa^{-\frac{p-1}{p}}$
$p > p_*$	∞

where $0 \le A \le N$ and

$$B > 0 \quad \text{if} \quad A = 0, \qquad B \in \mathbf{R} \quad \text{if} \quad 0 < A < N, \qquad B > 1 \quad \text{if} \quad A = N.$$

For simplicity of notation, we write T_κ instead of $T(\kappa\psi)$.

By Theorem 1.2, we obtain Tables 4 and 5. These tables show the behavior of the life span $T(\kappa\psi)$ as $\kappa \to +0$ when ψ is as in Theorem 1.2, that is, $\psi(x) = (1 + |x|)^{-A} (A > 0)$.

References

1. Arrieta, J.M., Rodríguez-Bernal, A.: Non well posedness of parabolic equations with supercritical nonlinearities. Commun. Contemp. Math. **6**, 733–764 (2004)
2. Arrieta, J.M., Carvalho, A.N., Rodríguez-Bernal, A.: Parabolic problems with nonlinear boundary conditions and critical nonlinearities. J. Differ. Equ. **156**, 376–406 (1999)
3. Deng, K., Fila, M., Levine, H.A.: On critical exponents for a system of heat equations coupled in the boundary conditions. Acta Math. Univ. Comenianae **63**, 169–192 (1994)

4. Dickstein, F.: Blowup stability of solutions of the nonlinear heat equation with a large life span. J. Differ. Equ. **223**, 303–328 (2006)
5. Fernández Bonder, J., Rossi, J.D.: Life span for solutions of the heat equation with a nonlinear boundary condition. Tsukuba J. Math. **25**, 215–220 (2001)
6. Filo, J., Kačur, J.: Local existence of general nonlinear parabolic systems. Nonlinear Anal. **24**, 1597–1618 (1995)
7. Galaktionov, V.A., Levine, H.A.: On critical Fujita exponents for heat equations with nonlinear flux conditions on the boundary. Israel J. Math. **94**, 125–146 (1996)
8. Gui, C., Wang, X.: Life spans of solutions of the Cauchy problem for a semilinear heat equation. J. Differ. Equ. **115**, 166–172 (1995)
9. Hisa, K., Ishige, K.: Existence of solutions for a fractional semilinear parabolic equation with singular initial data. Nonlinear Anal. **175**, 108–132 (2018)
10. Hisa, K., Ishige, K.: Solvability of the heat equation with a nonlinear boundary condition. SIAM J. Math. Anal. **51**, 565–594 (2019)
11. Ikeda, M., Sobajima, M.: Sharp upper bound for lifespan of solutions to some critical semilinear parabolic, dispersive and hyperbolic equations via a test function method. Nonlinear Anal. **182**, 57–74 (2019)
12. Ishige, K., Kawakami, T.: Global solutions of the heat equation with a nonlinear boundary condition. Calc. Var. Partial Differ. Equ. **39**, 429–457 (2010)
13. Ishige, K., Sato, R.: Heat equation with a nonlinear boundary condition and uniformly local L^r spaces. Discrete Contin. Dyn. Syst. **36**, 2627–2652 (2016)
14. Ishige, K., Sato, R.: Heat equation with a nonlinear boundary condition and growing initial data. Differ. Integr. Equ. **30**, 481–504 (2017)
15. Lee, T.Y., Ni, W.M.: Global existence, large time behavior and life span of solutions of a semilinear parabolic Cauchy problem. Trans. Am. Math. Soc. **333**, 365–378 (1992)
16. Mizoguchi, N., Yanagida, E.: Blowup and life span of solutions for a semilinear parabolic equation. SIAM J. Math. Anal. **29**, 1434–1446 (1998)
17. Mizoguchi, N., Yanagida, E.: Life span of solutions with large initial data in a semilinear parabolic equation. Indiana Univ. Math. J. **50**, 591–610 (2001)
18. Ozawa, T., Yamauchi, Y.: Life span of positive solutions for a semilinear heat equation with general non-decaying initial data. J. Math. Anal. Appl. **379**, 518–523 (2011)
19. Sato, S.: Life span of solutions with large initial data for a superlinear heat equation. J. Math. Anal. Appl. **343**, 1061–1074 (2008)
20. Yamauchi, Y.: Life span of solutions for a semilinear heat equation with initial data having positive limit inferior at infinity. Nonlinear Anal. **74**, 5008–5014 (2011)
21. Yamaguchi, M., Yamauchi, Y.: Life span of positive solutions for a semilinear heat equation with non-decaying initial data. Differ. Integr. Equ. **23**, 1151–1157 (2010)

Neutral Inclusions, Weakly Neutral Inclusions, and an Over-determined Problem for Confocal Ellipsoids

Yong-Gwan Ji, Hyeonbae Kang, Xiaofei Li, and Shigeru Sakaguchi

Abstract An inclusion is said to be neutral to uniform fields if upon insertion into a homogenous medium with a uniform field it does not perturb the uniform field at all. It is said to be weakly neutral if it perturbs the uniform field mildly. Such inclusions are of interest in relation to invisibility cloaking and effective medium theory. There have been some attempts lately to construct or to show existence of such inclusions in the form of core-shell structure or a single inclusion with the imperfect bonding parameter attached to its boundary. The purpose of this paper is to review recent progress in such attempts. We also discuss about the over-determined problem for confocal ellipsoids which is closely related with the neutral inclusion, and its equivalent formulation in terms of Newtonian potentials. The main body of this paper consists of reviews on known results, but some new results are also included.

Y.-G. Ji
Department of Mathematics and Institute of Applied Mathematics, Inha University, Incheon, South Korea
e-mail: 22151063@inha.edu

H. Kang (✉)
Department of Mathematics and Institute of Applied Mathematics, Inha University, Incheon, South Korea

Department of Mathematics and Statistics, Central South University, Changsha, Hunan, P.R. China
e-mail: hbkang@inha.ac.kr

X. Li
Department of Mathematics and Institute of Applied Mathematics, Inha University, Incheon, South Korea

College of Science, Zhejiang University of Technology, Hangzhou, P.R. China
e-mail: xiaofeili@zjut.edu.cn

S. Sakaguchi
Research Center for Pure and Applied Mathematics, Graduate School of Information Sciences, Tohoku University, Sendai, Japan
e-mail: sigersak@tohoku.ac.jp

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1 Introduction

This is a survey on recent progress in study on existence and construction of neutral and weakly neutral inclusions, and a related over-determined problem for confocal ellipsoids. The main body of the paper consists of reviews on known results with brief but coherent explanations. However, we include some new results as well.

To explain the problems related to the neutral and weakly neutral inclusion, let us consider the following conductivity problem:

$$\text{(CP)} \quad \begin{cases} \nabla \cdot \sigma \nabla u = 0 & \text{in } \mathbb{R}^d, \\ u(x) - a \cdot x = O(|x|^{-d+1}) & \text{as } |x| \to \infty, \end{cases}$$

for $d = 2$ or 3, where a is a constant vector so that $-a = -\nabla(a \cdot x)$ is the background uniform field and σ is a piecewise constant function representing the conductivity distribution.

We first consider the problem (CP) when the conductivity distribution σ is given by

$$\sigma = \sigma_c \chi(D) + \sigma_m \chi(\mathbb{R}^d \setminus D), \tag{1.1}$$

where D is a simply connected bounded domain in $\mathbb{R}^d$ whose boundary ∂D is Lipchitz continuous. Here $\chi(D)$ denotes the characteristic function of D ($\chi(\mathbb{R}^d \setminus D)$ likewise), and σ_c and σ_m are constants representing conductivities of D (the core) and $\mathbb{R}^d \setminus D$ (the matrix), respectively. In absence of the inclusion D, the solution to (CP) is nothing but $a \cdot x$. Thus, if we denote by u the solution to (CP) in presence of the inclusion, $u(x) - a \cdot x$ represents the perturbation occurred by insertion of the inclusion D into the homogeneous medium with the uniform field $-a$. As we see from Fig. 1, the uniform field is perturbed outside (and inside) the inclusion.

It is known (see, e.g., [2]) that the leading order term of the perturbation outside the inclusion can be expressed in terms of the dipolar expansion. In fact, we have the following expansion at infinity:

$$u(x) - a \cdot x = \frac{1}{\omega_d} \frac{\langle a, Mx \rangle}{|x|^d} + O(|x|^{-d}) \quad \text{as } |x| \to \infty, \tag{1.2}$$

where ω_d is the surface area of the unit sphere in $\mathbb{R}^d$ and $M = (M_{ij})$ is the $d \times d$ matrix determined by the domain D and the conductivity contrast σ_c/σ_m. The matrix M is called the polarization (or polarizability) tensor (PT in abbreviation,

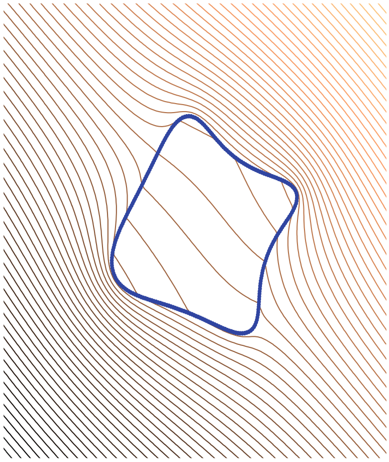


Fig. 1 Perturbation of the uniform field; inside and outside the inclusion (with the boundary in blue)

afterwards) associated with D. The PT is a signature of the existence of the inclusion D and has been effectively used to detect some properties of the inclusion D, for which we refer to [2]. It also plays an important role in the theory of composites and effective medium, for which we refer to [2, 31].

If D is a simply connected domain (or a union of simply connected domains), then M is positive-definite if $\sigma_c - \sigma_m > 0$ and negative-definite if $\sigma_c - \sigma_m < 0$. In fact, optimal bounds for PT, called the Hashin-Shtrikman bounds, are known, which will be explained in Sect. 2. Therefore, if D is simply connected, then there is $\hat{x} = x/|x|$ such that $\langle a, M\hat{x}\rangle \neq 0$ and for such x the following holds

$$|u(x) - a \cdot x| \geq C|x|^{-d+1} \quad \text{as } |x| \to \infty \tag{1.3}$$

for some $C > 0$. The dipolar expansion (1.2) shows that in general the solution u to (CP) admits the following:

$$u(x) - a \cdot x = O(|x|^{-d+1}) \quad \text{as } |x| \to \infty. \tag{1.4}$$

Furthermore, (1.3) shows that the decay rate $O(|x|^{-d+1})$ cannot be replaced by a faster rate, say $O(|x|^{-d})$.

However, if the inclusion is of a core-shell structure, then the situation can be quite different. Let D be a bounded domain and Ω be a bounded domain containing $\overline{D}$ so that (D, Ω) becomes a coated structure or a core-shell structure. Suppose that the conductivity distribution is given by

$$\sigma = \sigma_c \chi(D) + \sigma_s \chi(\Omega \setminus D) + \sigma_m \chi(\mathbb{R}^d \setminus \Omega). \tag{1.5}$$

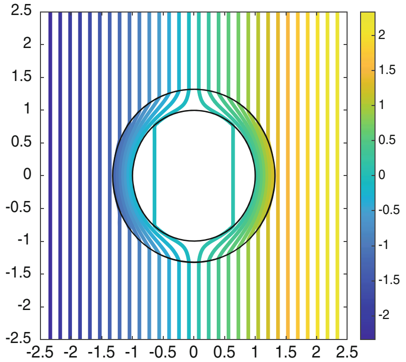


Fig. 2 Neutral inclusion: The uniform field is not perturbed

In particular, if (D, Ω) is a pair of concentric disks or balls, and if the conductivities σ_c, σ_s and σ_m are scalars and satisfy

$$(d - 1 + \sigma_c/\sigma_s)(\sigma_m/\sigma_s - 1) + f(1 - \sigma_c/\sigma_s)(\sigma_m/\sigma_s + d - 1) = 0 \tag{1.6}$$

for $d = 2$ or 3, where $f = |D|/|\Omega|$ (the volume fraction), then the solution u to (CP) satisfies

$$u(x) - a \cdot x \equiv 0 \quad \text{in } \mathbb{R}^d \setminus \Omega, \tag{1.7}$$

namely, the uniform field is not perturbed at all (see Fig. 2). In fact, with the conductivity given by (1.5), the solution u to (CP) is harmonic in $\mathbb{R}^d \setminus (\partial D \cup \partial \Omega)$, and along the interfaces ∂D and $\partial \Omega$ it satisfies the transmission conditions: continuity of the potential and continuity of the flux, namely,

$$\sigma_c \partial_\nu u|_- = \sigma_s \partial_\nu u|_+ \text{ on } \partial D, \quad \sigma_s \partial_\nu u|_- = \sigma_m \partial_\nu u|_+ \text{ on } \partial \Omega, \tag{1.8}$$

where the subscripts $+$ and $-$ indicate the limits from outside and inside D (or Ω), respectively. If D and Ω are concentric disks or balls, one can use spherical harmonics to find the solution explicitly to satisfy these interface conditions, and show that (1.6) implies (1.7).

This easy-to-prove fact was first discovered by Hashin [12], and significance of the discovery lies in its implications. Since insertion of inclusions does not perturb the outside uniform field, the effective conductivity of the assemblage filled with such inclusions of many different scales is the same as σ_m (the conductivity of the matrix) satisfying (1.6). It is also proved that such an effective conductivity is one of the Hashin-Shtrikman bounds on the effective conductivity of arbitrary two-phase composites [12, 13] (see also [31]).

The inclusion (D, Ω) of core-shell structure (or any other structure), which does not perturb the uniform field $-a$ upon its insertion, that is, satisfying (1.7), is said to be neutral to the field $-a$. If the inclusion is neutral to all uniform fields, it is said to be neutral to multiple uniform fields. The concentric disks (or balls) satisfying (1.6) is neutral to multiple uniform fields. If σ_m is anisotropic (σ_c and σ_s are scalars), then confocal ellipsoids whose common foci are determined by σ_m can be neutral to multiple fields. We include a proof of this fact in Sect. 3.

Now the question is if there are coated inclusions other than concentric disks or balls neutral to multiple uniform fields (or confocal ellipses or ellipsoids when σ_m is anisotropic), more generally, if we can coat a given domain D of general shape by a shell so that the resulting inclusion (D, Ω) is neutral to multiple uniform fields. The answer is proven to be no in two dimensions. In fact, it has been proved that if (D, Ω) is neutral to multiple uniform fields, then Ω and D are concentric disks if σ_m is isotropic, and confocal ellipses if σ_m is anisotropic (and the foci are determined by σ_m). This was proved by Milton-Serkov [32] when $\sigma_c = 0$ or ∞, and by Kang-Lee [15] when σ_c is finite. Since these two-dimensional results are proved using either the Riemann mapping or existence of harmonic conjugates, the methods of proofs cannot be extended to three dimensions. It is worth mentioning that there are many different shapes of coated inclusions neutral to a single uniform field as shown in two dimensions in [14, 32]. In three dimensions, it is proved in [19] that the coated inclusion (D, Ω) being neutral to multiple fields is equivalent to existence of a solution to a certain over-determined problem defined on $\Omega \setminus \overline{D}$. It is then proved as a consequence that if σ_m is isotropic, then the only inclusions of core-shell structure is a pair of concentric balls. Extension of this result to the anisotropic case has not been achieved and is open. We will review recent development on neutral inclusions and the related over-determined problem in Sect. 3. We also include in the same section a proof of their equivalence to a certain formulation in terms of Newtonian potentials.

Other than applications to the theory of composite as explained earlier, there is another interest in neutral inclusions in relation to invisibility cloaking. The neutrality condition (1.7) means that the uniform field is unperturbed at all outside the inclusion, namely, there is no difference of the field with or without the inclusion. It means that the inclusion is invisible from the probe by uniform fields. This was also observed in [24]. Recently, the idea of neutrally coated inclusions has been extended to construct multi-coated circular structures which are neutral not only to uniform fields but also to fields of higher degree [4]. It was proved there that the multi-coated structure combined with a transformation optics dramatically enhances the near cloaking proposed in [25].

Since there is no coated inclusion, other than concentric disks and balls if σ_m is isotropic, neutral to multiple fields (invisible by uniform fields), we may ask if there are inclusions which are vaguely visible by uniform fields. They are weakly neutral inclusions. In general, the solution u to (CP) satisfies the decay condition $u(x) - a \cdot x = O(|x|^{1-d})$ at ∞ and if the inclusion is neutral, then $u(x) - a \cdot x \equiv 0$ outside the inclusion. This property means that the field outside the inclusion is not perturbed even though the inclusion is inserted. The weakly neutral inclusions perturb the fields mildly:

$$u(x) - a \cdot x = O(|x|^{-d}) \quad \text{as } |x| \to \infty. \tag{1.9}$$

If (1.9) holds for all constant vectors a, then the inclusion is said to be weakly neutral to multiple uniform fields. According to (1.2), in order for (1.9) to hold for all a, the corresponding PT must vanish. Thus weakly neutral inclusions are also called PT-vanishing inclusions. We now formulate the weakly neutral inclusion problem:

Weakly Neutral Inclusion Problem. *Given a domain D can we find a domain Ω containing $\overline{D}$ so that the resulting inclusion of core-shell structure becomes weakly neutral to multiple uniform fields, or equivalently, its PT vanishes.*

In Sect. 4, we present two classes of domains which admit coatings so that the resulting inclusions of core-shell structure become weakly neutral to multiple uniform fields. One class is the collection of domains D such that the coefficients b_D vanish. Here b_D is the leading coefficient of the conformal mapping from the exterior of the unit disk onto the exterior of D (see (4.1)). For such domains we construct the coating explicitly. This is a new result. The other class is that of small perturbations of disks, for which it is proved in [20] that there are coatings so that the resulting inclusions become weakly neutral to multiple uniform fields.

There is yet another way, other than coating, to achieve weak neutrality. It is by introducing an imperfect bonding parameter on the boundary of the given domain. We review the result of [16] on this in Sect. 5.

This paper is organized in the following way. Section 2 is to review general properties of the PT, including the Hashin-Shtrikman bounds. In Sect. 3, we discuss problems and progress on neutral inclusions and related over-determined problem for confocal ellipsoids and an equivalent formulation in terms of the Newtonian potential. We also include a discussion on Neumann ovaloids. Section 4 is to discuss progress on the weakly neutral inclusion problem. Section 5 is for discussion on the construction of weakly neutral inclusion by imperfect bonding parameters.

2 Layer Potentials and Polarization Tensors

In this section we represent the PT appearing in the dipolar expansion (1.2) in terms of layer potentials and recall the optimal Hashin-Shtrikman (HS) bounds on traces of the PT and its inverse.

2.1 Layer Potentials

Let $\Gamma(x)$ be the fundamental solution to the Laplacian, that is, $\Gamma(x) = 1/(2\pi)\log|x|$ in two dimensions, and $\Gamma(x) = -(4\pi|x|)^{-1}$ in three dimensions. Let D be a bounded simply connected domain with a Lipschitz continuous boundary. The single layer potential of a function $\varphi \in H^{-1/2}(\partial D)$ (the L^2-Sobolev space of order $-1/2$ on ∂D) is defined by

$$\mathcal{S}_{\partial D}[\varphi](x) := \int_{\partial D} \Gamma(x-y)\varphi(y)\, dS(y), \quad x \in \mathbb{R}^d, \tag{2.1}$$

where dS is the line or surface element on ∂D. Let ∂_ν denote the outward normal derivative on ∂D. It is well known (see, for example, [2]) that the following jump relation holds:

$$\partial_\nu \mathcal{S}_{\partial D}[\varphi](x)\big|_{\pm} = \left(\pm\frac{1}{2}I + \mathcal{K}^*_{\partial D}\right)[\varphi](x), \quad \text{a.e. } x \in \partial D, \tag{2.2}$$

where I is the identity operator and $\mathcal{K}^*_{\partial D}$ is the operator defined by

$$\mathcal{K}^*_{\partial D}[\varphi](x) = \frac{1}{\omega_d}\int_{\partial D} \frac{\langle x-y, \nu(x)\rangle}{|x-y|^d}\varphi(y)\, dS(y).$$

Here, $\langle\, ,\, \rangle$ the scalar product in $\mathbb{R}^d$. The boundary integral operator $\mathcal{K}^*_{\partial D}$ is called the Neumann-Poincaré (NP) operator.

2.2 *Polarization Tensors*

Let u_l, $1 \le l \le d$, be the solution to (CP) when $a \cdot x = x_l$ and the conductivity distribution σ is given by (1.1). Then it is known (see, e.g., [2]) that u_l can be represented as

$$u_l(x) = x_l + \mathcal{S}_{\partial D}[\varphi^{(l)}](x), \quad x \in \mathbb{R}^d, \tag{2.3}$$

where $\varphi^{(l)}$ is the unique solution in $H_0^{-1/2}(\partial D)$ ($H^{-1/2}(\partial D)$ functions with the mean zero) to the integral equation

$$\left(\frac{\sigma_c + \sigma_m}{2(\sigma_c - \sigma_m)} I - \mathcal{K}^*_{\partial D}\right)[\varphi^{(l)}] = \nu_l, \tag{2.4}$$

where ν_l is the l-th component of the outward unit normal vector field ν on ∂D. By expanding out the term $\mathcal{S}_{\partial D}[\varphi^{(l)}](x)$ in (2.3) as $|x| \to \infty$, we see that the PT $M = M(D) = (m_{ll'})_{l,l'=1}^d$ in this case is given by

$$m_{ll'} = \int_{\partial D} x_{l'}\varphi^{(l)}\, dS. \tag{2.5}$$

If the conductivity distribution σ is given by (1.5), the solution u_l can be represented as

$$u_l(x) = x_l + \mathcal{S}_{\partial D}[\varphi_1^{(l)}](x) + \mathcal{S}_{\partial \Omega}[\varphi_2^{(l)}](x), \quad x \in \mathbb{R}^d,$$

where $(\varphi_1^{(l)}, \varphi_2^{(l)}) \in H_0^{-1/2}(\partial D) \times H_0^{-1/2}(\partial \Omega)$ is the unique solution to the system of integral equations

$$\begin{bmatrix} -\lambda I + \mathcal{K}_{\partial D}^* & \partial_\nu \mathcal{S}_{\partial \Omega} \\ \partial_\nu \mathcal{S}_{\partial D} & -\mu I + \mathcal{K}_{\partial \Omega}^* \end{bmatrix} \begin{bmatrix} \varphi_1^{(l)} \\ \varphi_2^{(l)} \end{bmatrix} = - \begin{bmatrix} \nu_l^{\partial D} \\ \nu_l^{\partial \Omega} \end{bmatrix}. \tag{2.6}$$

Here we denote the unit outward normal vector fields on ∂D and $\partial \Omega$ by $\nu^{\partial D}$ and $\nu^{\partial \Omega}$, respectively. The numbers λ and μ are given by

$$\lambda = \frac{\sigma_c + \sigma_s}{2(\sigma_c - \sigma_s)} \quad \text{and} \quad \mu = \frac{\sigma_s + \sigma_m}{2(\sigma_s - \sigma_m)}. \tag{2.7}$$

For unique solvability of the integral equation we refer to [20]. In this case, the PT $M = M(D, \Omega) = (m_{ll'})_{l,l'=1}^d$ of the core-shell structure (D, Ω) is given by

$$m_{ll'} = \int_{\partial D} x_{l'} \varphi_1^{(l)} \, dS + \int_{\partial \Omega} x_{l'} \varphi_2^{(l)} \, dS. \tag{2.8}$$

It is known that M is a symmetric matrix (see, e.g., [2]).

2.3 Hashin-Shtrikman Bounds

If the conductivity distribution is given by (1.1), then the following optimal bounds on traces of the PT M and its inverse hold: with $k = \sigma_c / \sigma_m$,

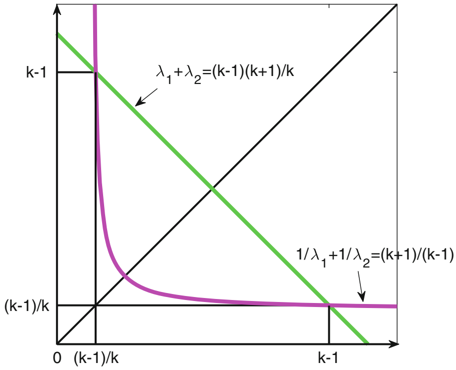
$$\mathrm{Tr}(M) < |D|(k-1)(d - 1 + \frac{1}{k}), \tag{2.9}$$

and

$$|D| \mathrm{Tr}(M^{-1}) \leq \frac{d - 1 + k}{k - 1}, \tag{2.10}$$

where Tr stands for trace. These bounds are obtained by Lipton [28] under the assumption of periodicity, and by Capdeboscq-Vogelius [7] without the assumption of periodicity, and called the Hashin-Shtrikman (HS) bounds after the names of scientists who found optimal bounds of effective properties of two phase composites, as described in the paragraph right after (1.8).

Fig. 3 Hashin-Shtrikman bounds for the PT



The first one is an upper bound (the green line in Fig. 3) and the second one is a lower one (the pink curve in Fig. 3). The upper bound is never attained by a domain, while the lower bound is attained by ellipses and ellipsoids, and the converse is also true. In fact, it is proved in [17, 18] that the simply connected domain whose PT satisfies the lower HS-bounds is an ellipse in two dimensions and an ellipsoid in three dimensions. This is an isoperimetric inequality for the PT and a generalized version of the Pólya-Szegö conjecture [35]. The original Pólya-Szegö conjecture asserts that the PT attains its minimal trace on and only on disks or balls. The constant trace lines of the PT are those parallel to the green line in Fig. 3. Thus the minimal trace is attained at the point of tangency of the line parallel to the green line to the pink curve. The generalized version asserts that if eigenvalues of the PT lies on the pink curve, then the domain must be an ellipse or an ellipsoid. The original Pólya-Szegö conjecture is now proved as a simple consequence of the generalized version. See Theorem 3.4 of this paper for more discussion on this. The bounds (2.9) and (2.10) are optimal in the sense that any matrix satisfying (2.9) and (2.10) is actually the PT associated with a domain (see [3, 6] for proofs).

3 Neutral Inclusions and an Over-determined Problem

In this section, the conductivity distribution σ is given by (1.5) with the inclusion (D, Ω) of core-shell structure. We assume that the conductivity of the matrix, σ_m, is in general anisotropic, i.e., a symmetric matrix. We review the result saying that if σ_m is isotropic, i.e., its eigenvalues are all the same, then the only inclusion of the core-shell structure neutral to multiple uniform fields is concentric balls. We also prove the equivalence of the neutral inclusion problem with an over-determined problem for confocal ellipsoids, and an equivalent formulation of the problem using the Newtonian potentials. In relation to these problems, we include at the end of this section a subsection on quadrature domains and Neumann ovaloids.

3.1 An Over-determined Problem for Confocal Ellipsoids

It was proved in [19] that if (D, Ω) is neutral to multiple uniform fields and $\sigma_c > \sigma_s$, then the following over-determined problem admits a solution:

$$\text{(ODP)} \begin{cases} \Delta w = 1 & \text{in } \Omega \setminus \overline{D}, \\ \nabla w = 0 & \text{on } \partial\Omega, \\ \nabla w(x) = Ax + b & \text{on } \partial D, \end{cases}$$

where A is a symmetric matrix and b is a constant vector, provided that ∂D is connected and $\mathbb{R}^3 \setminus \overline{D}$ is simply connected. This problem is over-determined since ∇w is prescribed on $\partial\Omega$ and ∂D. The matrix A is determined by σ_m. If σ_m is isotropic for example, so is A.

Let us briefly recall the proof. Suppose, after diagonalization, that

$$\sigma_m = \text{diag}[\sigma_{m,1}, \sigma_{m,2}, \sigma_{m,3}]. \tag{3.1}$$

Let e_j, $j = 1, 2, 3$, be the standard basis of $\mathbb{R}^3$ and let u_j be the solution to (CP) when $a = e_j$. The inclusion (D, Ω) being neutral to multiple uniform fields means that $u_j(x) - x_j = 0$ in $\mathbb{R}^3 \setminus \Omega$ for $j = 1, 2, 3$. Let

$$\beta_j := \frac{\sigma_{m,j}}{\sigma_s} - 1, \tag{3.2}$$

and

$$v = (\beta_1^{-1} u_1, \beta_2^{-1} u_2, \beta_3^{-1} u_3)^T. \tag{3.3}$$

The crux of the proof in [19] lies in proving that v is linear inside D. In fact, it is proved that $v(x) = c_0 x + b_0$ $(x \in D)$ for some constant c_0 and vector b_0. It is here where the assumption $\sigma_c > \sigma_s$ is needed.[1] It is then shown that ∇v is symmetric, and hence, thanks to the assumption that ∂D is connected and $\mathbb{R}^3 \setminus \overline{D}$ is simply connected, there is a function ψ in $\overline{\Omega} \setminus D$ such that $v = \nabla\psi$. Moreover, $\Delta\psi = \sum_{j=1}^3 \beta_j^{-1} + 1$ in $\Omega \setminus \overline{D}$. Thus w, defined by

$$w(x) := \psi(x) - \frac{1}{2} \sum_{j=1}^{3} \beta_j^{-1} x_j^2, \tag{3.4}$$

is the solution to (ODP) with $A = c_0 I - \text{diag}[\beta_1^{-1}, \beta_2^{-1}, \beta_3^{-1}]$ (I is the identity matrix). If σ_m is isotropic, so is A as mentioned before. The converse is also true,

[1] We believe it is true without this assumption even though we do not know how to prove it.

namely, if (ODP) admits a solution, then (D, Ω) is neutral. For this, see Theorem 3.5 below.

Remark The assumption that ∂D is connected and $\mathbb{R}^3 \setminus \overline{D}$ is simply connected in [19, Theorem 1.2] can be replaced with the weaker one that $\Omega \setminus \overline{D}$ is connected. Indeed, instead of using Stokes' theorem, by combining the formula [19, (2.18)] with the fact that $v(x) = c_0 x + b_0$ $(x \in D)$, we see that the function ψ is explicitly given by

$$\psi(x) = c_0 \left(1 - \frac{\sigma_c}{\sigma_s}\right) \int_D \Gamma(x-y)dy + \frac{1}{2} x \cdot Bx + \int_\Omega \Gamma(x-y)dy. \tag{3.5}$$

Hence the function $w = w(x)$ is given by

$$w(x) = c_0 \left(1 - \frac{\sigma_c}{\sigma_s}\right) \int_D \Gamma(x-y)dy + \int_\Omega \Gamma(x-y)dy. \tag{3.6}$$

For a domain D in three dimensions and a domain Ω containing $\overline{D}$, the assumption that $\Omega \setminus \overline{D}$ is connected is really more general than that ∂D is connected and $\mathbb{R}^3 \setminus \overline{D}$ is simply connected. In fact, this general assumption allows us to choose the genus of a closed surface ∂D arbitrarily. If the genus does not equal zero, $\mathbb{R}^3 \setminus \overline{D}$ is not simply connected, but $\Omega \setminus \overline{D}$ is connected.

Note that if Ω and D are concentric balls centered at the origin whose respective radii are r_e and r_i, then the solution w to (ODP) is given by

$$w(x) = \frac{r_e^3}{3|x|} + \frac{1}{6}|x|^2. \tag{3.7}$$

In this case, $b = 0$ and $A = \frac{1}{3}(-r_e^3/r_i^3 + 1)I$ which is isotropic. We emphasize that w is radial in this case.

It is shown in [19] that confocal ellipsoids admit a solution to (ODP). In fact, if ∂D is an ellipsoid given by

$$\frac{x_1^2}{c_1^2} + \frac{x_2^2}{c_2^2} + \frac{x_3^2}{c_3^2} = 1, \tag{3.8}$$

the confocal ellipsoidal coordinate ρ is given by

$$\frac{x_1^2}{c_1^2 + \rho} + \frac{x_2^2}{c_2^2 + \rho} + \frac{x_3^2}{c_3^2 + \rho} = 1, \tag{3.9}$$

and the confocal ellipsoid $\partial\Omega$ is given by $\rho = \rho_0$ for some $\rho_0 > 0$. Let

$$g(\rho) = (c_1^2 + \rho)(c_2^2 + \rho)(c_3^2 + \rho), \tag{3.10}$$

and

$$\varphi_j(\rho) = \int_\rho^\infty \frac{1}{(c_j^2 + s)\sqrt{g(s)}} ds, \quad j = 1, 2, 3. \tag{3.11}$$

Then the function w, defined by

$$w(x) = \frac{1}{2}\int_\rho^\infty \frac{1}{\sqrt{g(s)}} ds - \frac{1}{2}\sum_{j=1}^{3} \varphi_j(\rho) x_j^2 + \frac{1}{2}\sum_{j=1}^{3} \varphi_j(\rho_0) x_j^2, \tag{3.12}$$

is a solution of (ODP) with the equation $\Delta w = 1$ replaced by $\Delta w = 2/\sqrt{g(\rho_0)}$ and the matrix A given by

$$A = \mathrm{diag}[\varphi_1(\rho_0) - \varphi_1(0), \varphi_2(\rho_0) - \varphi_2(0), \varphi_3(\rho_0) - \varphi_3(0)].$$

Note that $b = 0$ and A is anisotropic.

The following problem arises naturally:

An over-determined problem for confocal ellipsoids. *Prove that if (ODP) admits a solution (in $H^1(\Omega \setminus D)$), then Ω and D are confocal ellipsoids (or ellipses) and the common foci (when the volumes are fixed) is determined by the eigenvalues of A.*

The two-dimensional problem can be solved using the conformal mapping between $\Omega \setminus \overline{D}$ and an annulus [1, Theorem 10, p. 255]. If A is isotropic, this problem is solved in three dimensions as the following theorem shows. The case of anisotropic A has not been solved and is open.

Theorem 3.1 (**[19]**) *Let D and Ω be bounded domains with Lipschitz boundaries in $\mathbb{R}^3$ with $\overline{D} \subset \Omega$. Suppose that $\Omega \setminus \overline{D}$ is connected. If (ODP) admits a solution for $A = cI$ for some constant c where I is the identity matrix in three dimensions, then D and Ω are concentric balls.*

As an immediate consequence, we have the following Corollary:

Corollary 3.2 *Suppose that $\sigma_c > \sigma_s$ and σ_m is isotropic in addition to hypotheses of Theorem 3.1. If (D, Ω) is neutral to multiple uniform fields, then D and Ω are concentric balls in three dimensions.*

Theorem 3.1 is proved as follows. Suppose that (ODP) admits a solution w for $A = cI$. Then, by (ODP), $|\Omega \setminus D| = -3c|D|$, and hence $c \neq 0$. Introduce the angular derivative:

$$A_{ij} = (x_j + \frac{b_j}{c})\partial_i - (x_i + \frac{b_i}{c})\partial_j, \quad i \neq j,$$

where $b = (b_1, b_2, b_3)$ is the constant vector appearing in (ODP) and ∂_j denotes the partial derivative with respect to x_j-variable. One can see that $\Delta A_{ij} w = A_{ij} \Delta w = 0$ in $\Omega \setminus \overline{D}$, $A_{ij} w = 0$ on $\partial\Omega$, and $A_{ij} w = 0$ on ∂D provided that $A = cI$. Hence $A_{ij} w = 0$ in $\Omega \setminus \overline{D}$, which implies that w is radial. Using this fact, one can prove Theorem 3.1. We emphasize that this argument using the angular derivative does not work if A is not isotropic.

3.2 The Newtonian Potential Formulation of the Problem

Consider the conductivity problem (CP) when the conductivity distribution σ is given by (1.1). As one can see from Fig. 4, the field inside D is uniform if D is an ellipse (or an ellipsoid). This a rather surprising fact that the field inside elliptic or ellipsoidal inclusions is uniform seems to have been known for long time and its proof goes back to Poisson (1826) and Maxwell (1873) (see [30]). The converse is also true as we explain it in the sequel. For doing so, we need to recall the notion of the Newtonian potential.

The Newtonian potential of the domain D, which we denote by N_D, is defined by

$$N_D(x) := \frac{1}{|D|} \int_D \Gamma(x - y) dy. \tag{3.13}$$

Usually the Newtonian potential is defined without the averaging factor $1/|D|$, but here it is more convenient to define it with the averaging factor. Since $\Delta N_D(x) = 1/|D|$ for $x \in D$, we have

$$N_D = \text{a quadratic part} + \text{a harmonic part} \quad \text{in } D.$$

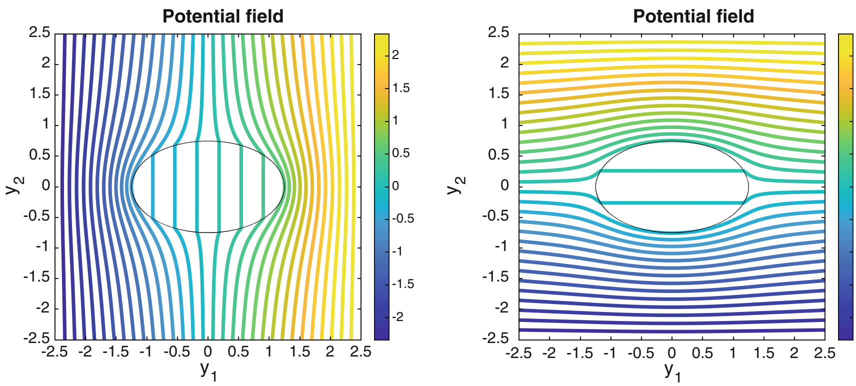


Fig. 4 Field inside an ellipse or an ellipsoid is uniform

If D is an ellipse or an ellipsoid, then the harmonic part of N_D is quadratic and so is N_D inside D. In fact, this is equivalent to the fact that the field inside elliptic or ellipsoidal inclusions is uniform. Moreover, this property of the Newtonian potential's being a quadratic function inside the domain characterizes the ellipsoid and the ellipse:

Theorem 3.3 *Let D be a simply connected bounded domain with the Lipschtiz boundary. If N_D is quadratic inside D, then D is an ellipse or an ellipsoid.*

This characterization of ellipsoids was proved by Dive in 1931 [9] and by Nikliborc in 1932 [34] (see also [8]). The reason why Dive and Nikliborc considered this problem was to prove the converse of a theorem due to Newton. Let D be a simply connected domain whose center of mass is $0 \in D$ and let λD be a dilation of D by $\lambda > 1$, i.e., $\lambda D = \{\lambda x : x \in D\}$. A theorem of Newton states that if D is an ellipsoid, then the gravitational force induced by the mass $\lambda D \setminus D$ is zero in D [23]. Dive and Nikliborc independently proved that the converse is true: If the gravitational force induced by the uniform mass on $\lambda D \setminus D$ is zero in D, then D must be an ellipsoid.

The following theorem was proved using the characterization of ellipsoids by Newtonian potentials.

Theorem 3.4 ([17, 18]) *The following are equivalent for a simply connected bounded domain D:*

(i) *The polarization tensor M associated with D attains the lower Hashin-Shtrikman bound (2.10).*
(ii) *The solution to the conductivity problem (CP) when the conductivity distribution σ is given by (1.1) is linear inside D.*
(iii) *D is an ellipse in two dimensions and an ellipsoid in three dimensions.*

This theorem resolves the generalized Pólya-Szegö conjecture explained before. That the linearity of the solution to (CP) when σ is given by (1.1) characterizes ellipsoids is known as the Eshelby's conjecture in the field of composites theory. Actually the Eshelby's conjecture (1961) [10] asserts that the inclusion inside which the field is uniform (or equivalently, the strain is constant) for a uniform loading is an ellipse or an ellipsoid. The corresponding conjecture for the electro-static case is proved by Theorem 3.4. The Eshelby's conjecture (for the elasto-static case) was proved by Sendecyj [37] in two dimensions and by Kang-Milton [18] and Liu [29] in three dimensions.

We now formulate the over-determined problem for the confocal ellipsoids in terms of the Newtonian potential. It is proved in [19] that the problem (ODP) admits a solution if and only if

$$N_\Omega(x) - N_D(x) = \begin{cases} 0, & x \in \mathbb{R}^3 \setminus \Omega, \\ \text{a quadratic polynomial}, & x \in D. \end{cases} \tag{3.14}$$

Now the problem is to show that D and Ω are confocal ellipsoids if (3.14) holds. If Ω and D are confocal ellipsoids, then both N_Ω and N_D are quadratic polynomials inside D, and so is $N_\Omega - N_D$. A proof of the fact that $N_\Omega = N_D$ outside Ω can be found in [30, p.61]. In the problem (3.14), the quadratic polynomial inside D determines the common foci of D and Ω. For example, one can see from Theorem 3.1 that if the quadratic polynomial is of the form $c|x|^2 +$ l.o.t, then Ω and D are concentric balls.

Now we can show the following theorem.

Theorem 3.5 *Suppose that ∂D is connected and $\mathbb{R}^3 \setminus \overline{D}$ is simply connected. Consider the following statements:*

(i) (D, Ω) *is neutral to multiple uniform fields for some σ given by (1.5).*
(ii) *The problem (ODP) admits a solution for some A and b.*
(iii) *The Newtonian potential formulation (3.14) holds.*

The following implications hold to be true:

$$\text{(i)} \Rightarrow \text{(ii) } \textit{if } \sigma_c > \sigma_s, \quad \text{(ii)} \Rightarrow \text{(iii)}, \quad \text{(iii)} \Rightarrow \text{(i)}. \tag{3.15}$$

Proof The first implication was proved in [19] and the proof is briefly reviewed at the beginning of this section. The second implication was proved in the same paper. We prove the third implication, namely, if (3.14) holds, then there are conductivities σ_c, σ_s and σ_m such that (D, Ω) is neutral to multiple uniform fields.

Let

$$w(x) := |\Omega|(N_\Omega(x) - N_D(x)). \tag{3.16}$$

By a rotation and a translation, if necessary, we may assume that

$$w(x) = \begin{cases} 0, & x \in \mathbb{R}^3 \setminus \Omega, \\ \sum_{j=1}^3 \alpha_j x_j^2 + \alpha, & x \in D, \end{cases} \tag{3.17}$$

for some constants $\alpha_1, \alpha_2, \alpha_3$ and α. In particular, there is no linear term in the quadratic function. Define u_j by

$$u_j(x) := \beta_j \partial_j \left(w(x) + \frac{1}{2} \sum_{j=1}^{3} \beta_j^{-1} x_j^2 \right), \tag{3.18}$$

where β_j's are defined by (3.2) with the conductivities to be determined later.

We claim that u_j is the solution to (CP) satisfying (1.7) with $a = e_j$. In fact, we see from the definition (3.16) of w that

$$\begin{aligned}\partial_j w(x) &= \int_\Omega \partial_{x_j}\Gamma(x-y)\,dy - f^{-1}\int_D \partial_{x_j}\Gamma(x-y)\,dy \\ &= -\int_\Omega \partial_{y_j}\Gamma(x-y)\,dy + f^{-1}\int_D \partial_{y_j}\Gamma(x-y)\,dy \\ &= -\int_{\partial\Omega} \Gamma(x-y)\,\nu_j(y)\,dS(y) + f^{-1}\int_{\partial D} \Gamma(x-y)\,\nu_j(y)\,dS(y),\end{aligned}$$

where the last equality follows from the divergence theorem. Here, f is the volume fraction, namely, $f = |D|/|\Omega|$. Thus,

$$\partial_j w(x) = -\mathcal{S}_{\partial\Omega}[\nu_j](x) + f^{-1}\mathcal{S}_{\partial D}[\nu_j](x). \tag{3.19}$$

Since the single layer potential is continuous across the boundary, u_j is continuous across the interfaces $\partial\Omega$ and ∂D.

Thanks to the jump relation (2.2) and (3.17), we have, on $\partial\Omega$,

$$\partial_\nu(\partial_j w)|_+ = -\left(\frac{1}{2}I + \mathcal{K}^*_{\partial\Omega}\right)[\nu_j] + f^{-1}\partial_\nu\mathcal{S}_{\partial D}[\nu_j] = 0,$$

and hence

$$\partial_\nu(\partial_j w)|_- = -\left(-\frac{1}{2}I + \mathcal{K}^*_{\partial\Omega}\right)[\nu_j] + f^{-1}\partial_\nu\mathcal{S}_{\partial D}[\nu_j] = \nu_j.$$

Thus,

$$\sigma_{m,j}\partial_\nu u_j|_+ = \sigma_{m,j}\nu_j,$$

and

$$\sigma_s\partial_\nu u_j|_- = \sigma_s(\beta_j + 1)\nu_j,$$

on $\partial\Omega$. Since $\sigma_{m,j} = \sigma_s(\beta_j + 1)$ by the definition (3.2) of β_j, we infer that

$$\sigma_{m,j}\partial_\nu u_j|_+ = \sigma_s\partial_\nu u_j|_- \quad \text{on } \partial\Omega. \tag{3.20}$$

Similarly, thanks to (2.2), we have from (3.17) and (3.19) that, on ∂D,

$$\partial_\nu(\partial_j w)|_- = -\partial_\nu\mathcal{S}_{\partial\Omega}[\nu_j] + f^{-1}\left(-\frac{1}{2}I + \mathcal{K}^*_{\partial D}\right)[\nu_j] = 2\alpha_j\nu_j,$$

and hence

$$\partial_\nu(\partial_j w)|_+ = -\partial_\nu \mathcal{S}_{\partial\Omega}[\nu_j] + f^{-1}\left(\frac{1}{2}I + \mathcal{K}^*_{\partial D}\right)[\nu_j] = (2\alpha_j + f^{-1})\nu_j.$$

Thus, we have

$$\sigma_c \partial_\nu u_j|_- = \sigma_c \beta_j (2\alpha_j + \beta_j^{-1})\nu_j$$

and

$$\sigma_s \partial_\nu u_j|_+ = \sigma_s \beta_j (2\alpha_j + f^{-1} + \beta_j^{-1})\nu_j.$$

Thus,

$$\sigma_s \partial_\nu u_j|_+ = \sigma_c \partial_\nu u_j|_- \quad \text{on } \partial D \tag{3.21}$$

if and only if

$$\sigma_c \beta_j (2\alpha_j + \beta_j^{-1}) = \sigma_s \beta_j (2\alpha_j + f^{-1} + \beta_j^{-1}),$$

or equivalently, by letting $\gamma := 1 - \sigma_c/\sigma_s$,

$$2\alpha_j \beta_j \gamma + f^{-1}\beta_j + \gamma = 0. \tag{3.22}$$

So if we choose γ and β_j (or σ_c, σ_s and $\sigma_{m,j}$) so that (3.22) holds, then (D, Ω) is neutral to multiple uniform fields.

There is yet another restriction when we solve (3.22) for γ and β_j; σ_c, σ_s and $\sigma_{m,j}$ should be positive. This condition can be easily fulfilled. In fact, the following relation follows easily from (3.22):

$$\sigma_{m,j} = \sigma_s \left(1 - \frac{\gamma}{2\alpha_j \gamma + f^{-1}}\right).$$

The quantity $2\alpha_j\gamma + f^{-1}$ in the above is nonzero since γ can be chosen small as we see shortly. Thus the positivity is achieved if

$$1 - \frac{\gamma}{2\alpha_j \gamma + f^{-1}} > 0$$

which in turn can be achieved by taking γ so that

$$|\gamma| \le \min_{1\le j\le 3} \frac{f^{-1}}{|1 - 2\alpha_j| + 1}.$$

This completes the proof. □

In the course of the proof, we derived the neutrality condition for confocal ellipsoids.

Corollary 3.6 *Let D and Ω be confocal ellipsoids whose boundaries are respectively given by (3.8) and (3.9). If the conductivity distribution σ given by (1.5) and (3.1) satisfies*

$$2\alpha_j f\left(\frac{\sigma_{m,j}}{\sigma_s}-1\right)\left(1-\frac{\sigma_c}{\sigma_s}\right)+\left(\frac{\sigma_{m,j}}{\sigma_s}-1\right)+f\left(1-\frac{\sigma_c}{\sigma_s}\right)=0, \quad j=1,2,3, \tag{3.23}$$

then (D,Ω) is neutral to multiple uniform fields. Here f is the volume fraction and α_j's are constants given by (3.17), i.e.,

$$\alpha_j=-\frac{1}{4}\int_0^{\rho_0}\frac{1}{c_j^2+s}\frac{\sqrt{(c_1^2+\rho_0)(c_2^2+\rho_0)(c_3^2+\rho_0)}}{\sqrt{(c_1^2+s)(c_2^2+s)(c_3^2+s)}}\,ds, \quad j=1,2,3. \tag{3.24}$$

Let us look into the neutrality condition (3.22) or (3.23) further. According to (3.16) and (3.17),

$$1-f^{-1}=\Delta w=2\sum_{j=1}^{3}\alpha_j \quad \text{in } D.$$

We then have from (3.22)

$$1-f^{-1}=-\frac{3f^{-1}}{\gamma}-\sum_{j=1}^{3}\frac{1}{\beta_j},$$

and hence

$$-1+\frac{3}{\gamma}+f\left(1+\sum_{j=1}^{3}\frac{1}{\beta_j}\right)=0.$$

Writing it in terms of conductivities, we have

$$\frac{2\sigma_s+\sigma_c}{\sigma_s-\sigma_c}+\frac{f}{3}\sum_{j=1}^{3}\frac{\sigma_{m,j}+2\sigma_s}{\sigma_{m,j}-\sigma_s}=0. \tag{3.25}$$

This is a necessary neutrality condition when σ_m is anisotropic. In particular, if σ_m is a scalar, namely, $\sigma_{m,j}=\sigma_m$, then it is exactly the neutrality condition (1.6) of concentric balls.

3.3 Quadrature Domains-Neumann Ovaloids

Let us look further into the Newtonian potential formulation (3.14) of the problem. The problem is to prove that if it holds, then D and Ω must be confocal ellipsoids. We show that the condition (3.14) in $\mathbb{R}^3 \setminus \Omega$ alone does not yield the answer.

The condition (3.14) in $\mathbb{R}^3 \setminus \Omega$ yields

$$\int_{\partial\Omega} N_\Omega(x) g(x)\, dS = \int_{\partial\Omega} N_D(x) g(x)\, dS$$

for any $g \in H^{-1/2}(\partial\Omega)$. By changing the order of integrations, we have

$$\frac{1}{|\Omega|}\int_\Omega u(x)dx = \frac{1}{|D|}\int_D u(x)dx, \tag{3.26}$$

where $u(x) = \mathcal{S}_{\partial\Omega}[g](x)$, $x \in \Omega$. Thus (3.26) holds for all $u \in H^1_h(\Omega)$ where subscript h means that it is a collection of harmonic functions in Ω. The condition (3.26) does not guarantee that D and Ω are confocal ellipsoids as will be seen in what follows, and the condition (3.14) in D should be utilized.

In fact, an open set $\Omega \subset \mathbb{R}^d$ is called a *quadrature domain* (see, e.g., [38, (4.1)], and also [11, 36]) if there exists a distribution μ with a compact support in Ω such that

$$\int_\Omega u(x)dx = \langle \mu, u\rangle \quad \text{for all } u \in H^1_h(\Omega). \tag{3.27}$$

The simplest class of quadrature domains may be balls: It is well known as the mean value property:

$$\int_\Omega u(x)dx = |\Omega| u(c) \quad \text{for all } u \in H^1_h(\Omega), \tag{3.28}$$

where c is the center of the ball. In this case the distribution μ is the point mass (the Dirac delta) multiplied by the volume of Ω.

Ellipsoids are also quadrature domains. Let

$$\Omega = \left\{ x \in \mathbb{R}^d \mid \sum_{i=1}^d \frac{x_i^2}{a_i^2} < 1 \right\} \quad (a_1 \geq a_2 \geq \cdots \geq a_{d-1} > a_d > 0),$$

and let

$$F = \left\{ x \in \mathbb{R}^{d-1} \mid \sum_{i=1}^{d-1} \frac{x_i^2}{a_i^2 - a_d^2} < 1 \right\}.$$

The lower dimensional set F is called the focal ellipsoid of Ω. The following quadrature identity holds (see, e.g., [26, 27]):

$$\int_\Omega u(x)dx = 2\prod_{i=1}^{d-1} \frac{a_i}{(a_i^2 - a_d^2)^{1/2}} \int_F \left(1 - \sum_{i=1}^{d-1} \frac{x_i^2}{a_i^2 - a_d^2}\right)^{1/2} u(x', 0)\, dx' \tag{3.29}$$

for all $u \in H_h^1(\Omega)$. Here, x' is $(x_1, \ldots, x_{d-1})$. Note that if D and Ω are confocal ellipsoids, then their focal ellipsoids are the same, and hence (3.26) holds.

There is yet another class of domains satisfying (3.26). A domain $\Omega \subset \mathbb{R}^d$ is called a *Neumann ovaloid* if it admits the following quadrature identity

$$\int_\Omega u(x)dx = C\,(u(p_1) + u(p_2)) \quad \text{for all } u \in H_h^1(\Omega), \tag{3.30}$$

where p_1 and p_2 are distinct points in $\mathbb{R}^d$ and $C > 0$ is a constant. If C is sufficiently small compared to $|p_1 - p_2|$, then a union of two balls of the same radius centered at p_1 and p_2 satisfies the identity (3.30). However, if C is sufficiently large, then there is an axially-symmetric domain satisfying (3.30). For example, if Ω is the domain in $\mathbb{R}^2$ bounded by the curve

$$\left(x_1^2 + x_2^2\right)^2 = \alpha^2 \left(x_1^2 + x_2^2\right) + 4\varepsilon^2 x_1^2, \tag{3.31}$$

where α and ε are some positive constants (see Fig. 5), then it admits a quadrature identity (3.30) with $C = |\Omega|/2$ (see [38, pp. 19–20] for a proof). In this case, the relation among $|\Omega|$, α and ϵ is given by $|\Omega| = \pi(\alpha^2 + 2\varepsilon^2)$. These two-dimensional Neumann ovals were discovered by C. Neumann [33]. The uniqueness of the Neumann oval in two dimensions was proved in [36]. The existence and uniqueness of the higher dimensional Neumann ovaloids are known (see [22] and references therein), but there is no known explicit expression except four-dimensional (and two-dimensional) ones, to the best of our knowledge. We refer to a recent paper [22]

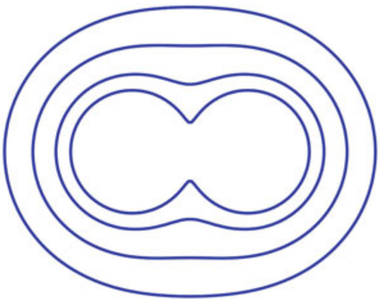


Fig. 5 Neumann ovals with same foci

for an explicit parametrization of a four-dimensional Neumann ovaloid. If (D, Ω) are Neumann ovaloids with same foci, then (3.26) holds.

4 Weakly Neutral Inclusions

We now consider the weakly neutral inclusion problem presented at the end of Introduction, namely, the problem of coating a given domain of general shape by another domain so that the resulting inclusion of core-shell structure satisfies the weak neutrality condition (1.9), namely, its polarization tensor vanishes. In this section we present two classes of domains for which the weakly neutral inclusion problem can be solved. The first one is defined by a conformal mapping from the exterior of the unit disk onto the exterior of the domain, and construction of the coating is explicit. This result is new. The other class are small perturbations of a disk for which existence of a coating is proved. This result is from [20, 21]. Note that neutral inclusions are weakly neutral. Thus concentric disks or balls can be realized as weakly neutral inclusions. However, no other examples of weakly neutral inclusions were known.

4.1 b_D-vanishing Domains

Let D be a bounded simply connected domain in $\mathbb{C} = \mathbb{R}^2$ with the Lipschitz continuous boundary, and let $z = \Phi(\zeta)$ is the Riemann mapping from $|\zeta| > 1$ ($\mathbb{C} \setminus \overline{U}$, where U is the unit disk) onto $\mathbb{C} \setminus \overline{D}$. The conformal mapping Φ takes the form

$$\Phi(\zeta) = b_{-1}\zeta + b_0 + \frac{b_1}{\zeta} + \text{h.o.t.}$$

By dilating and translating D if necessary, we assume that $b_{-1} = 1$ and $b_0 = 0$, and denote b_1 by b_D, so that the Riemann mapping Φ takes the form

$$\Phi(\zeta) = \zeta + \frac{b_D}{\zeta} + \text{h.o.t.} \tag{4.1}$$

The domains we consider in this subsection are such that $b_D = 0$. The following lemma shows that there are plenty of domains satisfying this condition.

Lemma 4.1 *Let D be a simply connected domain and suppose that the Riemann mapping Φ from $\mathbb{C} \setminus \overline{U}$ onto $\mathbb{C} \setminus \overline{D}$ is of the form (4.1). If D is invariant under the rotation around* 0 *by the angle $2\pi/n$ for a positive integer $n \geq 3$, then $b_D = 0$.*

Proof Let $\zeta_n = e^{i2\pi/n}$. Define the function Ψ by

$$\Psi(\zeta) := \zeta_n \Phi(\zeta_n^{-1}\zeta) = \zeta + \frac{b_D \zeta_n^2}{\zeta} + \text{h.o.t.}$$

From the rotational symmetry of D, Ψ is also the Riemann mapping from $\mathbb{C} \setminus \overline{U}$ onto $\mathbb{C} \setminus \overline{D}$. Hence, by the uniqueness of the Riemann mapping of the form (4.1), we infer that $\Psi = \Phi$. In particular,

$$b_D = b_D \zeta_n^2.$$

Since $n \geq 3$, $\zeta_n^2 \neq 1$, and hence $b_D = 0$. □

We want to coat D by an another bounded domain Ω so that the PT of the coated structure vanishes. Let the conductivity distribution σ be given by (1.5). Furthermore, we assume that $\sigma_c = 0$ or ∞. This assumption is required because we use the conformal mapping from $\mathbb{C} \setminus \overline{U}$ onto $\mathbb{C} \setminus \overline{D}$.

If $\sigma_c = \infty$, (CP) is of the form

$$(\text{CP})_\infty \quad \begin{cases} \nabla \cdot \sigma \nabla u = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ u = \lambda (\text{constant}) & \text{on } \partial D, \\ u(x) - a \cdot x = O(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases}$$

where $\sigma = \sigma_s \chi(\Omega \setminus D) + \sigma_m \chi(\mathbb{R}^2 \setminus \Omega)$. The constant λ is determined by the condition $\int_{\partial D} \partial_\nu u|_+ = 0$. If $\sigma_c = 0$, then the problem, which we denote by $(\text{CP})_0$, is $(\text{CP})_\infty$ with boundary condition on ∂D is replaced with $\partial_\nu u = 0$. The problem is to find σ_s, σ_m and Ω so that the solution u to either $(\text{CP})_\infty$ or $(\text{CP})_0$ satisfies the weak neutrality condition (1.9).

Since $u(x) - a \cdot x$ tends to 0 as $|x| \to \infty$ and D, Ω are simply connected, there are functions U_m and U_s analytic in $\mathbb{C} \setminus \overline{\Omega}$ and $\Omega \setminus \overline{D}$, respectively, such that $\Re U_m = u$ and $\Re U_s = u$ in their respective domains ($\Re$ stands for the real part). One can see using the Cauchy-Riemann equations that the transmission conditions to be satisfied by u on $\partial\Omega$ is equivalent to

$$(1 + \sigma_s/\sigma_m) U_s + (1 - \sigma_s/\sigma_m)\overline{U_s} = 2U_m + c \quad \text{on } \partial\Omega, \tag{4.2}$$

for some constant c. Moreover, U_m admits the following expansion at ∞:

$$U_m(z) = \alpha z + \frac{\alpha_1(\alpha)}{z} + \text{h.o.t.},$$

where $\alpha = a_1 - i a_2$. Thus the weak neutrality condition (1.9) is equivalent to

$$\alpha_1(\alpha) = 0 \quad \text{for all } \alpha \text{ (or equivalently, for } \alpha = 1, i). \tag{4.3}$$

With the conformal mapping Φ in (4.1), let $V_m^\alpha = U_m \circ \Phi$ and $V_s^\alpha = U_s \circ \Phi$. Then we have

$$V_m^\alpha(\zeta) = \alpha\Phi(\zeta) + \frac{\alpha_1(\alpha)}{\Phi(\zeta)} + \text{h.o.t.} = \alpha\zeta + \frac{\alpha b_D + \alpha_1(\alpha)}{\zeta} + \text{h.o.t.}\,. \tag{4.4}$$

Let U' be a simply connected domain containing $\overline{U}$ defined by

$$\Phi(\partial U') = \partial\Omega. \tag{4.5}$$

The transmission condition (4.2) is transformed by Φ to

$$(1 + \sigma_s/\sigma_m)V_s^\alpha + (1 - \sigma_s/\sigma_m)\overline{V_s^\alpha} = 2V_m^\alpha + c \quad \text{on } \partial U'. \tag{4.6}$$

If $b_D = 0$, then (4.4) takes the form

$$V_m^\alpha(\zeta) = \alpha\zeta + \frac{\alpha_1(\alpha)}{\zeta} + \text{h.o.t.}$$

Thus (4.3) is fulfilled if and only if V_m^α satisfies

$$|V_m^\alpha(\zeta) - \alpha\zeta| = O\left(|\zeta|^{-2}\right) \quad \text{as } |\zeta| \to \infty, \tag{4.7}$$

for $\alpha = 1, i$. Since $\Re(V^\alpha)$ is the solution to $(\text{CP})_\infty$ with D and Ω replaced by U and U', respectively, (4.7) is satisfied if (U, U') is a neutral inclusion. Since $\sigma_c = \infty$ and U is the unit disk, it suffices to take U' to be a disk of radius r and the conductivities σ_s, σ_m to satisfy the neutrality condition (1.6), which is

$$(\sigma_m/\sigma_s - 1) - r^2(\sigma_m/\sigma_s + 1) = 0 \tag{4.8}$$

if $\sigma_c = \infty$, and

$$(\sigma_m/\sigma_s - 1) + r^2(\sigma_m/\sigma_s + 1) = 0 \tag{4.9}$$

if $\sigma_c = 0$.

We arrive at the following theorem:

Theorem 4.2 *Let D be a simply connected domain such that $b_D = 0$ after rotation and translation and suppose that its conductivity σ_c is either ∞ or 0. If we take σ_s, σ_m and r to satisfy (4.8) if $\sigma_c = \infty$ and (4.9) if $\sigma_c = 0$, then (D, Ω) where Ω is defined by (4.5) with $\partial U'$ being the circle with the radius r is weakly neutral to multiple uniform fields.*

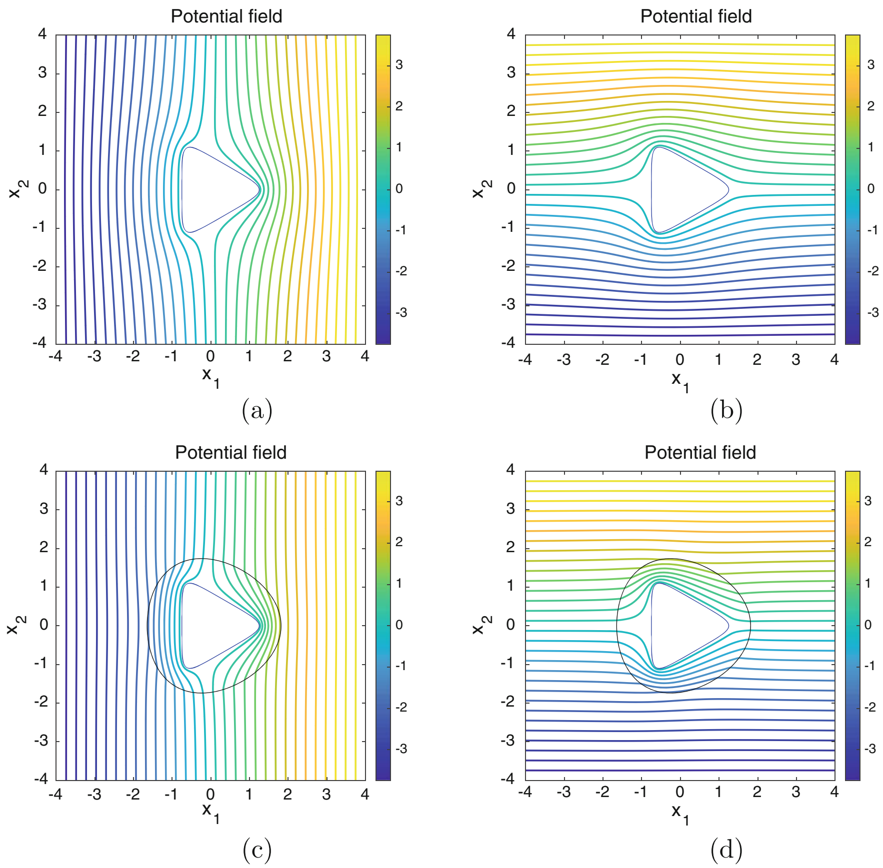


Fig. 6 The core-shell structure defined by the conformal mapping $\Phi(\zeta) = \zeta + \frac{1}{4\zeta^2}$. Field perturbations with the coating ((**c**) and (**d**)) are much weaker than those without it ((**a**) and (**b**))

We now present two results of numerical experiments. In Fig. 6, the conformal mapping for the domain D and the conductivity σ_s are given by

$$\Phi(\zeta) = \zeta + \frac{1}{4\zeta^2} \quad \text{and} \quad \sigma_s = 0.5. \tag{4.10}$$

It shows the domains D and its coating Ω determined by the method described above. Figure 7 is with the conformal mapping

$$\Phi(\zeta) = \zeta + \frac{1}{4\zeta^3} \quad \text{and} \quad \sigma_s = 0.3. \tag{4.11}$$

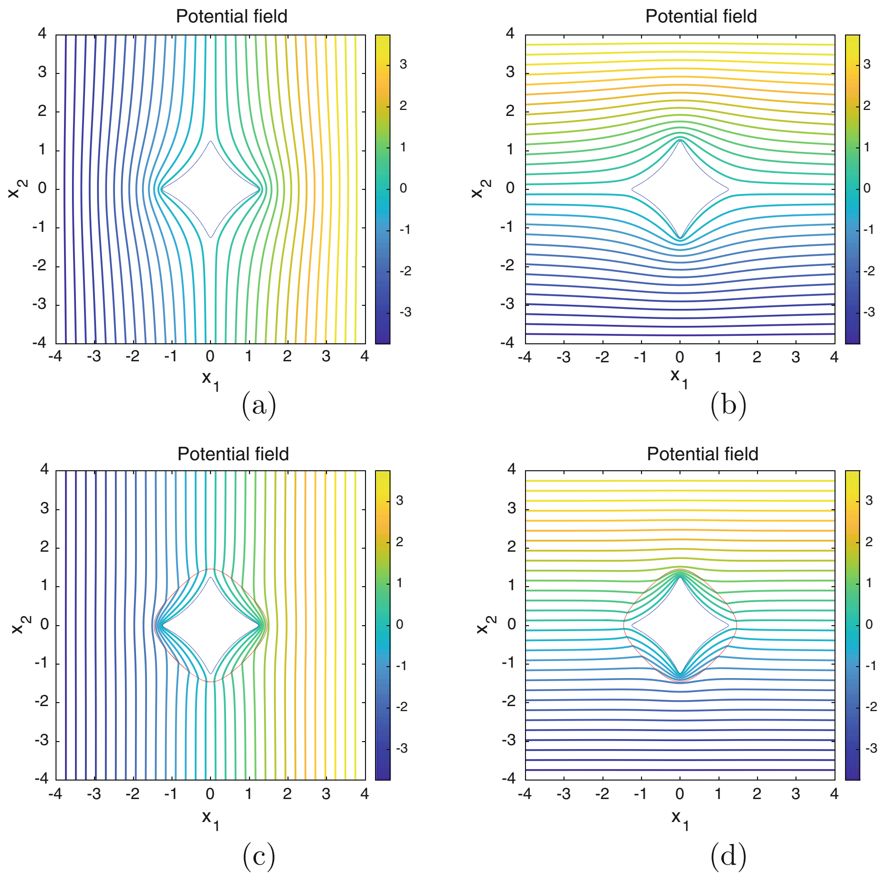


Fig. 7 The core-shell structure defined by the conformal mapping $\Phi(\zeta) = \zeta + \frac{1}{4\zeta^3}$. Field perturbations with the coating ((**c**) and (**d**)) are much weaker than those without it ((**a**) and (**b**))

In both examples, σ_m is chosen to be 1. They clearly show that field perturbation with the coating is much weaker than that without it.

4.2 Small Perturbation of Disks

We now review the result from [20] which shows that a small perturbation of a disk allows a coating such that the resulting core-shell structure is a weakly neutral inclusion, namely, a PT-vanishing structure. It is an existence result based on the implicit function theorem, so we do not know how small the perturbation can be.

Let D_0 be a disk of radius r_i centered at the origin. For a given function h on the unit circle T, the perturbation D_h of D_0 is defined to be

$$\partial D_h := \left\{ x \mid x = (r_i + h(\hat{x}))\hat{x}, \quad |\hat{x}| = 1 \right\}. \tag{4.12}$$

The function h defining the perturbed domain D_h belongs to the space $W^{2,\infty}(T)$, the class of functions on T whose derivatives up to order 2 are bounded.

To define domains for the coating, we let Ω_0 be the disk of radius r_e centered at the origin such that (D_0, Ω_0) be a neutral inclusions, namely, the radius and the conductivities are chosen so that the neutrality condition (1.6) is satisfied. We then define perturbations of Ω_0 as follows:

$$\partial \Omega_b := \left\{ x \mid x = (r_e + b(\hat{x}))\hat{x}, \quad |\hat{x}| = 1 \right\}, \tag{4.13}$$

where b is of the form

$$b(\theta) = b(\hat{x}) = b_0 + b_1 \cos 2\theta + b_2 \sin 2\theta. \tag{4.14}$$

Here b_0, b_1, b_2 are real constants.

If h and b are sufficiently small, then (D_h, Ω_b) defines an inclusion of the core-shell structure. Let $M(h, b) = M(D_h, \Omega_b)$ be the PT of (D_h, Ω_b) as defined in (2.8). Since M is symmetric, we may regard M as a three-dimensional vector-valued function. Since the collection of all b of the form (4.14) is of three dimensions, M can be regarded as a mapping from $U \times V$ into $\mathbb{R}^3$, where U and V are some neighborhoods of 0 in $W^{2,\infty}(T)$ and $\mathbb{R}^3$, respectively. Since (D_0, Ω_0) is neutral, we have $M(0, 0) = 0$. It is then proved that

$$\det \frac{\partial M}{\partial(b_0, b_1, b_2)}(0, 0) \neq 0. \tag{4.15}$$

Then an implicit function theorem is invoked to arrive at the following theorem.

Theorem 4.3 ([20]) *There is $\epsilon > 0$ such that for each $h \in W^{2,\infty}(T)$ with $\|h\|_{2,\infty} < \epsilon$ there is $b = b(h)$ of the form (4.14) such that*

$$M(h, b(h)) = 0, \tag{4.16}$$

namely, the inclusion $(D_h, \Omega_{b(h)})$ of the core-shell structure is weakly neutral to multiple uniform fields. The mapping $h \mapsto b(h)$ is continuous.

Proving (4.15) is quite technical. This two-dimensional theorem has been extended in [21] to three dimensions, which is even more technically complicated, to show that small perturbations of a sphere allow coatings so that the resulting inclusions of the core-shell structure are weakly neutral to multiple uniform fields.

For that, the functions b in (4.14) is replaced with

$$b(\hat{x}) = b_0 + \sum_{j=1}^{5} b_j Y_j^2(\hat{x}), \tag{4.17}$$

where $Y_j^2(\hat{x})$ are spherical harmonics of order 2 (there are five linearly independent ones). Then the PT is regarded as a local mapping from $W^{2,\infty}(S) \times \mathbb{R}^6$ (S is the unit sphere) into $\mathbb{R}^6$, and an analogy to (4.15) is proved.

5 Weakly Neutral Inclusions by Imperfect Bonding

So far we consider neutral or weakly neutral inclusions of the core-shell structure. There is yet another method to achieve neutrality: It is by introducing an imperfect bonding parameter on ∂D. The perfect bonding is characterized by the continuity of the flux and the potential along the interface ∂D as given in (1.8), while the imperfect bonding is characterized by either discontinuity of the potential or that of the flux along the interface. The former one is referred to as the low conductivity (LC) type, while the latter as the high conductivity (HC) type (see, e.g., [5]).

The LC type imperfect interface problem is described as follows:

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 & \text{in } D \cup (\mathbb{R}^d \setminus \overline{D}), \\ \beta(u|_+ - u|_-) = \sigma_m \partial_\nu u|_+ & \text{on } \partial D, \\ \sigma_c \partial_\nu u|_- = \sigma_m \partial_\nu u|_+ & \text{on } \partial D, \\ u(x) - a \cdot x = O(|x|^{-d+1}) & \text{as } |x| \to \infty. \end{cases} \tag{5.1}$$

Here, β is the interface parameter of the LC type, which is a non-negative function defined on the interface ∂D.

It is proved in [39] that if D is a disk (or a ball) of radius r and

$$\beta = \frac{1}{r} \frac{\sigma_c \sigma_m}{\sigma_c - \sigma_m}, \tag{5.2}$$

then the solution u to (5.1) satisfies

$$u(x) - a \cdot x \equiv 0 \quad \text{for all } x \in \mathbb{R}^d \setminus \Omega, \tag{5.3}$$

in other words, D with β is neutral. See Fig. 8.

It is proved in [16], based on the neutrality criterion obtained in [5], that the only neutral inclusions with the imperfect bonding parameters are disks (balls) with constant interface parameters if σ_m is isotropic, and ellipses (ellipsoids) if σ_m is anisotropic. In the same paper a way to construct an imperfect bonding parameter

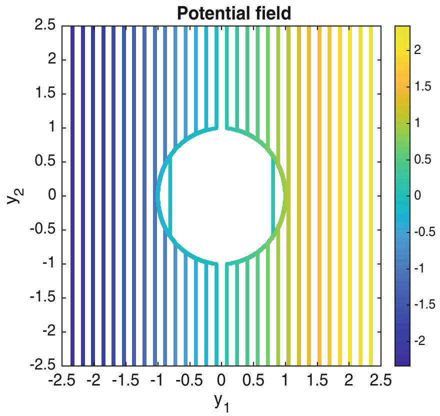


Fig. 8 Neutral inclusion by imperfect interface [39]

on the boundary of arbitrary domain has been investigated. For that purpose it is assumed that D is a perfect conductor, meaning that $\sigma_c = \infty$, which is to use the conformal mapping as in (4.1). Under this assumption, the problem (5.1) in two dimensions becomes the following one:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ \beta(u - \lambda) = \partial_\nu u|_+ & \text{on } \partial D, \\ u(x) - a \cdot x = O(|x|^{-1}) & \text{as } |x| \to \infty. \end{cases} \tag{5.4}$$

The following theorem is obtained.

Theorem 5.1 ([16]) *Let D be a bounded simply connected domain in $\mathbb{R}^2$ with the Lipschitz boundary which admits the conformal mapping Φ of the form (4.1). Assume that*

$$|b_D| \le 2 - \sqrt{3}. \tag{5.5}$$

Define β on ∂D by

$$\beta(z) = \left(\frac{1}{1+|b_D|} + \frac{1}{1-|b_D|} - 1 + \left(\frac{2}{1+|b_D|} - \frac{2}{1-|b_D|} \right) \cos 2\theta \right) \frac{1}{|\Phi'_D(e^{i\theta})|} \tag{5.6}$$

for $z = \Phi_D(e^{i\theta})$. Then the solution u to the problem (5.4) satisfies $u(x) - a \cdot x = O(|x|^{-2})$ as $|x| \to \infty$, namely, (D, β) is weakly neutral to multiple uniform fields.

It is helpful to mention that the condition (5.5) is imposed, even though the definition (5.6) makes sense without the condition, to guarantee the function β

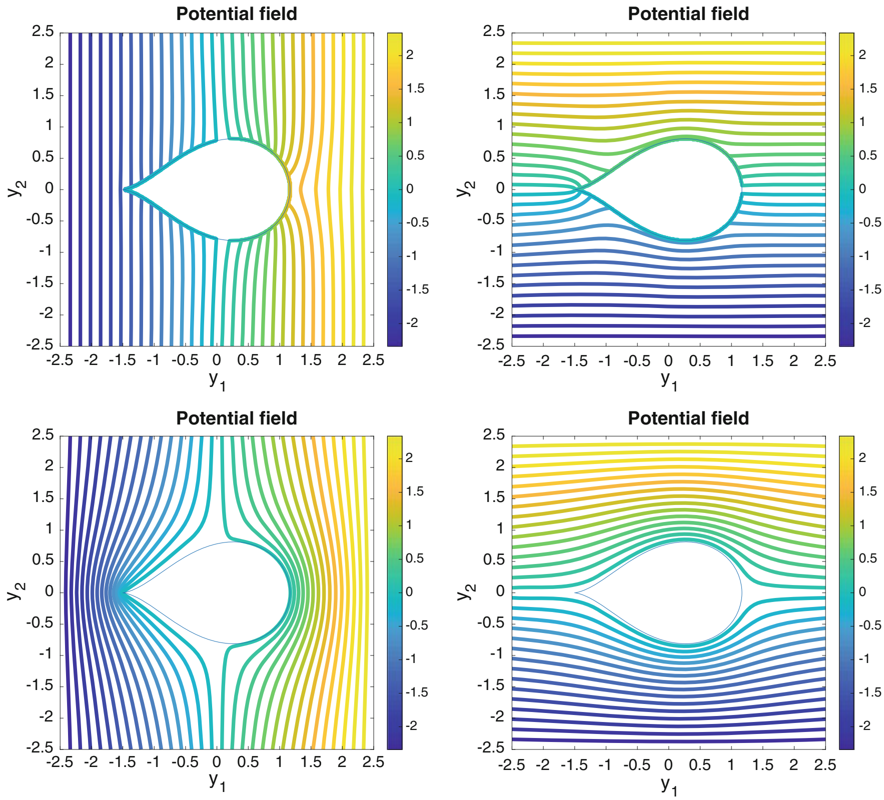


Fig. 9 Upper: the solution with the imperfect bonding parameter, Lower: without it. The solution in the upper one is less perturbed than that in the lower one

defined by (5.6) being positive. The positivity of β is required to assure uniqueness of the solution to (5.4). Figure 9 clearly shows that the field with the imperfect bonding parameter is less perturbed than that without it.

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References

1. Ahlfors, L.V.: Complex Analysis. An Introduction to the Theory of Analytic Functions of One Complex Variable. International Series in Pure and Applied Mathematics, 3rd edn. McGraw-Hill Book Co., New York (1978)
2. Ammari, H., Kang, H.: Polarization and Moment Tensors with Applications to Inverse Problems and Effective Medium Theory. Applied Mathematical Sciences, vol. 162. Springer, New York (2007)

3. Ammari, H., Capdeboscq, Y., Kang, H., Kim, E., Lim, M.: Attainability by simply connected domains of optimal bounds for polarization tensors. Euro. J. Appl. Math. **17**, 201–219 (2006)
4. Ammari, H., Kang, H., Lee, H., Lim, M.: Enhancement of near cloaking using generalized polarization tensors vanishing structures. Part I: the conductivity problem. Commun. Math. Phys. **317**, 253–266 (2013)
5. Benveniste, Y., Miloh, T.: Neutral inhomogeneities in conduction phenomena. J. Mech. Phys. Solid **47**, 1873–1892 (1999)
6. Capdeboscq, Y., Vogelius M.S.: Optimal asymptotic estimates for the volume of internal inhomogeneities in terms of multiple boundary measurements. Math. Modell. Num. Anal. **37**, 227–240 (2003)
7. Capdeboscq, Y., Vogelius M.S.: A review of some recent work on impedance imaging for inhomogeneities of low volume fraction. Contemp. Math. **362**, 69–87 (2005)
8. DiBenedetto, E., Friedman, A.: Bubble growth in Porous Media. Indiana Univ. Math. J. **35**(2), 573–606 (1986)
9. Dive, P.: Attraction des ellipsoides homogènes et réciproques d'un théorème de Newton. Bull. Soc. Math. France **59**, 128–140 (1931)
10. Eshelby, J.D.: Elastic inclusions and inhomogeneities. In: Sneddon, I.N., Hill, R. (eds.) Progress in Solid Mechanics, vol. II, pp. 87–140. North-Holland, Amsterdam (1961)
11. Gustafsson, B., Shapiro, H.S.: What is a quadrature domain? In: Quadrature Domains and Their Applications, pp. 1–25. Birkhäuser, Basel (2005)
12. Hashin, Z.: The elastic moduli of heterogeneous materials. J. Appl. Mech. **29**, 143–150 (1962)
13. Hashin, Z., Shtrikman, S.: A variational approach to the theory of the effective magnetic permeability of multiphase materials. J. Appl. Phys. **33**, 3125–3131 (1962)
14. Jarczyk, P., Mityushev, V.: Neutral coated inclusions of finite conductivity. Proc. R. Soc. A **468**, 954–970 (2012)
15. Kang, H., Lee, H.: Coated inclusions of finite conductivity neutral to multiple fields in two dimensional conductivity or anti-plane elasticity. Euro. J. Appl. Math. **25**(3), 329–338 (2014)
16. Kang, H., Li, X.: Construction of weakly neutral inclusions of general shape by imperfect interfaces. SIAM J. Appl. Math. **79**, 396–414 (2019)
17. Kang, H., Milton, G.W.: On conjectures of Pólya-Szegö and Eshelby. In: Kang, H., Ammari, H. (eds.) Inverse Problems, Multi-scale Analysis and Effective Medium Theory. Contemporary Math., vol. 408, pp. 75–80 (2006)
18. Kang, H., Milton, G.W.: Solutions to the Pólya-Szegö conjecture and the weak Eshelby conjecture. Arch. Rational Mech. Anal. **188**, 93–116 (2008)
19. Kang, H., Lee, H., Sakaguchi, S.: An over-determined boundary value problem arising from neutrally coated inclusions in three dimensions. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **XVI**, 1193–1208 (2016)
20. Kang, H., Li, X., Sakaguchi, S.: Existence of coated inclusions of general shape weakly neutral to multiple fields in two dimensions. Appl. Anal. (published online). arXiv:1808.01096
21. Kang, H., Li, X., Sakaguchi, S.: Polarization tensor vanishing structure of general shape: existence for small perturbations of balls. Asymptot. Anal. (published online). arXiv:1911.07250
22. Karp, L., Lundberg, E.: A four-dimensional Neumann ovaloid. Ark. Mat. **55**(1), 185–198 (2017)
23. Kellog, O.D.: Foundation of Potential Theory. Dover, New York (1953)
24. Kerker, M.: Invisible bodies. J. Opt. Soc. Am. **65**, 376–379 (1975)
25. Kohn, R.V., Shen, H., Vogelius, M.S., Weinstein, M.I.: Cloaking via change of variables in electric impedance tomography. Inverse Prob. **24**, article 015016 (2008)
26. Khavinson, D.: Holomorphic partial differential equations and classical potential theory. Departamento de análisis matemático, Universidad de la Laguna (1996)
27. Khavinson, D., Lundberg, E.: A tale of ellipsoids in potential theory. Notices Am. Math. Soc. **61**, 148–156 (2014)
28. Lipton, R.: Inequalities for electric and elastic polarization tensors with applications to random composites. J. Mech. Phys. Solids **41**, 809–833 (1993)
29. Liu, L.P.: Solutions to the Eshelby conjectures. Proc. R. Soc. A. **464**, 573–594 (2008)

30. MacMillan, W.D.: The Theory of the Potential. Dover, New York (1958)
31. Milton, G.W.: The Theory of Composites. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, Cambridge (2002)
32. Milton, G.W., Serkov, S.K.: Neutral coated inclusions in conductivity and anti-plane elasticity. Proc. R. Soc. Lond. A **457**, 1973–1997 (2001)
33. Neumann, C.: Uber das logarithmische Potential einer gewissen Ovalfläche, Zweite Mitteilung, ibib. vol. 60 (1908). Dritte Mitteilung, ibid, pp. 240–247
34. Nikliborc, W.: Eine Bemerkung über die Volumpotentiale. Math. Zeit. **35**, 625–631 (1932)
35. Pólya, G., Szegö, G.: Isoperimetric Inequalities in Mathematical Physics. Annals of Mathematical Studies, vol. 27. Princeton University Press, Princeton (1951)
36. Sakai, M.: Quadrature Domains. Lecture Notes in Mathematics, vol. 934. Springer, Berlin (1982)
37. Sendeckyj, G.P.: Elastic inclusion problems in plane elastostatics. Int. J. Solids Struct. **6**, 1535–1543 (1970)
38. Shapiro, H.S.: The Schwarz Function and Its Generalization to Higher Dimensions, vol. 4. Wiley, New York (1992)
39. Torquato, S., Rintoul, M.D.: Effect of the interface on the properties of composite media. Phys. Rev. Lett. **75**, 4067–4070 (1995)

Nonexistence of Radial Optimal Functions for the Sobolev Inequality on Cartan-Hadamard Manifolds

Tatsuki Kawakami and Matteo Muratori

Abstract It is well known that the Euclidean Sobolev inequality holds on any Cartan-Hadamard manifold of dimension $n \geq 3$, i.e. any complete, simply connected Riemannian manifold with nonpositive sectional curvature. Moreover, in the very special case of the Euclidean space itself, the optimal constant is achieved by the Aubin-Talenti functions. On a generic Cartan-Hadamard manifold $\mathbb{M}^n$, one may ask whether there exist at all optimal functions. Here we prove, with ad hoc arguments that do not take advantage of the validity of the so-called Cartan-Hadamard conjecture (claiming that such optimal constant is always Euclidean), that this is false at least for functions that are radially symmetric with respect to the geodesic distance from a fixed pole. More precisely, we show that if the optimum in the Sobolev inequality is achieved by some radial function, then $\mathbb{M}^n$ must be isometric to $\mathbb{R}^n$.

Keywords Sobolev inequality · Optimal constants · Radial functions · Cartan-Hadamard manifolds

1 Introduction

A Cartan-Hadamard manifold is a complete and simply connected Riemannian manifold $\mathbb{M}^n$ with everywhere nonpositive sectional curvature. By the Cartan-Hadamard theorem, any such manifold turns out to be topologically equivalent to the Euclidean space $\mathbb{R}^n$; more precisely, the exponential map centered at any point $o \in \mathbb{M}^n$ is a diffeomorphism. We refer to Sect. 2.1 for an account on this and further

T. Kawakami
Applied Mathematics and Informatics Course, Faculty of Advanced Science and Technology, Ryukoku University, Seta Oe-cho, Otsu, Shiga, Japan
e-mail: kawakami@math.ryukoku.ac.jp

M. Muratori (✉)
Dipartimento di Matematica, Politecnico di Milano, Milano, Italy
e-mail: matteo.muratori@polimi.it

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basic properties of Cartan-Hadamard manifolds. From the functional point of view, a remarkable and by now well-established fact is the validity, on every such manifold of dimension $n \geq 3$, of the *Euclidean Sobolev inequality*

$$\|f\|_{L^{2^*}(\mathbb{M}^n)} \leq C\,\|\nabla f\|_{L^2(\mathbb{M}^n)} \quad \forall f \in C_c^1(\mathbb{M}^n)\,, \qquad 2^* := \frac{2n}{n-2}\,, \tag{1.1}$$

for some positive constant $C > 0$. Here, with the term *Euclidean*, we simply mean that the exponent appearing in the left-hand side of (1.1) is exactly the same as the one corresponding to the case $\mathbb{M}^n \equiv \mathbb{R}^n$. It is possible to establish (1.1) through several techniques: see Sect. 2.4 for an explicit proof and for references to other arguments available in the literature. As concerns the value of the *optimal constant*, which will be denoted by $\mathfrak{C}$, the situation is more complicated. Indeed, it has been an open question whether $\mathfrak{C}$ coincides with the *Euclidean best constant* C_{E}, namely the one achieved in $\mathbb{R}^n$ by the celebrated Aubin-Talenti functions [3, 36]. It is plain, due to the local Euclidean structure of $\mathbb{M}^n$, that $\mathfrak{C}$ cannot be smaller than C_{E} (see Sect. 2.5). The fact that $\mathfrak{C} = C_{\mathrm{E}}$, as of this writing, is only known up to dimension $n = 4$ as a consequence of the validity of the so-called *Cartan-Hadamard conjecture*, a longstanding problem in geometric analysis. The latter asserts that the *isoperimetric inequality*, or equivalently the 1-Sobolev inequality

$$\|f\|_{L^{1^*}(\mathbb{M}^n)} \leq C_1\,\|\nabla f\|_{L^1(\mathbb{M}^n)} \quad \forall f \in C_c^1(\mathbb{M}^n)\,, \qquad 1^* := \frac{n}{n-1}\,, \tag{1.2}$$

holds with *Euclidean best constant* C_1 and the optimal functions are characteristic functions of Euclidean balls, i.e. equality is achieved if and only if $\mathbb{M}^n \equiv \mathbb{R}^n$ and $f = \chi_{B_r}$, $r > 0$, after a routine extension of (1.2) to the BV space. Once C_1 in (1.2) can be taken equal to the Euclidean isoperimetric constant, then a Schwarz-type symmetrization technique allows one to show that the same holds for (1.1), namely $\mathfrak{C} = C_{\mathrm{E}}$. We refer the reader to [24, Section 8] for an overview of the literature and the main techniques used until recently to attack the Cartan-Hadamard conjecture, along with its relation to p-Sobolev inequalities.

The aim of the present paper is to give a first contribution to the study of possible *optimal functions*, i.e. nontrivial functions attaining the identity in (1.1) with $C = \mathfrak{C}$. Indeed, regardless of the knowledge of the value of the optimal constant $\mathfrak{C}$, it is reasonable to ask whether (1.1) admits at all optimal functions and, in case of positive answer, what is the shape of the latter. We will work in the simplified *radially-symmetric* framework, that is we will consider functions $f(x) \equiv f(r(x))$ that depend only on the geodesic distance $r(x) := \mathrm{d}(x, o)$ from a fixed pole $o \in \mathbb{M}^n$, namely *radial functions*. This may appear as a strong restriction, nevertheless radial symmetry has proved to play a major role in the investigation of extremal functions for a wide class of Sobolev-type inequalities. The literature here is huge: without any claim of completeness, in addition to the pioneering papers [3, 36], we quote [12–14] and references therein for a thorough study of symmetry/symmetry-breaking issues in *Caffarelli-Kohn-Nirenberg inequalities*, the latter being functional inequalities of the type of (1.1) (possibly in interpolation form) with respect to power-type weights in $\mathbb{R}^n$. In fact (1.1), especially when restricted to radial functions, can be seen as a Euclidean *weighted* inequality. See in particular Sect. 2.2 below and [32].

Our main result is the following.

Theorem 1.1 *Let $\mathbb{M}^n$ ($n \geq 3$) be a Cartan-Hadamard manifold. Suppose that the Sobolev inequality* (1.1) *admits a (nontrivial) radial optimal function. Then $\mathbb{M}^n$ is isometric to $\mathbb{R}^n$.*

Clearly the above theorem can be interpreted both in terms of *nonexistence* and in terms of *rigidity*, in the sense that as soon as $\mathbb{M}^n \not\equiv \mathbb{R}^n$ there exists no (radial) optimal function to (1.1) and, should such a function exist, it is necessarily an Aubin-Talenti profile on $\mathbb{M}^n \equiv \mathbb{R}^n$. Note that optimal functions are naturally sought in $\dot{H}^1(\mathbb{M}^n)$, i.e. the closure of $C_c^1(\mathbb{M}^n)$ with respect to the $L^2(\mathbb{M}^n)$ norm of the gradient. We will give three different proofs of Theorem 1.1 in Sect. 3. In fact the third one, provided in Sect. 3.3, takes advantage of arguments similar to those used in [15, Theorem 1.3], where the same result was proved upon *assuming* the Cartan-Hadamard conjecture (see also [26] for analogous issues regarding Morrey-Sobolev inequalities). However, we want to emphasize that none of our proofs takes advantage of the Cartan-Hadamard conjecture; the strategies we employ only rely on classical Laplacian and volume-comparison tools (Sect. 2.2), along with the specific structure of the inequality in the radially-symmetric framework.

The investigation of optimal constants in functional inequalities has a long story. As we have already commented, the very first result dealing with the optimal functions for the Euclidean Sobolev inequality is due to two simultaneous and independent papers by T. Aubin [3] and G. Talenti [36]. In a series of subsequent articles [1, 2, 4], Aubin continued the analysis of Sobolev-type inequalities and optimality issues on Riemannian manifolds. Some improvements on [3] were then achieved by Hebey and Vaugon [25] and Hebey [23]. In [9, 27] it was shown, upon assuming curvature or volume-growth bounds *from below*, respectively, that a Riemannian manifold supporting the Sobolev inequality (1.1) with Euclidean constant is necessarily isometric to $\mathbb{R}^n$. For *topological rigidity* results in the same spirit, see also [34].

Concerning Poincaré inequalities, H.P. McKean [31] proved that, if on a Cartan-Hadamard manifold the sectional curvature is bounded from above by a negative constant $-k$, then in addition to (1.1) we have

$$\|f\|_{L^2(\mathbb{M}^n)} \leq \frac{2}{\sqrt{k}\,(n-1)} \, \|\nabla f\|_{L^2(\mathbb{M}^n)} \qquad \forall f \in C_c^1(\mathbb{M}^n)\,. \tag{1.3}$$

This is equivalent to the fact that the infimum of the spectrum of (minus) the Laplace-Beltrami operator on $\mathbb{M}^n$ is bounded from below by the constant $k(N-1)^2/4$ to $k(n-1)^2/4$, in other words $-\Delta$ has an explicit *spectral gap*. Moreover, such constant is sharp since it is attained on the *hyperbolic space* $\mathbb{H}^n$ of curvature $-k$. Also the requirement on the "nondegeneracy" of the curvature is, in some sense, sharp. Indeed, in [29] it was shown that, on any complete noncompact

Riemannian manifold, the (essential) spectrum of $-\Delta$ starts from zero as soon as the Ricci curvature vanishes at infinity. An alternative, and much simpler proof of (1.3) was carried out in [32], by means of one-dimensional techniques which are to some extent related to the arguments we develop in Sect. 3. Such paper deals with the validity of (radial) inequalities that interpolate between (1.1) and (1.3), under (power-type) bounds from above on the sectional curvature of $\mathbb{M}^n$. In the special, but significant case of the hyperbolic space, it is worth quoting the recent contributions [6], where the Poincaré inequality is established with *optimal remainder terms* of Hardy type, and [33], where the author proves a remarkable inequality on $\mathbb{H}^n$ yielding simultaneously the *optimal* Sobolev and Poincaré constants. In wider geometric settings, Hardy-type inequalities were also addressed in [8], for a class of nonstandard weights.

Finally, we recall that the Sobolev inequality (1.1), along with related Gagliardo-Nirenberg and Poincaré inequalities, was successfully exploited to prove (sharp) $L^1 - L^\infty$ smoothing effects for the *porous medium equation* [21] and finite-time extinction estimates for the *fast diffusion equation* [7] on Cartan-Hadamard manifolds, thus reinforcing the well-known connection between (nonlinear) diffusion equations and functional inequalities. In this regard, we also mention [20], where Faber-Krahn inequalities on Riemannian manifolds are investigated and consequent *heat-kernel* bounds are established.

2 Preliminary Material

In the following, we will provide an overview of the essential notions and tools that one needs to know when dealing with Cartan-Hadamard manifolds (Sects. 2.1–2.3), along with some well-established results regarding the Sobolev inequality, of which however we believe it is worth giving a direct proof, since we try to be as much as possible self contained (see in particular Sects. 2.4 and 2.5).

2.1 Basics of Cartan-Hadamard Manifolds

We recall that a Cartan-Hadamard manifold is an n-dimensional Riemannian manifold $(M, \mathfrak{g})$ which is complete, simply connected and has everywhere non-positive sectional curvature. This assumption entails a very strong topological (and geometric) consequence, due to the Cartan-Hadamard theorem (see e.g. [28, Theorem 1.10] or [10, Theorem II.6.2]): the cut-locus of *any* point $o \in M$ is empty, so that the exponential map $T_oM \equiv \mathbb{R}^n \ni y \mapsto \exp_o y \in M$ is actually a global diffeomorphism and therefore M is in particular a *manifold with a pole* (we refer to [16] for an excellent monograph on this class of manifolds). More than that: any point can play the role of a pole.

Before proceeding further, let us fix some notations. The (standard) symbol T_oM stands for the tangent space of M at $o \in M$, and we recall that $\exp_o$ is the map that to any element $y \in T_oM$ associates the point reached at time $t = 1$ by the constant-speed geodesic that starts from o at $t = 0$ with velocity y. In general the exponential map is well defined only for small y, but as we have just seen on Cartan-Hadamard manifolds it is in fact global.

We employ the symbol "$\equiv$" instead of "$=$" for identities that should be understood up to suitable (implicit) transformations. In the case of Riemannian manifolds, by $(M_1, \mathfrak{g}_1) \equiv (M_2, \mathfrak{g}_2)$ we mean that M_1 is isometric to M_2, i.e. there exists a diffeomorphism from M_1 onto M_2 which is also an isometry with respect to $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Finally, in order to lighten notations, an n-dimensional Cartan-Hadamard manifold is simply denoted by $\mathbb{M}^n$ and $\mathrm{d}(\cdot, \cdot)$ is the corresponding distance on $\mathbb{M}^n$ induced by its metric $\mathfrak{g}$.

At the level of curvatures, we denote by $\mathrm{Sect}(x)$ the sectional curvature at $x \in M$ with respect to a generic plane in the tangent space T_xM, whereas $\mathrm{Sect}_o(x)$ stands for the sectional curvature with respect to any plane in T_xM containing the radial direction, also known as *radial sectional curvature*. Similarly, we denote by $\mathrm{Ric}(x)$ the Ricci curvature at $x \in M$ as a quadratic form, whereas the number $\mathrm{Ric}_o(x)$ stands for the Ricci curvature evaluated in the radial direction, i.e. the *radial Ricci curvature*.

In the sequel, $o \in \mathbb{M}^n$ will tacitly be considered a *fixed* reference point elected as a pole, unless otherwise specified. In view of what we have recalled above, it is possible to exploit *radial coordinates* about o, namely to any $x \in \mathbb{M}^n \setminus \{o\}$ one can associate in a unique way a couple $(r, \theta) \in (0, \infty) \times \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ represents the $(n-1)$-dimensional unit sphere endowed with the usual round metric. Note that r is the distance between x and o, while θ is the starting direction of the geodesic that connects o to x. In this way, the metric $\mathfrak{g}$ of $\mathbb{M}^n$ at $x \equiv (r, \theta)$ can be written as follows:

$$\mathfrak{g} \equiv \mathrm{d}r^2 + \langle \mathsf{A}(r, \theta)\, \mathrm{d}\theta, \mathrm{d}\theta \rangle_\theta \,, \tag{2.1}$$

for a suitable linear map $\mathsf{A}(r, \theta)$ giving rise to a quadratic form in the tangent space of $\mathbb{S}^{n-1}$ at θ. Here the symbol $\langle \cdot, \cdot \rangle_\theta$ stands for the inner product of such tangent space that induces the norm $\| \cdot \|_\theta$, and in (2.1) we identify an element of the tangent space of $\mathbb{M}^n$ at $x \equiv (r, \theta)$ with $(\mathrm{d}r, \mathrm{d}\theta)$, where $\mathrm{d}r$ is an arbitrary real number that represents displacement in the radial direction and $\mathrm{d}\theta$ is an element of the tangent space of $\mathbb{S}^{n-1}$ at θ, that represents angular displacement.

To our purposes, a key role is played by the positive scalar function

$$A(r, \theta) := \sqrt{\det\big[\mathsf{A}(r, \theta)\big]} \qquad \forall (r, \theta) \in (0, \infty) \times \mathbb{S}^{n-1} \,.$$

In fact $A(r, \theta)$ coincides with the density of the volume measure of $\mathbb{M}^n$, which we denote by $d\mu$, with respect to the product measure $dr \otimes d\theta$. Here and below, with some abuse of notation, the symbol dr stands for the Lebesgue measure on $(0, \infty)$

and $d\theta$ for the volume (i.e. surface) measure of $\mathbb{S}^{n-1}$, still endowed with the standard round metric. It is plain that, since the metric of $\mathbb{M}^n$ is locally Euclidean, or more rigorously $\mathfrak{g}$ is differentiable on $\mathbb{M}^n$, in particular we have

$$\lim_{r\downarrow 0} \frac{A(r,\theta)}{r^{n-1}} = 1 \qquad \text{uniformly w.r.t. } \theta \in \mathbb{S}^{n-1} . \tag{2.2}$$

Let us denote by B_r the geodesic ball of radius $r > 0$, implicitly centered at o, i.e. the open set of points in $\mathbb{M}^n$ whose distance from o is less than r. If the center of the ball is another point $x \neq o$, we will write more explicitly $B_r(x)$. Similarly, the boundary of B_r, that is the geodesic sphere of all points at distance r from o, is denoted by S_r. Note that S_r itself is an $(n-1)$-dimensional Riemannian manifold embedded in $\mathbb{M}^n$. From the definition of $A(r,\theta)$, we infer that for any fixed $r > 0$ the function $\theta \mapsto A(r,\theta)$ is the density, with respect to $d\theta$, of the volume (i.e. surface) measure $d\sigma$ of S_r; as a result,

$$\sigma(S_r) = \int_{\mathbb{S}^{n-1}} A(r,\theta)\, d\theta . \tag{2.3}$$

2.2 Laplace-Beltrami Operator, Radial Functions and Sobolev Spaces

After the previous introductory section, we are in position to describe more precisely the functional setting in which we work. First of all, given a smooth function f on $\mathbb{M}^n$, the *Laplace-Beltrami operator* (also *Laplacian* for short) applied to f reads, in radial coordinates (see [17, Section 3] or [22, Section 2.2]),

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \mathsf{m}(r,\theta)\,\frac{\partial f}{\partial r} + \Delta_{S_r} f , \tag{2.4}$$

where Δ_{S_r} represents the Laplace-Beltrami operator on the submanifold S_r and

$$\mathsf{m}(r,\theta) := \frac{\partial}{\partial r}\big[\log A(r,\theta)\big] \qquad \forall x \equiv (r,\theta) \in (0,\infty) \times \mathbb{S}^{n-1} . \tag{2.5}$$

It is immediate to check that in fact $\mathsf{m}(r,\theta)$ coincides with the *Laplacian of the distance* function $r \equiv r(x) := \mathrm{d}(x,o)$, which is of key importance in the analysis of partial differential equations on manifolds due to crucial comparison results (see the next section). Note that, upon integrating (2.5) from a fixed $r_0 > 0$ to $r > r_0$, we obtain the identity

$$\int_{r_0}^{r} \mathsf{m}(s,\theta)\, ds = \log A(r,\theta) - \log A(r_0,\theta) \qquad \forall (r,\theta) \in (r_0,\infty) \times \mathbb{S}^{n-1} , \tag{2.6}$$

that is

$$A(r,\theta) = e^{\int_{r_0}^{r} \mathsf{m}(s,\theta)\,ds + c_\theta} \quad \forall (r,\theta) \in (r_0,\infty) \times \mathbb{S}^{n-1}\,, \qquad \text{where } c_\theta := \log A(r_0,\theta)\,.$$

Strictly related to the Laplacian is the *gradient* operator, which for $C^1(\mathbb{M}^n)$ functions reads (in radial coordinates)

$$\nabla f \equiv \left(\tfrac{\partial f}{\partial r}, \nabla_{S_r} f \right) \qquad \Longrightarrow \qquad |\nabla f|^2 = \left| \tfrac{\partial f}{\partial r} \right|^2 + \left\| \nabla_{S_r} f \right\|_\theta^2 ,$$

where ∇_{S_r} is in turn the gradient operator of the submanifold S_r. Clearly both Δ_{S_r} and ∇_{S_r} can explicitly be written in terms of $\mathsf{A}(r,\theta)$, which we avoid since we will only deal with *radial* functions, namely functions on $\mathbb{M}^n$ that depend solely on the radial coordinate, i.e. $f(r,\theta) \equiv f(r)$. In this special case, we adopt the simplified notation $\frac{\partial f}{\partial r} \equiv f'$.

Given a measurable function $f : \mathbb{M}^n \to \mathbb{R}$ and $p \in [1,\infty)$, we define its $L^p(\mathbb{M}^n)$ norm as

$$\|f\|_{L^p(\mathbb{M}^n)}^p := \int_{\mathbb{M}^n} |f|^p \, d\mu = \int_0^\infty \int_{\mathbb{S}^{n-1}} |f(r,\theta)|^p \; A(r,\theta)\, d\theta dr\,.$$

Analogously, for a $C^1(\mathbb{M}^n)$ function, the $L^2(\mathbb{M}^n)$ norm of its gradient is defined as

$$\|\nabla f\|_{L^2(\mathbb{M}^n)}^2 := \int_{\mathbb{M}^n} |\nabla f|^2 \, d\mu = \int_0^\infty \int_{\mathbb{S}^{n-1}} \left(\left| \tfrac{\partial f}{\partial r} \right|^2 + \left\| \nabla_{S_r} f \right\|_\theta^2 \right) A(r,\theta)\, d\theta dr\,.$$

In particular, upon setting

$$\psi_\star(r) := \left[\frac{\int_{\mathbb{S}^{n-1}} A(r,\theta)\, d\theta}{\left|\mathbb{S}^{n-1}\right|} \right]^{\frac{1}{n-1}} \qquad \forall r > 0\,, \tag{2.7}$$

where $\left|\mathbb{S}^{n-1}\right|$ is the total surface measure of the $(n-1)$-dimensional unit sphere, we deduce that for a $C^1(\mathbb{M}^n)$ radial function f the identities

$$\|f\|_{L^p(\mathbb{M}^n)}^p = \left|\mathbb{S}^{n-1}\right| \int_0^\infty |f(r)|^p \, \psi_\star(r)^{n-1} dr \tag{2.8}$$

and

$$\|\nabla f\|_{L^2(\mathbb{M}^n)}^2 = \left|\mathbb{S}^{n-1}\right| \int_0^\infty \left|f'(r)\right|^2 \psi_\star(r)^{n-1} dr \tag{2.9}$$

hold. The reason for the notation $\psi_\star$ in (2.7) will be clearer in the next subsection.

Finally, we denote by $\dot{H}^1(\mathbb{M}^n)$ the Sobolev space defined as the closure of $C_c^1(\mathbb{M}^n)$ with respect to $\|\nabla(\cdot)\|_{L^2(\mathbb{M}^n)}$, endowed with the latter norm. It is apparent that all the above formulas still hold for functions in $\dot{H}^1(\mathbb{M}^n)$, up to interpreting partial derivatives in the weak sense. Clearly the Sobolev inequality (1.1) extends to the whole $\dot{H}^1(\mathbb{M}^n)$, and it is (a priori) in this space that optimal functions should be sought.

2.3 Model Manifolds, Laplacian and Volume Comparison

A *model manifold* is an n-dimensional Riemannian manifold $(M, \mathfrak{g})$ with a pole $o \in M$ whose metric can be written, with respect to the radial coordinates about o, as (see [19, Section 3.10])

$$\mathfrak{g} = \mathrm{d}r^2 + \psi(r)^2 \, \|\mathrm{d}\theta\|_\theta^2 \, ,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a function belonging to the class

$$\mathcal{F} := \left\{ \psi \in C^\infty((0,\infty)) \cap C^1([0,\infty)) : \ \psi(0) = 0 \, , \ \psi(r) > 0 \ \forall r > 0 \, , \ \psi'(0) = 1 \right\} . \tag{2.10}$$

In other words, it corresponds to the particular case of (2.1) when $\mathsf{A}(r, \theta)$ is the identity times $\psi(r)^2$. Hence, it follows that $A(r, \theta) = \psi(r)^{n-1}$. For instance, the *Euclidean space* $\mathbb{R}^n$ corresponds to $\psi(r) = r$, while the *hyperbolic space* $\mathbb{H}^n$ corresponds to $\psi(r) = \sinh r$. Note that, in general, a model manifold need not be Cartan-Hadamard: the latter property is equivalent to requiring that ψ is in addition convex. Outside the class of Cartan-Hadamard manifolds, we recover the *unit sphere* $\mathbb{S}^{n-1}$ with the choice $\psi(r) = \sin r$, at least for r ranging in the bounded interval $[0, \pi)$.

Having introduced model manifolds, we can briefly recall some classical results that compare, in radial coordinates, the Laplacian of the distance function (w.r.t. to a given pole o) of a Cartan-Hadamard manifold $\mathbb{M}^n$ with the Laplacian of the distance function of the model manifold which attains the curvature bounds. More precisely, if

$$\mathrm{Sect}_o(x) \le -\frac{\psi''(r)}{\psi(r)} \qquad \forall (r, \theta) \equiv x \in \mathbb{M}^n \setminus \{o\} \tag{2.11}$$

for some function $\psi \in \mathcal{F}$, then

$$\mathsf{m}(r, \theta) \ge (n-1) \, \frac{\psi'(r)}{\psi(r)} \qquad \forall (r, \theta) \in (0, \infty) \times \mathbb{S}^{n-1} \, . \tag{2.12}$$

Similarly, if

$$\mathrm{Ric}_o(x) \geq -(n-1)\,\frac{\psi''(r)}{\psi(r)} \qquad \forall (r,\theta) \equiv x \in \mathbb{M}^n \setminus \{o\} \tag{2.13}$$

for another function $\psi \in \mathcal{F}$, then

$$\mathsf{m}(r,\theta) \leq (n-1)\,\frac{\psi'(r)}{\psi(r)} \qquad \forall (r,\theta) \in (0,\infty) \times \mathbb{S}^{n-1}\,.$$

We point out that the equality cases of the above inequalities do correspond to model manifolds, i.e. the radial sectional curvature of a model manifold coincides with the right-hand side of (2.11), and the same holds for the radial Ricci curvature in (2.13). Moreover, the Laplacian of the distance function on a model manifold is also a radial function that equals the right-hand side of (2.12). For further details, see e.g. [22, Section 2.2] and references therein. Our entire focus here is on Cartan-Hadamard manifolds. We mention, however, that the above comparison results do hold in much more general Riemannian frameworks, up to a possible weak interpretation of the inequalities: we refer the reader to [30, Sections 1.2.3 and 1.2.5] (see also [16, Section 2] or [17, Section 15]).

Because a Cartan-Hadamard manifold has everywhere nonpositive sectional curvature, by applying (2.11) and (2.12) with the trivial choice $\psi(r) = r$ we immediately deduce that

$$\mathsf{m}(r,\theta) \geq \frac{n-1}{r} \qquad \forall (r,\theta) \in (0,\infty) \times \mathbb{S}^{n-1}\,. \tag{2.14}$$

This simple inequality has a key consequence that will be crucial to our strategy, namely the fact that the volume measure of $\mathbb{M}^n$ is larger than the Euclidean one:

$$A(r,\theta) \geq r^{n-1} \qquad \forall (r,\theta) \in (0,\infty) \times \mathbb{S}^{n-1}\,. \tag{2.15}$$

To establish (2.15) let us notice that, by virtue of (2.6) and (2.14), for every $r_0 > 0$ it holds

$$\log\!\left(\frac{r^{n-1}}{r_0^{n-1}}\right) \leq \log\!\left(\frac{A(r,\theta)}{A(r_0,\theta)}\right) \qquad \forall (r,\theta) \in (r_0,\infty) \times \mathbb{S}^{n-1}\,,$$

so that by taking exponentials and letting $r_0 \downarrow 0$, using (2.2), we obtain (2.15).

We mention that (2.15) is the analogue, in the very special Cartan-Hadamard setting, of the celebrated Bishop-Gromov comparison theorem: see e.g. [24, Theorem 1.1] or [30, Theorem 1.13] for a more general statement. As a particular case of the latter, one deduces that the volume of geodesic balls of a Riemannian manifold with nonnegative *Ricci* curvature is at most Euclidean. On Cartan-Hadamard manifolds, given the nonpositive *sectional* curvature, we have the opposite inequality.

2.4 A Simple Proof of the Sobolev Inequality on Cartan-Hadamard Manifolds

For completeness, we provide an elementary proof of the validity of the Sobolev inequality on any n-dimensional ($n \geq 3$) Cartan-Hadamard manifold. A proof of the 1-Sobolev inequality, from which the standard Sobolev inequality (1.1) easily follows (see [24, Lemma 8.1]), can be found e.g. in [24, Theorem 8.3].

The argument we outline here exploits a comparison result between *heat kernels*, which is in fact a particular case of [18, Theorem 4.2]. That is, let $\mathcal{K}(x, y, t)$ be the heat kernel of $\mathbb{M}^n$, namely the (minimal) solution to

$$\begin{cases} \frac{\partial}{\partial t}\mathcal{K}(\cdot, y, \cdot) = \Delta\mathcal{K}(\cdot, y, \cdot) & \text{in } \mathbb{M}^n \times (0, +\infty)\,, \\ \mathcal{K}(\cdot, y, 0) = \delta_y & \text{in } \mathbb{M}^n\,, \end{cases} \tag{2.16}$$

where δ_y stands for the Dirac delta centered at a given but arbitrary $y \in \mathbb{M}^n$. Let $\mathcal{K}_{\mathrm{E}}$ denote the *Euclidean heat kernel*, that is

$$\mathcal{K}_{\mathrm{E}}(r, t) = \frac{e^{-\frac{r^2}{4t}}}{(4\pi t)^{\frac{n}{2}}} \qquad \forall (r, t) \in [0, \infty) \times (0, +\infty)\,,$$

which solves the analogue of (2.16) in $\mathbb{R}^n$ with r replaced by $|x - y|$. For each $y \in \mathbb{M}^n$, the function $\mathbb{M}^n \times (0, +\infty) \ni (x, t) \mapsto \mathcal{K}_{\mathrm{E}}(\mathrm{d}(x, y), t)$ turns out to be a *supersolution* to (2.16). Indeed, it is plain that $\frac{\partial}{\partial r}\mathcal{K}_{\mathrm{E}} \leq 0$; hence, from Laplacian comparison (recall (2.4) and (2.14)), we have:

$$\frac{\partial}{\partial t}\mathcal{K}_{\mathrm{E}} = \frac{\partial^2}{\partial r^2}\mathcal{K}_{\mathrm{E}} + \frac{n-1}{r}\frac{\partial}{\partial r}\mathcal{K}_{\mathrm{E}} \geq \frac{\partial^2}{\partial r^2}\mathcal{K}_{\mathrm{E}} + \mathsf{m}(r, \theta)\frac{\partial}{\partial r}\mathcal{K}_{\mathrm{E}}\,.$$

Upon setting $r \equiv r(x) := \mathrm{d}(x, y)$, the above inequality is equivalent to the fact that $(x, t) \mapsto \mathcal{K}_{\mathrm{E}}(\mathrm{d}(x, y), t)$ is a supersolution to the differential equation in (2.16). On the other hand, because the volume measure of $\mathbb{M}^n$ is locally Euclidean, i.e. (2.2) holds, it is straightforward to check that this function also attains a Dirac delta centered at y as $t \downarrow 0$. Hence, by the comparison principle and the arbitrariness of y, we infer that

$$\mathcal{K}(x, y, t) \leq \mathcal{K}_{\mathrm{E}}(\mathrm{d}(x, y), t) \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \qquad \forall (x, y, t) \in \mathbb{M}^n \times \mathbb{M}^n \times (0, +\infty)\,. \tag{2.17}$$

As concerns the just mentioned comparison principle, we limit ourselves to observing that the latter can rigorously be established by both approximating δ_y with a sequence of smooth radially decreasing data and filling $\mathbb{M}^n$ with a sequence of geodesic balls centered at y, solving the analogues of (2.16) with homogeneous Dirichlet boundary conditions.

Once (2.17) has been proved, (1.1) is then a consequence of well-known equivalence results between pointwise heat-kernel bounds and the validity of Sobolev-type inequalities: see e.g. [19, Corollary 14.23] or [11, Lemma 2.1.2 and Theorem 2.4.2].

However, we point out that in the above argument the optimality of the constants is not preserved in the passage from the bound (2.17) to (1.1). □

2.5 The Optimal Sobolev Constant Is Not Smaller Than the Euclidean One

The fact that the optimal constant $\mathfrak{C}$ in the Sobolev inequality (1.1) cannot be smaller than the Euclidean optimal constant C_{E}, which is attained in $\mathbb{R}^n$ by the Aubin-Talenti functions (see [3, 36])

$$f_b(x) \equiv f_b(|x|) := \left(1 + b\,|x|^2\right)^{-\frac{n-2}{2}} \quad \forall x \in \mathbb{R}^n\,, \quad \text{where } b > 0 \text{ is an arbitrary constant,} \tag{2.18}$$

is a consequence of the local Euclidean structure of $\mathbb{M}^n$, and it is actually true on *any* n-dimensional Riemannian manifold where (1.1) holds. Note that in (2.18) there should appear a further degree of freedom due to translations and another one due to multiplication by constants which we omit since it is inessential to our purposes (we only need scaling invariance). Because in Sect. 3 we will consistently take advantage of such inequality, we believe it is worth providing a direct (elementary and classical) proof.

To this end, first of all note that, thanks to (2.2) and (2.15), for every $\varepsilon \in (0, 1)$ there exists a positive constant $c(\varepsilon)$ such that

$$r^{n-1} \le A(r,\theta) \le (1 + c(\varepsilon))\, r^{n-1} \quad \forall (r,\theta) \in (0,\varepsilon) \times \mathbb{S}^{n-1}\,, \qquad \lim_{\varepsilon \downarrow 0} c(\varepsilon) = 0\,. \tag{2.19}$$

We can therefore exploit (2.19) along with the explicit expression of the Aubin-Talenti functions. Let us consider the following "truncated" versions of (2.18): given $\varepsilon \in (0, 1)$, we set

$$f_{b,\varepsilon}(x) \equiv f_{b,\varepsilon}(|x|) := \left[f_b(|x|) - f_b(\varepsilon)\right]^+ \qquad \forall x \in \mathbb{R}^n\,.$$

It is readily seen that

$$\lim_{b\to\infty} \frac{\left\| f_{b,\varepsilon} \right\|_{L^{2^*}(\mathbb{R}^n)}}{\left\| \nabla f_{b,\varepsilon} \right\|_{L^2(\mathbb{R}^n)}} = C_{\mathrm{E}} \qquad \forall \varepsilon \in (0,1)\,,$$

because f_b, and hence also $f_{b,\varepsilon}$, is concentrating at the origin as $b \to +\infty$. Consider now the function $g_{b,\varepsilon}(x) := f_{b,\varepsilon}(\mathrm{d}(x,o))$, which belongs to $\dot{H}^1(\mathbb{M}^n)$ and is supported by construction in $\overline{B}_\varepsilon$. Thanks to (2.19) and the fact that $g_{b,\varepsilon}$ is radial, recalling (2.8) and (2.9), for every $\varepsilon \in (0,1)$ we have

$$\left\| f_{b,\varepsilon} \right\|_{L^{2^*}(\mathbb{R}^n)} \le \left\| g_{b,\varepsilon} \right\|_{L^{2^*}(\mathbb{M}^n)} \le (1+c(\varepsilon))^{\frac{1}{2^*}} \left\| f_{b,\varepsilon} \right\|_{L^{2^*}(\mathbb{R}^n)} ,$$

$$\left\| \nabla f_{b,\varepsilon} \right\|_{L^{2}(\mathbb{R}^n)} \le \left\| \nabla g_{b,\varepsilon} \right\|_{L^{2}(\mathbb{M}^n)} \le (1+c(\varepsilon))^{\frac{1}{2}} \left\| \nabla f_{b,\varepsilon} \right\|_{L^{2}(\mathbb{R}^n)} .$$

As a consequence, since the definition of $\mathfrak{C}$ yields

$$\frac{\left\| f_{b,\varepsilon} \right\|_{L^{2^*}(\mathbb{R}^n)}}{(1+c(\varepsilon))^{\frac{1}{2}} \left\| \nabla f_{b,\varepsilon} \right\|_{L^{2}(\mathbb{R}^n)}} \le \frac{\left\| g_{b,\varepsilon} \right\|_{L^{2^*}(\mathbb{M}^n)}}{\left\| \nabla g_{b,\varepsilon} \right\|_{L^{2}(\mathbb{M}^n)}} \le \mathfrak{C} \qquad \forall b > 0\,,\ \forall \varepsilon \in (0,1)\,,$$

by letting $b \to +\infty$ we infer that

$$\frac{C_{\mathrm{E}}}{(1+c(\varepsilon))^{\frac{1}{2}}} \le \mathfrak{C} \qquad \forall \varepsilon \in (0,1)\,,$$

whence the thesis upon letting $\varepsilon \downarrow 0$. □

3 The Proof(s)

We provide three different proofs of Theorem 1.1. The conclusion of each of them will be that the volume measure of $\mathbb{M}^n$ is purely Euclidean, under the existence of an optimal radial profile for (1.1). For this reason, we first need a (rather intuitive) result ensuring that such property means that the Cartan-Hadamard manifold at hand is (isometric to) the Euclidean space.

Lemma 3.1 *Let $\mathbb{M}^n$ be a Cartan-Hadamard manifold. Suppose that its volume measure is Euclidean, that is*

$$A(r,\theta) = r^{n-1} \qquad \forall (r,\theta) \in (0,\infty) \times \mathbb{S}^{n-1}$$

with respect to radial coordinates about a fixed pole $o \in \mathbb{M}^n$. Then $\mathbb{M}^n \equiv \mathbb{R}^n$.

Proof We already know that the exponential map $\mathbb{R}^n \ni y \mapsto \exp_o y \in \mathbb{M}^n$ is a diffeomorphism, by the Cartan-Hadamard theorem (recall Sect. 2.1). Let us show that it is also an isometry. Given any two points $x_1 = \exp_o y_1$ and $x_2 = \exp_o y_2$, because a Cartan-Hadamard manifold is a CAT(0) space (see [5, Theorem 1.3.3] or [10, Exercise IV.12]) we have

$$\mathrm{d}(x_1,x_2) \ge |y_1 - y_2|\,, \tag{3.1}$$

i.e. the length of the side of a geodesic triangle in $\mathbb{M}^n$ opposite to the angle formed by the first two sides is not smaller than the length of side of the Euclidean triangle whose first two sides have the same length and angle. Our aim is to prove that (3.1) is in fact an identity. Suppose by contradiction that there exist $x_1, \tilde{x}_2 \in \mathbb{M}^n$ such that

$$r := \mathrm{d}(x_1, \tilde{x}_2) > |y_1 - \tilde{y}_2| .$$

It is plain that (3.1) yields

$$\left(\exp_o\right)^{-1}(B_r(x_1)) \subset B_r^{\mathrm{E}}(y_1) ,$$

where $B_r^{\mathrm{E}}(y_1)$ stands for the Euclidean ball of radius r centered at y_1. Hence, by continuity and the fact that the exponential map is a diffeomorphism, we deduce that actually there exists a nonempty open set $\Omega \subset B_r^{\mathrm{E}}(y_1)$ such that

$$\left(\exp_o\right)^{-1}(B_r(x_1)) \subset B_r^{\mathrm{E}}(y_1) \setminus \Omega .$$

Since, by assumption, the volume measure $d\mu$ of $\mathbb{M}^n$ is Euclidean, this would imply

$$\mu(B_r(x_1)) = \int_{\left(\exp_o\right)^{-1}(B_r(x_1))} dy \le \int_{B_r^{\mathrm{E}}(y_1)\setminus\Omega} dy < \left|B_r^{\mathrm{E}}(y_1)\right| ,$$

where dy denotes the n-dimensional Lebesgue measure and $|\cdot|$ the corresponding volume of measurable sets. However, due to volume comparison (see (2.15) in Sect. 2.3), this yields a contradiction since $\mu(B_r) \ge \left|B_r^{\mathrm{E}}\right|$ on any Cartan-Hadamard manifold, independently of the pole where B_r is centered. □

We are now in position to prove Theorem 1.1.

3.1 *First Proof: A Weighted Euclidean Inequality*

The starting point consists of exploiting a suitable modification of the radial change of variables introduced in [22, Section 7] (see also [37, Section 6] in the case of the hyperbolic space). That is, let us set

$$\frac{ds}{s^{n-1}} = \frac{dr}{\psi_\star(r)^{n-1}} ,$$

or more precisely

$$\frac{1}{(n-2)s^{n-2}} = \int_r^\infty \frac{dt}{\psi_\star(t)^{n-1}} , \tag{3.2}$$

where $\psi_\star$ is as in (2.7). It is not difficult to check that $\psi_\star$ belongs to the class $\mathcal{F}$ defined in (2.10). Moreover, $\psi_\star' \geq 1$ everywhere. Indeed, by combining (2.5), (2.14) and (2.15), we have

$$\psi_\star'(r) = \frac{\int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial r} A(r,\theta)\, d\theta}{(n-1)\left|\mathbb{S}^{n-1}\right|} \left[\frac{\int_{\mathbb{S}^{n-1}} A(r,\theta)\, d\theta}{\left|\mathbb{S}^{n-1}\right|} \right]^{\frac{1}{n-1}-1} \geq \frac{1}{r} \left[\frac{\int_{\mathbb{S}^{n-1}} A(r,\theta)\, d\theta}{\left|\mathbb{S}^{n-1}\right|} \right]^{\frac{1}{n-1}} \geq 1\,. \tag{3.3}$$

As a consequence,

$$\frac{1}{(n-2)s^{n-2}} \leq \int_r^\infty \frac{\psi_\star'(t)}{\psi_\star(t)^{n-1}}\, dt = \frac{1}{(n-2)\psi_\star(r)^{n-2}}\,,$$

that is

$$\rho(s) := \frac{\psi_\star(r(s))}{s} \leq 1 \qquad \forall s > 0\,. \tag{3.4}$$

Let us write Rayleigh quotients in terms of the new variable s. To this end, given a (nontrivial) radial function $f \equiv f(r) \in C_c^1(\mathbb{M}^n)$, we can construct another radial function $\hat{f} \equiv \hat{f}(s) := f(r(s)) \in C_c^1(\mathbb{R}^n)$, where $r(s)$ is obtained according to (3.2). It is plain that, for every $p \in [1,\infty)$, the following identities hold (recall (2.8)):

$$\frac{\|f\|_{L^p(\mathbb{M}^n)}^p}{\left|\mathbb{S}^{n-1}\right|} = \int_0^\infty |f(r)|^p\, \psi_\star(r)^{n-1} dr = \int_0^\infty \left|\hat{f}(s)\right|^p \rho(s)^{2(n-1)} s^{n-1} ds = \frac{\left\|\hat{f}\right\|_{L^p_\rho(\mathbb{R}^n)}^p}{\left|\mathbb{S}^{n-1}\right|},$$

where for a measurable function $g : \mathbb{R}^n \to \mathbb{R}$ we set

$$\|g\|_{L^p_\rho(\mathbb{R}^n)}^p := \int_{\mathbb{R}^n} |g(y)|^p\, \rho(|y|)^{2(n-1)} dy\,.$$

Similarly (recall (2.9)), we have:

$$\begin{aligned}
\frac{\|\nabla f\|_{L^2(\mathbb{M}^n)}^2}{\left|\mathbb{S}^{n-1}\right|} = \int_0^\infty \left|f'(r)\right|^2 \psi_\star(r)^{n-1} dr &= \int_0^\infty \left|\hat{f}'(s)\, \frac{s^{n-1}}{\psi_*(r(s))^{n-1}}\right|^2 \frac{\psi_*(r(s))^{2(n-1)}}{s^{n-1}}\, ds \\
&= \int_0^\infty \left|\hat{f}'(s)\right|^2 s^{n-1} ds \\
&= \frac{\left\|\nabla \hat{f}\right\|_{L^2(\mathbb{R}^n)}^2}{\left|\mathbb{S}^{n-1}\right|}.
\end{aligned}$$

Hence, by virtue of (3.4) and the Euclidean Sobolev inequality, we deduce that

$$\frac{\|\nabla f\|_{L^2(\mathbb{M}^n)}}{\|f\|_{L^{2^*}(\mathbb{M}^n)}} = \frac{\left\|\nabla \hat{f}\right\|_{L^2(\mathbb{R}^n)}}{\left\|\hat{f}\right\|_{L^{2^*}_{\rho}(\mathbb{R}^n)}} \geq \frac{\left\|\nabla \hat{f}\right\|_{L^2(\mathbb{R}^n)}}{\left\|\hat{f}\right\|_{L^{2^*}(\mathbb{R}^n)}} \geq \frac{1}{C_{\mathrm{E}}} \,. \tag{3.5}$$

Note that (3.5) yields equivalence between the (radial) Sobolev inequality on $\mathbb{M}^n$ and a (radial) weighted Euclidean Sobolev inequality. Clearly the latter can be extended to any nontrivial $f \in \dot{H}^1(\mathbb{M}^n)$ and therefore any nontrivial $\hat{f} \in \dot{H}^1(\mathbb{R}^n)$, still in the radial framework. Suppose now that $u \in \dot{H}^1(\mathbb{M}^n)$ is a radial optimal function for the Sobolev inequality in $\mathbb{M}^n$. Since we know from Sect. 2.5 that the corresponding best constant cannot be smaller than the Euclidean one, from (3.5) applied to $f = u$ we deduce that in fact equality holds, whence

$$\frac{\left\|\nabla \hat{u}\right\|_{L^2(\mathbb{R}^n)}}{\left\|\hat{u}\right\|_{L^{2^*}(\mathbb{R}^n)}} = \frac{1}{C_{\mathrm{E}}} \,.$$

This means that $\hat{u}$ is necessarily an Aubin-Talenti profile and

$$\left\|\hat{u}\right\|_{L^{2^*}_{\rho}(\mathbb{R}^n)} = \left\|\hat{u}\right\|_{L^{2^*}(\mathbb{R}^n)} \qquad \Longrightarrow \qquad \int_0^{\infty} \left|\hat{u}(s)\right|^{2^*} (1-\rho(s))\, ds = 0 \,.$$

Because $\hat{u}$ is everywhere positive (recall (2.18)) and $\rho(s) \leq 1$ for all $s > 0$, we infer that $\rho(s) = 1$ for all $s > 0$; from the definition of $\rho(s)$, it follows that $\psi_\star(r(s)) = s$ for all $s > 0$. In view of (3.2), this identity can be rewritten as

$$\psi_\star(r)^{n-2} = s(r)^{n-2} = \frac{1}{(n-2)\int_r^\infty \frac{dt}{\psi_\star(t)^{n-1}}} \qquad \forall r > 0 \,,$$

that is

$$\frac{d}{dr}\left(\int_r^\infty \frac{dt}{\psi_\star(t)^{n-1}}\right) = -\left[(n-2)\int_r^\infty \frac{dt}{\psi_\star(t)^{n-1}}\right]^{\frac{n-1}{n-2}} \qquad \forall r > 0 \,,$$

which upon integration yields

$$\int_r^\infty \frac{dt}{\psi_\star(t)^{n-1}} = \frac{1}{(n-2)r^{n-2}} \qquad \forall r > 0 \,,$$

so that $s(r) = r$ and therefore $\psi_\star(r) = r$ for all $r > 0$. Because $A(r,\theta) \geq r^{n-1}$ for all $r > 0$ and $\theta \in \mathbb{S}^{n-1}$, from the definition of $\psi_\star$ we can finally deduce that in fact $A(r,\theta) = r^{n-1}$, namely $\mathbb{M}^n \equiv \mathbb{R}^n$ thanks to Lemma 3.1. □

3.2 Second Proof: The Euler-Lagrange Equation

First of all let us observe that, by classical variational arguments (see e.g. [35, Chapter I]), we can assume with no loss of generality that a radial optimal function is nonnegative and satisfies, up to a multiplication by a constant, the Euler-Lagrange equation

$$-\Delta u = -u'' - \mathsf{m}(r,\theta)\, u' = u^{2^*-1} \qquad \text{in } \mathbb{M}^n \,, \tag{3.6}$$

where the spherical component Δ_{S_r} of the Laplace-Beltrami operator in (2.4) has been neglected since u is by assumption radial. Due to elliptic regularity (see again [35, Appendix B]), we deduce that u is at least $C^{1,\alpha}$ locally. Thanks to (2.5), note that (3.6) can be rewritten as

$$-\frac{1}{A(r,\theta)}\frac{\partial}{\partial r}\big(A(r,\theta)\, u'\big) = u^{2^*-1} \qquad \text{in } \mathbb{M}^n \,, \tag{3.7}$$

which immediately implies that u is strictly radially decreasing, in particular it is everywhere strictly positive and therefore $C^\infty(\mathbb{M}^n)$ still by elliptic (bootstrap) regularity. Hence, recalling (2.14), from (3.6) it follows

$$-u'' - \frac{n-1}{r}\, u' \le u^{2^*-1} \qquad \forall r > 0 \,. \tag{3.8}$$

As in Sect. 2.5 we have established that the optimal Sobolev constant $\mathfrak{C}$ cannot be smaller than the Euclidean one C_{E}, we have:

$$\begin{aligned}\left(\int_0^\infty \int_{\mathbb{S}^{n-1}} u(r)^{2^*} A(r,\theta)\, d\theta dr\right)^{\frac{2}{2^*}} &= \mathfrak{C}^2 \int_0^\infty \int_{\mathbb{S}^{n-1}} \left|u'(r)\right|^2 A(r,\theta)\, d\theta dr \\ &\ge C_{\mathrm{E}}^2 \int_0^\infty \int_{\mathbb{S}^{n-1}} \left|u'(r)\right|^2 A(r,\theta)\, d\theta dr \,.\end{aligned} \tag{3.9}$$

On the other hand, multiplying (3.7) by $u A(r,\theta)$ and integrating, we obtain:

$$\int_0^\infty \int_{\mathbb{S}^{n-1}} \left|u'(r)\right|^2 A(r,\theta)\, d\theta dr = \int_0^\infty \int_{\mathbb{S}^{n-1}} u(r)^{2^*} A(r,\theta)\, d\theta dr \,,$$

whence, in view of (3.9),

$$\left(\int_0^\infty \int_{\mathbb{S}^{n-1}} u(r)^{2^*} A(r,\theta)\, d\theta dr\right)^{\frac{2^*-2}{2^*}} \le \frac{1}{C_{\mathrm{E}}^2} \,. \tag{3.10}$$

Since $A(r, \theta) \geq r^{n-1}$, the radial profile u, now interpreted as a function in $\mathbb{R}^n$, is also an admissible competitor for the Euclidean Sobolev inequality, i.e.

$$\left(\int_0^\infty u(r)^{2^*} r^{n-1} \left|\mathbb{S}^{n-1}\right| dr\right)^{\frac{2}{2^*}} \leq C_{\mathrm{E}}^2 \int_0^\infty \left|u'(r)\right|^2 r^{n-1} \left|\mathbb{S}^{n-1}\right| dr \,. \tag{3.11}$$

By exploiting (3.8) as above, we deduce that

$$\int_0^\infty \left|u'(r)\right|^2 r^{n-1} \left|\mathbb{S}^{n-1}\right| dr \leq \int_0^\infty u(r)^{2^*} r^{n-1} \left|\mathbb{S}^{n-1}\right| dr \,. \tag{3.12}$$

Hence, upon combining (3.11) and (3.12), we end up with

$$\frac{1}{C_{\mathrm{E}}^2} \leq \left(\int_0^\infty u(r)^{2^*} r^{n-1} \left|\mathbb{S}^{n-1}\right| dr\right)^{\frac{2^*-2}{2^*}} . \tag{3.13}$$

Finally, (3.10) and (3.13) yield

$$\int_0^\infty \int_{\mathbb{S}^{n-1}} u(r)^{2^*} \left[A(r, \theta) - r^{n-1}\right] d\theta dr \leq 0 \,.$$

Since u is everywhere strictly positive and $A(r, \theta) \geq r^{n-1}$, this means that actually $A(r, \theta) = r^{n-1}$, namely $\mathbb{M}^n$ is isometric to the n-dimensional Euclidean space due to Lemma 3.1. □

3.3 Third Proof: The (Radial) Isoperimetric Inequality

We borrow the main ideas from the proof [24, Proposition 8.2], also taking advantage of the fact that the functions we consider are purely radial. This approach is in some sense the dual of the one carried out in Sect. 3.1, where starting from the optimal function u we constructed a Euclidean function $\hat{u}$ preserving the L^2 norm of the gradient and increasing the L^{2^*} norm. Conversely, here we aim at constructing a Euclidean function that has the same L^{2^*} norm but lowers the L^2 norm of the gradient. To our purpose, let $\Sigma, \Sigma_{\mathrm{E}} : (0, \infty) \to (0, \infty)$ be defined as follows:

$$\Sigma(v) := \int_{\mathbb{S}^{n-1}} A(R(v), \theta)\, d\theta \,, \qquad \Sigma_{\mathrm{E}}(v) := \left|\mathbb{S}^{n-1}\right|^{\frac{1}{n}} (nv)^{\frac{n-1}{n}} \qquad \forall v > 0 \,,$$

where $v \mapsto R(v)$ is the inverse function of $r \mapsto \mu(B_r)$. In other words, recalling formula (2.3), $\Sigma(v)$ is the surface measure of the geodesic sphere on $\mathbb{M}^n$ that encloses the geodesic ball of volume v, while $\Sigma_{\mathrm{E}}(v)$ is the surface measure of the Euclidean sphere that encloses the Euclidean ball of volume v. It is not difficult to check that $\Sigma(v) \geq \Sigma_{\mathrm{E}}(v)$ for all $v > 0$, namely that the radial *Euclidean isoperimetric inequality* holds in $\mathbb{M}^n$. Indeed, this is equivalent to showing that

$$\psi_\star(r) \geq \varrho(r) \qquad \forall r > 0\,, \tag{3.14}$$

where $\psi_\star$ is defined in (2.7) and $r \mapsto \varrho(r)$ is the function that to any $r > 0$ associates the radius of the Euclidean ball whose volume coincides with $\mu(B_r)$. Such a function can easily be computed by imposing

$$\int_0^r \psi_\star(t)^{n-1} dt = \int_0^{\varrho(r)} t^{n-1} dt \qquad \Longrightarrow \qquad \varrho(r) = \left(n \int_0^r \psi_\star(t)^{n-1} dt \right)^{\frac{1}{n}} \qquad \forall r > 0\,. \tag{3.15}$$

Hence (3.14) does hold by virtue of the property $\psi_\star' \geq 1$ (recall (3.3)):

$$\varrho(r) = \left(n \int_0^r \psi_\star(t)^{n-1} dt \right)^{\frac{1}{n}} \leq \left(n \int_0^r \psi_\star(t)^{n-1}\, \psi_\star'(t)\, dt \right)^{\frac{1}{n}} = \psi_\star(r) \qquad \forall r > 0\,.$$

Now let us consider a nonnegative radial function $f \equiv f(r) \in C^1(\mathbb{M}^n)$ and its corresponding transformed radial function $\tilde{f} \equiv \tilde{f}(\varrho) \in C^1(\mathbb{R}^n)$ according to the following implicit relation:

$$\mathcal{V}(\ell) := \mu(\{f \geq \ell\}) = \left| \left\{ \tilde{f} \geq \ell \right\} \right| \qquad \forall \ell > 0\,, \tag{3.16}$$

where in this case $|\cdot|$ stands for the Euclidean volume function. Of course (3.16) does not determine $\tilde{f}$ in a unique way unless $\tilde{f}$ is additionally required to be radially decreasing, which gives rise to an analogue of the well-established *Schwarz symmetrization*, originally employed by Talenti [36]. Note that, by construction, the functions f and $\tilde{f}$ share the same L^p norms (possibly infinite). Indeed, for any $p \in [1, \infty)$, by Fubini's theorem and (3.16) we have:

$$\begin{aligned} \|f\|_{L^p(\mathbb{M}^n)}^p = \int_{\mathbb{M}^n} f^p\, d\mu = \frac{1}{p} \int_{\mathbb{M}^n} \left(\int_0^f \ell^{p-1} d\ell \right) d\mu &= \frac{1}{p} \int_0^\infty \ell^{p-1} \left(\int_{f \geq \ell} d\mu \right) d\ell \\ &= \frac{1}{p} \int_0^\infty \ell^{p-1}\, \mathcal{V}(\ell)\, d\ell \\ &= \left\| \tilde{f} \right\|_{L^p(\mathbb{R}^n)}^p . \end{aligned} \tag{3.17}$$

Let us deal with gradients (i.e. radial derivatives). In the sequel, we additionally require that $f'(r) < 0$ for all $r > 0$, $\tilde{f}'(\varrho) < 0$ for all $\varrho > 0$ and $\inf f = 0$, so that in particular f and $\tilde{f}$ are strictly radially decreasing (therefore everywhere positive) and vanish at infinity. Note that, under such assumptions, we have $f(r) = \tilde{f}(\varrho(r))$, where $\varrho(r)$ is given in (3.15). In this case it is easy to check that $\ell \mapsto \mathcal{V}(\ell)$ is also a $C^1((0,c))$ function with $\mathcal{V}'(\ell) < 0$ for all $\ell \in (0,c)$, $c > 0$ being the maximum of f. Moreover, the following identities hold:

$$f'\Big(f^{-1}(\ell)\Big) = \frac{\Sigma(\mathcal{V}(\ell))}{\mathcal{V}'(\ell)} \qquad \text{and} \qquad \tilde{f}'\Big(\tilde{f}^{-1}(\ell)\Big) = \frac{\Sigma_{\mathrm{E}}(\mathcal{V}(\ell))}{\mathcal{V}'(\ell)} \qquad \forall \ell \in (0,c)\,. \tag{3.18}$$

In fact (3.18) is a simple consequence of the (radial) *co-area* formula

$$\int_{\mathbb{M}^n} g\, d\mu = \int_0^\infty g(r)\, \psi_\star(r)^{n-1} dr = -\int_0^c \frac{g\big(f^{-1}(\ell)\big)\, \psi_\star(R(\mathcal{V}(\ell)))^{n-1}}{f'\big(f^{-1}(\ell)\big)}\, d\ell\,, \tag{3.19}$$

valid for any measurable radial function $g \geq 0$, with the particular choice $g = \chi_{\{f \geq z\}}$ for each level $z \in (0,c)$. Clearly the same holds for $f \equiv \tilde{f}$ and $\mathbb{M}^n \equiv \mathbb{R}^n$. We point out that an analogue of (3.19) is available for a wider class of nonradial functions and general manifolds: see [24, Section 8.2] and [10, Chapter III]. However, in our simplified setting it follows directly from the change of variables $r = f^{-1}(\ell)$ inside the integral.

At this point we are in position to conclude the proof. Indeed, if a (nonnegative) radial optimal function $u \equiv u(r) \in \dot{H}^1(\mathbb{M}^n)$ exists, by virtue of the Euler-Lagrange equation (3.6) we know that it is smooth, positive and satisfies $u'(r) < 0$ for all $r > 0$ (see the beginning of the proof in Sect. 3.2). By choosing $f = u$ and $g = |\nabla u|^2 = |u'|^2$ in (3.19), using (3.18), we obtain:

$$\|\nabla u\|^2_{L^2(\mathbb{M}^n)} = -\int_0^c \frac{\Sigma(\mathcal{V}(\ell))^2}{\mathcal{V}'(\ell)}\, d\ell \geq -\int_0^c \frac{\Sigma_{\mathrm{E}}(\mathcal{V}(\ell))^2}{\mathcal{V}'(\ell)}\, d\ell = \|\nabla \tilde{u}\|^2_{L^2(\mathbb{R}^n)}\,,$$

where in the last passage we have exploited the radial isoperimetric inequality established in the beginning along with (3.18) and (3.19) also applied to $\tilde{f} = \tilde{u}$ and $\mathbb{M}^n = \mathbb{R}^n$. On the other hand, the optimality of u yields

$$\|\nabla u\|_{L^2(\mathbb{M}^n)} \leq \frac{\|u\|_{L^{2^*}(\mathbb{M}^n)}}{C_{\mathrm{E}}} \overset{(3.17)}{=} \frac{\|\tilde{u}\|_{L^{2^*}(\mathbb{R}^n)}}{C_{\mathrm{E}}} \leq \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^n)}\,.$$

Hence, by combining the last two formulas we end up with the identity

$$-\int_0^c \frac{\Sigma(\mathcal{V}(\ell)) - \Sigma_{\mathrm{E}}(\mathcal{V}(\ell))}{\mathcal{V}'(\ell)}\, d\ell = 0\,,$$

which yields $\Sigma(\mathcal{V}(\ell)) = \Sigma_{\mathrm{E}}(\mathcal{V}(\ell))$ for every $\ell \in (0,c)$ since $\Sigma \geq \Sigma_{\mathrm{E}}$, and it is readily seen that this implies $\psi_\star(r) = r$ for all $r > 0$, which proves the thesis in view of Lemma 3.1. □

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References

1. Aubin, T.: Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire (French). J. Math. Pures Appl. **55**, 269–296 (1976)
2. Aubin, T.: Espaces de Sobolev sur les variétés Riemanniennes (French). Bull. Sci. Math. **100**, 149–173 (1976)
3. Aubin, T.: Problèmes isopérimétriques et espaces de Sobolev (French). J. Differ. Geom. **11**, 573–598 (1976)
4. Aubin, T., Li, Y.Y.: On the best Sobolev inequality. J. Math. Pures Appl. **78**, 353–387 (1999)
5. Bačák, M.: Convex Analysis and Optimization in Hadamard Spaces. De Gruyter Series in Nonlinear Analysis and Applications, vol. 22. De Gruyter, Berlin (2014)
6. Berchio, E., Ganguly, D., Grillo, G.: Sharp Poincaré-Hardy and Poincaré-Rellich inequalities on the hyperbolic space. J. Funct. Anal. **272**, 1661–1703 (2017)
7. Bonforte, M., Grillo, G., Vázquez, J.L.: Fast diffusion flow on manifolds of nonpositive curvature. J. Evol. Equ. **8**, 99–128 (2008)
8. Carron, G.: Inégalités de Hardy sur les variétés riemanniennes non-compactes (French). J. Math. Pures Appl. **76**, 883–891 (1997)
9. Carron, G.: Inégalité de Sobolev et volume asymptotique (French). Ann. Fac. Sci. Toulouse Math. **21**, 151–172 (2012)
10. Chavel, I.: Riemannian Geometry. A Modern Introduction. Cambridge Studies in Advanced Mathematics, vol. 98. Cambridge University Press, Cambridge (2006)
11. Davies, E.B.: Heat Kernels and Spectral Theory. Cambridge Tracts in Mathematics, vol. 92. Cambridge University Press, Cambridge (1989)
12. Dolbeault, J., Esteban, M.J., Loss, M.: Rigidity versus symmetry breaking via nonlinear flows on cylinders and Euclidean spaces. Invent. Math. **206**, 397–440 (2016)
13. Dolbeault, J., Esteban M.J., Loss, M., Muratori, M.: Symmetry for extremal functions in subcritical Caffarelli-Kohn-Nirenberg inequalities. C. R. Math. Acad. Sci. Paris **355**, 133–154 (2017)
14. Dolbeault, J., Muratori, M., Nazaret, B.: Weighted interpolation inequalities: a perturbation approach. Math. Ann. **369**, 1237–1270 (2017)
15. Farkas, C., Kristály, A., Szakál, A.: Sobolev interpolation inequalities on Hadamard manifolds. In: 2016 IEEE 11th International Symposium on Applied Computational Intelligence and Informatics (SACI), pp. 161–166
16. Greene, R.E., Wu, H.: Function Theory on Manifolds which Possess a Pole. Lecture Notes in Mathematics, vol. 699. Springer, Berlin (1979)
17. Grigor’yan, A.: Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. Bull. Am. Math. Soc. **36**, 135–249 (1999)
18. Grigor’yan, A.: Heat Kernels on Weighted Manifolds and Applications. The Ubiquitous Heat Kernel. Contemp. Math., vol. 398. Amer. Math. Soc., Providence (2006)

19. Grigor'yan, A.: Heat Kernel and Analysis on Manifolds. AMS/IP Studies in Advanced Mathematics, vol. 47. American Mathematical Society, Providence; International Press, Boston (2009)
20. Grigor'yan, A., Saloff-Coste, L.: Surgery of the Faber-Krahn inequality and applications to heat kernel bounds. Nonlinear Anal. **131**, 243–272 (2016)
21. Grillo, G., Muratori, M.: Smoothing effects for the porous medium equation on Cartan-Hadamard manifolds. Nonlinear Anal. **131**, 346–362 (2016)
22. Grillo, G., Muratori, M., Vázquez, J.L.: The porous medium equation on Riemannian manifolds with negative curvature. The large-time behaviour. Adv. Math. **314**, 328–377 (2017)
23. Hebey, E.: Optimal Sobolev inequalities on complete Riemannian manifolds with Ricci curvature bounded below and positive injectivity radius. Am. J. Math. **118**, 291–300 (1996)
24. Hebey, E.: Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities. Courant Lecture Notes in Mathematics, vol. 5. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence (1999)
25. Hebey, E., Vaugon, M.: The best constant problem in the Sobolev embedding theorem for complete Riemannian manifolds. Duke Math. J. **79**, 235–279 (1995)
26. Kristály, A.: Sharp Morrey-Sobolev inequalities on complete Riemannian manifolds. Potential Anal. **42**, 141–154 (2015)
27. Ledoux, M.: On manifolds with non-negative Ricci curvature and Sobolev inequalities. Commun. Anal. Geom. **7**, 347–353 (1999)
28. Lee, J.M.: Riemannian Manifolds. An Introduction to Curvature. Graduate Texts in Mathematics, vol. 176. Springer, New York (1997)
29. Lu, Z., Zhou, D.: On the essential spectrum of complete non-compact manifolds. J. Funct. Anal. **260**, 3283–3298 (2011)
30. Mastrolia, P., Rigoli, M., Setti, A.G.: Yamabe-type Equations on Complete, Noncompact Manifolds. Progress in Mathematics, vol. 302. Birkhäuser/Springer, Basel (2012)
31. McKean, H.P.: An upper bound to the spectrum of Δ on a manifold of negative curvature. J. Differ. Geom. **4**, 359–366 (1970)
32. Muratori, M., Roncoroni, A.: Sobolev-type inequalities on Cartan-Hadamard manifolds and applications to some nonlinear diffusion equations. Potential Anal. https://doi.org/10.1007/s11118-021-09908-1
33. Nguyen, V.H.: The sharp Poincaré-Sobolev type inequalities in the hyperbolic space $\mathbb{H}^n$. J. Math. Anal. Appl. **462**, 1570–1584 (2018)
34. Pigola, S., Veronelli, G.: Lower volume estimates and Sobolev inequalities. Proc. Am. Math. Soc. **138**, 4479–4486 (2010)
35. Struwe, M.: Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Springer, Berlin (1990)
36. Talenti, G.: Best constant in Sobolev inequality. Ann. Mat. Pura Appl. **110**, 353–372 (1976)
37. Vázquez, J.L.: Fundamental solution and long time behavior of the porous medium equation in hyperbolic space. J. Math. Pures Appl. **104**, 454–484 (2015)

Semiconvexity of Viscosity Solutions to Fully Nonlinear Evolution Equations via Discrete Games

Qing Liu

Abstract In this paper, by using a discrete game interpretation of fully nonlinear parabolic equations proposed by Kohn and Serfaty (Commun Pure Appl Math 63(10):1298–1350, 2010), we show that the spatial semiconvexity of viscosity solutions is preserved for a class of fully nonlinear evolution equations with concave parabolic operators. We also reduce the game-theoretic argument to the viscous and inviscid Hamilton-Jacobi equations, categorizing the semiconvexity regularity of solutions in terms of semiconcavity of the Hamiltonian.

Keywords Discrete games · Semiconvexity of viscosity solutions · Fully nonlinear parabolic equations · Hamilton-Jacobi equations

1 Introduction

1.1 Background and Motivation

The convexity of solutions is known to be an important geometric property for elliptic and parabolic equations. It has been studied in [4, 17–19, 21, 24, 26, 29] for classical solutions of various equations and also in [1, 10, 15, 20, 23, 33] with viscosity techniques in different contexts.

In this work, we study semiconvexity preserving properties for the following fully nonlinear parabolic equation

$$u_t + F(x, t, \nabla u, \nabla^2 u) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \tag{1.1}$$

Q. Liu (✉)
Department of Applied Mathematics, Fukuoka University, Fukuoka, Japan
e-mail: qingliu@fukuoka-u.ac.jp

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with a bounded continuous initial value

$$u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

where $F : \mathbb{R}^n \times [0, T) \times \mathbb{R}^n \times \mathbf{S}^n \to \mathbb{R}$ is a (possibly degenerate) elliptic operator. Here $\mathbf{S}^n$ denotes the set of $n \times n$ symmetric real matrices. More precisely, we assume

(A0) $F(x, t, p, X_1) \leq F(x, t, p, X_2)$ for any $x, p \in \mathbb{R}^n$, $t \geq 0$, $X_1, X_2 \in \mathbf{S}^n$ with $X_1 \geq X_2$;

(A1) for any $R > 0$, there exists $L(R) > 0$ such that

$$|F(x, t, p_1, X_1) - F(x, t, p_2, X_2)| \leq L(R)(|p_1 - p_2| + |X_1 - X_2|)$$

for all $x \in \mathbb{R}^n$, $t \geq 0$, $p_1, p_2 \in \mathbb{R}^n$, $X_1, X_2 \in \mathbf{S}^n$ satisfying

$$|p_1| + |p_2| + |X_1| + |X_2| \leq R.$$

We here do not explicitly impose any assumptions on the continuity of F with respect to x and t. However, throughout this work, we assume that the comparison principle always holds for locally bounded semicontinuous sub- and supersolutions of (1.1):

(CP) Let u and v be respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution in $\mathbb{R}^n \times (0, \infty)$. Assume that u and v are bounded in $\mathbb{R}^n \times [0, T)$ for all $T > 0$. If $u(\cdot, 0) \leq v(\cdot, 0)$, then $u \leq v$ in $\mathbb{R}^n \times [0, \infty)$.

We refer to [8, 14, 15] for more assumptions on u_0 and F to guarantee the general wellposedness of fully nonlinear parabolic equations in the framework of viscosity solutions. We will also study related semiconvexity results for Hamilton-Jacobi equations in the form

$$u_t + F(x, t, \nabla u) - \sigma \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1.3}$$

with a given $\sigma \geq 0$; consult [2, 11] as well for existence and uniqueness of viscosity solutions in this case.

The classical convexity/concavity preserving properties for (1.1) with Lipschitz initial data and F independent of x and t is given in [15]; see also [14]. We here recall their main result.

Theorem 1.1 (Convexity Preserving Properties [15, Theorem 3.1]) *Assume that F is independent of x and t and satisfies (A0) and (A1). Assume also that $X \mapsto F(p, X)$ is concave. Let u_0 be Lipschitz in $\mathbb{R}^n$. If u_0 is convex in $\mathbb{R}^n$, then $u(\cdot, t)$ is also convex for all $t \geq 0$.*

The authors of [15] adapted the proof of so-called convexity/concavity maximum principle, proposed by Korevaar [29], Kawohl [25], and Kennington [26], to the framework of viscosity solutions. Such a convexity result also holds for the equations with mild singularities at $\nabla u = 0$, including the mean curvature flow equation and normalized p-Laplace equations ($1 < p \leq \infty$). See also [1, 15, 23, 38] for related results with PDE proofs on convexity of solutions to various nonlinear elliptic or parabolic problems.

An alternative proof is given in [33] to show the convexity preserving property for level set curvature flow equations and parabolic p-Laplace equations by adopting discrete game-theoretic interpretations provided by in [27, 34, 36, 37]. The main idea is to iteratively use the so-called dynamic programming principle to track the spatial convexity of the game value u^ε, which is a locally uniform approximation of the solution u.

1.2 Main Results

In this work, following the method in [33], we intend to generalize the notion of convexity that can be preserved by (1.1) via a unified game-theoretic approach. We are particularly interested in the question whether the same preserving property holds for semiconvexity; we refer the reader to the book [5] for an introduction of semiconcave/semiconvex functions and their applications in the study of Hamilton-Jacobi equations.

Let us briefly review the notion of semiconvex functions below. A function $f \in C(\mathbb{R}^n)$ is called c-convex for $c \in \mathbb{R}$ if

$$f(x+h) + f(x-h) \geq 2f(x) + c|h|^2 \quad \text{for all } x, h \in \mathbb{R}^n.$$

(In the literature this kind of property is sometimes called $\overline{c}$-convexity with $\overline{c} = -c$ if $c \leq 0$, which is exactly opposite to our terminology.) A continuous function f is said to be semiconvex if it is c-convex for some $c \leq 0$. See [41] for applications of semiconvex functions in optimal transport. Moreover, any c-convex function is convex if $c = 0$ and uniformly convex if $c > 0$. When $f \in C^2(\mathbb{R}^n)$, the above definition clearly yields $\nabla^2 f \geq cI$ in $\mathbb{R}^n$.

Our main result, Theorem 1.2 below, shows that the c-convexity with $c < 0$ is preserved by fully nonlinear equations as in (1.1), as long as F is concave in (x, p, X). We later need the following growth assumption on F for the game method to work.

(A2) There exist $C > 0$, $\sigma_1, \sigma_2 > 0$ such that

$$|F(x, t, p, X)| \leq C(1 + |p|^{\sigma_1} + |X|^{\sigma_2}) \;\text{ for all } x \in \mathbb{R}^n, t \geq 0, p \in \mathbb{R}^n \text{ and } X \in \mathbf{S}^n. \tag{1.4}$$

It implies that F is uniformly bounded in x, t. As mentioned above, we also assume that

(A3) F satisfies the following concavity: there exists $K \geq 0$ such that

$$F(x+h, t, p_1, X_1)+F(x-h, t, p_2, X_2) \leq 2F\left(x, t, \frac{p_1+p_2}{2}, \frac{X_1+X_2}{2}\right)+K|h|^2$$

for all $x, h \in \mathbb{R}^n$, $t \geq 0$, $p_1, p_2 \in \mathbb{R}^n$ and $X_1, X_2 \in \mathbf{S}^n$.

Theorem 1.2 (Semiconvexity Preserving Property) *Suppose that (A0)–(A3) hold. Assume that the comparison principle (CP) holds for* (1.1)–(1.2) *with initial value u_0 continuous and bounded in $\mathbb{R}^n$. Let u be the unique solution of* (1.1)–(1.2)*. If u_0 is c_0-convex in $\mathbb{R}^n$ for some $c_0 < 0$, then $u(\cdot, t)$ is $c(t)$-convex for all $t \geq 0$, where $c(t) = c_0 - Kt$.*

For first order Hamilton-Jacobi equations

$$u_t + F(x, t, \nabla u) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1.5}$$

semiconvexity of solutions are studied in [30, 31]; see also [22, 32], [9, Chapter 3.3] and [5, Chapter 1.6 and Chapter 5.3]. Our result is thus a generalization of these results for possibly degenerate parabolic equations.

As pointed out in [32] and [5], there are basically two classical methods to obtain such type of semiconvexity estimates for (1.5). The first combines the vanishing viscosity approach with an estimate of lower bound of $\nabla^2 u$ for the regularized equation. The other method is based on the Hopf-Lax formula, which gives an explicit representation of the solution. The first method is more flexible but requires heavy work on Hessian estimates of the solution. The second is more straightforward, but restrictive assumptions on the structure of H are needed.

As for semiconvexity/semiconcavity estimates for second order equations, the PDE method is applied much more widely; we refer to [12, 39] for more recent results on viscous Hamilton-Jacobi equations with Hessian estimates. Our method in this work somehow develops the idea of the second approach above in the context of fully nonlinear parabolic equations. In Sect. 3, following the strategy in [33], we give a short elementary proof of Theorem 1.2 via the dynamic programming principle arising in deterministic discrete games. The key idea is to establish semiconvexity estimates for the value function u^ε uniformly in $\varepsilon > 0$.

1.3 Generalizations

One may also consider a variant of Theorem 1.2 for preservation of uniform convexity, that is, we investigate the situation when $c_0 > 0$ and therefore u_0 cannot be bounded in $\mathbb{R}^n$. A similar result is expected to still hold; indeed, we show in Theorem 3.1 that the game value u^ε does preserve uniform convexity when $K = 0$. In order to conclude the estimate for the solution u instead of u^ε, we need a comparison principle to guarantee the convergence $u^\varepsilon \to u$ as $\varepsilon \to 0$. We choose not to discuss this case in detail in this work, since such a comparison principle needs to hold for unbounded solutions with a general growth at space infinity and will require more assumptions on F or u_0; see for example [3, 7, 14].

Besides the discussion for general second order equations, in Sect. 4 we reduce our game-theoretic interpretation to first order Hamilton-Jacobi equations ((1.3) with $\sigma = 0$) and use it to prove classical results on semiconvexity regularity of viscosity solutions. Although these results are more or less well known in the literature ([5] etc. for a comprehensive introduction), our approach in this note enables us to track a precise lower bound of the semiconvexity constant during the evolution even at the discrete level by choosing proper strategies of game players. In Sect. 5.1, by including a term of average integral in the dynamic programming equation, we also discuss similar results in the viscous case ($\sigma > 0$), which are consistent with those in [12, 39]. Our game-based estimates hold uniformly in $\sigma > 0$.

We conclude Sect. 5 by discussing how to handle the case when the general operator F depends also on the unknown function u. It turns out that our arguments in the previous sections still work provided that F is Lipschitz in the unknown.

The rest of the paper is organized in the following way. In Sect. 2, we review the game interpretations for fully nonlinear parabolic equations proposed in [28] and introduce an adaptation of the game to first order problems. In Sect. 3, we apply the game-theoretic approach to our study of semiconvexity preserving properties for second order evolution equations. We also include detailed analysis in Sect. 4 for semiconvexity of solutions to the first order time-dependent Hamilton-Jacobi equations via precise estimates on the associated game values. Section 5 is devoted to several remarks on how to further extend the methods in the previous sections to viscous Hamilton-Jacobi equations or general parabolic equations depending on the unknown.

2 The Game-Theoretic Interpretations

Let us give a brief review of the discrete games in relation to (1.1) introduced by Kohn and Serfaty [28]. With the comparison principle at hand, we may use this game interpretation to obtain the existence of viscosity solutions to (1.1)–(1.2).

2.1 Fully Nonlinear Parabolic Equations

Let $\alpha, \beta, \gamma \in (0, 1)$ satisfy

$$\gamma < \frac{1}{3}, \quad \alpha + \gamma < 1, \quad \beta + 2\gamma < 2, \quad \max\{\alpha\sigma_1, \beta\sigma_2\} < 2, \tag{2.1}$$

$$\beta < 1 - \gamma, \quad \alpha(\sigma_1 - 1) < \gamma + 1, \quad \beta(\sigma_2 - 1) < 2\gamma, \quad \beta\sigma_2 < 1 + \gamma. \tag{2.2}$$

We fix a step size $\varepsilon > 0$ and $(x, t) \in \mathbb{R}^n \times [0, \infty)$. Set the total steps of the game to be $N = [t/\varepsilon^2]$. Let $y_0 = x$.

At the k-th step ($k = 1, 2, \ldots, N$),

- Player I chooses $p_k \in \mathbb{R}^n$, $X_k \in \mathbf{S}^n$ satisfying $|p_k| \le \varepsilon^{-\alpha}$ and $|X_k| \le \varepsilon^{-\beta}$;
- Player II then picks $w_k \in \mathbb{R}^n$ with $|w_k| \le \varepsilon^{-\gamma}$;
- Once the choices of both players are determined, the game position moves from y_{k-1} to a new point $y_k = y_{k-1} + \sqrt{2}\varepsilon w_k$. Meanwhile, Player II pays the amount l_k to Player I, where

$$l_k = \sqrt{2}\varepsilon\langle p_k, w_k\rangle + \varepsilon^2 \left(\langle X_k w_k, w_k\rangle + F(y_{k-1}, k\varepsilon^2, p_k, X_k)\right). \tag{2.3}$$

Player II receives from Player I a terminal fee $u_0(y_N)$ after the last around. The game outcome for Player II at (x, t), determined by p_k, X_k, a_k, w_k for all $k = 1, \ldots, N$, is therefore

$$J^\varepsilon(x, t) = u_0(y_N) - \sum_{k=1}^{N} l_k. \tag{2.4}$$

Suppose that Player I wants to minimize $J^\varepsilon(x, t)$ while Player II intends to maximize the amount. The value function of the game is thus defined to be

$$u^\varepsilon(x, t) = \min_{p_1, X_1} \max_{w_1} \min_{p_2, X_2} \max_{w_2} \ldots \min_{p_N, X_N} \max_{w_N} J^\varepsilon(x, t). \tag{2.5}$$

It is obvious that the value function satisfies the so-called dynamic programming principle:

$$u^\varepsilon(x, t) = \min_{p,X} \max_{w} \left\{u^\varepsilon\left(x + \sqrt{2}\varepsilon w, t - \varepsilon^2\right) - \sqrt{2}\varepsilon\langle p, w\rangle - \varepsilon^2\langle Xw, w\rangle - \varepsilon^2 F(x, t, p, X)\right\} \tag{2.6}$$

for $(x, t) \in \mathbb{R}^n \times [\varepsilon^2, \infty)$, and

$$u^\varepsilon(x, t) = u_0(x) \qquad \text{for } (x, t) \in \mathbb{R}^n \times [0, \varepsilon^2). \tag{2.7}$$

The convergence of u^ε can be proved by using (2.6) above.

Theorem 2.1 (Game Convergence for General Parabolic Equations, [28, Theorem 2.2]) *Suppose that (A0), (A1) and (A2) hold. Assume that the comparison principle (CP) holds for* (1.1)*. Let* u^ε *be the value function defined as in* (2.5) *with* u_0 *continuous and bounded in* $\mathbb{R}^n$*. Then* $u^\varepsilon \to u$ *locally uniformly in* $\mathbb{R}^n \times [0, \infty)$ *as* $\varepsilon \to 0$*, where* u *is the unique viscosity solution to* (1.1)–(1.2)*.*

Remark 2.2 We obtain the same convergence result as in Theorem 2.1 if the objectives of both players are exchanged, i.e., the value function is defined by switching $\min_{p,X} \max_w$ to $\max_{p,X} \min_w$ in (2.5). In this case, the dynamic programming principle reads

$$u^\varepsilon(x, t) = \max_{p,X} \min_{w} \left\{u^\varepsilon \left(x + \sqrt{2}\varepsilon w, t - \varepsilon^2\right) - \sqrt{2}\varepsilon\langle p, w\rangle - \varepsilon^2\langle Xw, w\rangle - \varepsilon^2 F(x, t, p, X)\right\} \tag{2.8}$$

for $t \geq \varepsilon^2$.

Remark 2.3 This result can be generalized for spatially unbounded solutions that bear a growth condition provided that a comparison principle is available; see [6] for an approach based on stochastic games.

2.2 *First Order Hamilton-Jacobi Equations*

Since we are also interested in the time-dependent first order Hamilton-Jacobi equations, let us consider discrete games in the special case (1.5).

Suppose we can obtain a unique viscosity solution of (1.5) for any given bounded continuous initial condition (1.2). A game interpretation following Sect. 2.1 is given below.

We essentially take $\gamma = -1/2$ in our game setting in Sect. 2.1. Let $\alpha \in (0, 1)$ satisfy

$$\alpha(\sigma_1 - 1) < \frac{1}{2}.$$

Let $\varepsilon > 0$ and N be defined as in the game before. Set again $y_0 = x$. At the k-th step ($k = 1, 2, \ldots, N$),

- Player I chooses $p_k \in \mathbb{R}^n$ satisfying $|p_k| \leq \varepsilon^{-\alpha}$;
- Player II then picks $w_k \in \mathbb{R}^n$ with $|w_k| \leq \varepsilon^{1/2}$;
- Once the choices of both players are determined, the game position moves from y_k to a new point $y_k + \sqrt{2}\varepsilon w_k$. Meanwhile, Player II pays the amount l_k to Player I, where

$$l_k = \sqrt{2}\varepsilon\langle p_k, w_k\rangle + \varepsilon^2 F(y_{k-1}, k\varepsilon^2, p_k). \tag{2.9}$$

The total cost of the game and value function are defined as in (2.4) and (2.5) respectively.

In this case, the dynamic programming principle reads

$$u^\varepsilon(x,t) = \min_p \max_w \left\{ u^\varepsilon\left(x + \sqrt{2}\varepsilon w, t - \varepsilon^2\right) - \sqrt{2}\varepsilon\langle p, w\rangle - \varepsilon^2 F(x,t,p) \right\}. \tag{2.10}$$

Theorem 2.4 (Game Convergence for Hamilton-Jacobi Equations) *Suppose that F is independent of X and satisfies (A1) and (A2). Assume that the comparison principle (CP) holds for* (1.5)*. Let* u^ε *be the value function as defined above with* u_0 *bounded and continuous in* $\mathbb{R}^n$*. Then* $u^\varepsilon \to u$ *locally uniformly in* $\mathbb{R}^n \times [0,\infty)$ *as* $\varepsilon \to 0$*, where u is the unique viscosity solution to* (1.5) *with the initial condition* (1.2)*.*

We will not rigorously prove this theorem, since it streamlines the proof of Theorem 2.1. Instead, we give a quick formal proof below to show how this new game is related to the first order evolution equation (1.5).

Suppose that u^ε is smooth in $\mathbb{R}^n \times (0,\infty)$. Then by Taylor expansion on (2.10), we have

$$0 = \min_p \max_w \left\{ \sqrt{2}\varepsilon\langle \nabla u^\varepsilon(x,t) - p, w\rangle - \varepsilon^2 F(x,t,p) \right\} - \varepsilon^2 u_t^\varepsilon(x,t) + o(\varepsilon^2),$$

where we used the assumption that $|w| \leq \varepsilon^{1/2}$. It is clear that the maximum on the right hand side is attained when w has the same direction as $\nabla u^\varepsilon - p$ with $|w| = \varepsilon^{1/2}$, i.e.,

$$0 = \min_p \left\{ \sqrt{2}\varepsilon^{\frac{3}{2}}|\nabla u^\varepsilon(x,t) - p| - \varepsilon^2 F(x,t,p) \right\} - \varepsilon^2 u_t^\varepsilon(x,t) + o(\varepsilon^2).$$

Noticing that the right hand side is dominated by the term with $\varepsilon^{3/2}$, we find that the minimum is attained near $p = \nabla u^\varepsilon$ and therefore get

$$0 = -\varepsilon^2 F(x,t,\nabla u^\varepsilon(x,t)) - \varepsilon^2 u_t^\varepsilon(x,t) + o(\varepsilon^2).$$

Dividing this relation by ε^2 and sending $\varepsilon \to 0$, we see that the limit u satisfies (1.5).

Remark 2.5 Similar to the general case, we may consider the inverse game, whose value function also converges to the unique solution of (1.5) with (1.2). The dynamic programming equation for the inverse game is

$$u^\varepsilon(x,t) = \max_p \min_w \left\{ u^\varepsilon\left(x + \sqrt{2}\varepsilon w, t - \varepsilon^2\right) - \sqrt{2}\varepsilon\langle p, w\rangle - \varepsilon^2 F(x,t,p) \right\} \tag{2.11}$$

for all $t \geq \varepsilon^2$.

We conclude this section by remarking that for our later application, in Sect. 5.1 we will introduce an alternative game approach, based on slight modification of the game in Sect. 2.2, to the viscous Hamilton-Jacobi equation (1.3).

3 Semiconvexity Preserving for Parabolic Equations

Let us now consider the c-convexity preserving properties for nonlinear parabolic equations by games. We now use the dynamic programming principle (2.6) to prove Theorem 1.2.

Theorem 3.1 (Semiconvexity Preserving for Parabolic Game Values) *Suppose that (A0)–(A3) hold. Let u^ε be the value function defined as in (2.5) with $u_0 \in C(\mathbb{R}^n)$. If u_0 is c_0-convex in $\mathbb{R}^n$ for some $c_0 \in \mathbb{R}$, then $u^\varepsilon(\cdot, t)$ is $c_\varepsilon(t)$-convex for all $\varepsilon > 0$ and $t \geq 0$, where $c_\varepsilon(t) = c_0 - KN\varepsilon^2$ with $N = [t/\varepsilon^2]$.*

It is clear that by Theorem 2.1 and Remark 2.2, Theorem 1.2 is an immediate consequence of Theorem 3.1.

Proof of Theorem 3.1 Similar to the proof for convexity preserving properties as in [33], our proof here is based on iterations of (2.6). Fix $h \in \mathbb{R}^n$ and $\varepsilon > 0$ arbitrarily. By (2.6), for any $x \in \mathbb{R}^n$, there exist $p_\pm \in \mathbb{R}^n$, $X_\pm \in \mathbf{S}^n$ with $|p_\pm| \leq \varepsilon^{-\alpha}$ and $|X_\pm| \leq \varepsilon^{-\beta}$

$$u^\varepsilon(x \pm h, \varepsilon^2) \geq u_0\left(x \pm h + \sqrt{2}\varepsilon w\right) - \sqrt{2}\varepsilon\langle p_\pm, w\rangle - \varepsilon^2\langle X_\pm w, w\rangle - \varepsilon^2 F(x \pm h, \varepsilon^2, p_\pm, X_\pm) \tag{3.1}$$

for any $w \in \mathbb{R}^n$ with $|w| \leq \varepsilon^{-\gamma}$. Summing up these two inequalities and applying the c_0-convexity of u_0, we get

$$\begin{aligned} u^\varepsilon(x+h, \varepsilon^2) + u^\varepsilon(x-h, \varepsilon^2) \geq{}& 2u_0(x + \sqrt{2}\varepsilon w) + c_0|h|^2 - \sqrt{2}\varepsilon\langle (p_+ + p_-), w\rangle \\ & - \varepsilon^2\langle (X_+ + X_-)w, w\rangle - \varepsilon^2\left(F(x+h, \varepsilon^2, p_+, X_+) + F(x-h, \varepsilon^2, p_-, X_-)\right) \end{aligned} \tag{3.2}$$

for any $w \in \mathbb{R}^n$ with $|w| \leq \varepsilon^{-\gamma}$.

In view of (A3), it follows that for any w,

$$\begin{aligned} u^\varepsilon(x+h, \varepsilon^2) + u^\varepsilon(x-h, \varepsilon^2) \geq{}& 2u_0(x + \sqrt{2}\varepsilon w) + c_0|h|^2 - \sqrt{2}\varepsilon\langle p_+ + p_-, w\rangle \\ & - \varepsilon^2\langle (X_+ + X_-)w, w\rangle - 2\varepsilon^2 F\left(x, \varepsilon^2, \frac{p_+ + p_-}{2}, \frac{X_+ + X_-}{2}\right) - \varepsilon^2 K|h|^2. \end{aligned}$$

Since

$$\left|\frac{p_+ + p_-}{2}\right| \le \varepsilon^{-\alpha}, \quad \left|\frac{X_+ + X_-}{2}\right| \le \varepsilon^{-\beta},$$

we have

$$u^\varepsilon(x+h, \varepsilon^2) + u^\varepsilon(x-h, \varepsilon^2) \ge 2u^\varepsilon(x, \varepsilon^2) + (c_0 - K\varepsilon^2)|h|^2$$

for any $x \in \mathbb{R}^n$.

Iterating this argument, we end up with

$$u^\varepsilon(x+h, t) + u^\varepsilon(x-h, t) \ge 2u^\varepsilon(x, t) + (c_0 - KN\varepsilon^2)|h|^2$$

for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$. □

Remark 3.2 One may also prove semiconcavity preserving property provided that F is convex. In this case, we can take the original game instead of the inverse one and apply an argument symmetric to the proof above.

Remark 3.3 We may prove a similar result for generalized semiconvexity. Indeed, for some $1 < \theta \le 2$ and $c_0 < 0$, when u_0 satisfies

$$u_0(x+h) + u_0(x-h) \ge 2u_0(x) + c_0|h|^\theta,$$

for all $x, h \in \mathbb{R}^n$, we can apply the same argument to show that

$$u(x+h, t) + u(x-h, t) \ge 2u(x, t) + c_0|h|^\theta$$

for all $x, h \in \mathbb{R}^n$ and $t \ge 0$. This inequality can be viewed as a half of Hölder-Zygmund regularity (equivalent to $C^{1,\theta-1}$ class) in space; consult [40] for an introduction.

The concavity of $p \mapsto F(x, t, p, X)$ turns out to be necessary to deduce c-convexity preserving, as suggested in the following example.

Example 3.4 Consider the following one-dimensional time-dependent eikonal equation

$$u_t + |u_x| = 0 \quad \text{in } \mathbb{R}$$

with initial condition

$$u_0(x) = -\sqrt{x^2 + \delta^2} \quad x \in \mathbb{R}$$

for some $\delta > 0$. It is clear that u_0 enjoys c-convexity with $c = -1/\delta$.

Using an optimal control interpretation [2], one can express the unique solution by

$$u(x,t) = \min_{|x-y|\le t} u_0(y), \quad (x,t) \in \mathbb{R}^n \times [0,\infty)$$

and therefore

$$u(x,t) = \begin{cases} -\sqrt{(x+t)^2+\delta^2} & \text{for } x \ge 0; \\ -\sqrt{(x-t)^2+\delta^2} & \text{for } x \le 0, \end{cases}$$

it is not difficult to see that

$$u_x(x,t) = \begin{cases} -(x+t)((x+t)^2+\delta^2)^{-1/2} & \text{for } x > 0, \\ -(x-t)((x-t)^2+\delta^2)^{-1/2} & \text{for } x < 0, \end{cases}$$

which indicates a breakdown of the semiconvexity at $x = 0$ and any $t > 0$.

The initial value u_0 in the current example is not bounded, but it can be easily modified to be bounded without essentially changing the example.

On the other hand, it is not clear to us if the concavity of $X \mapsto F(p, X)$ implied by (A3) is necessary, although this assumption is usually imposed in the classical concavity preserving results [15, 23]. It would be interesting to find an example showing solutions fail to preserve concavity without the concavity of $F(p, X)$ with respect to X.

The Hamiltonian in Example 3.4 is not semiconcave. We study first order equations with semiconcave Hamiltonians in Sect. 4, where a local-in-time semiconvexity preserving property can be obtained; Theorems 4.2 and 4.5.

Combining Theorem 1.2 with the results in [15], we can show preservation of spatial $C^{1,1}$ regularity of concave solutions to (1.1)–(1.2) when F is independent of x and t.

Corollary 3.5 ($C^{1,1}$ **Regularity Preserving for Spatially Concave Solutions)** *Suppose that (A0)–(A3) hold. Assume further that* $F = F(p, X)$ *is affine in* X*; namely,*

$$\frac{1}{2}F(p, X_1) + \frac{1}{2}F(p, X_2) = F\left(p, \frac{X_1 + X_2}{2}\right)$$

for all $p_1, p_2 \in \mathbb{R}^n$ *and* $X_1, X_2 \in \mathbf{S}^n$*. Assume that* $u_0 \in C^{1,1}(\mathbb{R}^n)$ *is Lipschitz and satisfies*

$$cI \le \nabla^2 u_0 \le 0 \quad \textit{a.e. in } \mathbb{R}^n$$

for some $c \leq 0$. Let u be the unique solution of (1.1)–(1.2)*. Then $u(\cdot, t) \in C^{1,1}(\mathbb{R}^n)$ and*

$$(c - Kt)I \leq \nabla^2 u(\cdot, t) \leq 0 \quad \textit{a.e. in } \mathbb{R}^n$$

holds for all $t \geq 0$.

Proof Since u_0 is Lipschitz, we can use [15, Theorem 2.1] to show that the solution u satisfies the comparison principle (for unbounded solutions) and is Lipschitz in space. We thus can apply Theorem 1.2 to get the $(c - Kt)$-convexity of $u(\cdot, t)$ for any $t \geq 0$. On the other hand, the symmetric version of Theorem 1.1 (for concavity) implies that $u(\cdot, t)$ is concave in $\mathbb{R}^n$ for all $t \geq 0$. The $C^{1,1}$ regularity in space follows immediately. □

This result can be viewed as a parabolic version of that in [16]. A typical example that satisfies the assumptions above is the semilinear equations in the form

$$u_t - \mathrm{tr}(A\nabla^2 u) + H(\nabla u) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

with $H : \mathbb{R}^n \to \mathbb{R}$ concave and $A \in \mathbf{S}^n$ positive semidefinite.

4 Semiconvexity Preserving for Hamilton-Jacobi Equations

Let us use the game-theoretic method to investigate the semiconvexity of the solution of the first order Hamilton-Jacobi equation (1.5) with (1.2). In this special case, we are able to quantify the change of semiconvexity of the solution in space during the evolution.

4.1 Semiconvexity Preserving at the Discrete Level

We hereafter adopt an assumption on the semiconcavity of F:

(A4) There exist $K_1 \geq 0$, $K_2, K_3 \in \mathbb{R}$ such that

$$F(x+h, t, p+ch) + F(x-h, t, p-ch) \leq 2F(x, t, p) + |h|^2(K_1 + 2K_2 c + K_3 c^2)$$

for all $x, p, h \in \mathbb{R}^n$, $c \in \mathbb{R}$ and $t \geq 0$.

When F is smooth in x and p, a sufficient condition of (A4) is

$$\begin{pmatrix} \nabla_x^2 F & \nabla_x \nabla_p F \\ \nabla_x \nabla_p F & \nabla_p^2 F \end{pmatrix} \leq \begin{pmatrix} K_1 I & K_2 I \\ K_2 I & K_3 I \end{pmatrix}$$

for all $(x, t, p) \in \mathbb{R}^n \times [0, \infty) \times \mathbb{R}^n$.

Theorem 4.1 (Semiconvexity of Game Values for HJ Equations) *Suppose that F is independent of X and satisfies (A2) and (A4). Let u^ε be the value function of the inverse game for* (1.5) *with $u_0 \in C(\mathbb{R}^n)$. Assume that u_0 is c_0-convex in $\mathbb{R}^n$ for some $c_0 \in \mathbb{R}$. Then u^ε satisfies*

$$u^\varepsilon(x+h,t)+u^\varepsilon(x-h,t) \geq 2u^\varepsilon(x,t)+c_\varepsilon(t)|h|^2+S_\varepsilon(t)\varepsilon \tag{4.1}$$

for all $x \in \mathbb{R}^n$, $t \geq \varepsilon^2$, $h \in \mathbb{R}^n$ and $\varepsilon > 0$, where $c_\varepsilon : [0,\infty) \to \mathbb{R}$ is defined by

$$\begin{cases} c_\varepsilon(t) = c_\varepsilon(t-\varepsilon^2) - K_1\varepsilon^2 - c_\varepsilon^2(t-\varepsilon^2)K_3\varepsilon^2 - 2c_\varepsilon(t-\varepsilon^2)K_2\varepsilon^2 \quad \text{if } t \geq \varepsilon^2, & (4.2)\\ c_\varepsilon(t) = c_0 \qquad \text{if } 0 \leq t < \varepsilon^2, & (4.3)\end{cases}$$

and $S_\varepsilon : [0,\infty) \to \mathbb{R}$ is given by

$$\begin{cases} S_\varepsilon(t) = 2\varepsilon^2 \displaystyle\sum_{k=0}^{[t/\varepsilon^2]-1} \min\left\{c_\varepsilon(k\varepsilon^2), 0\right\} \quad \text{if } t \geq \varepsilon^2, & (4.4)\\ S_\varepsilon(t) = 0 \qquad \text{if } 0 \leq t < \varepsilon^2. & (4.5)\end{cases}$$

Proof The proof is again based on an iteration of DPP (2.11). Take $h \in \mathbb{R}^n$ arbitrarily. In view of (2.11), there exists $p_0 \in \mathbb{R}^n$ with $|p_0| \leq \varepsilon^{-\alpha}$ such that

$$u^\varepsilon(x,\varepsilon^2) = \min_w \left\{u_0(x+\sqrt{2}\varepsilon w) - \sqrt{2}\varepsilon\langle p_0, w\rangle - \varepsilon^2 F(x,\varepsilon^2,p_0)\right\}. \tag{4.6}$$

Also, for $p = p_0 \pm c_0 h \in \mathbb{R}^n$, there exist $w_\pm \in \mathbb{R}^n$ with $|w_\pm| \leq \varepsilon^{\frac{1}{2}}$ such that

$$u^\varepsilon(x\pm h,\varepsilon^2) \geq u_0(x\pm h+\sqrt{2}\varepsilon w_\pm) - \sqrt{2}\varepsilon\langle p_0 \pm c_0 h, w_\pm\rangle - \varepsilon^2 F(x\pm h,\varepsilon^2,p_0\pm c_0 h).$$

Summing these two inequalities and using c_0-convexity of u_0, we have

$$\begin{aligned} u^\varepsilon(x+h,\varepsilon^2)+u^\varepsilon(x-h,\varepsilon^2) \geq{}& 2u_0\left(x+\frac{\sqrt{2}}{2}\varepsilon(w_+ + w_-)\right)\\ &+ c_0\left|h+\frac{\sqrt{2}\varepsilon}{2}(w_+ - w_-)\right|^2 - \sqrt{2}\varepsilon\langle p_0 + c_0 h, w_+\rangle - \sqrt{2}\varepsilon\langle p_0 - c_0 h, w_-\rangle\\ &- \varepsilon^2\left(F(x+h,\varepsilon^2,p_0+c_0h) + F(x-h,\varepsilon^2,p_0-c_0h)\right)\\ \geq{}& 2u_0\left(x+\frac{\sqrt{2}}{2}\varepsilon(w_+ + w_-)\right) + c_0|h|^2 + \frac{c_0}{2}\varepsilon^2|w_+ - w_-|^2\\ &-\sqrt{2}\varepsilon\langle p_0, w_+ + w_-\rangle - \varepsilon^2\left(F(x+h,\varepsilon^2,p_0+c_0h) + F(x-h,\varepsilon^2,p_0-c_0h)\right). \end{aligned}$$

In view of (A4) and the size of $|w_\pm|$, it follows that

$$u^\varepsilon(x+h,\varepsilon^2)+u^\varepsilon(x-h,\varepsilon^2)\geq 2u_0\left(x+\frac{\sqrt{2}}{2}\varepsilon(w_++w_-)\right)+c_0|h|^2$$
$$-\sqrt{2}\varepsilon\langle p_0, w_++w_-\rangle-2\varepsilon^2F(x,\varepsilon^2,p_0)-\varepsilon^2|h|^2(K_1+2c_0K_2+c_0^2K_3)+2\varepsilon^3\min\{c_0,0\}.$$

By (4.6), we get

$$\begin{aligned}&u^\varepsilon(x+h,\varepsilon^2)+u^\varepsilon(x-h,\varepsilon^2)\\&\geq 2u^\varepsilon(x,\varepsilon^2)+\left(c_0-K_1\varepsilon^2-2c_0K_2\varepsilon^2-c_0^2K_3\varepsilon^2\right)|h|^2+2\varepsilon^3\min\{c_0,0\}\\&\geq 2u^\varepsilon(x,\varepsilon^2)+c_\varepsilon(\varepsilon^2)|h|^2+S_\varepsilon(\varepsilon^2)\varepsilon,\end{aligned}$$

where $c_\varepsilon(t)$ and $S_\varepsilon(t)$ are given respectively by (4.2)–(4.3) and by (4.4)–(4.5). Iterating the argument above yields (4.1). □

4.2 Semiconvexity in the Continuum Limit

In order to track the change of semiconvexity of $u(\cdot,t)$ for all $t\geq 0$, we pass to the limit of (4.2) as $\varepsilon\to 0$. It suffices to solve a first order ordinary differential equation in the form

$$c'(t)=-K_1-2K_2c(t)-K_3c^2(t), \tag{4.7}$$

where $c(t)$ is the limit of $c_\varepsilon(t)$ as $\varepsilon\to 0$.

In fact, if there is a unique C^1 solution $c(t)$ in $[0,T]$ with $c(0)=c_0$ for some $T>0$, then $c_\varepsilon\to c$ uniformly in $[0,T]$ as $\varepsilon\to 0$. As a result, in $[0,T]$ the Riemann sum $S_\varepsilon(t)$ is bounded uniformly for all $\varepsilon>0$ small and consequently the error term $S_\varepsilon(t)\varepsilon\to 0$ as $\varepsilon\to 0$. Hence, under the comparison principle (CP), the $c(t)$-convexity of $u(\cdot,t)$ for $t\in[0,T]$ can be deduced immediately by passing to the limit of (4.1).

Let us discuss several different cases below. Set

$$D=K_2^2-K_1K_3.$$

4.2.1 The Case $D<0$

In this case, we can only obtain local-in-time semiconvexity estimates. Since we assume $K_1\geq 0$, it is clear that $K_1, K_3>0$.

We can solve (4.7) for $c(t)$ in $[0, T)$, where

$$T = \frac{1}{\sqrt{-D}}\left(\frac{\pi}{2} + \arctan\left(\frac{c_0K_3 + K_2}{\sqrt{-D}}\right)\right) \tag{4.8}$$

and

$$c(t) = \frac{\sqrt{-D}}{K_3}\tan\left(\arctan\left(\frac{c_0K_3 + K_2}{\sqrt{-D}}\right) - \sqrt{-D}t\right) - \frac{K_2}{K_3}. \tag{4.9}$$

Hence, we get the following result.

Theorem 4.2 (Semiconvexity Preserving for HJ Equations, Part I) *Suppose that F is independent of X and satisfies (A2) and (A4) with $K_1 > 0$ and $K_1K_3 > K_2^2$. Assume that the comparison principle (CP) holds for* (1.5)*. Let $D = K_2^2 - K_1K_3$. Let u be the unique viscosity solution of* (1.5) *with a bounded continuous initial value* (1.2)*. If u_0 is c_0-convex in $\mathbb{R}^n$ for some $c_0 < 0$, then $u(\cdot, t)$ is $c(t)$-convex in $\mathbb{R}^n$ for $t \in [0, T)$, where T and $c(t)$ are given by* (4.8) *and* (4.9) *respectively.*

In particular, if $K_1, K_3 > 0$ and $K_2 = 0$, we have

$$T = \frac{1}{\sqrt{K_1K_3}}\left(\frac{\pi}{2} + \arctan\left(\sqrt{\frac{K_3}{K_1}}c_0\right)\right) \tag{4.10}$$

and

$$c(t) = \sqrt{\frac{K_1}{K_3}}\tan\left(\arctan\left(\sqrt{\frac{K_3}{K_1}}c_0\right) - \sqrt{K_1K_3}t\right). \tag{4.11}$$

4.2.2 The Case $D > 0$

We also assume that $K_3 \neq 0$.

Let us again solve (4.7). We have

$$c(t) = \frac{(\sqrt{D} - K_2)(c_0K_3 + K_2 + \sqrt{D}) + (\sqrt{D} + K_2)(c_0K_3 + K_2 - \sqrt{D})e^{-2\sqrt{D}t}}{K_3(c_0K_3 + K_2 + \sqrt{D}) - K_3e^{-2\sqrt{D}t}(c_0K_3 + K_2 - \sqrt{D})} \tag{4.12}$$

for all $t \geq 0$ if

$$\frac{c_0K_3 + K_2 - \sqrt{D}}{c_0K_3 + K_2 + \sqrt{D}} < 1. \tag{4.13}$$

The condition (4.13) holds when $c_0K_3 + K_2 + \sqrt{D} > 0$.

Note that the latter essentially does not require any semiconvexity assumption on u_0. We will therefore discuss the smoothing effect of the Hamilton-Jacobi equation under the assumptions that $K_3 < 0$ and $K_2^2 > K_1 K_3$; see Corollary 4.4.

On the other hand, if

$$\frac{c_0 K_3 + K_2 - \sqrt{D}}{c_0 K_3 + K_2 + \sqrt{D}} > 1, \tag{4.14}$$

then $u(\cdot, t)$ is $c(t)$-convex for $0 \le t < T$ with

$$T = \frac{1}{2\sqrt{D}} \log \frac{c_0 K_3 + K_2 - \sqrt{D}}{c_0 K_3 + K_2 + \sqrt{D}} \tag{4.15}$$

and $c(t)$ given as in (4.12).

Theorem 4.3 (Semiconvexity Preserving for HJ Equations, Part II) *Suppose that F is independent of X and satisfies (A2) and (A4) with $K_3 \neq 0$ and $K_1 K_3 < K_2^2$. Assume that (CP) holds for* (1.5)*. Let $D = K_2^2 - K_1 K_3$. Let u be the unique viscosity solution of* (1.5) *with a bounded continuous initial value* (1.2)*. If u_0 is c_0-convex in $\mathbb{R}^n$ for some $c_0 < 0$, then $u(\cdot, t)$ is $c(t)$-convex in $\mathbb{R}^n$ for $t \in [0, T)$, where $c(t)$ is given by* (4.12) *and*

(1) *$T = \infty$ if* (4.13) *holds;*
(2) *T is finite and given by* (4.15) *if* (4.14) *holds.*

Moreover, in the case (1), there holds

$$c(t) \to \frac{\sqrt{D} - K_2}{K_3} \quad \text{as } t \to \infty.$$

Since any bounded continuous function on $\mathbb{R}^n$ can be uniformly approximated by a semiconvex function via sup-convolutions, Theorem 4.3 implies the following result on a regularizing effect of the Hamiltonian $F(x, t, p)$ that is uniformly concave in p. We assume that $K_1 > 0$ and $K_3 < 0$, which implies that (4.13) holds for $c_0 < 0$.

Corollary 4.4 (Regularizing Effect) *Suppose that F is independent of X and satisfies (A2) and (A4) with $K_1 > 0$ and $K_3 < 0$. Assume that (CP) holds for* (1.5)*. Let u be the unique viscosity solution of* (1.5) *with a bounded continuous initial value* (1.2)*. Then the unique solution u of* (1.5) *with* (1.2) *satisfies*

$$u(x+h, t) + u(x-h, t) - 2u(x, t) \ge \frac{(\sqrt{D} - K_2) + (\sqrt{D} + K_2) e^{-2\sqrt{D}t}}{K_3 (1 - e^{-2\sqrt{D}t})} |h|^2 \tag{4.16}$$

for all $t > 0$, where $D = K_2^2 - K_1 K_3 > 0$.

Proof Let us take the sup-convolution on u_0, getting

$$u_{0,\theta}(x) = \sup_{y \in \mathbb{R}^n} \left\{ u_0(y) - \frac{|x-y|^2}{2\theta} \right\} \tag{4.17}$$

for any $\theta > 0$. It is not difficult to see that $u_{0,\theta}$ is $(-1/\theta)$-convex. We use $u_{0,\theta}$ as a new initial value of Eq. (1.5) and apply Theorem 4.3 to obtain $c(t,\theta)$-convexity of $u(\cdot, t)$ for all $t \geq 0$, where $c(t,\theta)$ is given as in (4.12) with c_0 replaced by $-1/\theta$. Now for every $t > 0$, letting $\theta \to 0$, we have

$$c(t,\theta) \to \frac{(\sqrt{D} - K_2) + (\sqrt{D} + K_2)e^{-2\sqrt{D}t}}{K_3(1 - e^{-2\sqrt{D}t})}$$

as $\theta \to 0$. This completes the proof. □

The result above has applications in asymptotic analysis. In fact, for a particular F that satisfies the assumptions in Corollary 4.4, the large-time profile of the solution to (1.5) with a Lipschitz initial value was shown in [13] to be semiconvex via a PDE method. We here provided an alternative proof about the semiconvexity based on discrete games.

4.2.3 The Case $D = 0$

Let us finally discuss this critical case when $D = 0$. In principle, one can consider it as a limit case of the others by perturbing K_i $(i = 1, 2, 3)$. Instead of categorizing all of the subcases, we only consider two simple but important special cases. We omit the proofs of the results below, since they are again based on solving (4.7) as in the proof of Theorem 4.2.

Theorem 4.5 (Local-in-time Semiconvexity Preserving for Semiconcave HJ Equations) *Suppose that F is independent of X and satisfies (A2) and (A4) with $K_1 = K_2 = 0$ and $K_3 > 0$. Assume that the comparison principle (CP) holds for* (1.5)*. Let u be the unique viscosity solution of* (1.5) *with a bounded continuous initial value* (1.2)*. If u_0 is c_0-convex in $\mathbb{R}^n$ for some $c_0 < 0$, then $u(\cdot, t)$ is $c(t)$-convex in $\mathbb{R}^n$ for $t \in [0, T)$, where*

$$T = -\frac{1}{c_0 K_3} \tag{4.18}$$

and

$$c(t) = \frac{c_0}{1 + c_0 K_3 t} \leq 0. \tag{4.19}$$

This result corresponds to the formation of shocks in one-dimensional Burgers' equation such as

$$u_t + \frac{1}{2}(u^2)_x = 0, \tag{4.20}$$

whose space integral gives rise to the eikonal equation

$$u_t + \frac{1}{2}(u_x)^2 = 0 \quad \text{in } \mathbb{R} \times (0, \infty).$$

Then the semiconvexity estimate as in Theorem 4.5 implies a Lipschitz estimate for (4.20). More precisely, given a Lipschitz initial value for (4.20) with space derivative greater than $c_0 < 0$, the spatial Lipschitz continuity will not break down before the moment T given as in (4.18); see [9] for a classical proof via the Hopf-Lax formula.

Let us turn to the case when $K_1 = K_2 = 0$ and $K_3 < 0$.

Theorem 4.6 (Global-in-time Semiconvexity Preserving for Concave HJ Equations) *Suppose that F is independent of X and satisfies (A2) and (A4) with $K_1 = K_2 = 0$ and $K_3 < 0$. Assume that (CP) holds for* (1.5)*. Let u be the unique viscosity solution of* (1.5) *with a bounded continuous initial value* (1.2)*. If u_0 is c_0-convex in $\mathbb{R}^n$ for some $c_0 < 0$, then $u(\cdot, t)$ is $c(t)$-convex in $\mathbb{R}^n$ for any $t \geq 0$, where $c(t)$ is given by* (4.19)*.*

Following the proof of Corollary 4.4, we can use Theorem 4.6 to get the classical semiconcavity estimate for the simplest Hamilton-Jacobi equation

$$u_t + H(\nabla u) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{4.21}$$

as shown in [30] (also presented in [9, Lemma 4, Chapter 3.3]). In order to be consistent with those classical results, we drop the boundedness of u_0 but assume it to be Lipschitz, which is sufficient to keep the comparison principle valid in this case.

Corollary 4.7 ([9, 30]) *Suppose that $H : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz and satisfies*

$$H(p+h) + H(p-h) \leq 2H(p) + K_3|h|^2$$

for some $K_3 < 0$. Let u_0 be Lipschitz in $\mathbb{R}^n$. Then the unique solution u of (4.21) *with* (1.2) *satisfies*

$$u(x+h, t) + u(x-h, t) - 2u(x, t) \geq -\frac{1}{K_3 t}|h|^2 \tag{4.22}$$

for all $t > 0$.

Proof We again take the sup-convolution $u_{0,\theta}$ as in (4.17) for $\theta > 0$. We use $u_{0,\theta}$ as a new initial value of Eq. (1.5) and obtain a unique solution u_θ, which is $c(t, \theta)$-

convex with

$$c(t, \theta) = -\frac{1}{\theta - K_3 t}$$

by an analogous argument in Theorem 4.5, especially (4.19) with $c_0 = -1/\theta$. Sending $\theta \to 0$, we end up with (4.22) for all $t > 0$. □

5 Further Generalizations

We give several remarks on possible generalizations of our argument in the previous sections.

5.1 Viscous Hamilton-Jacobi Equations

Let us first extend our results in Sect. 4 to the viscous Hamilton-Jacobi equation (1.3). To this end, instead of using the general game setting in Sect. 2.1, we slightly modify the game in Sect. 2.2 to interpret the solution of (1.3) with σ satisfying

$$\sigma < \frac{1}{2(n+2)}. \tag{5.1}$$

Our game below is no longer deterministic. In fact, the stochastic game setting is analogous to the Tug-of-War game with noise proposed in [34, 35]. Suppose that F satisfies the same assumptions as in Sect. 2.2. Let α also be given as in Sect. 2.2. Let

$$\mu_1 = 2\sigma(n+2), \qquad \mu_2 = 1 - \mu_1. \tag{5.2}$$

Following (2.11), we first state the dynamic programming principle

$$u^\varepsilon(x,t) = \max_{|p| \le \varepsilon^{-\alpha}} \min_{|w| \le \varepsilon^{1/2}} \left\{ \mu_1 u^\varepsilon\left(x + \sqrt{2}\varepsilon w, t - \varepsilon^2\right) - \sqrt{2}\varepsilon\mu_1 \langle p, w\rangle - \varepsilon^2 F(x,t,p) \right.$$
$$\left. + \mu_2 \fint_{B_\varepsilon(x)} u^\varepsilon\left(y, t - \varepsilon^2\right) dy \right\} \quad \text{for } x \in \mathbb{R}^n \text{ and } t \ge \varepsilon^2 \tag{5.3}$$

with initial value (2.7). Here $\fint_A f(y)\,dy$ denotes the average value of $f \in C(\mathbb{R}^n)$ in a bounded Lebesgue measurable set $A \subset \mathbb{R}^n$, that is,

$$\fint_A f(y)\,dy = \frac{1}{|A|}\int_A f(y)\,dy,$$

where $|A|$ denotes the Lebesgue measure of A.

By the formal expansion as shown in Sect. 2.2, we can easily observe that this DPP does lead to the Cauchy problem for (1.3).

Theorem 5.1 (Game Convergence for Viscous Hamilton-Jacobi Equations) *Suppose that* $\sigma > 0$ *satisfies* (5.1)*, and* F *is independent of* X *and satisfies (A1) and (A2). Assume that* u_0 *is bounded and continuous in* $\mathbb{R}^n$*. Assume that (CP) holds. Let* μ_1, μ_2 *be given by* (5.2) *and* u^ε *satisfy the dynamic programming principle* (5.3) *and* (2.7)*. Then* $u^\varepsilon \to u$ *locally uniformly in* $\mathbb{R}^n \times [0, \infty)$ *as* $\varepsilon \to 0$*, where* u *is the unique viscosity solution to* (1.3) *with the initial condition* (1.2)*.*

We now briefly introduce the game rules corresponding to the dynamic programming equation (5.3). Fix $x \in \mathbb{R}^n$ and $t \geq 0$. As before, let $N = [t/\varepsilon^2]$.

At the k-th step ($k = 1, 2, \ldots, N$),

- We toss a biased coin with probabilities μ_1 and μ_2 to respectively get a head and a tail.
- If it is a head, then we play the deterministic game as described in Sect. 2.2; that is, Player I chooses $p_k \in \mathbb{R}^n$ with $|p_k| \leq \varepsilon^{-\alpha}$, then Player II takes $w_k \in \mathbb{R}^n$ with $|w_k| \leq \varepsilon^{1/2}$, and the game state moves from y_k to a new point $y_k + \sqrt{2}\varepsilon w_k$. Player I receives from Player II the following amount of payment:

$$l_k = \sqrt{2}\varepsilon\mu_1\langle p_k, w_k\rangle + \varepsilon^2 F(y_k, k\varepsilon^2, p_k).$$

- On the other hand, if we get a tail, the game state moves randomly according to the uniform probability density to a point in the ball $B_\varepsilon(y_k)$.

We define $u^\varepsilon(x, t)$ to be the expected value of the cost $J^\varepsilon(x, t)$ as in (2.4), optimized by both players.

We remark that although it is somehow obvious that the value function u^ε should satisfy (5.3) under the game rules above, its proof is not trivial at all. Since in this work we are more interested in the connection between (5.3) and the equation (1.3), we refer the reader to [34] for detailed derivation of dynamic programming equations of stochastic discrete games.

We now study the semiconvexity property for (1.3) by using the associated dynamic programming principle (5.3). It turns out that, despite the presence of an extra mean value part in (5.3), the same argument on semiconvexity in Sect. 4 still holds. The linear term Δu essentially maintains the semiconvexity property of the original first order equation. Our game estimate holds uniformly for all $\sigma > 0$.

Theorem 5.2 (Semiconvexity of Game Values for Viscous HJ Equation) *Suppose that $\sigma > 0$ satisfies (5.1), F is independent of X and satisfies (A1), (A2) and (A4). Assume that $u_0 \in C(\mathbb{R}^n)$. Let μ_1, μ_2 be given by (5.2) and u^ε satisfy the dynamic programming principle (5.3) with (2.7). If u_0 is c_0-convex in $\mathbb{R}^n$ for some $c_0 \in \mathbb{R}$, then u^ε satisfies (4.1) for all $\sigma > 0$, $x \in \mathbb{R}^n$, $t \geq \varepsilon^2$, $h \in \mathbb{R}^n$ and $\varepsilon > 0$, where $c_\varepsilon, S_\varepsilon : [0, \infty) \to \mathbb{R}$ are given respectively by (4.2)–(4.3) and by (4.4)–(4.5).*

The proof is essentially the same as that of Theorem 4.1. The only difference lies at the averaged integral term. However, since we have

$$\mu_2 \fint_{B_\varepsilon(x)} u_0(y+h)\, dy + \mu_2 \fint_{B_\varepsilon(x)} u_0(y-h)\, dy \geq 2\mu_2 \fint_{B_\varepsilon(x)} u_0(y)\, dy + c_0\mu_2|h|^2$$

provided that u_0 is c_0-convex in $\mathbb{R}^n$, we can easily adapting the iterative argument in Theorem 4.1 to the current case.

Remark 5.3 We emphasize that the semiconvexity estimate in Theorem 5.2 does not depend on $\sigma > 0$. Moreover, the assumption (5.1) is only technical. Indeed, for any $\lambda > 0$, if we rescale the solution u by taking $v(x, t) = u(x, \lambda t)$, then v solves

$$v_t + \lambda F(x, \lambda t, \nabla v) - \lambda\sigma \Delta v = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

Letting $\tilde{F}(x, t, p) = \lambda F(x, \lambda t, p)$, one is thus able to study (1.3) with F replaced by $\tilde{F}$ and with a general σ. We then can adopt Theorem 5.2 with a new set of K_i ($i = 1, 2, 3$) in (A4) in this general case.

As a consequence of Theorems 5.2 and 5.1, we can obtain, in the viscous case, all of the results similar to those in Sect. 4.2 including Theorems 4.2, 4.3 and Corollary 4.4 uniformly in $\sigma > 0$. The proof will also be the same; according to the sign of D, we divide our discussion into different cases and repeat the same arguments when passing to the limit as $\varepsilon \to 0$. In particular, the viscous version of Theorem 4.3 generalizes the results in [12, 39] and our game-theoretic method is very different from theirs.

5.2 *Parabolic Operators Involving the Unknown*

We finally mention how to handle the case when the operator F involves the unknown function u. Our method here can apply to a general class of fully nonlinear equations including the u-dependent versions of those equations discussed in the previous sections. Let us only generalize our results for (1.1); that is, we consider

$$u_t + F(x, t, u, \nabla u, \nabla^2 u) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \tag{5.4}$$

with a bounded continuous initial value (1.2). We still assume that F is degenerate elliptic, locally Lipschitz etc. as in (A0)–(A2), i.e.,

(B0) $F(x, t, r, p, X_1) \leq F(x, t, r, p, X_2)$ for any $x, p \in \mathbb{R}^n$, $r \in \mathbb{R}$ $t \geq 0$, $X_1, X_2 \in \mathbf{S}^n$ with $X_1 \geq X_2$;

(B1) for any $R > 0$, there exists $L(R) > 0$ such that

$$|F(x, t, r, p_1, X_1) - F(x, t, r, p_2, X_2)| \leq L(R)(|p_1 - p_2| + |X_1 - X_2|)$$

for all $r \in \mathbb{R}$, $x \in \mathbb{R}^n$, $t \geq 0$ and $p_1, p_2 \in \mathbb{R}^n$, $X_1, X_2 \in \mathbf{S}^n$ satisfying

$$|p_1| + |p_2| + |X_1| + |X_2| \leq R;$$

(B2) there exist $C > 0$, $\sigma_1, \sigma_2 > 0$ such that

$$|F(x, t, p, X)| \leq C(1 + |p|^{\sigma_1} + |X|^{\sigma_2})$$

for all $x \in \mathbb{R}^n$, $r \in \mathbb{R}$, $t \geq 0$, $p \in \mathbb{R}^n$ and $X \in \mathbf{S}^n$.

Moreover, we assume that

(B3) there exists $K \geq 0$ such that

$$F(x + h, t, r_1, p_1, X_1) + F(x - h, t, r_2, p_2, X_2) \leq 2F\left(x, t, \frac{r_1 + r_2}{2}, \frac{p_1 + p_2}{2}, \frac{X_1 + X_2}{2}\right) + K|h|^2$$

for all $x, h \in \mathbb{R}^n$, $t \geq 0$, $r_1, r_2 \in \mathbb{R}$, $p_1, p_2 \in \mathbb{R}^n$ and $X_1, X_2 \in \mathbf{S}^n$

and

(B4) $r \mapsto F(x, t, r, p, X)$ is Lipschitz, i.e., there exists $\Lambda_1, \Lambda_2 \in \mathbb{R}$ with $\Lambda_1 \leq \Lambda_2$ such that

$$\Lambda_1(r_1 - r_2) \leq F(x, t, r_1, p, X) - F(x, t, r_2, p, X) \leq \Lambda_2(r_1 - r_2)$$

for all $r_1 \geq r_2$, $x, p \in \mathbb{R}^n$, $t \geq 0$ and $X \in \mathbf{S}^n$.

Besides, we assume that the comparison principle holds for (5.4). As in the previous sections, we need a game-theoretic scheme for this general equation in order to study the evolution of semiconvexity.

Although the associated discrete game in this case is studied in [28], we do not directly use their dynamic programming equation but instead consider the following variant of (2.6):

$$u^\varepsilon(x,t) = \min_{p,X}\max_{w}\left\{u^\varepsilon\left(x+\sqrt{2}\varepsilon w, t-\varepsilon^2\right) - \sqrt{2}\varepsilon\langle p, w\rangle \right.$$
$$\left. -\varepsilon^2\langle Xw, w\rangle - \varepsilon^2 F\left(x, t, u^\varepsilon(x, t-\varepsilon^2), p, X\right)\right\} \tag{5.5}$$

with $u^\varepsilon(\cdot, t) = u_0$ in $\mathbb{R}^n$ for $t < \varepsilon^2$, where the sets of p, X, w are taken as in Sect. 2.1.

Take $v(x,t) = e^{\Lambda_2 t}u(x,t)$. Then v is the unique viscosity solution of

$$v_t + \overline{F}(x, t, v, \nabla v, \nabla^2 v) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \tag{5.6}$$

with $v(\cdot, 0) = u_0$ in $\mathbb{R}^n$, where

$$\overline{F}(x,t,r,p,X) = -\Lambda_2 r + e^{\Lambda_2 t}F\left(x, t, re^{-\Lambda_2 t}, pe^{-\Lambda_2 t}, Xe^{-\Lambda_2 t}\right). \tag{5.7}$$

It is clear that

$$(\Lambda_1 - \Lambda_2)(r_1 - r_2) \le \overline{F}(x,t,r_1,p,X) - \overline{F}(x,t,r_2,p,X) \le 0 \tag{5.8}$$

for all $r_1 \ge r_2$, $x, p \in \mathbb{R}^n$, $t \ge 0$ and $X \in \mathbf{S}^n$, due to (B4).

If we take $v^\varepsilon(x,t) = e^{\Lambda_2 t}u^\varepsilon(x,t)$, then the DPP corresponding to v^ε becomes

$$v^\varepsilon(x,t) = \min_{p,X}\max_{w}\left\{v^\varepsilon\left(x+\sqrt{2}\varepsilon w, t-\varepsilon^2\right) - \sqrt{2}\varepsilon\langle p, w\rangle \right.$$
$$\left. -\varepsilon^2\langle Xw, w\rangle - \varepsilon^2 \overline{F}\left(x, t, v^\varepsilon(x, t-\varepsilon^2), p, X\right)\right\}. \tag{5.9}$$

We next use (5.9) to show a result similar to Theorem 3.1 for (5.4). Suppose u_0 is c_0-convex with $c_0 \le 0$. Let us fix $h \in \mathbb{R}^n$ and $\varepsilon > 0$ arbitrarily. Fix $x \in \mathbb{R}^n$. In view of (5.9), there exist $p_\pm \in \mathbb{R}^n$, $X_\pm \in \mathbf{S}^n$ with $|p_\pm| \le \varepsilon^{-\alpha}$ and $|X_\pm| \le \varepsilon^{-\beta}$ such that for any $w \in \mathbb{R}^n$ with $|w| \le \varepsilon^{-\gamma}$, we have

$$v^\varepsilon(x \pm h, \varepsilon^2) \ge u_0\left(x \pm h + \sqrt{2}\varepsilon w\right) - \sqrt{2}\varepsilon\langle p_\pm, w\rangle - \varepsilon^2\langle X_\pm w, w\rangle$$
$$-\varepsilon^2\overline{F}(x \pm h, \varepsilon^2, u_0(x \pm h), p_\pm, X_\pm). \tag{5.10}$$

By (B3) and the c_0-convexity of u_0 together with the monotonicity of $\overline{F}$ in (5.8), we get

$$
\begin{aligned}
& v^\varepsilon(x+h,\varepsilon^2)+v^\varepsilon(x-h,\varepsilon^2) \\
& \geq 2u_0(x+\sqrt{2}\varepsilon w)+c_0|h|^2-\sqrt{2}\varepsilon\langle (p_+ + p_-), w\rangle - \varepsilon^2\langle (X_+ + X_-)w, w\rangle \\
& \quad - 2\varepsilon^2\overline{F}\left(x,\varepsilon^2,\frac{u_0(x+h)+u_0(x-h)}{2},\frac{p_+ + p_-}{2},\frac{X_+ + X_-}{2}\right)-\varepsilon^2 K e^{\Lambda_2\varepsilon^2}|h|^2 \\
& \geq 2u_0(x+\sqrt{2}\varepsilon w)+c_0|h|^2-\sqrt{2}\varepsilon\langle (p_+ + p_-), w\rangle - \varepsilon^2\langle (X_+ + X_-)w, w\rangle \\
& \quad - 2\varepsilon^2\overline{F}\left(x,\varepsilon^2,u_0(x)+\frac{c_0}{2}|h|^2,\frac{p_+ + p_-}{2},\frac{X_+ + X_-}{2}\right)-\varepsilon^2 K e^{\Lambda_2\varepsilon^2}|h|^2
\end{aligned}
\tag{5.11}
$$

for any $w \in \mathbb{R}^n$ with $|w| \leq \varepsilon^{-\gamma}$. By (5.8) again, we have

$$
\begin{aligned}
& v^\varepsilon(x+h,\varepsilon^2)+v^\varepsilon(x-h,\varepsilon^2) \\
& \geq 2u_0(x+\sqrt{2}\varepsilon w)+c_0|h|^2-\sqrt{2}\varepsilon\langle (p_+ + p_-), w\rangle - \varepsilon^2\langle (X_+ + X_-)w, w\rangle \\
& \quad - 2\varepsilon^2\overline{F}\left(x,\varepsilon^2,u_0(x),\frac{p_+ + p_-}{2},\frac{X_+ + X_-}{2}\right)+\varepsilon^2 c_0(\Lambda_2-\Lambda_1)|h|^2-\varepsilon^2 K e^{\Lambda_2\varepsilon^2}|h|^2,
\end{aligned}
\tag{5.12}
$$

which implies that $v^\varepsilon(\cdot,\varepsilon^2)$ is $c_\varepsilon(\varepsilon^2)$-convex with

$$
c_\varepsilon(\varepsilon^2)=c_0\left(1+\varepsilon^2(\Lambda_2-\Lambda_1)\right)-\varepsilon^2 K e^{\Lambda_2\varepsilon^2}.
$$

By iterating the argument above, we have the following result.

Theorem 5.4 (Semiconvexity Preserving for the Discrete Scheme Involving u) *Suppose that (B0)–(B4) hold. Let v^ε satisfy Eq. (5.9) with initial value $v^\varepsilon(\cdot,t) = u_0 \in C(\mathbb{R}^n)$ for $t < \varepsilon^2$ and $\overline{F}$ given by (5.7). If u_0 is c_0-convex in $\mathbb{R}^n$ for some $c_0 \leq 0$, then $v^\varepsilon(\cdot,t)$ is $c_\varepsilon(t)$-convex for all $\varepsilon > 0$ and $t \geq \varepsilon^2$, where $c_\varepsilon(t) = c_0$ for $0 \leq t < \varepsilon^2$ and*

$$
c_\varepsilon(t)=c_\varepsilon(t-\varepsilon^2)\left(1+\varepsilon^2(\Lambda_2-\Lambda_1)\right)-\varepsilon^2 K\varepsilon^{\Lambda_2 t}
\tag{5.13}
$$

for $t \geq \varepsilon^2$.

To investigate the semiconvexity of u, we should consider the limit of $c_\varepsilon(t)e^{-\Lambda_2 t}$, but for our simplicity of calculations, we send $\varepsilon \to 0$ in (5.13) directly. We see that the limit $\overline{c}$ of c_ε satisfies

$$
\overline{c}'(t)=(\Lambda_2-\Lambda_1)\overline{c}(t)-Ke^{\Lambda_2 t}
$$

for all $t > 0$ and therefore,

$$\overline{c}(t) = \left(c_0 + \frac{K}{\Lambda_1}\right) e^{(\Lambda_2 - \Lambda_1)t} - \frac{K}{\Lambda_1} e^{\Lambda_2 t}$$

if $\Lambda_1 \neq 0$ and

$$\overline{c}(t) = c_0 - Kt$$

if $\Lambda_1 = 0$.

Since $u^\varepsilon \to u$ as $\varepsilon \to 0$ under (CP) as shown in [28], we are immediately led to the following consequence.

Theorem 5.5 (Semiconvexity Preserving for Parabolic Equations Depending on u) *Suppose that (B0)–(B4) hold. Assume that the comparison principle holds (CP) for* (5.4)*. Let u be the unique solution of* (5.4) *and* (1.2)*. If u_0 is c_0-convex in $\mathbb{R}^n$ for some $c_0 < 0$, then $u(\cdot, t)$ is $c(t)$-convex for all $t \geq 0$, where*

$$c(t) = \begin{cases} (c_0 + K/\Lambda_1)\, e^{-\Lambda_1 t} - K/\Lambda_1 & \text{if } \Lambda_1 \neq 0, \\ (c_0 - Kt) e^{-\Lambda_2 t} & \text{if } \Lambda_1 = 0. \end{cases}$$

Semiconvexity preserving properties for viscous or inviscid Hamilton-Jacobi equations can be similarly obtained by using the argument here to generalize the results in Sect. 4.

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References

1. Alvarez, O., Lasry, J.-M., Lions, P.-L.: Convex viscosity solutions and state constraints. J. Math. Pures Appl. (9) **76**(3), 265–288 (1997)
2. Bardi, M., Capuzzo-Dolcetta, I.: Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations. Systems & Control: Foundations & Applications. Birkhäuser, Boston (1997). With appendices by Maurizio Falcone and Pierpaolo Soravia
3. Barles, G., Biton, S., Bourgoing, M., Ley, O.: Uniqueness results for quasilinear parabolic equations through viscosity solutions' methods. Calc. Var. Partial Differ. Equ. **18**(2), 159–179 (2003)
4. Bian, B., Guan, P.: A microscopic convexity principle for nonlinear partial differential equations. Invent. Math. **177**(2), 307–335 (2009)

5. Cannarsa, P., Sinestrari, C.: Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control. Progress in Nonlinear Differential Equations and Their Applications, vol. 58. Birkhäuser, Boston (2004)
6. Cheridito, P., Soner, H.M., Touzi, N., Victoir, N.: Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs. Commun. Pure Appl. Math. **60**(7), 1081–1110 (2007)
7. Crandall, M.G., Newcomb, R., Tomita, Y.: Existence and uniqueness viscosity solutions of degenerate quasilinear elliptic equations in R^n. Technical Summary Report, Wisconsin Univ-Madison Center for Mathematical Sciences (1988)
8. Crandall, M.G., Ishii, H., Lions, P.-L.: User's guide to viscosity solutions of second order partial differential equations. Bull. Am. Math. Soc. (N.S.) **27**(1), 1–67 (1992)
9. Evans, L.C.: Partial Differential Equations. Graduate Studies in Mathematics, vol. 19. American Mathematical Society, Providence (1998)
10. Evans, L.C., Spruck, J.: Motion of level sets by mean curvature. I. J. Differ. Geom. **33**(3), 635–681 (1991)
11. Fleming, W.H., Soner, H.M.: Controlled Markov Processes and Viscosity Solutions. Stochastic Modelling and Applied Probability, vol. 25, 2nd edn. Springer, New York (2006)
12. Fujita, Y.: Hessian estimates for viscous Hamilton-Jacobi equations with the Ornstein-Uhlenbeck operator. Differ. Integr. Equ. **18**(12), 1383–1396 (2005)
13. Fujita, Y., Loreti, P.: Long-time behavior of solutions to Hamilton-Jacobi equations with quadratic gradient term. Nonlinear Differ. Equ. Appl. **16**(6), 771–791 (2009)
14. Giga, Y.: Surface Evolution Equations. A Level Set Approach. Monographs in Mathematics, vol. 99. Birkhäuser, Basel (2006).
15. Giga, Y., Goto, S., Ishii, H., Sato, M.-H.: Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains. Indiana Univ. Math. J. **40**(2), 443–470 (1991)
16. Imbert, C.: Convexity of solutions and $C^{1,1}$ estimates for fully nonlinear elliptic equations. J. Math. Pures Appl. (9) **85**(6), 791–807 (2006)
17. Ishige, K., Salani, P.: Is quasi-concavity preserved by heat flow? Arch. Math. **90**, 450–460 (2008)
18. Ishige, K., Salani, P.: Convexity breaking of the free boundary for porous medium equations. Interfaces Free Bound. **12**, 75–84 (2010)
19. Ishige, K., Nakagawa, K., Salani, P.: Power concavity in weakly coupled elliptic and parabolic systems. Nonlinear Anal. **131**, 81–97 (2016)
20. Ishige, K., Liu, Q., Salani, P.: Parabolic Minkowski convolutions and concavity properties of viscosity solutions to fully nonlinear equations. J. Math. Pures Appl. (9) **141**, 342–370 (2020)
21. Ishige, K., Nakagawa, K., Salani, P.: Spatial concavity of solutions to parabolic systems. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **20**(1), 291–313 (2020)
22. Ishii, H.: Uniqueness of unbounded viscosity solution of Hamilton-Jacobi equations. Indiana Univ. Math. J. **33**(5), 721–748 (1984)
23. Juutinen, P.: Concavity maximum principle for viscosity solutions of singular equations. Nonlinear Differ. Equ. Appl. **17**(5), 601–618 (2010)
24. Kawohl, B.: Rearrangements and Convexity of Level Sets in PDE. Lecture Notes in Mathematics, vol. 1150. Springer, Berlin (1985)
25. Kawohl, B.: Qualitative properties of solutions to semilinear heat equations. Expo. Math. **4**(3), 257–270 (1986)
26. Kennington, A.U.: Power concavity and boundary value problems. Indiana Univ. Math. J. **34**(3), 687–704 (1985)
27. Kohn, R.V., Serfaty, S.: A deterministic-control-based approach to motion by curvature. Commun. Pure Appl. Math. **59**(3), 344–407 (2006)
28. Kohn, R.V., Serfaty, S.: A deterministic-control-based approach to fully nonlinear parabolic and elliptic equations. Commun. Pure Appl. Math. **63**(10), 1298–1350 (2010)
29. Korevaar, N.J.: Convex solutions to nonlinear elliptic and parabolic boundary value problems. Indiana Univ. Math. J. **32**(4), 603–614 (1983)

30. Kružkov, S.N.: Generalized solutions of nonlinear equations of the first order with several independent variables. II. Mat. Sb. (N.S.) **72**(114), 108–134 (1967)
31. Kružkov, S.N.: Generalized solutions of Hamilton-Jacobi equations of eikonal type. I. Statement of the problems; existence, uniqueness and stability theorems; certain properties of the solutions. Mat. Sb. (N.S.) **98**(140)(3(11)), 450–493, 496 (1975)
32. Lions, P.-L.: Generalized Solutions of Hamilton-Jacobi Equations. Research Notes in Mathematics, vol. 69. Pitman (Advanced Publishing Program), Boston (1982)
33. Liu, Q., Schikorra, A., Zhou, X.: A game-theoretic proof of convexity preserving properties for motion by curvature. Indiana Univ. Math. J. **65**, 171–197 (2016)
34. Manfredi, J.J., Parviainen, M., Rossi, J.D.: An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games. SIAM J. Math. Anal. **42**(5), 2058–2081 (2010)
35. Manfredi, J.J., Parviainen, M., Rossi, J.D.: An asymptotic mean value characterization for p-harmonic functions. Proc. Am. Math. Soc. **138**(3), 881–889 (2010)
36. Peres, Y., Sheffield, S.: Tug-of-war with noise: a game-theoretic view of the p-Laplacian. Duke Math. J. **145**(1), 91–120 (2008)
37. Peres, Y., Schramm, O., Sheffield, S., Wilson, D.B.: Tug-of-war and the infinity Laplacian. J. Am. Math. Soc. **22**(1), 167–210 (2009)
38. Sakaguchi, S.: Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14**(3), 403–421 (1987/1988)
39. Strömberg, T.: Semiconcavity estimates for viscous Hamilton-Jacobi equations. Arch. Math. (Basel) **94**(6), 579–589 (2010)
40. Triebel, H.: Theory of Function Spaces. II. Monographs in Mathematics, vol. 84. Birkhäuser, Basel (1992)
41. Villani, C.: Optimal Transport. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338. Springer, Berlin (2009). Old and new

An Interpolating Inequality for Solutions of Uniformly Elliptic Equations

Rolando Magnanini and Giorgio Poggesi

Abstract We extend an inequality for harmonic functions, obtained in Magnanini and Poggesi (Calc Var Partial Differ Equ 59(1):Paper No. 35, 2020) and Poggesi (The Soap Bubble Theorem and Serrin's problem: quantitative symmetry, PhD thesis, Università di Firenze, 2019), to the case of solutions of uniformly elliptic equations in divergence form, with merely measurable coefficients. The inequality for harmonic functions turned out to be a crucial ingredient in the study of the stability of the radial symmetry for Alexandrov's Soap Bubble Theorem and Serrin's problem. The proof of our inequality is based on a mean value property for elliptic operators stated and proved in Caffarelli (The Obstacle Problem. Lezioni Fermiane. [Fermi Lectures]. Accademia Nazionale dei Lincei, Rome; Scuola Normale Superiore, Pisa, 1998) and Blank and Hao (Commun Anal Geom 23(1):129–158, 2015).

Keywords Interpolation inequality · Elliptic operators in divergence form · Mean value property · Serrin's overdetermined problem · Alexandrov Soap Bubble Theorem · Stability · Quantitative estimates

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and denote its boundary by Γ. The volume of Ω and the $(N-1)$-dimensional Hausdorff measure of Γ will be denoted, indifferently, by $|\Omega|$ and $|\Gamma|$. Let $A(x)$ be an $N \times N$ symmetric matrix whose

R. Magnanini
Dipartimento di Matematica ed Informatica "U. Dini", Università di Firenze, Firenze, Italy
e-mail: magnanini@unifi.it
http://web.math.unifi.it/users/magnanin

G. Poggesi (✉)
Department of Mathematics and Statistics, The University of Western Australia, Perth, WA, Australia
e-mail: giorgio.poggesi@uwa.edu.au
https://research-repository.uwa.edu.au/en/persons/giorgio-poggesi-2

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entries $a_{ij}(x)$, $i, j = 1, \dots, N$, are measurable functions in Ω. We assume that $A(x)$ satisfies the (uniform) ellipticity condition:

$$\lambda\,|\xi|^2 \le \langle A(x)\,\xi, \xi\rangle \le \Lambda\,|\xi|^2 \ \text{ for any } \ x \in \Omega, \ \xi \in \mathbb{R}^N. \tag{1.1}$$

Here, λ and Λ are positive constants. Associated to $A(x)$ we consider a uniformly elliptic linear operator L in divergence form, defined formally by

$$Lv = \operatorname{div}[A(x)\,\nabla v], \tag{1.2}$$

for every $x \in \Omega$.

In what follows, we shall use two scaling invariant quantities: for $1 \le p \le \infty$ the number $\|v\|_{p,\Omega}$ will denote the L^p-norm of a measurable function $v : \Omega \to \mathbb{R}$ with respect to the normalized Lebesgue measure $dx/|\Omega|$ and, for $0 < \alpha \le 1$, we define the scaling invariant Hölder seminorm

$$[v]_{\alpha,\Omega} = \sup\left\{\left(\frac{d_\Omega}{2}\right)^{\alpha} \frac{|v(x_1) - v(x_2)|}{|x_1 - x_2|^{\alpha}} \ : \ x_1, x_2 \in \overline{\Omega}, \ x_1 \ne x_2\right\}, \tag{1.3}$$

where d_Ω is the diameter of Ω. Also, the mean value of v on Ω will be indicated by v_Ω.

We let $\Sigma_\alpha(\Omega)$ be the set of weak solutions v of class $C^{0,\alpha}(\overline{\Omega})$ of $Lv = 0$ in Ω. We denote by B_r and S_r the ball and sphere of radius r centered at the origin. To avoid unessential technicalities, we state here our main result in the case in which Ω is a ball. The case of general domains will be treated later on.

Theorem 1.1 *Take $p \in [1, \infty)$. There exists a positive constant K, which only depends on N, p, α, λ, and Λ, such that, for any $v \in \Sigma_\alpha(B_r)$, the following holds:*

$$\max_{S_r} v - \min_{S_r} v \le K\,[v]_{\alpha,B_r}^{\frac{N}{N+\alpha\,p}}\,\|v - v_{B_r}\|_{p,B_r}^{\frac{\alpha p}{N+\alpha p}}. \tag{1.4}$$

Moreover, (1.4) is optimal in the sense that the equality sign holds for some $v \in \Sigma_\alpha(B_r)$.

We recall that, by De Giorgi-Nash-Moser's theorem, we have that a solution of $Lu = 0$ is locally of class $C^{0,\alpha}(\Omega)$ for some $\alpha \in (0, 1]$ that depends on N, λ and Λ. Moreover that regularity can be extended up to the boundary provided u is Hölder-continuous on Γ and Γ is sufficiently smooth—e.g., Γ satisfies a uniform exterior cone condition (see [10, Theorem 8.29]) or, more in general, condition (A) defined in [12, pag. 6] (see [12, Theorem 1.1 of Chapter 4]).

The reader's attention should be focused on the quantitative character of (1.4). This says that the oscillation of a solution of an elliptic equation can be controlled, up to the boundary, by its L^p-norm in the domain, provided some a priori information is given on its Hölder seminorm.

The effectiveness of an inequality like (1.4) can be understood from an important application of it, that was first given in [14], and then refined in [15–18] (see also [13] for a survey on those issues). There, rougher versions of (1.4) for harmonic functions were used to obtain quantitative rigidity estimates for the spherical symmetry in two celebrated problems in differential geometry and potential theory: Alexandrov's Soap Bubble Theorem and Serrin's overdetermined problem. For the reader's convenience, we recall these two rigidity results. Alexandrov's Soap Bubble Theorem [1–3] states that, if a compact hypersurface, embedded in the Euclidean space, has constant mean curvature, then it must be a sphere. Serrin's symmetry result [19] asserts that a solution of the Poisson's equation $\Delta u = N$ in a bounded domain, subject to constant Dirichlet and Neumann boundary conditions exists if and only if the domain is a ball. In [16] and [15] the two results are shown to be intimately connected. Moreover, this fact and the rougher version of (1.4) help to obtain quantitative estimates of spherical symmetry in both problems. Another application of an inequality like (1.4) can be found in [9].

Theorem 1.1 improves the result obtained in [18, Lemma 3.14] (and hence the previous ones) from various points of view. As already mentioned, it extends the analogous estimates obtained for harmonic functions to the case of a uniformly elliptic linear operator in divergence form with *merely* measurable coefficients. Moreover, it removes the restriction of smallness of the term $\|v - v_\Omega\|_{p,B}$ that was present in the previous inequalities. In doing so, it clears up which are the essential ingredients to consider to obtain a best possible bound. Finally, It also relaxes the former Lipschitz assumption on the solutions to a weaker Hölder continuous a priori information.

The proof of the existence of the optimal constant K in (1.4) is obtained by a quite standard variational argument. The necessary compactness of the optimizing sequence is derived from a rougher version of (1.4), which is proved in Lemma 2.2. More precisely, Lemma 2.2 shows that an inequality like (1.4) holds for sub-solutions of the equation $Lv = 0$ and it provides an explicit upper bound for the constant K. The proof of this lemma extends the arguments, first used in [14] and refined in [15, 16, 18] for (sub-)harmonic functions, to the case of an elliptic operator. The crucial ingredient to do so is a mean value theorem for elliptic equations in divergence form (see Theorem 2.1) the proof of which is sketched in [8, Remark at page 9] and given with full details in [7, Theorem 6.3].

The proof of Theorem 1.1 is given in Sect. 2. There, we also provide a proof for the case of smooth domains. In this case, the constant K also depends on the ratio between the diameter and the radius of a uniform interior touching ball for the relevant domain. In Sect. 3, we show that the proof's scheme can be extended to two instances of non-smooth domains: those satisfying either the uniform interior cone condition or the so-called local John's condition. The dependence of K on the relevant parameters follows accordingly.

2 The Inequality in a Ball and in Smooth Domains

We recall the already mentioned result introduced by L. Caffarelli [8, Remark on page 9], the proof of which is provided in full details in [7, Theorem 6.3]. In what follows, $B_r(x_0)$ denotes the ball of radius r centered at x_0.

Theorem 2.1 (Mean Value Property for Elliptic Operators) *Let Ω be an open subset of $\mathbb{R}^N$. Let L be the elliptic operator defined by (1.1)–(1.2) and pick any $x_0 \in \Omega$. Then, there exist two constants c, C that only depend on N, λ and Λ, and, for $0 < r < r_0$ with $r_0 \geq \operatorname{dist}(x_0, \Gamma)/C$, an increasing family of domains $D_r(x_0)$ which satisfy the properties:*

(i) $B_{cr}(x_0) \subset D_r(x_0) \subset B_{Cr}(x_0)$*;*
(ii) for any v satisfying $Lv \geq 0$, we have that

$$v(x_0) \leq \frac{1}{|D_r(x_0)|} \int_{D_r(x_0)} v(y)\, dy \leq \frac{1}{|D_\rho(x_0)|} \int_{D_\rho(x_0)} v(y)\, dy, \tag{2.1}$$

for any $0 < r < \rho < r_0$.

Issues related to this theorem and the study of the geometric properties of the sets $D_r(x_0)$ have been recently studied by I. Blank and his collaborators in [4–6].

2.1 The Inequality for a Ball

We begin our presentation by considering the case of a ball. This will avoid extra technicalities. We will later show how to extend our arguments to other types of domains.

The following lemma gives a rough estimate for sub-solutions of the elliptic equation $Lv = 0$.

Lemma 2.2 *Take $p \geq 1$. Let $v \in C^{0,\alpha}(\overline{B_r})$, $0 < \alpha \leq 1$, be a weak solution of $Lv \geq 0$ in B_r. Then we have that*

$$\max_{S_r} v - \min_{S_r} v \leq 2\left(1 + \frac{\alpha p}{N}\right)\left(\frac{N}{\alpha p}\right)^{\frac{\alpha p}{N+\alpha p}} \left(\frac{C}{c}\right)^{\frac{\alpha N}{N+\alpha p}} [v]_{\alpha, B_r}^{\frac{N}{N+\alpha\, p}} \, \|v - v_{B_r}\|_{p, B_r}^{\frac{\alpha p}{N+\alpha p}}. \tag{2.2}$$

Proof Without loss of generality, we can assume that $v_{B_r} = 0$. Let x_1 and x_2 be points on S_r that respectively minimize and maximize v on S_r and, for $0 < \sigma < r$, define the two points $y_j = x_j - \sigma x_j / r$, $j = 1, 2$. Notice that x_j / r is the exterior unit normal vector to S_r at the point x_j.

By (1.3) and the fact that $2r$ is the diameter of B_r, we have that

$$|v(x_j)| \le |v(y_j)| + [v]_{\alpha, B_r} \left(\frac{\sigma}{r}\right)^{\alpha}, \ j = 1, 2. \tag{2.3}$$

Being as $0 < \sigma < r$, we have that $B_\sigma(y_j) \subset \Omega$. Thus, we apply Theorem 2.1 by choosing $x_0 = y_j$, $j = 1, 2$, and $r = \sigma/C$. By item (i), we have that

$$B_{\frac{c}{C}\sigma}(y_j) \subset D_{\frac{\sigma}{C}}(y_j) \subset B_\sigma(y_j) \subset B_r, \quad j = 1, 2. \tag{2.4}$$

Also, item (ii) gives that

$$\begin{aligned} |v(y_j)| \le \frac{1}{|D_{\frac{\sigma}{C}}(y_j)|} \int_{D_{\frac{\sigma}{C}}(y_j)} |v| \, dy \le \\ \frac{1}{|D_{\frac{\sigma}{C}}(y_j)|^{1/p}} \left[\int_{D_{\frac{\sigma}{C}}(y_j)} |v|^p \, dy\right]^{1/p} \\ \le |B|^{-\frac{1}{p}} \left(\frac{C}{c\,\sigma}\right)^{N/p} \left(\int_{B_r} |v|^p \, dy\right)^{1/p}. \end{aligned} \tag{2.5}$$

The second inequality is a straightforward application of Hölder's inequality and, in the last inequality, we used (2.4), that also gives that

$$|D_{\frac{\sigma}{C}}(y_j)| \ge |B| \left(\frac{c}{C}\right)^N \sigma^N.$$

Putting together (2.3) and (2.5) yields

$$\max_{S_r} v - \min_{S_r} v \le 2 \left[\left(\frac{C}{c}\right)^{N/p} \|v\|_{p, B_r} \left(\frac{\sigma}{r}\right)^{-N/p} + [v]_{\alpha, B_r} \left(\frac{\sigma}{r}\right)^{\alpha}\right], \tag{2.6}$$

for every $0 < \sigma < r$.

Therefore, in order to minimize the right-hand side of the last inequality, we can conveniently choose

$$\frac{\sigma^*}{r} = \left[\frac{N}{\alpha p} \left(\frac{C}{c}\right)^{N/p} \frac{\|v\|_{p, B_r}}{[v]_{\alpha, B_r}}\right]^{p/(N+\alpha p)} \tag{2.7}$$

and obtain (2.2) if $\sigma^* < r$.

On the other hand, if $\sigma^* \geq r$, by (1.3) we can write:

$$\max_{S_r} v - \min_{S_r} v \leq 2^\alpha \, [v]_{\alpha, B_r} \leq 2^\alpha \, [v]_{\alpha, B_r} \left(\frac{\sigma^*}{r} \right)^\alpha .$$

Thus, (2.7) gives

$$\max_{S_r} v - \min_{S_r} v \leq 2^\alpha \left(\frac{N}{\alpha p} \right)^{\frac{\alpha p}{N+\alpha p}} \left(\frac{C}{c} \right)^{\frac{\alpha N}{N+\alpha p}} [v]_{\alpha, B_r}^{\frac{N}{N+\alpha p}} \, \|v\|_{p, B_r}^{\frac{\alpha p}{N+\alpha p}} .$$

Therefore, (2.2) always holds true, since $2^\alpha \leq 2(1 + \alpha p/N)$. □

Proof of Theorem 1.1 Lemma 2.2 tells us that (1.4) holds with

$$K = \sup \left\{ \max_{S_r} v - \min_{S_r} v : v \in \Sigma_\alpha(B_r) \text{ with } [v]_{\alpha, B_r}^{\frac{N}{N+\alpha p}} \, \|v - v_{B_r}\|_{p, B_r}^{\frac{\alpha p}{N+\alpha p}} \leq 1 \right\},$$

and

$$K \leq 2 \left(1 + \frac{\alpha p}{N} \right) \left(\frac{N}{\alpha p} \right)^{\frac{\alpha p}{N+\alpha p}} \left(\frac{C}{c} \right)^{\frac{\alpha N}{N+\alpha p}} .$$

We are thus left to prove the existence of a $v \in \Sigma_\alpha(B_r)$ that attains the supremum. Again, we assume that $v_{B_r} = 0$ in the supremum and take a maximizing sequence of functions v_n, that is

$$[v_n]_{\alpha, B_r}^{\frac{N}{N+\alpha p}} \, \|v_n\|_{p, B_r}^{\frac{\alpha p}{N+\alpha p}} \leq 1 \text{ and } \max_{S_r} v_n - \min_{S_r} v_n \to K \text{ as } n \to \infty.$$

Observe that

$$\|v_n\|_{p, B_r} \leq 2^{\frac{\alpha N}{N+\alpha p}}, \; n \in \mathbb{N},$$

since

$$\|v\|_{p, B_r} = \|v - v_{B_r}\|_{p, B_r} \leq 2^\alpha [v]_{\alpha, B_r}, \; v \in \Sigma_\alpha(B_r).$$

We can then extract a subsequence of functions, that we will still denote by v_n, that weakly converges in $L^p(B_r)$ to a function $v \in L^p(B_r)$. By the mean value property of Theorem 2.1, the sequence converges uniformly to v on the compact subsets of B_r, and hence v satisfies the mean value property of Theorem 2.1 in B_r. The converse of the mean value theorem (see, e.g., [4, Theorem 1.2]) then gives that $Lv = 0$ in B_r.

Next, we fix $x_1, x_2 \in B_r$ with $x_1 \neq x_2$. Notice that

$$r^\alpha \frac{|v_n(x_1) - v_n(x_2)|}{|x_1 - x_2|^\alpha} \leq [v_n]_{\alpha,B_r} \leq \|v_n\|_{p,B_r}^{-\frac{\alpha p}{N}},$$

where the second inequality follows from the normalizations of the maximizing sequence that we assumed above. Thus, the local uniform convergence and the semicontinuity of the L^p-norm with respect to weak convergence give that

$$r^\alpha \frac{|v(x_1) - v(x_2)|}{|x_1 - x_2|^\alpha} \leq \|v\|_{p,B_r}^{-\frac{\alpha p}{N}}.$$

Since x_1 and x_2 are arbitrary, we infer that $[v]_{\alpha,B_r} \|v\|_{p,B_r}^{\frac{\alpha p}{N}} \leq 1$. This means that v extends to a function of class $C^{0,\alpha}(\overline{B_r})$.

If we now prove that $v_n \to v$ uniformly on S_r, we will have that

$$K = \lim_{n\to\infty} \left(\max_{S_r} v_n - \min_{S_r} v_n \right) = \max_{S_r} v - \min_{S_r} v,$$

and the proof would be complete. For any $x \in S_r$ and $y \in B_r$, we can easily show that

$$\limsup_{n\to\infty} |v_n(x) - v(x)| \leq r^{-\alpha} |x - y|^\alpha \limsup_{n\to\infty} [v_n]_{\alpha,B_r} + |v(y) - v(x)| \leq$$
$$r^{-\alpha} |x - y|^\alpha \|v\|_{p,B_r}^{-\frac{\alpha p}{N}} + |v(y) - v(x)|.$$

Since $y \in B_r$ is arbitrary and v is continuous up to S_r, the right-hand side can be made arbitrarily small, and hence we infer that v_n converges to v pointwise on S_r. The convergence turns out to be uniform on S_r. In fact, if $x_n \in S_r$ maximizes $|v_n - v|$ on S_r then by compactness $x_n \to x$ as $n \to \infty$ for some $x \in S_r$, modulo a subsequence. Thus,

$$\max_{S_r} |v_n - v| = |v_n(x_n) - v(x_n)| \leq$$
$$r^{-\alpha} |x_n - x|^\alpha [v_n]_{\alpha,B_r} + |v_n(x) - v(x)| + |v(x) - v(x_n)|,$$

and the right-hand side vanishes as $n \to \infty$, by the continuity of v and the pointwise convergence of v_n. The proof is complete. □

2.2 The Inequality for Smooth Domains

The extension of Theorem 1.1 to the case of bounded domains with boundary Γ of class C^2 is not difficult. We recall that such domains satisfy a *uniform interior sphere condition*. In other words, there exists $r_i > 0$ such that for each $z \in \Gamma$ there is a ball of radius r_i contained in Ω the closure of which intersects Γ only at z.

Theorem 2.3 *Take $p \in [1, \infty)$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary Γ of class C^2 and let L be the elliptic operator defined by* (1.1)–(1.2)*.*

If $v \in \Sigma_\alpha(\Omega)$, then

$$\max_\Gamma v - \min_\Gamma v \le K\, [v]_{\alpha,\Omega}^{\frac{N}{N+\alpha\, p}}\, \|v - v_\Omega\|_{p,\Omega}^{\frac{\alpha p}{N+\alpha p}} \tag{2.8}$$

for some optimal constant K, which only depends on N, p, α, λ, Λ, and d_Ω/r_i.

Proof The proof runs similarly to that of Theorem 1.1. We just have to make some necessary changes to the proof of Lemma 2.2,

We take x_1 and x_2 in Γ that respectively minimize and maximize v on Γ and define the corresponding y_1, y_2 by $y_j = x_j - \sigma\nu(x_j)$, $j = 1, 2$, where $\nu(x_j)$ is the exterior unit normal vector to Ω at the point x_j. This time we use the restriction $0 < \sigma < r_i$, so that $B_\sigma(y_j) \subset \Omega$, $j = 1, 2$.

Next, we must replace (2.3) by

$$|v(x_j)| \le |v(y_j)| + [v]_{\alpha,\Omega} \left(\frac{2\sigma}{d_\Omega}\right)^\alpha, \quad j = 1, 2, \tag{2.9}$$

and (2.5) by

$$|v(y_j)| \le |B|^{-\frac{1}{p}} \left(\frac{C}{c\,\sigma}\right)^{N/p} \left(\int_\Omega |v|^p\, dy\right)^{1/p}, \quad j = 1, 2.$$

Thus, we arrive at

$$\max_\Gamma v - \min_\Gamma v \le 2\left[\left(\frac{C}{c}\right)^{N/p} \|v\|_{p,\Omega} \left(\frac{2\sigma}{d_\Omega}\right)^{-N/p} + [v]_{\alpha,\Omega} \left(\frac{2\sigma}{d_\Omega}\right)^\alpha\right] \tag{2.10}$$

for $0 < \sigma < r_i$, in place of (2.6). Here, we used that $|\Omega| \le |B|\,(d_\Omega/2)^N$.

In order to minimize the right-hand side of (2.10), this time we can choose

$$\frac{2\sigma^*}{d_\Omega} = \left[\frac{N}{\alpha p} \left(\frac{C}{c}\right)^{N/p} \frac{\|v\|_{p,\Omega}}{[v]_{\alpha,\Omega}}\right]^{p/(N+\alpha p)},$$

and obtain that (2.8) holds true with

$$K \le \max\left[2\left(1+\frac{\alpha p}{N}\right), \left(\frac{d_\Omega}{r_i}\right)^{\alpha}\right]\left(\frac{N}{\alpha p}\right)^{\frac{\alpha p}{N+\alpha p}}\left(\frac{C}{c}\right)^{\frac{\alpha N}{N+\alpha p}}, \tag{2.11}$$

if $\sigma^* < r_i$.

On the other hand, if $\sigma^* \ge r_i$, (1.3) gives:

$$\max_\Gamma v - \min_\Gamma v \le 2^\alpha [v]_{\alpha,\Omega} \le \left(\frac{2\sigma^*}{r_i}\right)^{\alpha}[v]_{\alpha,\Omega} = \left(\frac{N}{\alpha p}\right)^{\frac{\alpha p}{N+\alpha p}}\left(\frac{C}{c}\right)^{\frac{\alpha N}{N+\alpha p}}\left(\frac{d_\Omega}{r_i}\right)^{\alpha}[v]_{\alpha,\Omega}^{\frac{N}{N+\alpha p}}\|v\|_{p,\Omega}^{\frac{\alpha p}{N+\alpha p}}.$$

Again, (2.8) and (2.11) hold true. □

Remark 2.4 Theorem 2.3 can be compared with [18, Lemma 3.14], that was proved for the Laplace operator. In that case, we have that $c = C = 1$ and the seminorm in (1.3) can be replaced by the maximum of $(d_\Omega/2)\,|\nabla v|$ on Γ.

3 The Inequality for Two Classes of Non-smooth Domains

In this section, for future reference, we consider and carry out some details for two cases of domains with non-smooth boundary.

3.1 Domains with Corners

Given $\theta \in [0, \pi/2]$ and $h > 0$, we say that Ω satisfies the (θ, h)*-uniform interior cone condition*, if for every $x \in \Gamma$ there exists a finite right spherical cone $\mathcal{C}_x$ (with vertex at x and axis in some direction e_x), having opening width θ and height h, such that

$$\mathcal{C}_x \subset \overline{\Omega} \quad \text{and} \quad \overline{\mathcal{C}}_x \cap \Gamma = \{x\}.$$

Theorem 3.1 *Take $p \in [1, \infty)$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain satisfying the (θ, h)-uniform interior cone condition and let L be the elliptic operator defined by* (1.1)–(1.2).

If $v \in \Sigma_\alpha(\Omega)$, then (2.8) *holds true for some optimal constant K, which only depends on N, p, α, λ, Λ, d_Ω/h, and θ.*

Proof The proof runs similarly to that of Theorem 2.3. We just have to take care of the bound for K.

Let x_1 and x_2 be the usual extremum points for v on Γ. This time, instead, we define the two points $y_j = x_j - \sigma e_{x_j}$, $j = 1, 2$, for

$$0 < \sigma < \frac{h}{1 + \sin\theta}.$$

Notice that, in view of the (θ, h)-uniform interior cone condition, the ball $B_{\sigma \sin\theta}(y_j)$ is contained in Ω. Thus, by proceeding as in the proof of Theorem 2.3 (this time applying Theorem 2.1 with $r = \frac{\sin\theta}{C}\sigma$ and $x_0 = y_j$, $j = 1, 2$), we arrive at the inequality

$$\max_\Gamma v - \min_\Gamma v \le 2\left[\left(\frac{C}{c\,\sin\theta}\right)^{N/p} \|v\|_{p,\Omega} \left(\frac{2\sigma}{d_\Omega}\right)^{-N/p} + [v]_{\alpha,\Omega}\left(\frac{2\sigma}{d_\Omega}\right)^{\alpha}\right],$$

for every $0 < \sigma < h/(1 + \sin\theta)$. Hence, this time we can choose

$$\frac{2\sigma^*}{d_\Omega} = \left[\frac{N}{\alpha p}\left(\frac{C}{c\,\sin\theta}\right)^{N/p} \frac{\|v\|_{p,\Omega}}{[v]_{\alpha,\Omega}}\right]^{p/(N+\alpha p)},$$

and obtain that (2.8) holds true with

$$K \le \max\left[2\left(1 + \frac{\alpha p}{N}\right), \left(\frac{d_\Omega}{h}\right)^{\alpha}(1 + \sin\theta)^{\alpha}\right]\left(\frac{N}{\alpha p}\right)^{\frac{\alpha p}{N+\alpha p}}\left(\frac{C}{c\,\sin\theta}\right)^{\frac{\alpha N}{N+\alpha p}}, \tag{3.1}$$

if $\sigma^* < h/(1 + \sin\theta)$.

On the other hand, if $\sigma^* \ge h/(1 + \sin\theta)$, by (1.3) we have that

$$\max_\Gamma v - \min_\Gamma v \le 2^{\alpha}\,[v]_{\alpha,\Omega} \le \left(\frac{2\sigma^*}{h}\right)^{\alpha}(1 + \sin\theta)^{\alpha}\,[v]_{\alpha,\Omega} =$$

$$\left(\frac{N}{\alpha p}\right)^{\frac{\alpha p}{N+\alpha p}}\left(\frac{C}{c\,\sin\theta}\right)^{\frac{\alpha N}{N+\alpha p}}\left(\frac{d_\Omega}{h}\right)^{\alpha}(1 + \sin\theta)^{\alpha}\,[v]_{\alpha,\Omega}^{\frac{N}{N+\alpha\,p}}\,\|v\|_{p,\Omega}^{\frac{\alpha p}{N+\alpha p}}.$$

Again, (2.8) and (3.1) hold true. □

3.2 Locally John's Domains

Following [11, Definition 3.1.12], we say that a bounded domain $\Omega \subset \mathbb{R}^N$ satisfies the (b_0, R)-*local John condition* if there exist two constants, $b_0 > 1$ and $R > 0$, with the following properties. For every $x \in \Gamma$ and $r \in (0, R]$ we can find $x_r \in B_r(x) \cap \Omega$

such that $B_{r/b_0}(x_r) \subset \Omega$. Also, for each z in the set $\Delta_r(x)$ defined by $B_r(x) \cap \Gamma$, we can find a rectifiable path $\gamma_z : [0, 1] \to \overline{\Omega}$, with length $\leq b_0 r$, such that $\gamma_z(0) = z$, $\gamma_z(1) = x_r$, and

$$\mathrm{dist}(\gamma_z(t), \Gamma) > \frac{|\gamma_z(t) - z|}{b_0} \quad \text{for any} \quad t > 0. \tag{3.2}$$

The constants b_0, R, the point x_r, and the curve γ_z are respectively called *John's constants, John's center (of* $\Delta_r(x)$*), and John's path*. The class of domains satisfying the local John condition is huge and contains, among others, the so-called *non-tangentially accessible* domains (see [11, Lemma 3.1.13]).

Theorem 3.2 *Take* $p \in [1, \infty)$*. Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain satisfying the* (b_0, R)*-local John condition and let* L *be the elliptic operator defined by* (1.1)–(1.2)*.*

If $v \in \Sigma_\alpha(\Omega)$*, then* (2.8) *holds true for some optimal constant* K*, which only depends on* N*,* p*,* α*,* λ*,* Λ*,* d_Ω/R*, and* b_0*.*

Proof Let x be one of the usual extremum points for v on Γ. Let γ_x be a John's path from x to the John's center x_R of $\Delta_R(x)$. Since $B_{R/b_0}(x_R) \subset \Omega$ we have that

$$|x - x_R| \geq \mathrm{dist}(x_R, \Gamma) > \frac{R}{b_0}.$$

Thus, for $0 < \sigma < R/b_0$, we can find a point y on the John's curve γ_x such that $|x - y| = \sigma$. Hence, by (1.3) we have that (2.9) still holds true.

In view of (3.2) we have that $B_{\sigma/b_0}(y) \subset \Omega$. Thus, proceeding as we did to obtain Eq. (2.5) (this time applying Theorem 2.1 with $r = \sigma/(C b_0)$ and $x_0 = y$), we get that

$$|v(y)| \leq \left(\frac{|\Omega|}{|B|}\right)^{\frac{1}{p}} \left[\frac{C\, b_0}{c\, \sigma}\right]^{N/p} \|v\|_{p,\Omega}.$$

This, (2.9), and the inequality $|\Omega| \leq |B|\,(d_\Omega/2)^N$ then yield that

$$\max_\Gamma v - \min_\Gamma v \leq 2 \left[\left(\frac{C\, b_0}{c}\right)^{N/p} \left(\frac{2\,\sigma}{d_\Omega}\right)^{-N/p} \|v\|_{p,\Omega} + [v]_{\alpha,\Omega} \left(\frac{2\,\sigma}{d_\Omega}\right)^{\alpha}\right],$$

for every $0 < \sigma < R/b_0$. Hence, this time we can choose

$$\frac{2\,\sigma^*}{d_\Omega} = \left[\frac{N}{\alpha\, p} \left(\frac{C\, b_0}{c}\right)^{N/p} \frac{\|v\|_{p,\Omega}}{[v]_{\alpha,\Omega}}\right]^{p/(N+\alpha p)}, \tag{3.3}$$

and have that (2.8) holds true with

$$K \le \max\left[2\left(1+\frac{\alpha p}{N}\right), \left(\frac{d_\Omega b_0}{R}\right)^{\alpha}\right]\left(\frac{N}{\alpha p}\right)^{\frac{\alpha p}{N+\alpha p}} \left(\frac{C b_0}{c}\right)^{\frac{\alpha N}{N+\alpha p}}, \tag{3.4}$$

if $\sigma^* < R/b_0$.

On the other hand if $\sigma^* \ge R/b_0$, since by (1.3) we have that

$$\max_\Gamma v - \min_\Gamma v = v(x_1) - v(x_2) \le 2^\alpha\, [v]_{\alpha,\Omega} \left(\frac{\sigma^* b_0}{R}\right)^{\alpha} \le [v]_{\alpha,\Omega}\left(2\,\sigma^*\right)^{\alpha}\left(\frac{b_0}{R}\right)^{\alpha},$$

by (3.3) we immediately get

$$\max_\Gamma v - \min_\Gamma v \le \left(\frac{d_\Omega b_0}{R}\right)^{\alpha}\left(\frac{N}{\alpha p}\right)^{\frac{\alpha p}{N+\alpha p}} \left(\frac{C b_0}{c}\right)^{\frac{\alpha N}{N+\alpha p}} [v]_{\alpha,\Omega}^{\frac{N}{N+\alpha\, p}}\, \|v\|_{p,\Omega}^{\alpha p/(N+\alpha p)}.$$

Hence, (2.8) and (3.4) still hold true. □

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References

1. Aleksandrov, A.D.: Uniqueness theorems for surfaces in the large. V. Vestn. Leningr. Univ. **13**(19), 5–8 (1958)
2. Aleksandrov, A.D.: Uniqueness theorems for surfaces in the large. V. Am. Math. Soc. Transl. (2) **21**, 412–416 (1962)
3. Alexandrov, A.D.: A characteristic property of spheres. Ann. Mat. Pura Appl. (4) **58**, 303–315 (1962)
4. Armstrong, N., Blank, I: Nondegenerate motion of singular points in obstacle problems with varying data. J. Differ. Equ. **267**(9), 5370–5387 (2019)
5. Aryal, A., Blank, I.: Geometry of mean value sets for general divergence form uniformly elliptic operators. Potential Anal. **50**(1), 43–54 (2019)
6. Benson, B., Blank, I., LeCrone, J.: Mean value theorems for Riemannian manifolds via the obstacle problem. J. Geom. Anal. **29**(3), 2752–2775 (2019)
7. Blank, I., Hao, Z.: The mean value theorem and basic properties of the obstacle problem for divergence form elliptic operators. Commun. Anal. Geom. **23**(1), 129–158 (2015)
8. Caffarelli, L.A.: The Obstacle Problem. Lezioni Fermiane. [Fermi Lectures]. Accademia Nazionale dei Lincei, Rome; Scuola Normale Superiore, Pisa (1998)
9. Dipierro, S., Poggesi, G, Valdinoci, E.: A Serrin-type problem with partial knowledge of the domain. Nonlinear Analysis 208 (2021)

10. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Classics in Mathematics. Springer, Berlin (2001). Reprint of the 1998 edition
11. Hofmann, S., Mitrea, M., Taylor, M.: Singular integrals and elliptic boundary problems on regular Semmes–Kenig–Toro domains. Int. Math. Res. Notices **2010**(14), 2567–2865 (2010)
12. Ladyzhenskaya, O.A., Ural'tseva, N.N.: Linear and Quasilinear Elliptic Equations. Academic, New York (1968). Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis
13. Magnanini, R.: Alexandrov, Serrin, Weinberger, Reilly: symmetry and stability by integral identities. In: Bruno Pini Mathematical Analysis Seminar 2017. Bruno Pini Math. Anal. Semin., vol. 8, pp. 121–141. Univ. Bologna, Alma Mater Stud., Bologna (2017)
14. Magnanini, R., Poggesi, G.: On the stability for Alexandrov's Soap Bubble theorem. J. Anal. Math. **139**(1), 179–205 (2019)
15. Magnanini, R., Poggesi, G.: Nearly optimal stability for Serrin's problem and the Soap Bubble theorem. Calc. Var. Partial Differ. Equ. **59**(1), Paper No. 35 (2020)
16. Magnanini, R., Poggesi, G.: Serrin's problem and Alexandrov's soap bubble theorem: enhanced stability via integral identities. Indiana Univ. Math. J. **69**(4), 1181–1205 (2020)
17. Poggesi, G.: Radial symmetry for p-harmonic functions in exterior and punctured domains. Appl. Anal. **98**(10), 1785–1798 (2019)
18. Poggesi, G.: The Soap Bubble Theorem and Serrin's problem: quantitative symmetry. PhD thesis, Università di Firenze, defended on February 2019. Preprint (2019), arxiv:1902.08584
19. Serrin, J.: A symmetry problem in potential theory. Arch. Rational Mech. Anal. **43**, 304–318 (1971)

Asymptotic Behavior of Solutions for a Fourth Order Parabolic Equation with Gradient Nonlinearity via the Galerkin Method

Nobuhito Miyake and Shinya Okabe

Abstract In this paper we consider the initial-boundary value problem for a fourth order parabolic equation with gradient nonlinearity. The problem is regarded as the L^2-gradient flow for an energy functional which is unbounded from below. We first prove the existence and the uniqueness of solutions to the problem via the Galerkin method. Moreover, combining the potential well method with the Galerkin method, we study the asymptotic behavior of global-in-time solutions to the problem.

Keywords Fourth order parabolic equation · Gradient nonlinearity · Epitaxial growth · Galerkin method

1 Introduction

We consider the following initial-boundary value problem for a fourth order parabolic equation with gradient nonlinearity:

$$\begin{cases} \partial_t u + (-\Delta)^2 u = -\nabla \cdot (|\nabla u|^{p-2} \nabla u) & \text{in} \quad \Omega \times (0, T), \\ \partial_\nu u = \partial_\nu \Delta u = 0 & \text{on} \quad \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0(\cdot) & \text{in} \quad \Omega. \end{cases} \tag{P}$$

Here, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a smooth bounded domain, $u_0 \in L^2(\Omega)$, $p > 2$, $T > 0$, $\partial_t := \partial/\partial t$ and ∂_ν denotes the outer normal derivative to $\partial\Omega$. In this paper we show the existence and the uniqueness of local-in-time solutions to problem (P) and consider the asymptotic behavior of global-in-time solutions to problem (P) via the Galerkin method.

N. Miyake · S. Okabe (✉)
Mathematical Institute, Tohoku University, Aoba, Sendai, Japan
e-mail: nobuhito.miyake.t2@dc.tohoku.ac.jp; shinya.okabe@tohoku.ac.jp

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Fourth order parabolic equations with gradient nonlinearity appear in a model of thin film growth. King–Stein–Winkler [8] studied the following continuum model for epitaxial thin film growth proposed by Ortiz–Repetto–Si [11], based on phenomenological considerations by Zangwill [15]:

$$\begin{cases} \partial_t u + (-\Delta)^2 u = \nabla \cdot f(\nabla u) + g & \text{in} \quad \Omega \times (0,T), \\ \partial_\nu u = \partial_\nu \Delta u = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0(\cdot) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

In the paper [8], they assumed that (1) has a gradient structure and the corresponding energy is bounded from below (for example, $f(z) = |z|^{p-2}z - z$ and $g \equiv 0$). Under these conditions, they studied the existence of global-in-time solutions and large time behavior of solutions to (1). Recently problem (1) was studied in the mathematical literature (e.g., see [4, 17, 18]). However, the approaches which were used in these papers cannot be applied directly to problem (P). Indeed, problem (P) is regarded as the L^2-gradient flow for the energy functional

$$E(u) := \frac{1}{2}\int_\Omega |\Delta u|^2\,dx - \frac{1}{p}\int_\Omega |\nabla u|^p\,dx$$

and the functional E is unbounded from below due to $p > 2$.

On the other hand, problem (P) was studied by Sandjo–Moutari–Gningue [13] via the semigroup approach. They showed the existence of local-in-time solutions to problem (P) under the condition $3 < p < 4$. The assumption for p was required for the Lipschitz continuity of the nonlinear term and hence this approach cannot be adapted for the case $2 < p < 3$ in problem (P). However, as in the result [7] which is the whole space case for problem (P), the restriction $p > 3$ should be eliminated.

In this paper we prove the existence of local-in-time solutions to problem (P) in the case

(a) $u_0 \in H^2_{\mathcal{N}}(\Omega)$ and $2 < p < p_S$, or (b) $u_0 \in L^2_{\mathcal{N}}(\Omega)$ and $2 < p < p_*$,

where

$$L^2_{\mathcal{N}}(\Omega) := \left\{ v \in L^2(\Omega) \;\middle|\; \int_\Omega v\,dx = 0 \right\} \subset L^2(\Omega),$$
$$H^2_{\mathcal{N}}(\Omega) := \left\{ v \in H^2(\Omega) \cap L^2_{\mathcal{N}}(\Omega) \mid \partial_\nu v = 0 \text{ on } \partial\Omega \right\} \subset H^2(\Omega),$$

and

$$p_* := 2 + \frac{4}{N+2}, \qquad p_S := \begin{cases} \dfrac{2N}{N-2} & \text{if} \quad N \geq 3, \\ \infty & \text{if} \quad N = 2. \end{cases}$$

Moreover, we study the asymptotic behavior of global-in-time solutions to problem (P) in the case (a). In order to formulate a definition of the solution to problem (P), we set

$$\mathscr{V} := \left\{ \varphi \in H^1(0,T; L^2_{\mathscr{N}}(\Omega)) \cap L^2(0,T; H^2_{\mathscr{N}}(\Omega)) \mid \nabla\varphi \in (L^p(0,T; L^p(\Omega)))^N \right\}.$$

Definition 1.1 Let $u_0 \in L^2_{\mathscr{N}}(\Omega)$ and $T > 0$. We say that a function

$$u \in C([0,T]; L^2_{\mathscr{N}}(\Omega)) \cap L^2(0,T; H^2_{\mathscr{N}}(\Omega)) \quad \text{with} \quad \nabla u \in (L^p(0,T; L^p(\Omega)))^N$$

is a solution to problem (P) in $\Omega \times [0,T]$ if u satisfies

$$\int_\Omega [u(T)\varphi(T) - u_0\varphi(0)]\,dx \\ - \int_0^T \int_\Omega u\partial_t\varphi\,dx\,dt + \int_0^T \int_\Omega \left[\Delta u \Delta\varphi - |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\right] dx\,dt = 0 \tag{2}$$

for $\varphi \in \mathscr{V}$. Moreover, we say that u is a global-in-time solution to problem (P) if u is a solution to problem (P) in $\Omega \times [0,T']$ for all $T' > 0$.

The first and the second main results of this paper are the existence and the uniqueness of local-in-time solutions to problem (P) in the case (a) and (b) respectively:

Theorem 1.1 *Let $u_0 \in H^2_{\mathscr{N}}(\Omega)$ and assume that $2 < p < p_S$. Then the following hold*:

(i) *There exist $T > 0$ and a solution u to problem* (P) *in $\Omega \times [0,T]$. Moreover, the solution u satisfies*

$$u \in H^1(0,T; L^2_{\mathscr{N}}(\Omega)) \cap C_{\mathrm{w}}([0,T]; H^2_{\mathscr{N}}(\Omega)) \quad \textit{with} \quad \nabla u \in (C([0,T]; L^p(\Omega)))^N.$$

(ii) *If u_1 and u_2 are solutions to problem* (P) *in $\Omega \times [0,T]$ for some $T > 0$ and satisfy*

$$\nabla u_1, \nabla u_2 \in (L^\infty(0,T; L^p(\Omega)))^N,$$

then it holds that $u_1 \equiv u_2$ in $\Omega \times [0,T]$.

Theorem 1.2 *Let $u_0 \in L^2_{\mathscr{N}}(\Omega)$ and assume that $2 < p < p_*$. Then there exist $T > 0$ and a unique solution u to problem* (P) *in $\Omega \times [0,T]$.*

The precise definition of the space $C_{\mathrm{w}}([0,T]; H^2_{\mathscr{N}}(\Omega))$, see Sect. 2.

The other purpose of this paper is to study the asymptotic behavior of global-in-time solutions to problem (P) in the case (a). In order to state our main result on this topic, we introduce several notations. Let $0 < \mu_1 < \mu_2 < \cdots$ be the strictly monotone increasing divergent sequence of all eigenvalues of the boundary value problem

$$\begin{cases} -\Delta\psi = \mu\psi & \text{in} \quad \Omega, \\ \partial_\nu \psi = 0 & \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega \psi\, dx = 0. \end{cases} \tag{3}$$

Let P_0 be the zero map. For each $k \in \mathbb{N}$, we define P_k as the projection from $L^2_{\mathcal{N}}(\Omega)$ to the subspace spanned by the eigenfunctions corresponding to μ_1, μ_2, $\cdots$, μ_k. Let $k_* \in \mathbb{N}$ be the number such that

$$\mu_{k_*}^2 < (p-1)\mu_1^2 \le \mu_{k_*+1}^2. \tag{4}$$

We note that the condition $p > 2$ implies the existence of the number k_*. As a class of initial data, we set

$$W := \left\{ v \in H^2_{\mathcal{N}}(\Omega) \mid E(v) < d,\ I(v) > 0 \right\},$$

where I denotes the Nehari functional given by

$$I(v) := \int_\Omega |\Delta v|^2\, dx - \int_\Omega |\nabla v|^p\, dx, \quad v \in H^2_{\mathcal{N}}(\Omega),$$

and

$$d := \left(\frac{1}{2} - \frac{1}{p}\right) S_p^{\frac{p}{p-2}} > 0, \quad S_p := \inf_{v \in H^2_{\mathcal{N}}(\Omega), v \neq 0} \frac{\|\Delta v\|^2_{L^2(\Omega)}}{\|\nabla v\|^2_{L^p(\Omega)}} > 0. \tag{5}$$

Then the third main result of this paper is stated as follows:

Theorem 1.3 *Let $u_0 \in W$ and assume that $2 < p < p_S$. Then problem* (P) *possesses the unique global-in-time solution u such that*

$$\|\Delta u(t)\|_{L^2(\Omega)} = O(e^{-\mu_1^2 t}) \quad \text{as} \quad t \to \infty. \tag{6}$$

Moreover, it holds that

$$\|u(t) - P_{k-1}u(t)\|_{L^2(\Omega)} = O(e^{-\mu_k^2 t}) \qquad \text{as} \quad t \to \infty, \quad 1 \le k \le k_*, \tag{7}$$

$$\|u(t) - P_{k_*}u(t)\|_{L^2(\Omega)} = O(e^{-(1-\varepsilon)(p-1)\mu_1^2 t}) \quad \text{as} \quad t \to \infty, \quad 0 < \varepsilon < 1, \tag{8}$$

where k_* is the positive integer satisfying (4).

One of the main ingredients of the strategy in the proof of Theorem 1.3 is the potential well method introduced by Sattinger [14] and Payne–Sattinger [12]. The assumption $u_0 \in W$ implies that E is bounded from below along the orbit of the solution to problem (P) starting from u_0. Then, combining the Galerkin method, one can prove that problem (P) has a global-in-time solution. Indeed, [5] proved the existence of global-in-time solutions to a related problem by the same strategy. However, the strategy does not show the existence of local-in-time solutions to problem (P) with more general initial data. Moreover, it is not clear how to derive the asymptotic behavior of global-in-time solutions to problem (P), because useful mathematical tools such as the comparison principle do not hold for fourth order parabolic problems. In Theorems 1.1 and 1.2, making use of the Galerkin method and the Aubin–Lions–Simon compactness theorem, we prove the existence of local-in-time solutions to problem (P) without using the potential well method. To the best of our knowledge, this is the first paper to prove the solvability of problem (P) for $u_0 \in L^2(\Omega)$ and $2 < p < 3$. Moreover, our argument based on the Galerkin method and the potential well method enable to derive the precise asymptotic behavior of the global-in-time solutions as in Theorem 1.3.

This paper is organized as follows. We introduce some notations and collect several useful propositions in Sect. 2. We construct a solution to problem (P) with the Galerkin method in Sect. 3. In Sect. 4, we give the characterization of stable sets and study the asymptotic behavior of global-in-time solutions which converge to 0. In appendix, we prove the uniqueness of solutions to problem (P) under the assumption in Theorems 1.1 and 1.2.

2 Preliminaries

In this section we collect several notations and propositions which are used in this paper. In what follows, we rewrite the norm $\|\cdot\|_{L^q(\Omega)}$ as $\|\cdot\|_q$ for $q \in [1, \infty]$ and the L^2-inner product $(\cdot, \cdot)_{L^2(\Omega)}$ as $(\cdot, \cdot)_2$.

We mention several remarks on $L^2_{\mathscr{N}}(\Omega)$ and $H^2_{\mathscr{N}}(\Omega)$. As stated in [8], the map $\Delta\colon H^2_{\mathscr{N}}(\Omega) \to L^2_{\mathscr{N}}(\Omega)$ is a homeomorphism and hence there exists a constant $c_1 = c_1(N) > 0$ such that

$$c_1^{-1} \sum_{k=0}^{2} \|\nabla^k v\|_2^2 \leq \|v\|_{H^2_{\mathscr{N}}}^2 := \|\Delta v\|_2^2, \quad v \in H^2_{\mathscr{N}}(\Omega). \tag{9}$$

Moreover, $H^2_{\mathscr{N}}(\Omega)$ is a Hilbert space with the inner product

$$(v, w)_{H^2_{\mathscr{N}}} := \int_\Omega \Delta v \Delta w \, dx, \quad v, w \in H^2_{\mathscr{N}}(\Omega).$$

Let X be a real Banach space and X^* denote the dual space of X. We denote $C_{\rm w}([0,T];X)$ by the set of all X-valued weak continuous functions. Here, we say that u is an X-valued weak continuous function if for all $F \in X^*$ the function

$$[0,T] \ni s \mapsto {}_{X^*}\langle F, u(s)\rangle_X \in \mathbb{R}$$

is continuous.

By (5) we see that S_p is the best constant for the following inequality:

$$S_p \|\nabla v\|_p^2 \le \|\Delta v\|_2^2, \quad v \in H^2_{\mathcal{N}}(\Omega). \tag{10}$$

We collect several useful propositions. The Gagliard–Nirenberg inequality (cf. [1, Theorems 5.2 and 5.8]) and (9) lead the following interpolation inequality:

Proposition 2.1 *Let $2 < p < p_S$. Then there exists a positive constant c_2 depending only on N, Ω and p such that*

$$\|\nabla v\|_p \le c_2 \|\Delta v\|_2^{\theta} \|v\|_2^{1-\theta}, \quad v \in H^2_{\mathcal{N}}(\Omega),$$

where

$$\theta := \frac{N(p-2)}{4p} + \frac{1}{2} \in \left(\frac{1}{2}, 1\right). \tag{11}$$

Next we mention the property of the space of weak continuous functions (cf. [2, Lemma II.5.9]).

Proposition 2.2 *Let $T > 0$, X be a separable and reflexive real Banach space and Y be a real Banach space such that the embedding $X \subset Y$ is continuous. Then*

$$L^\infty(0,T;X) \cap C_{\rm w}([0,T];Y) = C_{\rm w}([0,T];X).$$

We introduce the Aubin–Lions–Simon compactness theorem (cf. [2, Theorem II.5.16]).

Proposition 2.3 *Let X_0, X_1, X_2 be Banach spaces with the following properties*:

- *The embedding $X_0 \subset X_1$ is compact.*
- *The embedding $X_1 \subset X_2$ is continuous.*

Let $T > 0$, $q, r \in [1,\infty]$ and set

$$E_{q,r} := \left\{ v \in L^q(0,T;X_0) \;\middle|\; \partial_t v \in L^r(0,T;X_2) \right\}.$$

Then the following hold:

- *If $q \neq \infty$, then the embedding $E_{q,r} \subset L^q(0,T;X_1)$ is compact.*
- *If $q = \infty$ and $r > 1$, then the embedding $E_{q,r} \subset C([0,T];X_1)$ is compact.*

We close Sect. 2 with a remark on the assumption on initial data.

Remark 2.1 Without loss of generality, we may assume that u_0 and the solution u to problem (P) satisfy

$$\int_\Omega u\,dx = \int_\Omega u_0\,dx = 0, \quad t \in (0, T).$$

Indeed, setting

$$\tilde{u}(x,t) := u(x,t) - \frac{1}{|\Omega|}\int_\Omega u_0\,dx \quad \text{and} \quad \tilde{u}_0(x) := u_0(x) - \frac{1}{|\Omega|}\int_\Omega u_0\,dx,$$

we see that $\tilde{u}$ is a solution to problem (P) with $\tilde{u}_0$. Formally, integrating the both side of the equation in problem (P), we have

$$\frac{d}{dt}\int_\Omega u(x,t)\,dx = \frac{d}{dt}\int_\Omega \tilde{u}(x,t)\,dx = 0$$

for $t \in (0, T)$. Hence it holds that

$$\begin{aligned} &u \text{ is a solution to problem (P)} \\ &\iff \tilde{u} \text{ is a solution to problem (P) replaced } u_0 \text{ by } \tilde{u}_0. \end{aligned}$$

3 Proof of Theorems 1.1 and 1.2

In this section we prove Theorems 1.1 and 1.2. We first construct approximate solutions we use in the both of proofs. We consider the eigenvalue problem for the Laplace equation under Neumann boundary condition:

$$\begin{cases} -\Delta\psi = \lambda\psi & \text{in} \quad \Omega, \\ \partial_\nu\psi = 0 & \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega \psi\,dx = 0. \end{cases} \tag{12}$$

Let $\{(\lambda_k, \psi_k)\}_{k=1}^\infty$ be a family of pairs with the following properties:

$$\begin{cases} \text{For each } k \in \mathbb{N},\ (\lambda, \psi) = (\lambda_k, \psi_k) \text{ satisfies (12)}, \\ 0 < \lambda_1 \le \lambda_2 \le \cdots, \\ \{\psi_k\}_{k=1}^\infty : \text{orthonormal basis of } L^2_{\mathcal{N}}(\Omega), \\ \{\lambda_k^{-1}\psi_k\}_{k=1}^\infty : \text{orthonormal basis of } H^2_{\mathcal{N}}(\Omega). \end{cases} \tag{13}$$

For each $m \in \mathbb{N}$, we consider the solution $a^m = (a_1^m, \cdots, a_m^m)$ to the following ODE system:

$$\begin{cases} \dfrac{da_k^m}{dt}(t) + \lambda_k^2 a_k^m(t) = \displaystyle\int_\Omega \Big| \sum_{i=1}^m a_i^m(t) \nabla \psi_i \Big|^{p-2} \sum_{j=1}^m a_j^m(t) \nabla \psi_j \cdot \nabla \psi_k \, dx \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad k \in \{1, \cdots, m\},\ t \in (0, T), \\ a_k^m(0) = (u_0, \psi_k)_2 \qquad\qquad\qquad\quad \text{for} \quad k \in \{1, \cdots, m\}. \end{cases} \tag{14}$$

We remark that there exists a unique solution $a^m \in C^1([0, T]; \mathbb{R}^m)$ to (14) for some $T > 0$ in the classical sense. Moreover, the solution can be uniquely extended if $|a_k^m|$ is bounded for each $k \in \{1, \cdots, m\}$. Define $T_m > 0$ as the maximal existence time of the solution to (14), that is,

$$T_m := \sup \left\{ \tau > 0 \;\middle|\; \exists a^m \in C^1([0, \tau]; \mathbb{R}^m) \colon \text{unique classical solution to (14)} \right\}.$$

Define u^m as

$$u^m(x, t) := \sum_{k=1}^m a_k^m(t) \psi_k(x), \quad (x, t) \in \Omega \times [0, T_m).$$

Since $\{\psi_k\}_{k=1}^\infty$ satisfies (12) and (13), u^m satisfies

$$\frac{d}{dt}(u^m(t), \psi_k)_2 + (\Delta u^m(t), \Delta \psi_k)_2 = \int_\Omega |\nabla u^m(t)|^{p-2} \nabla u^m(t) \cdot \nabla \psi_k \, dx \tag{15}$$

for $k \in \{1, \cdots, m\}$ and $t \in [0, T_m)$. Multiplying $\partial_t a_k^m$ by (15), summing k from 1 to m and integrating it on (t', t), we have

$$E(u^m(t)) + \int_{t'}^t \|\partial_t u^m(\tau)\|_2^2 \, d\tau = E(u^m(t')), \quad 0 \le t' \le t < T_m. \tag{16}$$

Similarly, multiplying a_k^m by (15), we obtain

$$\|u^m(t)\|_2^2 + \int_{t'}^t I(u^m(\tau)) \, d\tau = \|u^m(t')\|_2^2, \quad 0 \le t' \le t < T_m. \tag{17}$$

In particular, (16) implies that $E(u^m(t))$ is non-increasing with respect to t.

We prove Theorems 1.1 and 1.2 in Sects. 3.1 and 3.2, respectively.

3.1 Proof of Theorem 1.1

By a standard argument in [10] we can prove the uniqueness of solutions to problem (P). Thus we postpone the proof of the uniqueness to Sect. 5.1.

We divide the proof into three steps.

Step 1: *We derive a priori estimate of* $\{u^m\}_{m=1}^{\infty}$. Since $u_0 \in H^2_{\mathcal{N}}(\Omega)$, (10) and (13) imply that

$$K = K(u_0) := \sup_{m \in \mathbb{N}} E(u^m(0)) < \infty, \quad E(u^m(0)) \to E(u_0) \quad \text{as} \quad m \to \infty.$$

Moreover, it follows from (10), (12) and (13) that

$$\begin{aligned} \|\nabla u^m(0)\|_p^2 &\le S_p^{-1} \|\Delta u^m(0)\|_2^2 \\ &= S_p^{-1} \sum_{k=1}^{m} \lambda_k^2 (u_0, \psi_k)_2^2 = S_p^{-1} \sum_{k=1}^{m} (u_0, \lambda_k^{-1}\psi_k)_{H^2_{\mathcal{N}}}^2 \le S_p^{-1} \|\Delta u_0\|_2^2. \end{aligned} \tag{18}$$

Let $L > S_p^{-p/2} \|\Delta u_0\|_2^p$ and set

$$T_{L,m} := \sup \left\{ \tau > 0 \,\middle|\, \sup_{0 \le t \le \tau} \|\nabla u^m(t)\|_p^p \le L \right\} \le T_m.$$

Since $\nabla u^m \in (C([0, T_m); L^p(\Omega)))^N$, we deduce from (18) that $T_{L,m} > 0$. By (16) and the definition of E we have

$$\|\Delta u^m(t)\|_2^2 = 2E(u^m(t)) + \frac{2}{p} \|\nabla u^m(t)\|_p^p \le 2K + \frac{2}{p} L, \tag{19}$$

$$E(u^m(t)) = \frac{1}{2} \|\Delta u^m(t)\|_2^2 - \frac{1}{p} \|\nabla u^m(t)\|_p^p \ge -\frac{1}{p} L, \tag{20}$$

$$\int_0^t \|\partial_t u^m(\tau)\|_2^2 \, d\tau = E(u^m(0)) - E(u^m(t)) \le K + \frac{1}{p} L, \tag{21}$$

for $t \in [0, T_{L,m}]$, where we used (20) in (21).

Step 2: *We show a lower estimate for* $T_{L,m}$. It follows from (21) that

$$\begin{aligned} \|u^m(t) - u^m(t')\|_2^2 &= \int_\Omega \left(\int_{t'}^{t} \partial_t u^m(\tau) \, d\tau \right)^2 dx \\ &\le (t - t') \int_{t'}^{t} \|\partial_t u^m(\tau)\|_2^2 \, d\tau \le \left(K + \frac{1}{p} L \right) (t - t') \end{aligned} \tag{22}$$

for $0 \le t' \le t \le T_{L,m}$. Moreover, we observe from Proposition 2.1, (19) and (22) that

$$\begin{aligned}\|\nabla u^m(t) - \nabla u^m(t')\|_p &\le c_2 \|\Delta u^m(t) - \Delta u^m(t')\|_2^{\theta} \|u^m(t) - u^m(t')\|_2^{1-\theta} \\ &\le 2^{3\theta/2} c_2 \left(K + \frac{L}{p}\right)^{1/2} (t-t')^{(1-\theta)/2}\end{aligned} \tag{23}$$

for $0 \le t' \le t \le T_{L,m}$, where c_2 is obtained in Proposition 2.1 and θ is as in (11). Hence by (18) and (23) we see that

$$\begin{aligned}\|\nabla u^m(t)\|_p &\le \|\nabla u^m(0)\|_p + 2^{3\theta/2} c_2 \left(K + \frac{L}{p}\right)^{1/2} t^{(1-\theta)/2} \\ &\le S_p^{-1/2} \|\Delta u_0\|_2 + 2^{3\theta/2} c_2 \left(K + \frac{L}{p}\right)^{1/2} T_{L,m}^{(1-\theta)/2}\end{aligned}$$

for $t \in [0, T_{L,m}]$. Combining this estimate with the definition of $T_{L,m}$, we have

$$S_p^{-1/2} \|\Delta u_0\|_2 + 2^{3\theta/2} c_2 \left(K + \frac{L}{p}\right)^{1/2} T_{L,m}^{(1-\theta)/2} \ge L^{1/p},$$

that is,

$$T_{L,m} \ge T_* := \left[\frac{p}{2^{3\theta} c_2^2 (Kp + L)} \left(L^{1/p} - S_p^{-1/2} \|\Delta u_0\|_2\right)^2\right]^{1/(1-\theta)} \tag{24}$$

for $m \in \mathbb{N}$.

Step 3: *We construct a solution to problem* (P). We observe from (19), (21) and (24) that

$$\sup_{t \in [0, T_*]} \|\Delta u^m(t)\|_2^2 \le 2K + \frac{2}{p} L, \qquad \int_0^{T_*} \|\partial_t u^m(\tau)\|_2^2 \, d\tau \le K + \frac{1}{p} L.$$

Then, up to subsequence, there exists $u \in L^\infty(0, T_*; H^2_{\mathcal{N}}(\Omega)) \cap H^1(0, T_*; L^2_{\mathcal{N}}(\Omega))$ such that

$$u^m \overset{*}{\rightharpoonup} u \quad \text{weakly-* in} \quad L^\infty(0, T_*; H^2_{\mathcal{N}}(\Omega)) \quad \text{as} \quad m \to \infty, \tag{25}$$

$$u^m \rightharpoonup u \quad \text{weakly} \quad \text{in} \quad H^1(0, T_*; L^2_{\mathcal{N}}(\Omega)) \quad \text{as} \quad m \to \infty. \tag{26}$$

Moreover, by Proposition 2.3 we have

$$\nabla u^m \to \nabla u \quad \text{in} \quad (C([0, T_*]; L^p(\Omega)))^N \quad \text{as} \quad m \to \infty.$$

By Proposition 2.2 we see that u belongs to $C_{\mathrm{w}}([0, T_*]; H^2_{\mathscr{N}}(\Omega))$.

Recalling that u^m satisfies

$$\int_0^{T_*} \int_\Omega \left[\partial_t u^m \psi_k \zeta + \Delta u^m \Delta \psi_k \zeta - |\nabla u^m|^{p-2} \nabla u^m \cdot \nabla \psi_k \zeta \right] dt\, dx = 0$$

for $m \in \mathbb{N}$, $k \in \mathbb{N}$ and $\zeta \in C^\infty([0, T_*])$, we deduce from (25) and (26) that

$$\int_\Omega [u(T_*)\zeta(T_*) - u_0 \zeta(0)]\, \eta\, dx + \int_0^{T_*} \int_\Omega \left[-u\eta \partial_t \zeta + \Delta u \Delta \eta \zeta - |\nabla u|^{p-2} \nabla u \cdot \nabla \eta \zeta \right] dx\, dt = 0$$

for $\eta \in H^2_{\mathscr{N}}(\Omega)$ and $\zeta \in C^\infty([0, T_*])$. By a density argument (cf. [16, Proposition 23.23 (iii)]), we see that u satisfies (2) for $\varphi \in \mathscr{V}$. Hence we complete the proof of Theorem 1.1.

3.2 *Proof of Theorem 1.2*

Along the same line as in Sect. 3.1, we postpone the proof of the uniqueness of solutions to Sect. 5.2. We divide the proof of Theorem 1.2 into three steps.

Step 1: *We derive a priori estimate of* $\{u^m\}_{m=1}^\infty$. Since $u_0 \in L^2_{\mathscr{N}}(\Omega)$, relation (13) implies that $\|u^m(0)\|_2^2 \le \|u_0\|_2^2$ for $m \in \mathbb{N}$. Let $L > 0$ and set

$$\tilde{T}_{L,m} := \sup \left\{ \tau > 0 \,\middle|\, \int_0^\tau \|\nabla u^m(t)\|_p^p\, dt \le L \right\} \in (0, T_m].$$

By (17) and the definition of I we have

$$\|u^m(t)\|_2^2 + \int_0^t \|\Delta u^m(\tau)\|_2^2\, d\tau = \|u^m(0)\|_2^2 + \int_0^t \|\nabla u^m(\tau)\|_p^p\, d\tau \le \|u_0\|_2^2 + L \tag{27}$$

for $t \in [0, \tilde{T}_{L,m}]$.

Step 2: *We show a lower estimate for* $\tilde{T}_{L,m}$. It follows from Proposition 2.1 that

$$\begin{aligned}&\int_0^t \|\nabla u^m(\tau)\|_p^p\,d\tau\\&\le c_2^p\int_0^t \|\Delta u^m(\tau)\|_2^{\theta p}\|u^m(\tau)\|_2^{(1-\theta)p}\,d\tau\\&\le c_2^p\left(\sup_{\tau\in[0,t]}\|u^m(\tau)\|_2^2\right)^{(1-\theta)p/2}\int_0^t\|\Delta u^m(\tau)\|_2^{\theta p}\,d\tau\\&\le c_2^p\left(\sup_{\tau\in[0,t]}\|u^m(\tau)\|_2^2\right)^{(1-\theta)p/2}\left(\int_0^t\|\Delta u^m(\tau)\|_2^2\,d\tau\right)^{\theta p/2}t^{1-\theta p/2}\end{aligned}\tag{28}$$

for $t\in[0,\tilde{T}_{L,m}]$, where θ is as in (11). Here, we note that the condition $2<p<p_*$ is equivalent to $0<\theta p<2$. If $\tilde{T}_{L,m}<\infty$, it follows from (27) and (28) that

$$L=\int_0^{\tilde{T}_{L,m}}\|\nabla u^m(\tau)\|_p^p\,d\tau\le c_2^p(\|u_0\|_2^2+L)^{p/2}\tilde{T}_{L,m}^{1-\theta p/2},$$

that is,

$$\tilde{T}_{L,m}\ge\tilde{T}_*:=\left(\frac{L}{c_2^p(\|u_0\|_2^2+L)^{p/2}}\right)^{-(1-\theta p)/2}.$$

Step 3: *We construct a solution to problem* (P). We observe from (27) that

$$\sup_{t\in[0,T_*]}\|u^m(t)\|_2^2\le\|u_0\|_2^2+L,\qquad\int_0^{T_*}\|\Delta u^m(\tau)\|_2^2\,d\tau\le\|u_0\|_2^2+L.\tag{29}$$

These clearly imply that

$$\sup_{t\in[0,T_*]}\sum_{l=1}^m|(u^m(t),\psi_l)_2|^2\le\|u_0\|_2^2+L.\tag{30}$$

Moreover, by (15) and (29) we have

$$\begin{aligned}&|(u^m(t),\psi_k)_2-(u^m(t'),\psi_k)|\\&\quad\le\int_{t'}^t\|\Delta u^m(\tau)\|_2\|\Delta\psi_k\|_2\,d\tau+\int_{t'}^t\int_\Omega\|\nabla u^m(\tau)\|_p^{p-1}\|\nabla\psi_k\|_p\,d\tau\\&\quad\le(\|u_0\|_2^2+L)^{1/2}\|\Delta\psi_k\|_2(t-t')^{1/2}+L^{(p-1)/p}\|\nabla\psi_k\|_p(t-t')^{1/p}\end{aligned}$$

for $1 \le k \le m$ and $0 \le t' \le t \le \tilde{T}_*$. Then, extracting a subsequence, we find $c_k \in C([0, \tilde{T}_*])$ such that

$$(u^m(t), \psi_k)_2 \to c_k(t) \quad \text{uniformly in} \quad [0, \tilde{T}_*] \quad \text{as} \quad m \to \infty$$

for $k \in \mathbb{N}$. Set

$$u(t) := \sum_{k=1}^{\infty} c_k(t)\psi_k, \quad t \in [0, \tilde{T}_*].$$

Then it follows from (13), (29) and (30) that $\|u(t)\|_2^2 \le \|u_0\|_2^2 + L$ and

$$\begin{aligned}
&|(u^m(t) - u(t), \eta)_2| \\
&\le \sum_{k=1}^{M} |(\eta, \psi_k)_2| \left|(u^m(t) - u(t), \psi_k)_2\right| + \left|\left(u^m(t) - u(t), \sum_{k=M+1}^{\infty} (\eta, \psi_k)_2 \psi_k\right)_2\right| \\
&\le \sum_{k=1}^{M} |(\eta, \psi_k)_2| \left|(u^m(t) - u(t), \psi_k)_2\right| + 2^{1/2}(\|u_0\|_2^2 + L)^{1/2} \left(\sum_{k=M+1}^{\infty} (\eta, \psi_k)_2^2\right)^{1/2}
\end{aligned}$$

for $\eta \in L^2_{\mathcal{N}}(\Omega)$. Hence we see that $(u^m(t), \eta)_2$ converges to $(u(t), \eta)_2$ uniformly in $[0, \tilde{T}_*]$ for $\eta \in L^2_{\mathcal{N}}(\Omega)$ and $u \in C_{\mathrm{w}}([0, \tilde{T}_*]; L^2_{\mathcal{N}}(\Omega))$. Along the same argument as in step 3 in Sect. 3.1, we see that

$$\begin{aligned}
u^m &\rightharpoonup u &&\text{weakly in} \quad L^2(0, T_*; H^2_{\mathcal{N}}(\Omega)) \quad \text{as} \quad m \to \infty, \\
\frac{\partial u^m}{\partial x_j} &\rightharpoonup \frac{\partial u}{\partial x_j} &&\text{weakly in} \quad L^p(0, T_*; L^p(\Omega)) \quad \text{as} \quad m \to \infty \quad \text{for} \quad 1 \le j \le N
\end{aligned}$$

and u satisfies (2) for $\varphi \in \mathscr{V}$. Moreover, we observe from the same argument as in [10, Chapter III, Section 4] that $u \in C([0, \tilde{T}_*]; L^2_{\mathcal{N}}(\Omega))$. Hence we complete the proof of Theorem 1.2.

4 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. We first show two lemmas.

Lemma 4.1 *Let $v \in H^2_{\mathcal{N}}(\Omega) \setminus \{0\}$ and assume that $2 < p < p_S$. Then*

$$\lambda_* := \left(\frac{\|\Delta v\|_2^2}{\|\nabla v\|_p^p}\right)^{\frac{1}{p-2}} > 0$$

is the unique maximum point of the function

$$(0,\infty) \ni \lambda \mapsto E(\lambda v) \in \mathbb{R}$$

and λ_* *satisfies*

$$E(\lambda_* v) = \left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{\|\Delta v\|_2^2}{\|\nabla v\|_p^2}\right)^{\frac{p}{p-2}}, \qquad I(\lambda v)\begin{cases} > 0 & \textit{if} \quad \lambda \in (0,\lambda_*), \\ = 0 & \textit{if} \quad \lambda = \lambda_*, \\ < 0 & \textit{if} \quad \lambda \in (\lambda_*,\infty). \end{cases}$$

Moreover, it holds that

$$d = \inf_{v \in H^2_{\mathcal{N}}(\Omega), v \neq 0} E(\lambda_* v),$$

where d *is defined in* (5)*.*

Lemma 4.1 follows from a direct calculation. Thus we omit the proof.

Lemma 4.2 *Let* $u_0 \in W$ *and assume that* $2 < p < p_S$*. Then there exists* $m_* \in \mathbb{N}$ *such that* $T_m = \infty$ *and* $u^m(t) \in W$ *for* $t \in (0,\infty)$ *if* $m \geq m_*$*, where* $\{u^m(t)\}_{m=1}^{\infty}$ *denotes the family of functions constructed in the proof of Theorem* 1.1*.*

Proof Since $u_0 \in W$, we find $m_* \in \mathbb{N}$ such that

$$u^m(0) \in W \quad \text{if} \quad m \geq m_*.$$

This together with (16) implies that $E(u^m(t)) < d$ for $m \geq m_*$ and $t \in (0, T_m)$. From now on, we let $m \geq m_*$. Assume that $I(u^m(t_*)) = 0$ holds for some $t_* \in (0, T_m)$. Then by Lemma 4.1 we have

$$E(u^m(t_*)) = \left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{\|\Delta u^m(t_*)\|_2^2}{\|\nabla u^m(t_*)\|_p^2}\right)^{\frac{p}{p-2}} \geq d$$

and it contradicts to $E(u^m(t_*)) < d$. Hence $u^m(t) \in W$ for $t \in (0, T_m)$. Moreover, this together with (17) implies that

$$\sum_{k=1}^{m} (a_k^m(t))^2 = \|u^m(t)\|_2^2 = \|u^m(0)\|_2^2 - \int_0^t I(u^m(\tau))\,d\tau \leq \|u_0\|_2^2$$

for $t \in (0, T_m)$. Thus we have $T_m = \infty$.

We are in a position to prove Theorem 1.3.

Proof (Theorem** 1.3**) Let $\{u^m(t)\}_{m=1}^{\infty}$ be the family of functions constructed in the proof of Theorem 1.1 and $\{\mu_k\}_{k=1}^{\infty}$ be as in (3). By the variational characterization of eigenvalues, we have

$$\mu_k^2 \|v - P_{k-1}v\|_2^2 \le \|\Delta(v - P_{k-1}v)\|_2^2, \quad k \in \mathbb{N}, \ v \in H^2_{\mathcal{N}}(\Omega), \tag{31}$$

where P_k is as in Sect. 1.

Let $m_* \in \mathbb{N}$ be the number obtained in Lemma 4.2. From now on, we let $m \ge m_*$. By Lemma 4.2 we have

$$E(u^m(t)) < d, \quad t > 0. \tag{32}$$

Moreover, since

$$E(u^m(t)) = \left(\frac{1}{2} - \frac{1}{p}\right) \|\Delta u^m(t)\|_2^2 + \frac{1}{p} I(u^m(t)), \quad t > 0, \tag{33}$$

it follows from Lemma 4.2 that

$$\|\Delta u^m(t)\|_2^2 < \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} E(u^m(t)), \quad t > 0. \tag{34}$$

Hence, combining (32) with (34), we have

$$\|\Delta u^m(t)\|_2^2 < S_p^{p/(p-2)}, \quad t > 0, \tag{35}$$

where we use (5) in (35).

In the following, we consider the decaying estimate for u^m. We divide the proof into five steps.

Step 1: *We show that* $\|\Delta u^m(t)\|_2 \to 0$ *as* $t \to \infty$. The argument in Step 1 is based on [6] (see also [9]). Combining with (10), (16) and (34), we see that

$$\begin{aligned}
\|\nabla u^m(t)\|_p^p &\le S_p^{-\frac{p}{2}} \|\Delta u^m(t)\|_2^p \\
&< S_p^{-\frac{p}{2}} \left(\frac{1}{2} - \frac{1}{p}\right)^{-\frac{p-2}{2}} E(u^m(t))^{\frac{p-2}{2}} \|\Delta u^m(t)\|_2^2 \\
&\le S_p^{-\frac{p}{2}} \left(\frac{1}{2} - \frac{1}{p}\right)^{-\frac{p-2}{2}} E(u^m(0))^{\frac{p-2}{2}} \|\Delta u^m(t)\|_2^2 \\
&= \left(\frac{E(u^m(0))}{d}\right)^{\frac{p-2}{2}} \|\Delta u^m(t)\|_2^2
\end{aligned}$$

for $t > 0$. This implies that

$$I(u^m(t)) > \gamma \|\Delta u^m(t)\|_2^2, \quad t > 0, \tag{36}$$

where

$$\gamma := 1 - \sup_{m \ge m_*} \left(\frac{E(u^m(0))}{d} \right)^{\frac{p-2}{2}} \in (0,1).$$

Hence by (33) and (36) we obtain

$$E(u^m(t)) < \left[\frac{1}{\gamma} \left(\frac{1}{2} - \frac{1}{p} \right) + \frac{1}{p} \right] I(u^m(t)), \quad t > 0. \tag{37}$$

On the other hand, we deduce from (17), (31) and (34) that

$$\int_t^\infty I(u^m(\tau))\, d\tau \le \|u(t)\|_2^2 \le \mu_1^{-2} \|\Delta u^m(t)\|_2^2 < \mu_1^{-2} \left(\frac{1}{2} - \frac{1}{p} \right)^{-1} E(u^m(t)) \tag{38}$$

for $t > 0$. Thus it follows from (37) and (38) that

$$\int_t^\infty E(u^m(\tau))\, d\tau < A E(u^m(t)), \quad \text{i.e.,} \quad \frac{d}{dt} \left[e^{\frac{t}{A}} \int_t^\infty E(u^m(\tau))\, d\tau \right] < 0, \tag{39}$$

for $t > 0$, where

$$A := \frac{1}{\mu_1^2} \left(\frac{1}{\gamma} + \frac{2}{p-2} \right).$$

Therefore, combining (32) with (39), we have

$$\int_t^\infty E(u^m(\tau))\, d\tau < e^{1-\frac{t}{A}} \int_A^\infty E(u^m(\tau))\, d\tau < A e^{1-\frac{t}{A}} E(u^m(A)) < A d e^{1-\frac{t}{A}} \tag{40}$$

for $t > A$. Since it follows from $u^m(t) \in W$ and (33) that $E(u^m(t)) > 0$, by (16) and (34) we see that

$$\begin{aligned} \int_t^\infty E(u^m(\tau))\, d\tau &\ge \int_t^{A+t} E(u^m(\tau))\, d\tau \\ &\ge A E(u^m(A+t)) > A \left(\frac{1}{2} - \frac{1}{p} \right) \|\Delta u^m(A+t)\|_2^2 \end{aligned} \tag{41}$$

for $t > 0$. This together with (40) implies that

$$\|\Delta u^m(t)\|_2^2 \le e^2 S_p^{\frac{p}{p-2}} e^{-\frac{t}{A}}, \quad t > 2A.$$

Therefore, recalling (35), we obtain

$$\|\Delta u^m(t)\|_2^2 \le c_A e^{-\frac{t}{A}}, \quad t > 0, \tag{42}$$

where c_A is a constant which is independent of m and t.

Step 2: *We derive a modified decay rate.* Fix $\varepsilon \in (0, 1)$ arbitrarily. We first prove the following: there exists $c_\varepsilon > 0$ such that

$$\|u^m(t)\|_2 \le c_\varepsilon e^{-(1-\varepsilon)\mu_1^2 t}, \quad t > 0. \tag{43}$$

Multiplying a_k^m by (15) and summing k from 1 to m, we see that

$$\frac{1}{2}\frac{d}{dt}\|u^m(t)\|_2^2 + \|\Delta u^m(t)\|_2^2 = \|\nabla u^m(t)\|_p^p, \quad t > 0. \tag{44}$$

This together with (10) implies that

$$\frac{1}{2}\frac{d}{dt}\|u^m(t)\|_2^2 + \|\Delta u^m(t)\|_2^2 \le S_p^{-p/2}\|\Delta u^m(t)\|_2^p, \quad t > 0. \tag{45}$$

Thanks to (42), we find $T_\varepsilon > 0$ such that

$$S_p^{-p/2}\|\Delta u^m(t)\|_2^{p-2} < \varepsilon, \quad t > T_\varepsilon. \tag{46}$$

Thus we observe from (45) and (46) that

$$\frac{d}{dt}\|u^m(t)\|_2^2 + 2(1-\varepsilon)\|\Delta u^m(t)\|_2^2 \le 0, \quad t > T_\varepsilon.$$

Hence by (31), we have

$$\frac{d}{dt}\left(e^{2(1-\varepsilon)\mu_1^2 t}\|u^m(t)\|_2^2\right) \le 0, \quad \text{i.e.,} \quad e^{2(1-\varepsilon)\mu_1^2 t}\|u^m(t)\|_2^2 \le e^{2(1-\varepsilon)\mu_1^2 T_\varepsilon}\|u^m(T_\varepsilon)\|_2^2$$

for $t > T_\varepsilon$. This clearly implies (43), because it follows from (9) and (35) that $\|u^m(t)\|_2^2 < c_1 S_p^{p/(p-2)}$ for $t > 0$.

Moreover, it follows from (17), (37) and (43) that for $\varepsilon \in (0, 1)$

$$\int_t^\infty E(u^m(\tau))\,d\tau < c_\varepsilon \left[\frac{1}{\gamma}\left(\frac{1}{2} - \frac{1}{p}\right) + \frac{1}{p}\right] e^{-2(1-\varepsilon)\mu_1^2 t}, \quad t > 0.$$

Along the same argument as in (41), we have

$$\int_t^\infty E(u^m(\tau))\,d\tau \geq \left(\frac{1}{2}-\frac{1}{p}\right)\|\Delta u^m(t+1)\|_2^2, \quad t>0.$$

Thus we find that $\tilde{c}_\varepsilon > 0$ such that

$$\|\Delta u^m(t)\|_2 \leq \tilde{c}_\varepsilon e^{-(1-\varepsilon)\mu_1^2 t}, \quad t>0. \tag{47}$$

Step 3: *We derive the precise decay rate*: *there exist* $c_*, \tilde{c}_* > 0$ *such that*

$$\|u^m(t)\|_2 \leq c_* e^{-\mu_1^2 t}, \quad t>0, \tag{48}$$

$$\|\Delta u^m(t)\|_2 \leq \tilde{c}_* e^{-\mu_1^2 t}, \quad t>0. \tag{49}$$

Fix $\varepsilon \in (0,1)$ arbitrarily. By (31), (44), (47) and Proposition 2.1 we have

$$\frac{d}{dt}\|u^m(t)\|_2^2 + 2\mu_1^2\|u^m(t)\|_2^2 \leq 2c_2^p\tilde{c}_\varepsilon^{\theta p}e^{-\theta p(1-\varepsilon)\mu_1^2 t}\|u^m(t)\|_2^{(1-\theta)p}, \quad t>0,$$

that is,

$$\frac{d}{dt}\left(e^{2\mu_1^2 t}\|u^m(t)\|_2^2\right) \leq 2c_2^p\tilde{c}_\varepsilon^{\theta p}e^{(\theta p\varepsilon-(p-2))\mu_1^2 t}\left(e^{2\mu_1^2 t}\|u^m(t)\|_2^2\right)^{(1-\theta)p/2}, \quad t>0,$$

where θ is as in (11). Since $p>2$ implies that

$$(1-\theta)p = \frac{2N-(N-2)p}{4} \in (0,1),$$

we have

$$\left(e^{2\mu_1^2 t}\|u^m(t)\|_2^2\right)^{1-(1-\theta)p/2} \leq \|u_0\|_2^{2-(1-\theta)p} + \frac{4c_2^p\tilde{c}_\varepsilon^{\theta p}}{2-(1-\theta)p}\int_0^t e^{(\theta p\varepsilon-(p-2))\mu_1^2 s}\,ds$$

for $t>0$. Taking $\varepsilon>0$ sufficiently small, we see that (48) holds for some positive constant $c_*>0$. Similarly to (47), we obtain (49).

Step 4: *We show the asymptotic behavior of* u^m. Fix $\varepsilon \in (0,1)$ and $1 \leq k \leq k_*$ arbitrarily. We first prove the following: there exists $c_{\varepsilon,k} > 0$ such that

$$\|u^m(t) - P_{k-1}u^m(t)\|_2 \leq c_{\varepsilon,k}e^{-(1-\varepsilon)\mu_k^2 t}, \quad t>0, \tag{50}$$

where μ_k, P_k are as in Sect. 1. Similarly to (44), we see that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u^m(t) - P_{k-1}u^m(t)\|_2^2 + \|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2^2\\&\quad = \int_\Omega |\nabla u^m(t)|^{p-2}\nabla u^m(t)\cdot\nabla(u^m(t) - P_{k-1}u^m(t))\,dx, \quad t > 0.\end{aligned} \tag{51}$$

Combining (51) with (10) and (49) we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u^m(t) - P_{k-1}u^m(t)\|_2^2 + \|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2^2\\&\quad\le \|\nabla u^m(t)\|_p^{p-1}\|\nabla(u^m(t) - P_{k-1}u^m(t))\|_p\\&\quad\le S_p^{-p/2}\tilde{c}_*^{p-1}e^{-(p-1)\mu_1^2 t}\|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2\\&\quad\le \varepsilon\|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2^2 + \frac{S_p^{-p}\tilde{c}_*^{2(p-1)}}{4\varepsilon}e^{-2(p-1)\mu_1^2 t}\end{aligned}$$

for $t > 0$. Hence by (31) we obtain

$$\frac{d}{dt}\left(e^{2(1-\varepsilon)\mu_k^2 t}\|u^m(t) - P_{k-1}u^m(t)\|_2^2\right) \le \frac{S_p^{-p}\tilde{c}_*^{2(p-1)}}{2\varepsilon}e^{-2((p-1)\mu_1^2-\mu_k^2)t}$$

and we can find the constant $c_{\varepsilon,k} > 0$ satisfying (50). Along the same line as the above argument, we find $c_{\varepsilon,k_*+1} > 0$ such that

$$\|u^m(t) - P_{k_*}u^m(t)\|_2 \le c_{\varepsilon,k_*+1}e^{-(1-\varepsilon)(p-1)\mu_1^2 t}, \quad t > 0. \tag{52}$$

On the other hand, we observe from (10), (49), (50) and (51) that

$$\begin{aligned}&\int_t^\infty \|\Delta(u^m(\tau) - P_{k-1}u^m(\tau))\|_2^2\,dt\\&\le \frac{1}{2}\|u^m(t) - P_{k-1}u^m(t)\|_2^2 + \int_t^\infty \|\nabla u^m(\tau)\|_p^{p-1}\|\nabla(u^m(t) - P_{k-1}u^m(t))\|_p\,d\tau\\&\le \frac{1}{2}c_{\varepsilon,k}e^{-2(1-\varepsilon)\mu_k^2 t} + \tilde{c}_*^{p-1}S_p^{-p/2}\int_t^\infty e^{-(p-1)\mu_1^2\tau}\|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2\,d\tau\\&\le \frac{1}{2}c_{\varepsilon,k}e^{-2(1-\varepsilon)\mu_k^2 t} + \frac{\tilde{c}_*^{2(p-1)}S_p^{-p}}{2}\int_t^\infty e^{-2(p-1)\mu_1^2\tau}\,d\tau\\&\quad + \frac{1}{2}\int_t^\infty \|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2^2\,d\tau\end{aligned}$$

and hence there exists $\tilde{c}_{\varepsilon,k} > 0$ such that

$$\int_t^\infty \|\Delta(u^m(\tau) - P_{k-1}u^m(\tau))\|_2^2\, dt \le \tilde{c}_{\varepsilon,k} e^{-2(1-\varepsilon)\mu_k^2 t}, \quad t > 0. \tag{53}$$

We improve the decay rate in (50). By (10), (31), (48) and (51) we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u^m(t) - P_{k-1}u^m(t)\|_2^2 + \mu_k^2\|u^m(t) - P_{k-1}u^m(t)\|_2^2 \\
&\quad \le S_p^{-p/2}\tilde{c}_*^{p-1} e^{-(p-1)\mu_1^2 t}\|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{d}{dt}\left(e^{2\mu_k^2 t}\|u^m(t) - P_{k-1}u^m(t)\|_2^2\right) \\
&\le 2S_p^{-p/2}\tilde{c}_*^{p-1} e^{(2\mu_k^2-(p-1)\mu_1^2)t}\|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2 \\
&\le S_p^{-p}\tilde{c}_*^{2(p-1)} e^{2((1+\delta)\mu_k^2-(p-1)\mu_1^2)t} + e^{2(1-\delta)\mu_k^2 t}\|\Delta(u^m(t) - P_{k-1}u^m(t))\|_2^2
\end{aligned}$$

for $\delta \in (0, 1)$. Integrating this inequality, we see that

$$\begin{aligned}
&e^{2\mu_k^2 t}\|u^m(t) - P_{k-1}u^m(t)\|_2^2 \\
&\le \|u_0\|_2^2 + S_p^{-p}\tilde{c}_*^{2(p-1)}\int_0^t e^{2((1+\delta)\mu_k^2-(p-1)\mu_1^2)\tau}\, d\tau \\
&+ \int_0^t e^{2(1-\delta)\mu_k^2\tau}\|\Delta(u^m(\tau) - P_{k-1}u^m(\tau))\|_2^2\, d\tau \\
&\le \|u_0\|_2^2 + S_p^{-p}\tilde{c}_*^{2(p-1)}\int_0^t e^{2((1+\delta)\mu_k^2-(p-1)\mu_1^2)\tau}\, d\tau \\
&+ \int_0^\infty \|\Delta(u^m(\tau) - P_{k-1}u^m(\tau))\|_2^2\, d\tau \\
&+ 2(1-\delta)\mu_k^2\int_0^t e^{2(1-\delta)\mu_k^2\tau}\int_\tau^\infty \|\Delta(u^m(\eta) - P_{k-1}u^m(\eta))\|_2^2\, d\eta\, d\tau.
\end{aligned}$$

Taking $\varepsilon > 0$, $\delta > 0$ sufficiently small, by (53) we find $\tilde{c}_k > 0$ such that

$$\|u^m(t) - P_{k-1}u^m(t)\|_2 \le \tilde{c}_k e^{-\mu_k^2 t}, \quad t > 0. \tag{54}$$

Step 5: *Letting $m \to \infty$, we obtain the conclusion.* It follows from the proof of Theorem 1.1 and Lemma 4.2 that u^m satisfies

$$u^m \overset{*}{\rightharpoonup} u \quad \text{weakly-* in} \quad L^\infty(0,T;H^2_{\mathcal{N}}(\Omega)) \quad \text{as} \quad m \to \infty, \tag{55}$$

$$u^m \to u \quad \text{in} \quad C([0,T];L^2_{\mathcal{N}}(\Omega)) \quad \text{as} \quad m \to \infty, \tag{56}$$

for $T \in (0,\infty)$. Combining (56) with (52) and (54), we obtain (7) and (8).

We prove (6). Fix $t > 0$ arbitrarily. Then (49) and (55) imply that

$$\operatorname*{ess\,sup}_{\tau \in (t,t+1)} \|\Delta u(\tau)\|_2 \le \liminf_{m\to\infty} \operatorname*{ess\,sup}_{\tau\in(t,t+1)} \|\Delta u^m(\tau)\|_2 \le \tilde{c}_* e^{-\mu_1^2 t}$$

and hence there exists $N = N_t \subset (t,t+1)$ of measure zero such that

$$\|\Delta u(\tau)\|_2 \le \tilde{c}_* e^{-\mu_1^2 t}, \qquad \tau \in (t,t+1) \setminus N_t.$$

Let $\{\tau_k\}_{k=1}^\infty \subset (t,\infty) \setminus N_t$ such that $\tau_k \to t$ as $k \to \infty$. Since $u \in C_{\mathrm{w}}([0,\infty); W^{2,2}_{\mathcal{N}}(\Omega))$, we have

$$\|\Delta u(t)\|_2 \le \liminf_{k\to\infty} \|\Delta u(\tau_k)\|_2 \le \tilde{c}_* e^{-\mu_1^2 t}.$$

Since $t > 0$ is arbitrary, we obtain (6). Therefore, Theorem 1.3 follows.

5 Appendix

In this section, we prove the uniqueness of solutions to problem (P). Let u_1 and u_2 be solutions to problem (P). In what follows, by the letter C we denote generic positive constants (which may depend on u_1 and u_2) and they may have different values also within the same line.

Let $t_0, t_1 \in (0,T)$ with $t_0 < t_1$, $h \in (0, \min\{t_0, T - t_1, t_1 - t_0\}/2)$. Define

$$W_h(x,t) := \frac{1}{h}\int_t^{t+h} \chi_{[t_0,t_1]}(\tau) w(x,\tau)\,d\tau, \qquad \tilde{W}_h(x,t) := \frac{1}{h}\int_{t-h}^{t} W_h(x,\tau)\,d\tau,$$

for $(x,t) \in \Omega \times [0,T]$, where $w := u_1 - u_2$ and

$$\chi_{[t_0,t_1]}(t) := \begin{cases} 1 & \text{if} \quad t \in [t_0,t_1], \\ 0 & \text{if} \quad t \in \mathbb{R} \setminus [t_0,t_1]. \end{cases}$$

We remark that $\tilde{W}_h$ belongs to $\mathscr{V}$ and satisfies the following properties (cf. [3]):

$$\tilde{W}_h \to \chi_{[t_0,t_1]} w \quad \text{as} \quad h \searrow 0 \quad \text{in} \quad L^2(0,T; H^2_{\mathscr{N}}(\Omega)), \tag{57}$$

$$\nabla \tilde{W}_h \to \chi_{[t_0,t_1]} \nabla w \quad \text{as} \quad h \searrow 0 \quad \text{in} \quad (L^p(0,T; L^p(\Omega)))^N. \tag{58}$$

Taking $\varphi = \tilde{W}_h$ in (2), we see that

$$\begin{aligned}
&- \int_0^T \int_\Omega w \partial_t \tilde{W}_h \, dx \, dt + \int_0^T \int_\Omega \Delta w \Delta \tilde{W}_h \, dx \, dt \\
&= \int_0^T \int_\Omega (|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2) \cdot \nabla \tilde{W}_h \, dx \, dt \\
&\le C \int_0^T (\|\nabla u_1(t)\|_p^{p-2} + \|\nabla u_2(t)\|_p^{p-2}) \|\nabla w(t)\|_p \|\nabla \tilde{W}_h(t)\|_p \, dt.
\end{aligned} \tag{59}$$

On the other hand, setting

$$w_h(x,t) := \frac{1}{h} \int_t^{t+h} w(x,\tau) \, d\tau,$$

we see that

$$\begin{aligned}
&- \int_0^T \int_\Omega w \partial_t \tilde{W}_h \, dx \, dt \\
&= \int_0^T \int_\Omega \partial_t w_h(x,t) W_h(x,t) \, dx \, dt \\
&= \int_{t_0}^{t_1-h} \int_\Omega \partial_t w_h(x,t) w_h(x,t) \, dx \, dt + \left(\int_{t_0-h}^{t_0} + \int_{t_1-h}^{t_1} \right) \int_\Omega \partial_t w_h(x,t) W_h(x,t) \, dx \, dt \\
&= \frac{1}{2} (\|w_h(t_1-h)\|_2^2 - \|w_h(t_0)\|_2^2) + \left(\int_{t_0-h}^{t_0} + \int_{t_1-h}^{t_1} \right) \int_\Omega \partial_t w_h(x,t) W_h(x,t) \, dx \, dt \\
&\to \frac{1}{2} (\|w(t_1)\|_2^2 - \|w(t_0)\|_2^2) \quad \text{as} \quad h \searrow 0.
\end{aligned}$$

This together with (57), (58) and (59) implies that

$$\begin{aligned}
&\frac{1}{2} (\|w(t_1)\|_2^2 - \|w(t_0)\|_2^2) + \int_{t_0}^{t_1} \|\Delta w(t)\|_2^2 \, dt \\
&\quad \le C \int_{t_0}^{t_1} (\|\nabla u_1(t)\|_p^{p-2} + \|\nabla u_2(t)\|_p^{p-2}) \|\nabla w(t)\|_p^2 \, dt.
\end{aligned} \tag{60}$$

Since $t_0, t_1 \in (0,T)$ are arbitrary, (60) holds for $t_0, t_1 \in [0,T]$.

We prove the uniqueness of solutions to problem (P) in Theorems 1.1 and 1.2 in Sect. 5.1 and Sect. 5.2, respectively.

5.1 Uniqueness of Solutions in Theorem 1.1

Since $\nabla u_1, \nabla u_2 \in (L^\infty(0,T;L^p(\Omega)))^N$ and $w(0)=0$, we observe from (60) that

$$\|w(t)\|_2^2 + 2\int_0^t \|\Delta w(\tau)\|_2^2\,d\tau \le C\int_0^t \|\nabla w(\tau)\|_p^2\,d\tau$$

for $t \in [0,T]$. By Proposition 2.1 and the Young inequality we have

$$\|w(t)\|_2^2 \le C\int_0^t \|w(\tau)\|_2^2\,d\tau, \quad \text{i.e.,} \quad \frac{d}{dt}\left(e^{-Ct}\int_0^t \|w(\tau)\|_2^2\,d\tau\right) \le 0.$$

This implies that $w \equiv 0$ and we complete the proof of the uniqueness in Theorem 1.1.

5.2 Uniqueness of Solutions in Theorem 1.2

Fix $t_0, t_1 \in [0,T]$ arbitrarily. By Proposition 2.1 and $u_j \in C([0,T];L^2_{\mathcal{N}}(\Omega))$ we have

$$\begin{aligned}&\int_{t_0}^{t_1} \|\nabla u_j(t)\|_p^{p-2}\|\nabla w(t)\|_p^2\,dt \\ &\quad\le C\left(\sup_{t\in[t_0,t_1]}\|w(t)\|_2^2\right)^{1-\theta}\int_{t_0}^{t_1}\|\Delta u_j(t)\|_2^{(p-2)\theta}\|\Delta w(t)\|_2^{2\theta}\,dt\end{aligned} \tag{61}$$

for $j=1,2$. Moreover, since $2<p<p_*$ is equivalent to $0<\theta p<2$, it holds that

$$\begin{aligned}&\int_{t_0}^{t_1}\|\Delta u_j(t)\|_2^{(p-2)\theta}\|\Delta w(t)\|_2^{2\theta}\,dt \\ &\le C(t_1-t_0)^{1-\theta p/2}\left(\int_{t_0}^{t_1}\|\Delta u_j(t)\|_2^2\,dt\right)^{(p-2)\theta/2}\left(\int_{t_0}^{t_1}\|\Delta w(t)\|_2^2\,dt\right)^{\theta} \\ &\le C(t_1-t_0)^{1-\theta p/2}\left(\int_{t_0}^{t_1}\|\Delta w(t)\|_2^2\,dt\right)^{\theta}\end{aligned}$$

for $j = 1, 2$. This together with (60) and (61) we have

$$\begin{aligned}
&\frac{1}{2}(\|w(t_1)\|_2^2 - \|w(t_0)\|_2^2) + \int_{t_0}^{t_1} \|\Delta w\|_2^2\, dt \\
&\quad \le C(t_1 - t_0)^{1-\theta p/2} \left(\int_{t_0}^{t_1} \|\Delta w(t)\|_2^2\, dt \right)^{\theta} \left(\sup_{t\in[t_0,t_1]} \|w(t)\|_2^2 \right)^{1-\theta} \\
&\quad \le \frac{1}{2} \int_{t_0}^{t_1} \|\Delta w(t)\|_2^2\, dt + C(t_1 - t_0)^{(2-\theta p)/2(1-\theta)} \sup_{t\in[t_0,t_1]} \|w(t)\|_2^2,
\end{aligned}$$

that is,

$$\|w(t_1)\|_2^2 \le \|w(t_0)\|_2^2 + c_0(t_1 - t_0)^{(2-\theta p)/2(1-\theta)} \sup_{t\in[t_0,t_1]} \|w(t)\|_2^2, \tag{62}$$

where $c_0 > 0$ is a constant depending on N, p, Ω, u_1 and u_2. Let $\delta > 0$ be such that

$$c_0 \delta^{(2-\theta p)/2(1-\theta)} < \frac{1}{2}.$$

Since $w(0) = 0$, we observe from (62) and $0 < \theta p < 2$ that $w(t) = 0$ for $t \in [0, \delta]$. Iterating this argument, we obtain $w \equiv 0$ in $[0, T]$ and we complete the proof of the uniqueness in Theorem 1.2.

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References

1. Adams, R.A., Fournier, J.F.F.: Sobolev Spaces. Pure and Applied Mathematics, vol. 140, 2nd edn. Academic Press, Amsterdam (2003)
2. Boyer, F., Fabrie, P.: Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models. Applied Mathematical Sciences, vol. 183. Springer, New York (2013)
3. Cazenave, T., Haraux, A.: An Introduction to Semilinear Evolution Equations. Oxford Lecture Series in Mathematics and Its Applications, vol. 13. The Clarendon Press, Oxford University Press, New York (1998)
4. Duan, N., Zhao, X.P.: Global attractor for a fourth-order parabolic equation modeling epitaxial thin film growth. Bull. Polish Acad. Sci. Math. **60**(3), 259–268 (2012)
5. Han, Y.: A class of fourth-order parabolic equation with arbitrary initial energy. Nonlinear Anal. Real World Appl. **43**, 451–466 (2018)

6. Ikehata, R., Suzuki, T.: Semilinear parabolic equations involving critical Sobolev exponent: local and asymptotic behavior of solutions. Differ. Integr. Equ. **13**(7–9), 869–901 (2000)
7. Ishige, K., Miyake, N., Okabe, S.: Blowup for a fourth-order parabolic equation with gradient nonlinearity. SIAM J. Math. Anal. **52**(1), 927–953 (2020)
8. King, B.B., Stein, O., Winkler, M.: A fourth-order parabolic equation modeling epitaxial thin film growth. J. Math. Anal. Appl. **286**(2), 459–490 (2003)
9. Komornik, V.: Exact Controllability and Stabilization, the Multiplier Method, RAM: Research in Applied Mathematics. Masson, Paris; John Wiley & Sons, Ltd., Chichester (1994)
10. Ladyženskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N.: Linear and Quasi-linear Equations of Parabolic Type. Translations of Mathematical Monographs, vol. 23. American Mathematical Society, Providence (1968)
11. Ortiz, M., Repetto, E.A., Si, H.: A continuum model of kinetic roughening and coarsening in thin films. J. Mech. Phys. Solids **47**(4), 697–730 (1999)
12. Payne, L.E., Sattinger, D.H.: Saddle points and instability of nonlinear hyperbolic equations. Israel J. Math. **22**(3–4), 273–303 (1975)
13. Sandjo, A.N., Moutari, S., Gningue, Y.: Solutions of fourth-order parabolic equation modeling thin film growth. J. Differ. Equ. **259**(12), 7260–7283 (2015)
14. Sattinger, D.H.: On global solution of nonlinear hyperbolic equations. Arch. Rational Mech. Anal. **30**, 148–172 (1968)
15. Zangwill, A.: Some causes and a consequence of epitaxial roughening. J. Cryst. Growth **163**(1), 8–21 (1996)
16. Zeidler, E.: Nonlinear Functional Analysis and Its Applications II/A, Linear Monotone Operators. Springer, New York (1990). Translated from the German by the author and Leo F. Boron
17. Zhao, X., Liu, C.: The existence of global attractor for a fourth-order parabolic equation. Appl. Anal. **92**(1), 44–59 (2013)
18. Zheng, S.: Long-time behavior of the global solutions to an epitaxial thin film model. Adv. Math. Sci. Appl. **21**(2), 519–534 (2011)

A Note on Radial Solutions to the Critical Lane-Emden Equation with a Variable Coefficient

Daisuke Naimen and Futoshi Takahashi

Abstract In this note, we consider the following problem

$$\begin{cases} -\Delta u = (1 + g(x))u^{\frac{N+2}{N-2}}, \ u > 0 \text{ in } B, \\ u = 0 \text{ on } \partial B, \end{cases}$$

where $N \geq 3$ and $B \subset \mathbb{R}^N$ is the unit ball centered at the origin and $g(x)$ is a radial Hölder continuous function such that $g(0) = 0$. We prove the existence and nonexistence of radial solutions by the variational method with the concentration compactness analysis and the Pohozaev identity.

Keywords Elliptic equation · Variational method · Critical problem

1 Introduction

We study the following problem

$$\begin{cases} -\Delta u = (1 + g(x))u^{\frac{N+2}{N-2}}, \ u > 0 \text{ in } B \\ u = 0 \text{ on } \partial B, \end{cases} \tag{1.1}$$

where $B \subset \mathbb{R}^N$ is the unit ball centered at the origin with $N \geq 3$, g is a locally Hölder continuous function in $\overline{B}$ and radial, i.e., $g(x) = g(|x|)$. We note that a typical case is given by $g(x) = |x|^\beta$ with $\beta \geq 0$. We will show some existence and nonexistence results for (1.1).

D. Naimen (✉)
Muroran Institute of Technology, Muroran, Japan
e-mail: naimen@mmm.muroran-it.ac.jp

F. Takahashi
Osaka City University, Osaka, Japan

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First let us consider the next basic problem which is extensively investigated by many authors;

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}}, \ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 3$. Since the nonlinearity $u^{\frac{N+2}{N-2}}$ has the critical growth, as is well-known, due to the lack of the compactness of the associated Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$, the existence/nonexistence of solutions of (1.2) becomes a very delicate and interesting question. In fact, in contrast to the subcritical case, we can prove that (1.2) has no smooth solution if Ω is a star-shaped domain by the Pohozaev identity [15] (see also [6]). Hence in order to ensure the existence of solutions of (1.2), we need some "perturbation" to (1.2). A celebrated work in this direction is given by [6]. They add a lower order term λu^q $(1 \leq q < (N+2)/(N-2))$ to the critical nonlinearity $u^{\frac{N+2}{N-2}}$ (i.e., replace $u^{\frac{N+2}{N-2}}$ by $u^{\frac{N+2}{N-2}} + \lambda u^q$) and successfully show the existence of solutions of (1.2). After that, [4, 5, 8] prove that the topological perturbation to the domain can also induce solutions to (1.2). See also [10, 14] for the effect of the geometric perturbation to the domain. Furthermore, another perturbation is found by Ni [13]. He considers a variable coefficient $|x|^\alpha$ with $\alpha > 0$ on $u^{\frac{N+2}{N-2}}$. More precisely he investigates

$$\begin{cases} -\Delta u = |x|^\alpha u^p, \ u > 0 \text{ in } B, \\ u = 0 \text{ on } \partial B, \end{cases} \tag{1.3}$$

where $\alpha > 0$ and $p \in (1, \frac{N+2+2\alpha}{N-2})$. The crucial role of the variable coefficient $|x|^\alpha$ appears in the following compactness lemma for radially symmetric functions in $H_0^1(B)$. Here we define $H_r(B)$ is a subspace of $H_0^1(B)$ which consists of all radial functions.

Lemma 1.1 ([13]) *The map $u \mapsto |x|^m u$ from $H_r(B)$ to $L^p(B)$ is compact, for $p \in [1, \tilde{m})$ where*

$$\tilde{m} = \begin{cases} \frac{2N}{N-2-2m} \ \textit{if}\ m < \frac{N-2}{2} \\ \infty \ \ \textit{otherwise}\ . \end{cases}$$

Applying this, one successfully obtains the existence of a mountain pass solution of (1.3) for all $p \in \left(1, \frac{N+2+2\alpha}{N-2}\right)$. The exponent p can be supercritical (i.e., $p > \frac{N+2}{N-2}$) if $\alpha > 0$. We here note that, for the critical or supercritical case in (1.3), the essential point to assure the existence seems that $u^{\frac{N+2}{N-2}}$ has a variable coefficient which is radial and attains 0 at the origin (see Example 2.1 in [17]). In view of this it is an interesting question that whether it is possible to ensure the existence of solutions in the case where the coefficient does not attain 0 at the origin. Very recently, Ai-Cowan [2] study another problem including our problem (1.1). Applying their dynamical system approach, which is developed in [1], the authors in [2] can confirm the existence of radially symmetric solutions of (1.1) for the case $g(x) = |x|^\beta$ with $\beta \in (0, N-2)$. An interesting point in this case is that the coefficient $(1 + g(x))$ attains a local minimum at the origin that is not 0. Hence we cannot apply Lemma 1.1 directly. Then it is an interesting question to investigate how the coefficient can exclude the non-compactness of their nonlinearity. Motivated by this, we investigate (1.1) via the variational method. Our aim is to give a variational interpretation on the results in [2] and further, to extend their results to a more general coefficient which has a local minimum at the origin.

Now in order to explain our main results, we give a comment on the results in [2]. In the variational point of view, it seems better to write the right hand side of the equation of (1.1) as $u^{\frac{N+2}{N-2}} + g(x)u^{\frac{N+2}{N-2}}$. Then the first term is actually noncompact. On the other hand, the second one becomes compact by Lemma 1.1 if $g(x)$ behaves like $|x|^\beta$ with $\beta > 0$. Then we clearly expect that it would play the role of the subcritical perturbation λu^q with $1 \le q < (N+2)/(N-2)$ in [6] mentioned above.

Then, it is natural to consider the next more general problem. (See also the generalization in [2].)

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} + \lambda k(x) f(u), \ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases} \tag{1.4}$$

where $\lambda > 0$ is a parameter and $k : \overline{B} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ satisfy some of the next assumptions.

(k1) $k(x) \not\equiv 0$ is a nonnegative Hölder continuous function on $\overline{B}$ and radial, i.e., $k(x) = k(|x|)$.
(k2) $k(x) = O(|x|^\beta)$ $(|x| \to 0)$ for some $\beta > 0$.
(k3) There exist constants $\gamma \ge \beta > 0$ and $C, \delta > 0$ such that $k(|x|) \ge C|x|^\gamma$ for all $|x| \in (0, \delta)$.
(f1) $f(t)$ is locally Hölder continuous function on $[0, \infty]$ and $f(t) \ge 0$ for all $t > 0$ and $f(t) = 0$ for all $t \le 0$.
(f2) $\lim_{t\to 0} \frac{f(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{f(t)}{t^q} = 0$ for $q = (N+2+2\beta)/(N-2)$.
(f3) There exists a constant $\theta > 2$ such that $f(t)t \ge \theta F(t)$ for all $t \ge 0$ where $F(t) := \int_0^t f(s)ds$.

Now, we give our main results.

Theorem 1.2 *We have the following.*

(i) If k, f satisfy (k1), (k2), (k3), (f1), (f2), (f3) and further,

(f4) $\lim_{t\to\infty} \frac{f(t)}{t^p} = \infty$ *for* $p = \max\left\{1, \frac{2\gamma+6-N}{N-2}\right\}$,

then (1.4) admits a radially symmetric solution for all $\lambda > 0$.

(ii) If k, f *verify (k1), (k2), (f1), (f2), (f3) and further,*

(k4) there exists a point $x_0 \in \overline{B}$ *such that* $k(x_0) > 0$ *and,*
(f5) there exists a constant $c > 0$ *such that* $f(t) > 0$ *for all* $t \in (0, c)$*,*

then, there exists a constant $\lambda^* > 0$ *such that* (1.4) *has a radially symmetric solution for all* $\lambda > \lambda^*$.

Remark 1.3 The hypothesis in Theorem 1.2 (i) permits the case where $k(x) = |x|^\beta$ for $\beta > 0$ and $f(u) = u_+^q$ with any $q \in (\max\{1, (2\beta+6-N)/(N-2)\}, (N+2+2\beta)/(N-2))$. The condition $q > \max\{1, (2\beta+6-N)/(N-2)\}$ is assumed to lower the mountain pass energy down to the level for which the local compactness of the Palais-Smale sequences is valid. See Lemmas 2.3 and 2.4 for the detail. On the other hand, (ii) is valid for $f(u) = u_+^q$ with any $q \in (1, (N+2+2\beta)/(N-2))$.

Remark 1.4 A similar problem is considered in [7] and [9]. The existence and nonexistence for the linear perturbation case with $k(r) = r^\beta$ for $\beta > 0$ and $f(t) = t_+$ are completed by [7]. Furthermore, the superlinear perturbation case with $k(r) = r^\beta$ for $\beta > 0$ and $f(t) = t_+^q$ with $q \in (1, (N+2+2\beta)/(N-2))$ is treated in [9]. Our theorem gives a generalization of a part of their results.

A nonexistence result on (1.4) is given by the Pohozaev identity as follows.

Theorem 1.5 *Let* $\lambda \in \mathbb{R}$, $k(x) = |x|^\beta$ *with* $\beta \geq 0$, $f(u) = u_+^q$ *and* $q \geq 1$. *Then* (1.4) *admits no solution if one of the following is true;*

(i) $q \in [1, (2\beta+N+2)/(N-2)]$ *and* $\lambda \leq 0$, *or*
(ii) $q \geq \frac{2\beta+N+2}{N-2}$ *and* $\lambda \geq 0$, *or,*
(iii) $\beta = 0$ *and* $q = (N+2)/(N-2)$.

Remark 1.6 The same conclusion holds even if we replace the domain B by any star-shaped domain. See the argument in Sect. 3.

Now we come back to our main question on (1.1). The desired existence results are given as a corollary of (i) of Theorem 1.2.

Corollary 1.7 *We assume*

(g1) $g(x)$ *is Hölder continuous and* $g \geq -1$ *on* $\overline{B}$ *and radial, i.e.,* $g(x) = g(|x|)$*,*
(g2) $g(0) = 0$*, and*
(g3) there exist constants $\gamma \in (0, N-2)$*,* $\delta \in (0, 1]$ *and* $C > 0$ *such that* $g(|x|) \geq C|x|^\gamma$ *for all* $|x| \in (0, \delta)$.

Then (1.1) *admits at least one radially symmetric solution.*

Remark 1.8 This theorem generalizes Theorem 2 in [2] for the case $g(|x|, u) = g(|x|)$. To see this, note first that their condition (6) in [2] implies (g2) and (g3). Furthermore, since (g3) is a condition for the behavior of g only near the origin, we can easily construct an example which satisfies (g2) and (g3), but not (6). In addition, they prove Theorem 2 in [2] by dynamical system approach while we shall prove it via the variational method with the concentration compactness analysis.

Hence our proof can give a variational interpretation and a generalization of their theorem.

By Corollary 1.7, we have the existence of solution of (1.1) if $g(x) = \lambda|x|^\beta$ with $\beta \in (0, N-2)$ and $\lambda > 0$. For the case including $\beta \geq N-2$, we have the next corollary as a direct consequence of (ii) in Theorem 1.2.

Corollary 1.9 *Let $\lambda > 0$, $g(x) = \lambda k(x)$ and $k(x)$ is a nonnegative Hölder continuous function in $\overline{B}$ such that $k(0) = 0$ and $k(x) = k(|x|)$. Furthermore, assume there exists a point $x_0 \in \overline{B}$ such that $k(x_0) > 0$. Then there exists a constant $\lambda^* > 0$ such that (1.1) admits at least one radially symmetric solution for all $\lambda > \lambda^*$.*

Remark 1.10 Corollary 1.9 implies that if $g(x) = \lambda|x|^\beta$ with $\beta > 0$, a radially symmetric solution exists for all sufficiently large $\lambda > 0$. Furthermore, we remark that this generalizes Theorem 1 of [2].

The existence results above are best possible in the following sense. We have the following nonexistence result.

Theorem 1.11 *Let $g(x) = \lambda|x|^\beta$ with $\beta \geq 0$ and $\lambda \in \mathbb{R}$. Then (1.1) does not admit any radially symmetric solution if $\beta = 0$ and $\lambda \in \mathbb{R}$, or $\beta \geq 0$ and $\lambda \leq 0$. In addition if $\beta \geq N-2$, there exists a constant $\lambda_* > 0$ which depends on β and N such that (1.1) has no radially symmetric solution for all $\lambda \in [0, \lambda_*]$.*

Remark 1.12 In our computation, we can choose

$$\lambda_* = \begin{cases} \frac{2(N-1)}{N-2} \text{ if } \beta = N-2, \\ \frac{2(N-1)}{N-2} \left(\frac{2N-2+\beta}{\beta-N+2}\right)^{\frac{\beta-N+2}{N-2}} \text{ if } \beta > N-2. \end{cases}$$

For the detail, see the proof of Theorem 1.11 in Sect. 3.

Organization of This Paper

This paper consists of three sections with an appendix. In Sect. 2, we give the proof of the existence results. In Sect. 3, we show the nonexistence assertions by the Pohozaev identity. Lastly in Appendix A, we give a remark on the proof for the critical case for the reader's convenience. Throughout this paper we define $H_r(B)$ as a subspace of $H_0^1(B)$ which consists of all the radial functions. Furthermore we put $2^* = 2N/(N-2)$ and define the Sobolev constant $S > 0$ as usual by

$$S := \inf_{u \in H_0^1(B)\setminus\{0\}} \frac{\|u\|^2}{\int_B |u|^{2^*} dx}$$

where $\|u\|^2 = \int_B |\nabla u|^2 dx$. Finally we define $B_s(0)$ as a N dimensional ball centered at the origin with radius $s > 0$.

2 Existence Results

In this section, we give a proof of the existence results of our main theorems and corollaries. In the following we always suppose (k1), (k2), (f1) and (f2). For the problem (1.4), we define the associated energy functional

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\int_B u_+^{2^*} dx - \int_B kF(u)dx \quad (u \in H_r(B)).$$

Then noting our assumptions and Lemma 1.1, it is standard to see that $I(u)$ is well-defined and continuously differentiable on $H_r(B)$. In addition, by (k1) and (f1), the usual elliptic theory and the strong maximum principle ensure that every critical point of I is a solution of (1.4). Hence our aim becomes to look for critical points of I. We first prove the mountain pass geometry of I [3].

Lemma 2.1 *We have*

(a) $\exists \rho, a > 0$ *such that* $I(u) \geq a$ *for all* $u \in H_r(B)$ *with* $\|u\| = \rho$, *and*
(b) for all $u \in H_r(B) \setminus \{0\}$, $I(tu) \to -\infty$ *as* $t \to \infty$,

for all $\lambda > 0$.

Proof First note that by (f1) and (f2), we have that for any $\varepsilon > 0$, there exists a constant $C > 0$ such that $|f(t)| \leq \varepsilon t + Ct^p$ for all $t \geq 0$ and some $p \in (1, (N + 2 + 2\beta)/(N - 2))$. Then Lemma 1.1 and the Sobolev inequality give

$$I(u) \geq \left(\frac{1}{2} - \frac{\lambda\varepsilon}{\mu_1}\right)\|u\|^2 - \lambda C\|u\|^{p+1} - C\|u\|^{2^*}$$

for all $u \in H_r(B)$. Taking $\varepsilon \in (0, \mu_1/(4\lambda))$, we get (a) for all $\lambda \in (0, \infty)$.

Next, since $k(x)f(u) \geq 0$ for all $x \in \overline{B}$ and $u \in \mathbb{R}$, we have for all $t > 0$ and $u \in H_r(B) \setminus \{0\}$ that

$$I(tu) \leq \frac{t^2}{2}\|u\|^2 - \frac{t^{2^*}}{2^*}\int_B u_+^{2^*} dx.$$

Since $2 < 2^*$, we obtain $I(tu) \to -\infty$ as $t \to \infty$, which shows (b). This finishes the proof. □

Noting Lemma 2.1, we define

$$\Gamma := \{\gamma \in C([0, 1], H_r(B)) \mid \gamma(0) = 0,\ \gamma(1) = e\}$$

with $e \in H_r(B)$ satisfying $\|e\| > \rho$ and $I(e) \le 0$. Then we put

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} I(u).$$

We next show the local compactness property of the Palais-Smale sequences of I. Here, as usual, we say that $(u_n) \subset H_r(B)$ is a $(\mathrm{PS})_c$ sequence for I if $I(u_n) \to c$ for some $c \in \mathbb{R}$ and $I'(u_n) \to 0$ in $H_r^{-1}(B)$ as $n \to \infty$ where $H_r^{-1}(B)$ is the dual space of $H_r(B)$.

Lemma 2.2 *Suppose f satisfies (f3) and $\lambda > 0$. If $(u_n) \subset H_r(B)$ is a $(PS)_c$ sequence for a value $c < S^{N/2}/N$, then (u_n) contains a subsequence which strongly converges in $H_r(B)$ as $n \to \infty$.*

Proof By (f3), we obtain that

$$\begin{aligned} c + o(1) &= I(u_n) - \frac{1}{\min\{2^*, \theta\}} \langle I'(u_n), u_n \rangle + o(1)\|u_n\| \\ &\ge \left(\frac{1}{2} - \frac{1}{\min\{2^*, \theta\}} \right) \|u_n\|^2 + o(1)\|u_n\| \end{aligned}$$

This shows the boundedness of (u_n) in $H_r(B)$. Hence noting (f1), (f2) and Lemma 1.1, we have that, up to a subsequence, there exists a nonnegative function $u \in H_r(B)$ such that

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } H_0^1(B), \\ \int_B kf(u_n)dx \to \int_B kf(u)dx, \\ \int_B r^\beta |u_n|^{s+1} dx \to \int_B r^\beta |u|^{s+1} dx \text{ for any } s \in [1, (N+2+2\beta)/(N-2)), \\ u_n \to u \text{ a.e. on } B, \end{cases} \tag{2.1}$$

as $n \to \infty$. Furthermore, since $(u_n) \subset H_r(B)$, the concentration compactness lemma (Lemma I.1 in [12]) implies that there exist values $\nu_0, \mu_0 \ge 0$ such that

$$\begin{aligned} |\nabla u_n|^2 &\rightharpoonup d\mu \ge |\nabla u|^2 + \mu_0 \delta_0, \\ (u_n)_+^{2^*} &\rightharpoonup d\nu = u^{2^*} + \nu_0 \delta_0, \end{aligned}$$

in the measure sense where δ_0 denotes the Dirac measure with mass 1 which concentrates at $0 \in \mathbb{R}^N$ and

$$S \nu_0^{\frac{2}{2^*}} \le \mu_0. \tag{2.2}$$

Let us show $\nu_0 = 0$. If not, we define a smooth test function ϕ in $\mathbb{R}^N$ such that $\phi = 1$ on $B(0, \varepsilon)$, $\phi = 0$ on $B(0, 2\varepsilon)^c$ and $0 \le \phi \le 1$ otherwise. We also assume

$|\nabla\phi| \le 2/\varepsilon$. Then noting (f1), (f2) and using (k1), (k2), (2.1) and Lemma 1.1, we get

$$\begin{aligned}0 &= \lim_{n\to\infty}\langle I'(u_n), u_n\phi\rangle\\ &= \lim_{n\to\infty}\left(\int_B \nabla u_n\cdot\nabla(u_n\phi)dx - \int_B (u_n)_+^{2^*}\phi dx - \lambda\int_B kf(u_n)u_n\phi dx\right)\\ &= \lim_{n\to\infty}\left(\int_B |\nabla u_n|^2\phi dx - \int_B (u_n)_+^{2^*}\phi dx - \lambda\int_B kf(u_n)u_n\phi dx + \int_B u_n\nabla u_n\cdot\nabla\phi dx\right)\\ &= \int_{\overline{B}}\phi d\mu - \int_{\overline{B}}\phi d\nu + o(1)\end{aligned}$$

where $o(1)\to 0$ as $\varepsilon\to 0$. It follows that

$$0 \ge \mu_0 - \nu_0.$$

Then by (2.2), we obtain

$$\nu_0 \ge S^{\frac{N}{2}}.$$

Using this estimate, we have by (f3) that

$$\begin{aligned}c &= \lim_{n\to\infty}\left(I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle\right)\\ &\ge \frac{1}{N}\lim_{n\to\infty}\int_{\overline{B}} d\nu\\ &\ge \frac{S^{\frac{N}{2}}}{N}\end{aligned}$$

which contradicts our assumption. It follows that

$$\lim_{n\to\infty}\int_B (u_n)_+^{2^*}dx = \int_B u^{2^*}dx.$$

Then the usual argument proves $u_n \to u$ in $H_r(B)$. We finish the proof. □

Next we estimate the mountain pass energy c_λ. To do this, we use the Talenti function $U_\varepsilon(x) := \frac{\varepsilon^{\frac{N-2}{2}}}{(\varepsilon^2+|x|^2)^{\frac{N-2}{2}}}$ [16]. Moreover we define a cut off function $\psi \in C_0^\infty(B)$ such that $\psi(x) = \psi(|x|)$, $\mathrm{supp}\{\psi\} \subset B_\delta(0)$ and $\psi = 1$ on $B_\eta(0)$ for some $\eta \in (0,\delta)$. We set $u_\varepsilon := \psi U_\varepsilon$ and $v_\varepsilon := u_\varepsilon/\|u_\varepsilon\|_{L^{2^*}(B)} \in H_r(B)$. Then, if

$q > \max(2\gamma + 6 - N)/(N - 2)$, a similar calculation with that in [6] shows that

$$\begin{cases} \|v_\varepsilon\|^2 = S + O(\varepsilon^{N-2}) \\ \|v_\varepsilon\|_{L^{2^*}(B)} = 1, \\ \int_B k v_\varepsilon^{q+1} dx \geq C \int_B |x|^\gamma v_\varepsilon^{q+1} dx = C' \varepsilon^a + O(\varepsilon^{N-2}) \end{cases} \tag{2.3}$$

where $a = \gamma + N - \frac{(N-2)(q+1)}{2}$ and $C, C' > 0$ are constants. Let us prove the next lemma (Cf. Lemma 2.1 in [6]).

Lemma 2.3 *Assume that k verifies (k3). Then if*

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma+2} \int_0^{\varepsilon^{-1}} F\left[\left(\frac{\varepsilon^{-1}}{1+r^2}\right)^{\frac{N-2}{2}}\right] r^{\gamma+N-1} dr = \infty \tag{2.4}$$

holds, we have $c_\lambda < S^{N/2}/N$ *for all* $\lambda > 0$.

Proof Let $v_\varepsilon \in H_r(B)$ as above. Then from Lemma 2.1, we find a constant $t_\varepsilon > 0$ such that $I(t_\varepsilon v_\varepsilon) = \max_{t \geq 0} I(t v_\varepsilon)$. Since

$$0 = \frac{d}{dt}|_{t=t_\varepsilon} I(t v_\varepsilon) = t_\varepsilon \|v_\varepsilon\|^2 - t_\varepsilon^{2^*-1} - \int_B k f(t_\varepsilon v_\varepsilon) v_\varepsilon dx$$

and $\int_B k f(v_\varepsilon) v_\varepsilon dx \geq 0$ by (k1) and (f1), we have

$$t_\varepsilon \leq \|v_\varepsilon\|^{\frac{2}{2^*-2}} =: T_\varepsilon.$$

Since $T_\varepsilon = \|v_\varepsilon\|^{\frac{2}{2^*-2}}$ is the maximum point of the map $t \mapsto \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*}}{2^*}$, we get by (2.3) that for any $t > 0$

$$\begin{aligned} I(t v_\varepsilon) &\leq I(t_\varepsilon v_\varepsilon) \\ &\leq \frac{T_\varepsilon^2}{2}\|v_\varepsilon\|^2 - \frac{T_\varepsilon^{2^*}}{2^*} - \int_B k F(t_\varepsilon v_\varepsilon) dx \\ &\leq \frac{S^{\frac{N}{2}}}{N} - \int_B k F(t_\varepsilon v_\varepsilon) dx + O(\varepsilon^{N-2}). \end{aligned}$$

Therefore once we prove

$$\lim_{\varepsilon \to 0} \varepsilon^{-(N-2)} \int_B k F(t_\varepsilon v_\varepsilon) dx = \infty, \tag{2.5}$$

we conclude $c_\lambda \leq I(t_\varepsilon v_\varepsilon) < S^{N/2}/N$ for all small $\varepsilon > 0$. This completes the proof. Lastly let us ensure (2.5). To do this, we first claim that $\lim_{\varepsilon \to 0} t_\varepsilon = S^{(N-2)/4}$.

Indeed, using (f2), for any $\delta > 0$, there exists a constant $C_\delta > 0$ such that

$$\int_B \frac{kf(t_\varepsilon v_\varepsilon)v_\varepsilon}{t_\varepsilon}dx \le t_\varepsilon^{q-1}\delta \int_B |x|^\beta v_\varepsilon^{q+1}dx + C_\delta \int_B |x|^\beta v_\varepsilon^2 dx.$$

Since $t_\varepsilon \le T_\varepsilon = O(1)$, $\int_B |x|^\beta v_\varepsilon^{q+1}dx = O(1)$ by $q = (N+2+2\beta)/(N-2)$ and $\int_B |x|^\beta v_\varepsilon^2 dx = o(1)$ as $\varepsilon \to 0$, we get

$$0 \le \limsup_{\varepsilon\to 0} \int_B \frac{kf(t_\varepsilon v_\varepsilon)v_\varepsilon}{t_\varepsilon}dx \le O(1)\delta.$$

Since $\delta > 0$ can be chosen arbitrarily small, we conclude that

$$\lim_{\varepsilon\to 0} \int_B \frac{kf(t_\varepsilon v_\varepsilon)v_\varepsilon}{t_\varepsilon}dx = 0.$$

Then since $\langle I'(t_\varepsilon v_\varepsilon), v_\varepsilon\rangle = 0$, we have

$$t_\varepsilon = \left(\|v_\varepsilon\|^2 - \int_B \frac{kf(t_\varepsilon v_\varepsilon)v_\varepsilon}{t_\varepsilon}dx\right)^{\frac{1}{2^*-2}}.$$

This with (2.3) proves that $\lim_{\varepsilon\to 0} t_\varepsilon = S^{(N-2)/4}$. In particular, t_ε converges to a positive value as $\varepsilon \to 0$. Now we calculate by (k3) that

$$\begin{aligned}
\varepsilon^{-(N-2)} \int_B kF(t_\varepsilon v_\varepsilon)dx &\ge C_1\varepsilon^{-(N-2)} \int_0^\eta F\left[t_\varepsilon\left(\frac{\varepsilon}{\varepsilon^2+r^2}\right)^{\frac{N-2}{2}}\right] r^{\gamma+N-1}dr \\
&\ge C_2\varepsilon^{\gamma+2} \int_0^{\frac{\eta}{\varepsilon}} F\left[t_\varepsilon\left(\frac{\varepsilon^{-1}}{1+r^2}\right)^{\frac{N-2}{2}}\right] r^{\gamma+N-1}dr \\
&\ge C_3\varepsilon^{\gamma+2} \int_0^{\frac{D}{\varepsilon}} F\left[\left(\frac{\varepsilon^{-1}}{1+r^2}\right)^{\frac{N-2}{2}}\right] r^{\gamma+N-1}dr
\end{aligned}$$

for some constant $C_1, C_2, C_3, D > 0$ where in the last inequality we replace $\varepsilon/t_\varepsilon^{(N-2)/2}$ by ε which does not change the conclusion below. If $D \ge 1$, we clearly get (2.5) by our assumption (2.4). If $D < 1$, we obtain

$$\begin{aligned}
\varepsilon^{\gamma+2} \int_0^{\frac{D}{\varepsilon}} F\left[\left(\frac{\varepsilon^{-1}}{1+r^2}\right)^{\frac{N-2}{2}}\right] r^{\gamma+N-1}dr =&\varepsilon^{\gamma+2} \int_0^{\frac{1}{\varepsilon}} F\left[\left(\frac{\varepsilon^{-1}}{1+r^2}\right)^{\frac{N-2}{2}}\right] r^{\gamma+N-1}dr \\
&- \varepsilon^{\gamma+2} \int_{\frac{D}{\varepsilon}}^{\frac{1}{\varepsilon}} F\left[\left(\frac{\varepsilon^{-1}}{1+r^2}\right)^{\frac{N-2}{2}}\right] r^{\gamma+N-1}dr.
\end{aligned}$$

Finally, note that (f2) shows

$$\varepsilon^{\gamma+2}\int_{\frac{D}{\varepsilon}}^{\frac{1}{\varepsilon}} F\left[\left(\frac{\varepsilon^{-1}}{1+r^2}\right)^{\frac{N-2}{2}}\right] r^{\gamma+N-1}dr = o(1)$$

where $o(1) \to 0$ as $\varepsilon \to 0$. This finishes the proof. □

The next lemma confirms that under our assumptions, $f(t)$ satisfies (2.4).

Lemma 2.4 *Assume (k3). Then, if f satisfies (f4), then* (2.4) *holds true.*

Proof By (f4), for any $M > 0$, there exists a constant $R > 0$ such that $f(t) \geq Mt^p$ where $p = \max\{1, \frac{2\gamma+6-N}{N-2}\}$. Furthermore, note that if $r \leq C\varepsilon^{-1/2}$ for $C = (2R)^{-(N-2)/2}$, we get

$$\left(\frac{\varepsilon^{-1}}{1+r^2}\right)^{\frac{N-2}{2}} \geq R$$

for all small $\varepsilon > 0$. It follows that

$$\begin{aligned}\varepsilon^{\gamma+2}\int_0^{\varepsilon^{-1}} F\left[\left(\frac{\varepsilon^{-1}}{1+r^2}\right)^{\frac{N-2}{2}}\right] r^{\gamma+N-1}dr &\geq \varepsilon^{\gamma+2}\int_0^{C\varepsilon^{-\frac{1}{2}}} F\left[\left(\frac{\varepsilon^{-1}}{1+r^2}\right)^{\frac{N-2}{2}}\right] r^{\gamma+N-1}dr \\ &\geq \varepsilon^{\gamma+2}\frac{M}{p+1}\int_0^{C\varepsilon^{-\frac{1}{2}}} \left(\frac{\varepsilon^{-1}}{1+r^2}\right)^{\frac{(N-2)(p+1)}{2}} r^{\gamma+N-1}dr \\ &\to \infty\end{aligned}$$

as $\varepsilon \to 0$. This completes the proof. □

Lemma 2.5 *If k, f satisfy (k4) and (f5), we have a constant $\lambda^* > 0$ such that $c_\lambda < S^{N/2}/N$ for all $\lambda > \lambda^*$.*

Proof Since $k(x_0) > 0$ by (k4), there exist constants $0 < r_1 < |x_0| < r_2 < 1$ such that $k > 0$ on $\overline{B(0,r_2)} \setminus B(0,r_1)$. Then we choose a radial function $u \in C_0^\infty(B)\setminus\{0\}$ such that $u \geq 0$ and $\operatorname{supp}\{u\} \subset \overline{B(0,r_2)} \setminus B(0,r_1)$. Then by Lemma 2.1, we have a constant $t_\lambda > 0$ such that $I(t_\lambda u) = \max_{t>0} I(tu)$. Since $\frac{d}{dt}|_{t=t_\lambda} I(tu) = 0$, we get

$$\|u\|^2 - t_\lambda^{2^*-2}\int_B u_+^{2^*}dx - \lambda\int_B \frac{kf(t_\lambda u)u}{t_\lambda}dx = 0$$

It follows that $t_\lambda \to 0$ as $\lambda \to \infty$. If not, there exists a sequence $(\lambda_n) \subset (0,\infty)$ such that $\lambda_n \to \infty$ and $t_{\lambda_n} \to t_0 > 0$ for some value $t_0 > 0$ as $n \to \infty$. But this is impossible in view of the previous formula and (f5). Then it follows from (k1) and (f1) that

$$c_\lambda \le I(t_\lambda u) \le t_\lambda^2 \|u\|^2 \to 0$$

as $\lambda \to \infty$. This finishes the proof. □

Then we prove the existence assertions of main theorems.

Proof of Theorem 1.2 First note that under the assumption in Lemma 2.1 and the mountain pass theorem ([3], see also Theorem 2.2 in [6]), there exists a $(\mathrm{PS})_{c_\lambda}$ sequence $(u_n) \subset H_r(B)$ of I. Hence our aim is to see that (u_n) has a subsequence which strongly converges in $H_r(B)$. This fact follows from Lemmas 2.1, 2.2, 2.3 and 2.4, which proves (i). The proof of (ii) is completed by Lemmas 2.1, 2.2 and 2.5. This completes the proof of Theorem 1.2. □

Proof of Corollary 1.7 The proof is clear from (i) of Theorem 1.2. □

Remark 2.6 Here we remark on (g1) and (g2). We first note that non-negativity of k in (k1) is needed only to apply the maximum principle. Hence it is clear that in the present case it can be weakened to $g \ge -1$ in (g1). Furthermore, by (g1), the associated energy functional

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\int_B (1+g)|u|^{2^*} dx$$

is always well-defined. Hence we can weaken (k2) in Theorem 1.2 to the condition $k(0) = 0$. Finally, in the present case, since we do not assume $k(|x|) = O(|x|^\beta)$ for $\beta > 0$, in principle, we cannot use Lemma 1.1 directly in the proof of Lemma 2.2. Although the modification is trivial, we will give the modified proof in Appendix A for the readers' convenience.

Proof of Corollary 1.9 The proof is immediate by (ii) of Theorem 1.2. □

3 Nonexistence Results

In this section, we prove the nonexistence results by the Pohozaev identity. Since some results still hold true for the star-shaped domain, we first consider the problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + g|u|^{q-1}u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ is a bounded smooth domain, $q \geq 1$ and g is a $C^1(\overline{\Omega})$-function. Now, let us recall the formula

$$\int_\Omega \left\{ \frac{x \cdot \nabla g}{q+1} + \left(\frac{N}{q+1} - \frac{N-2}{2} \right) g \right\} |u|^{q+1} dx = \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) |\nabla u|^2 ds_x \tag{3.2}$$

holds for any solution $u \in C^1(\overline{\Omega})$. This is the Pohozaev identity for (3.1).

Theorem 3.1 *Let $\lambda \in \mathbb{R}$ and $g(x) = \lambda |x|^\beta$ with $\beta \geq 0$. Then if Ω is a strictly star-shaped domain,* (3.1) *has no C^1 solution if either one of the following holds;*

(i) $\lambda \leq 0$ *and* $q \leq (N + 2 + 2\beta)/(N - 2)$ *or,*
(ii) $\lambda \geq 0$ *and* $q \geq (N + 2 + 2\beta)/(N - 2)$ *or otherwise,*
(iii) $\beta = 0$, $\lambda \in \mathbb{R}$ *and* $q = (N + 2)/(N - 2)$.

Proof Let $u \in C^1(\overline{\Omega})$ be a solution of (3.1). Then under the assumption in the theorem, we get by (3.2) that

$$\lambda \int_\Omega \left(\frac{\beta + N}{q+1} - \frac{N-2}{2} \right) |x|^\beta |u|^{q+1} dx = \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) |\nabla u|^2 ds_x.$$

Then if one of (i)–(iii) holds, the left hand side is nonpositive. It is easy to obtain the conclusion if the left hand side is strictly negative, since u is zero outside the origin, and hence also at the origin by continuity. On the other hand, if the left hand side is zero, since $x \cdot \nu > 0$ by our assumption, we have $|\nabla u| \equiv 0$ on $\partial\Omega$. Then from the principle of unique continuation we must have $u \equiv 0$ in Ω. This shows the proof. □

Proof of Theorem 1.5 The proof is a direct consequence of Theorem 3.1. □

Lastly let us show the proof of Theorem 1.11. To do this, we assume $q \geq 1$ and $u = u(r)$ $(r \in [0, 1])$ is a solution of

$$\begin{cases} -u'' - \dfrac{(N-1)}{r} u' = |u|^{\frac{4}{N-2}} u + g|u|^{q-1} u \text{ in } (0, 1), \\ u'(0) = 0 = u(1). \end{cases} \tag{3.3}$$

with a C^1 function $g(r)$ on $[0, 1]$. In addition, we suppose $\psi(r)$ $(r \in [0, 1])$ is a smooth test function such that $\psi(0) = 0$. Then we have the following. (See [6] and also [11].)

Theorem 3.2 *If u is a solution of* (3.3)*, we get*

$$\psi(1)|u'(1)|^2 = \frac{1}{2}\int_0^1 u^2 r^{N-4}\left\{r^3\psi''' - (N-1)(N-3)r\psi' + (N-1)(N-3)\psi\right\}dr \\ + \frac{2(N-1)}{N}\int_0^1 |u|^{2^*}(r^{N-1}\psi' - r^{N-2}\psi)dr \\ + \frac{1}{q+1}\int_0^1 |u|^{q+1}\left\{(q+3)gr^{N-1}\psi' - (q-1)(N-1)gr^{N-2}\psi + 2g'r^{N-1}\psi\right\}dr. \tag{3.4}$$

Proof Multiplying the equation in (3.3) by $r^{N-1}\psi u'$ gives

$$\psi(1)|u'(1)|^2 - \int_0^1 |u'|^2\left\{r^{N-1}\psi' - (N-1)r^{N-2}\psi\right\}dr \\ = \frac{N-2}{N}\int_0^1 |u|^{2^*}\left\{r^{N-1}\psi' + (N-1)r^{N-2}\psi\right\}dr \\ + \frac{\lambda(q+1)}{2}\int_0^1 |u|^{q+1}\left\{g'r^{N-1}\psi + r^{N-1}g\psi' + (N-1)r^{N-2}g\psi\right\}dr. \tag{3.5}$$

On the other hand, we multiply the equation in (3.3) by $(r^{N-1}\psi' - (N-1)r^{N-2}\psi)u$ and compute

$$\int_0^1 |u'|^2\left\{r^{N-1}\psi' - (N-1)r^{N-2}\psi\right\}dr \\ - \frac{1}{2}\int_0^1 u^2\left\{r^{N-1}\psi''' + (N-1)(N-3)r^{N-4}(\psi - r\psi')\right\}dr \\ = \int_0^1 |u|^{2^*}\left\{r^{N-1}\psi' - (N-1)r^{N-2}\psi\right\}dr \\ + \lambda\int_0^1 g(r)|u|^{q+1}\left\{r^{N-1}\psi' - (N-1)r^{N-2}\psi\right\}dr. \tag{3.6}$$

Combining (3.5) and (3.6), we complete the proof. □

Proof of Theorem 1.11 The first assertion follows from Theorem 3.1. Let us prove the second assertion. To do this, assume $\lambda > 0$ and $u = u(r)$ $(r \in [0, 1])$ is a radially symmetric solution of (1.1). Then it satisfies

$$\begin{cases} -u'' - \frac{(N-1)}{r}u' = (1+g)|u|^{\frac{4}{N-2}}u \text{ in } (0,1), \\ u'(0) = u(1) = 0, \end{cases} \tag{3.7}$$

where we put $g(r) = \lambda r^\beta$. Again choose a smooth test function ψ such that $\psi(0) = 0$. Then by Theorem 3.2, we have

$$\begin{aligned}&\frac{1}{2}\int_0^1 u^2 r^{N-4}\left\{r^3\psi''' - (N-1)(N-3)r\psi' + (N-1)(N-3)\psi\right\} dr\\&= \psi(1)|u'(1)|^2\\&+ \frac{1}{N}\int_0^1 |u|^{2^*}\left\{-(N-2)g' r^{N-1}\psi + 2(N-1)(1+g(r))(r^{N-2}\psi - r^{N-1}\psi')\right\} dr.\end{aligned} \tag{3.8}$$

We fix $\beta \geq N-2$ and then select $\psi(r) = ar^{N-1} + br$ so that $r^3\psi''' - (N-1)(N-3)r\psi' + (N-1)(N-3)\psi = 0$ and $\psi(0) = 0$. This ODE has an explicit solution $\psi(r) = ar^{N-1} + br + cr^{-(N-3)}$ where $a, b, c \in \mathbb{R}$ are arbitrary constants. Since we assume $\psi(0) = 0$, we must have $c = 0$, i.e., $\psi(r) = ar^{N-1} + br$. Then we get

$$\begin{aligned}&\psi(1)|u'(1)|^2\\&+ \frac{1}{N}\int_0^1 |u|^{2^*}\left\{-(N-2)g' r^{N-1}\psi + 2(N-1)(1+g(r))(r^{N-2}\psi - r^{N-1}\psi')\right\} dr = 0.\end{aligned} \tag{3.9}$$

Substituting $\psi(r) = ar^{N-1} + br$ into

$$h(r) := -(N-2)k' r^{N-1}\psi + 2(N-1)(1+k)(r^{N-2}\psi - r^{N-1}\psi'),$$

we see

$$\begin{aligned}&h(r) = r^{2N-3}\times\\&\left[-\lambda a(N-2)\left\{2(N-1)+\beta\right\} r^\beta - \lambda b\beta(N-2)r^{\beta-N+2} - 2a(N-1)(N-2)\right].\end{aligned}$$

Finally, we choose $a < 0$ and $b = |a| > 0$. In particular, we have $\psi(1) = a + b = 0$. Then some elementary calculations show that if we set

$$\lambda_* = \begin{cases} \frac{2(N-1)}{N-2} \text{ if } \beta = N-2, \\ \frac{2(N-1)}{N-2}\left(\frac{2N-2+\beta}{\beta-N+2}\right)^{\frac{\beta-N+2}{N-2}} \text{ if } \beta > N-2, \end{cases}$$

we assure that $h \neq 0$ and $h \geq 0$ for all $\lambda \in [0, \lambda_*]$. Therefore in view of (3.9), we reach to a contradiction if $\lambda \in [0, \lambda_*]$. This finishes the proof. □

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A Critical Case

In this appendix, we give a proof of Lemma 2.2 under the assumption in Corollary 1.7 for the readers' convenience. Especially we will use only the condition (g2) which is weaker than (k2).

Lemma A.1 *Assume (g1), (g2) and* $(u_n) \subset H_r(B)$ *is a* $(PS)_c$ *sequence of*

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\int_B (1+g)u_+^{2^*}\,dx.$$

Then if $c < S^{\frac{N}{2}}/N$, (u_n) *has a subsequence which strongly converges in* $H_r(B)$.

Proof From the definition we have

$$\begin{aligned} c + o(1) &= I(u_n) - \frac{1}{2^*}\langle I'(u_n), u_n\rangle + o(1)\|u_n\| \\ &\geq \frac{1}{N}\|u_n\|^2 + o(1)\|u_n\|. \end{aligned}$$

This implies (u_n) is bounded in $H_r(B)$. Then we can assume that there exists a nonnegative function $u \in H_r(B)$ such that

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } H_r(B), \\ u_n \to u \text{ a.e. on } B, \end{cases}$$

up to a subsequence. By the concentration compactness lemma, we can suppose that there exist values $\mu_0, \nu_0 \geq 0$ such that

$$\begin{cases} |\nabla u_n|^2 \rightharpoonup d\mu \geq |\nabla u|^2 + \mu_0\delta_k, \\ u_n \to u \text{ in } L^p(B) \text{ for all } p \in (1, 2N/(N-2)), \\ (u_n)_+^{2^*} \rightharpoonup d\nu = u_+^{2^*} + \nu_0\delta_0, \end{cases}$$

in the measure sense, where δ_0 denotes the Dirac delta measure concentrated at the origin with mass 1 as before. Furthermore, we have

$$S\nu_0^{\frac{2}{2^*}} \leq \mu_0. \tag{A.1}$$

We show $\nu_0 = 0$. To this end, we assume $\nu_0 > 0$ on the contrary. Then, for small $\varepsilon > 0$, we define a smooth test function ϕ as in the proof of Lemma 2.2. Since $I'(u_n) \to 0$ in $H^{-1}(B)$ and (u_n) is bounded, we have

$$\begin{aligned}
0 &= \lim_{n\to\infty} \langle I'(u_n), u_n\phi\rangle \\
&= \lim_{n\to\infty} \left(\int_B \nabla u_n \cdot \nabla(u_n\phi)dx - \int_B (1+g)(u_n)_+^{2^*}\phi dx \right) \\
&= \lim_{n\to\infty} \left(\int_B |\nabla u_n|^2 \phi dx - \int_B (1+g)(u_n)_+^{2^*}\phi dx + \int_B u_n \nabla u_n \cdot \nabla\phi dx \right) \\
&= \int_{\overline{B}} \phi d\mu - \int_{\overline{B}} (1+g)\phi d\nu + o(1)
\end{aligned}$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Taking $\varepsilon \to 0$ and noting $g(0) = 0$, we obtain

$$0 \geq \mu_0 - \nu_0.$$

Then using (A.1), we get

$$\nu_0 \geq S^{\frac{N}{2}}.$$

Finally, noting this estimate, we see

$$\begin{aligned}
c &= \lim_{n\to\infty} \left(I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \right) \\
&= \frac{1}{N} \lim_{n\to\infty} \int_{\overline{B}} (1+g) d\nu \\
&\geq \frac{S^{\frac{N}{2}}}{N}
\end{aligned}$$

since $g(0) = 0$, which is a contradiction. It follows that

$$\lim_{n\to\infty} \int_B (1+g)(u_n)_+^{2^*} dx = \int_B (1+g) u_+^{2^*} dx.$$

Then a standard argument shows that $u_n \to u$ in $H_r(B)$. This completes the proof.

□

References

1. Ai, S.: Multipulse-like orbits for a singularly perturbed nearly integrable system. J. Differ. Equ. **179**, 384–432 (2002)
2. Ai, A., Cowan, C.: Critical elliptic equations via a dynamical system approach. Nonlinear Anal. **182**, 97–112 (2019)
3. Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**, 349–381 (1973)
4. Bahri, A.: Critical Points at Infinity in Some Variational Problems. Pitman Research Notes in Mathematics Series, vol. 182, vi+I15+307 pp. Longman Scientific and Technical, Harlow; co-published in the United States with Wiley, New York (1989)
5. Bahri, A., Coron, J.-M.: On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain. Commun. Pure Appl. Math. **41**(3), 253–294 (1988)
6. Brezis, H., Nirenberg, L.: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Commun. Pure Appl. Math. **36**, 437–477 (1983)
7. Clément, P., de Figueiredo, D.G., Mitidieri, E.: Quasilinear elliptic equations with critical exponents. Topol. Methods Nonlinear Anal. **7**, 133–170 (1996)
8. Coron, J.M.: Topologie et cas limite des injections de Sobolev, (French. English summary) [Topology and limit case of Sobolev embeddings]. C. R. Acad. Sci. Paris Sér. I Math. **299**(7), 209–212 (1984)
9. de Figueiredo, D.G., Gonçalves, J.V., Miyagaki, O.H.: On a class of quasilinear elliptic problems involving critical exponents. Commun. Contemp. Math. **2**, 47–59 (2000)
10. Ding, W.: Positive solutions of $\Delta u + u^{\frac{(n+2)}{(n-2)}} = 0$ on contractible domains. J. Partial Differ. Equ. **2**, 83–88 (1989)
11. Gladiali, F., Grossi, M.: Linear perturbations for the critical Hénon problem. Differ. Integr. Equ. **28**, 733–752 (2015)
12. Lions, P.L.: The concentration-compactness principle in the calculus of variations. The limit case. Part 1. Rev. Mat. Iberoamericana **1**, 145–201 (1985)
13. Ni, W.: A nonlinear Dirichlet problem on the unit ball and its applications. Indiana Univ. Math. J. **31**(6), 801–807 (1982)
14. Passaseo, D.: Multiplicity of positive solutions of nonlinear elliptic equations with critical Sobolev exponent in some contractible domains. Manuscr. Math. **65**, 147–165 (1989)
15. Pohozaev, S.I.: Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. Sov. Math. Dokl. **6**, 1408–1411 (1965)
16. Talenti, G.: Best constant in Sobolev inequality. Ann. Mat. Pure Appl. **110**, 353–372 (1976)
17. Wang, X.: Existence of positive solutions to nonlinear elliptic equations involving critical Sobolev exponents. Acta Math. Sin. (N.S.) **8**, 273–291 (1992). A Chinese summary appears in Acta Math. Sin. **36**(4), 574 (1993)

Remark on One Dimensional Semilinear Damped Wave Equation in a Critical Weighted L^2-space

Motohiro Sobajima and Yuta Wakasugi

Abstract We study the Cauchy problem of the semilinear damped wave equation in one space dimension. We show the existence of global solutions in the critical case with small initial data in weighted L^2-spaces. This problem in multidimensional cases was dealt with in Sobajima (Differ Integr Equ 32:615–638, 2019) via the weighted Hardy inequality which is false in one-dimension. The crucial idea of the proof is the use of an incomplete version of Hardy inequality.

Keywords Semilinear damped wave equations · Critical case · Global existence · One dimension

1 Introduction

In this paper we consider the Cauchy problem of the one-dimensional semilinear damped wave equation

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) + \partial_t u(x,t) = |u(x,t)|^{p-1}u(x,t), \quad (x,t) \in \mathbb{R} \times (0,T) \tag{1.1}$$

with $1 < p < \infty$ and the initial data

$$(u_0, u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \tag{1.2}$$

M. Sobajima (✉)
Department of Mathematics, Faculty of Science and Technology, Tokyo University of Science, Noda-shi, Chiba, Japan

Y. Wakasugi
Graduate School of Engineering, Hiroshima University, Higashi-Hiroshima, Hiroshima, Japan
e-mail: wakasugi@hiroshima-u.ac.jp

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satisfying the following condition

$$\langle x\rangle^{\lambda}u_0,\ \langle x\rangle^{\lambda}\partial_x u_0,\ \langle x\rangle^{\lambda}u_1 \in L^2(\mathbb{R}) \tag{1.3}$$

for some $\lambda \in (0, 1/2)$, that is, we do not require $u_0, u_1 \in L^1(\mathbb{R})$. The motivation for studying the linear problem

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) + \partial_t u(x,t) = 0, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), & x \in \mathbb{R}^N \end{cases} \tag{1.4}$$

can be found in the literature (see Matsumura [7] and Nishihara [9]). It is well known that the asymptotic behavior of the solution to (1.4) is similar to the one of the heat equation

$$\begin{cases} \partial_t v(x,t) - \Delta v(x,t) = 0, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ v(x,0) = u_0(x) + u_1(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.5}$$

The following nonlinear problem related to (1.5) is also studied;

$$\begin{cases} \partial_t v(x,t) - \Delta v(x,t) = v(x,t)^p, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ v(x,0) = v_0(x) \geq 0, & x \in \mathbb{R}^N. \end{cases} \tag{1.6}$$

In the pioneering work by Fujita [2], it is proved that if $1 < p < 1 + \frac{2}{N}$, then the problem (1.6) does not have non-trivial global-in-time solutions and if $1+\frac{2}{N} < p < \infty$, then the problem (1.6) possesses global-in-time solutions. The constant $1 + \frac{2}{N}$ is called the Fujita exponent. Global existence for (1.6) with slowly decaying initial data was discussed in Weissler [14]. In [14], it is shown that (1.6) possesses the global-in-time solution when $p \geq 1 + \frac{2r}{N}$ and $\|v_0\|_{L^r(\mathbb{R}^N)}$ is sufficiently small (with a restriction when $p > 1 + \frac{2r}{N}$ for solvability). For the detail, see Quittner–Souplet [10].

In the analysis of global existence of weak (H^1) solutions to the semilinear problem (1.1) with (1.2) small enough (in a suitable sense), Nakao–Ono [8] found global existence in the N-dimensional situation

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) + \partial_t u(x,t) = |u(x,t)|^{p-1}u(x,t), & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}^N), \quad \partial_t u(x,0) = u_1(x) \in L^2(\mathbb{R}^N) \end{cases} \tag{1.7}$$

when $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ ($N \geq 3$) and $1 + \frac{4}{N} \leq p < \infty$ ($N = 1, 2$). After that, in Todorova–Yordanov [13] it is proved via weighted energy estimates that, if the initial data is compactly supported, smooth and sufficiently small, then $p > 1+\frac{2}{N}$ provides the existence of global-in-time solutions of (1.7). On the other hand, in the opposite range of p ($1 < p \le 1 + \frac{2}{N}$) with the nonlinearity $|u|^p$, the smallness does not give global existence (see also Zhang [15]). So, the Fujita exponent $1 + \frac{2}{N}$ also divides the situation between the existence and non-existence of global-in-time solutions of (1.7). The problem (1.7) with slowly decaying (L^r) initial data is considered in Ikehata–Ohta [6]. They proved global existence with initial data belonging to $(H^1 \cap L^r) \times (L^2 \cap L^r)$ when $p > 1 + \frac{2r}{N}$ and nonexistence for $1 < p < 1 + \frac{2r}{N}$.

By using Fourier analysis, in Hayashi–Kaikina–Naumkin [3], the existence of global-in-time (H^α) solutions to (1.7) with $p > 1 + \frac{2}{N}$ in the class of initial data $(H^{\alpha,0} \cap H^{0,\delta}) \times (H^{\alpha-1,0} \cap H^{0,\delta})$ is shown when

$$\delta > \frac{N}{2} \ \& \ \begin{cases} [\alpha] \le p, \alpha \ge \frac{N}{2} - \frac{1}{p-1} & \text{if } N \ge 2, \\ \frac{1}{2} - \frac{1}{2(p-1)} \le \alpha < 1 & \text{if } N = 1, \end{cases}$$

where $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ and

$$H^{\ell,m} = \{u \in L^2(\mathbb{R}^N)\ ;\ \|\langle x \rangle^m \langle i\nabla \rangle^\ell u\|_{L^2} < \infty\}.$$

It should be emphasized that $H^{0,\delta} \subset L^1(\mathbb{R}^N)$ if $\delta > \frac{N}{2}$. Their argument has been generalized to the case of slowly decaying initial data in Ikeda–Inui–Wakasugi [5]. More precisely, in [5], global existence is proved for $p > 1 + \frac{2r}{N}$ in the class of initial data $(H^{\alpha,0} \cap H^{0,\delta}) \times (H^{\alpha-1,0} \cap H^{0,\delta})$ with $\delta > \frac{N}{2}(\frac{1}{r} - \frac{1}{2})$; note that this restriction gives $H^{0,\delta} \subset L^r(\mathbb{R}^N)$.

Recently in Sobajima [11], by using weighted energy estimates involving confluent Hypergeometric functions, global existence for (1.7) (in exterior domain) with

$$\begin{cases} 1 + \frac{2}{1+\lambda} \le p < \infty & \text{if } N = 2, \\ 1 + \frac{4}{N+2\lambda} \le p \le \frac{N}{N-2} & \text{if } N \ge 3 \end{cases}$$

and $\lambda \in [0, N/2)$ is shown under the smallness of the size of the following factors:

$$\langle x \rangle^\lambda u_0, \langle x \rangle^\lambda \nabla u_0, \langle x \rangle^\lambda u_1 \in L^2(\mathbb{R}^N) \quad \Big((u_0, u_1) \in H^{1,\lambda} \times H^{0,\lambda}\Big). \tag{1.8}$$

Although the assumption on ∇u_0 is stronger than [5], the critical case $p = 1 + \frac{4}{N+2\lambda}$ can be dealt with. On the contrary, if $1 < p < 1 + \frac{4}{N+2\lambda}$ then (1.7)–(1.8) with

$$u_0(x) + u_1(x) \ge c_0 |x|^{-\frac{N}{2}-\lambda}, \quad |x| \gg 1$$

and $c_0 > 0$ does not have global-in-time solutions (see [5] for the case of whole space and [11] for the case of exterior domains). It is also shown in [11] that if

$N = 1$ and $1 + \frac{4}{1+2\lambda} < p < \infty$, the problem (1.1)–(1.3) possesses global-in-time solutions. Therefore the case of $N = 1$ and $p = 1 + \frac{4}{1+2\lambda}$ remains open whether the problem (1.1)–(1.3) possesses global-in-time solutions or not. The difficulty comes from the lack of validity of the weighted Hardy inequality

$$\int_{\mathbb{R}^N} \langle x \rangle^{2\lambda-2} |w|^2 \, dx \le C \int_{\mathbb{R}^N} \langle x \rangle^{2\lambda} |\nabla w|^2 \, dx, \quad \frac{2-N}{2} < \lambda < \infty, \ w \in H^{1,\lambda},$$

which is quite important in the N-dimensional case for controlling the nonlinear effect by the linear profile.

The purpose of the present paper is to discuss the one-dimensional critical case $p = 1 + \frac{4}{1+2\lambda}$. To state the result, we first introduce the precise definition of solutions to (1.1) in the present paper.

Definition 1.1 (Weak Solutions) The function $u : \mathbb{R} \times [0, T) \to \mathbb{R}$ is called a weak solution of (1.1) in $(0, T)$ if u belongs to the class

$$S_T = \{u \in C([0, T); H^1(\mathbb{R})) \cap C^1([0, T); L^2(\mathbb{R})) \ ; \ |u|^{p-1}u \in C([0, T); L^2(\mathbb{R}))\}$$

and $U = (u(t), \partial_t u(t))$ satisfies the following integral equation in $\mathcal{H} = H^1(\mathbb{R}) \times L^2(\mathbb{R})$:

$$U(t) = e^{-t\mathcal{A}} U_0 + \int_0^t e^{-(t-s)\mathcal{A}} [\mathcal{N}(U(s))] \, ds, \quad t \in [0, T),$$

where $U_0 = (u_0, u_1)$, $\mathcal{A} = \begin{pmatrix} 0 & -1 \\ -\partial_x^2 & 1 \end{pmatrix}$ with a domain $D(\mathcal{A}) = H^2(\mathbb{R}) \times H^1(\mathbb{R})$, $\mathcal{N}(u, v) = (0, |u|^{p-1}u)$ and $\{e^{-t\mathcal{A}}\}$ is the C_0-semigroup on $\mathcal{H}$ generated by $-\mathcal{A}$.

The following proposition is about the existence of local-in-time solutions. It is well-known but quite important to discuss the global existence.

Proposition 1.1 *Assume* (1.2) *and* $1 < p < \infty$. *Then there exists a positive constant* $T > 0$ *depending only on* $p, \|u_0\|_{H^1}, \|u_1\|_{L^2}$ *such that there exists a unique weak solution* u *of* (1.1) *in* $(0, T)$.

We are now in a position to state the main result of the present paper.

Theorem 1.2 *Let* $\lambda \in (0, 1/2)$. *Assume that* $p = 1 + \frac{4}{1+2\lambda}$ *and* (1.3) *is satisfied. Then there exists a positive constant* $\varepsilon_0 > 0$ *such that if*

$$\|\langle x \rangle^\lambda u_0\|_{L^2(\mathbb{R})}^2 + \|\langle x \rangle^\lambda \partial_x u_0\|_{L^2(\mathbb{R})}^2 + \|\langle x \rangle^\lambda u_1\|_{L^2(\mathbb{R})}^2 \le \varepsilon$$

for some $\varepsilon \in (0, \varepsilon_0]$, *then the problem* (1.1)–(1.2) *admits a unique global-in-time weak solution*

$$u \in S_\infty = C([0, \infty); H^1(\mathbb{R})) \cap C^1([0, \infty); L^2(\mathbb{R}))$$

satisfying the following estimates:

$$(1+t)\int_{\mathbb{R}}\Big(|\partial_x u(x,t)|^2+|\partial_t u(x,t)|^2\Big)(1+t+|x|^2)^{\lambda}\,dx$$
$$+\int_{\mathbb{R}}|u(x,t)|^2(1+t+|x|^2)^{\lambda}\,dx\le C\varepsilon, \qquad (1.9)$$

$$\int_0^t\int_{\mathbb{R}}\Big(|\partial_x u(x,s)|^2+(1+s)|\partial_t u(x,s)|^2\Big)(1+s+|x|^2)^{\lambda}\,dx\,ds\le C\varepsilon, \qquad (1.10)$$

where the positive constant C is independent of ε. In particular, one has

$$\Big\|u(x,t)(1+t+|x|^2)^{\lambda/2}\Big\|_{L^\infty(\mathbb{R})}\le C'\sqrt{\varepsilon}(1+t)^{-\frac{1}{4}}, \qquad (1.11)$$

where the positive constant C' is also independent of ε.

Remark 1.1 The previous works [3, 6] and [5] exclude the critical case $p=1+\frac{4}{1+2\lambda}$. Global existence for the (one-dimensional) critical case is now established. The class of solutions to (1.1) in Theorem 1.2 is slightly different from those of [3, 5, 6]. Global existence for the low regularity solutions treated in [3] and [5] is still open when $p=1+\frac{4}{1+2\lambda}$.

Remark 1.2 The same strategy also works for another nonlinearity $|u|^p$ or $-|u|^{p-1}u$. In the latter case, the detailed profile of global solutions such as an asymptotic behavior has been studied in Hayashi–Kaikina–Naumkin [4] for $p=1+\frac{2}{N}$ $(N=1,2,3)$.

The conclusion is, as a result, the same as the N-dimensional situation. In particular, in the one-dimensional case, the Sobolev space $H^1(\mathbb{R})$ is embedded into $C_b(\mathbb{R})$, that is, the set of all bounded continuous functions. The last estimate (1.11) is a consequence of this embedding.

From a viewpoint of an asymptotic profile of linear equations, it is expected that the global-in-time solution of (1.1) behaves like that of the heat equation with suitable initial value.

The next theorem asserts that under some condition which is slightly stronger than that of Theorem 1.2, the global-in-time solution u is bounded by an explicit function depending on λ.

Theorem 1.3 *Let u be a global-in-time solution of* (1.1) *in Theorem 1.2. Assume further that*

$$\|\langle x\rangle^{\lambda+1}\partial_x u_0\|^2_{L^2(\mathbb{R})}+\|\langle x\rangle^{\lambda+1}u_1\|^2_{L^2(\mathbb{R})}\le\varepsilon.$$

Then u satisfies

$$\int_{\mathbb{R}} \Big(|\partial_x u(x,t)|^2 + |\partial_t u(x,t)|^2\Big)(1+t+|x|^2)^{\lambda+1}\,dx \le C''\varepsilon, \tag{1.12}$$

$$\int_0^t \int_{\mathbb{R}} |\partial_t u(x,s)|^2 (1+s+|x|^2)^{\lambda+1}\,dx\,ds \le C''\varepsilon, \tag{1.13}$$

$$\Big\| u(x,t)(1+t+|x|^2)^{\frac{1+2\lambda}{4}} \Big\|_{L^\infty(\mathbb{R})} \le C''\sqrt{\varepsilon}, \tag{1.14}$$

where the positive constant C'' is independent of ε.

Remark 1.3 Actually, the function $(1+t+|x|^2)^{-\frac{1+2\lambda}{4}} = (1+t+|x|^2)^{-\frac{1}{p-1}}$ is bounded below and above by the particular self-similar solution $\Phi_{\frac{1}{p-1}}$ of the heat equation:

$$c\Big(1+t+|x|^2\Big)^{-\frac{1+2\lambda}{4}} \le \Phi_{\frac{1}{p-1}}(x,t;1) \le C\Big(1+t+|x|^2\Big)^{-\frac{1+2\lambda}{4}},$$

where Φ_β is defined in Sect. 2.1. Here β is the parameter describing scaling structure of the heat equation. The parameter $\beta = \frac{1}{p-1}$ appears naturally in the analysis of semilinear heat equations with power nonlinearity.

The crucial idea of the proof of Theorem 1.2 is to use an incomplete version of Hardy inequality

$$\int_{\mathbb{R}} \langle x\rangle^{2\lambda-1}|w|^2\,dx \le C\left(\int_{\mathbb{R}} \langle x\rangle^{2\lambda}|\partial_x w|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} \langle x\rangle^{2\lambda}|w|^2\,dx\right)^{\frac{1}{2}}, \quad \lambda > 0,\ w \in H^{1,\lambda}$$

(see Lemma 2.5 below) in the weighted energy estimate with confluent hypergeometric functions for the semilinear problem (1.1). This enables us to control the nonlinearity $|u|^{p-1}u$.

This paper is organized as follows. In Sect. 2, we give the definition of self-similar solutions involving Kummer's confluent hypergeometric functions and state some basic properties of them. We also prove some important functional inequalities such as incomplete version of Hardy inequality mentioned above. Section 3 is devoted to the proof of global existence (Theorem 1.2), and Sect. 4 is to find an upper bound of solutions under a slightly stronger assumption.

2 Preliminaries

2.1 Weight Functions for Energy Functionals

For $t_0 \geq 1$ and $\beta \geq 0$, define

$$\Phi_\beta(x,t:t_0) = (t_0+t)^{-\beta} e^{-\frac{|x|^2}{4(t_0+t)}} M\left(\frac{1}{2}-\beta, \frac{1}{2}; \frac{x^2}{4(t_0+t)}\right), \quad (x,t) \in \mathbb{R}\times[0,\infty),$$

$$\Psi(x,t:t_0) = t_0+t+\frac{x^2}{4}, \quad (x,t) \in \mathbb{R}\times[0,\infty),$$

where $M(a,c;z)$ is Kummer's confluent hypergeometric function defined as

$$M(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\frac{z^n}{n!}$$

with the Pochhammer symbol $(d)_0 = 1$ and $(d)_n = \prod_{k=1}^{n}(d+k-1)$. For the details of Kummer's hypergeometric functions, see e.g., Beals–Wong [1]. These functions are given by Sobajima–Wakasugi [12] as a family of self-similar solutions of the linear heat equation $\partial_t\Phi - \partial_x^2\Phi = 0$. Then we have the following lemma.

Lemma 2.1 ([12]) *Let t_0 be a positive constant. Then the family $\{\Phi_\beta(\cdot,\cdot:t_0)\}_{\beta\geq 0}$ has the following properties.*

(i) *for every $\beta \geq 0$, $\partial_t\Phi_\beta(x,t:t_0) = \partial_x^2\Phi_\beta(x,t:t_0)$ for $(x,t)\in\mathbb{R}\times[0,\infty)$,*
(ii) *for every $\beta \geq 0$, $\partial_t\Phi_\beta(x,t:t_0) = -\beta\Phi_{\beta+1}(x,t:t_0)$ for $(x,t)\in\mathbb{R}\times[0,\infty)$,*
(iii) *for every $\beta \geq 0$, there exists a positive constant C_β such that*

$$|\Phi_\beta(x,t:t_0)| \leq C_\beta \Psi(x,t:t_0)^{-\beta},$$

(iv) *for every $0 \leq \beta < \frac{1}{2}$, there exists a positive constant c_β such that*

$$\Phi_\beta(x,t:t_0) \geq c_\beta \Psi(x,t:t_0)^{-\beta}.$$

In the N-dimensional situation, the function $\Phi_\beta^{-1+2\delta}$ (for some small δ) can be used as the weight function in the energy functional. This fact can be seen in [11]. The following definition is valid also for the one-dimensional case.

Definition 2.1 For $t_0 \geq 1$ and $\beta \geq 0$, we define $\widetilde{\Phi}_\beta$ as

$$\widetilde{\Phi}_\beta(x,t:t_0) = \left(2-\frac{1}{t_0+t}\right)\Phi_\beta(x,t:t_0), \quad (x,t)\in\mathbb{R}\times[0,\infty).$$

The properties of $\widetilde{\Phi}$, which are used later, are listed in the following lemma. The assertions directly follow from Lemma 2.1.

Lemma 2.2 *Let $t_0 \geq 1$ be fixed. Then the family $\{\widetilde{\Phi}_\beta(\cdot, \cdot : t_0)\}_{0<\beta<\frac{1}{2}}$ satisfies the following properties.*

(i) *for every* $0 < \beta < \frac{1}{2}$,

$$\partial_t \widetilde{\Phi}_\beta(x, t : t_0) \geq \partial_x^2 \widetilde{\Phi}_\beta(x, t : t_0) + \frac{1}{2}(t_0 + t)^{-2} \widetilde{\Phi}_\beta(x, t : t_0),$$

(ii) *for every* $0 < \beta < \frac{1}{2}$,

$$|\partial_t \widetilde{\Phi}_\beta(x, t : t_0)| \leq \frac{\widetilde{C}'_\beta}{t_0 + t} \widetilde{\Phi}_\beta(x, t : t_0),$$

(iii) *for every* $0 < \beta < \frac{1}{2}$,

$$c_\beta \Psi(x, t : t_0)^{-\beta} \leq \widetilde{\Phi}_\beta(x, t : t_0) \leq C_\beta \Psi(x, t : t_0)^{-\beta}.$$

2.2 Functional Inequalities

Here we state some functional inequalities for the weighted energy estimate.

The first inequality is crucial to obtain the weighted energy estimate for the linear damped wave equation.

Lemma 2.3 ([11, Lemma 2.5]) *Assume that $\Phi \in C^2(\mathbb{R})$ is positive and $\delta \in (0, \frac{1}{2})$. Then for every $u \in H^2(\mathbb{R})$ having a compact support,*

$$\int_{\mathbb{R}} u(\partial_x^2 u)\Phi^{-1+2\delta}\, dx \leq -\frac{\delta}{1-\delta} \int_{\mathbb{R}} |\partial_x u|^2 \Phi^{-1+2\delta}\, dx + \frac{1-2\delta}{2} \int_{\mathbb{R}} u^2 (\partial_x^2 \Phi)\Phi^{-2+2\delta}\, dx.$$

Next we use a weighted version of Gagliardo–Nirenberg inequality in the following lemma to control the nonlinearity.

Lemma 2.4 *Let $\lambda \in (0, \frac{1}{2})$ and $2 < q \leq 2 + \frac{1}{\lambda}$. For every function w satisfying $\langle x\rangle^\lambda w, \langle x\rangle^\lambda \partial_x w \in L^2(\mathbb{R})$,*

$$\int_{\mathbb{R}} |w|^q \Psi^\lambda\, dx \leq \left[2(\tfrac{2}{\lambda})^\lambda\right]^{q-2} \left(\int_{\mathbb{R}} |\partial_x w|^2 \Psi^\lambda\, dx\right)^{(q-2)\frac{1+2\lambda}{4}} \left(\int_{\mathbb{R}} |w|^2 \Psi^\lambda\, dx\right)^{\frac{q}{2}-(q-2)\frac{1+2\lambda}{4}},$$

where $\Psi = \Psi(x, t; t_0)$.

To prove Lemma 2.4, we need the following inequality which is a kind of incomplete Hardy inequality.

Lemma 2.5 *Let* $\lambda \in (0, \frac{1}{2})$. *For every* $w \in C_c^1(\mathbb{R})$,

$$\int_{\mathbb{R}} |w|^2 \Psi^{\lambda-\frac{1}{2}}\, dx \le \frac{2}{\lambda}\left(\int_{\mathbb{R}} |\partial_x w|^2 \Psi^{\lambda}\, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |w|^2 \Psi^{\lambda}\, dx\right)^{\frac{1}{2}}. \tag{2.1}$$

Proof Observe that

$$\partial_x\Big(\frac{x}{2}\Psi^{\lambda-\frac{1}{2}}\Big) = \frac{1}{2}\Psi^{\lambda-\frac{1}{2}} - \left(\frac{1}{2}-\lambda\right)\frac{|x|^2}{4}\Psi^{\lambda-\frac{3}{2}} \ge \lambda\Psi^{\lambda-\frac{1}{2}}.$$

Therefore by integration by parts we have

$$\begin{aligned}
\lambda\int_{\mathbb{R}} |w|^2\Psi^{\lambda-\frac{1}{2}}\, dx &\le \int_{\mathbb{R}} |w|^2 \partial_x\Big(\frac{x}{2}\Psi^{\lambda-\frac{1}{2}}\Big)\, dx \le -2\int_{\mathbb{R}} w(\partial_x w)\Big(\frac{x}{2}\Psi^{\lambda-\frac{1}{2}}\Big)\, dx \\
&\le 2\int_{\mathbb{R}} |w|\,|\partial_x w|\Psi^{\lambda}\, dx \\
&\le 2\left(\int_{\mathbb{R}} |\partial_x w|^2\Psi^{\lambda}\, dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} |w|^2\Psi^{\lambda}\, dx\right)^{\frac{1}{2}}.
\end{aligned}$$

This yields the desired inequality. □

By using Lemma 2.5, we deduce the weighted L^∞-estimate.

Lemma 2.6 *Let* $\lambda \in (0, \frac{1}{2})$. *For every* $w \in C_c^1(\mathbb{R})$,

$$\big\|w\Psi^{\frac{\lambda}{2}}\big\|_{L^\infty(\mathbb{R})} \le 2\left(\int_{\mathbb{R}} |\partial_x w|^2\Psi^{\lambda}\, dx\right)^{\frac{1}{4}}\left(\int_{\mathbb{R}} |w|^2\Psi^{\lambda}\, dx\right)^{\frac{1}{4}}.$$

Proof Noting that

$$\partial_x\big(w^2\Psi^{\lambda}\big) = 2w\partial_x w\Psi^{\lambda} + \lambda w^2\frac{x}{2}\Psi^{\lambda-1},$$

we see from the compactness of the support of w that

$$\begin{aligned}
|w(x,t)|^2\Psi(x,t)^{\lambda} &= \int_{-\infty}^{x} \partial_y\big(w(y,t)^2\Psi(y,t)^{\lambda}\big)\, dy \\
&\le 2\int_{\mathbb{R}} |w|\,|\partial_x w|\Psi^{\lambda}\, dx + \lambda\int_{\mathbb{R}} |w|^2\Psi^{\lambda-\frac{1}{2}}\, dx.
\end{aligned}$$

Applying Lemma 2.5, we deduce the desired inequality. □

Then we prove a (space-time) weighted version of Gagliardo–Nirenberg inequality by employing the weighted L^∞ estimate in Lemma 2.6.

Proof of Lemma 2.4 By Hölder inequality, we have

$$\begin{aligned}\int_{\mathbb{R}} |w|^q \Psi^\lambda \, dx &\le \|w\Psi^{\frac{\lambda}{2}}\|_{L^\infty(\mathbb{R})}^{q-2} \int_{\mathbb{R}} |w|^2 \Psi^{\lambda-\frac{q-2}{2}\lambda} \, dx \\ &\le \|w\Psi^{\frac{\lambda}{2}}\|_{L^\infty(\mathbb{R})}^{q-2} \left(\int_{\mathbb{R}} |w|^2 \Psi^{\lambda-\frac{1}{2}} \, dx\right)^{(q-2)\lambda} \left(\int_{\mathbb{R}} |w|^2 \Psi^{\lambda} \, dx\right)^{1-(q-2)\lambda}.\end{aligned}$$

Then using Lemmas 2.5 and 2.6, we obtain the desired inequality. □

3 Proof of Theorem 1.2

To prove global existence, the following proposition (so-called the blowup–alternative) is essential.

Proposition 3.1 *Assume* $1 < p < \infty$. *Let* u *be the weak solution of* (1.1) *in* $(0, T_*)$ *with the corresponding lifespan* T_*. *If* $T_* < \infty$, *then one has*

$$\lim_{t \uparrow T_*} (\|u\|_{H^1(\mathbb{R})} + \|u\|_{L^2(\mathbb{R})}) = \infty.$$

From the viewpoint of the class of initial data, we will use the following energy functional in the proof of Theorem 1.2.

Definition 3.1 Let $\lambda \in (0, \frac{1}{2})$ and let u be the weak solution of (1.1) in $(0, T_*)$. Define for $t_0 \ge 1$ and $t \in [0, T_*)$,

$$\begin{aligned}m_\lambda(t : t_0) &:= (t_0 + t) \int_{\mathbb{R}} \left(|\partial_x u(x,t)|^2 + |\partial_t u(x,t)|^2\right) \Psi(x, t : t_0)^\lambda \, dx \\ &\quad + \int_{\mathbb{R}} |u(x,t)|^2 \Psi(x, t : t_0)^\lambda \, dx, \end{aligned} \tag{3.1}$$

$$Y_\lambda(t : t_0) := \int_0^t \int_{\mathbb{R}} |\partial_x u(x,s)|^2 \Psi(x, s : t_0)^\lambda \, dx \, ds, \tag{3.2}$$

$$Z_\lambda(t : t_0) := \int_0^t (t_0 + s) \int_{\mathbb{R}} |\partial_t u(x,s)|^2 \Psi(x, s : t_0)^\lambda \, dx \, ds. \tag{3.3}$$

Proposition 3.2 *There exist positive constants* $t_0^* \geq 1$ *and* $\eta^* > 0$ *such that*

$$\begin{aligned}
&\eta^* \left(m_\lambda(t : t_0^*) + Y_\lambda(t : t_0^*) + Z_\lambda(t : t_0^*)\right) \\
&\leq m_\lambda(0 : t_0^*) + \int_{\mathbb{R}} |u_0|^{p+1} \Psi_*(0)^\lambda \, dx \\
&\quad + (t_0^* + t) \int_{\mathbb{R}} |u(t)|^{p+1} \Psi_*(t)^\lambda \, dx + \int_0^t \int_{\mathbb{R}} |u(s)|^{p+1} \Psi_*(s)^\lambda \, dx \, ds,
\end{aligned}$$

where $\Psi_*(t) = \Psi(\cdot, t : t_0^*)$.

Sketch of the Proof of Proposition 3.2 Set the following two (weighted) energy functionals:

$$E_\lambda(t : t_0) := (t_0 + t) \int_{\mathbb{R}} \left(|\partial_x u(x,t)|^2 + |\partial_t u(x,t)|^2\right) \Psi(x, t : t_0)^\lambda \, dx, \quad t \in [0, T_*), \tag{3.4}$$

$$\widetilde{E}_\lambda(t : t_0) := \int_{\mathbb{R}} \left(2u(x,t)\partial_t u(x,t) + |u(x,t)|^2\right) \widetilde{\Phi}_\beta(x, t : t_0)^{-1+2\delta} \, dx, \quad t \in [0, T_*), \tag{3.5}$$

where $\beta = \frac{2\lambda}{1+2\lambda} \in (0, \frac{1}{2})$ and $\delta = \frac{1-2\lambda}{4} \in (0, \frac{1}{4})$. The parameter t_0 will be fixed later (we often use E_λ, $\widetilde{E}_\lambda$ if there is no confusion). Differentiating E_λ and using integration by parts, we see from $|\partial_x \Psi| \leq \Psi^{\frac{1}{2}}$ and $t_0 + t \leq \Psi$ that

$$\begin{aligned}
\frac{d}{dt} E_\lambda &\leq 2(t_0 + t) \int_{\mathbb{R}} \partial_t u(t)\left(\partial_t^2 u(t) - \partial_x^2 u(t)\right) \Psi(t)^\lambda \, dx + (t_0 + t) \int_{\mathbb{R}} |\partial_t u|^2 \Psi(t)^\lambda \, dx \\
&\quad + (\lambda^2 + \lambda + 1) \int_{\mathbb{R}} \left(|\partial_x u(t)|^2 + |\partial_t u(t)|^2\right) \Psi(t)^\lambda \, dx.
\end{aligned}$$

Using the equation in (1.1), we have

$$\begin{aligned}
\frac{d}{dt} E_\lambda &\leq \frac{2}{p+1} \frac{d}{dt} \left[(t_0 + t) \int_{\mathbb{R}} |u|^{p+1} \Psi(t)^\lambda \, dx\right] - (t_0 + t) \int_{\mathbb{R}} |\partial_t u(t)|^2 \Psi(t)^\lambda \, dx \\
&\quad + (\lambda^2 + \lambda + 1) \int_{\mathbb{R}} \left(|\partial_x u(t)|^2 + |\partial_t u(t)|^2\right) \Psi(t)^\lambda \, dx.
\end{aligned}$$

On the one hand, differentiating $\widetilde{E}_\lambda$ and using the equation in (1.1), we deduce

$$\begin{aligned}
\frac{d}{dt} \widetilde{E}_\lambda &= 2 \int_{\mathbb{R}} |\partial_t u(t)|^2 \widetilde{\Phi}_\beta(t)^{-1+2\delta} \, dx + 2 \int_{\mathbb{R}} u(t)\left(\partial_x^2 u(t) + |u(t)|^{p-1} u(t)\right) \widetilde{\Phi}_\beta(t)^{-1+2\delta} \, dx, \\
&\quad - (1 - 2\delta) \int_{\mathbb{R}} \left(2u(t)\partial_t u(t) + u(t)^2\right) \widetilde{\Phi}_\beta(t)^{-2+2\delta} \partial_t \widetilde{\Phi}_\beta(t) \, dx.
\end{aligned}$$

Applying Lemma 2.3 and Lemma 2.2 **(i)** and **(ii)**, we have

$$\frac{d}{dt}\widetilde{E}_\lambda \le 2\int_{\mathbb{R}} |\partial_t u(t)|^2 \widetilde{\Phi}_\beta(t)^{-1+2\delta}\,dx - \frac{2\delta}{1-\delta}\int_{\mathbb{R}} |\partial_x u(t)|^2 \widetilde{\Phi}_\beta(t)^{-1+2\delta}\,dx$$
$$- \frac{(1-2\delta)}{2(t_0+t)^2}\int_{\mathbb{R}} |u(t)|^2 \widetilde{\Phi}_\beta(t)^{-1+2\delta}\,dx + 2\int_{\mathbb{R}} |u(t)|^{p+1} \widetilde{\Phi}_\beta(t)^{-1+2\delta}\,dx$$
$$+ \frac{2\widetilde{C}'_\beta(1-2\delta)}{t_0+t}\int_{\mathbb{R}} |u(t)|\,|\partial_t u(t)| \widetilde{\Phi}_\beta(t)^{-1+2\delta}\,dx.$$

The Schwarz inequality and Lemma 2.2 with $\delta_1 = \frac{2\delta}{1-\delta}$ provide

$$\frac{d}{dt}\widetilde{E}_\lambda \le K_1\int_{\mathbb{R}} |\partial_t u(t)|^2 \Psi(t)^\lambda\,dx - \delta_1\int_{\mathbb{R}} |\partial_x u(t)|^2 \Psi(t)^\lambda\,dx + K_2\int_{\mathbb{R}} |u(t)|^{p+1} \Psi(t)^\lambda\,dx$$

for some positive constants $K_1, K_2 > 0$. Taking $\nu = \frac{\lambda^2+\lambda+2}{\delta_1}$, we find

$$\frac{d}{dt}\Big[E_\lambda + \nu\widetilde{E}_\lambda\Big] \le \frac{2}{p+1}\frac{d}{dt}\left[(t_0+t)\int_{\mathbb{R}} |u|^{p+1}\Psi(t)^\lambda\,dx\right] - \int_{\mathbb{R}} |\partial_x u(t)|^2 \Psi(t)^\lambda\,dx$$
$$- (t_0 - K_3 + t)\int_{\mathbb{R}} |\partial_t u|^2 \Psi(t)^\lambda\,dx + K_2\nu\int_{\mathbb{R}} |u(t)|^{p+1}\Psi(t)^\lambda\,dx, \tag{3.6}$$

where $K_3 = \lambda^2 + \lambda + 1 + K_1\nu$. Noting that

$$\left|\widetilde{E}_\lambda - \frac{1}{2}\int_{\mathbb{R}} |u(t)|^2 \widetilde{\Phi}_\beta(t)^{-1+2\delta}\,dx\right| \le 2c_\beta^{-1+2\delta}\int_{\mathbb{R}} |\partial_t u(t)|^2 \Psi(t)^\lambda\,dx,$$

by choosing $t_0^* = \max\{K_3, 2\nu c_\beta^{-1+2\delta}\}+1$ and integrating (3.6) over $[0,t]$, we obtain the desired inequality. □

Here we prove Theorem 1.2.

Proof of Theorem 1.2 Applying Lemma 2.4 with $q = p+1 (= 2+\frac{4}{1+2\lambda} < 2+\frac{1}{\lambda})$,

$$\int_{\mathbb{R}} |u(t)|^{p+1}\Psi_*(t)^\lambda\,dx \le C\int_{\mathbb{R}} |\partial_x u(t)|^2 \Psi(t)^\lambda\,dx \left(\int_{\mathbb{R}} |u(t)|^2 \Psi(t)^\lambda\,dx\right)^{\frac{p-1}{2}}.$$

This gives the following two estimates

$$(t_0^*+t)\int_{\mathbb{R}} |u(t)|^{p+1}\Psi_*(t)^\lambda\,dx \le C\big(m_\lambda(t:t_0^*)\big)^{\frac{p+1}{2}},$$
$$\int_0^t\int_{\mathbb{R}} |u(s)|^{p+1}\Psi_*(s)^\lambda\,dx\,ds \le CY_\lambda(t)\left(\sup_{0\le s\le t}\big(m_\lambda(s:t_0^*)\big)\right)^{\frac{p-1}{2}}.$$

Therefore setting

$$\widetilde{M}_\lambda(t) = \sup_{0\le s\le t} \big(m_\lambda(s : t_0^*)\big) + Y_\lambda(t),$$

we see from Proposition 3.2 that

$$\eta^* \widetilde{M}_\lambda(t) \le m_\lambda(0 : t_0^*) + \frac{1}{p+1}\int_{\mathbb{R}} |u_0|^{p+1}\Psi_*(0)^\lambda\, dx + 2C\big(\widetilde{M}_\lambda(t)\big)^{\frac{p+1}{2}}.$$

There exists $\varepsilon_0 > 0$ such that if

$$m_\lambda(0 : t_0^*) + \frac{1}{p+1}\int_{\mathbb{R}} |u_0|^{p+1}\Psi_*(0)^\lambda\, dx \le \varepsilon$$

for some $\varepsilon \in (0, \varepsilon_0]$, then by continuity of $\widetilde{M}_\lambda(\cdot)$, we obtain the desired boundness

$$\sup_{0<t<T_*} \widetilde{M}_\lambda(\cdot) \le C\varepsilon.$$

In view of Proposition 3.1 (blowup alternative), we obtain global existence with (1.9) and (1.10); note that the weighted L^∞-estimate (1.11) follows from the use of Lemma 2.6 with (1.9) and (1.10). □

4 On Initial Data with a Slightly Stronger Assumption

Proof of Theorem 1.3 Let u be a global-in-solution of (1.1) in Theorem 1.2. Then we already have

$$\int_{\mathbb{R}} |\partial_x u(x,t)|^2 \Psi(x, t : t_0^*)^\lambda\, dx + \int_0^t\int_{\mathbb{R}} |\partial_x u(x,s)|^2 \Psi(x, s : t_0^*)^\lambda\, dx\, ds \le C\varepsilon.$$

Therefore using integration by parts, we see from (1.1) and $|\partial_x \Psi_*| \le \Psi_*^{\frac{1}{2}}$ that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} \Big(|\partial_t u|^2 + |\partial_x u|^2 - \frac{2}{p+1}|u|^{p+1}\Big)\Psi_*^{\lambda+1}\, dx \\ &\le -1\int_{\mathbb{R}} |\partial_t u|^2\Psi_*^{\lambda+1}\, dx + (\lambda+1)\int_{\mathbb{R}} |\partial_t u|^2\Psi_*^{\lambda}\, dx + (\lambda+1)(\lambda+2)\int_{\mathbb{R}} |\partial_x u|^2\Psi_*^{\lambda}\, dx.\end{aligned}$$

Recalling that $t_0^* \geq \lambda + 1$, we deduce

$$\frac{d}{dt}\int_{\mathbb{R}}\Big(|\partial_t u|^2 + |\partial_x u|^2 - \frac{2}{p+1}|u|^{p+1}\Big)\Psi_*^{\lambda+1}\,dx \\ \leq -\int_{\mathbb{R}}|\partial_t u|^2\Psi_*^{\lambda+1}\,dx + (\lambda+1)(\lambda+2)\int_{\mathbb{R}}|\partial_x u|^2\Psi_*^{\lambda}\,dx.$$

Combining the last inequality with (1.10) yields

$$\begin{aligned}\int_{\mathbb{R}}\Big(|\partial_t u|^2 + |\partial_x u|^2\Big)\Psi_*^{\lambda+1}\,dx &\leq \frac{2}{p+1}\int_{\mathbb{R}}|u|^{p+1}\Psi_*^{\lambda+1}\,dx \\ &\quad + \int_{\mathbb{R}}\Big(|u_1|^2 + |\partial_x u_0|^2\Big)\Psi_*^{\lambda+1}(0)\,dx + C\varepsilon \\ &\leq \frac{2}{p+1}\int_{\mathbb{R}}|u|^{p+1}\Psi_*^{\lambda+1}\,dx + ((t_0^*)^{\lambda+1} + C)\varepsilon. \end{aligned} \tag{4.1}$$

To control the term $\int_{\mathbb{R}}|u|^{p+1}\Psi_*^{\lambda+1}\,dx$, we use the inequality

$$\frac{1}{2}\int_{\mathbb{R}}|w|^2\Psi_*^{\lambda}\,dx \leq \int_{\mathbb{R}}|\partial_x w|^2\Psi_*^{\lambda+1}\,dx \tag{4.2}$$

(see [11, Lemma 2.2]). We see from $\partial_x(u^2\Psi_*^{\mu}) = 2u\partial_x u\Psi_*^{\mu} + \mu\frac{x}{2}u^2\Psi_*^{\mu-1}$ that

$$\begin{aligned}\|u^2\Psi_*^{\lambda+\frac{1}{2}}\|_{L^\infty(\mathbb{R})} &\leq 2\int_{\mathbb{R}}|u|\,|\partial_x u|\Psi_*^{\lambda+\frac{1}{2}}\,dx + \Big(\lambda+\frac{1}{2}\Big)\int_{\mathbb{R}}|u|^2\Big(\frac{|x|}{2}\Psi_*^{\lambda-\frac{1}{2}}\Big)\,dx \\ &\leq 2\Big(\int_{\mathbb{R}}|\partial_x u|^2\Psi^{\lambda_*+1}\,dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}}|u|^2\Psi_*^{\lambda}\,dx\Big)^{\frac{1}{2}} \\ &\quad + \Big(\lambda+\frac{1}{2}\Big)\int_{\mathbb{R}}|u|^2\Psi_*^{\lambda}\,dx \\ &\leq \Big(2\sqrt{2}+1+2\lambda\Big)\int_{\mathbb{R}}|\partial_x u|^2\Psi_*^{\lambda+1}\,dx. \end{aligned} \tag{4.3}$$

Therefore using (4.2) again, we have

$$\begin{aligned}\int_{\mathbb{R}}|u|^{p+1}\Psi_*^{\lambda+1}\,dx &\leq \|u^2\Psi_*^{\lambda+\frac{1}{2}}\|_{L^\infty(\mathbb{R})}^{\frac{p-1}{2}}\int_{\mathbb{R}}|u|^2\Psi_*^{\lambda+1-(p-1)\frac{1+2\lambda}{4}}\,dx \\ &\leq \|u^2\Psi_*^{\lambda+\frac{1}{2}}\|_{L^\infty(\mathbb{R})}^{\frac{p-1}{2}}\int_{\mathbb{R}}|u|^2\Psi_*^{\lambda}\,dx \\ &\leq C\varepsilon\Big(2\sqrt{2}+1+2\lambda\Big)^{\frac{p-1}{2}}\Big(\int_{\mathbb{R}}|\partial_x u|^2\Psi_*^{\lambda+1}\,dx\Big)^{\frac{p+1}{2}}. \end{aligned}$$

The above inequality and (4.1) imply the boundness $\int_{\mathbb{R}} |\partial_x u|^2 \Psi_*^{\lambda+1}\, dx \le C\varepsilon$ via a continuity argument. Then by (4.3) we obtain $\|u^2 \Psi_*^{\lambda+\frac{1}{2}}\|_{L^\infty(\mathbb{R})} \le C\sqrt{\varepsilon}$. The proof is complete. □

References

1. Beals, R., Wong, R.: Special Functions, A Graduate Text. Cambridge Studies in Advanced Mathematics, vol. 126. Cambridge University Press, Cambridge (2010)
2. Fujita, H.: On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo Sect. I **13**, 109–124 (1966)
3. Hayashi, N., Kaikina, E.I., Naumkin, P.I.: Damped wave equation with super critical nonlinearities. Differ. Integr. Equ. **17**, 637–652 (2004)
4. Hayashi, N., Kaikina, E.I., Naumkin, P.I.: Damped wave equation with a critical nonlinearity. Trans. Am. Math. Soc. **358**, 1165–1185 (2006)
5. Ikeda, M., Inui, T., Wakasugi, Y.: The Cauchy problem for the nonlinear damped wave equation with slowly decaying data. Nonlinear Differ. Equ. Appl. **24**(2), Art. 10, 53 pp. (2017)
6. Ikehata, R., Ohta, M.: Critical exponents for semilinear dissipative wave equations in R^N. J. Math. Anal. Appl. **269**, 87–97 (2002)
7. Matsumura, A.: On the asymptotic behavior of solutions of semi-linear wave equations. Publ. Res. Inst. Math. Sci. **12**, 169–189 (1976)
8. Nakao, M., Ono, K.: Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations. Math. Z. **214**, 325–342 (1993)
9. Nishihara, K.: L^p-L^q estimates of solutions to the damped wave equation in 3-dimensional space and their application. Math. Z. **244**, 631–649 (2003)
10. Quittner, P., Souplet, P.: Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States. Birkhäuser Advanced Texts: Basler Lehrbücher [Birkhäuser Advanced Texts: Basel Textbooks], xii+584 pp. Birkhäuser, Basel (2007)
11. Sobajima, M.: Global existence of solutions to semilinear damped wave equation with slowly decaying initial data in exterior domain. Differ. Integr. Equ. **32**, 615–638 (2019)
12. Sobajima, M., Wakasugi, Y.: Weighted energy estimates for wave equation with space-dependent damping term for slowly decaying initial data. Commun. Contemp. Math. **21**, 1850035, 30 pp. (2019)
13. Todorova, G., Yordanov, B.: Critical exponent for a nonlinear wave equation with damping. J. Differ. Equ. **174**, 464–489 (2001)
14. Weissler, F.B.: Existence and nonexistence of global solutions for a semilinear heat equation. Israel J. Math. **38**, 29–40 (1981)
15. Zhang, Q.S.: A blow-up result for a nonlinear wave equation with damping: the critical case. C. R. Acad. Sci. Paris S'er. I Math. **333**, 109–114 (2001)