Residual Based Method for Sediment Transport

P. Poullet, P. Ramsamy, and M. Ricchiuto

Abstract This contribution deals with a high order Residual Distribution (RD) numerical scheme to simulate sediment transport. The morphodynamic model that has been used, couples shallow–water equations for the fluid flow and the Exner law for the sediment part. Thus, the choice of the approach by a non-conservative hyperbolic system has been made. Different schemes have already been applied to approximate the entropic solution for several test cases [10]. The one proposed in this paper resorts to RD-method, TVD Runge Kutta [27, 31] and stabilisation upwind methods [13], with limiters. It can be viewed as an improvement of the generalized approximate Roe method [8, 14, 29] with some other good properties (Path-conservative, well-balanced...). Numerical results show the ability of the model in 1D to compute accurate solutions and to reproduce some classical test problems. The best results that we obtained, use MinMod flux limiters.

1 Introduction

This work is incorporated within the framework of the study of a sediment transport modelling. One aim of this contribution, is to provide first, an useful simulation tool, in the context of a 1D space-time problem. Considering the geophysical aspect,

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the sediment transport can be divided into several categories, in this paper we will be interested in bedload transport. The governing equations consist of a system of three equations, modelling the interaction between fluid and sediment in a river. The hydrodynamical component is described by the shallow–water equations (SWE) and the morphodynamical component, is given by a solid transport discharge due to the Exner law with the Grass formula. In this way, the model can be depicted by a non-conservative and non-linear hyperbolic system, and our main objective is to seek numerical solutions in accordance with these specific aspects. In particular, a good scheme for that model, must comply with the well-balanced property and the path-conservative character. Moreover, as the fluid interacts very weakly with the sediment and characteristic velocities being such a different magnitude, long time simulation and high order accuracy (at least second order) are needed.

To treat hyperbolic problem, finite volumes are the most popular methods as the Godunov scheme. For conservation laws, first attempts to propose approximate solver for hyperbolic systems in non-conservative form, were due to Roe [28]. After that, several approaches have been introduced like approximate finite volume Roe with characteristic flux scheme [14], schemes based on exact or incomplete Riemann solvers [5], WENO schemes [32], generalized Roe methods with or without WENO reconstruction [8–10], or kinetic schemes [24].

In parallel to finite volume, another family of methods called Residual Distribution (RD) methods that emerged from Roe's works in the 80s, is used in this paper [11, 28]. Combining advantages from finite volume and finite element methods, their construction allows them nowadays to be monotone, conservative, well-balanced and easily high order accurate [1, 2, 19, 20, 27, 31].

In this contribution, a RD scheme, viewed as a recast of the approximate Riemann solver, called FV-Roe approximation scheme is proposed with simulations. Often used to solve the shallow–water problem, the residual based method is adapted here to solve the coupling problem with sediment transport. And for that the use of a TVD Runge Kutta procedure, but also upwinding and a flux limiter procedures, have been added, to compute weak entropic solution, considering Lax entropy.

To introduce our scheme, the present paper is organized according to the following outline: In Sect. 2, the governing equations are introduced. In Sect. 3, our scheme is proposed. And finally, numerical tests are given, in order to show how accurate the scheme is, but also its well-balanced preserving aspect, for the lake at rest.

2 A Sediment Transport in a Shallow Water

In the context of the study of sediment transport in shallow water, several morphodynamical models can be found in the litterature depending on the way of considering the displacement mode. In the case of bedload transport, among several models [12, 18, 21, 30], the discharge is written by the Grass formula [15] for simplicity. It involves that the critical shear stress is neglected, then the sediment is viewed as starting its own movement as soon as the fluid starts to move. About the hydrody-

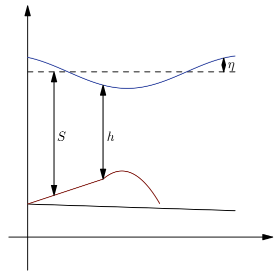

Fig. 1 A sediment layer in a shallow water

namical part, shallow–water equations are considered. The result is an hyperbolic system of three equations.

The governing system for the 1D space-time problem, is as follows

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial q}{\partial x} = 0 \\ \dfrac{\partial q}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{q^2}{h} + \dfrac{1}{2}gh^2\right) = gh\dfrac{\partial S}{\partial x} - ghS_f \\ \dfrac{\partial S}{\partial t} - \zeta\dfrac{\partial q_b}{\partial x} = 0, \end{cases} \tag{1}$$

where x denote the horizontal variable at the axis of the channel and t the time variable (see [10]). By $h(x,t)$ we denote the height of the water column, $q(x,t)$ is the discharge, g is the gravity constant, S_f models the friction term and ζ a parameter linked to the sediment porosity ($\zeta = 1/(1-\rho_0)$ with ρ_0 the porosity). The third equation of the model describes the sediment transport by the expression of the sediment volume equation, q_b being the solid transport discharge obtained by Grass formula (here, $q_b = A_g(q/h)^3$ with A_g related to the interaction between the fluid and the sediment). The variable $S(x,t)$ is the distance from a given reference level to the bottom layer. A schematic description is provided (see Fig. 1), η denoting the extra height of the water column.

Neglecting S_f in this study, a classical approach is to treat the system (1) as a hyperbolic system with a non-conservative term B:

$$\frac{\partial W}{\partial t} + \mathcal{A}(W)\frac{\partial W}{\partial x} = 0, \tag{2}$$

with $W = W(x,t)$ and $(x,t) \in \mathbf{R} \times \mathbf{R}^+$. In fact, the vector of unknowns is

$$W = (h,\ q,\ S)^T\ ,$$

F is the flux function and, $\mathcal{A}(W)$ equals to the difference between the Jacobian matrix of F and the non-conservative part:

$$\mathcal{A}(W) = \frac{\partial F}{\partial W}(W) - B(W).$$

More precisely,

$$F(W)=\begin{pmatrix} q \\ \dfrac{q^2}{h}+\dfrac{1}{2}gh^2 \\ -\zeta q_b \end{pmatrix},\quad B=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & gh \\ 0 & 0 & 0 \end{pmatrix},\quad A(W)=\begin{pmatrix} 0 & 1 & 0 \\ -\dfrac{q^2}{h^2}+gh & 2\dfrac{q}{h} & 0 \\ -\zeta\dfrac{\partial q_b}{\partial h} & -\zeta\dfrac{\partial q_b}{\partial q} & 0 \end{pmatrix},$$

$$\text{and } \mathcal{A}(W)=\begin{pmatrix} 0 & 1 & 0 \\ -\dfrac{q^2}{h^2}+gh & 2\dfrac{q}{h} & -gh \\ -\zeta\dfrac{\partial q_b}{\partial h} & -\zeta\dfrac{\partial q_b}{\partial q} & 0 \end{pmatrix}.$$

3 A Residual Based Predictor-Corrector Upwind Discretization for 1D Space-Time Sediment Transport

Following the introduction, the scheme built in this work, can be viewed as an high order recast of the approximate FV-Roe as introduced by Ghidaglia *et al.* in [14]. Instead of using a high order reconstruction, our approach exploits the residual based approach discussed in [26, 27].

To present the scheme, one neglects the friction term, and one considers the hyperbolic system with a non-conservative term. Also, to simplify writing in this section, in integrals the term dx will be omitted (all are space integrals).

One will consider the intervals (computing cells) defined by $I_i=[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}]$, $i\in\mathbb{Z}$, but also the intervals $I_{i+\frac{1}{2}}=[x_i,x_{i+1}]$. Let the step $\Delta x_i=x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}$ and that $x_{i-\frac{1}{2}}=\sum_{k=1}^{i-1}\Delta x_k$ is the intercell located at the middle of $I_{i-1}\cup I_i$. Δt is the time step and $t^n=n\Delta t$. As usual, we denote by W_i^n the approximate mean value of W in node x_i and at time t^n. The RD procedure consists of making the computation of the residual, called global, on a cell I_i and then distributing fractions of this quantity to each of its verteces. Under these assumptions, one gets for a linear approximation of $W(x,t^n)$,

$$W_i^n:=\frac{1}{\Delta x_i}\int_{I_i}W(x,t^n)\approx W(x_i,t^n).$$

Then, the residual–based predictor corrector method developed in [26, 27] can be written as follows

$$W_i^{n+1}=W_i^n-\frac{\Delta t}{\Delta x}\left\{\frac{1}{2}\Theta_i(W^n)+\frac{1}{2}\Theta_i(W^*)\right\}+\Psi_{i-\frac{1}{2}}+\Psi_{i+\frac{1}{2}},\tag{3}$$

with $\Theta_i(W)$ corresponds to the approximate FV-Roe scheme discrete evolution operator (without the additional term)

$$\Theta_i(W) = \mathcal{P}^+(\mathcal{A}_{i-\frac{1}{2}})\phi_{i-\frac{1}{2}} + \mathcal{P}^-(\mathcal{A}_{i+\frac{1}{2}})\phi_{i+\frac{1}{2}}.$$

Concerning the non-conservative terms, one proceeds by linearization along the path joining W_i and $W_{i\pm1}$, to compute an approximate value of $W_{i\pm\frac{1}{2}} = \frac{1}{2}(W_i + W_{i\pm1})$.

We then denote by $A_{i\pm\frac{1}{2}} = A(W_{i\pm\frac{1}{2}})$, $B_{i\pm\frac{1}{2}} = B(W_{i\pm\frac{1}{2}})$. Also, the fluctuations are:

$$\phi_{i-\frac{1}{2}} = F_i - F_{i-1} - B_{i-\frac{1}{2}}(W_i - W_{i-1}) \text{ and } \phi_{i+\frac{1}{2}} = F_{i+1} - F_i - B_{i+\frac{1}{2}}(W_{i+1} - W_i),$$

and the projectors,

$$\mathcal{P}^\pm(\mathcal{A}_{i\mp\frac{1}{2}}) = \frac{1}{2}(I \pm \text{sign}\big(\mathcal{A}_{i\mp\frac{1}{2}}\big)),$$

with the sign of a matrix computed by eigen-decomposition

$$\text{sign}(\mathcal{A}_{i\mp\frac{1}{2}}) = \mathcal{K}_{i\mp\frac{1}{2}}\text{sign}(\mathcal{L}_{i\mp\frac{1}{2}})\mathcal{K}^{-1}_{i\mp\frac{1}{2}}.$$

The matrix $\mathcal{K}$ gathering the eigenvectors of the Roe matrix (along the path), sign($\mathcal{L}$) is the diagonal matrix whose coefficients are the sign of the eigenvalues (see [23] for example). We denote by W^*, a predicted value of the solution that has been obtained, from the upwind scheme,

$$W_i^* = W_i^n - \frac{\Delta t}{\Delta x_i}\Theta_i(W^n).$$

Actually, if we introduce a parallel approach using Galerkin method by seeking a piecewise linear solution, in a domain $\Omega =]0, L[$,

$$W_h(x,t) = \sum_i \varphi_i(x) W_i(x_i, t), \qquad \text{and} \quad F_h = F(W_h),$$

with φ_i representing the standard Lagrange basis fonctions associated to the node x_i. One replaces the unknown solution by its approximate solution by finite element from the variational form

$$\int_\Omega \varphi_i \partial_t W_h + \int_\Omega \varphi_i \partial_x F_h - \int_\Omega \varphi_i B(W_h)\partial_x W_h = 0. \tag{4}$$

And one can notice that by linear approximation,

$$\int_{\Omega} \varphi_i \partial_x F_h = \int_{I_{i-\frac{1}{2}}} \varphi_i \partial_x F_h + \int_{I_{i+\frac{1}{2}}} \varphi_i \partial_x F_h \approx \frac{F_i - F_{i-1}}{2} + \frac{F_{i+1} - F_i}{2}. \tag{5}$$

Hence, the resulting non-stabilized method reads

$$\int_{\Omega} \varphi_i \partial_t W_h + \frac{1}{2}\phi_{i+\frac{1}{2}} + \frac{1}{2}\phi_{i-\frac{1}{2}} = 0, \tag{6}$$

with the approximation of the first term $\Delta x_i \dfrac{dW_i}{dt}$, one recovers the mass lumping process.

But as it is known that Galerkin method suffers from lack of stability, one adds a residual based stabilization in the spirit of the streamline upwind method or Streamline Upwind Petrov Galerkin (SUPG) [17, 26, 27].

For a node x_i, the stabilization operator $\mathcal{S}_i$ reads

$$\mathcal{S}_i = \int_{\Omega} \mathcal{A}(W_h)\partial_x \varphi_i \, \mathcal{T} \, \tilde{r} \quad \text{with} \quad \tilde{r} = \partial_t W_h + \partial_x F_h - B(W_h)\partial_x W_h,$$

the matrix $\mathcal{T}$ being a scaling factor guaranteeing the uniform boundedness of the stabilization w.r.t. the residual. As before, explicit computable expressions are obtained when introducing the linear finite element approximation and introducing appropriate mean value linearizations of the matrices that appear. Recalling that for a linear approximation $\partial_x \varphi_i \big|_{I_{i\pm\frac{1}{2}}} = \mp 1/\Delta x_{i\pm\frac{1}{2}}$, the stabilization term can be evaluated as

$$\mathcal{S}_i = \mathcal{A}_{i+\frac{1}{2}} \mathcal{T}_{i+\frac{1}{2}} \Big(\int_{I_{i+\frac{1}{2}}} \partial_t W_h + \phi_{i+\frac{1}{2}} \Big) - \mathcal{A}_{i-\frac{1}{2}} \mathcal{T}_{i-\frac{1}{2}} \Big(\int_{I_{i-\frac{1}{2}}} \partial_t W_h + \phi_{i-\frac{1}{2}} \Big) \tag{7}$$

In one dimension, a typical definition of the scaling matrix $\mathcal{T}$, also used here, being the following

$$\mathcal{T}_{i\pm\frac{1}{2}} = \frac{1}{2}\Delta x_{i\pm\frac{1}{2}} \mathrm{sign}(\mathcal{A}_{i\pm\frac{1}{2}}) \mathcal{A}^{-1}_{i\pm\frac{1}{2}} = \frac{\Delta x_{i\pm\frac{1}{2}}}{2} |\mathcal{A}_{i\pm\frac{1}{2}}|^{-1}, \tag{8}$$

the complete semi-discrete (in space) equations read

$$\int_{I_{i-\frac{1}{2}}} \Big(\varphi_i + \mathrm{sign}(\mathcal{A}_{i-\frac{1}{2}})\Big) \partial_t W_h + \int_{I_{i+\frac{1}{2}}} \Big(\varphi_i - \mathrm{sign}(\mathcal{A}_{i+\frac{1}{2}})\Big) \partial_t W_h$$
$$= -\Big(P^+(\mathcal{A}_{i-\frac{1}{2}})\phi_{i-\frac{1}{2}} + P^-(\mathcal{A}_{i+\frac{1}{2}})\phi_{i+\frac{1}{2}}\Big) . \tag{9}$$

As it has been explained in [27], a simplified stabilization step can be added in the Galerkin process with the Runge-Kutta scheme without any loss of accuracy (here, an explicit RK2 of second order accuracy is used). Thus, the residual

$$r^{n+1} := \frac{W_h^{n+1} - W_h^n}{\Delta t} + \frac{1}{2}(\partial_x F_h - B_h \partial_x W_h)^* + \frac{1}{2}(\partial_x F_h - B_h \partial_x W_h)^n \tag{10}$$

can be replaced with a simplified residual

$$r^* := \frac{W_h^* - W_h^n}{\Delta t} + \frac{1}{2}(\partial_x F_h - B_h \partial_x W_h)^* + \frac{1}{2}(\partial_x F_h - B_h \partial_x W_h)^n, \tag{11}$$

to design the scheme by computing

$$\int_\Omega \varphi_i r^{n+1} + \int_{I_{i-\frac{1}{2}} \cup I_{i+\frac{1}{2}}} A(W_h)\partial_x \varphi_i \mathcal{T} r^* = 0. \tag{12}$$

Using an explicit scheme by performing mass lumping in the predictor step (6), and with midpoint rule to evaluate the integrals, one obtains after few calculations and recast Eq. (3), that one recalls

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\left\{\frac{1}{2}\Theta_i(W^n) + \frac{1}{2}\Theta_i(W^*)\right\} + \Psi_{i-\frac{1}{2}} + \Psi_{i+\frac{1}{2}},$$

with the aim to define $\Psi_{i\pm\frac{1}{2}}$ as:

$$\begin{aligned}\psi_{i\pm\frac{1}{2}} =& \frac{\Delta t}{\Delta x}\int_{I_{i\pm\frac{1}{2}}}\Big(\varphi_i - \frac{1}{2}(I \mp \text{sign}(\mathcal{A}_{i\pm\frac{1}{2}}))\Big)\frac{W_h^* - W_h^n}{\Delta t} \\ \approx& \frac{1}{2}\Big(W_i^* - W_i^n - (I \mp \text{sign}(\mathcal{A}_{i\pm\frac{1}{2}}))(W_{i\pm\frac{1}{2}}^* - W_{i\pm\frac{1}{2}}^n)\Big).\end{aligned}$$

However, comparing to the FV-Roe scheme, the additional terms $\Psi_{i\pm\frac{1}{2}}$ that derive from residual stabilization are of second order, that cannot always match with non regular solution. To avoid their effects accross discontinuous features, cell based limiters have been introduced and finally,

$$\Psi_{i\pm\frac{1}{2}} = \delta_{i\pm\frac{1}{2}}\psi_{i\pm\frac{1}{2}}$$

with $\delta_{i\pm\frac{1}{2}}$ computed by means of a standard finite volume limiters using values of the MUSCL MinMod flux limiter function [7].

4 Numerical Results

We present the numerical results of several reference problems. In this section, we aim at validating our numerical scheme and highlighting its characteristic properties against classical tests. The first test consists in proving an approximate well-balanced property of our scheme. We then underline the ability of our scheme to simulate a parabolic sediment transport until a discontinuous solution is obtained. We also prove its high order accuracy by means of an order test problem that has been discussed in [10]. Finally, we prove that our scheme is capable of faithfully reproducing a dam-break problem over a wet bottom topography [4].

4.1 Test of Well-Balanced Property

To check the property, the following numerical test is used [25]. It deals with the ability of the scheme to reproduce the behaviour of the steady state. Thus, if the numerical scheme is well-balanced, a small difference should be observed between the initial solution and the solution obtained at the final instant.

For this, the interval [0, 10] is assumed as the physical domain and the simulation is performed up to $T = 0.5\,\text{s}$ with 100 and 200 cells. A discontinuity in the bed is assumed as the initial condition, so the thickness of the sediment layer is considered,

$$z_b(x,0) = \begin{cases} 4 & \text{if} \quad 4 \leq x \leq 8 \\ 0 & \text{elsewhere} \end{cases} \tag{13}$$

and

$$q(x,0) = 0, \quad h(x,0) + z_b(x,0) = 10.$$

The results in Table 1, show that the scheme preserves the approximate well-balanced property [22]. The differences between the initial solution and the solution at final time are very small. More precisely, the ratio is of around 1.5 between the doubled gridpoints and the coarser one. The accuracy seems to be of order 1.5 (Table 1).

Table 1 Accuracy of the scheme for the well-balanced test property

Precision	L^2-error h	Ratio	L^2-error q	Ratio
100	8.6052×10^{-16}	-	7.1712×10^{-15}	-
200	5.2050×10^{-16}	1.65	5.8743×10^{-15}	1.22

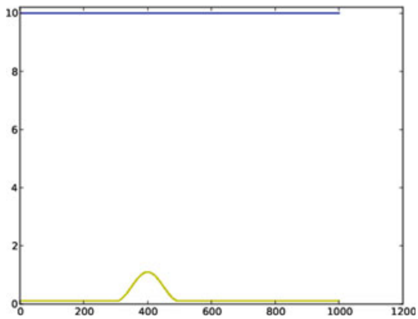

Fig. 2 The dune at initial time

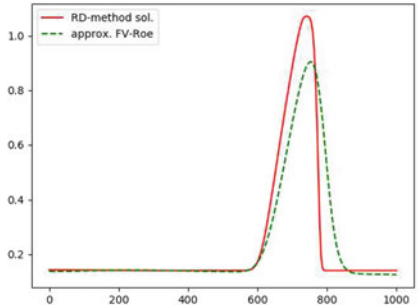

Fig. 3 z_b at 700 s for RD scheme and FV-Roe

4.2 A 1D Space-Time Dune Test Case

To verify the shock capturing property, the classical transport of parabolic sediment layer has been taken. For this case proposed in [7, 10, 16], the interval [0, 1000] is assumed as the physical domain and a strong interaction between the fluid and the sediment is taken ($A_g = 1$). The initial conditions are given as follows (see Fig. 2): the bathymetry is of 10 m, $q(x, t) = 10$ for the discharge of the fluid, $h(x, t) = 10 - z_b(x, t)$ for the water column height with the sediment layer thickness ,

$$z_b(x, 0) = \begin{cases} 0.1 + \sin^2\left(\frac{\pi(x-300)}{200}\right) & \text{if} \quad 300 \leq x \leq 500 \\ 0.1 & \text{elsewhere} \end{cases}. \qquad (14)$$

Numerical solutions are generated up to $T = 700$ s. As shown in Fig. 3, the solution of our scheme (solid line) is compared to the solution of the FV-Roe approximation (dotted line), and we note that the RD scheme is less diffusive than the first and the shock is more visible (thanks to the limiters). Therefore, the results show that our numerical scheme seems to satisfy the shock capturing property for this test. It can be noted that it is also stable, more precisely the RD scheme does not generate oscillations even though the interaction between the fluid and the sediment is strong (somewhat artificial with the initial conditions chosen [16]). The comparison between the evolution of the dune between the RD scheme and the SRNH scheme (a second-

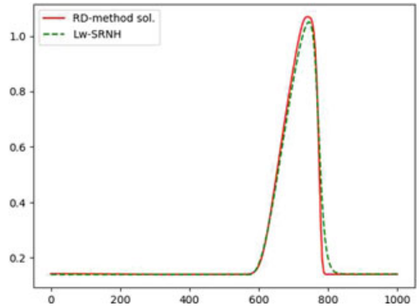

Fig. 4 z_b at 700 s for RD scheme and SRNH

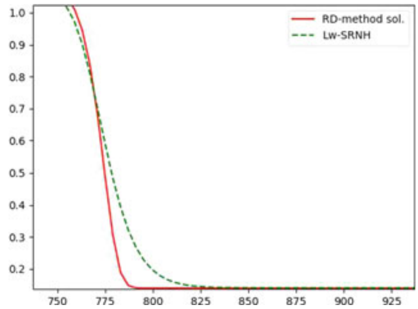

Fig. 5 z_b at 700 s for RD scheme and SRNH: zoom on the shock

order scheme extracted from [6], see Figs. 4 and 5) seems to confirm that our scheme is of the second order and is less diffusive than the other.

4.3 A Test of Order

To check the accuracy of the numerical scheme, let us introduce the following one-dimensional problem for which initial conditions are:

$$q(0, x) = 0, \qquad h(0, x) = 2 - 0.1 \exp(-x^2), \qquad z_b(0, x) = 0.1 - 0.01 \exp(-x^2). \tag{15}$$

This test problem has been considered in a previous work by Castro Diaz *et al.* [10], and as the exact solution is unknown, we use a reference solution obtained by a fine mesh of 5120 volumes (as it has been done in their work) with a medium interaction ($A_g = 0.3$).

Numerically, the results for h, q and the sediment layer thickness (z_b) and different error norms, show that the second order of accuracy seems to be obtained (see Tables 2, 3 and 4).

Table 2 Accuracy of the RD-scheme with distributed residual scheme for h

# gridpts	L^1-error	Order	L^2-error	Order	L^∞-error	Order
20	5.628×10^{-3}	-	8.600×10^{-3}	-	2.088×10^{-2}	-
40	2.421×10^{-3}	1.22	4.159×10^{-3}	1.05	1.086×10^{-2}	0.94
80	7.918×10^{-4}	1.61	1.408×10^{-3}	1.56	4.036×10^{-3}	1.43
160	2.072×10^{-4}	1.93	3.815×10^{-4}	1.88	1.211×10^{-3}	1.74
320	5.257×10^{-5}	1.98	9.693×10^{-5}	1.98	3.159×10^{-4}	1.94
640	1.314×10^{-5}	2.00	2.423×10^{-5}	2.00	7.927×10^{-5}	1.99

Table 3 Accuracy of the RD-scheme with distributed residual scheme for q, the discharge

# gridpts	L^1-error	Order	L^2-error	Order	L^∞-error	Order
20	2.350×10^{-2}	-	3.771×10^{-2}	-	9.248×10^{-2}	-
40	1.042×10^{-2}	1.17	1.835×10^{-2}	1.04	4.783×10^{-2}	0.95
80	3.435×10^{-3}	1.60	6.205×10^{-3}	1.56	1.783×10^{-2}	1.42
160	8.995×10^{-4}	1.93	1.681×10^{-3}	1.88	5.349×10^{-3}	1.73
320	2.281×10^{-4}	1.98	4.269×10^{-4}	1.98	1.397×10^{-3}	1.94
640	5.697×10^{-5}	2.00	1.067×10^{-4}	2.00	3.510×10^{-4}	1.99

Table 4 Accuracy of the RD-scheme with distributed residual scheme for z_b, the height of the sediment layer

# gridpts	L^1-error	Order	L^2-error	Order	L^∞-error	Order
20	2.969×10^{-5}	-	5.443×10^{-5}	-	1.423×10^{-4}	-
40	1.265×10^{-5}	1.23	2.739×10^{-5}	0.99	8.168×10^{-5}	0.80
80	4.469×10^{-6}	1.50	9.014×10^{-6}	1.60	2.861×10^{-5}	1.51
160	1.200×10^{-6}	1.90	2.536×10^{-6}	1.83	8.578×10^{-6}	1.74
320	3.019×10^{-7}	1.99	6.409×10^{-7}	1.98	2.217×10^{-6}	1.95
640	7.516×10^{-8}	2.01	1.596×10^{-7}	2.00	5.571×10^{-7}	1.99

4.4 A Dam Break Test over a Wet Bottom Topography

In this classical test case ([4]) a dam break is considered over a flat wet bottom, in a channel of 10m long. A low interaction between the fluid and the sediment is taken ($A_g = 0.005$), and the initial conditions are,

$$h(x,0) = \begin{cases} 2 & \text{if } \ x \leq 5m \\ 0.125 & \text{if } \ x > 5m \end{cases}, \tag{16}$$

$q(x,0) = 0m/s$ and the bottom topography $z_b = 0m$. The numerical test is performed until $T = 1s$. The results confirm that our scheme keep the stability as attempted for the coupled approaches (see Fig. 6, 7), in comparison the approach by splitting [3]. The accuracy of our RD scheme is of course better than those obtained by the FV-Roe scheme (8). More precisely, as expected and considering

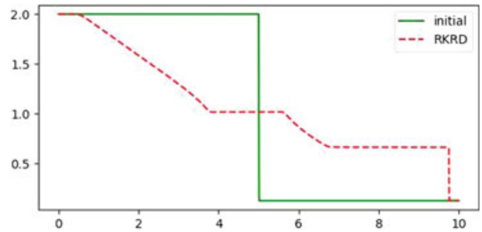

Fig. 6 Dam break over a wet bottom: h at $t = 1s$ for RD scheme with 1000 cells

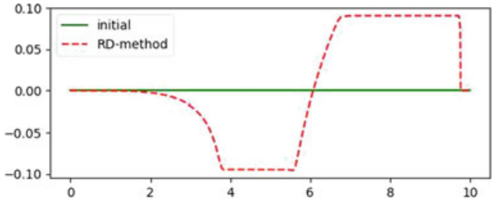

Fig. 7 Dam break over a wet bottom: z_b at $t = 1s$ for RD scheme with 1000 cells

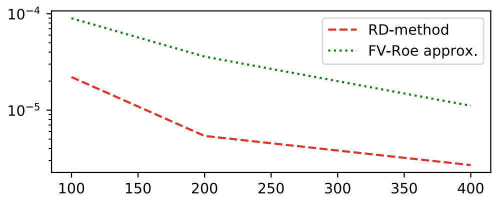

Fig. 8 Dam break over a wet bottom: error analysis for RD scheme

the bottom firstly (Fig. 6) the solution computed with the RD scheme forms two plates without any oscillation (in a critical region includes the interval [4, 6] with $-0.1 < z_b < -0.05$, and in a region where $x > 6.5$ with $z_b > 0.5$). Then, for the free surface (Fig. 7), the solution of the RD scheme (dashed line) is decreasing along the time, and a plate is reached without oscillation in the critical region [4, 6]. For both unknowns, a comparison with the FV-Roe approximation (dotted line) is proposed, underlying that the results are quite similar. However the RD scheme is more accurate for the bottom (see Fig. 8 for which the L^2 errors are produced from a referent solution computed with 2000 elements grid).

5 Concluding Remarks

This contribution proposed a new predictor-corrector scheme, based on the residual, to simulate a sediment transport problem. Numerical tests have highlighted its high order accuracy, its approximate well-balancedness property and its stability for some test problems. In particular, the solutions obtained for the problem of dam-break with wet topography are sharp. Work is in progress to take into account dry bottoms, for example. The extension of this model to take into account a coastal configuration (2D physical domain), by parallel programming, will also be done in a future contribution.

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