# Dynamics of a Chemostat-Type Model with Impulsive Effects in a Polluted Karst Environment

Jianjun Jiao(✉)

School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang 550025, People's Republic of China

**Abstract.** In this paper, we present a chemostat-type model with impulsive effects in a polluted karst environment. The globally asymptotically stable sufficient condition are gained for a microorganism-extinction periodic solution. System permanent condition are also presented. The results are illustrated by simulations.

**Keywords:** Chemostat-type model · Impulsive diffusing · Pulse inputting · Polluted karst environments · Extinction

## 1 Introduction

There are wide underground rivers in the underground rock gaps of karst areas. It easy for pollutants and nutrient of soils to come into aquifers for the thin soil layers of karst areas [1]. Many authors [2–4] employed SDE,PDE,ODE and IDE to study chemostat systems. References [5,6] devoted themselves to the effects of toxicants on population. But there are few authors devoted themselves to karst environmental chemostat system with impulsive diffusing and pulse inputting. We mark a notation as $N = nT, N + L = (n + l)T$.

## 2 The Model

In this paper, we present a chemostat-type model with impulsive effects in a polluted karst environment

Supported by NNSFC (No.11791019,11361014), the Science Technology Foundation of Guizhou Education Department (No.QJK[KY]2018019).

© ICST Institute for Computer Sciences, Social Informatics and Telecommunications Engineering 2021
Published by Springer Nature Switzerland AG 2021. All Rights Reserved
H. Song and D. Jiang (Eds.): SIMUtools 2020, LNICST 370, pp. 63–78, 2021.
https://doi.org/10.1007/978-3-030-72795-6_6

$$\begin{cases} \left.\begin{aligned} &\frac{dx_1}{dt} = D_1(x_1^0 - x_1), \\ &\frac{dx_2}{dt} = D_2(x_2^0 - x_2) \\ &\qquad -\frac{1}{\delta} \times \frac{\eta x_2 Y}{\alpha + x_2}, \\ &\frac{dY}{dt} = -D_2 \\ &\qquad +\frac{\eta x_2 Y}{\alpha + x_2} - \beta c_1 Y, \\ &\frac{dc_1}{dt} = f_1 c_2 - (g_1 + m_1) c_1, \\ &\frac{dc_2}{dt} = -h_1 c_2, \end{aligned}\right\} t \neq (N+L), t \neq (N+1), \\ \left.\begin{aligned} &x_1^+ = x_1, \\ &x_2^+ = x_2, \\ &Y^+ = Y, \\ &c_1^+ = c_1, \\ &c_2^+ = (1-\theta_1) c_2, \end{aligned}\right\} t = (N+L), \\ \left.\begin{aligned} &x_1^+ = (1-d)x_1 + dx_2 + \theta_2, \\ &x_2^+ = dx_1(1-d)x_2, \\ &Y^+ = Y, \\ &c_1^+ = (1-h_3)c_1, \\ &c_2^+ = c_2 + \theta, \end{aligned}\right\} t = (N+1), \end{cases} \tag{1}$$

where system (1) is constructed by two patches, patch (1) is a non-polluted environment and patch (2) is a polluted environment. The meanings of the variables and parameters can be consulted from reference [3–5].

## 3 The Foundation

If $Y = 0$, there are two subsystems of system (1) as

$$\begin{cases} \left.\begin{aligned} &\frac{dx_1}{dt} = D_1(x_1^0 - x_1), \\ &\frac{dx_2}{dt} = D_2(x_2^0 - x_2), \end{aligned}\right\} t \neq N, \\ \left.\begin{aligned} &x_1^+ = (1+d)x_1 + dx_2 + \theta_1, \\ &x_2^+ = dx_1 + (1-d)x_2, \end{aligned}\right\} t = N, \end{cases} \tag{2}$$

and

$$\left\{\begin{array}{l} \left.\begin{array}{l} \frac{dc_1}{dt} = f_1 c_2 - (g_1 + m_1)c_1, \\ \frac{dc_2}{dt} = -h_1 c_2, \end{array}\right\} t \neq N, t \neq (N+1), \\ \left.\begin{array}{l} \triangle c_1 = 0, \\ \triangle c_2 = -\theta_1 c_2, \end{array}\right\} t = (N+L), \\ \left.\begin{array}{l} \triangle c_1 = -\theta_3 c_1, \\ \triangle c_2 = \theta, \end{array}\right\} t = (N+1). \end{array}\right. \tag{3}$$

*i*) Integrating on system (2)

$$x_i(t) = x_i^0 - [x_i^0 - x_i(nT^+)]e^{-D_i(t-nT)}. \tag{4}$$

Then,

$$\left\{\begin{array}{l} x_1((n+1)T^+) = (1-d)e^{-D_1T}x_1(nT^+) + de^{-D_2T}x_2(nT^+) \\ \qquad\qquad +(1-d)(1-e^{-D_1T})x_1^0 + d(1-e^{-D_2T})x_2^0 + \theta_1, \\ x_2((n+1)T^+) = de^{-D_1T}x_1(nT^+) + (1-d)e^{-D_2T}x_2(nT^+) \\ \qquad\qquad +d(1-e^{-D_1T})x_1^0 + (1-d)(1-e^{-D_2T})x_2^0. \end{array}\right. \tag{5}$$

From system (5), we gain

$$\left\{\begin{array}{l} x_1^* = \frac{K_1 + K_1'}{K_2 + K_2'}, \\ x_2^* = \frac{K_3 + K_3'}{K_4 + K_4'}, \end{array}\right. \tag{6}$$

where $K_1 = [(1-d)(1-e^{-D_1T}) - (1-2d)(1-e^{-D_2T})e^{-D_1T}]x_1^0 - d(1-e^{-D_2T})x_2^0 + (1-e^{-D_2T}$, $K_1' = de^{-D_2T})\theta_1$, $K_2 = 1-(1-d)e^{-D_1T} - (1-d)e^{-D_2T}$ $K_2' = -(1-2d)e^{-(D_1+D_2)T}$, $K_3 = -d(1-e^{-D_1T})x_1^0 + [(1-d)(1-e^{-D_2T}) - (1-2d)(1-e^{-D_1T})e^{-D_2T}]x_2^0$, $K_3' = de^{-D_1T}\theta_1$, and $K_4 = 1-(1-d)e^{-D_2T} - (1-d)e^{-D_1T}$, $K_4' = -(1-2d)e^{-(D_2+D_1)T}$.

The map $G: R_+^2 \to R_+^2$ coming from system (5) is presented as

$$\left\{\begin{array}{l} G_1(x) = G_{11}x_1 + G_{12}x_2 + G_{13}, \\ G_2(x) = G_{21}x_1 + G_{22}x_2, \end{array}\right. \tag{7}$$

where $x = (x_1, x_2) \in R_+^2$, and $G_{11} = (1-d)e^{-D_1T}, G_{12} = de^{-D_2T}, G_{13} = (1-d)(1-e^{-D_1T})x_1^0 + d(1-e^{-D_2T})x_2^0 + \theta_1, G_{21} = de^{-D_1T}, G_{22} = (1-d)e^{-D_2T}, G_{23} = +d(1-e^{-D_1T})x_1^0 + (1-d)(1-e^{-D_2T})x_2^0$.

Being similar to reference [7], two lemmas are gotten.

**Lemma 1.** $G^n(x) \to (x_1^*, x_2^*)$ (as $n \to \infty$) holds for $(x_1, x_2)$ of system (7).

**Lemma 2.** The solution $(\widetilde{x_1}, \widetilde{x_2})$, which is defined as

$$\begin{cases} \widetilde{x_1} = x_1^0 - (x_1^0 - x_1^*)e^{-D_1(t-nT)}, N < t \leq (N+1), \\ \widetilde{x_2} = x_2^0 - (x_2^0 - x_2^*)e^{-D_2(t-nT)}, N < t \leq (N+1), \end{cases} \tag{8}$$

is globally stable.

*ii*) Integrating on system (3), we have

$$\begin{cases} c_1((n+1)T^+) = c_1(nT^+)e^{-(g_1+m_1)T} \\ \qquad + \dfrac{c_2(nT^+)f_1(e^{-(g_1+m_1)(1-l)T} - e^{-(h_1-g_1-m_1)lT-(g_1+m_1)(1-l)T})}{(h_1-g_1-m_1)} \\ \qquad + \dfrac{(1-\theta_1)c_2(nT^+)f_1(e^{-h_1 lT} - e^{-(h_1-g_1-m_1)T)})}{(h_1-g_1-m_1)}, \\ c_2((n+1)T^+) = (1-\theta_1)e^{-h_1 T}c_2(nT^+) + \theta. \end{cases} \tag{9}$$

From (9), we have $c_1^*$ and $c_2^*$ with

$$\begin{cases} c_1^* = \dfrac{1-\theta_3}{1-(1-\theta_3)e^{-(g_1+m_1)T}} \\ \qquad \times [\dfrac{\theta f_1(e^{-(g_1+m_1)(1-l)T} - e^{-(h_1-g_1-m_1)lT-(g_1+m_1)(1-l)T})}{(h_1-g_1-m_1)(1-(1-\theta_1)e^{-h_1 T})(1-e^{-(g_1+m_1)T})} \\ \qquad + \dfrac{(1-\theta_1)\theta f_1(e^{-h_1 lT} - e^{-(h_1-g_1-m_1)T)})}{(h_1-g_1-m_1)(1-(1-\theta_1)e^{-h_1 T})(1-e^{-(g_1+m_1)T})}], \\ c_2^* = \dfrac{\theta}{1-(1-\theta_1)e^{-h_1 T}}. \end{cases} \tag{10}$$

Obviously, for (9), $(c_1^*, c_2^*)$ is globally asymptotically stable.

**Lemma 3.** The globally asymptotically stable $(\widetilde{c_1}, \widetilde{c_2})$ of (3) exists. and $\widetilde{c_1}, \widetilde{c_2})$ are in reference [8], $c_1^*, c_2^*$ are as (11), and $c_1^{**} = c_1^* e^{-(g_1+m_1)lT} + \frac{f_1 c_2^*(1-e^{-(h_1-g_1-m_1)lT})}{(h_1-g_1-m_1)}$, $c_2^{**} = (1-\theta_1)e^{-h_1 lT}c_2^*$.

**Remark 4.** There exist positive constants $m_0, M_0, m_e, M_e$, it is easy to get $m_0 \leq c_o(t) \leq M_0$ and $m_e \leq c_e(t) \leq M_e$.

Similar to reference [7], we can obtain

**Lemma 5.** There exists a positive constant $\lambda > 0$, we can easily have $x_1(t) \leq [\delta(D_1 x_1^0 + D_2 x_2^0) + \frac{\theta \exp(\lambda T)}{\exp(\lambda T)-1}]e^{\lambda T_1}$, $x_2(t) \leq [\delta(D_1 x_1^0 + D_2 x_2^0) + \frac{\theta \exp(\lambda T)}{\exp(\lambda T)-1}]e^{\lambda T_1}$, $Y(t) \leq \delta(D_1 x_1^0 + D_2 x_2^0) + \frac{\theta \exp(\lambda T)}{\exp(\lambda T)-1}$, $c_0(t) \leq \delta(D_1 x_1^0 + D_2 x_2^0) + \frac{\theta \exp(\lambda T)}{\exp(\lambda T)-1}$, and $c_e(t) \leq \delta(D_1 x_1^0 + D_2 x_2^0) + \frac{\theta \exp(\lambda T)}{\exp(\lambda T)-1}$.

## 4 Dynamical Analysis

**Theorem 1.** Suppose

$$d > \frac{1}{2}, \tag{11}$$

and

$$
\begin{aligned}
&(\eta - D_2)T - \frac{\alpha\eta}{D_2(\alpha + x_2^0)} \ln \frac{\alpha e^{-D_2T} + (e^{-D_2T} - 1)x_2^0 + x_2^*}{\alpha + x_2^*} \\
&+ \frac{c_1^*(1 - e^{-(g_1+m_1)lT})}{g_1 + m_1} + \frac{f_1 c_2^* lT}{h_1 - g_1 - m_1} - \frac{f_1 c_2^*(1 - e^{-(h_1-g_1-m_1)lT})}{(h_1 - g_1 - m_1)^2} \\
&\quad + \frac{c_1^{**}(e^{-(g_1+m_1)lT} - e^{-(g_1+m_1)T})}{g_1 + m_1} + \frac{f_1 c_2^{**}(1 - l)T}{h_1 - g_1 - m_1} \\
&\quad\quad - \frac{f c_2^{**}(e^{-(h_1-g_1-m_1)lT} - e^{-(h_1-g_1-m_1)T})}{(h_1 - g_1 - m_1)^2} < 0,
\end{aligned}
\tag{12}
$$

hold, the globally asymptotically stable $(\widetilde{x_1}, \widetilde{x_2}, 0, \widetilde{c_1}, \widetilde{c_2})$ exists, and $x_2^*$, $x_2^{**}$ are with (6), and $c_1^*, c_1^{**}$,$c_2^*, c_2^{**}$ are with in (10).

**Proof.** Doing it as $y_1 = x_1 - \widetilde{x_1}, y_2 = x_2 - \widetilde{x_2}, Y = Y, z_1 = c_1 - \widetilde{c_1}, z_2 = c_2 - \widetilde{c_2}$, then, linear system with considering for one periodic solution $(\widetilde{x_1}, \widetilde{x_2}, 0, \widetilde{c_1}, \widetilde{c_2})$ is presented as

$$
\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dY}{dt} \\ \frac{dz_1}{dt} \\ \frac{dz_2}{dt} \end{pmatrix} = \begin{pmatrix} -D_1 & 0 & 0 & 0 & 0 \\ 0 & -D_2 & K_5 & 0 & 0 \\ 0 & 0 & K_5' & 0 & 0 \\ 0 & 0 & 0 & -(g_1 + m_1) & f_1 \\ 0 & 0 & 0 & 0 & -h_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ Y \\ z_1 \\ z_2 \end{pmatrix},
$$

where $K_5 = -\frac{1}{\delta} \times \frac{\eta \widetilde{x_2(t)}}{\alpha + \widetilde{x_2(t)}}, K_5' = -[D_2 + \beta \widetilde{c_1(t)} - \frac{\eta \widetilde{x_2(t)}}{\alpha + \widetilde{x_2(t)}}]$. Then, the fundamental solution matrix is

$$
\Phi(t) = \begin{pmatrix} e^{-D_1 t} & 0 & 0 & 0 & 0 \\ 0 & e^{-D_2 t} & \dagger & 0 & 0 \\ 0 & 0 & K_6 & 0 & 0 \\ 0 & 0 & 0 & K_6' & \ddagger \\ 0 & 0 & 0 & 0 & e^{-h_1 t} \end{pmatrix},
$$

where $K_6 = e^{\int_0^t (-D_2 + \frac{\eta \widetilde{x_2(\xi)}}{\alpha + \widetilde{x_2(\xi)}} - \beta \widetilde{c_1(\xi)}) d\xi}, K_6' = e^{-(g_1+m_1)t}$. There is no need for computing $\dagger, \ddagger$.

When $t = (n + l)T$, we get

$$\begin{pmatrix} y_1(nT^+) \\ y_2(nT^+) \\ Y(nT^+) \\ z_1(nT^+) \\ z_2(nT^+) \end{pmatrix} = \begin{pmatrix} 1\,0\,0\,0 & 0 \\ 0\,1\,0\,0 & 0 \\ 0\,0\,1\,0 & 0 \\ 0\,0\,0\,1 & 0 \\ 0\,0\,0\,0 & 1-\theta_1 \end{pmatrix} \begin{pmatrix} y_1(nT) \\ y_2(nT) \\ Y(nT) \\ z_1(nT) \\ z_2(nT) \end{pmatrix}.$$

When $t = (n + 1)T$, we also get

$$\begin{pmatrix} y_1(nT^+) \\ y_2(nT^+) \\ Y(nT^+) \\ z_1(nT^+) \\ z_2(nT^+) \end{pmatrix} = \begin{pmatrix} 1-d & d & 0 & 0 & 0 \\ d & 1-d & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1-\theta_3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1(nT) \\ y_2(nT) \\ Y(nT) \\ z_1(nT) \\ z_2(nT) \end{pmatrix}.$$

The stability of $(\widetilde{x_1}, \widetilde{x_1}, 0, \widetilde{c_1}, \widetilde{c_2})$ is decided by eigenvalues of

$$M = \begin{pmatrix} 1-d & d & 0 & 0 & 0 \\ d & 1-d & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1-\theta_3 & 0 \\ 0 & 0 & 0 & 0 & 1-\theta_1 \end{pmatrix} \Phi(\tau).$$

From condition (12), $e^{-D_1\tau} < 1$, and $e^{-D_2\tau} < 1$, the eigenvalues of $M$ are presented as

$$\gamma_1 = \frac{K_7 + \sqrt{K_7'}}{2} << 1,$$

$$\gamma_2 = \frac{K_7 - \sqrt{K_7'}}{2} < (1-d)e^{-D_2 T} < 1,$$

$$\gamma_3 = e^{\int_0^T (-D_2 + \frac{\eta \widetilde{x_2(\xi)}}{\alpha + \widetilde{x_2(\xi)}} - \beta \widetilde{c_1(\xi)})d\xi},$$

$$\gamma_4 = (1-\theta_3)e^{-(g_1+m_1)\tau} < 1,$$

$$\gamma_5 = (1-\theta_1)e^{-h_1\tau} < 1.$$

where $K_7 = (1-d)(e^{-D_1T} + e^{-D_2T}), K_7' = (1-d)^2(e^{-D_1T} + e^{-D_2T})^2 - 4(1-2d)e^{-(D_1+D_2)T}$. For the Floquet theory [6] and condition (14), it is easily to have $\gamma_3 < 1$, then, the locally stable $(\widetilde{x_1}, \widetilde{x_2}, 0, \widetilde{c_1}, \widetilde{c_2})$ exists.

Choosing $\varepsilon > 0$, we have

$$\rho_1 = \exp[\int_0^\tau K_8 - \beta(\widetilde{c_1(t)} - \varepsilon)dt] < 1.$$

where $K_8 = -D_2 + \frac{\eta(\widetilde{x_2}+\varepsilon)}{\alpha+(\widetilde{x_2}+\varepsilon)}$. We get $\frac{dx_2}{dt} \leq D_2(x_2^0 - x_2)$ with considering system (1). So

$$\begin{cases} \left.\begin{array}{l} \dfrac{dm_1}{dt} = D_1(x_1^0 - m_1), \\ \dfrac{dm_2}{dt} = D_2(x_2^0 - m_2), \end{array}\right\} t \neq (n+1)T, \\ \left.\begin{array}{l} \Delta m_1 = d(m_2 - m_1) + \theta_2, \\ \Delta m_2 = d(m_1 - m_2), \end{array}\right\} t = (n+1)T. \end{cases} \tag{13}$$

From lemma (2), and [8], we get

$$\begin{cases} x_1 \leq m_1 \leq \widetilde{x_1} + \varepsilon, \\ x_2 \leq m_2 \leq \widetilde{x_2} + \varepsilon. \end{cases} \tag{14}$$

and

$$\begin{cases} \widetilde{c_1} - \varepsilon \leq c_1 \leq \widetilde{c_1} + \varepsilon, \\ \widetilde{c_2} - \varepsilon \leq c_2 \leq \widetilde{c_2} + \varepsilon. \end{cases} \tag{15}$$

Therefore,

$$\frac{dY}{dt} \leq Y[-D_2 + \frac{\eta(\widetilde{x_2} + \varepsilon)}{\alpha + (\widetilde{x_2} + \varepsilon)} - \beta(\widetilde{c_1} - \varepsilon)]. \tag{16}$$

Then, $Y(t) \leq Y(0^+)\exp[\int_0^t(-D_2 + \frac{\eta(\widetilde{x_2}+\varepsilon)}{\alpha+(\widetilde{x_2}+\varepsilon)} - \beta(\widetilde{c_1} - \varepsilon))ds]$, thus

$$\begin{aligned} Y((n+1)T) &\leq Y(nT^+) \\ &\times \exp[\int_{nT}^{(n+1)T}(-D_2 + \frac{\eta(\widetilde{x_2(s)} + \varepsilon)}{\alpha + (\widetilde{x_2(s)} + \varepsilon)} - \beta(\widetilde{c_1(s)} - \varepsilon))ds]. \end{aligned} \tag{17}$$

Hence $Y(n\tau) \leq Y(0^+)\rho_1^n$ and $Y(nT) \to 0$ as $n \to \infty$. So $Y(t) \to 0$ as $t \to \infty$.

For $\varepsilon > 0$, it has a $t_0 > 0$ such that $0 < Y < \varepsilon$ for all $t \geq t_0$. It is no difficulty to gain

$$\frac{\delta\alpha D_2 + \eta\varepsilon}{\delta(\alpha + M)}[\frac{\delta(\alpha + M)D_2x_2^0}{\delta(\alpha + M)D_2 + \eta\varepsilon} - x_2] \leq \frac{dx_2}{dt} \leq D_2(x_2^0 - x_2). \tag{18}$$

$(v_1, v_2)$, $(n_1, n_2)$ are of

$$\begin{cases} \left.\begin{aligned} &\frac{dv_1}{dt} = D_1(x_1^0 - v_1), \\ &\frac{dv_2}{dt} = \frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}[\frac{\delta(\alpha+M)D_2x_2^0}{\delta(\alpha+M)D_2+\eta\varepsilon} - v_2], \end{aligned}\right\} t \neq nT, \\ \left.\begin{aligned} &\Delta v_1 = d(v_2 - v_1) + \theta_2, \\ &\Delta v_2 = d(v_1 - v_2), \end{aligned}\right\} t = nT, \end{cases} \tag{19}$$

and

$$\begin{cases} \left.\begin{aligned} &\frac{dn_1}{dt} = D_1(x_1^0 - n_1), \\ &\frac{dn_2}{dt} = D_2(x_2^0 - n_2), \end{aligned}\right\} t \neq nT, \\ \left.\begin{aligned} &\Delta n_1 = d(n_2 - n_1) + \theta_2, \\ &\Delta n_2 = d(n_1 - n_2), \end{aligned}\right\} t = nT, \end{cases} \tag{20}$$

respectively, while

$$\begin{cases} \widetilde{v_1} = x_1^0 - (x_1^0 - v_1^*)e^{-D_1(t-nT)}, \\ \widetilde{v_2} = \dfrac{\delta(\alpha+M)D_2x_2^0}{\delta(\alpha+M)D_2+\eta\varepsilon} \\ \qquad\qquad -(\dfrac{\delta(\alpha+M)D_2x_2^0}{\delta(\alpha+M)D_2+\eta\varepsilon} - v_2^*)e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}(t-nT)}, \end{cases} \tag{21}$$

with

$$\begin{cases} v_1^* \\ = \dfrac{[(1-d)(1-e^{-D_1T}) - (1-2d)(1-e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}T})e^{-D_1T}]x_1^0 - d(1-e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}T})\frac{\delta(\alpha+M)D_2x_2^0}{\delta(\alpha+M)D_2+\eta\varepsilon}}{1-(1-d)e^{-D_1T} - (1-d)e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}\tau} - (1-2d)e^{-(D_1+\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)})T}} \\ \qquad + \dfrac{(1-e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}\tau} + de^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}\tau})\theta_2}{1-(1-d)e^{-D_1\tau} - (1-d)e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}\tau} - (1-2d)e^{-(D_1+\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)})T}}, \\ v_2^* \\ = \dfrac{-d(1-e^{-D_1\tau})x_1^0 + [(1-d)(1-e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}T}) - (1-2d)(1-e^{-D_1T})e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}\tau}]\frac{\delta(\alpha+M)D_2x_2^0}{\delta(\alpha+M)D_2+\eta\varepsilon}}{1-(1-d)e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}\tau} - (1-d)e^{-D_1T} - (1-2d)e^{-(\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}+D_1)T}} \\ \qquad + \dfrac{+de^{-D_1T}\theta_2}{1-(1-d)e^{-\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}T} - (1-d)e^{-D_1T} - (1-2d)e^{-(\frac{\delta(\alpha+M)D_2+\eta\varepsilon}{\delta(\alpha+M)}+D_1)T}}, \end{cases} \tag{22}$$

For $\varepsilon_1 > 0$, existing a $t_1, t > t_1$ such that

$$\widetilde{v_1} - \varepsilon_1 < x_1 < \widetilde{n_1} + \varepsilon_1,$$

and

$$\widetilde{v_2} - \varepsilon_1 < x_2 < \widetilde{n_2} + \varepsilon_1.$$

One will gain the followings with considering $\varepsilon \to 0$,

$$\widetilde{x_1} - \varepsilon_1 < x_1 < \widetilde{x_1} + \varepsilon_1,$$

and

$$\widetilde{x_2} - \varepsilon_1 < x_2 < \widetilde{x_2} + \varepsilon_1.$$

This completes the proofs.

**Theorem 2.** Suppose

$$d > \frac{1}{2}, \tag{23}$$

and

$$\begin{aligned}(\eta - D_2)T - \frac{\alpha\eta}{D_2(\alpha + x_2^0)} \ln \frac{\alpha e^{-D_2T} + (e^{-D_2T} - 1)x_2^0 + x_2^*}{\alpha + x_2^*} \\ + \frac{c_1^*(1 - e^{-(g_1+m_1)lT})}{g_1 + m_1} + \frac{f_1 c_2^* lT}{h_1 - g_1 - m_1} - \frac{f_1 c_2^*(1 - e^{-(h_1-g_1-m_1)lT})}{(h_1 - g_1 - m_1)^2} \\ + \frac{c_1^{**}(e^{-(g_1+m_1)lT} - e^{-(g_1+m_1)T})}{g_1 + m_1} + \frac{f_1 c_2^{**}(1 - l)T}{h_1 - g_1 - m_1} \\ - \frac{f c_2^{**}(e^{-(h_1-g_1-m_1)lT} - e^{-(h_1-g_1-m_1)T})}{(h_1 - g_1 - m_1)^2} > 0,\end{aligned} \tag{24}$$

hold, system permanence, and $x_2^*$ and $x_2^{**}$ are with (6), $c_1^*, c_1^{**}, c_2^*, c_2^{**}$ are with (10).

**Proof.** Owing to remark (4), and lemma (5), we have obtain that $(x_1, x_2, Y, c_1, c_2)$ is bounded. It can be easily obtained that $c_1 \geq m_o$ and $c_2 \geq m_e$ with considering with remark (4).

Therefore,

$$\begin{cases} \left.\begin{aligned} \frac{dx_1}{dt} &= D_1(x_1^0 - x_1), \\ \frac{dx_2}{dt} &\geq D_2(x_2^0 - x_2) - \frac{\eta M}{\delta\alpha} x_2, \end{aligned}\right\} t \neq nT, \\ \left.\begin{aligned} \triangle x_1 &= d(-x_1 + x_2) + \theta_2, \\ \triangle x_2 &= d(x_1 - x_2), \end{aligned}\right\} t = nT, \end{cases} \tag{25}$$

with its comparison system

$$\begin{cases} \left.\begin{aligned} \frac{dw_1}{dt} &= D_1(x_1^0 - w_1), \\ \frac{dw_2}{dt} &= \frac{\delta\alpha D_2 + \eta M}{\delta\alpha}[\frac{\delta\alpha D_2 x_2^0}{\delta\alpha D_2 + \eta M} - w_2], \end{aligned}\right\} t \neq nT, \\ \left.\begin{aligned} \triangle w_1 &= d(w_2 - w_1) + \theta_2, \\ \triangle w_2 &= d(w_1 - w_2), \end{aligned}\right\} t = nT. \end{cases} \tag{26}$$

With considering (2) and (8), $(\widetilde{w_1}, \widetilde{w_2})$ of (26) is

$$\begin{cases} \widetilde{w_1} = x_1^0 - (x_1^0 - w_1^*)e^{-D_1(t-nT)}, nT < t \le (n+1)T, \\ \widetilde{w_2} = \dfrac{\delta\alpha D_2 x_2^0}{\delta\alpha D_2 + \eta M} - (\dfrac{\delta\alpha D_2 x_2^0}{\delta\alpha D_2 + \eta M} - w_2^*)e^{-\frac{\delta\alpha D_2+\eta M}{\delta\alpha}(t-nT)}, \\ \qquad nT < t \le (n+1)T, \end{cases} \tag{27}$$

with

$$\begin{cases} w_1^* = \dfrac{[(1-d)(1-e^{-D_1T}) - (1-2d)(1-e^{-\frac{\delta\alpha D_2+\eta M}{\delta\alpha}T})e^{-D_1T}]x_1^0 - d(1-e^{-\frac{\delta\alpha D_2+\eta M}{\delta\alpha}T})\frac{\delta\alpha D_2 x_2^0}{\delta\alpha D_2+\eta M}}{1-(1-d)e^{-D_1T} - (1-d)e^{-\frac{\delta\alpha D_2+\eta M}{\delta\alpha}T} - (1-2d)e^{-(D_1+\frac{\delta\alpha D_2+\eta M}{\delta\alpha})T}} \\ \qquad + \dfrac{+(1-e^{-\frac{\delta\alpha D_2+\eta M}{\delta\alpha}T} + de^{-\frac{\delta\alpha D_2+\eta M}{\delta\alpha}T})\theta_2}{1-(1-d)e^{-D_1T} - (1-d)e^{-\frac{\delta\alpha D_2+\eta M}{\delta\alpha}T} - (1-2d)e^{-(D_1+\frac{\delta\alpha D_2+\eta M}{\delta\alpha})T}}, \\ w_2^* = \dfrac{-d(1-e^{-D_1T})x_1^0 + [(1-d)(1-e^{-\frac{\delta\alpha D_2+\eta M}{\delta\alpha}T}) - (1-2d)(1-e^{-D_1\tau})e^{-\frac{\delta\alpha D_2+\eta M}{\delta\alpha}T}]\frac{\delta\alpha D_2 x_2^0}{\delta\alpha D_2+\eta M}}{1-(1-d)e^{-\frac{\delta\alpha D_2+\eta M}{\delta\alpha}T} - (1-d)e^{-D_1T} - (1-2d)e^{-(\frac{\delta\alpha D_2+\eta M}{\delta\alpha}+D_1)\tau}} \\ \qquad + \dfrac{+de^{-D_1\tau}\theta_2}{1-(1-d)e^{-\frac{\delta\alpha D_2+\eta M}{\delta\alpha}\tau} - (1-d)e^{-D_1T} - (1-2d)e^{-(\frac{\delta\alpha D_2+\eta M}{\delta\alpha}+D_1)T}}. \end{cases} \tag{28}$$

Furthermore, $(\widetilde{w_1}, \widetilde{w_2})$ of (27) is globally asymptotic stable. Then, it exists a $\varepsilon > 0$ such that $x_1 \ge w_1 \ge \widetilde{w_1} - \varepsilon \ge w_1^* - \varepsilon = k_1$ and $x_2 \ge w_2 \ge \widetilde{w_2} - \varepsilon \ge w_2^* - \varepsilon = k_2$.

Since

$$\begin{aligned} &(\eta - D_2)T + \frac{\eta\alpha}{D_2(\alpha + x_2^*)} \ln \frac{\alpha e^{-D_2T} + (e^{-D_2T} - 1)x_2^0 + x_2^*}{\alpha + x_2^*} \\ &+ \frac{c_1^*(1 - e^{-(g_1+m_1)l\tau})}{g_1 + m_1} + \frac{fc_2^* lT}{h_1 - g_1 - m_1} - \frac{f_1 c_2^*(1 - e^{-(h_1-g_1-m_1)lT})}{(h_1 - g_1 - m_1)^2} \\ &+ \frac{c_1^{**}(e^{-(g_1+m_1)lT} - e^{-(g_1+m_1)T})}{g_1 + m_1} + \frac{f_1 c_2^{**}(1-l)T}{h_1 - g_1 - m_1} \\ &- \frac{f_1 c_2^{**}(e^{-(h_2-g_2-m_2)lT} - e^{-(h_1-g_1-m_1)T})}{(h_1 - g_1 - m_1)^2} > 0, \end{aligned} \tag{29}$$

$m_3 > 0$ and $\varepsilon_1 > 0$ can be selected to do as

$$\begin{aligned} \sigma = &(\eta - \frac{\delta\alpha D_2 + \eta m_3}{\delta\alpha} - \varepsilon)T - \frac{\eta\alpha}{D_2(\alpha - \varepsilon + \frac{\delta\alpha D_2}{\delta\alpha D_2+\eta m_3}x_2^0)} \\ &\times \ln \frac{(\alpha - \varepsilon + \frac{\delta\alpha D_2}{\delta\alpha D_2+\eta m_3})e^{-\frac{\delta\alpha D_2+\eta m_3}{\delta\alpha}} - \frac{\delta\alpha D_2}{\delta\alpha D_2+\eta m_3}x_2^0) + k_2^*}{\alpha - \varepsilon + k_2^*} \\ &+ \frac{c_1^*(1 - e^{-(g_1+m_1)lT})}{g_1 + m_1} + \frac{f_1 c_2^* l\tau}{h_1 - g_1 - m_1} - \frac{f_1 c_2^*(1 - e^{-(h_1-g_1-m_1)lT})}{(h_1 - g_1 - m_1)^2} \\ &+ \frac{c_1^{**}(e^{-(g_1+m_1)lT} - e^{-(g_1+m_1)T})}{g_1 + m_1} + \frac{f_1 c_2^{**}(1-l)T}{h_1 - g_1 - m_1} \\ &- \frac{f_1 c_2^{**}(e^{-(h_1-g_1-m_1)lT} - e^{-(h_1-g_1-m_1)T})}{(h_1 - g_1 - m_1)^2} > 0, \end{aligned} \tag{30}$$

where $k_2^*$ is defined as (34)

$Y < m_3$ will be proved that it can not be held for $t \geq 0$. Otherwise,

$$\begin{cases} \left.\begin{aligned} &\frac{dx_1}{dt} = D_1(x_1^0 - x_1), \\ &\frac{dx_2}{dt} \geq D_2(x_2^0 - x_2(t)) - \frac{\eta m_3}{\delta\alpha} x_2, \end{aligned}\right\} t \neq nT, \\ \left.\begin{aligned} &\triangle x_1 = d(-x_1 + x_2) + \theta_2, \\ &\triangle x_2 = d(x_1 - x_2), \end{aligned}\right\} t = nT. \end{cases} \tag{31}$$

with its comparison system

$$\begin{cases} \left.\begin{aligned} &\frac{dk_1}{dt} = D_1(x_1^0 - k_1), \\ &\frac{dk_2}{dt} \\ &\quad = \frac{\delta\alpha D_2 + \eta m_3}{\delta\alpha}[\frac{\delta\alpha D_2}{\delta\alpha D_2 + \eta m_3} x_2^0 - k_2], \end{aligned}\right\} t \neq nT, \\ \left.\begin{aligned} &\triangle k_1 = d(k_2 - k_1) + \mu_2, \\ &\triangle k_2 = d(k_1 - _2), \end{aligned}\right\} t = nT. \end{cases} \tag{32}$$

By lemma (2), we gain $x_1 \geq k_1, x_2 \geq k_2$ and $k_1 \to \overline{k_1}, k_2(t) \to \overline{k_2}$, as $t \to \infty$, and

$$\begin{cases} \overline{k_1} = x_1^0 - [x_1^0 - k_1^*]e^{-D_1(t-nT)}, nT < t \leq (n+1)T, \\ \overline{k_2} = \frac{\delta\alpha D_2}{\delta\alpha D_2 + \eta m_3} x_2^0 - [\frac{\delta\alpha D_2}{\delta\alpha D_2 + \eta m_3} x_2^0 - k_2^*]e^{-\frac{\delta\alpha D_2 + \eta m_3}{\delta\alpha}(t-nT)}, \\ \qquad nT < t \leq (n+1)T, \end{cases} \tag{33}$$

with

$$\begin{cases} k_1^* = \frac{[(1-d)(1-e^{-D_1T}) - (1-2d)(1-e^{-\frac{\delta\alpha D_2+\eta m_3}{\delta\alpha}\tau})e^{-D_1T}]x_1^0 - d(1-e^{-\frac{\delta\alpha D_2+\eta m_3}{\delta\alpha}\tau})\frac{\delta\alpha D_2}{\delta\alpha D_2+\eta m_3}x_2^0}{1-(1-d)e^{-D_1\tau} - (1-d)e^{-\frac{\delta\alpha D_2+\eta m_3}{\delta\alpha}T} - (1-2d)e^{-(D_1+\frac{\delta\alpha D_2+\eta m_3}{\delta\alpha})\tau}} \\ \qquad + \frac{+(1-e^{-\frac{\delta\alpha D_2+\eta m_3}{\delta\alpha}T} + de^{-\frac{\delta\alpha D_2+\eta m_3}{\delta\alpha}T})\theta_2}{1-(1-d)e^{-D_1T} - (1-d)e^{-\frac{\delta\alpha D_2+\eta m_3}{\delta\alpha}T} - (1-2d)e^{-(D_1+\frac{\delta\alpha D_2+\eta m_3}{\delta\alpha})T}}, \\ k_2^* = \frac{-d(1-e^{-D_1T})x_1^0 + [(1-d)(1-e^{-\frac{\delta\alpha D_2+\eta m_3}{\delta\alpha}T}) - (1-2d)(1-e^{-D_1T})e^{-\frac{\delta\alpha D_2+\eta m_3}{\delta\alpha}T}]\frac{\delta\alpha D_2}{\delta\alpha D_2+\eta m_3}x_2^0}{1-(1-d)e^{-\frac{\delta\alpha D_2+\eta m_3}{\delta\alpha}T} - (1-d)e^{-D_1T} - (1-2d)e^{-(\frac{\delta\alpha D_2+\eta m_3}{\delta\alpha}+D_1)T}} \\ \qquad + \frac{+de^{-D_1T}\theta_2}{1-(1-d)e^{-\frac{\delta\alpha D_2+\eta m_3}{\delta\alpha}T} - (1-d)e^{-D_1T} - (1-2d)e^{-(\frac{\delta\alpha D_2+\eta m_3}{\delta\alpha}+D_1)T}}, \end{cases} \tag{34}$$

Therefore,

$$\begin{cases} x_1 \geq k_1 \geq \overline{k_1} - \varepsilon_1, \\ x_2 \geq k_2 \geq \overline{k_2} - \varepsilon_1. \end{cases} \tag{35}$$

For $t \geq T_1$

$$\frac{dY}{dt} \geq [-D_2 + \frac{\eta(\overline{k_2} - \varepsilon_1)}{\alpha + (\overline{k_2} - \varepsilon_1)} - \beta(\widetilde{c_1} + \varepsilon_1)]Y, \tag{36}$$

Let $K_9 \in N^+$ and $K_9\tau > T_1$, integrating (36) on $(nT, (n+1)T), n \geq K_9$, we have

$$Y((n+1)T) \geq Y(nT^+)\exp(\int_{nT}^{(n+1)T}[-D_2 + \frac{\eta(\overline{k_1}-\varepsilon_1)}{\alpha+(\overline{k_1}-\varepsilon_1)} - \beta(\overline{k_2}+\varepsilon_1)]dt)$$
$$= x(nT)e^{\sigma},$$

then $Y((K_9+k)T) \geq Y(K_9T^+)e^{k\sigma} \to \infty$, as $k \to \infty$, it is an illogicality with the bounded $Y$. Hence $Y \geq m_3$.

## 5 Discussion

In this work, we present a chemostat-type model with impulsive effects in a polluted karst environment. If it is supposed that the variables are shown in the table below:

| $x_1(0)$ | $x_2(0)$ | $Y(0)$ | $c_1(0)$ | $c_2(0)$ | $D_1$ | $D_2$ | $x_1^0$ | $x_2^0$ | $\delta$ | $\eta$ | $\alpha$ | $\beta$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta$ | $f_1$ | $g_1$ | $m_1$ | $h_1$ | $d$ | $T$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 2 | 0.13 | 0.15 | 2 | 0.2 | 2 | 3 | 1 | 0.5 | 2 | 0.5 | 0.5 | 0.8 | 0.6 | 0.01 | 0.4 | 0.3 | 0.1 | 0.4 | 0.2 | 1 |

system (1) is permanent (one can see Fig. 1). If it is supposed that another variables are shown in the table below:

| $x_1(0)$ | $x_2(0)$ | $Y(0)$ | $c_1(0)$ | $c_2(0)$ | $D_1$ | $D_2$ | $x_1^0$ | $x_2^0$ | $\delta$ | $\eta$ | $\alpha$ | $\beta$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta$ | $f_1$ | $g_1$ | $m_1$ | $h_1$ | $d$ | $T$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 2 | 0.13 | 0.15 | 2 | 0.2 | 2 | 3 | 1 | 0.5 | 2 | 0.5 | 0.5 | 0.8 | 0.1 | 0.01 | 0.4 | 0.3 | 0.1 | 0.4 | 0.2 | 1 |

there exists a globally asymptotically stable solution $(\widetilde{x_1(t)}, \widetilde{x_2(t)}, 0, \widetilde{c_1(t)}, \widetilde{c_2(t)})$ of system (1) (one can see Fig. 2). If it is supposed that another variables are shown in the table below:

| $x_1(0)$ | $x_2(0)$ | $Y(0)$ | $c_1(0)$ | $c_2(0)$ | $D_1$ | $D_2$ | $x_1^0$ | $x_2^0$ | $\delta$ | $\eta$ | $\alpha$ | $\beta$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta$ | $f_1$ | $g_1$ | $m_1$ | $h_1$ | $d$ | $T$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 2 | 0.13 | 0.15 | 2 | 0.16 | 2 | 3 | 1 | 0.5 | 2 | 0.5 | 0.5 | 0.8 | 0.1 | 0.01 | 0.4 | 0.3 | 0.1 | 0.4 | 0.2 | 1 |

then, system (1) is permanent (see Fig. 3).

The simulations show that parameters $0 < \theta_3 < 1$ and $D_2$ are very important for system (1). The parameters $D_1, \theta_1, \theta_2, \theta$ and $d$ of system (1) can also be discussed. The results will guide us how to manage the source of water management in karst areas.

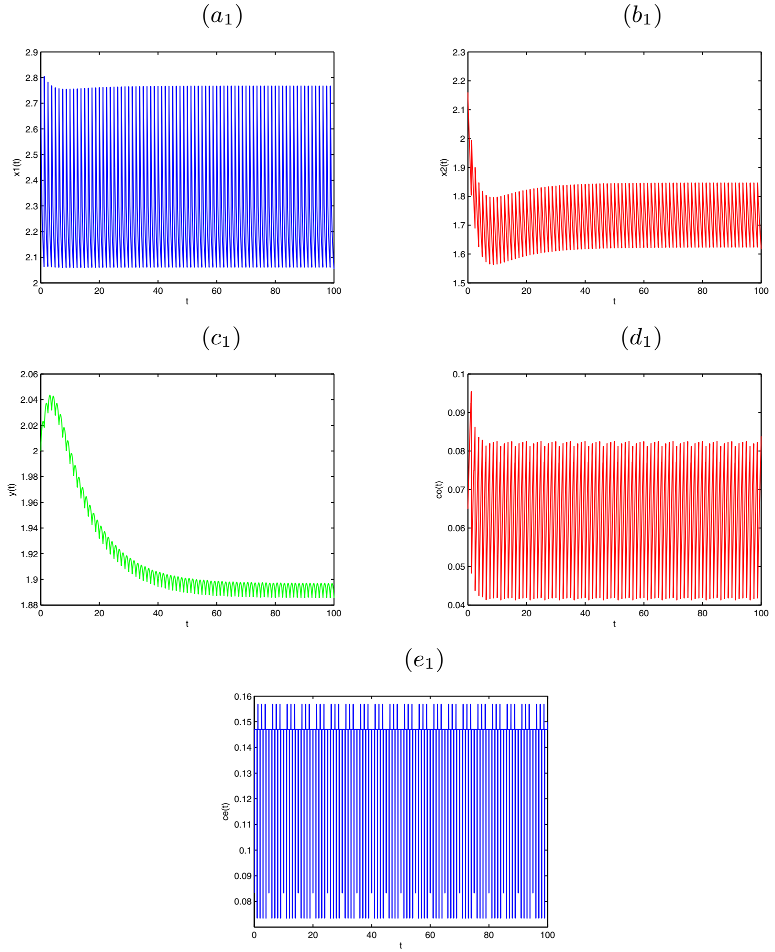


**Fig. 1.** The permanence of system (1) with parameters in the first table.

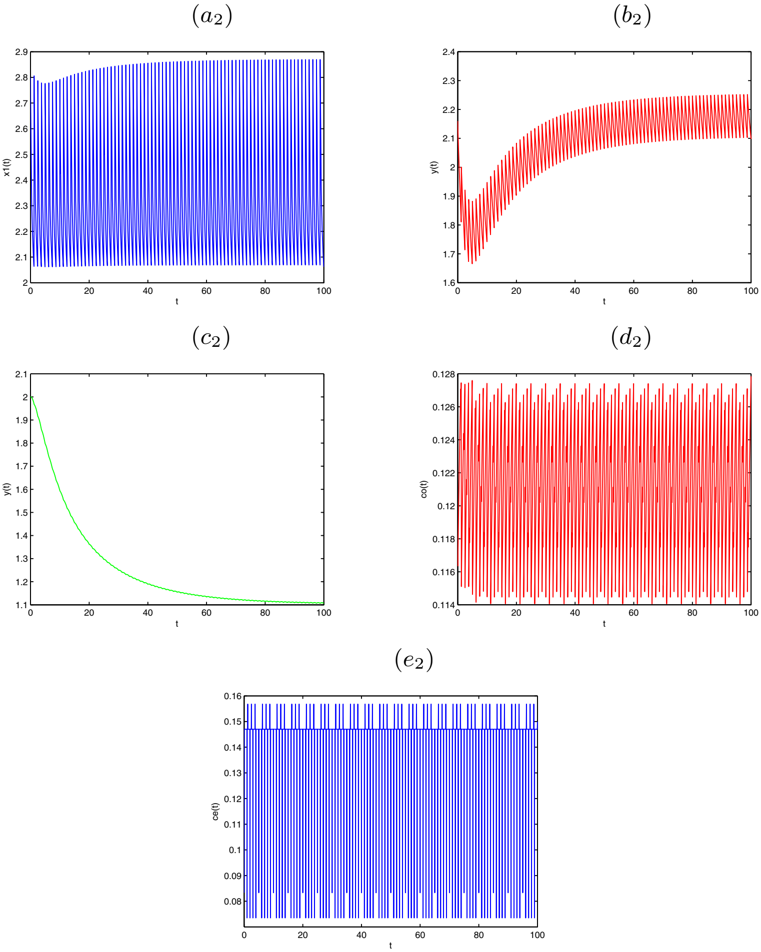


**Fig. 2.** The dynamics of the globally asymptotically stable microorganism-extinction with parameters in the second table.

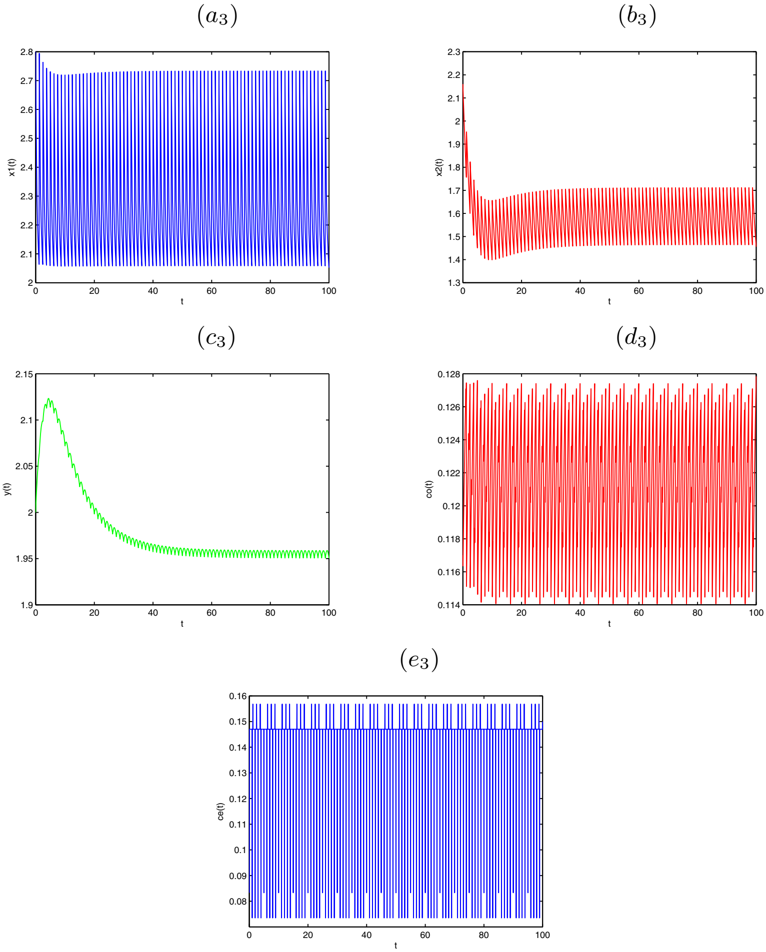


**Fig. 3.** The dynamics of the globally asymptotically stable microorganism-extinction with parameters in the third table.

## References

1. Bo, L., Yi-Fan, Z., Bei-Bei, Z., Xian-Qing, W.: A risk evaluation model for karst groundwater pollution based on geographic information system and artificial neural network applications. Environ. Earth Sci. **77**(9), 1–14 (2018). https://doi.org/10.1007/s12665-018-7539-7
2. Zhao, D., Yuan, S.: Noise-induced bifurcations in the stochastic chemostat model with general nutrient uptake functions. Appl. Math. Lett. **103**, 106180 (2020)
3. Amster, P., et al.: Existence of -periodic solutions for a delayed chemostat with periodic inputs. Nonlinear Anal. Real World Appl. **55**, 103134 (2020)
4. Yuan, H., Zhang, C., Li, Y.: Existence and stability of coexistence states in a competition unstirred chemostat. Nonlinear Anal. Real World Appl. **35**, 441–456 (2017)

5. Liu, B., Chen, L.S., Zhang, Y.J.: The effects of impulsive toxicant input on a population in a polluted environment. J. Biol. Syst. **11**, 265–287 (2003)
6. Bainov, D., Simeonov, P.: Impulsive differential equations: periodic solutions and applications. Longman, 66 (1993)
7. Jiao, J., Chen, L.: Global attractivity of a stage-structure variable coefficients predator-prey system with time delay and impulsive perturbations on predators. Int. J. Biomathematics **1**, 197–208 (2008)
8. Jiao, J., Li, Q.: Dynamics of a stochastic eutrophication-chemostat model with impulsive dredging and pulse inputting on environmental toxicant. Adv. Differ. Equ. **2020**(1), 1–16 (2020). https://doi.org/10.1186/s13662-020-02905-5