Block-Diagonal and Anti-block-Diagonal Splitting Iteration Method for Absolute Value Equation

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Abstract. In this paper, the absolute value equation (AVE) is equivalently reformulated as a nonlinear equation in the form of 2 times 2 blocks. A block diagonal inverse block diagonal iteration method based on block-diagonal and anti-block-diagonal splitting (BAS) is proposed. Theoretical analysis shows that BAS is convergent, and numerical experiments show that the method is effective.

Keywords: Absolute value equation · Block-diagonal and anti-block-diagonal splitting · Convergence

1 Introduction

For the given matrix $A \in \mathbb{R}^{n\times n}$ and the given vector $b \in \mathbb{R}^n$, we consider the iterative solution of the absolute value equation (AVE)

$$Ax - |x| = b, \tag{1}$$

where $|\cdot|$ is the absolute value. Like linear programming, quadratic programming, bimatrix games and quasi complementarity problems [1–5]), AVE has been widely concerned as a practical optimization tool. AVE (1) is a NP hard problem, because there is a nonlinear and non differentiable term $|x|$, which makes AVE (1) nonlinear and non differentiable. When the $|x|$ in (1) disappears, the AVE (1) will be reduced to a linear system, see [13,14,16,18,24–29].

In recent years, the numerical solutions of AVE can be obtained by iterative methods, including the successful linearization algorithm [2], Picard and Picard-HSS algorithm [7,8], sign accord algorithm [6] and hybrid algorithm [12], interval algorithm [19], preconditioned AOR iterative algorithm [22], the generalized Newton algorithm [9,20,21], and so on.

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For solving the AVE (1), the generalized Newton (GN) method in [9] works below

$$x^{k+1} = (A - D(x^k))^{-1}b, k = 0, 1, \ldots, \tag{2}$$

where $D(x^k) = diag(sign(x^k))$. Here, the $sign(x)$ is a vector composed of $1, 0, -1$, which determines whether the value of x is greater than zero, equal to zero or less than zero.

In [9], the convergence of the GN method is given by Mangasarian under appropriate conditions. In numerical experiments, the GN method is superior to the successive linearization method in [2]. After that, the GN method is extended to solve the GAVE [10,11] related to the second-order cone.

On the basis of the previous work in [14], in [15], on the basis of Hermitian and skew Hermitian splitting (HSS in [13]) of matrix A in (1), a nonlinear HSS (NHSS) method for ave (1) is proposed.

The NHSS method. Let $\alpha > 0$ and $x^{(0)} \in \mathbb{R}^n$ be an arbitrary initial value. For $k = 0, 1, 2, \ldots$ until the iterative sequences $\{x^{(k)}\}_{k=0}^{\infty}$ is convergent, calculate $x^{(k+1)}$ by the following procedure:

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + |x^k| + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + |x^{(k+\frac{1}{2})}| + b, \end{cases} \tag{3}$$

where $H = \frac{1}{2}(A + A^T)$ and $S = \frac{1}{2}(A - A^T)$, A^* stands for the transpose of the matrix A.

Compared with GN method, NHSS method can avoid variable coefficient matrix $A - D(x^k)$, which is an advantage of this method. However, it is worth noting that in each iteration step using the NHSS iterative method, both matrices $\alpha I + H$ and $\alpha I + S$ need to be calculated. It is well known that the coefficient matrix of a linear system $\alpha I + S$ is skew-Hermitian, and generally it is difficult to obtain its solution. See [16] for more details.

It is well known that different iterative methods are suitable for different matrix splittings. On this basis, based on the block-diagonal and anti-block-diagonal splitting (BAS) of linear term coefficient matrix in AVE, the block-diagonal and anti-block-diagonal splitting (BAS) iterative methods for AVE (1) are designed. Theoretical analysis shows that BAS method is convergent under mild conditions.

The remainder of the paper lays out below. In Sect. 2, for solving the AVE (1), the BAS iteration method is established and its convergence properties are studied in detail. In Sect. 3, numerical experiments are given to confirm the effectiveness and feasibility of the proposed method. In Sect. 4, some conclusions are given to end the paper.

2 The BAS Method

In this section, to solve the AVE (1), the BAS iteration method is introduced. To this end, we reformulate equivalently AVE (1) as a nonlinear equation with two-by-two block form. That is to say, let $y = |x|$, then the AVE is equal to

$$\begin{cases} Ax - y = b, \\ -|x| + y = 0, \end{cases}$$

that is,

$$\mathcal{A}z \equiv \begin{bmatrix} A & -I \\ -\hat{D} & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}, \tag{4}$$

where $\hat{D} = D(x) = diag(sign(x)), x \in \mathbb{R}^n$.

A block-diagonal and anti-block-diagonal splitting (BAS) of matrix $\mathcal{A}$ can be constructed as follows

$$\mathcal{A} = \begin{bmatrix} A & -I \\ -\hat{D} & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} & -I \\ -\hat{D} & 0 \end{bmatrix}.$$

Further, matrix $\mathcal{A}$ can be expressed as

$$\mathcal{A} = \begin{bmatrix} A & -I \\ -\hat{D} & I \end{bmatrix} = \begin{bmatrix} \alpha I + A & 0 \\ 0 & \alpha I + I \end{bmatrix} - \begin{bmatrix} \alpha I & I \\ \hat{D} & \alpha I \end{bmatrix},$$

where α is a given appropriate constant. This splitting naturally leads to the BAS iteration method for solving the nonlinear equation (4).

The BAS iteration method: Let $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ be a nonsingular. Given an initial pair vector $(x^{(0)}y^{(0)})$, for $k = 0, 1, 2, ...$, until the iteration sequence $\{x^{(k)}, y^{(k)}\}_{k=0}^{+\infty}$ is convergent, compute

$$\begin{bmatrix} \alpha I + A & 0 \\ 0 & \alpha I + I \end{bmatrix} \begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = \begin{bmatrix} \alpha I & I \\ \hat{D} & \alpha I \end{bmatrix} \begin{bmatrix} x^{(k)} \\ y^{(k)} \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix}, \tag{5}$$

or

$$\begin{cases} x^{(k+1)} = (\alpha I + A)^{-1}(\alpha x^{(k)} + y^{(k)} + b), \\ y^{(k+1)} = \dfrac{1}{1+\alpha}(\hat{D}x^{(k)} + \alpha y^{(k)}), \end{cases} \tag{6}$$

where α is a given appropriate constant.

Lemma 1. [18] *Let λ be any root of the quadratic equation $x^2 - bx + d = 0$ with $b, d \in \mathbb{R}$. Then $|\lambda| < 1$ if and only if $|d| < 1$ and $|b| < 1 + d$.*

Let (x^*, y^*) be the solution pair of the Eq. (4) and the iteration errors

$$e_k^x = x^* - x^{(k)}, e_k^y = y^* - y^{(k)},$$

where $(x^{(k)}, y^{(k)})$ is generated by the iteration method (5) or (6). Then we give the following main result with respect to the BAS iteration method (5) or (6).

Theorem 1. *Let $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ be nonsingular. Denote*

$$\beta = \|(\alpha I + A)^{-1}\|,$$

where $\|\cdot\|$ denotes the Euclid norm.

If

$$\beta(1+\alpha) < 1, \tag{7}$$

then

$$|||(e_{k+1}^x, e_{k+1}^y)||| < |||(e_k^x, e_k^y)|||, k = 0, 1, \ldots,$$

where

$$|||(e_k^x, e_k^y)||| = \sqrt{\|e_k^x\|^2 + \|e_k^y\|^2}.$$

This implies that the BAS iteration method is convergent.

Proof. Based on (5) and (6),

$$\begin{cases} e_{k+1}^x = \alpha(\alpha I + A)^{-1} e_k^x + (\alpha I + A)^{-1} e_k^y, \\ e_{k+1}^y = \dfrac{1}{1+\alpha}(\hat{D} e_k^x + \alpha e_k^y). \end{cases} \tag{8}$$

From (8), we can get

$$\begin{aligned} \|e_{k+1}^x\| &= \|\alpha(\alpha I + A)^{-1} e_k^x + (\alpha I + A)^{-1} e_k^y\| \\ &\le \alpha\|(\alpha I + A)^{-1} e_k^x\| + \|(\alpha I + A)^{-1} e_k^y\| \\ &\le \alpha\|(\alpha I + A)^{-1}\| \cdot \|e_k^x\| + \|(\alpha I + A)^{-1}\| \cdot \|e_k^y\| \\ &= \alpha\beta\|e_k^x\| + \beta\|e_k^y\|. \end{aligned}$$

and

$$\begin{aligned} \|e_{k+1}^y\| &= \|\frac{1}{1+\alpha}(\hat{D} e_k^x + \alpha e_k^y)\| \\ &\le \|\frac{1}{1+\alpha}\hat{D} e_k^x\| + \|\frac{\alpha}{1+\alpha} e_k^y\| \\ &= \frac{1}{1+\alpha}\|\hat{D} e_k^x\| + \frac{\alpha}{1+\alpha}\|e_k^y\| \\ &\le \frac{1}{1+\alpha}\|\hat{D}\| \cdot \|e_k^x\| + \frac{\alpha}{1+\alpha}\|e_k^y\| \\ &\le \frac{1}{1+\alpha}\|e_K^x\| + \frac{\alpha}{1+\alpha}\|e_k^y\|. \end{aligned}$$

Further,

$$\begin{aligned} \begin{pmatrix} \|e_{k+1}^x\| \\ \|e_{k+1}^y\| \end{pmatrix} &\le \begin{pmatrix} \alpha\beta & \beta \\ \frac{1}{1+\alpha} & \frac{\alpha}{1+\alpha} \end{pmatrix} \begin{pmatrix} \|e_k^x\| \\ \|e_k^y\| \end{pmatrix} \\ &\le \begin{pmatrix} \alpha\beta & \beta \\ \frac{1}{1+\alpha} & \frac{\alpha}{1+\alpha} \end{pmatrix}^2 \begin{pmatrix} \|e_{k-1}^x\| \\ \|e_{k-1}^y\| \end{pmatrix} \\ &\cdots \\ &\le \begin{pmatrix} \alpha\beta & \beta \\ \frac{1}{1+\alpha} & \frac{\alpha}{1+\alpha} \end{pmatrix}^k \begin{pmatrix} \|e_0^x\| \\ \|e_0^y\| \end{pmatrix} \end{aligned}$$

Let

$$T = \begin{pmatrix} \alpha\beta & \beta \\ \frac{1}{1+\alpha} & \frac{\alpha}{1+\alpha} \end{pmatrix}$$

Clearly, if $\rho(T) < 1$, then $\lim_{k\to\infty} T^k = 0$. This implies

$$\lim_{k\to\infty} \|e_k^x\| = 0 \text{ and } \lim_{k\to\infty} \|e_k^y\| = 0.$$

In this way, the iteration sequence $\{x^{(k)}\}$ produced by the BAS iteration method (5) or (6) can achieve to the unique solution of the AVE (1).

Next, we just need to get the sufficient conditions for $\rho(T) < 1$. Let λ represent an eigenvalue of the matrix T. Then λ satisfies

$$(\lambda - \alpha\beta)(\lambda - \frac{\alpha}{1+\alpha}) - \frac{\beta}{1+\alpha} = 0,$$

which is equal to

$$\lambda^2 - (\alpha\beta + \frac{\alpha}{1+\alpha})\lambda - (1-\alpha)\beta = 0. \tag{9}$$

Applying Lemma 1 to Eq. (9), $|\lambda| < 1$ if and only if

$$|(1-\alpha)\beta| < 1$$

and

$$|\alpha\beta + \frac{\alpha}{1+\alpha}| < 1 - (1-\alpha)\beta.$$

Therefore, if the condition (7) holds, then $\rho(T) < 1$. This completes the proof. □

Theorem 2. *Let $\lambda_{\min}$ denote the smallest eigenvalue of matrix A, where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite. If*

$$\alpha < \lambda_{\min},$$

then

$$|||(e_{k+1}^x, e_{k+1}^y)||| < |||(e_k^x, e_k^y)|||, k = 0, 1, \ldots,$$

where

$$|||(e_k^x, e_k^y)||| = \sqrt{\|e_k^x\|^2 + \|e_k^y\|^2}.$$

This implies that the BAS method is convergent.

Proof. By simple calculation, we have

$$\begin{aligned} \beta(1+\alpha) &= (1+\alpha)\|(\alpha I + A)^{-1}\| \\ &= (1+\alpha)\|(\alpha I + A)^{-1}\| \\ &= \frac{1+\alpha}{1+\lambda_{\min}}. \end{aligned}$$

Obviously, when $\alpha < \lambda_{\min}$, $\beta(1+\alpha) < 1$. This complete the proof. □

Corollary 1. *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and $b \in \mathbb{R}^n$. If*

$$||A^{-1}|| \leq \frac{1}{1+2\alpha},$$

then

$$|||(e_{k+1}^x, e_{k+1}^y)||| < |||(e_k^x, e_k^y)|||, k = 0, 1, \ldots,$$

where

$$|||(e_k^x, e_k^y)||| = \sqrt{\|e_k^x\|^2 + \|e_k^y\|^2}.$$

This implies that the BAS iteration method is convergent.

Proof. Based on the Banach perturbation lemma in [23], we obtain

$$\beta(1+\alpha) \leq \frac{(1+\alpha)\|A^{-1}\|}{1-\alpha\|A^{-1}\|}.$$

Obviously, when

$$||A^{-1}|| \leq \frac{1}{1+2\alpha},$$

$\beta(1+\alpha) < 1$. This complete the proof. □

3 Numerical Experiments

In this section, to demonstrate the performance of the BAS method for solving the AVE (1), some numerical experiments are given. To this end, we compare the BAS method with the GN method [9] and the NHSS method in [14,15].

In our computations, we chose zero vector as all initial vectors and all iterations are stopped once the relative residual error meets

$$\frac{\|Ax^{(k)} - |x^{(k)}| - b\|_2}{\|b\|_2} \leq 10^{-6}$$

or if the prescribed iteration number 500 is exceeded. The vector b in (1) is properly chosen such that the vector $x = (x_1, x_2, ..., x_n)^T$ with

$$x_i = (-1)^i i, i = 1, 2, \ldots, n,$$

is the exact solution of the AVE (1). The coefficient matrix $\alpha I + H$ of the first subsystems in the NHSS method is symmetric positive definite and can be solved by the Cholesky factorization, and the coefficient matrix $\alpha I + S$ of the second subsystems in the NHSS method can be solved by the LU factorization. All tests were completed in MATLAB 7.0.

In our numerical experiments, the experimentally found optimal parameters α_{exp} are employed, which result in the least numbers of the BAS and NHSS iterations. Therefore, the optimal parameters employed in the BAS and NHSS iteration methods are used experimentally. As mentioned in [14] the computation

of the optimal parameter is generally difficult to be gained and often problem-dependent.

In our numerical experiments, we consider the two-dimensional convection-diffusion equation

$$\begin{cases} -(u_{xx} + u_{yy}) + q(u_x + u_y) + pu = f(x, y), \ (x, y) \in \Omega, \\ u(x, y) = 0, \ (x, y) \in \partial\Omega, \end{cases} \tag{10}$$

where $p \in \mathbb{R}$ and $q \in \mathbb{R}$ is used to measure the magnitude of the diffusive term, $\Omega = (0, 1) \times (0, 1)$, and $\partial\Omega$ is its boundary, see [15]. On the unit square Ω, using the five-point finite difference technique for the diffusive terms and the central difference technique for the convective terms with the mesh-size $h = 1/(m + 1)$, we obtain the linear equations $Cx = d$, where C is of the form

$$C = T_x \otimes I_m + I_m \otimes T_y + pI_n,$$

and its order is $n = m^2$, $\otimes$ stands for the Kronecker product, I_m and I_n are the identity matrices of order m and n, respectively,

$$T_x = \text{tridiag}(-1 - R_e, 4, -1 + R_e), T_y = \text{tridiag}(-1 - R_e, 0, -1 + R_e)$$

and $R_e = \frac{qh}{2}$ is the mesh Reynolds number. In our numerical experiments, we define the matrix A in AVE (1) by

$$A = C + 2(L - L^T)$$

with L being the strictly lower part of C. In Tables 1, 2 and 3, for different values of n, p and q, the numerical results are listed.

Table 1. Numerical results of $(q, p) = (0, -1)$

	n	400	900	1600	2500	3600
BAS	IT	41	53	69	87	107
	RES	1.8048e–7	2.7662e–7	3.1273e–7	6.2514e–7	3.0472e–7
	α_{exp}	0	0	0	0	0
NHSS	IT	51	70	89	111	137
	RES	9.2033e–7	8.8056e–7	9.9765e–7	9.9698e–7	8.4863e–7
	α_{exp}	7	7	6.9	6.8	7
GN	IT	–	–	–	–	–
	RES	–	–	–	–	–

In Tables 1, 2 and 3, it is easy to find that the number steps of the BAS and NHSS methods increase with the mesh size n increasing. When the GN method is used to solve the AVE, we find that it does not converge in 500 iterations

Table 2. Numerical results of $(q, p) = (0, -0.5)$

	n	400	900	1600	2500	3600
BAS	IT	35	45	55	67	83
	RES	8.7387e–7	4.5606e–7	4.7165e–7	8.1872e–7	7.8321e–7
	α_{exp}	0	0	0	0	0
NHSS	IT	38	51	64	78	92
	RES	6.0528e–7	7.2623e–7	8.9910e–7	8.1588e–7	8.5619e–7
	α_{exp}	5	5.2	4.9	5.4	5.5
GN	IT	–	–	–	–	–
	RES	–	–	–	–	–

Table 3. Numerical results of $(q, p) = (1, -1)$

	n	400	900	1600	2500	3600
BAS	IT	39	51	67	85	105
	RES	4.9489e–7	5.6453e–7	4.3071e–7	8.6319e–7	4.0652e–7
	α_{exp}	0	0	0	0	0
NHSS	IT	51	69	88	110	136
	RES	6.8367e–7	9.7396e–7	9.6429e–7	8.3406e–7	5.8932e–7
	α_{exp}	6.8	6.7	6.8	6.6	7.3
GN	IT	–	–	–	–	–
	RES	–	–	–	–	–

(denoted by '–' in tables). Compared the BAS method with the NHSS method, the number of iterations of the former are less than that of the latter. This implies that when the BAS and NHSS iteration methods are employed, the BAS method overmatches the NHSS method in terms of the number of iterations. From the numerical results in Tables 1, 2 and 3, the BAS method has better computing efficiency, compared with the GN and NHSS methods.

4 Conclusions

In this paper, based on the block-diagonal and anti-block-diagonal splitting (BAS) of the coefficient matrix of the equal two-by-two block nonlinear equation of the AVE, a block-diagonal and anti-block-diagonal splitting (BAS) iteration method is introduced. Some convergence conditions are obtained. Numerical experiments confirm that the BAS method is feasible, robust and efficient for the AVE.

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