A Modified Gauss-Seidel Iteration Method for Solving Absolute Value Equations

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Abstract. The iterative Gauss-Seidel method is an effective and practical method for solving the absolute value equations. However, the solution efficiency of this method usually decreases, and even the equation cannot be solved even when the problem reaches a certain large scale. To improve the efficiency of the Gauss-Seidel method for solving absolute value equations, a modified Gauss-Seidel (MGS) iteration method is presented in this paper. In the our method, we create a diagonal matrix Ω with nonnegative diagonal elements in the Gauss-Seidel matrix splitting. Under the given constraints the convergence theory of the MGS method have been studied. The numerical results show that the method is effective. It can be noted that with the increase in the scale of the problem, the setting effect of the matrix Ω is more obvious.

Keywords: Absolute value equation · Gauss-Seidel splitting · MGS iteration method · Convergence theory

1 Introduction

Let matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, we consider the absolute value equation with the following form

$$Ax - |x| = b, \tag{1.1}$$

where $|x|$ denotes the absolute value of the vector x.

Absolute value equalization is a special, non-differentiatable optimization problem proposed by O.L. Mangasarian in 2006 [1]. Absolute value equalization is widely distributed in the optimization field There are many optimization problems that can be transformed into Eqs. (1.1), such as linear programming

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problems, convex quadratic optimization problems and general linear complementarity problems [2–6]. Formally, the Eq. (1.1) is relatively simple, but in fact the equation is NP-hard in general [1].

Since the absolute equation has been proposed, there are many different methods to solve the Eq. (1.1) from different perspectives have been proposed. In recent years we have found that there are many iterative methods to solve the Eq. (1.1). In [7], D.K. Salkuyeh proposed the iterative method picard-HSS, which is used, to the formula $Ax-|x|=b$, where $A \in \mathbb{C}^{n\times n}$ and $b \in \mathbb{C}^n$. In [8], by the Gauss-Seidel matrix splitting, Edalatpour et al. established the generalized Gauss-Seidel (GGS) iteration method for solving the Eq. (1.1). The Eq. (1.1) is used on the SOR-like iterative method in two non-linear equations with two blocks in both [9] and [10] block converted, Ke et al. and Guo et al. presented the SOR-like iteration methods to solve the Eq. (1.1) respectively.

It is worth mentioning that Edalatpour et al. establishes an iterative Gauss-Seidel method based on the Gauss-Seidel splitting, and analyses its convergence from a specific angle in [8]. However, we find that the GGS method cannot solve the equation when the problem size n becomes larger. Therefore this paper improves the GGS method to improve the efficiency of theGauss-Seidel method in solving the Eq. (1.1). Inspired by the work of Edalatpour et al. in [8], by introducing a diagonal matrix Ω whose diagonal elements are all nonnegative in the splitting of the matrix A, we propose a modified iterative modified Gauss-Seide (MGS) to solve the Eq. (1.1). Some convergence theories of the method are maintained and given limitations proven. In the last phase we give some examples to illustrate the effectiveness of the iterative MGS method.

In this article the rest of the organization is as follows. In the Second section, we list some necessary results in the form of symbols, definitions and frames. The third section, the MGS method for solving the Eq. (1.1) is determined and its convergence under given conditions has been proven. In fourth section numerical examples are given to illustrate the effectiveness of the MGS method and the results of the comparison between the MGS method and the GGS method are given.

2 Preliminaries

For the sake of the subsequent convergence discussions, we list some summary results in this section.

For the given matrices $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $B = (b_{ij}) \in \mathbb{R}^{m\times n}$, then $A \geq B (A > B)$ if $A - B \geq 0 (A - B > 0)$. The absolute value of the matrix A is denoted by $|A| = (|a_{ij}|)$. The infinity norm of the matrix A is defined as

$$\|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|.$$

Moreover, the definitions of the absolute value and infinity norm of the matrices can be applied to the vectors.

Suppose $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, then the comparison matrix of A is defined as $\langle A\rangle = (\langle a_{ij}\rangle)$, where

$$\langle a_{ij}\rangle = \begin{cases} |a_{ij}|, & for \quad i=j, \\ -|a_{ij}|, & for \quad i\neq j, \end{cases} \quad i,j=1,2,\cdots,n.$$

The above results can be seen in [11].

Next, we will list some special matrices for the sequel discussions. Assume that the matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, then A is said to be [12]

(1) a Z-matrix if $a_{ij} \leq 0$ holds for all $i \neq j$.
(2) an M-matrix if $A^{-1} \geq 0$ and A is a Z-matrix.
(3) an H-matrix if $\langle A\rangle$ is an M-matrix;
(4) an H_+-matrix if A is an H-matrix and $a_{ij} > 0$ holds for all $i = j$.

Lemma 1. *[13] If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ be two matrices which satisfy $B \geq A$ and $b_{ij} \leq 0$ for any $i \neq j$. Then the matrix B is an M-matrix if A is an M-matrix.*

Lemma 2. *[14] If A be an H-matrix. Then $\langle A\rangle^{-1} \geq |A^{-1}|$.*

Lemma 3. *[11] If A be a $n\times n$ matrix and $A = M - N$ be a regular splitting of the matrix A. Then A is nonsingular with $A^{-1} \geq 0$ if and only if $\rho(M^{-1}N) < 1$.*

Lemma 4. *[11] If A be a $n\times n$ nonnegative matrix. Then $I - A$ is nonsingular with $(I-A)^{-1} \geq 0$ if and only if $\rho(A) < 1$.*

Lemma 5. *[11] Let x, y be two vectors $\in \mathbb{R}^n$. Then $\|x-y\|_\infty \geq \||x|-|y|\|_\infty$.*

3 The MGS Method

For the Eq. (1.1), we make the following matrix splitting

$$A = D - L - U = (\Omega + D - L) - (\Omega + U),$$

where D, L and U, respectively, are the diagonal, the strictly lower-triangular and the strictly upper-triangular matrices of A, and Ω is a $n\times n$ nonnegative diagonal matrix.

Based on the above splitting, we can convert the Eq. (1.1) to the fixed-point equation with the form

$$(\Omega + D - L)x - |x| = (\Omega + U)x + b. \tag{3.1}$$

Then, we present a modified Gauss-Seidel (MGS) iteration method to solve the Eq. (1.1) which has the iterative scheme as follows

$$(\Omega + D - L)x^{(k+1)} - |x^{(k+1)}| = (\Omega + U)x^{(k)} + b, k = 0, 1, 2, \ldots, \tag{3.2}$$

where the initial vector $x^{(0)}$ is given in advance by the experimenter.

By adjusting the matrix Ω, we expect to be able to improve the solving efficiency of the Gauss-Seidel method for the Eq. (1.1). It is easy to see that the MGS method just reduces to the GGS method when $\Omega = 0$.

In the following, we will discuss the convergence properties for the MGS method.

Theorem 1. *Assume that the Eq. (1.1) is solvable. Suppose the diagonal entries of the matrix A are all greater than 1, the matrix Ω is a $n \times n$ nonnegative diagonal matrix, the matrix I is the $n \times n$ identity matrix and the matrix $\Omega + D - L - I$ be strictly row diagonally dominant. If*

$$\|(\Omega + D - L)^{-1}(\Omega + U)\|_\infty < 1 - \|(\Omega + D - L)^{-1}\|_\infty, \tag{3.3}$$

then the iteration sequence $\{x^{(k)}\}_{k=0}^{\infty}$ generated by (3.2) converges to the unique solution x^* of the Eq. (1.1) for any initial vector $x^{(0)} \in \mathbb{R}^n$.

Proof. By the conditions of the theorem, we can obtain that the diagonal entries of the matrix D are all greater than 1.

Firstly, we will prove that $\|(\Omega + D - L)^{-1}\|_\infty < 1$.

For $L = 0$, because Ω is a nonnegative diagonal matrix and the diagonal entries of D are all greater than 1, it can be shown that

$$\|(\Omega + D - L)^{-1}\|_\infty = \|(\Omega + D)^{-1}\|_\infty < 1.$$

We assume that $L \neq 0$ in the following. By the assumption of the theory, one can get

$$0 \leq |L|e < (\Omega + D - I)e,$$

or equivalently,

$$(\Omega + D)^{-1}e < (I - |F|)e, \tag{3.4}$$

where $e = (1, 1, \cdots, 1)^T$ and $F = (\Omega + D)^{-1}L$. In addition, we have

$$\begin{aligned} 0 \leq |(I - F)^{-1}| &= |I + F + F^2 + \cdots + F^{n-1}| \\ &\leq (I + |F| + |F|^2 + \cdots + |F|^{n-1}) = (I - |F|^{-1}). \end{aligned} \tag{3.5}$$

Hence, from (3.4) and (3.5), we obtain

$$\begin{aligned} |(\Omega + D - L)^{-1}|e &= |(I - F)^{-1}(\Omega + D)^{-1}|e \\ &\leq |(I - F)^{-1}||(\Omega + D)^{-1}|e \\ &< |(I - |F|)^{-1}||I - |F||e = e. \end{aligned}$$

Therefore,

$$\|(\Omega + D - L)^{-1}\|_\infty < 1. \tag{3.6}$$

Next, we are going to show that the Eq. (1.1) has an unique solution. Assume that x^* and y^* are two different solutions of the Eq. (1.1). From the Eq. (3.1), we have

$$x^* = (\Omega + D - L)^{-1}|x^*| + (\Omega + D - L)^{-1}[(\Omega + U)x^* + b], \tag{3.7}$$
$$y^* = (\Omega + D - L)^{-1}|y^*| + (\Omega + D - L)^{-1}[(\Omega + U)y^* + b],$$

then

$$x^* - y^* = (\Omega + D - L)^{-1}(|x^*| - |y^*|) + (\Omega + D - L)^{-1}(\Omega + U)(x^* - y^*).$$

By taking infinity norm on both sides of the latter equation, it holds from Lemma 5 and the Eq. (3.3) that

$$\begin{aligned}\|x^* - y^*\|_\infty \le& \|(\Omega + D - L)^{-1}\|_\infty \||x^*| - |y^*|\|_\infty \\ &+ \|(\Omega + D - L)^{-1}(\Omega + U)\|_\infty \|x^* - y^*\|_\infty \\ <& \|(\Omega + D - L)^{-1}\|_\infty \|x^* - y^*\|_\infty \\ &+ (1 - \|(\Omega + D - L)^{-1}\|_\infty)\|x^* - y^*\|_\infty \\ =& \|x^* - y^*\|_\infty,\end{aligned}$$

which is a contradiction. Thus, $x^* = y^*$.

Finally, we will prove that the iteration sequence $\{x^{(k)}\}_{k=0}^\infty$ generated by (3.2) converges to the unique solution x^* of the Eq. (1.1). From (3.2), we get

$$x^{(k+1)} = (\Omega + D - L)^{-1}|x^{(k+1)}| + (\Omega + D - L)^{-1}[(\Omega + U)x^{(k)} + b]. \tag{3.8}$$

From (3.7) and (3.8), it holds that

$$x^{(k+1)} - x^* = (\Omega + D - L)^{-1}(|x^{(k+1)}| - |x^*|) + (\Omega + D - L)^{-1}(\Omega + U)(x^{(k)} - x^*).$$

Taking infinity norm on both sides of the latter equation. By similarly calculations, we get the following results

$$\begin{aligned}\|x^{(k+1)} - x^*\|_\infty =& \|(\Omega + D - L)^{-1}(|x^{(k+1)}| - |x^*|) \\ &+ (\Omega + D - L)^{-1}(\Omega + U)(x^{(k)} - x^*)\|_\infty \\ \le& \|(\Omega + D - L)^{-1}(|x^{(k+1)}| - |x^*|)\|_\infty \\ &+ \|(\Omega + D - L)^{-1}(\Omega + U)(x^{(k)} - x^*)\|_\infty \\ \le& \|(\Omega + D - L)^{-1}\|_\infty \||x^{(k+1)}| - |x^*|\|_\infty \\ &+ \|(\Omega + D - L)^{-1}(\Omega + U)\|_\infty \|x^{(k)} - x^*\|_\infty,\end{aligned}$$

which equivalent to

$$\begin{aligned}\|x^{(k+1)} - x^*\|_\infty - \|(\Omega + D - L)^{-1}\|_\infty \||x^{(k+1)}| - |x^*|\|_\infty \\ \le \|(\Omega + D - L)^{-1}(\Omega + U)\|_\infty \|x^{(k)} - x^*\|_\infty.\end{aligned}$$

By Lemma 5, we obtain the following formula

$$\|x^{(k+1)} - x^*\|_\infty - \|(\Omega + D - L)^{-1}\|_\infty \|x^{(k+1)} - x^*\|_\infty \\ \leq \|(\Omega + D - L)^{-1}(\Omega + U)\|_\infty \|x^{(k)} - x^*\|_\infty.$$

By a simple calculation, one can see that the above formula is equivalent to

$$(1 - \|(\Omega + D - L)^{-1}\|_\infty)\|x^{(k+1)} - x^*\|_\infty \leq \|(\Omega + D - L)^{-1}(\Omega + U)\|_\infty \|x^{(k)} - x^*\|_\infty.$$

Since $\|(\Omega + D - L)^{-1}\|_\infty < 1$, then

$$\|x^{(k+1)} - x^*\|_\infty \leq \frac{\|(\Omega + D - L)^{-1}(\Omega + U)\|_\infty}{1 - \|(\Omega + D - L)^{-1}\|_\infty} \|x^{(k)} - x^*\|_\infty.$$

By the above inequality, we can find that the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ converges to the unique solution x^* when the condition (3.3) is fulfilled.

The convergence theory of the MGS method to solve the Eq. (1.1) is proved.

By the Theorem 1, one can obtain the following corollary easily.

Corollary 1. *Assume that the Eq. (1.1) is solvable. Suppose the matrix $A - I$ is a strictly row diagonally dominant matrix with positive diagonal entries and the matrix Ω is a nonnegative diagonal matrix. If*

$$\|(\Omega + D - L)^{-1}(\Omega + U)\|_\infty < 1 - \|(\Omega + D - L)^{-1}\|_\infty, \tag{3.9}$$

then the iteration sequence $\{x^{(k)}\}_{k=0}^{\infty}$ generated by (3.2) converges to the unique solution x^* of the Eq. (1.1) for any initial vector $x^{(0)} \in \mathbb{R}^n$.

Following, we will demonstrate the convergence property of the MGS method when the matrix A is an H-matrix.

Theorem 2. *Assume that the Eq. (1.1) is solvable. Suppose the matrix $A - I$ is an H_+-matrix and the matrix Ω is a nonnegative diagonal matrix. Then the iteration sequence $\{x^{(k)}\}_{k=0}^{\infty}$ obtained from (3.2) converges to the unique solution x^* for any initial vector $x^{(0)}$.*

Proof. From the mentioned splitting in this section

$$A = (\Omega + D - L) - (\Omega + U),$$

by the definition of the comparison matrix, we can get

$$\langle A \rangle = \langle \Omega + D - L \rangle - |\Omega + U|.$$

Then, we obtain

$$\langle A - I \rangle \leq \langle A \rangle \leq \langle \Omega + D - L \rangle \leq \mathrm{diag}(\Omega + D - L) = \Omega + D.$$

Therefore, by Lemma 1, it can be easy to know that $\Omega + D - L$ is an H-matrix. With that, we get the following formula by Lemma 2

$$|(\Omega + D - L)^{-1}| \leq \langle \Omega + D - L \rangle^{-1}. \tag{3.10}$$

Since x^* is the solution of the Eq. (1.1), from (3.2) and (3.7) we obtain

$$x^{(k+1)} - x^* = (\Omega + D - L)^{-1}(|x^{(k+1)}| - |x^*|) + (\Omega + D - L)^{-1}(\Omega + U)(x^{(k)} - x^*).$$

By the inequality (3.10), by taking absolute values on both sides of the latter equation, we get

$$\begin{aligned} |x^{(k+1)} - x^*| \leq& |(\Omega + D - L)^{-1}||x^{(k+1)} - x^*| + |(\Omega + D - L)^{-1}||\Omega + U||x^{(k)} - x^*| \\ \leq& \langle \Omega + D - L \rangle^{-1}|x^{(k+1)} - x^*| + \langle \Omega + D - L \rangle^{-1}|\Omega + U||x^{(k)} - x^*|. \end{aligned} \tag{3.11}$$

Let $G = \langle \Omega + D - L \rangle^{-1}$. Because of the matrix $\Omega + D - L$ is an H-matrix, we can know that the matrix $\langle \Omega + D - L \rangle$ is an M-matrix with $\langle \Omega + D - L \rangle^{-1} \geq 0$. In addition, it is obviously shown that $\rho(G) < 1$ because $A - I$ is an H_+-matrix. By Lemma 4, one can derive that $I - G$ is nonsingular and $(I - G)^{-1} \geq 0$. Therefore, it follows from the Eq. (3.11) that

$$\begin{aligned} |x^{(k+1)} - x^*| \leq& (I - \langle \Omega + D - L \rangle^{-1})^{-1} \langle \Omega + D - L \rangle^{-1} |\Omega + U||x^{(k)} - x^*| \\ \leq& (\langle \Omega + D - L \rangle - I)^{-1}|\Omega + U||x^{(k)} - x^*|. \end{aligned}$$

Let $\widetilde{M} = \langle \Omega + D - L \rangle - I$ and $\widetilde{N} = |\Omega + U|$. Assume that $\widetilde{A} = \widetilde{M} - \widetilde{N}$, it just be a matrix splitting of $\widetilde{A}$, then $\widetilde{G} = \widetilde{M}^{-1}\widetilde{N}$ is the iteration matrix corresponding to the splitting. As we all know that the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ converges to x^* if $\rho(\widetilde{G}) < 1$. Since the matrix $A - I$ is an H_+-matrix, the diagonal entries of A are all greater than one. It is obviously that $\widetilde{A} = \langle A \rangle - I = \langle A - I \rangle$ and $\widetilde{M} = \langle \Omega + D - L - I \rangle$. Hence, by Lemma 1, we know that $\widetilde{A}$ is an M-matrix and then $\widetilde{M}$ is also an M-matrix because of $\widetilde{M} > \widetilde{A}$. Moreover, the matrix splitting $\widetilde{A} = \widetilde{M} - \widetilde{N}$ is a regular splitting because of $\widetilde{N} \geq 0$. Then, by Lemma 3, it holds true that

$$\rho(\widetilde{G}) = \rho(\widetilde{M}^{-1}\widetilde{N}) < 1.$$

The conclusion is obtained.

4 Numerical Examples

In this section, three examples are used to verify the efficiency of the MGS method to solve the Eq. (1.1). We compare the MGS method with the GGS method in the iteration steps (IT), CPU time in seconds (CPU).

In the examples, the right-hand-side vector b is uniquely determined by substituting the vector $x^* = (x_1, x_2, ..., x_n)^T (x_i = (-1)^i, i = 1, 2, ..., n)$ into the Eq. (1.1). Let $\Omega = \alpha I$ with $\alpha \geq 0$, the values of α is obtained by the experiments. For the sake of clarity, we denote the experimental optimum parameter as α_{exp}

in the following examples. All of the runs terminated if the current iteration satisfies $IT \geq 500$ or $ERR < 10^{-9}$, where

$$ERR = \frac{\|Ax^{(k)} - |x^{(k)}| - b\|_2}{\|b\|_2}.$$

Example 1. [15] Let $m \in \mathbb{N}^+$ and $n = m^2$, we consider the Eq. (1.1) in which $A = M + \mu I \in \mathbb{R}^{n \times n}$, where

$$M = tridiag(-I, S, -I) \in \mathbb{R}^{n \times n},$$

with

$$\begin{aligned} S &= tridiag(-1, 4, -1) \\ &= \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 4 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 4 \end{pmatrix} \in \mathbb{R}^{m \times m}, \end{aligned}$$

and I being the m order identity matrix. $b = Ax^* - |x^*| \in \mathbb{R}^n$, here $x^* = (-1, 1, ..., (-1)^n)^T$.

Table 1. Numerical results for Example 1

M	Method	$\mu = 0$			$\mu = -0.5$		
		α_{exp}	IT	CPU	α_{exp}	IT	CPU
20	GGS	–	112	0.1875	–	68	0.1719
	MGS	0.10	111	0.1563	0	68	0.1719
40	GGS	–	112	3.8125	–	73	4.4063
	MGS	0.20	102	3.4688	0	73	4.4063
60	GGS	–	115	22.0469	–	75	25.7813
	MGS	0.20	100	19.1406	0	75	25.7813
80	GGS	–	119	121.5781	–	76	81.2188
	MGS	0.20	98	125.5938	0.10	75	47.2188
100	GGS	–	122	315.4375	–	77	133.1875
	MGS	0.20	97	321.0781	0.10	75	127.8438

We use the methods GGS and MGS to use the Eq. (1.1) of Example 1 separately. Table 1 lists the calculation results of the CPU time of the above two methods for different problem sizes of n are listed when $\mu = 0$ and $\mu = -0.5$.

For $\mu = 0$, we found that the MGS method is smaller than the GGS method when the parameter α_{exp} is correctly specified. In other words, the matrix $\Omega = \alpha I$ plays an important regulating role in our method, and with the magnification of the problem the adaptation effect of the matrix Ω is more obvious. For $\mu = -0.5$, we can see that the parameter $\alpha_{exp} = 0$ when the scale of the problem is small ($m = 20, 40, 60$), and the MGS method is reduced exactly to GGS. It is not difficult to see that the MGS method is also superior to the GGS method when we choose $\alpha_{exp} = 0.1$ with the larger problem size ($m = 80, 100$).

Example 2. [15] Let $m \in \mathbb{N}^+$ and $n = m^2$, we consider the Eq. (1.1) in which $A = M + \mu I \in \mathbb{R}^{n\times n}$, where

$$M = tridiag(-1.5I, S, -0.5I) \in \mathbb{R}^{n\times n},$$

with

$$S = tridiag(-1.5, 4, -0.5) = \begin{pmatrix} 4 & -0.5 & 0 & \cdots & 0 & 0 \\ -1.5 & 4 & -0.5 & \cdots & 0 & 0 \\ 0 & -1.5 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & 4 & -0.5 \\ 0 & 0 & \ddots & \ddots & -1.5 & 4 \end{pmatrix} \in \mathbb{R}^{m\times m},$$

and I being the m order identity matrix. $b = Ax^* - |x^*| \in \mathbb{R}^n$, here $x^* = (-1, 1, ..., (-1)^n)^T$.

Table 2. Numerical results for Example 2

M	Method	$\mu = 0$			$\mu = -0.5$		
		α_{exp}	IT	CPU	α_{exp}	IT	CPU
20	GGS	–	143	0.6563	–	Fail	–
	MGS	0	143	0.6563	0.20	263	1.1719
40	GGS	–	Fail	–	–	Fail	–
	MGS	0.075	284	12.8594	0.29	263	12.0156
60	GGS	–	Fail	–	–	Fail	–
	MGS	0.19	444	91.1875	0.30	255	55.9375
80	GGS	–	Fail	–	–	Fail	–
	MGS	–	Fail	–	0.30	249	157.1563
100	GGS	–	Fail	–	–	Fail	–
	MGS	–	Fail	–	0.30	244	366.7656

For Example 2, we also list the calculation data related to the Example 2 for different problem sizes of n when $\mu = 0$ and $\mu = -0.5$ in Table 2. The table shows $'Fail'$ that the method cannot find a solution if the abortion condition is met, such as Example 4.2.

Table 2 shows that the GGS method can solve the problem of $\mu = 0$ when $m = 20$ and and the GGS method is $'Fail'$ when $m > 20$. Compared with the GGS method, the MGS method can solve the problem when $m \leq 60$. For $\mu = -0.5$, it is obvious that the GGS method cannot find the solution for all the given m in Table 2. Nevertheless, the MGS method can solve the problems for all the given m when the the parameters α_{exp} are suitable choice. That is to say, the regulating role of the matrix $\Omega = \alpha I$ becomes more obvious as the problem size has increased.

Example 3. Let $m \in \mathbb{N}^+$ and $n = m^2$, we consider the Eq. (1.1) in which $A = M + \mu I \in \mathbb{R}^{n\times n}$, where

$$M = tridiag(-0.5I, S, -1.5I) \in \mathbb{R}^{n\times n},$$

with

$$S = tridiag(-0.5, 4, -1.5) = \begin{pmatrix} 4 & -1.5 & 0 & \cdots & 0 & 0 \\ -0.5 & 4 & -1.5 & \cdots & 0 & 0 \\ 0 & -0.5 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & 4 & -1.5 \\ 0 & 0 & \ddots & \ddots & -0.5 & 4 \end{pmatrix} \in \mathbb{R}^{m\times m},$$

and I being the m order identity matrix. $b = Ax^* - |x^*| \in \mathbb{R}^n$, here $x^* = (-1, 1, ..., (-1)^n)^T$.

Table 3. Numerical results for Example 3

M	Method	$\mu = 0$			$\mu = -0.5$		
		α_{exp}	IT	CPU	α_{exp}	IT	CPU
20	GGS	–	143	0.7344	–	Fail	–
	MGS	0	143	0.7344	0.20	263	1.2344
40	GGS	–	Fail	–	–	Fail	–
	MGS	0.075	284	13.8438	0.29	263	11.0469
60	GGS	–	Fail	–	–	Fail	–
	MGS	0.19	444	90.2563	0.30	255	45.4688
80	GGS	–	Fail	–	–	Fail	–
	MGS	0.20	462	220.3634	0.30	249	154.3366
100	GGS	–	Fail	–	–	Fail	–
	MGS	–	Fail	–	0.30	243	328.6675

To better observe the adaptation effect of the matrix $\Omega = \alpha I$ in MGS method, we construct Example 3. Similar to the Example 1 and Example 2, we also list the calculation data for different problem sizes of n when $\mu = 0$ and $\mu = -0.5$ in Table 3. In the table, $'Fail'$ still denotes that the method cannot find the solution for Example 3 when the termination conditions are satisfied.

As shown in Table 3, if the parameter α_{exp} is the right choice, the MGS method can solve more problems than the GGS method. The results also show that Matrix Ω plays an important role in the solution.

5 Conclusion

In this paper, by introducing a non-negative diagonal matrix Ω, a new split of the matrix A in the absolute Eq. (1.1) is given firstly. And then, based on the new splitting, we have presented the MGS method for solving the Eq. (1.1) and discussed the convergence theory of the method. The numerical results show that the MGS method is better than the GGS method if the matrix Ω is appropriate. In general, the matrix Ω plays an important role in our method. It can effectively improve not only the convergence rate, but also the iterative Gauss-Seidel method for solving Eqs. (1.1). It creates some uncertainty that the parameter α in the matrix $\Omega = \alpha I$ is acquired by the experiments. This affects the efficiency of the algorithm to some extent. In the following research, we will try to determine the optimal parameters α through the theoretical derivation and further improve the efficiency of the method.

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