Estimates for Lipschitz and BMO Norms of Operators on Differential Forms

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Abstract In this paper, we introduce the generalized Lipschitz and BMO norms of differential forms and establish the upper bound estimates for the generalized Lipschitz and BMO norms of operators applied to differential forms. We also demonstrate applications of our main results using examples.

1 Introduction

The main purpose of this paper is to establish the upper bound estimates for the generalized Lipschitz and BMO norms of the iterated operators $D^k G^k$ and $D^{k+1} G^k$ applied to differential forms u defined in $\mathbb{R}^n$ in terms of the L^p norms of u, where k is a positive integer; G is Green's operator and $D = d + d^\star$ is the Hodge-Dirac operator on differential forms. The Dirac operator D and Green's operator G are very well studied and widely used in many fields of mathematics and physics. They play a critical role in the study of the nonlinear problems in PDEs and nonlinear potential theory. For example, in the case $k = 1$, the composition $D^2 G$ is used to define the well-known Poisson's equation $D^2 G(u) = u - H(u)$ (or $\Delta G(u) = u - H(u)$), where H is the harmonic projection operator. In the same sense as the L^p theory, the estimates for the BMO norms of differential forms and the related operators are also decisive on the investigation of the solution properties of PDEs, especially on the study of Harnack's inequality for solutions

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to certain partial differential equations, see [1] for example. Some estimates for the BMO norm and local Lipschitz norm of differential forms or related operators can be found in [2–5]. We should notice that D^kG^k and $D^{k+1}G^k$ are more general operators which include the composite operator DG as a special case, see [5] where DG has been investigated. However, there is no systematic study on the BMO norm and local Lipschitz norm of the iterated operators D^kG^k and $D^{k+1}G^k$ for the case $k > 1$ in the literature. Hence, we are motivated to establish the upper bound estimates for the generalized Lipschitz and BMO norms of the composite operator in this paper. We first extend the definitions of the classical locLip_α and BMO norms into the generalized locLip_α^s and BMO^s norms, respectively. Then, we study the relationship between these two norms and L^p norms. The estimates for norms and comparisons of norms are very important in the investigation of the corresponding spaces in analysis. For example, it is well known that the BMO space, the dual of Hardy space, is a substitute of L^∞ space and has been playing a very indispensable role in harmonic analysis and exterior differential analysis, as well as in the study of the characterization of singular integral operators since it was set forth by John and Nirenberg in 1961. We refer the readers to Chapter IV in [6] and [1, 7] for the function case of the BMO space, and Chapter 9 in [2] and [8–11] for the case to differential forms. Our main results are presented and proved in Section 3. These results will enrich the theory of operators on differential forms.

Unless stated otherwise, we keep using the traditional notation and symbols throughout this paper. Let Ω be a smoothly bounded domain without the boundary in $\mathbb{R}^n$, $n \geq 2$, and $B = B(x, \rho)$ be the ball in $\mathbb{R}^n$ with radius ρ centered at x, which satisfies $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$. Let the direct sum $\Lambda = \Lambda(\mathbb{R}^n) = \oplus_{l=0}^{n} \Lambda^l(\mathbb{R}^n)$ be a graded algebra with respect to the exterior product, and $\Lambda^l = \Lambda^l(\mathbb{R}^n)$ be the space of l-covectors in $\mathbb{R}^n$, which is spanned by the dual orthogonal basis $dx_{i_1}, \cdots, dx_{i_l}$, where $x_{i_1}, \cdots, x_{i_l}$ are the coordinate functions on $\mathbb{R}^n$. For the set Λ, we denote the pointwise inner product by $< \cdot, \cdot >$ and the module by $|\cdot|$. Then, every differential form $u(x) \in \Lambda^l(\mathbb{R}^n)$ can be uniquely written as

$$u(x) = \sum_I u_I(x) dx_I = \sum u_{i_1 i_2 \cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l},$$

where the coefficients $u_{i_1 i_2 \cdots i_l}(x)$ are differentiable functions and $I = (i_1, i_2, \cdots, i_l)$, $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. Actually, differential forms are the generalizations of the functions, which include functions as their special cases (functions are called 0-forms). The Hodge-star operator $\star : \Lambda^l(\mathbb{R}^n) \to \Lambda^{n-l}(\mathbb{R}^n)$ is defined by the rule that $\star 1 = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ and $\alpha \wedge \star\beta = < \alpha, \beta > (\star 1)$ for every $\alpha, \beta \in \Lambda^l$, $l = 0, 1, \cdots, n$. By this definition, it induces that $\star$ is an isometric isomorphism on Λ^l. The linear operator $d : \mathbb{D}'(\Omega, \Lambda^l) \to \mathbb{D}'(\Omega, \Lambda^{l+1})$, $0 \leq l \leq n-1$, is called the exterior differential and $d^\star = (-1)^{nl+1} \star d \star : \mathbb{D}'(\Omega, \Lambda^{l+1}) \to \mathbb{D}'(\Omega, \Lambda^l)$, the formal adjoint of d, is known as Hodge codifferential. The interested readers could see [10–13] for further introduction and appropriate properties. Also, we use $L^p(\Omega, \Lambda)$ to denote the classical L^p space for differential forms, $1 < p < \infty$, equipped with the norm $\|u\|_{p,\Omega} = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}} = \left(\int_\Omega (\sum_I |u_I|^2)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}$. $W^{1,p}(\Omega, \Lambda)$ is the classical Sobolev space for differential forms with the norm $\|u\|_{W^{1,p}(\Omega)} = (diam(\Omega))^{-1}\|u\|_{p,\Omega} + \|\nabla u\|_{p,\Omega}$. $W_d^p(\Omega, \Lambda^l)$ is the space of differential l-forms such that $du \in L^p(\Omega, \Lambda^l)$. Analogously, $W_{d^\star}^p(\Omega, \Lambda^l)$ is the space of differential l-forms such that $d^\star u \in L^p(\Omega, \Lambda^l)$. Inspired by these classical spaces for differential forms, we generalize the BMO space and local Lipschitz space as follows.

Definition 1.1 For every $\omega \in L^s_{loc}(\Omega, \Lambda^l)$, $s \geq 1$, we say $\omega \in \mathrm{BMO}^s(\Omega, \Lambda^l)$ with the norm defined by

$$\|\omega\|_{*,s,\Omega} = \sup_{\sigma Q \subset \Omega} |Q|^{-1/s} \|\omega - \omega_Q\|_{s,Q}, \tag{1.1}$$

if ω satisfies $\sup_{\sigma Q \subset \Omega} |Q|^{-1/s}\|\omega - \omega_Q\|_{s,Q} < \infty$, where $l = 0, 1, \cdots, n$ and $\sigma > 1$ is some expansion factor.

Definition 1.2 For every $\omega \in L^s_{loc}(\Omega, \Lambda^l)$, $s \geq 1$, $l = 0, 1, \cdots, n$ and $0 < \alpha \leq 1$, we call $\omega \in \mathrm{loc}Lip^s_\alpha(\Omega, \Lambda^l)$ with the norm denoted by

$$\|\omega\|_{locLip^s_\alpha(\Omega)} = \sup_{\sigma Q \subset \Omega} |Q|^{-(n+\alpha s)/sn} \|\omega - \omega_Q\|_{s,Q}, \tag{1.2}$$

if ω satisfies $\sup_{\sigma Q \subset \Omega} |Q|^{-(n+\alpha s)/sn}\|\omega - \omega_Q\|_{s,Q} < \infty$, where $\sigma > 1$ is some expansion factor.

Especially, for the case $s = 1$, the BMO^s norm and locLip^s_α norm just reduce to the following classical BMO norm and locLip_α norm given in [10] by C. Nolder, respectively.

$$\|\omega\|_{*,1,\Omega} = \|\omega\|_{*,\Omega} = \sup_{\sigma Q \subset \Omega} |Q|^{-1} \|\omega - \omega_Q\|_{1,Q} \tag{1.3}$$

and

$$\|\omega\|_{locLip^1_\alpha(\Omega)} = \|\omega\|_{locLip_\alpha(\Omega)} = \sup_{\sigma Q \subset \Omega} |Q|^{-(n+\alpha)/n} \|\omega - \omega_Q\|_{1,Q} \tag{1.4}$$

Furthermore, notice that $|Q|^{\alpha/n} \leq |\Omega|^{1/n}$ since $0 < \alpha \leq 1$ and $n \geq 1$, which results in that

$$|Q|^{-1/s} = |Q|^{\alpha/n}|Q|^{-1/s-\alpha/n} \leq |\Omega|^{1/n}|Q|^{-1/s-\alpha/n}$$

So, similarly as the result in [14], we have that there is a constant $C > 0$, independent of ω, such that

$$\|\omega\|_{*,s,\Omega} \leq C\|\omega\|_{locLip^s_\alpha(\Omega)} \tag{1.5}$$

for every $\omega \in W^{1,s}(\Omega, \Lambda^l)$, which enables us to compare the locLip^s_α norm and the BMO^s norm for $D^k G^k$ and $D^{k+1}G^k$ simply. In addition, from now on, we point out that the constants C and C_i employed in this paper, $i = 1, 2, \cdots$, may differ from one line to the next.

2 Local Poincaré-Type Inequalities

In this section, as preparation for the principle assertion, we show the explicit formulas of $D^k G^k$ and $D^{k+1}G^k$ and the Poincaré-type inequalities of $D^k G^k$ and $D^{k+1}G^k$ by applying the explicit representation in Lemma 2.4 and Lemma 2.5. First, let us start with the brief review of Green's operator G. For any fixed integer $l = 0, 1, \cdots, n$, let $\mathbb{H}$ be the harmonic l-field denoted by

$$\mathbb{H} = \{u \in W(\Omega, \Lambda) : du = d^\star u = 0,\ u \in L^p,\ \text{for some } 1 < p < \infty\}.$$

In the meantime, we take the operator $\delta : L^p(\Omega, \Lambda) \bigcap \mathbb{H}^\perp \to W^{1,p}(\Omega, \Lambda) \bigcap \mathbb{H}^\perp$ defined by Morrey in [15], which satisfies that for every $u \in L^p(\Omega, \Lambda) \bigcap \mathbb{H}^\perp$, $\delta(u)$ is the unique form in $W^{1,p}(\Omega, \Lambda) \bigcap \mathbb{H}^\perp$ such that $\Delta\delta(u) = u$, where $\triangle = D^2 = dd^\star + d^\star d$ is the Laplace operator, and $\mathbb{H}^\perp$ is the complement space of harmonic field $\mathbb{H}$. Therefore, we are given the definition as follows.

Definition 2.1 ([16]) Green's operator $G : L^p(\Omega, \Lambda) \to W^{1,p}(\Omega, \Lambda) \cap \mathbb{H}^\perp$, $1 < p < \infty$, is defined by

$$G(u) = \delta(u - H(u))$$

for every $u \in L^p(\Omega, \Lambda)$, where $H : L^p(\Omega, \Lambda) \to \mathbb{H}$ is the projection operator. Moreover, observe that $\Delta\delta(u) = u$, so we have that

$$\Delta G(u) = u - H(u). \tag{2.1}$$

By employing the classical dominated convergence theorem, C. Scott in [16] further gave the upper bound estimate of Green's operator G.

Lemma 2.2 *Let $u \in L^s(\Omega, \Lambda)$, $1 < s < \infty$, be a differential form defined in the domain Ω. Then, there exists a positive constant C, independent of u, such that*

$$\|dd^\star G(u)\|_{s,B} + \|d^\star dG(u)\|_{s,B} + \|dG(u)\|_{s,B} + \|d^\star G(u)\|_{s,B} + \|G(u)\|_{s,B} \leq C(s)\|u\|_{s,\sigma B} \tag{2.2}$$

for any ball $B \subset \sigma B \subset \Omega$ with some constant $\sigma > 1$, where Ω is a smoothly bounded domain without boundary.

Remark 1 For any $v \in L^p(\Omega, \Lambda) \bigcap \mathbb{H}^{\perp}$, by the definition of the projection operator H, it is easy to obtain that $H(v) = 0$. Since $G(u) \in W^{1,p}(\Omega, \Lambda) \cap \mathbb{H}^{\perp}$ for every $u \in L^p(\Omega, \Lambda)$, replacing v with $G(u)$ yields that $H(v) = HG(u) = 0$. In other words, the harmonic projection of Green's operator G on $L^p(\Omega, \Lambda)$ is always equal to zero.

Remark 2 Also, applying Lemma 2.2 repeatedly, it is obvious to achieve that there is a constant $C > 0$, independent of u, such that

$$\|G^m(u)\|_{p,B} \leq C\|u\|_{p,\sigma B}. \tag{2.3}$$

In particular, if $u \in W_d^p(\Omega, \Lambda)$ (or $u \in W_{d^\star}^p(\Omega, \Lambda)$), we know that Green's operator G can commute with d (or $d^\star$), which implies that

$$dG(u) = G(du) \text{ or } d^\star G(u) = G(d^\star u).$$

Similarly to the method employed in (2.3), we have that

$$\|dG^m(u)\|_{p,B} \leq C\|du\|_{p,\sigma B} \quad \text{or} \quad \|d^\star G^m(u)\|_{p,B} \leq C\|d^\star u\|_{p,\sigma B} \tag{2.4}$$

for any integer $m \geq 1$, where $\sigma > 1$ is some constant.

Meanwhile, to facilitate the upcoming argument about the Poincaré-type estimates in Theorem 2.6 and Theorem 2.7, we need the following results as well.

Lemma 2.3 ([17]) *Let $v \in L^p_{loc}(\Omega, \Lambda^l)$, $1 < p < \infty$, be a differential form defined in Ω and $T : L^p(\Omega, \Lambda^l) \to W^{1,p}(\Omega, \Lambda^{l-1})$ be the homotopy operator, $l = 1, 2, \cdots, n$. Then, we have*

$$v = d(Tv) + T(dv), \tag{2.5}$$

$$\|\nabla(Tv)\|_{p,\Omega} \leq C|\Omega|\|v\|_{p,\Omega} \quad \text{and} \quad \|Tv\|_{p,\Omega} \leq C|\Omega|diam(\Omega)\|v\|_{p,\Omega} \tag{2.6}$$

hold for any bounded and convex domain Ω.

Before starting the primary argument in this section, it is worth to note that the explicit representations in Lemma 2.4 and Lemma 2.5 are the essential steps for the argument of the Poincaé-type inequalities. In precise, if our attention is only to estimate $\|D^kG^k(u)\|_{locLip^s_\alpha}$ (or $\|D^{k+1}G^k(u)\|_{locLip^s_\alpha(\Omega)}$) in terms of the L^p norm $\|u\|_{p,\Omega}$, we can prove it directly with the aid of the higher imbedding inequality given in [18]. Otherwise, while we are concerned on the upper boundedness of $\|D^kG^k(u)\|_{locLip^s_\alpha(\Omega)}$ (or $\|D^{k+1}G^k(u)\|_{locLip^s_\alpha(\Omega)}$) in terms of the BMO^s norm $\|u\|_{*,s,\Omega}$, the higher imbedding result is not valid for this case any more. Thus, to overcome this difficulty, the key tools in our approach are Lemma 2.4 and Lemma 2.5, which are established by adapting the technique developed in [19] with the inductive method.

Lemma 2.4 *Let $u \in L^p_{loc}(\Omega, \Lambda)$, $1 < p < \infty$, be a differential form in the domain Ω, D be the Hodge-Dirac operator and G be Green's operator. Then, we have that*

$$D^k G^k(u) = G^m(u), \tag{2.7}$$

$$D^{k+1} G^k(u) = dG^m(u) + d^\star G^m(u), \tag{2.8}$$

for every even integer $k = 2m$ and $m = 1, 2, \cdots$.

Proof First, since $\Delta = D^2 = (d + d^\star)^2 = dd^\star + d^\star d$, we know that it holds

$$u = \Delta G(u) + H(u) = dd^\star G(u) + d^\star dG(u) + H(u) \tag{2.9}$$

for every $u \in L^p_{loc}(\Omega, \Lambda)$, which also implies that

$$dd^\star G(u) + d^\star dG(u) = u - H(u). \tag{2.10}$$

Due to the fact that $HG(u) = 0$ always holds by Remark 1, replacing u with $G^m(u)$ in (2.10) gives that

$$dd^\star G(G^m(u)) + d^\star dG(G^m(u)) = G^m(u) \tag{2.11}$$

whenever the positive integer $m \geq 1$.

Now, we will assert the representation (2.7) by using the inductive method. In the case of $k = 2$ and $m = 1$, we have

$$D^2 G^2(u) = (d + d^\star)^2 G^2(u) = dd^\star G(G(u)) + d^\star dG(G(u)). \tag{2.12}$$

Substituting 2.11 with $m = 1$ into (2.12) yields that $D^2 G^2(u) = G(u)$. Assume that the desired result holds for any $k = 2(m-1)$, $m = 2, 3, \cdots$, that is,

$$D^k G^k(u) = D^{2(m-1)} G^{2(m-1)}(u) = G^{m-1}(u). \tag{2.13}$$

Then, when k is taken as $2m$, it continues with (2.13) and (2.11) that

$$\begin{aligned} D^k G^k(u) &= D^2 D^{2(m-1)} G^{2(m-1)}(G^2(u)) = D^2 G^{m+1}(u) \\ &= dd^\star G(G^m(u)) + d^\star dG(G^m(u)) = G^m(u). \end{aligned} \tag{2.14}$$

So, the desired result (2.7) holds. Moreover, for the operator $D^{k+1} G^k(u)$, making use of (2.7) and the fact $D = d + d^\star$, we obtain that

$$D^{k+1} G^k(u) = D(D^k G^k(u)) = D(G^m(u)) = dG^m(u) + d^\star G^m(u).$$

Therefore, we finish the proof of Lemma 2.4. □

In analogue to the method developed in Lemma 2.4, we also derive the following results for the case $k = 2m + 1$.

Lemma 2.5 *Let $u \in L^p_{loc}(\Omega, \Lambda)$, $1 < p < \infty$, be a differential form defined in the domain Ω, D be the Hodge-Dirac operator and G be Green's operator. Then, we derive that*

$$D^k G^k(u) = dG^{m+1}(u) + d^\star G^{m+1}(u), \tag{2.15}$$

$$D^{k+1} G^k(u) = G^m(u) \tag{2.16}$$

for every odd integer $k = 2m + 1$ and $m = 1, 2, \cdots$.

Now, we are ready to give the local Poincaré-type estimates of the iterated operator $D^k G^k$ and $D^{k+1} G^k$ in terms of the L^p norms of du and $d^\star u$, respectively.

Theorem 2.6 *Assume that the differential form u is of the Sobolev class $W^{1,p}_{loc}(\Omega, \Lambda)$, $1 < p < \infty$, D is the Hodge-Dirac operator and G is Green's operator. Then, for any even integer $k = 2m$, $m = 1, 2, \cdots$, there exists a constant $C > 0$, independent of u, such that*

$$\|D^k G^k(u) - (D^k G^k(u))_B\|_{p,B} \le C|B|^{1+1/n}\|du\|_{p,\sigma B}, \tag{2.17}$$

$$\|D^{k+1} G^k(u) - (D^{k+1} G^k(u))_B\|_{p,B} \le C|B|^{1+1/n}\|d^\star u\|_{p,\sigma B} \tag{2.18}$$

for all balls $B \subset \sigma B \subset \Omega$ with some constant $\sigma > 1$.

Proof Initially, to prove (2.17), applying the decomposition (2.5) to $D^k G^k(u)$, we have

$$D^k G^k(u) = dT(D^k G^k(u)) + Td(D^k G^k(u)). \tag{2.19}$$

Since $dT(D^k G^k(u)) = (D^k G^k(u))_B$, for every $p > 1$, using (2.19), (2.7) and (2.6), it follows that

$$\begin{aligned} \|D^k G^k(u) - (D^k G^k(u))_B\|_{p,B} &= \|Td(D^k G^k(u))\|_{p,B} \\ &\le C_1|B|diam(B)\|d(D^k G^k(u))\|_{p,B} \\ &= C_1|B|diam(B)\|d(G^m(u))\|_{p,B} \\ &\le C_2|B|^{1+1/n}\|d(G^m(u))\|_{p,B}. \end{aligned} \tag{2.20}$$

Due to the definition of the Sobolev space and the facts that $\|du\|_{p,\Omega'} \le \|\nabla u\|_{p,\Omega'} < \infty$ and $\|d^\star u\|_{p,\Omega'} \le \|\nabla u\|_{p,\Omega'} < \infty$ for any $\Omega' \subset\subset \Omega$, one may readily see that Green's operator G can commute with d and $d^\star$. Then, combining (2.20) with (2.4) follows that

$$\|D^kG^k(u) - (D^kG^k(u))_B\|_{p,B} \le C_3|B|^{1+1/n}\|du\|_{p,\sigma_1 B}$$

for any even integer $k > 0$. Thus, we have (2.17) always holds for all balls $B \subset \sigma B \subset \Omega$ with some constant $\sigma_1 > 1$.

Now, we turn to the proof of the inequality (2.18). First, applying the commute property between G and $d^\star$ and (2.2), we have

$$\|dd^\star G^k(u)\|_{p,B} = \|dG^k(d^\star(u))\|_{p,B} \le C_4\|d^\star u\|_{p,\sigma_2 B}. \tag{2.21}$$

Making use of the similar treatment as in the proof of D^kG^k with (2.8) and (2.21), we attain that

$$\begin{aligned}\|D^{k+1}G^k(u) - (D^{k+1}G^k(u))_B\|_{p,B} &= \|Td(D^{k+1}G^k(u))\|_{p,B}\\ &\le C_5|B|diam(B)\|d(D^{k+1}G^k(u))\|_{p,B}\\ &\le C_6|B|^{1+1/n}\|d(dG^m(u) + d^\star G^m(u))\|_{p,B}\\ &= C_6|B|^{1+1/n}\|dd^\star G^m(u)\|_{p,B}\\ &\le C_7|B|^{1+1/n}\|d^\star u\|_{p,\sigma_2 B}\end{aligned} \tag{2.22}$$

for every even integer $k > 0$ and some constant $\sigma_2 > 1$ with all balls $B \subset \sigma_2 B \subset \Omega$. Therefore, the proof of Theorem 2.6 is ended. □

Next, it is natural to take the case of the odd integer $k > 1$ into account. Using the same process as the case $k = 2m$ by Lemma 2.5 instead of Lemma 2.4, we derive the results for the odd integer $k = 2m + 1$. Considering the length of the paper, we only state the results of Theorem 2.7.

Theorem 2.7 *Assume that the differential form u is of the Sobolev class $W^{1,p}_{loc}(\Omega, \Lambda)$, $1 < p < \infty$, D is the Hodge-Dirac operator and G is Green's operator. Then, for any odd integer $k = 2m + 1$, $m = 1, 2, \cdots$, there exists a constant $C > 0$, independent of u, such that*

$$\|D^kG^k(u) - (D^kG^k(u))_B\|_{p,B} \le C|B|^{1+1/n}\|d^\star u\|_{p,\sigma B}, \tag{2.23}$$

$$\|D^{k+1}G^k(u) - (D^{k+1}G^k(u))_B\|_{p,B} \le C|B|^{1+1/n}\|du\|_{p,\sigma B} \tag{2.24}$$

for all balls $B \subset \sigma B \subset \Omega$ with some constant $\sigma > 1$.

Remark 3 It should be noticed that the results in Theorem 2.6 and Theorem 2.7 will play a significant role in latter discussion. Specifically, just because of the right terms du and $d^\star u$ in Theorem 2.6 and Theorem 2.7, it provides us an effective way to derive the upper boundedness of the iterated operators D^kG^k and $D^{k+1}G^k$ in terms of the BMO^s norm for the conjugate A-harmonic tensors u and v.

3 Estimates for BMO^s and locLip_α^s Norms

In this section, we present our principal results about the estimates for BMO^s norm and locLip_α^s norm for D^kG^k and $D^{k+1}G^k$ applied to differential forms u and v associated with some conjugate A-harmonic equation.

During the recent years, the study in the conjugate A-harmonic tensors is of growing interest and has made much progress, see [2, 10, 20, 21] for examples. Here, we consider the conjugate A-harmonic tensors of the form as follows.

Definition 3.1 ([10]) Differential forms $u \in W^{1,p}(\Omega, \Lambda)$ and $v \in W^{1,q}(\Omega, \Lambda)$ are called the conjugate A-harmonic tensors if u and v satisfy the conjugate A-harmonic equation of the form

$$A(du) = d^\star v, \tag{3.1}$$

where the operator $A : \Lambda(\Omega) \to \Lambda(\Omega)$ is restricted by the following structural assumptions:

(i) the mapping $\xi \to A(\xi)$ is continuous;
(ii) $|A(\xi)| \leq a_1|\xi|^{p-1}$, $< A(\xi), \xi > \geq b_1|\xi|^p$;
(iii) $A(\lambda\xi) = \lambda|\lambda|^{p-2}A(\xi)$ whenever $\lambda \in \mathbb{R}, \lambda \neq 0$;
(iv) the monotonicity inequality: $| < A(\xi) - A(\eta), \xi - \eta > | \geq L_1(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}|\xi - \eta|^2$.

for all $\xi \in \Lambda(\mathbb{R}^n)$. Here, a_1, b_1 and $L_1 > 0$ are the positive constants and $1 < p, q < \infty$ are the conjugate exponents with $1/p + 1/q = 1$.

According to Definition 3.1, together with the facts $dd = 0$ and $d^\star d^\star = 0$, it is obvious to see that such a differential form u in (3.1) is also a solution to the A-harmonic equation

$$d^\star A(du) = 0. \tag{3.2}$$

Moreover, if the operator A is invertible, in view of the isometric property of the Hodge-star operator $\star$, there exists an operator B such that the differential form v in (3.1) meanwhile satisfies

$$d^\star B(d(\star v)) = 0, \tag{3.3}$$

where the operator $B : \Lambda(\Omega) \to \Lambda(\Omega)$ is given the similar conditions i)–iv) that

(b-i) the mapping $\xi \to B(\xi)$ is continuous on $\Lambda(\mathbb{R}^n)$;
(b-ii) $|B(\xi)| \leq a_2|\xi|^{q-1}$, $< B(\xi), \xi > \geq b_2|\xi|^q$;
(b-iii) $B(\kappa\xi) = \kappa|\kappa|^{q-2}B(\xi)$ whenever $\kappa \in \mathbb{R}, \kappa \neq 0$;
(b-iv) the monotonicity inequality: $| < B(\xi) - B(\eta), \xi - \eta > | \geq L_2(|\xi|^2 + |\eta|^2)^{\frac{q-2}{2}}|\xi - \eta|^2$.

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here, a_2, b_2 and L_2 are the positive constants and $1 < q < \infty$ is associated with (3.3).

Observe that A-harmonic equation is a special case of Dirac-harmonic equation, so we derive the Caccippoli inequality and the weak reverse Hölder inequality, respectively, by Corollary 2.3 and Theorem 4.3 in [22].

Lemma 3.2 *Let $u \in W^{1,p}(\Omega, \Lambda)$ and $v \in W^{1,q}(\Omega, \Lambda)$ satisfy the conjugate A-harmonic equation (3.1), and the operator A be invertible, where $1 < p,\ q < \infty$ are the given conjugate exponents with $1/p + 1/q = 1$. Then, there exists a constant $C > 0$, independent of u and v, such that*

$$\|du\|_{p,B} \le C|B|^{-1/n}\|u - c\|_{p,\sigma B}, \tag{3.4}$$

$$\|d^\star v\|_{q,B} \le C|B|^{-1/n}\| \star v - c^\star\|_{q,\sigma B} \tag{3.5}$$

for some constant $\sigma > 1$ and any ball $B \subset \sigma B \subset \Omega$, where c and $c^\star$ are both closed forms.

Lemma 3.3 *Let $\omega \in W^{1,p}(\Omega, \Lambda)$ be a solution to the homogenous A-harmonic equation, $1 < p < \infty$. Then, for every $0 < s, t < \infty$, there exists a constant $C > 0$, independent of ω, such that*

$$\|\omega\|_{s,B} \le C|B|^{1/s-1/t}\|\omega\|_{t,\sigma B}, \tag{3.6}$$

where all balls $B \subset \sigma B \subset \Omega$ and $\sigma > 1$ is some constant.

In addition, the local higher order inequality is also necessary for our latter argument.

Lemma 3.4 *Let $u \in L^p_{loc}(\Omega, \Lambda)$, $1 < p < \infty$, be a differential form, D be the Hodge-Dirac operator and G be Green's operator. Then, for any positive integer $k \ge 1$, we have that*

(i) if $1 < p < n$, for any real number $0 < s < np/(n-p)$, there exists a constant $C > 0$, independent of u, such that

$$\|D^kG^k(u) - (D^kG^k(u))_B\|_{s,B} \le C|B|^{1+1/n+1/s-1/p}\|u\|_{p,\sigma B}, \tag{3.7}$$

$$\|D^{k+1}G^k(u) - (D^{k+1}G^k(u))_B\|_{p,B} \le C|B|^{1+1/n+1/s-1/p}\|u\|_{p,\sigma B} \tag{3.8}$$

(ii) if $p \ge n$, for any real number $s > 0$, there is a constant $C > 0$, independent of u, such that

$$\|D^kG^k(u) - (D^kG^k(u))_B\|_{s,B} \le C|B|^{1+1/n+1/s-1/p}\|u\|_{p,\sigma B}, \tag{3.9}$$

$$\|D^{k+1}G^k(u) - (D^{k+1}G^k(u))_B\|_{p,B} \le C|B|^{1+1/n+1/s-1/p}\|u\|_{p,\sigma B} \tag{3.10}$$

for all balls $B \subset \sigma B \subset \Omega$ with some constant $\sigma > 1$.

Now, with these facts in mind, let us first prove Theorem 3.5.

Theorem 3.5 *Let $u \in L^p(\Omega, \Lambda)$, $1 < p < n$, be a differential form defined on the smoothly bounded domain Ω, D be the Hodge-Dirac operator and G be Green's operator. Then, for any positive integer $k > 1$ and any real number $0 < s < np/(n-p)$, there exist two constants $C_1, C_2 > 0$, independent of u, such that*

$$\|D^k G^k(u)\|_{*,s,\Omega} \le C_1 \|D^k G^k(u)\|_{locLip^s_\alpha(\Omega)} \le C_2 \|u\|_{p,\Omega}, \tag{3.11}$$

$$\|D^{k+1} G^k(u)\|_{*,s,\Omega} \le C_1 \|D^{k+1} G^k(u)\|_{locLip^s_\alpha(\Omega)} \le C_2 \|u\|_{p,\Omega}, \tag{3.12}$$

where $0 < \alpha \le 1$ is some constant.

Proof First, we notice that $1 + \frac{1}{n} - \frac{1}{p} - \frac{\alpha}{n} = \left(1 - \frac{1}{p}\right) + \left(\frac{1}{n} - \frac{\alpha}{n}\right) > 0$ because $0 < \alpha \le 1$ and $1 < p < \infty$. Then, for any ball $B \subset \Omega$, we have

$$|B|^{1+1/n-1/p-\alpha/n} \le |\Omega|^{1+1/n-1/p-\alpha/n}. \tag{3.13}$$

In the meantime, by replacing ω with $D^k G^k(u)$ and $D^{k+1} G^k(u)$ in (1.5), respectively, it is immediate to achieve that

$$\|D^k G^k(u)\|_{*,s,\Omega} \le C_1 \|D^k G^k(u)\|_{locLip^s_\alpha(\Omega)}, \tag{3.14}$$

$$\|D^{k+1} G^k(u)\|_{*,s,\Omega} \le C_1 \|D^{k+1} G^k(u)\|_{locLip^s_\alpha(\Omega)}. \tag{3.15}$$

Thus, to estimate (3.11), applying (3.7) and (3.13) gives

$$\begin{aligned}
\|D^k G^k(u)\|_{locLip^s_\alpha(\Omega)} &= \sup_{\sigma_2 B \subset \Omega} |B|^{-\frac{n+\alpha s}{sn}} \|D^k G^k(u) - (D^k G^k(u))_B\|_{s,B} \\
&\le \sup_{\sigma_2 B \subset \Omega} |B|^{-1/s-\alpha/n} C_2 |B|^{1+1/s+1/n-1/p} \|u\|_{p,\sigma_1 B} \\
&= \sup_{\sigma_2 B \subset \Omega} C_2 |B|^{1+1/n-1/p-\alpha/n} \|u\|_{p,\sigma_1 B} \\
&\le \sup_{\sigma_2 B \subset \Omega} C_2 |\Omega|^{1+1/n-1/p-\alpha/n} \|u\|_{p,\sigma_1 B} \\
&\le C_3 \sup_{\sigma_2 B \subset \Omega} \|u\|_{p,\sigma_1 B} \\
&\le C_4 \|u\|_{p,\Omega},
\end{aligned} \tag{3.16}$$

where the constants $\sigma_2 > \sigma_1 > 1$ and all balls $B \subset \sigma_1 B \subset \sigma_2 B \subset \Omega$. So, according to (3.14) and (3.16), we have that (3.11) holds as desired. Moreover, using the same treatment to the operator $D^{k+1} G^k(u)$ with (3.8) and (3.15), the inequality (3.12) holds as well. Therefore, the proof of Theorem 3.5 is finished. □

For the case $p \geq n$, repeating the process as in Theorem 3.5 with (3.9) and (3.10), we obtain the analogue results.

Theorem 3.6 *Let $u \in L^p(\Omega, \Lambda)$, $p \geq n$, be a differential form defined on the smoothly bounded domain Ω, D be the Hodge-Dirac operator and G be Green's operator. Then, for any positive integer $k > 1$ and any real number $s > 0$, there exist two constants $C_1, C_2 > 0$, independent of u, such that*

$$\|D^k G^k(u)\|_{*,s,\Omega} \leq C_1 \|D^k G^k(u)\|_{locLip^s_\alpha(\Omega)} \leq C_2 \|u\|_{p,\Omega}, \tag{3.17}$$

$$\|D^{k+1} G^k(u)\|_{*,s,\Omega} \leq C_1 \|D^{k+1} G^k(u)\|_{locLip^s_\alpha(\Omega)} \leq C_2 \|u\|_{p,\Omega}, \tag{3.18}$$

where $0 < \alpha \leq 1$ is some constant.

Next, we begin to establish our principle relationship between BMO^s norm and locLip$^s_\alpha$ norm of the iterated operators in terms of the norms of the conjugate harmonic tensors u and v. From Theorem 3.7 and Theorem 3.8 to Corollary 3.9 and Corollary 3.10 below, we always assume that $\Omega \subset \mathbb{R}^n$ is smoothly bounded domain without boundary, the operator A in (3.1) is invertible.

Theorem 3.7 *Let $u \in W^{1,p}(\Omega, \Lambda)$ and $v \in W^{1,q}(\Omega, \Lambda)$, $1 < p, q < \infty$ with $1/p + 1/q = 1$, be the conjugate A-harmonic tensors satisfying the Equation (3.1), D be the Hodge-Dirac operator and G be Green's operator. Then, for every integer $k = 2m$ and any real number $s > 0$, $m = 1, 2, \cdots$, there are two constants $C_1, C_2 > 0$, independent of u and v, such that*

$$\|D^k G^k(u)\|_{*,s,\Omega} \leq C_1 \|D^k G^k(u)\|_{locLip^s_\alpha(\Omega)} \leq C_2 \|u\|_{*,p,\Omega}, \tag{3.19}$$

$$\|D^{k+1} G^k(v)\|_{*,s,\Omega} \leq C_1 \|D^{k+1} G^k(v)\|_{locLip^s_\beta(\Omega)} \leq C_2 \| \star v\|_{*,p,\Omega}, \tag{3.20}$$

where $0 < \alpha, \beta \leq 1$ are the expansion factors.

Proof First, without loss of generality, we assume that the conjugate A-harmonic tensor u is a solution to the A-harmonic equation (3.2). Then, it is natural to view the corresponding v as a solution to Equation (3.3). Next, we will divide our proof into two parts.

(i) For every $1 < p < \infty$, applying (2.17) into Definition 1.2, we have that

$$\begin{aligned} \|D^k G^k(u)\|_{locLip^s_\alpha(\Omega)} &= \sup_{\sigma_1 B \subset \Omega} |B|^{-1/s-\alpha/n} \|D^k G^k(u) - (D^k G^k(u))_B\|_{s,B} \\ &\leq \sup_{\sigma_1 B \subset \Omega} C_1 |B|^{-1/s-\alpha/n} |B|^{1+1/n} \|du\|_{s,\sigma_2 B} \\ &\leq C_1 \sup_{\sigma_1 B \subset \Omega} |B|^{1+1/n-1/s-\alpha/n} \|du\|_{s,\sigma_2 B} \end{aligned} \tag{3.21}$$

Observe that du is a solution for the A-harmonic equation since du is a closed form. Then, for any real number $s > 0$, using Lemma 3.3 yields that

$$\|du\|_{s,\sigma_2 B} \le C_2|B|^{1/s-1/p}\|du\|_{p,\sigma_3 B}. \tag{3.22}$$

where $\sigma_3 > \sigma_2 > 1$. Under the assumption, we know that u satisfies the Caccippoli inequality (3.4). Especially, choosing $c = u_B$ in (3.4) follows

$$\|du\|_{p,\sigma_3 B} \le C_3|B|^{-1/n}\|u - u_B\|_{p,\sigma_4 B} \tag{3.23}$$

for some constant $\sigma_4 > \sigma_3 > 1$ with any ball $\sigma_3 B \subset \sigma_4 B \subset \Omega$. Moreover, combining (3.22) and (3.23) gives

$$\|du\|_{s,\sigma_2 B} \le C_4|B|^{1/s-1/n-1/p}\|u - u_B\|_{p,\sigma_4 B} \tag{3.24}$$

So, substituting (3.24) into (3.21), together with Definition 1.1, yields that

$$\begin{aligned}\|D^k G^k(u)\|_{locLip^s_\alpha(\Omega)} &\le C_5 \sup_{\sigma_1 B\subset\Omega} |B|^{1+1/n-1/s-\alpha/n}|B|^{1/s-1/n-1/p}\|u-u_B\|_{p,\sigma_4 B}\\ &\le C_5 \sup_{\sigma_1 B\subset\Omega} |B|^{1-1/p-\alpha/n}\|u - u_B\|_{p,\sigma_4 B}\\ &\le C_5 \sup_{\sigma_1 B\subset\Omega} |\Omega|^{1-\alpha/n}|B|^{-1/p}\|u - u_B\|_{p,\sigma_4 B}\\ &\le C_6 \sup_{\sigma_1 B\subset\Omega} |B|^{-1/p}\|u - u_B\|_{p,\sigma_4 B}\\ &\le C_6\|u\|_{*,p,\Omega},\end{aligned} \tag{3.25}$$

where the constants $\sigma_1 > \sigma_4 > 1$. Therefore, we have that (3.19) holds for any even integer $k > 1$ and any real number $s > 0$.

The proof of (3.20) is similar to that of (3.19). Next, we only present the different steps.

(ii) For every conjugate A-harmonic tensor $v \in W^{1,q}(\Omega, \Lambda)$, employing the same treatment used in the proof of (3.19), along with (2.18), we have that

$$\|D^k G^k(v)\|_{locLip^s_\beta(\Omega)} \le C_7 \sup_{\eta_1 B\subset\Omega} |B|^{1+1/n-1/s-\beta/n}\|d^\star v\|_{s,\eta_2 B}. \tag{3.26}$$

According to the isometric property of the Hodge-star operator $\star$, we know that $|d^\star v| = |d\star v|$. Notice that $d\star v$ is a closed form satisfying A-harmonic equation. So, for any real number $s > 0$, using Lemma 3.3 again, we derive that

$$\|d^\star v\|_{s,\eta_2 B} = \|d \star v\|_{s,\eta_2 B} \le C_8|B|^{1/s-1/q}\|d \star v\|_{q,\eta_3 B} \tag{3.27}$$

Also, by the comments after Definition 3.1, it implies that $\star v$ is a solution to the A-harmonic equation (3.3). Then, by Lemma 3.2, letting $c^{\star} = (\star v)_B$ shows that

$$\|d \star v\|_{q,\eta_3 B} \le C_9 |\eta_3 B|^{-1/n} \| \star v - (\star v)_B \|_{q,\eta_4 B}. \tag{3.28}$$

So, combining (3.27) with (3.28) and plugging it into (3.26), we have that

$$\begin{aligned}
\|D^{k+1}G^k(v)\|_{locLip^s_\beta(\Omega)} &\le \sup_{\eta_1 B \subset \Omega} |B|^{1+1/n-1/s-\beta/n} \|d^{\star} v\|_{s,\eta_2 B} \\
&\le \sup_{\eta_1 B \subset \Omega} C_{10} |B|^{1+1/n-1/s-\beta/n} |B|^{1/s-1/q-1/n} \\
&\quad \| \star v - (\star v)_B \|_{q,\eta_4 B} \\
&\le \sup_{\eta_1 B \subset \Omega} C_{10} |\Omega|^{1-\beta/n} |B|^{-1/q} \| \star v - (\star v)_B \|_{q,\eta_4 B} \\
&\le C_{11} \sup_{\eta_1 B \subset \Omega} |B|^{-1/q} \| \star v - (\star v)_B \|_{q,\eta_4 B} \\
&= C_{11} \| \star v \|_{*,q,\Omega}
\end{aligned} \tag{3.29}$$

as desired, where the constants $\eta_1 > \eta_4 > \eta_3 > \eta_2 > 1$.

□

Now, in the odd case $k = 2m + 1$, we have the similar estimates as follows. It should be pointed out that the proof of Theorem 3.8 is the analogue of Theorem 3.7, so we only state the results and leave the proof of the odd case $k > 1$ to the readers.

Theorem 3.8 *Let $u \in W^{1,p}(\Omega, \Lambda)$ and $v \in W^{1,q}(\Omega, \Lambda)$, $1 < p, q < \infty$ with $1/p + 1/q = 1$, be the conjugate A-harmonic tensors satisfying the Equation (3.1), D be the Hodge-Dirac operator and G be Green's operator. Then, for every odd integer $k = 2m + 1$ and any real number $s > 0$, $m = 1, 2, \cdots$, there are two constants $C_1, C_2 > 0$, independent of u and v, such that*

$$\|D^k G^k(v)\|_{*,s,\Omega} \le C_1 \|D^k G^k(v)\|_{locLip^s_\beta(\Omega)} \le C_2 \| \star v\|_{*,q,\Omega}, \tag{3.30}$$

$$\|D^{k+1} G^k(u)\|_{*,s,\Omega} \le C_1 \|D^{k+1} G^k(u)\|_{locLip^s_\alpha(\Omega)} \le C_2 \|u\|_{*,p,\Omega}, \tag{3.31}$$

where $0 < \alpha, \beta \le 1$ are the expansion factors.

In particular, if $p(\alpha - 1) = q(\eta - 1)$, as a consequence of Theorem 3.7 and 3.8, the following estimates are established simply by means of Theorem 6.6 in [10]. It is worth to notice that the treatment applied in Corollary 3.9 and Corollary 3.10 are very similar, so we only give the complete proof of Corollary 3.9 in details.

Corollary 3.9 *Let $u \in W^{1,p}(\Omega, \Lambda)$ and $v \in W^{1,q}(\Omega, \Lambda)$, $1 < p, q < \infty$ with $1/p + 1/q = 1$, be the conjugate A-harmonic tensors satisfying the Equation (3.1),*

D be the Dirac operator and G be the Green's operator. If $0 < \alpha, \beta \leq 1$ satisfy $p(\alpha - 1) = q(\beta - 1)$, for any real $s > 0$, then there exist two constants $C_1, C_2 > 0$, independent of u and v, such that

$$\|D^k G^k(u)\|_{*,s,\Omega} \leq C_1 \|D^k G^k(u)\|_{locLip^s_\alpha(\Omega)} \leq C_2 \| \star v\|^{q/p}_{locLip^q_\beta(\Omega)}, \tag{3.32}$$

$$\|D^{k+1} G^k(v)\|_{*,s,\Omega} \leq C_1 \|D^{k+1} G^k(v)\|_{locLip^s_\beta(\Omega)} \leq C_2 \|u\|^{p/q}_{locLip^p_\alpha(\Omega)}, \tag{3.33}$$

whenever $k = 2m$, $m = 1, 2, \cdots$.

Proof First, combining (1.5) and Theorem 6.6 in [10], we have

$$\|u\|_{*,s,\Omega} \leq C_1 \|u\|_{locLip^s_\alpha(\Omega)} \leq C_2 \| \star v\|^{q/p}_{locLip^q_\beta(\Omega)}, \tag{3.34}$$

$$\| \star v\|_{BMO,\Omega} \leq C_3 \| \star v\|_{locLip_\beta(\Omega)} \leq C_4 \|u\|^{p/q}_{locLip_\alpha(\Omega)}. \tag{3.35}$$

Then, substituting (3.34) into (3.19) and (3.35) into (3.20), respectively, it yields that

$$\|D^k G^k(u)\|_{*,s,\Omega} \leq C_5 \|D^k G^k(u)\|_{locLip^s_\alpha(\Omega)} \leq C_6 \| \star v\|^{q/p}_{locLip^q_\beta(\Omega)},$$

$$\|D^{k+1} G^k(v)\|_{*,s,\Omega} \leq C_7 \|D^{k+1} G^k(v)\|_{locLip^s_\beta(\Omega)} \leq C_8 \|u\|^{p/q}_{locLip^p_\alpha(\Omega)}$$

as desired. □

Corollary 3.10 *Suppose that $0 < \alpha, \beta \leq 1$ satisfy $p(\alpha - 1) = q(\beta - 1)$, for any real number $s > 0$, then there exist two constants $C_1, C_2 > 0$, independent of u and v, such that*

$$\|D^k G^k(v)\|_{*,s,\Omega} \leq C_1 \|D^k G^k(v)\|_{locLip^s_\beta(\Omega)} \leq C_2 \|u\|^{p/q}_{locLip^p_\alpha(\Omega)}, \tag{3.36}$$

$$\|D^{k+1} G^k(u)\|_{*,s,\Omega} \leq C_1 \|D^{k+1} G^k(u)\|_{locLip^s_\alpha(\Omega)} \leq C_2 \| \star v\|^{q/p}_{locLip^q_\beta(\Omega)}, \tag{3.37}$$

whenever $k = 2m + 1$, $m = 1, 2, \cdots$.

What is more, for each pair of conjugate A-harmonic tensors u and v, in accord to the facts that $|du|^p \leq |d^\star v|^q \leq a_1^q |du|^p$ and $|d^\star v| = |d \star v|$, one may easily establish such a useful L^p-equivalence with respect to u and v as follows:

$$\|du\|_{p,\Omega'} \leq \|d \star v\|^{q/p}_{q,\Omega'} \leq a_1^{q/p} \|du\|_{p,\Omega'}, \tag{3.38}$$

whenever $\Omega' \subset \Omega$, where $1 < p, q < \infty$ are the conjugate Hölder exponents. In view of the equivalence (3.38), if u and v are the conjugate A-harmonic tensors, it further reveals the relations (3.39)–(3.42) below. Namely, when k is any positive even integer, there exist two constants $C_1, C_2 > 0$, independent of u and v, such that

$$\|D^k G^k(u)\|_{*,s,\Omega} \le C_1 \|D^k G^k(u)\|_{locLip^s_\alpha(\Omega)} \le C_2 \|d \star v\|^{q/p}_{q,\Omega}, \tag{3.39}$$

$$\|D^{k+1} G^k(v)\|_{*,s,\Omega} \le C_1 \|D^{k+1} G^k(v)\|_{locLip^s_\beta(\Omega)} \le C_2 \|du\|^{p/q}_{p,\Omega}. \tag{3.40}$$

for any real number $s > 0$. As such, when $k > 1$ is any odd integer, there also exist two constants $C_1, C_2 > 0$, independent of u and v, such that

$$\|D^k G^k(v)\|_{*,s,\Omega} \le C_1 \|D^k G^k(v)\|_{locLip^s_\beta(\Omega)} \le C_2 \|du\|^{p/q}_{p,\Omega}, \tag{3.41}$$

$$\|D^{k+1} G^k(u)\|_{*,s,\Omega} \le C_1 \|D^{k+1} G^k(u)\|_{locLip^s_\alpha(\Omega)} \le C_2 \|d \star v\|^{q/p}_{q,\Omega}. \tag{3.42}$$

It should be pointed out that the proof of the above assertions are parallel to the those of Theorem 3.7. Therefore, we omit the details.

4 Applications

In this section, we use some concrete examples to illustrate the applications of the main results obtained in Section 3.

Let the mapping $f : \Omega \to \mathbb{R}^n$, $f = (f^1, \cdots, f^n)$, be of Sobolev class $W^{1,p}_{loc}(\Omega, \Lambda)$ and $J(x, f) = det(Df(x))$ be the Jacobian determinant of f. Then, we have that

$$u = J(x_{i_1}, x_{i_2}, \cdots, x_{i_l}; f^{j_1}, f^{j_2}, \cdots, f^{j_l}) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}, \tag{4.1}$$

is a differential l-form, where $J(x_{i_1}, x_{i_2}, \cdots, x_{i_l}; f^{j_1}, f^{j_2}, \cdots, f^{j_l})$ is the subdeterminant of $J(x, f)$ of the form:

$$J(x_{i_1}, x_{i_2}, \cdots, x_{i_l}; f^{j_1}, f^{j_2}, \cdots, f^{j_l}) = \begin{vmatrix} f^{j_1}_{x_{i_1}} & f^{j_1}_{x_{i_2}} & \cdots & f^{j_1}_{x_{i_l}} \\ f^{j_2}_{x_{i_1}} & f^{j_2}_{x_{i_2}} & \cdots & f^{j_2}_{x_{i_l}} \\ \cdots & \cdots & \cdots & \cdots \\ f^{j_l}_{x_{i_1}} & f^{j_l}_{x_{i_2}} & \cdots & f^{j_l}_{x_{i_l}} \end{vmatrix}$$

Referring to Chapter 1 in [2], we find that Theorem 3.5 and Theorem 3.6 are applicable to such sort of the differential form u. Here, take a special case of 2-dimensional Euclidean space for instance.

Example 4.1 Assume that $u = J(x, y; f^1, f^2)dx \wedge dy$ is the differential 2-form defined on the domain $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < r^2\}$, where the mapping $f : \Omega \to \mathbb{R}^2$ is of the Sobolev class $W^{1,p}_{loc}(\Omega, \Lambda)$ denoted by

$$f(x, y) = (f^1(x, y), f^2(x, y)) = \left(\frac{x}{(x^2 + y^2)^{1/8}}, \frac{y}{(x^2 + y^2)^{1/8}} \right) \quad (4.2)$$

for any $r > 0$ and $p > 1$. After a simple calculation, one may derive that

$$u = J(x, y; f^1, f^2)dx \wedge dy = \frac{3}{4}(x^2 + y^2)^{-1/4}dx \wedge dy.$$

Thus, by the spherical coordinate transformation, it is easy to see that $u \in L^p(\Omega, \Lambda^2)$ for any $p < 4$. For example, choosing $p = 3/2$, we know that $u \in L^{3/2}(\Omega, \Lambda^2)$. However, by the direct integral calculation with Definition 1.1 and Definition 1.2, it is quite hard to infer the higher order boundedness of BMO^s norm and locLip$^s_\alpha$ norm with respect to $D^k G^k(u)$ and $D^{k+1}G^k(u)$. Then, applying Theorem 3.5 to $D^k G^k$ and $D^{k+1}G^k$, for any $0 < s < np/(n - p) = \frac{2\cdot 3/2}{2-3/2} = 6$, we have that $D^k G^k(u) \in \mathrm{BMO}^s(\Omega, \Lambda^2)$ and $D^{k+1}G^k(u) \in \mathrm{BMO}^s(\Omega, \Lambda^2)$. Moreover, there are two constants $C_1, C_2 > 0$, independent of u, such that

$$\|D^k G^k(u)\|_{*,s,\Omega} \le C_1\|D^k G^k(u)\|_{locLip^s_\alpha(\Omega)} \le C_2 r^{5/4}, \quad (4.3)$$

$$\|D^{k+1}G^k(u)\|_{*,s,\Omega} \le C_1\|D^{k+1}G^k(u)\|_{locLip^s_\alpha(\Omega)} \le C_2 r^{5/4} \quad (4.4)$$

for every $0 < \alpha \le 1$ and all positive integer $k \ge 1$.

Especially, if the homeomorphism $f : \Omega \to \mathbb{R}^n$ of Sobolev class $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$, as mentioned above, is the K-quasiregular mapping, $K \ge 1$. From [23], we know that

$$u = f^l df^1 \wedge \cdots \wedge df^{l-1} \quad \text{and} \quad v = \star f^{l+1}df^{l+2} \wedge \cdots \wedge df^n$$

are the conjugate A-harmonic tensors, whenever $l = 1, 2, \cdots, n-1$. Here, consider the 4-dimensional space as an example.

Example 4.2 Let $f = (f^1, f^2, f^3, f^4)$ be the K-quasiregular mapping defined on the domain $\Omega = \{(x_1, x_2, x_3, x_4) : |x_i| < a, i = 1, 2, 3, 4\} \subset \mathbb{R}^4$, and choose the conjugate A-harmonic tensors as follows:

$$u = f^2 df^1 \quad \text{and} \quad v = \star f^3 df^4.$$

where $0 < a < \infty$ is some real number. If $u \in \mathrm{BMO}^p(\Omega, \Lambda)$ and $\star v \in \mathrm{BMO}^q(\Omega, \Lambda)$, where p and q are conjugate exponents with $1/p + 1/q = 1$, by applying Theorem 3.7 and Theorem 3.8, respectively, we have that for any even integer $k = 2m$ and any real number $s > 0$, $m = 1, 2, \cdots$, there are two constants $C_1, C_2 > 0$, such that

$$\|D^k G^k(u)\|_{*,s,\Omega} \le C_1 \|D^k G^k(u)\|_{locLip^s_\alpha(\Omega)} \le C_2 \|f^2 df^1\|_{*,p,\Omega},$$

$$\|D^{k+1} G^k(v)\|_{*,s,\Omega} \le C_1 \|D^{k+1} G^k(v)\|_{locLip^s_\beta(\Omega)} \le C_2 \|f^3 df^4\|_{*,q,\Omega},$$

where $0 < \alpha, \beta \le 1$ are two factors. While the integer $k = 2m + 1$, $m = 1, 2, \cdots$, it holds that

$$\|D^k G^k(v)\|_{*,s,\Omega} \le C_1 \|D^k G^k(v)\|_{locLip^s_\beta(\Omega)} \le C_2 \|f^3 df^4\|_{*,q,\Omega},$$

$$\|D^{k+1} G^k(u)\|_{*,s,\Omega} \le C_1 \|D^{k+1} G^k(u)\|_{locLip^s_\alpha(\Omega)} \le C_2 \|f^2 df^1\|_{*,p,\Omega},$$

for any real number $s > 0$, where $0 < \alpha, \beta \le 1$ are two factors.

Remark 4 In general, all results we establish here provide us an impressive description about the relation between BMO^s norm and locLip^s_α norm for the iterated operators. Also, from the results, one may realize that locLip^s_α-norm estimates for differential forms are fairly essential for the process to derive the BMO^s estimate with respect to $D^k G^k$ and $D^{k+1} G^k$ for differential forms.

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