Invariance Solutions and Blow-Up Property for Edge Degenerate Pseudo-Hyperbolic Equations in Edge Sobolev Spaces

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Abstract This article is dedicated to study of the initial-boundary value problem of edge pseudo-hyperbolic system with damping term on the manifold with edge singularity. First, we will discuss about the invariance of solution set of a class of edge degenerate pseudo-hyperbolic equations on the edge Sobolev spaces. Then, by using a family of modified potential wells and concavity methods, it is obtained existence and nonexistence results of global solutions with exponential decay and is shown the blow-up in finite time of solutions on the manifold with edge singularities.

1 Introduction

Initial-boundary value problems written for hyperbolic semilinear partial differential equations emerged in several applications to physics, mechanics and engineering sciences [9, 24, 25]. Interesting phenomena are often connected with geometric singularities, for instance, in mechanics or cracks in a medium are described by hypersurfaces with a boundary. In this cases, configurations of that kind belong to the category of spaces (manifolds) with geometric singularities, here with edges. Also, when one asks physics to calculate the self-energy of an electron, or the structure of space time at the center of a black hole, one encounter with mathematical bad behaviour, that is the singularities from the point view of mathematics. In recent years, from a mathematical point of view, the analysis on such (in general, stratified) spaces has become a mathematical structure theory with many deep relations with geometry, topology, and mathematical physics [10, 15, 23, 25]. In [21], Melrose,Vasy and Wunsch investigated the geometric propagation and

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diffraction of singularities of solutions to the wave equation on manifolds with edge singularities. Let X be an n-dimensional manifold with boundary, where the boundary ∂X is endowed with a fibration $Z \to \partial X$ and $\partial X \to Y$ where Y, Z are without boundary. By an edge metric g on X, we mean a metric g on the interior of X which is a smooth 2-cotensor up to the boundary but which degenerates there in a way compatible with the fibration. A manifold with boundary equipped with such an edge metric also is called an edge manifold or a manifold with edge structure. If Z is point, then an edge metric on X is simply a metric in the usual sense, smooth up to the boundary, while if Y is a point, X is conic manifold [4]. A simple example of a more general edge metric is obtained by performing a real blowup on a submanifold B of a smooth, boundaryless manifold A. The blowup operation simply introduces polar coordinates near B, i.e., it replaces B by its spherical normal bundle, thus yielding a manifold X with boundary. The pullback of a smooth metric on A to X is then an edge metric [21].

Up to now, elliptic boundary value problems in domains with point singularities have been thoroughly investigated [1–4, 7, 8, 14]. The natures of the solutions to these equations have been investigated by several means. For instance, problems with the Dirichlet boundary conditions were investigated in [1, 2, 7, 10, 14] in which the unique existence, the multiplicity, the regularity and the asymptotic behaviour near the conical points of the solutions are established. Finite time blow-up of solutions of generalized hyperbolic equations have been studied by many authors [1, 2, 5, 7, 18, 28]. In these references, the authors consider problems either for negative energy or for weaker conditions than a condition of negative initial energy. Other authors have assumed a condition of positive energy under other two conditions on the initial functions. However, the mentioned authors have not studied the compatibility of these conditions, which is come times hard to understand. These authors have used the classic concavity Levine's method [17]. In this article, we use the edge Sobolev inequality and Poincaré inequality and modified method in [7, 8] to prove on the global well-posedness of solutions to initial-boundary value problems for semilinear degenerate pseudo-hyperbolic equations with dissipative term on manifolds with edge singularities. More precisely, we study the following initial-boundary value problem for semilinear hyperbolic equation

$$\begin{cases} \partial_t^2 u - \Delta_{\mathbb{E}} u + V(z)u + \gamma \Delta_{\mathbb{E}} \partial_t u = g_t(z)|u|^{p-1}u, & z \in int\mathbb{E}, t > 0, \\ u(z,0) = u_0(z), \qquad \partial_t u(z,0) = u_1(z), & z \in int\mathbb{E} \\ u(z,t) = 0, & z \in \partial\mathbb{E}, t \geq 0, \end{cases} \tag{1}$$

where, $2 < p+1 < \frac{2n}{n-2} = 2^*$ is the critical cone Sobolev exponents, $z = (r, x, y)$, $u = u(z,t)$ is unknown function and γ is a non-negative parameter. Also, $u_0 \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$, $u_1 \in L_2^{\frac{n+1}{2}}(\mathbb{E})$, $N = 1+n+q \geq 3$ is a dimension of $\mathbb{E}$ and coordinates $z := (r, x, y) = (r, x_1, \ldots, x_n, y_1, \ldots, y_q) \in \mathbb{E}$. Here the domain $\mathbb{E}$ is $[0,1) \times X \times Y$, X is an $(n-1)$-dimensional closed compact manifold, $Y \subset \mathbb{R}^q$ is a bounded domain, which is regarded as the local model near the edge points on manifolds with edge singularities, and $\partial\mathbb{E} = \{0\} \times X \times Y$. Moreover, the operator $\Delta_{\mathbb{E}}$ in 1 is defined by $(r\partial_r)^2 + \partial_{x_1}^2 + \ldots + \partial_{x_n}^2 + (r\partial_{y_1})^2 + \ldots + (r\partial_{y_q})^2$, which is an elliptic operator with totally characteristic degeneracy on the boundary $r = 0$, we also call it Fuchsian type edge-Laplace operator, and the corresponding gradient operator by $\nabla_{\mathbb{E}} := (r\partial_r, \partial_{x_1}, \ldots, \partial_{x_n}, r\partial_{y_1}, \ldots, r\partial_{y_q})$. In the Equation 1, we assume that $V(z) \in L^{\frac{n+1}{4}}(int\mathbb{E}) \cap C(int\mathbb{E})$ is a positive potential function such that $\inf_{z\in\mathbb{E}} V(z) > 0$. For every $t \geq 0$, we suppose that $g_t : \mathbb{E} \to \mathbb{R}$ is a non-negative function which $g_t(z) := g(z,t)$ for every $z \in int\mathbb{E}$ and $g(z,t) \in L^\infty(int\mathbb{E}) \cap C^1(int\mathbb{E})$. The through of this paper we consider the following constants:

$$C_* = \inf\left\{ \frac{\|\sqrt{V(z)}u(z)\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})}}{\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{B})}} \quad ; \quad u \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \right\},$$

$$C_{**} = \sup\left\{ \frac{\|g_t(z)^{\frac{1}{p+1}}u\|_{L_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}}{\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})}} \quad ; \quad u \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \right\}.$$

Our investigation is in fact provoked by the study of [20] and we shall apply a potential method which was established by Sattinger [26]. So based on edge Sobolev spaces [10, 27], we study the existence and non-existence global weak solutions for semilinear pseudo-hyperbolic differential equations with respect to variable time with a positive potential function and a non-negative weighted function. The well-known operator $(\Delta_{\mathbb{E}} + V(x) + \Delta_{\mathbb{E}}\partial_t)u$ and other special types of it (see [11]) appears naturally in the nonlinear heat and wave equations [25], nonlinear Schrödinger equation with potential function [12] and the references therein for a complete description of the model. In the sitting of parabolic type system, the authors [6, 18] studied global existence, exponential decay and finite time blow-up of solutions for a class of semilinear pseudo-parabolic equations with conical degeneration. Also, our problem can be seen as a class of degenerate hyperbolic type equations in case that $V(z) = 0$ and $g_t(z) \equiv 1$ then the problem 1 is reduced to problem 1.1 in [13] and in the classical sense our problem include the classical problem

$$\begin{cases} \partial_t^2 u - \Delta u + \gamma\partial_t u = f(u), & x \in \Omega, t > 0, \\ u(x,0) = u_0(x), \qquad \partial_t u(x,0) = u_1(x), & x \in \Omega \\ u(t,x) = 0, & x \in \partial\Omega, t \geq 0, \end{cases} \tag{2}$$

where Ω is bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and Δ is the standard Laplace operator and f is a suitable function [13, 17, 19]. It is well-known that problem 2 has been studied by many authors, for example [19, 20] and the references therein.

In Section 2, we recall the definition of the edge Sobolev space and the corresponding properties. In Section 3, we will give some properties of potential wells for problem 1 on the manifold with edge singularity, which is very useful in the process of our main results. In Section 4, we give the proofs of the results of global existence and non-existence, exponential decay and finite time blowing-up of problem 1.

2 Edge Sobolev Spaces

Consider X as a closed compact C^∞-manifold of dimension n of the unit sphere in $\mathbb{R}^{n+1}$. We define an infinite cone in $\mathbb{R}^{n+1}$ as a quotient space $X^\Delta = \frac{\bar{\mathbb{R}}_+ \times X}{\{0\}\times X}$, with base X. The cylindrical coordinates $(r, \theta) \in X^\Delta - \{0\}$ in $\mathbb{R}^{n+1} - \{0\}$ are the standard coordinates. This gives us the description of $X^\Delta - \{0\}$ in the form $\mathbb{R}_+ \times X$. Then the stretched cone can be defined as $\bar{\mathbb{R}}_+ \times X = X^\wedge$. Now, consider $B = X^\Delta$ with a conical point, then by the similar way in [8, 10, 27], one can define the stretched manifold $\mathbb{B}$ with respect to B as a C^∞-manifold with smooth boundary $\partial\mathbb{B} \cong X(0)$, where $X(0)$ is the cross section of singular point zero such that there is a diffeomorphism $B - \{0\} \cong \mathbb{B} - \partial\mathbb{B}$, the restriction of which to $U - \{0\} \cong V - \partial\mathbb{B}$ for an open neighborhood $U \subset B$ near the conic point zero and a collar neighborhood $V \subset \mathbb{B}$ with $V \cong [0, 1) \times X(0)$. Therefore, we can take $\mathbb{B} = [0, 1) \times X \subset \bar{\mathbb{R}}_+ \times X = X^\wedge$. In order to consider another type of a manifold with singularity of order one so-called wedge manifold, we consider a bounded domain Y in $\mathbb{R}^q$. Set $W = X^\Delta \times Y = B \times Y$. Then W is a corresponding wedge in $\mathbb{R}^{1+n+q}$. Therefore, the stretched wedge manifold $\mathbb{W}$ to W is $X^\wedge \times Y$ which is a manifold with smooth boundary $\{0\} \times X \times Y$. Set $(r, x) \in X^\wedge$. In order to define a finite wedge, it sufficient to consider the case $r \in [0, 1)$. Thus, we define a finite wedge as

$$E = \frac{[0, 1) \times X}{\{0\} \times X} \times Y \subset X^\Delta \times Y = W.$$

The stretched wedge manifold with respect to E is

$$\mathbb{E} = [0, 1) \times X \times Y = \mathbb{B} \times Y \subset X^\wedge \times Y = W^\wedge,$$

with smooth boundary $\partial\mathbb{E} = \{0\} \times X \times Y$.

Definition 1 For $(r, x, y) \in \mathbb{R}^N_+$ with $N = 1 + n + q$, assume that $u(r, x, y) \in \mathscr{D}'(\mathbb{R}^N_+)$. We say that $u(r, x, y) \in L_p(\mathbb{R}^N_+; d\mu)$ if

$$\|u\|_{L_p} = \left(\int_{\mathbb{R}^N_+} r^N |u(r, x, y)|^p d\mu \right)^{\frac{1}{p}} < +\infty,$$

where $d\mu = \frac{dr}{r} dx_1 \ldots .dx_n \frac{dy_1}{r} \ldots \frac{dy_q}{r}$ and for $1 \le p < \infty$.

Moreover, the weighted L_p spaces with wight $\gamma \in \mathbb{R}$ is denoted by $L_p^{\gamma}(\mathbb{R}_+^N; d\mu)$, which consists of function $u(r, , y)$ such that

$$\|u\|_{L_p^{\gamma}} = \left(\int_{\mathbb{R}_+^N} r^N |r^{-\gamma} u(r, x, y)|^p d\mu \right)^{\frac{1}{p}} < +\infty.$$

Now, we can define the weighted p-Sobolev spaces with natural scale for all $1 \leq p < \infty$ on $\mathbb{R}_+^{N=1+n+q}$.

Definition 2 For $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and $N = 1 + n + q$, the spaces

$$\mathscr{H}_p^{m,\gamma}(\mathbb{R}_+^N) = \left\{ u \in \mathscr{D}'(\mathbb{R}_+^N) \quad | \quad r^{\frac{N}{p}-\gamma} (r\partial_r)^k \partial_x^{\alpha} (r\partial_y)^{\beta} u \in L_p(\mathbb{R}_+^N; d\mu) \right\}$$

for $k \in \mathbb{N}$, multi-indices $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^q$ with $k + |\alpha| + |\beta| \leq m$. In other words, if $u(r, x, y) \in \mathscr{H}_p^{m,\gamma}(\mathbb{R}_+^N)$ then $(r\partial_r)^k \partial_x^{\alpha} (r\partial_y)^{\beta} u \in L_p^{\gamma}(\mathbb{R}_+^N; d\mu)$. Therefore, $\mathscr{H}_p^{m,\gamma}(\mathbb{R}_+^N)$ is a Banach space with the following norm

$$\|u\|_{\mathscr{H}_p^{m,\gamma}(\mathbb{R}_+^N)} = \sum_{k+|\alpha|+|\beta| \leq m} \left(\int_{\mathbb{R}_+^N} r^N |r^{-\gamma} (r\partial_r)^k \partial_x^{\alpha} (r\partial_y)^{\beta} u|^p d\mu \right)^{\frac{1}{p}}.$$

Moreover, the subspace $\mathscr{H}_{p,0}^{m,\gamma}(\mathbb{R}_+^N)$ of $\mathscr{H}_p^{m,\gamma}(\mathbb{R}_+^N)$ denotes the closure of $C_0^{\infty}(\mathbb{R}_+^N)$ in $\mathscr{H}_p^{m,\gamma}(\mathbb{R}_+^N)$. Now, similarly to the definitions above, we can introduce the following weighted p-Sobolev spaces on $X^{\wedge} \times Y$, where $X^{\wedge} = \mathbb{R}_+ \times X$ and $X^{\wedge} \times Y$ is an open stretched wedge.

$$\mathscr{H}_p^{m,\gamma}(X^{\wedge} \times Y) := \left\{ u \in \mathscr{D}'(X^{\wedge} \times Y) \quad | \quad r^{\frac{N}{p}-\gamma} (r\partial_r)^k \partial_x^{\alpha} (r\partial_y)^{\beta} u \in L_p(X^{\wedge} \times Y; d\mu) \right\}$$

for $k \in \mathbb{N}$, multi-indices $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^q$ with $k + |\alpha| + |\beta| \leq m$.

Then $\mathscr{H}_p^{m,\gamma}(X^{\wedge} \times Y)$ is a Banach space with the following norm

$$\|u\|_{\mathscr{H}_p^{m,\gamma}(X^{\wedge} \times Y)} = \sum_{k+|\alpha|+|\beta| \leq m} \left(\int_{X^{\wedge} \times Y} r^N |r^{-\gamma} (r\partial_r)^k \partial_x^{\alpha} (r\partial_y)^{\beta} u|^p d\mu \right)^{\frac{1}{p}}.$$

The subspace $\mathscr{H}_{p,0}^{m,\gamma}(X^{\wedge} \times Y)$ of $\mathscr{H}_p^{m,\gamma}(X^{\wedge} \times Y)$ is defined as the closure of $C_0^{\infty}(X^{\wedge} \times Y)$.

Definition 3 Let $\mathbb{E}$ be the stretched wedge to the finite wedge E, then $\mathscr{H}_p^{m,\gamma}(\mathbb{E})$ for $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$ denotes the subset of all $u \in W_{loc}^{m,p}(int\mathbb{E})$ such that $\omega u \in$

$\mathscr{H}_p^{m,\gamma}(X^\wedge \times Y)$ for any cut-off function ω, supported by a collar neighborhood of $(0,1) \times \partial\mathbb{E}$. Moreover, the subspace $\mathscr{H}_{p,0}^{m,\gamma}(\mathbb{E})$ of $\mathscr{H}_p^{m,\gamma}(\mathbb{E})$ is defined as follows

$$\mathscr{H}_{p,0}^{m,\gamma}(\mathbb{E}) := [\omega]\mathscr{H}_{p,0}^{m,\gamma}(X^\wedge \times Y) + [1-\omega]W_0^{m,p}(int\mathbb{E})$$

where the classical Sobolev space $W_0^{m,p}(int\mathbb{E})$ denotes the closure of $C_0^\infty(int\mathbb{E})$ in $W^{m,p}(\tilde{\mathbb{E}})$ for $\tilde{\mathbb{E}}$ that is a closed compact C^∞ manifold with boundary.

If $u \in L_p^{\frac{n+1}{p}}(\mathbb{E})$ and $v \in L_{p'}^{\frac{n+1}{p'}}(\mathbb{E})$ with $p, p' \in (1,\infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then one can obtain the following edge type Hölder inequality

$$\int_{\mathbb{E}} r^q |uv| d\mu \leq \left(\int_{\mathbb{E}} r^q |u|^p d\mu \right)^{\frac{1}{p}} \left(\int_{\mathbb{E}} r^q |v|^{p'} d\mu \right)^{\frac{1}{p'}}.$$

In the case $p = 2$, we have the corresponding edge type Schwartz inequality

$$\int_{\mathbb{E}} r^q |uv| d\mu \leq \left(\int_{\mathbb{E}} r^q |u|^2 d\mu \right)^{\frac{1}{2}} \left(r^q |v|^2 d\mu \right)^{\frac{1}{2}}.$$

In the sequel, for convenience we denote

$$(u, v)_2 = \int_{\mathbb{E}} r^q uv d\mu, \qquad \|u\|_{L_p^{\frac{n+1}{p}}(\mathbb{E})} = \left(\int_{\mathbb{E}} r^q |u|^p d\mu \right)^{\frac{1}{p}}.$$

Proposition 1 (Poincaré Inequality [7]) *Let $\mathbb{E} = [0,1) \times X \times Y$ be a stretched edge manifold, $\gamma \in \mathbb{R}$ and $p \in (1,\infty)$. If $u \in \mathscr{H}_p^{1,\gamma}(\mathbb{E})$ then*

$$\|u(z)\|_{L_p^\gamma(\mathbb{E})} \leq c \|\nabla_{\mathbb{E}} u(z)\|_{L_p^\gamma(\mathbb{E})} \tag{3}$$

where $\nabla_{\mathbb{E}} := (r\partial_r, \partial_{x_1}, \ldots, \partial_{x_n}, r\partial_{y_1}, \ldots, r\partial_{y_q})$ and the constant c depending only on $\mathbb{E}$.

Proposition 2 ([7]) *For $1 < p < 2^*$ the embedding $\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \hookrightarrow \mathscr{H}_{p,0}^{0,\frac{n+1}{p}}(\mathbb{E})$ is continuous.*

Proposition 3 ([7]) *There exist $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_j \leq \ldots$, and $\lambda_j \to \infty$ such that for all $j \geq 1$, the following Dirichlet problem*

$$\begin{cases} -\Delta_{\mathbb{E}} \phi_j = \lambda_j \phi_j, & x \in int\mathbb{E}, \\ \phi_j = 0, & x \in \partial\mathbb{E}, \end{cases} \tag{4}$$

admits non-trivial solution in $\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$*. Moreover, we can choose positive* $\{\phi_j\}_{j\geq 1}$ *which constitute an orthonormal basis of Hilbert space* $\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$*, and the inequality*

$$\lambda_1^{\frac{1}{2}}\|u(z)\|_{L_2^{\frac{n}{2}}(\mathbb{E})} \leq \|\nabla_{\mathbb{E}}u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})},$$

holds.

3 Some Auxiliary Results

In this section we give some results about the potential wells for problem 1 and we obtain some properties of energy functional that we will use to prove the main results in Section 4.

Similar to the classical case, we introduce the following functionals on the cone Sobolev space $\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$:

$$J(u) = \frac{1}{2}\int_{\mathbb{E}} r^q|\nabla_{\mathbb{E}}u|^2 d\mu + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|u|^2 d\mu - \frac{1}{p+1}\int_{\mathbb{E}} r^q g_t(z)|u|^{p+1} d\mu,$$

$$K(u) = \int_{\mathbb{E}} r^q|\nabla_{\mathbb{E}}u|^2 d\mu + \int_{\mathbb{E}} r^q V(x)|u|^2 d\mu - \int_{\mathbb{E}} r^q g_t(z)|u|^{p+1} d\mu.$$

Then $J(u)$ and $K(u)$ are well-defined and belong to space $C^1\Big(\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}), \mathbb{R}\Big)$. Now we define

$$\mathscr{N} = \Big\{u \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \quad ; \quad K(u) = 0, \quad \int_{\mathbb{E}} r^q|\nabla_{\mathbb{E}}u|^2 d\mu \neq 0\Big\},$$

$$d = \inf\Big\{\sup_{\lambda\geq 0} J(\lambda u) \quad ; \quad u \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}), \quad \int_{\mathbb{E}} r^q|\nabla_{\mathbb{E}}u|^2 d\mu \neq 0\Big\}.$$

Thus, similar to the results in [20] we obtain that $0 < d = \inf_{u\in\mathscr{N}} J(u)$. For $0 < \delta$ we define

$$K_\delta(u) = \delta\Big[\int_{\mathbb{E}} r^q|\nabla_{\mathbb{E}}u|^2 d\mu + \int_{\mathbb{E}} r^q V(z)|u|^2 d\mu\Big] - \int_{\mathbb{E}} r^q g_t(z)|u|^{p+1} d\mu,$$

$$\mathcal{N}_\delta = \left\{ u \in \mathscr{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{B}) \quad ; \quad K_\delta(u) = 0, \quad \int_{\mathbb{E}} r^q |\nabla_{\mathbb{E}}|^2 d\mu \neq 0 \right\},$$

$$d(\delta) = \inf_{u \in \mathcal{N}_\delta} J(u).$$

Proposition 4 *If* $0 < \|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} < r(\delta)$ *where* $r(\delta) = \left(\frac{(C_*^2+1)\delta}{C_{**}^{p+1}}\right)^{\frac{1}{p-1}}$, *then* $K_\delta(u) > 0$. *In particular, if*

$$0 < \|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} < r(1)$$

then $K(u) > 0$.

Proof We conclude the following

$$\|g_t(z)^{\frac{1}{p+1}} u\|^{p+1}_{L_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})} = \int_{\mathbb{E}} r^q |g_t(z)^{\frac{1}{p+1}} u(z)|^{p+1} d\mu = \int_{\mathbb{E}} r^q |g_t(z)||u(z)|^{p+1} d\mu \leq$$

$$\|g_t\|_{L^\infty} \int_{\mathbb{E}} r^q |u|^{p+1} d\mu \quad \Rightarrow$$

$$\|g_t(z)^{\frac{1}{p+1}} u\|^{p+1}_{L_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})} \leq C_g \|u\|^{p+1}_{L_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}. \tag{5}$$

Also from definition of C_*:

$$\|V(z)^{\frac{1}{2}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \geq C_*^2 \|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}. \tag{6}$$

Then by definition of K_δ and using the assumption we get that

$$\begin{aligned} K_\delta(u) &= \delta\left[\int_{\mathbb{E}} r^q |\nabla_{\mathbb{E}} u|^2 d\mu + \int_{\mathbb{E}} r^q V(z)|u|^2 d\mu\right] - \int_{\mathbb{E}} r^q g_t(z)|u|^{p+1} d\mu \\ &\geq \delta(1 + C_*^2)\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - C_{**}^{p+1} \|\nabla_{\mathbb{E}} u\|^{p+1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &= \left(\delta(1 + C_*^2) - C_{**}^{p+1} \|\nabla_{\mathbb{E}} u\|^{p-1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\right) \|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} > 0. \end{aligned}$$

In case that $\delta = 1$ then by definition of functional K we obtain that $K(u) > 0$.

Proposition 5 *If* $K_\delta(u) < 0$, *then* $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} > r(\delta)$. *In particular, if* $K(u) < 0$, *then* $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} > r(1)$.

Proof Since $K_\delta(u) < 0$, then by definition of $K_\delta(u)$, we get that $\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$. Now, we have

$$\begin{aligned}
\delta\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} &< \int_{\mathbb{E}} r^q g_t(z)|u(z)|^{p+1} d\mu - \delta \int_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu \\
&\leq \|g_t(x)^{\frac{1}{p+1}} u\|^{p+1}_{L_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})} - \delta\|V(z)^{\frac{1}{2}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&< C_{**}^{p+1} \|\nabla_{\mathbb{E}} u\|^{p-1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \delta C_*^2 \|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.
\end{aligned}$$

Therefore,

$$\|\nabla_{\mathbb{E}} u\|^{p-1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})} > \left(\frac{\delta(1 + C_*^2)}{C_{**}^{p+1}}\right) = r^{p-1}(\delta).$$

Corollary 1 *Let* $u \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$, $K_\delta(u) = 0$ *and* $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$. *Then* $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \geq r(\delta)$. *In particular, if* $K(u) = 0$ *and* $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$, *then* $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \geq r(1)$.

Lemma 1 *(i) The functional* $J(\lambda u)$ *admits its maximum for* $\lambda = \lambda_*$ *where*

$$\lambda_* = \left(\frac{\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu}{\int_{\mathbb{E}} r^q g_t(z)|u(z)|^{p+1} d\mu}\right)^{\frac{1}{p-1}}.$$

Also for $0 \leq \lambda < \lambda_*$, $J(\lambda u)$ *is strictly increasing and for* $\lambda_* < \lambda$, *it is strictly decreasing.*

(ii) $K(\lambda_* u) = 0$ *and* $K(\lambda u) > 0$ *if* $0 < \lambda < \lambda_*$. *Also if* $\lambda_* < \lambda$ *then* $K(\lambda u) < 0$.

(iii) By results in i *and* ii *we obtain that*

$$d = \inf\left\{\sup_{\lambda \geq 0} J(\lambda u) \quad ; \quad u \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}), \quad \int_{\mathbb{E}} |\nabla_{\mathbb{E}} u|^2 d\mu \neq 0\right\}$$

$$= \frac{p-1}{2(p+1)}(1 + C_*^2)^{\frac{p+1}{p-1}} C_{**}^{-2\frac{p+1}{1-p}} \|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.$$

Proof For proof of i and ii we obtain the following conclusions. Let $u \in \mathscr{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E})$ and $\int_{\mathbb{E}} |\nabla_{\mathbb{E}} u|^2 d\mu \neq 0$. Then by definition of J we obtain that

$$\lim_{\lambda\to+\infty} J(\lambda u) = \lim_{\lambda\to+\infty}\Bigg[\frac{1}{2}\int_{\mathbb{E}} r^q|\nabla_{\mathbb{E}}\lambda u|^2 d\mu + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|\lambda u(z)|^2 d\mu$$

$$-\frac{1}{p+1}\int_{\mathbb{E}} r^q g_t(z)|\lambda u(x)|^{p+1} d\mu\Bigg]$$

$$= \lim_{\lambda\to+\infty}\Bigg[\frac{1}{2}\|\nabla_{\mathbb{E}}\lambda u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\|V(z)^{\frac{1}{2}}\lambda u(z)\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \frac{1}{p+1}\|g_t(z)^{\frac{1}{p+1}}\lambda u(z)\|^{p+1}_{L_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}\Bigg]$$

$$= \lim_{\lambda\to+\infty}\Bigg[\frac{\lambda^2}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{\lambda^2}{2}\|V(z)^{\frac{1}{2}} u(z)\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \frac{\lambda^{p+1}}{p+1}\|g_t(z)^{\frac{1}{p+1}} u(z)\|^{p+1}_{L_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}\Bigg]$$

$$\geq \lim_{\lambda\to+\infty}\Bigg[\frac{\lambda^2}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{\lambda^2}{2}C_*^2\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \frac{\lambda^{p+1}}{p+1}C_{**}^{p+1}\|\nabla_{\mathbb{E}} u\|^{p+1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\Bigg]$$

$$= \lim_{\lambda\to+\infty}\Bigg[\frac{\lambda^2}{2} + \frac{\lambda^2}{2}C_*^2 - \frac{\lambda^{p+1}}{p+1}C_{**}^{p+1}\|\nabla_{\mathbb{E}} u\|^{p-1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\Bigg]\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = -\infty.$$

Also we have

$$J(\lambda u) = \frac{1}{2}\int_{\mathbb{E}} |\nabla_{\mathbb{E}}\lambda u|^2 d\mu + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|\lambda u(z)|^2 d\mu - \frac{1}{p+1}\int_{\mathbb{E}} r^q g_t(z)|\lambda u(z)|^{p+1}$$

$$= \frac{\lambda^2}{2}\int_{\mathbb{E}} |\nabla_{\mathbb{E}} u|^2 d\mu + \frac{\lambda^2}{2}\int_{\mathbb{E}} V(z)|u(z)|^2 d\mu - \frac{\lambda^{p+1}}{p+1}\int_{\mathbb{E}} g_t(z)|u(z)|^{p+1} d\mu.$$

Then

$$\frac{\partial J(\lambda u)}{\partial\lambda} = \lambda\int_{\mathbb{E}} |\nabla_{\mathbb{E}} u|^2 d\mu + \lambda\int_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu - \lambda^p\int_{\mathbb{E}} r^q g_t(z)|u(z)|^{p+1} d\mu$$

$$= \lambda\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \lambda\|V(z)^{\frac{1}{2}} u(z)\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \lambda^p\|g_t(z)^{\frac{1}{p+1}} u(z)\|^{p+1}_{L_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})}.$$

Now, $\frac{\partial J(\lambda u)}{\partial\lambda} = 0$, it follows that

$$\lambda_* := \left(\frac{\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu}{\int_{\mathbb{E}} r^q g_t(z)|u(z)|^{p+1} d\mu}\right)^{\frac{1}{p-1}}$$

is a maximum of $J(\lambda u)$ since $\frac{\partial^2(J(\lambda u))}{\partial\lambda^2}|_{\lambda=\lambda_*} < 0$.

(iii) Using of i and ii $\sup\limits_{\lambda\geq 0} J(\lambda u) = J(\lambda_* u)$. Thus,

$$J(\lambda_* u) = \frac{1}{2}\int\limits_{\mathbb{E}} r^q |\nabla_{\mathbb{E}} \lambda_* u|^2 d\mu + \frac{1}{2}\int\limits_{\mathbb{E}} r^q V(z)|\lambda_* u(z)|^2 d\mu$$

$$- \frac{1}{p+1}\int\limits_{\mathbb{E}} r^q g_t(z)|\lambda_* u|^{p+1} d\mu$$

$$= \lambda_*^2\Big[\frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int\limits_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu - \frac{\lambda_*^{p-1}}{p+1}\int\limits_{\mathbb{E}} r^q g_t(z)|u|^{p+1} d\mu\Big]$$

$$= \lambda_*^2\Big[\frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int\limits_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu$$

$$- \frac{1}{p+1}\Bigg(\frac{\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu}{\int_{\mathbb{E}} r6q g_t(z)|u(z)|^{p+1} d\mu}\Bigg)^{\frac{p-1}{p-1}} \Big(\int\limits_{\mathbb{E}} r^q g_t(z)|u|^{p+1} d\mu\Big)\Big]$$

$$= \lambda_*^2\Big[\frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int\limits_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu - \frac{1}{p+1}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}$$

$$- \frac{1}{p+1}\int\limits_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu\Big]$$

$$= \lambda_*^2\Big[\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|\nabla u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int\limits_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu\Big]$$

$$= \Bigg(\frac{\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu}{\int_{\mathbb{E}} r^q g_t(z)|u(z)|^{p+1} d\mu}\Bigg)^{\frac{2}{p-1}}$$

$$\times \Bigg(\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int\limits_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu\Bigg) \times \frac{p-1}{2(p+1)}$$

$$\geq \Bigg(\frac{\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + C_*^2\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}}{\int_{\mathbb{E}} r^q g_t|u|^{p+1} d\mu}\Bigg)^{\frac{2}{p-1}}\Big[\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + C_*^2\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\Big]\frac{p-1}{2(p+1)}$$

$$\geq \frac{p-1}{2(p+1)}(1 + C_*^2)^{\frac{p+1}{p-1}} C_{**}^{\frac{-2(p+1)}{p-1}} \|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.$$

Therefore,

$$d = \inf_{u \in \mathscr{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E}), \|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0} J(\lambda_* u)$$

$$= \frac{p-1}{2(p+1)} (1 + C_*^2)^{\frac{p+1}{p-1}} C_{**}^{\frac{-2(p+1)}{p-1}} \|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.$$

Proposition 6 *Let* $0 < \delta < \frac{p+1}{2}$, *then* $d(\delta) \geq a(\delta) r^2(\delta)$ *where* $a(\delta) = \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\left(1 + C_*^2\right)$. *Moreover, we have*

$$d(\delta) = \inf_{u \in \mathscr{N}_\delta} J(u) = d\ \lambda(\delta)^2\ a(\delta)[1 + c_*^2]^{-1}\ \frac{2(p+1)}{p-1}.$$

Proof Let $u \in \mathscr{N}_\delta$, so by Proposition 5 we get that $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} > r(\delta)$. Then by definition of J and K_δ we obtain that

$$J(u) = \frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu - \frac{1}{p+1}\int_{\mathbb{E}} r^q g_t(z)|u(z)|^{p+1} d\mu$$

$$= \frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|u(z)|^2 r^q$$

$$- \frac{1}{p+1}\left(\delta\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - K_\delta(u) + \delta\int_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu\right).$$

Since $K_\delta(u) = 0$,

$$J(u) \geq (\frac{1}{2} - \frac{\delta}{p+1})\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{\delta(p-1)}{2(p+1)}\|V(x)^{\frac{1}{2}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}$$

$$\geq (\frac{1}{2} - \frac{\delta}{p+1})\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{\delta(p-1)}{2(p+1)} C_*^2 \|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}$$

$$= \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\left(1 + C_*^2\right)\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.$$

Since $\|\nabla u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \geq r^2(\delta)$ then,

$$d(\delta) \geq a(\delta) r^2(\delta).$$

Now, we prove the second part of the assertion. By definition of $\mathcal{N}_\delta$ and $\mathcal{N}$, for $\bar{u} \in \mathcal{N}_\delta$ and $\lambda\bar{u} \in \mathcal{N}$, we obtain

$$\lambda^2 \|\nabla_{\mathbb{E}}\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \lambda^2 \int\limits_{\mathbb{E}} r^q V(z)|\bar{u}|^2 d\mu = \lambda^{p+1} \int\limits_{\mathbb{E}} r^q g_t(z)|\bar{u}|^{p+1} d\mu, \tag{7}$$

and

$$\delta \|\nabla_{\mathbb{E}}\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \delta \int\limits_{\mathbb{E}} r^q V(z)|\bar{u}|^2 d\mu = \int\limits_{\mathbb{E}} r^q g_t(z)|\bar{u}|^{p+1} d\mu. \tag{8}$$

Then 7 gives

$$\lambda = \left(\frac{\|\nabla_{\mathbb{E}}\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int\limits_{\mathbb{E}} r^q V(z)|\bar{u}|^2 d\mu}{\int\limits_{\mathbb{E}} r^q g_t(z)|\bar{u}|^{p+1} d\mu} \right)^{\frac{1}{p-1}}, \tag{9}$$

and 8 gives that

$$\delta = \frac{\int\limits_{\mathbb{E}} r^q g_t(z)|\bar{u}|^{p+1} d\mu}{\|\nabla_{\mathbb{E}}\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int\limits_{\mathbb{E}} r^q V(z)|\bar{u}|^2 d\mu}. \tag{10}$$

By 10 and 9, we define

$$\lambda = \lambda(\delta) = \left(\frac{1}{\delta}\right)^{\frac{1}{p-1}}. \tag{11}$$

Moreover, for such λ, $\lambda\bar{u} \in \mathcal{N}$, so by definition of d we get that

$$\begin{aligned}
d \le J(\lambda\bar{u}) &= \frac{1}{2}\|\nabla_{\mathbb{E}}\lambda\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int\limits_{\mathbb{E}} r^q V(z)|\lambda\bar{u}|^2 d\mu \\
&\quad - \frac{1}{p+1}\int\limits_{\mathbb{E}} r^q g_t(z)|\lambda\bar{u}|^{p+1} d\mu \\
&= \frac{\lambda^2}{2}\|\nabla_{\mathbb{E}}\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{\lambda^2}{2}\int\limits_{\mathbb{E}} r^q V(z)|\bar{u}|^2 d\mu \\
&\quad - \frac{1}{p+1}\left[\|\nabla_{\mathbb{E}}\lambda\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int\limits_{\mathbb{E}} r^q V(z)|\lambda\bar{u}|^2 d\mu - K(\lambda\bar{u})\right]
\end{aligned}$$

$$= \lambda^2 \Big[\frac{1}{2} \|\nabla_{\mathbb{E}} \bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2} \int_{\mathbb{E}} r^q V(z) |\bar{u}|^2 d\mu - \frac{1}{p+1} \|\nabla_{\mathbb{E}} \bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \frac{1}{p+1} \int_{\mathbb{E}} r^q V(z) |\bar{u}|^2 d\mu \Big]$$

$$\leq (\frac{1}{\delta})^{\frac{2}{p-1}} \Big[\frac{p-1}{2(p+1)} \|\nabla_{\mathbb{E}} \bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \frac{(1-p)C_*^2}{2(p+1)} \|\nabla_{\mathbb{E}} \bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \Big].$$

On the other hand,

$$d(\delta) = J(\bar{u}) = \frac{1}{2} \|\nabla_{\mathbb{E}} \bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2} \int_{\mathbb{E}} r^q V(z) |\bar{u}|^2 d\mu - \frac{1}{p+1} \int_{\mathbb{E}} r^q g_t(z) |\bar{u}|^{p+1} d\mu$$

$$= \frac{1}{2} \|\nabla_{\mathbb{E}} \bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2} \int_{\mathbb{E}} r^q V(z) |\bar{u}|^2 d\mu - \frac{1}{p+1} \Big(\delta \|\nabla_{\mathbb{E}} \bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \delta \int_{\mathbb{E}} r^q V(z) |\bar{u}|^2 d\mu - K_\delta(\bar{u}) \Big)$$

$$= (\frac{1}{2} - \frac{\delta}{p+1}) \|\nabla_{\mathbb{E}} \bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + (\frac{1}{2} - \frac{\delta}{p+1}) \int_{\mathbb{E}} r^q V(z) |\bar{u}|^2 d\mu$$

$$\geq \Big(\frac{1}{2} - \frac{\delta}{p+1} \Big) (1 + C_*^2) \|\nabla_{\mathbb{E}} \bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = a(\delta) \|\nabla_{\mathbb{E}} \bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.$$

Indeed,

$$d \leq J(\lambda \bar{u}) \leq \big(\frac{1}{\delta}\big)^{\frac{2}{p-1}} \Big[\frac{p-1}{2(p+1)} \, (1 + C_*^2) \Big] \frac{d(\delta)}{a(\delta)}.$$

Hence,

$$d(\delta) \geq a(\delta) \, \big(\frac{1}{\delta}\big)^{-\frac{2}{p-1}} \, [1 + C_*^2]^{-1} \, [\frac{2(p+1)}{p-1}] d.$$

Now, we let $0 < \delta$ and $\tilde{u} \in \mathcal{N}$ is minimizer of d that is

$$d = J(\tilde{u}) = \frac{1}{2} \|\nabla_{\mathbb{E}} \tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2} \int_{\mathbb{E}} r^q V(z) |\tilde{u}|^2 d\mu - \frac{1}{p+1} \int_{\mathbb{E}} r^q g_t(z) |\tilde{u}|^{p+1} d\mu.$$

we define $\lambda = \lambda(\delta)$ by

$$\delta \|\nabla_{\mathbb{E}} \lambda \tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \delta \int_{\mathbb{E}} r^q V(z)|\lambda \tilde{u}|^2 d\mu = \int_{\mathbb{E}} r^q g_t(z)|\lambda \tilde{u}|^{p+1} d\mu.$$

Then for any $0 < \delta$, there exists a unique λ which satisfies

$$\lambda = \delta^{\frac{1}{p-1}}.$$

Hence, for such λ, $\lambda \tilde{u} \in \mathcal{N}_\delta$ by definition of $d(\delta)$ we get that

$$\begin{aligned} d &= \frac{1}{2}\|\nabla_{\mathbb{E}} \lambda \tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|\lambda \tilde{u}|^2 d\mu \\ &- \frac{1}{p+1}\left(\delta \|\nabla_{\mathbb{E}} \lambda \tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \delta \int_{\mathbb{E}} r^q V(z)|\lambda \tilde{u}|^2 d\mu - K_\delta(\lambda \tilde{u})\right) \\ &= (\frac{1}{2} - \frac{\delta}{p+1})\|\nabla_{\mathbb{E}} \lambda \tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + (\frac{1}{2} - \frac{\delta}{p+1})\int_{\mathbb{E}} r^q V(z)|\lambda \tilde{u}|^2 d\mu \\ &\geq \left[\frac{1}{2} - \frac{\delta}{p+1} + C_*^2(\frac{1}{2} - \frac{\delta}{p+1})\right]\|\nabla_{\mathbb{E}} \lambda \tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}. \end{aligned}$$

On the other hand,

$$\begin{aligned} d(\delta) &\leq J(\lambda \tilde{u}) = \frac{1}{2}\|\nabla_{\mathbb{E}} \lambda \tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|\lambda \tilde{u}|^2 d\mu - \frac{1}{p+1}\int_{\mathbb{E}} r^q g_t(z)|\lambda \tilde{u}|^{p+1} d\mu \\ &= \frac{\lambda^2}{2}\|\nabla_{\mathbb{E}} \tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{\lambda^2}{2}\int_{\mathbb{E}} r^q V(z)|\tilde{u}|^2 d\mu \\ &- \frac{1}{p+1}\left(\delta \|\nabla_{\mathbb{E}} \lambda \tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \delta \int_{\mathbb{E}} r^q V(z)|\lambda \tilde{u}|^2 d\mu - K_\delta(\lambda \tilde{u})\right) \\ &= \lambda^2\left[(\frac{1}{2} - \frac{\delta}{p+1})\|\nabla_{\mathbb{E}} \tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - (\frac{\delta}{p+1} - \frac{1}{2})\int_{\mathbb{E}} r^q V(z)|\tilde{u}|^2 d\mu\right] \\ &\leq \delta^{\frac{2}{p-1}}\left[(\frac{1}{2} - \frac{\delta}{p+1})\|\nabla_{\mathbb{E}} \tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - (\frac{\delta}{p+1} - \frac{1}{2})C_*^2\|\nabla_{\mathbb{E}} \tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\right] \\ &\leq \delta^{\frac{2}{p+1}}\left[\frac{1}{2} - \frac{\delta}{p+1} + (\frac{1}{2} - \frac{\delta}{p+1})C_*^2\right]\|\nabla_{\mathbb{E}} \tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = \delta^{\frac{2}{p-1}} a(\delta)\|\nabla_{\mathbb{E}} \tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}. \end{aligned}$$

Then,

$$d(\delta) \leq \delta^{\frac{2}{p-1}}\ a(\delta)\ d\ [1 + C_*^2]^{-1}\ \frac{2(p+1)}{p-1}.$$

Therefore,

$$d(\delta) = \inf J(u)_{u \in \mathcal{N}_\delta} = \delta^{\frac{2}{p-1}}\ a(\delta)\ d\ [1 + C_*^2]^{-1}\ \frac{2(p+1)}{p-1}.$$

Remark 1 According to $d(\delta)$ in Proposition 6, we obtain that

(i) $\lim_{\delta \to 0} d(\delta) = 0.$

(ii) $d(\delta) = d\frac{2(p+1)}{p-1}\left[\frac{1}{2}\delta^{\frac{2}{p-1}} - \frac{1}{p+1}\delta^{\frac{p+1}{p-1}}\right]$. Then

$$d'(\delta) = \frac{d2(p+1)}{(p-1)^2}\delta^{\frac{2}{p-1}}\left[\delta^{-1} - 1\right] = 0 \Rightarrow \delta = 1.$$

Hence, if $0 < \delta < 1$ then $d(\delta)$ is strictly increasing function and if $\delta > 1$ then $d(\delta)$ is strictly decreasing function.

4 Invariance of the Solutions

Now, we introduce the following potential wells

$$W = \left\{ u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \quad ; \quad K(u) > 0, \quad J(u) < d \right\} \cup \{0\},$$

$$W_\delta = \left\{ u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \quad ; \quad K_\delta(u) > 0, \quad J(u) < d(\delta) \right\} \cup \{0\},$$

for $0 < \delta$, and corresponding potentials outside of the set that defined as above

$$E = \left\{ u \in u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \quad ; \quad K(u) < 0, \quad J(u) < d \right\},$$

$$E_\delta = \left\{ u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \quad ; \quad K_\delta(u) < 0, \quad J(u) < d(\delta) \right\}$$

for any $0 < \delta$. According to the definition of potential wells W_δ and potential outside E_δ one can get the following inclusions:

(I) $W_{\delta_1} \subset W_{\delta_2}$ whenever $0 < \delta_1 < \delta_2 \leq 1$,

(II) $E_{\delta_1} \subset E_{\delta_2}$ whenever $1 \le \delta_2 < \delta_1 < \frac{p+1}{2}$. Furthermore, from the above results on can define the following sets

$$V_\delta = \left\{u \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \quad : \quad \|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} < r(\delta)\right\}$$

$$\bar{V_\delta} = V_\delta \cup \partial V_\delta = \left\{u \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \quad : \quad \|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \le r(\delta)\right\}$$

$$V_\delta^c = \left\{u \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \quad : \quad \|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} > r(\delta)\right\}.$$

Then for every $0 < \delta < \frac{p+1}{2}$ one gets that

$$V_{t(\delta)} \subset W_\delta \subset V_{s(\delta)}, \qquad E_\delta \subset V_\delta^c$$

where

$$V_{t(\delta)} = \left\{u \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \quad : \quad \|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} < \min\{r^2(\delta), r_0^2(\delta)\}\right\}$$

$$V_{s(\delta)} = \left\{u \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \quad : \quad \|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} < \frac{d(\delta)}{a(\delta)}\right\}$$

where $r_0(\delta)$ is the unique real root of equation $\frac{r^2}{2} = d(\delta)$.

Definition 4 Suppose that $u(t)$ is a weak solution of problem 1. $T_{\max}$ is called maximal existence time of solution $u(t)$ if one the following conditions hold:

(1) If $u(t)$ exists for every $0 \le t < +\infty$ then $T_{\max} = +\infty$. In this case, we say that the solution is global.
(2) If there exists a $t_0 \in (0, \infty)$ such that $u(t)$ exists for every $0 \le t < t_0$, but does not exist at $t = t_0$, then $T_{\max} = t_0$.

Definition 5 $u = u(z,t) \in L^\infty\Big(0, T_{\max}; \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\Big)$ with $\partial_t u \in L^\infty\Big(0, T_{\max}; L_2^{\frac{n+1}{2}}(\mathbb{E})\Big)$ is called a weak solution of the problem 1 on $int\mathbb{E} \times [0, T_{\max})$ if

$$\begin{aligned}(u_t, v)_2 + \gamma(\nabla_{\mathbb{E}} u, \nabla_{\mathbb{E}} v)_2 + \int_0^t (\nabla_{\mathbb{E}} u, \nabla_{\mathbb{E}} v)_2 d\tau + \int_0^t (V(x)u, v)_2 d\tau \\ = \int_0^t (g_t(z)|u|^{p-1}u, v)_2 d\tau \\ + (\gamma u_0, v)_2 + (u_1, v)_2 \qquad \forall v \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}),\end{aligned}$$

$u(z,0)=u_0$ in $\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ and hold the following energy inequality

$$I(t)+\gamma\int_0^t \|\nabla_{\mathbb{E}}(\partial_\tau u)\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\,d\tau \le I(0),\quad \forall t\in(0,T_{\max}),$$

where $0\le T_{\max}\le\infty$ and

$$I(t)=\frac{1}{2}\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\frac{1}{p+1}\int_0^t \|(\frac{d}{d\tau}g_\tau(z))^{\frac{1}{p+1}}u\|^{p+1}_{L_p^{\frac{n+1}{p+1}}(\mathbb{E})}\,d\tau + J(u).$$

We note, since $u\in L^\infty\Big(0,T_{\max};\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\Big)$ and $\partial_t u\in L^\infty\Big(0,T_{\max};L_2^{\frac{n+1}{2}}(\mathbb{E})\Big)$ from the first equation of the problem 1 as similar in [13], one can obtain that $\partial_t^2 u\in L^\infty\Big(0,T_{\max};\mathscr{H}_{2,0}^{-1,\frac{n+1}{2}}(\mathbb{E})\Big)$.

Now we discuss the invariance of some sets corresponding to the problem 1.

Proposition 7 *Let $0<J(u)<d$ for $u\in\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$. Suppose that $\delta_1<1<\delta_2$ be roots of equation $d(\delta)=J(u)$. Then $K_\delta(u)$ has no change in its sign for $\delta\in(\delta_1,\delta_2)$.*

Proof Since $0<J(u)<d$ then by Propositions 1 and 2 we can assume that $\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\neq 0$. We assume that there exists a $\delta_0\in(\delta_1,\delta_2)$ for which $K_{\delta_0}(u)=0$. Hence, by definition of $d(\delta)$ we have $J(u)\ge d(\delta)$. But, we have two cases the following for δ_0

$$\begin{cases}\delta_1<\delta_0<1<\delta_2\\ \delta_1<1<\delta_0<\delta_2\end{cases}$$

Now, by Remark 1 We get that $d(\delta_1)<d(\delta_0)$ or $d(\delta_2)<d(\delta_0)$ then we obtain that $d(\delta_1)=d(\delta_2)=J(u)<d(\delta_0)$ that this is contradiction .

Theorem 1 *Let $u_0\in\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$, $0<e<d$. Suppose that $\delta_1<\delta_2$ are roots of equations $d(\delta)=e$ then*

(i) all solutions of problem 1 with $0<J(u_0)\le e$ belong to set W_δ for $\delta_1<\delta<\delta_2$ provided $K(u_0)>0$ or $\|\nabla_{\mathbb{E}}u_0\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}=0$.

(ii) all solutions of problem 1 with $0<J(u_0)\le e$ belong to E_δ for $\delta\in(\delta_1,\delta_2)$ provided $K(u_0)<0$.

Proof

(i) Let $u(t)$ be a solution of the problem 1 with initial value u_0 for which satisfies in conditions $0<J(u_0)\le e<d$, $K(u_0)>0$ or $\|\nabla_{\mathbb{E}}u_0\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}=0$. Let T

be existence time for solution $u(t)$. If $\|\nabla_{\mathbb{E}} u_0\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = 0$, then since u_0 has compact support $u_0 = 0$, so by definition of W_δ we obtain that $u_0 \in W_\delta$. If $K(u_0) > 0$ then by assumption we have

$$0 < J(u_0) \leq e = d(\delta_1) = d(\delta_2) < d(\delta) \leq d$$

for $\delta_1 < \delta < \delta_2$. Hence, $K_\delta(u_0(t)) > 0$ for $\delta_1 < \delta < \delta_2$, by Proposition 7. Therefore, by definition of W_δ, $u_0 \in W_\delta$ for $\delta_1 < \delta < \delta_2$. Now, we have to show that for $\delta_1 < \delta < \delta_2$ and $0 < t < T$, $u(t) \in W_\delta$. Suppose that, there exist $t_0 \in (0, T)$ such that for $\delta_1 < \delta < \delta_2$, $u(t_0) \in \partial W_\delta$. Then we can imply that, $K_\delta(u(t_0)) = 0$ and $\|\nabla_{\mathbb{E}} u_0\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$, or by definition of W_δ, $J(u(t_0)) = d(\delta)$. Since $u(t_0)$ is a solution of problem 1, so it satisfies in energy inequality i.e.

$$\frac{1}{2}\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{p+1}\int_0^t \|(\frac{d}{d\tau} g_\tau(z))^{\frac{1}{p+1}} u\|^{p+1}_{L_{p+1}^{\frac{n+1}{p+1}}(\mathbb{Z})} d\tau + J(u(t))$$
$$+ \gamma \int_0^t \|\nabla_{\mathbb{E}}(\partial_\tau u)\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau$$
$$\leq I(0) = J(u_0) \leq e < d(\delta),$$

for any $\delta \in (\delta_1, \delta_2)$ and $t \in (0, T)$. Therefore, the equality $J(u(t_0)) = d(\delta)$ for any $\delta \in (\delta_1, \delta_2)$ and $t \in (0, T)$ is not possible. If $K_\delta(u(t_0)) = 0$ and $\|\nabla_{\mathbb{E}} u_0\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$, then by definition of $d(\delta)$ we get that $d(\delta) \leq J(u_0(t))$, that is in contradiction with energy inequality. Therefore, $u(t) \in W_\delta$ for any $\delta \in (\delta_1, \delta_2)$ and $t \in (0, T)$.

(ii) similar to first case it can be prove that $u_0 \in E_\delta$ for $\delta \in (\delta_1, \delta_2)$ provided $K_\delta(u_0) < 0$. Now, we should prove $u(t) \in E_\delta$ for any $\delta \in (\delta_1, \delta_2)$ and $t \in (0, T)$. Suppose that there exist $t_0 \in (0, T)$, such that for $t \in [0\ ,\ t_0)$, $u(t) \in E_\delta$ and $u(t_0) \in \partial E_\delta$, that is, $K_\delta(u_0) = 0$ or $J(u(t_0)) = d(\delta)$ for $\delta \in (\delta_1, \delta_2)$. According to energy inequality the equality $J(u(t_0)) = d(\delta)$ is not possible similar to first case. Hence, we assume that $K_\delta(u(t_0)) = 0$, then $K_\delta(u(t)) < 0$ for $t \in (0, t_0)$, since for $t \in [0, t_0)$, $u(t) \in E_\delta$, then by definition of E_δ, $K_\delta(u(t)) < 0$. Now, using the Proposition 5 we obtain that $\|\nabla_{\mathbb{E}} u(t)\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} > r(\delta)$ and $\|\nabla_{\mathbb{E}} u(t_0)\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} > r(\delta) \neq 0$. Hence by definition of $d(\delta)$, $J(u(t_0)) \geq d(\delta)$ which is in contradiction with energy inequality.

Remark 2 suppose that all assumptions in Theorem 1 hold. Then for any $\delta \in (\delta_1, \delta_2)$ both seta W_δ and E_δ are invariant. Moreover, both sets

$$W_{\delta_1\delta_2} = \bigsqcup_{\delta_1<\delta<\delta_2} W_\delta, \qquad E_{\delta_1\delta_2} = \bigsqcup_{\delta_1<\delta<\delta_2} E_\delta$$

are invariant respectively under flow of the problem 1. Hence, we can get for all weak solutions of the problem 1

$$u(t) \notin \mathcal{N}_{\delta_1\delta_2} = \bigsqcup_{\delta_1<\delta<\delta_2} \mathcal{N}_\delta.$$

To discuss about the invariant of the solutions with negative level energy, we introduce the following results.

Proposition 8 *Let* $u_0 \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ *and* $u_1 \in L_2^{\frac{n+1}{2}}(\mathbb{E})$. *Suppose that* $I(0) = 0$ *and* $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$. *Then all weak solutions of the problem 1 satisfy*

$$\|\nabla_{\mathbb{E}} u\|^{p-1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \geq M = \frac{(p+1)(1+C_*^2)}{2C_{**}^{p+1}}.$$

Proof Let us consider $u \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ as a weak solution of the problem 1. According to the Definition 5

$$I(t) + \gamma \int_0^t \|\nabla_{\mathbb{E}}(\partial_\tau u)\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau \leq I(0) = 0.$$

Therefore, by definition of constants C_* and C_{**}

$$\begin{aligned}\frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{C_*^2}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} &\leq \frac{1}{2}\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &\quad + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|u|^2 d\mu \\ &\leq \frac{C_{**}^{p+1}}{p+1}\|\nabla_{\mathbb{E}} u\|^{p+1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.\end{aligned}$$

Hence,

$$\|\nabla_{\mathbb{E}} u\|^{p-1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \geq \frac{(p+1)(1+C_*^2)}{2C_{**}^{p+1}} = M.$$

Theorem 2 *Let* $u_0 \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ *and* $u_1 \in L_2^{\frac{n+1}{2}}(\mathbb{E})$. *Suppose that either* $I(0) < 0$ *or* $I(0) = 0$ *and* $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$. *Then all weak solutions of the problem 1 belong to* E_δ *for any* $\delta > 0$.

Proof Let $u(t)$ be an arbitrary weak solution of the problem 1 with expressed assumptions in face of the Theorem and T be the existence time of $u(t)$. From Definition 5, for every $\delta > 0$ and $t \in [0, T)$, we can obtain

$$
\begin{aligned}
\frac{1}{2}\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} &+ a(\delta)\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{p+1}K_\delta(u) \\
&= \frac{1}{2}\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \left((\frac{1}{2} - \frac{\delta}{p+1})(1 + C_*^2)\right)\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&+ \frac{1}{p+1}\Big(\delta\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \delta\int_{\mathbb{E}} r^q V(z)|u|^2 d\mu \\
&\qquad - \int_{\mathbb{E}} r^q g_t(z)|u|^{p+1} d\mu\Big) \\
&= \frac{1}{2}\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \frac{\delta C_*^2}{p+1}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&+ \frac{C_*^2}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{\delta}{p+1}\int_{\mathbb{E}} r^q V(z)|u|^2 d\mu \\
&\qquad - \frac{1}{p+1}\int_{\mathbb{E}} r^q g_t(z)|u|^{p+1} d\mu \\
&= \frac{1}{2}\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \frac{\delta C_*^2}{p+1}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&+ \frac{C_*^2}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - (\frac{1}{2} - \frac{\delta}{p+1})\int_{\mathbb{E}} r^q V(z)|u|^2 d\mu \\
&\qquad + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|u|^2 d\mu \\
&- \frac{1}{p+1}\int_{\mathbb{E}} r^q g_t(z)|u|^{p+1} d\mu \le \frac{1}{2}\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + J(u) \\
&+ \left(\frac{C_*^2}{2} - \frac{\delta C_*^2}{p+1} - C_*^2(\frac{1}{2} - \frac{\delta}{p+1})\right)\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&+ \frac{1}{p+1}\int_0^t \|(\frac{d}{d\tau} g_\tau(x))^{\frac{1}{p+1}} u\|^{p+1}_{L_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})} \\
&\qquad + \gamma\int_0^t \|\nabla_{\mathbb{E}}(\partial_\tau u)\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \le I(0).
\end{aligned} \tag{12}
$$

If $I(0) < 0$, then 12 implies that $K_\delta(u) < 0$ and $J(u) < 0 < d(\delta)$ for every $\delta > 0$ and $t \in [0, T)$. If $I(0) = 0$ and $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$, then Proposition 8 gives $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \geq M$ for $t \in [0, T)$. Again by relation 12 we get $K_\delta(u) < 0$ and $J(u) < 0 < d(\delta)$ for $\delta > 0$ and $t \in [0, T)$. Therefore, for two cases discussed above, for every $\delta > 0$ and $t \in [0, T)$, we have $u \in E_\delta$.

5 Global Existence and Finite-Time of the Solutions

In this section, we prove the global existence and nonexistence of solutions and give a sharp condition for global existence of solutions for problem 1 with $I(0) < d$.

Theorem 3 *Let $\gamma \geq 0$, $u_0 \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ and $u_1 \in L_2^{\frac{n+1}{2}}(\mathbb{E})$. Suppose that $I(0) < d$, $K(u_0) > 0$ or $\|\nabla_{\mathbb{E}} u_0\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = 0$. Then problem 1 admits a global weak solution $u(t) \in L^\infty\Big(0, \infty; \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\Big)$ with $\partial_t u \in L_2^{\frac{n+1}{2}}(\mathbb{E})$ and $u(t) \in W$ for $t \in [0, \infty)$.*

Proof By Proposition 3 we can choose $\{w_j(z)\}$ as orthonormal basis of space $\mathscr{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$. Then we construct approximation solution $u_m(z, t)$ similar to [20] as following:

$$u_m(z, t) = \sum_{j=1}^{m} h_{jm}(t) w_j(z),$$

for $m = 1, 2, \ldots$ that satisfies in problem 1 then,

$$(\partial_t^2 u_m, w_k)_2 + (\nabla_{\mathbb{E}} u_m, \nabla_{\mathbb{E}} w_k)_2 + (V(z) u_m, w_k)_2 + \gamma\, (\nabla_{\mathbb{E}}(\partial_t u_m), \nabla_{\mathbb{E}} w_k)_2 = (g_t(z) u_m |u_m|^{p-1}, w_k)_2, \tag{13}$$

$$u_m(z, 0) = \sum_{j=1}^{m} h_{jm}(0) w_j(z) \to u_0(z), \tag{14}$$

in $\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ and

$$\partial_t u_m(z, 0) = \sum_{j=1}^{m} h'_{jm}(0) w_j(z) \to u_1(z), \tag{15}$$

in $L_2^{\frac{n+1}{2}}(\mathbb{E})$. Multiplying 13, 14 and 15 by $h'_{km}(t)$ and forming the crossmarklogo sum on $k = 1, 2, \ldots,$

$$\sum_{k=1}^{m}(\partial_t^2 u_m, w_k)_2 h'_{km}(t) + (\nabla_{\mathbb{E}} u_m, \nabla_{\mathbb{E}} w_k)_2 h'_{km}(t) + (V(z)u_m, w_k)_2 h'_{km}(t) + \sum_{k=1}^{m}\gamma\,(\nabla_{\mathbb{E}}(\partial_t u_m), \nabla_{\mathbb{E}} w_k)_2 h'_{km}(t) = \sum_{k=1}^{m}(g_t(z)u_m|u_m|^{p-1}, w_k)_2 h'_{km}(t),$$

for $m = 1, 2, 3, \ldots$. Therefore,

$$\int_{\mathbb{E}} r^q \partial_t^2 u_m \partial_t u_m d\mu + \int_{\mathbb{E}} r^q \nabla_{\mathbb{E}} u_m \partial_t \nabla_{\mathbb{E}} u_m d\mu + \int_{\mathbb{E}} r^q V(z) u_m \partial_t u_m d\mu + \gamma \int_{\mathbb{E}} r^q \nabla_{\mathbb{E}}(\partial_t u_m) \nabla_{\mathbb{E}}(\partial_t u_m) d\mu = \int_{\mathbb{E}} r^q g_t(z) u_m |u_m|^{p-1} \partial_t u_m d\mu. \tag{16}$$

Using The Leibniz rule one can get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{E}} r^q |\partial_t^2 u_m|^2 d\mu + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{E}} r^q |\nabla_{\mathbb{E}} u_m|^2 d\mu + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{E}} r^q V(z)|u_m|^2 d\mu + \gamma \int_{\mathbb{E}} r^q |\nabla_{\mathbb{E}}(\partial_t u_m)|^2 d\mu = \frac{1}{p+1}\frac{d}{dt}\int_{\mathbb{E}} r^q g_t(z)|u_m|^{p+1} d\mu - \frac{1}{p+1}\int_{\mathbb{E}} r^q (\frac{d}{dt} g_t(z))|u_m|^{p+1} d\mu. \tag{17}$$

By integration of the relation 17 with respect to t

$$\frac{1}{2}\|\partial_t u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\|\nabla_{\mathbb{E}} u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int r^q V(z)|u_m|^2 d\mu + \gamma \int_0^t \|\nabla_{\mathbb{E}}(\partial_\tau u_m)\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau - \frac{1}{p+1}\int_{\mathbb{E}} r^q g_t(z)|u_m|^{p+1} d\mu + \frac{1}{p+1}\int_0^t \|(\frac{d}{d\tau} g_\tau(z))^{p+1} u_m\|^{\frac{1}{p+1}} d\tau = I(t) + \gamma \int_0^t \|\nabla_{\mathbb{E}}(\partial_\tau u_m)\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau \le I(0) < d, \tag{18}$$

where the last equal is upon Definition 5. Hence, for sufficiently large m and $0 \le t < \infty$ we obtain that $u_m \in W$ by Proposition 1. Using 18 and definition of functional K,

$$
\begin{aligned}
J(u_m) &= \frac{1}{2}\|\nabla_{\mathbb{E}} u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|u_m|^2 d\mu - \frac{1}{p+1}\int_{\mathbb{E}} r^q g_t(z)|u_m|^{p+1} d\mu \\
&= \frac{1}{2}\|\nabla_{\mathbb{E}} u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|u_m|^2 d\mu \\
&- \frac{1}{p+1}\left(\|\nabla_{\mathbb{E}} u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z)|u_m|^2 d\mu - K(u_m)\right) \\
&\geq (\frac{p-1}{2(p+1)}\left[\|\nabla_{\mathbb{E}} u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z)|u_m|^2 d\mu\right] \\
&\geq \frac{p-1}{2(p+1)}(1+C_*^2)\|\nabla_{\mathbb{E}} u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_0^t \frac{1}{2}\|\partial_\tau u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau &+ \frac{1}{p+1}\int_0^t \|(\frac{d}{d\tau} g_\tau)^{\frac{1}{p+1}} u_m\|^{p+1}_{L_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})} d\tau \\
&+ \frac{p-1}{2(p+1)}(1+C_*^2)\|\nabla_{\mathbb{E}} u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&\leq I(t) + \gamma \int_0^t \|\nabla_{\mathbb{E}}(\partial_\tau u_m)\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau \leq I(0) < d. \quad (19)
\end{aligned}
$$

for $t \in [0, \infty)$ and sufficiently large m. Now, by relation 19 we can get that

$$
\|\nabla_{\mathbb{E}} u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} < \frac{2(p+1)}{p-1}\,(1+C_*^2)^{-1}\, d, \tag{20}
$$

for $t \in [0, \infty)$ and

$$
\frac{1}{2}\int_0^t \|\partial_\tau u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau + \frac{1}{p+1}\int_0^t \|(\frac{d}{d\tau} g_\tau)^{\frac{1}{p+1}} u_m\|^{p+1}_{L_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})} d\tau < d, \tag{21}
$$

for $t \in [0, \infty)$. Also we obtain that

$$
\begin{aligned}
\int\limits_{\mathbb{E}} r^q |g_t(z)^{\frac{p}{p+1}} u_m |u_m|^{p-1}|^{\frac{p+1}{p}} d\mu = \int\limits_{\mathbb{E}} r^q g_t(z)|u_m|^{p+1} d\mu &\leq C_{**}^{p+1}\|\nabla_{\mathbb{E}} u_m\|^{p+1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&< C_{**}^{p+1}(\frac{2(p+1)}{p-1}\,(1+C_*^2)^{-1}\, d)^{\frac{p+1}{2}} \quad (22)
\end{aligned}
$$

and

$$\int_{\mathbb{E}} r^q |V(z)^{\frac{1}{2}} u_m|^2 d\mu = \int_{\mathbb{E}} r^q V(z) |u_m|^2 d\mu \le C_*^2 \|\nabla_{\mathbb{E}} u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}$$
$$< C_*^2 \Big(\frac{2(p+1)}{p-1}(1+C_*^2)^{-1}\, d\,\Big)^2. \tag{23}$$

From 20, 21, 22 and 23, it follows that there exists u and a subsequence still denotes $\{u_m\}$ for which as $m \to \infty$, $u_m \to u$ in $L^\infty\Big(0,\infty\,;\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\Big)$ weakly star and a.e. in $int\mathbb{E}\times[0,\infty)$, $\partial_t u_m \to \partial_t u$ in $L^2\Big(0,\infty\,;L_2^{\frac{n+1}{2}}(\mathbb{E})\Big)$, weakly star. Also we have $V(z)|u_m|^2 \to V(z)|u|^2$ in $L^\infty\Big(0,\infty\,;\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\Big)$ weakly star and a.e. in $int\mathbb{E}\times[0,\infty)$, and $g_t(z)u_m|u_m|^{p-1} \to g_t(z)u|u|^{p-1}$ in $L^\infty\Big(0,\infty\,;\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\Big)$ weakly star and a.e. in $int\mathbb{E}\times[0,\infty)$. Therefore, in 13 for k fixed and $m\to\infty$ we get that

$$(\gamma u, w_k)_2 + (u_t, w_k)_2 + \int_0^t (\nabla_{\mathbb{E}} u, \nabla_{\mathbb{E}} w_k)_2 d\tau + \int_0^t (V(z)u, w_k)_2 d\tau$$
$$= \int_0^t (g_t(z)u|u|^{p-1}, w_k)_2 d\tau$$
$$+ (\gamma u_0, w_k)_2 + (u_1, w_k)_2.$$

On the other hand, from the relation 14, $u(z,0) = u_0(z)$ in $\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ and from 15 $\partial_t u(z,0) = u_1$ in $L_2^{\frac{n+1}{2}}(\mathbb{E})$. By density we obtain $u \in L^\infty\Big(0,\infty\,;\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\Big)$ with $\partial_t u \in L^2\Big(0,\infty\,;L_2^{\frac{n+1}{2}}(\mathbb{E})\Big)$ is global weak solution of problem 1. Since u satisfies problem 1, so by definition of K we have $K(u)=0$. Hence, $u(t)\in W$ for $0\le t<\infty$.

Corollary 2 *If we replace the assumption $I(0)<d$, $K(u_0)>0$ by $0<I(0)<d$, $K_{\delta_2}(u_0)>0$ where (δ_1,δ_2) is the maximal interval including $\delta=1$, (see Remark 1) such that $I(0)<d(\delta)$ for $\delta\in(\delta_1,\delta_2)$. Then problem 1 admits a global weak solution $u(t)\in L^\infty\Big(0,\infty;\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\Big)$ with $\partial_t u \in L^\infty\Big(0,\infty,L_2^{\frac{n+1}{2}}(\mathbb{E})\Big)$ and $u(t)\in W_\delta$ for $\delta\in(\delta_1,\delta_2)$, $t\in[0,\infty)$.*

Proof It is immediately implied form Theorems 1 and 3.

Corollary 3 *If we replace the assumption $K_{\delta_2}(u_0)>0$ or $\|\nabla_{\mathbb{E}} u_0\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})}=0$, by $\|\nabla_{\mathbb{E}} u_0\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})}<r(\delta_2)$, then problem 1 admits a global weak solution $u(t)\in L^\infty\Big(0,\infty;\mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\Big)$ with $\partial_t u(t)\in L^\infty\Big(0,\infty;L_2^{\frac{n+1}{2}}(\mathbb{E})\Big)$ satisfying*

$$\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \leq \frac{I(0)}{a(\delta_1)}, \quad \|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \leq 2I(0), \quad 0 \leq t \leq \infty \tag{24}$$

Proof From assumption $\|\nabla_{\mathbb{E}}u_0\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} < r(\delta_2)$, we can get that $K_{\delta_2}(u_0) > 0$ or $\|\nabla_{\mathbb{E}}u_0\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = 0$. Then it follows from Corollary 2 that problem 1 admits a global weak solution such that for any $\delta_1 < \delta < \delta_2$, $0 \leq t < \infty$, $u(t) \in L^{\infty}\Big(0, \infty; \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\Big)$ with $\partial_t u \in L^{\infty}\Big(0, \infty; L_2^{\frac{n+1}{2}}(\mathbb{E})\Big)$ and $u(t) \in W_{\delta}$. Moreover, similar of the proof Theorem 2 for every $\delta_1 < \delta < \delta_2$, $0 \leq t < \infty$,

$$\frac{1}{2}\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + a(\delta)\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{p+1}K_{\delta}(u) \leq I(0).$$

If we tend δ to δ_1 then we achieve 24.

Now we discuss the global non-existence of solutions of the problem 1.

Theorem 4 *Let* $0 \leq \gamma \leq (p-1)\sqrt{1 + C_*^2\lambda_1^{\frac{1}{2}}}$, $u_0 \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$, $u_1 \in L_2^{\frac{n+1}{2}}(\mathbb{E})$. *Suppose that* $I(0) < d$ *and* $K(u_0) < 0$. *Then the existence time of solution for problem 1 is finite, where* λ_1 *is the first eigenvalue in Proposition 3 i.e.*

$$\lambda_1^{\frac{1}{2}} = \inf_{u \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}),\, \|\nabla_{\mathbb{E}}u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0} \frac{\|\nabla_{\mathbb{E}}u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})}}{\|u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})}}.$$

Proof Let $u(t)$ be any weak solution of problem 1 with $I(0) < d$ and $K(u_0) < 0$, T be the maximal existence time of $u(t)$. We will prove $T < \infty$ by contradiction. Let $M(t) := \|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}$, then

$$\dot{M}(t) = \frac{d}{dt}\int_E r^q |u(z,t)|^2 d\mu = 2(\partial_t u, u)_2,$$

from definition of functional K,

$$\ddot{M}(t) = 2\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + 2(\partial_t^2 u, u)_2 = 2\|\partial_t u\|^2_{L_2^{\frac{+1}{2}}(\mathbb{E})} - 2\gamma(\nabla_{\mathbb{E}}(\partial_t u), \nabla_{\mathbb{E}}u)_2 - 2K(u). \tag{25}$$

Using proof of Theorem 2 we can get,

$$\frac{1}{2}\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + a(1)\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{p+1}K(u)$$

$$
\begin{aligned}
&= \frac{1}{2}\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \left((\frac{1}{2} - \frac{1}{p+1})(1 + C_*^2) \right) \|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&+ \frac{1}{p+1} \left(\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z)|u|^2 d\mu - \int_{\mathbb{E}} r^q g_t(z)|u|^{p+1} d\mu \right) \\
&\leq \frac{1}{2}\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&\qquad + + \left[(\frac{1}{2} - \frac{1}{p+1}) + \frac{1}{p+1} \right] \int_{\mathbb{E}} r^q V(x)|u|^2 d\mu \\
&- \frac{1}{p+1} \int_{\mathbb{E}} r^q g_t(z)|u|^{p+1} d\mu \ \leq \frac{1}{2}\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&\qquad + \ \frac{1}{p+1} \int_0^t \ \|(\frac{d}{d\tau} g_\tau(z))^{\frac{1}{p+1}} u\|^{p+1}_{L_{p+1}^{\frac{n+1}{p+1}}(\mathbb{E})} \\
&+ \ J(u) \ + \ \gamma \int_0^t \ \|\nabla_{\mathbb{E}}(\partial_\tau u)\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&\qquad = I(t) + \gamma \int_0^t \ \|\nabla_{\mathbb{E}}(\partial_t u)\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \ \leq I(0).
\end{aligned}
\tag{26}
$$

Thus inequality 26 implies that

$$
\begin{aligned}
\ddot{M}(t) &\geq 2\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&- 2\gamma(\partial_t u, u)_2 - 2(p+1)\Big[I(0) - \frac{1}{2}\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&\qquad - \frac{p-1}{2(p+1)}(1 + C_*^2)\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \Big] \\
&= (p+3)\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + (p-1)(1 + C_*^2)\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&- 2\gamma(\nabla_{\mathbb{E}}(\partial_t u), \nabla_{\mathbb{E}} u)_2 - 2(p+1)I(0).
\end{aligned}
\tag{27}
$$

In first, let us consider $I(0) \leq 0$. Then,

$$
\ddot{M}(t) \geq (p+3)\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + (p-1)(1+C_*^2)\lambda_1\|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - 2\gamma(\nabla_{\mathbb{E}}(\partial_t u), \nabla_{\mathbb{E}} u)_2.
$$

condition $\gamma < (p-1)(1+C_*^2)\lambda_1$ implies that, there exists a constant $\epsilon \in \Big(0 \quad , \quad (p-1)(1+C_*^2)\Big)$ such that

$$\gamma^2 < (p-1-\epsilon)(1+C_*^2)\lambda_1.$$

Therefore,

$$\ddot{M}(t) \geq (4+\epsilon)\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + (p-1-\epsilon)\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - 2\gamma(\nabla_{\mathbb{E}}(\partial_t u), \nabla_{\mathbb{E}} u)_2$$
$$+ (p-1)(1+C_*^2)\lambda_1^2\|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}. \quad (28)$$

On the other hand,

$$2\gamma(\nabla_{\mathbb{E}}(\partial_t u), \nabla_{\mathbb{E}} u)_2 \leq (p-1-\epsilon)\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{\gamma^2}{p-1-\epsilon}\|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}$$
$$\leq (p-1-\epsilon)\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + (p-1)(1+C_*^2)\lambda_1^2\|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}. \quad (29)$$

From 28 and 29, we can get that

$$\ddot{M}(t) \geq (4+\epsilon)\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}. \quad (30)$$

By Edge Hölder inequality we get

$$M(t)\ddot{M}(t) - \frac{4+\epsilon}{4}\dot{M}(t) \geq (4+\epsilon)\Big(\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - (\nabla_{\mathbb{E}}(\partial_t u), \nabla_{\mathbb{E}} u)_2\Big) \geq 0,$$

$$(M^{-\alpha})'' = \frac{-\alpha}{M^{\alpha+2}(t)}\Big(M(t)\ddot{M}(t) - (\alpha+1)\dot{M}(t)^2\Big) \leq 0,$$

for $\alpha = \frac{\epsilon}{4}$ and $0 \leq t < \infty$. Hence, there exists a $T_1 > 0$ such that

$$\lim_{t\to T_1} M^{-\alpha}(t) = 0$$

and $\lim_{t\to T_1} M(t) = +\infty$, which is contradicts $T_{\max} = +\infty$.

In second case, we consider $0 < I(0) < d$. In this case from Theorem 1 we have $u \in E_\delta$ for $0 \leq t < \infty$ and $\delta \in (1 \quad , \quad \delta_2)$ (see Remark 1) where $(\delta_1 \quad , \quad \delta_2)$ is the maximal interval including $\delta = 1$ such that $d(\delta) > I(0)$ for $\delta \in (\delta_1 \quad , \quad \delta_2)$.

Therefore, $K_\delta(u) < 0$ and $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} > r(\delta)$ for $1 < \delta < \delta_2$, $0 \le t < \infty$. Consequent, $K_\delta(u) \le 0$ and $\|\nabla_{\mathbb{E}} u\| \ge r(\delta)$ for $0 \le t < \infty$. From 25,

$$\frac{d}{dt}(e^{\gamma t}\dot{M}(t)) = e^{\gamma t}\Big(\gamma \dot{M}(t) + \ddot{M}(t)\Big) = 2e^{\gamma t}\Big(\|\partial_t u\|^2_{L_2^{\frac{n}{2}}(\mathbb{E})}\| - K(u)\Big)$$

$$= 2e^{\gamma t}\Big(\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\| + (\delta_2 - 1)\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - K_{\delta_2}(u)\Big)$$

$$\ge 2e^{\gamma t}(\delta_2 - 1)r^2\delta_2 = C\ \delta_2 e^{\gamma t}.$$

Hence,

$$e^{\gamma t}\dot{M}(t) \ge C\ \delta_2 \int_0^t e^{\gamma \tau} d\tau + \dot{M}(0) = \frac{C\ \delta_2}{\gamma}(e^{\gamma t} - 1) + \dot{M}(0),$$

$$\dot{M}(t) \ge \frac{C\ \delta_2}{\gamma}(1 - e^{-\gamma t}) + e^{-\gamma t}\dot{M}(0).$$

Hence there exists $t_0 > 0$ for which

$$\dot{M}(t) \ge \frac{C\ \delta_2}{2\gamma} \quad \forall t \ge t_0$$

and

$$M(t) \ge \frac{C\ \delta_2}{2\gamma}(t - t_0) + M(t_0) \ge \frac{C\ \delta_2}{2\gamma}(t - t_0), \qquad t \ge t_0. \tag{31}$$

From assumption $\gamma < (p-1)(1 + C_*^2)\lambda_1$, it follows there exists a constant

$$\epsilon \in \Big(0 \quad , \quad (p-1)(1 + C_*^2)\Big)$$

such that

$$\gamma^2 < (p - 1 - \epsilon)\Big[(p-1)(1 + C_*^2)\lambda_1 - \epsilon\Big].$$

From 27,

$$\ddot{M}(t) \ge (p+3)\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - 2\gamma(\nabla_{\mathbb{E}}(\partial_t u), \nabla_{\mathbb{E}} u)_2 + (p-1)(1 + C_*^2)\lambda_1^2 \|u\|^2_{L_2^{\frac{n}{2}}(\mathbb{E})}$$
$$-2(p+1)I(0)$$

$$= \|(4+\epsilon)\|\partial_t u\|^2_{L_2^{\frac{n}{2}}(\mathbb{E})} + (p-1-\epsilon)\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - 2\gamma(\nabla_{\mathbb{E}}(\partial_t u), \nabla_{\mathbb{E}} u)_2$$

$$+ [(p-1)(1+C_*^2)\lambda_1^2 - \epsilon]\|\partial_t u\|^2_{L_2^{\frac{n}{2}}(\mathbb{E})} + \epsilon M(t) - 2(p+1)I(0). \tag{32}$$

Also we can obtain

$$\begin{aligned} 2\gamma(\nabla_{\mathbb{E}}(\partial_t u), \nabla_{\mathbb{E}} u)_2 &\le (p-1-\epsilon)\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{\gamma^2}{p-1-\epsilon}\|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &\le (p-1-\epsilon)\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &\quad +[(p-1)(1+C_*^2)\lambda_1^2 - \epsilon]\|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}. \end{aligned} \tag{33}$$

From 32 and 33 we get

$$\ddot{M}(t) \ge (4+\epsilon)\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \epsilon M(t) - 2(p+1)I(0). \tag{34}$$

From 31, it follows that there exists a $t_1 > 0$ such that

$$\epsilon M(t) > 2(p+1)I(0) \qquad \forall t > t_1,$$

and then

$$\ddot{M}(t) > (4+\epsilon)\|\partial_t u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}, \qquad \forall t > t_1.$$

Now, similar to first case we can obtain a contradiction. Hence we always have $T_{\max} < \infty$.

From Theorems 13 and 4 we can obtain the following theorem for global existence and non-existence of solutions for problem 1.

Theorem 5 *Let* $0 \le \gamma \le (p-1)\sqrt{1+C_*^2}\lambda_1^{\frac{1}{2}}$, $u_0 \in \mathscr{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ *and* $u_1 \in L_2^{\frac{n+1}{2}}(\mathbb{E})$. *Suppose that* $I(0) < 0$. *Then, when* $K(u_0) > 0$, *problem 1 admits a global weak solution and when* $K(u_0) < 0$, *problem 1 does not admits any global weak solution.*

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