Nonlinear Dynamics of the KdV-B Equation and Its Biomedical Applications

Michail A. Xenos and Anastasios C. Felias

Abstract In recent years there is an incremental degree of bridging open questions in biomechanics with the help of applied mathematics and nonlinear analysis. Recent advancements concerning the cardiac dynamics pose important questions about the cardiac waveform. A governing equation, namely the KdV-B equation (Korteweg–de Vries–Burgers),

$$\frac{\partial u}{\partial t} + \gamma u \frac{\partial u}{\partial x} - \alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^3 u}{\partial x^3} = 0, \quad u = u(t, x), \quad \alpha,\ \beta,\ \gamma \in \mathbb{R}, \tag{1}$$

is a partial differential equation utilized to answer several of those questions. The cardiac dynamics mathematical model features both solitary and shock wave characteristics due to the dispersion and dissipation terms, as occurring in the arterial tree. In this chapter a focus is given on describing cardiac dynamics. It is customarily difficult to solve nonlinear problems, especially by analytical techniques. Therefore, seeking suitable solving methods, exact, approximate or numerical, is an active task in branches of applied mathematics. The phase plane of the KdV–B equation is analyzed and its qualitative behavior is derived. An asymptotic expansion is presented and traveling wave solutions under both shock and solitary profiles are sought. Numerical solutions are obtained for the equation, by means of the Spectral Fourier analysis and are evolved in time by the Runge–Kutta method. This whole analysis provides vital information about the KdV–B equation and its connection to cardiac hemodynamics. The applications of KdV–B, presented in this chapter, highlight its essence to human hemodynamics.

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1 Introduction

1.1 Background Information for KdV–B

In the last few decades, much attention from a rather diverse group of scientists such as physicists, engineers and applied mathematicians has been attracted to two contrasting themes: (a) the theory of dynamical systems, most popularly associated with the study of chaos, and (b) the theory of integrable (or nonintegrable) systems associated, among other things, with the study of solitary waves.

It is common knowledge that many physical phenomena, such as nonlinear shallow-water waves and wave motion in plasma, can be described by the Korteweg–de Vries (KdV) equation [29]. It is well known that solitons and solitary waves are the class of special solutions of the KdV equation. In order to study propagation of undular bores in shallow water [6, 27], liquid flow containing gas bubbles [54], fluid flow in elastic tubes [28], crystal lattice theory, nonlinear circuit theory and turbulence [20, 30, 51], the governing equation can be reduced to the so-called Kortweg–de Vries–Burgers equation (KdV–B) as follows [10],

$$\frac{\partial u}{\partial t} + \gamma u \frac{\partial u}{\partial x} - \alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^3 u}{\partial x^3} = 0, \quad u = u(t, x), \quad \alpha,\ \beta,\ \gamma \neq 0. \tag{2}$$

This is a nonintegrable equation in the sense that its spectral problem is nonexistent [19]. Multiplying t, x and u, by constants can be used to make the coefficients of any of the above four terms equal to any given nonzero constant. Therefore, we focus on the case where $\alpha \geq 0$, $\beta > 0$ and $\gamma \neq 0$.

This equation is equivalent to the KdV equation with the addition of a viscous dissipation term $(\alpha \frac{\partial^2 u}{\partial x^2})$. The studies of the KdV equation [29] ($\alpha = 0$) and the Burgers equation [9] ($\beta = 0$) have been undertaken, but the exact solution for the general case of equation (2) ($\alpha \geq 0$, $\beta > 0$, $\gamma \neq 0$) has still not been completed.

1.2 Biomechanical Applications

Solitons are mathematical entities appearing as solutions of nonlinear wave equations [8]. They are waves of stable and steady form, although internal oscillations may occur, exhibiting unique characteristics when colliding with other solitary waves as described by Ablowitz and Segur [1]. During the last decade, soliton profiles are found when studying nonlinear optics, condensed matter Physics and quantum theory of matter and gravity [43]. Lately, an increasing number of studies focuses on describing the cardiac pulse as a soliton, due to the features those two seem to share. The pulsatility synchronization of the smooth arterial muscle allows the consideration of solitary profiles in cardiac hemodynamics [34].

Theoretical investigation for the blood waves have been developed by many researchers through the use of weakly nonlinear theories. The theoretical investigation of pulse wave propagation in human arteries has a long history starting from ancient times until today. Over the past decade, the scientific efforts have been concentrated on theoretical investigations of nonlinear wave propagation in arteries with a variable radius. The question "How local imperfections appeared in the artery can disturb the arterial wall deformation?" is important for understanding the nature and main features of various cardiovascular diseases, such as stenoses and aneurysms. Rowlands (1982) reported some extraordinary features of the cardiac pulse, leading to his conception of the arterial flows as a solitary motion [44]. A few years later, Otwinowski and collaborators presented a nonlinear differential equation whose solutions exhibited similar characteristics with those reported by Rowlands [38].

Based on those evolutionary theories, adding the inertial behavior of blood vessel in an one-dimensional cardiovascular model, researchers concluded that the KdV equation is a seemingly reliable tool in modelling cardiac dynamics. It was supported that the solitary wave formulation fits much better in describing the arterial pulse wave experimental results than the wave equation proposed by the majority of researchers [57]. An additional reason to support the above formulation is the peaking and steepening features of the pressure pulse, which coincide with the structure of soliton profiles of KdV [11].

The majority of studies on the wave propagation in blood flow is mainly based on linear waves. The linearized theories proposed by Resal, Witzig, Womersly, McDonald and others, consider the vessel as a straight, infinite, circular elastic tube filled with an isotropic and Newtonian fluid, blood [55]. Blood is studied as an incompressible fluid, a characterization justified by its compressibility being rather insignificant, compared to the dilation of the blood vessels. In 1958, Lambert based on the Euler equations of fluid motion, proposed the Method of Characteristics for the calculations concerning the nonlinear blood flow. All theories presented to model nonlinear blood flow are one-dimensional, meaning that both pressure and flow velocity are seen as functions of the axial distance along the vessel in time. Contributions in nonlinear modulation were done by Rudinger, Skalak, Rockwell, Hawley and Anliker. The suggested equations are basically the equations of continuity and motion coupled with an extra equation to describe the vessel wall distensibility [3, 43, 45, 49]. Sakanishi and Hasegawa proposed a soliton profile pulsatile wave modeling, based on the nonlinear elasticity of the vessel wall [46]. Yomosa and collaborators proposed a theory describing solitons in long arteries, where the viscous effects, the reflective effects caused by the arterial branch as well as the effects of the peripheral resistance are neglectible. For the above reasons, the latter modulation is unable to describe the pressure drop caused when moving away from the heart. Nevertheless, it points out that it does make some sense to attribute the special features of the pulsatile wave, including the "sudden steepening" and the change in the phase velocity, to the solitary profile [57]. While the pulsatile wave travels to arteries with smaller radius, viscosity seems to play a vital role in both the flow decay and the widening of the wave width [57].

Antar and Demiray studied the propagation of weak nonlinear waves in a thin elastic tube, under an initial stress distribution, due to the flow of an incompressible viscous fluid [4]. The propagation of pressure pulses in dilatable tubes has been studied by various researchers [22, 41]. Most of those studies, consider waves of small width, neglecting the nonlinear characteristics and focusing on their dispersive character [5, 12, 42]. It is widely accepted that a long-term evolution of weak nonlinear waves of either dispersion or dissipation, can be modeled by nonlinear dispersive equations. Two classical simplified and indicative examples are the Burgers equation and the KdV equation, exhibiting balance between nonlinearity and dissipation and nonlinearity and dispersion, respectively. On the other hand, when a balance is exhibited among nonlinearity, dissipation and dispersion, the simplest and most representative dispersive equation is the KdV–B equation, combining the KdV and Burgers equations. Via asymptotic methods, the propagation of small, but with finite width, waves in dilatable tubes has been studied sufficiently [4].

Hashizume and Yomosa showed that propagation, in the case of weak nonlinear waves in a thin and nonlinear elastic tube for incompressible flow, is determined by the KdV equation [57]. Erbay and collaborators, examining the propagation of weak nonlinear waves in a thin viscoelastic tube filled with fluid, were lead to the Burgers, KdV and KdV–B equations, depending on the parameters considered [15]. Demiray studied the propagation of slightly nonlinear waves in thin elastic and viscoelastic tubes for an incompressible fluid and finally concluded to the KdV and KdV–B equations, respectively. In all the above studies, an inviscid fluid was considered and the axial movement of the tube wall was neglected. However, regarding biological applications, blood is an incompressible and viscous fluid. So, Antar and Demiray formulated their mathematical model toward this direction [4].

In this chapter, an emphasis is given to the theoretical and numerical analysis of the KdV–B equation and its applications, providing vital information about the KdV–B equation and its connection to cardiac hemodynamics. More precisely, in the next section the phase plane of the KdV–B equation is analyzed and its qualitative behavior is derived. Furthermore, an asymptotic expansion is presented and traveling wave solutions under both shock and solitary profiles are derived. Finally, numerical solutions are obtained for the KdV–B equation, by means of spectral Fourier analysis and are evolved in time by the well known explicit 4th order Runge–Kutta method.

2 Phase Plane Analysis of KdV–B

In this section, the phase plane of the KdV–B equation is analyzed and its qualitative behavior is derived and further described. The wave variable ζ is introduced as [25, 47],

$$\zeta = x - \lambda t, \tag{3}$$

with λ being the wave velocity. Then equation (2) using (3), can be written as,

$$(\gamma u-\lambda)\frac{du}{d\zeta}-\alpha\frac{d^2u}{d\zeta^2}+\beta\frac{d^3u}{d\zeta^3}=0,\quad u=u(t,x)=u(x-\lambda t)=u(\zeta)\,. \tag{4}$$

The so-called traveling–wave solution, $u = u(\zeta)$, shall be considered here. By integrating equation (4) with respect to ζ, a nonlinear differential equation can be obtained as follows,

$$\frac{d^2u}{d\zeta^2}+c_1\frac{du}{d\zeta}+c_2u^2+c_3u=c_0, \tag{5}$$

where $c_1=-\frac{\alpha}{\beta}$, $c_2=\frac{\gamma}{2\beta}$, $c_3=-\frac{\lambda}{\beta}$ and the integral constant $c_0>-\frac{\lambda^2}{2\beta}$.

In the case where $c_0 \neq 0$, a simple translation transformation,

$$u=u'+c_0',\quad c_0'=\frac{-c_3\pm\sqrt{c_3^2+4c_0c_2}}{2c_2},$$

can be made, with u' satisfying the following equation,

$$\frac{d^2u'}{d\zeta^2}+c_1\frac{du'}{d\zeta}+c_2u'^2+(c_3+2c_2c_0')u'=0.$$

Without loss of generality, we shall confine ourselves to the consideration of $c_0 = 0$ alone from now on. It can be further assumed that $\lambda \geq 0$, because the discussion on $\lambda' = -\lambda$ can be made in the same manner for $\lambda < 0$.

Equation (5) can be written as an autonomous system of first-order equations,

$$\begin{cases}\dfrac{du}{d\zeta}=v,\\ \dfrac{dv}{d\zeta}=-\dfrac{u}{\beta}(\gamma\dfrac{u}{2}-\lambda)+\dfrac{\alpha}{\beta}v.\end{cases}$$

Now, we study the above system according to the qualitative theory of ordinary differential equations. Initially, we find the system's singular points, setting,

$$f_1(u,v)=v,\quad f_2(u,v)=-\frac{u}{\beta}(\gamma\frac{u}{2}-\lambda)+\frac{\alpha}{\beta}v.$$

The following conditions should be met,

$$f_1(u,v)=f_2(u,v)=0$$

$$\Leftrightarrow \begin{cases} v = 0 \\ -\frac{u}{\beta}(\gamma \frac{u}{2} - \lambda) + \frac{\alpha}{\beta} v = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} v = 0 \\ u = 0, \quad u = \frac{2\lambda}{\gamma}. \end{cases}$$

Therefore, the singular points are,

$$\begin{cases} P_1 = (0, 0), \\ P_2 = (\frac{2\lambda}{\gamma}, 0). \end{cases}$$

Next, we are to find the eigenvalues of the linearization matrices, defined for our singular points, as follows,

$$A(P_1) = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix}, \ (P_1) = \begin{bmatrix} 0 & 1 \\ \frac{\lambda}{\beta} & \frac{\alpha}{\beta} \end{bmatrix},$$

so, for its eigenvalues we get,

$$det(A_{P_1} - sI_2) = 0 \Leftrightarrow s^2 - \frac{\alpha}{\beta} s - \frac{\lambda}{\beta} = 0$$

$$\Leftrightarrow \begin{cases} s_1 = \dfrac{\frac{\alpha}{\beta} + \frac{1}{\beta}\sqrt{\alpha^2 + 4\lambda\beta}}{2} > 0 \\ s_2 = \dfrac{\frac{\alpha}{\beta} - \frac{1}{\beta}\sqrt{\alpha^2 + 4\lambda\beta}}{2} < 0 \end{cases} .$$

$$A(P_2) = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix}, \ (P_2) = \begin{bmatrix} 0 & 1 \\ -\frac{\lambda}{\beta} & \frac{\alpha}{\beta} \end{bmatrix},$$

so, for its eigenvalues we get,

$$det(A_{P_2} - sI_2) = 0 \Leftrightarrow s^2 - \frac{\alpha}{\beta} s + \frac{\lambda}{\beta} = 0$$

$$\Leftrightarrow \begin{cases} s_1 = \dfrac{\dfrac{\alpha}{\beta} + \dfrac{1}{\beta}\sqrt{\alpha^2 - 4\lambda\beta}}{2} > 0 \\ s_2 = \dfrac{\dfrac{\alpha}{\beta} - \dfrac{1}{\beta}\sqrt{\alpha^2 - 4\lambda\beta}}{2} > 0 \end{cases} , \quad \alpha \geq 2\sqrt{\lambda\beta} \, .$$

$$\begin{cases} s_1 = \dfrac{\dfrac{\alpha}{\beta} + i\dfrac{1}{\beta}\sqrt{4\lambda\beta - \alpha^2}}{2} \\ s_2 = \dfrac{\dfrac{\alpha}{\beta} - i\dfrac{1}{\beta}\sqrt{4\lambda\beta - \alpha^2}}{2} \end{cases} , \quad \alpha \in (0, 2\sqrt{\lambda\beta}) \, .$$

$$\begin{cases} s_1 = i\sqrt{\dfrac{\lambda}{\beta}} \\ s_2 = -i\sqrt{\dfrac{\lambda}{\beta}} \end{cases} \quad \alpha = 0 \, .$$

We conclude that (0, 0) is invariably a saddle point, whereas $(\frac{2\lambda}{\gamma}, 0)$ has three cases depending on the values of α, β, λ [37],

A. a source for $\alpha \geq 2\sqrt{\lambda\beta}$,
B. a spiral source for $\alpha \in (0, 2\sqrt{\lambda\beta})$,
C. a central point for $\alpha = 0$ (KdV).

Regarding the geometric nature of the above characterizations, we have the following [37],

1. (0, 0) being a saddle point, means that it's an unstable node and phase trajectories tend to move around it in hyperbolas, defined by the separatrices (i.e. straight lines directed along the two eigenvectors of the linearization matrix).
2. $((\frac{2\lambda}{\gamma}, 0) : \alpha \geq 2\sqrt{\lambda\beta})$ being a source, means that it's an unstable node from where phase trajectories diverge away without any (or relatively little) rotation.
3. $((\frac{2\lambda}{\gamma}, 0) : \alpha \in (0, 2\sqrt{\lambda\beta}))$ being a spiral source, means that it's an unstable focus where phase trajectories tend to spiral around before eventually diverge away from it.
4. $((\frac{2\lambda}{\gamma}, 0) : \alpha = 0)$ being a central point, means that the phase trajectories tend to move in ellipses around the point, describing periodic motion of a point in the phase space.

The phase plots of Fig. 1 depict our three cases, where we have set for convenience $\gamma = 1$, since it is not related to any of α, β, λ in effecting the stabillity statuses.

An essential tool in studying the phase portrait of nonlinear autonomous systems, like the above, is the Hartman–Grobman Theorem [22, 23, 37],

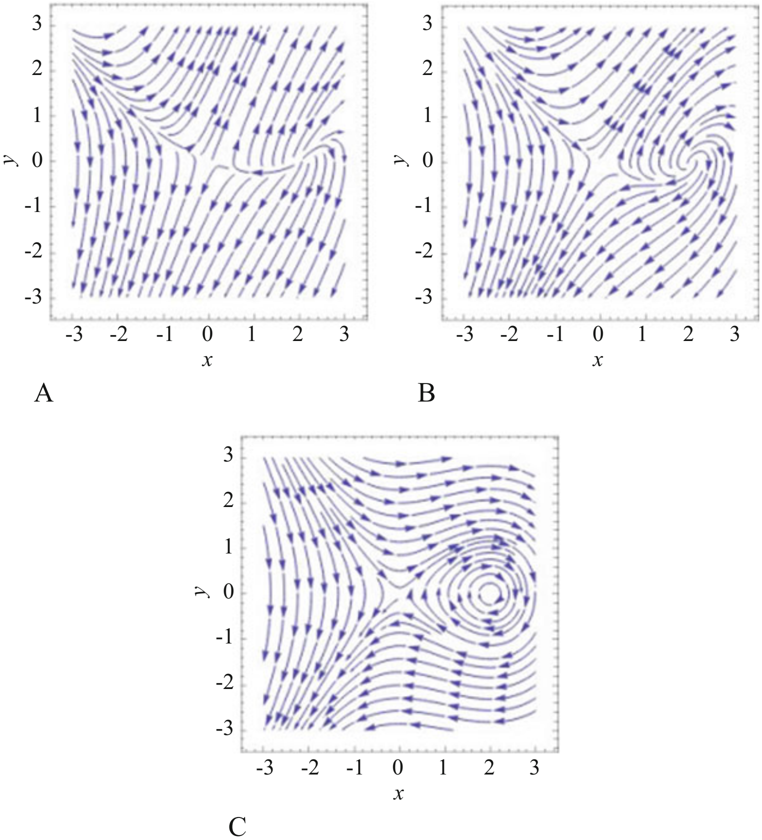

Fig. 1 The point (0, 0) is invariably a saddle point whereas (2, 0) is a source point in **a**, a spiral source point in **b** and a central point in **c**. (**a**) ($\alpha = 2, \beta = \lambda = 1$). (**b**) ($\alpha = \beta = \lambda = 1$). (**c**) ($\alpha = 0, \beta = \lambda = 1$)

Theorem 1 (Hartman-Grobman) *Consider a two-dimensional nonlinear autonomous system with a continuously differentiable field* $\bar{f}$,

$$\bar{x}' = \bar{f}(\bar{x})$$

and consider its linearization at a hyperbolic critical point $\bar{x}_0$ *(that is the Jacobian matrix has eigenvalues with non-zero real part),*

$$\bar{u}' = (Df_0)(\bar{u}).$$

Then there is a neighborhood of the hyperbolic critical point where all the solutions of the linear system can be transformed into solutions of the nonlinear system by a continuous, invertible transformation.

Remark 1 The above theorem implies that the phase portrait of the linear system in a neighborhood of the hyperbolic critical point can be transformed to the phase portrait of the nonlinear system by a continuous, invertible transformation. When that happens, we say that the two phase portraits are topologically equivalent.

Additional information about the phase plane of KdV–B equation can be found in [14].

3 Asymptotic Expansion for KdV–B

In the study of ordinary differential equations and their applications, an asymptotic expansion is of high importance. It would be very useful to understand thoroughly the property of the solution to the KdV–B equation. The asymptotic expansion would provide a reliable basis for estimating the advantages and disadvantages when seeking and applying numerical methods to our equation.

Here, by means of variable transformation and the qualitative theory of ordinary differential equations, the asymptotic behavior of the traveling wave solutions to the KdV–B equation is presented. The asymptotic expansion is real and continuous, if the argument is greater than a certain value.

The following variable transformation can be made [36, 48],

$$u = -e^{-\frac{c_1(1-k)\zeta}{2}} \frac{c_1 k^2}{c_2} y(\xi), \quad \xi = e^{-c_1 k \zeta}, \quad k = \sqrt{1 - \frac{4c_3}{c_1^2}} = \sqrt{1 + \frac{4\beta\lambda}{\alpha^2}} \geq 1. \tag{6}$$

Equation (5) ($c_0 = 0$), can be reduced to the Emden–Fowler equation [35],

$$\frac{d^2 y}{d\xi^2} = \xi^\sigma y^2, \quad \sigma = \frac{1-5k}{2k}. \tag{7}$$

It is obvious that,

$$\begin{cases} \sigma = -2, \quad \lambda = 0, \\ \sigma \in [\frac{-5}{2}, -2), \quad \lambda > 0. \end{cases} \tag{8}$$

Some characteristics of the KdV–B equation can be derived from equation (7).

Next we demonstrate some essential results that will help us in deriving the asymptotic expansion of KdV–B.

First, we show that the KdV–B equation, Equation (5) has finite isolated zero points only. Since y satisfies (7), y'' does not change sign for $\xi \in (0, \infty)$, so Equation (7) has finite zero points only, except that it identically vanishes for some intervals. Equation (7) has finite zero points only. This indicates that the solution of

KdV–B is consistently positive negative or zero for large arguments, which depends upon the condition of infinite point.

Our next important tool is the Integral Rule of asymptotic formulae [31, 36, 48].

Lemma 1 (Integral Rule of Asymptotic Formulae) *Let,*

$$\phi(t) \sim f(t),$$

where $f \neq 0$ *and* f *does not change in sign. Then,*

$$\begin{cases} \int_{t_0}^{t} \phi(t)dt \sim \int_{t_0}^{t} f(t)dt, & if \quad \int_{t_0}^{\infty} |f(t)|dt = \infty, \\ \int_{t}^{\infty} \phi(t)dt \sim \int_{t}^{\infty} f(t)dt, & if \quad \int_{t_0}^{\infty} |f(t)|dt < \infty. \end{cases}$$

Following the above result, we demonstrate the character of asymptotic expansion [31, 36, 48].

Lemma 2 (Character of Asymptotic Expansion) *If* $f(t) > 0$ *and* f' *is continuous and non-negative as* $t \geq t_0$*, then,*

$$f' \leq f^{1+\epsilon}$$

for any $t \geq t_0$ *and for any* $\epsilon > 0$*, except perhaps in a set of intervals of finite total length, which depends upon* ϵ*.*

The final necessary result will be Hardy's Theorem [31, 36, 48].

Theorem 2 (Hardy) *Any solution of an equation,*

$$\frac{df}{dt} = \frac{P(f,t)}{Q(f,t)},$$

which is continuous for $t \geq t_0$*, is ultimately monotonic, together with all of its derivatives, and satisfies one of the following relations,*

$$f \sim at^b e^{E(t)},$$

or

$$f \sim at^b (\ln t)^c,$$

where $E(t)$ *is a polynomial in time and* a, b, c *are constants.*

Now, all the above three results can be adopted to derive the asymptotic expansion of KdV–B, for the different values of λ.

Claim (Shu [48]) Let $\lambda > 0$. The negative asymptotic expansion of KdV–B has the following form,

$$u = -\frac{2k^2 v^2 U_\infty}{\delta} e^{-\frac{(k-1)v\zeta}{2\delta}} - \frac{8k^4 v^2 U_\infty^2}{(k-1)(3k-1)\delta} e^{-\frac{(k-1)v\zeta}{\delta}} [1+O(1)], \quad \zeta \to \infty, \tag{9}$$

where $k = \sqrt{1 + \frac{4\lambda\delta}{v^2}}$ and $U_\infty > 0$ is a constant.

In order to prove that, we consider $\lambda > 0$ for which $\sigma \in (-\frac{5}{2}, -2)$. If u has a negative asymptotic expansion then y has a positive asymptotic expansion. Since,

$$\frac{d^2 y}{d\xi^2} = \xi^\sigma y^2 > 0, \quad \xi > 0,$$

y' must be strictly monotonically increasing for $\xi > 0$ and y must be a monotone function for large ξ. Thus y' has three possible cases as $\xi \to \infty$,

1. $y' \to 0$
2. $y' \to y_0' = const > 0$
3. $y' \to \infty$

Let us show that case (2) cannot hold.
If

$$y' \to y_0' = const > 0,$$

then,

$$y \sim y_0' \xi$$

and from equation (7),

$$y'' = y^2 \xi^\sigma \sim y_0'^2 \xi^{\sigma+2} > \frac{1}{2} y_0'^2 \xi^{\sigma+2},$$

whose integration yields,

$$y' > \frac{y_0'^2}{2(\sigma+3)} \xi^{\sigma+3} \to \infty,$$

for large ξ, which leads to a contradiction. Then it will be shown that case (3) leads to a contradiction as well.
If

$$y' \to \infty,$$

then,

$$y' > M,$$

for large ξ and some $M > 0$, and hence,

$$y > M\xi \ .$$

Reverting to equation (7),

$$y'' = \xi^{\sigma} y^2 > M^2 \xi^{\sigma+2},$$

so,

$$y > \frac{M^2}{(\sigma+3)(\sigma+4)} \xi^{\sigma+4},$$

for large ξ. Continuing in this fashion,

$$y > y_0 \xi^5,$$

can be obtained for large ξ and the constant y_0. Hence, from equation (7),

$$y'' = \xi^{\sigma} y^2 > \sqrt{y_0} y^{\frac{3}{2}},$$

for large ξ. Since y' is positive,

$$y' y'' > \sqrt{y_0} y^{\frac{3}{2}} y',$$

whose integration yields,

$$y' > \frac{2 y_0^{\frac{1}{4}}}{\sqrt{5}} y^{\frac{3}{4}},$$

which is impossible due to Lemma (2).

Consequently, we are left with case (1). Since $y' < 0$ is strictly monotone increasing for $\xi > 0$, and y is strictly monotone decreasing for $\xi > 0$. Since $y > 0$ for large ξ, y has a finite limit $U_{\infty} \geq 0$ as $\xi \to \infty$. Now, let us show that $U_{\infty} \neq 0$. If $U_{\infty} = 0$, $y(\xi_0) = \delta > 0$ is set to be small. Since y is strictly monotone decreasing,

$$\delta = y(\xi_0) = \int_{\xi_0}^{\infty} \left(\int_t^{\infty} \tau^{\sigma} y^2 d\tau \right) dt < \delta^2 \int_{\xi_0}^{\infty} \left(\int_t^{\infty} \tau^{\sigma} d\tau \right) dt$$

or

$$\delta > \frac{(\sigma+1)(\sigma+2)}{\xi_0^{\sigma+2}},$$

which leads to the contradiction for δ sufficiently small.

Then let,

$$y(\infty) = U_\infty > 0, \quad y(\xi) = U_\infty + O(1), \quad \xi \to \infty .$$

Then

$$y'(\xi) = -\int_\xi^\infty y'' dt = -\int_\xi^\infty t^\sigma y^2 dt = \frac{U_\infty^2}{\sigma+1}\xi^{\sigma+1}[1+O(1)]$$

and thus,

$$y(\xi) = U_\infty - \int_\xi^\infty y' dt = U_\infty + \frac{U_\infty^2}{(\sigma+1)(\sigma+2)}\xi^{\sigma+2}[1+O(1)] .$$

The latter proves our claim.

Claim (Shu [48]) Let $\lambda = 0$. The negative asymptotic expansion of the KdV–B equation has the following form,

$$u = -\frac{2kv}{\zeta} e^{\frac{-(k-1)v\zeta}{2\delta}}, \quad \zeta \to \infty, \tag{10}$$

where $k = \sqrt{1 + \frac{4\lambda\delta}{v^2}}$.

For the proof, we consider $\lambda = 0$, which gives us $\sigma = -2$. If u has a negative asymptotic expansion, y has a positive asymptotic expansion. Let $\xi = e^s$, obtaining from equation (7),

$$\frac{d^2y}{ds^2} - \frac{dy}{ds} - y^2 = 0. \tag{11}$$

If $\frac{dy}{ds} = 0$ at s_0, then,

$$\frac{d^2y}{ds^2} = y^2 > 0$$

and y can only have a minimum at s_0. Hence, y is a monotone function for large ξ. Thus y has three possible cases as $s \to \infty$:

1. $y \to 0$

2. $y \to y_0 = const > 0$
3. $y \to \infty$

Let us show that case (2) cannot hold.
If

$$y \to y_0 = const > 0,$$

then

$$\frac{d^2y}{ds^2} - \frac{dy}{ds} \sim y^2 .$$

Integrating, we get,

$$\frac{dy}{ds} - y \sim y_0^2 s .$$

Since $y \to y_0$, this implies,

$$\frac{dy}{ds} \sim y_0^2 s$$

from which,

$$y \sim \frac{1}{2} y_0^2 s^2,$$

which contradicts $y \to y_0$. Next, we will show that case (3) is impossible.
If $y \to \infty$, let,

$$p = \frac{dy}{ds}.$$

Then equation (11) becomes,

$$p\frac{dp}{dy} - p - y^2 = 0. \tag{12}$$

Since $y \to y_0$, we have,

$$p\frac{dp}{dy} > 0 .$$

Now, Theorem (2) indicates that p has two possible cases for large y,

1. $p \sim ay^b e^{E(y)}$,
2. $p \sim ay^b (\ln y)^c$,

where $E(y)$ is a polynomial in y and $a > 0$, b, c are constants. We also show that case (1) is impossible.

If $E(y) \to -\infty$, then,

$$p \to 0, \quad \frac{dp}{dy} \to 0$$

which leads to a contradiction by referring to equation (12).

If $E(y) \to \infty$, then,

$$p > y^2,$$

for large y, which contradicts Lemma (2). Hence,

$$E(y) = const\ .$$

If $b > 1$, then,

$$p > y^{\frac{b+1}{2}},$$

for large y, which is impossible due to Lemma (2).
If $b \leq 1$, then,

$$p\frac{dp}{dy} \sim y^2,$$

is obtained from equation (12). By integration,

$$\frac{1}{2}p^2 \sim \frac{1}{3}y^3,$$

is obtained, so that $b = \dfrac{3}{2} > 1$, which leads to a contradiction. Let us now show that case (2) is also impossible.

If $b > 1$, then,

$$p > y^{\frac{b+1}{2}},$$

for large y, which is impossible due to Lemma (2).

If $b \leq 1$, then,

$$\frac{1}{2}p^2 \sim \frac{1}{3}y^3,$$

is obtained, so that $b = \dfrac{3}{2} > 1$, which leads to a contradiction.

Consequently, we are left with case (1), where $y \to 0$. Let $v = \frac{1}{y}$ and $w = \frac{dv}{ds}$, obtaining from equation (12),

$$w\frac{dw}{dv} - \frac{2w^2}{v} - w + 1 = 0. \tag{13}$$

Since $y \to 0$, $v \to \infty$ and $\frac{dv}{ds} < 0$, we have,

$$w = \frac{dv}{ds} = -\frac{1}{y^2}\frac{dy}{ds} > 0,$$

is obtained. Theorem (2) indicates that w has two possible cases for large v,

1. $w \sim av^b e^{E(v)}$,
2. $w \sim av^b (\ln v)^c$,

where $E(v)$ is a polynomial in v and $a > 0$, b, c are constants. It is now shown that if case (3) is satisfied, $E(v) = const$ and $b = 0$. Similar to above, $E(v) = const$ and $b \leq 1$.

If $b = 1$, then,

$$\frac{dw}{dv} \sim a > 0.$$

From equation (13), $a = -1$ is obtained, which leads to a contradiction.

If $b \in (0, 1)$, then,

$$\frac{dw}{ds} \sim 1,$$

is obtained from equation (13). By integrating, we get,

$$w \sim v,$$

so that $b = 1$, which also leads to a contradiction.

If $b < 0$, then,

$$w\frac{dw}{dv} \sim -1,$$

is obtained from equation (13). By integrating, we get,

$$\frac{1}{2}w^2 \sim -v,$$

which leads to a contradiction.

Let us now show that if case (2) is satisfied, then $b = c = 0$. Similar to above, either $b = 1, c \neq 0$ or $b = 0$.
If $b = 1$, $c < 0$ or $c > 0$, then,

$$\frac{dw}{dv} \sim 1 \quad or \quad \frac{dw}{dv} \sim \frac{2w}{v},$$

is obtained from equation (13). By integrating, we get,

$$w \sim v \quad or \quad w \sim v^2,$$

is obtained, so that $c = 0$, which leads to a contradiction. Hence $b = 0$.
If $c < 0$, then,

$$w\frac{dw}{dv} \sim -1,$$

is obtained from equation (13). By integrating, we get,

$$\frac{1}{2}w^2 \sim -v,$$

is obtained, which leads to a contradiction.

If $c > 0$, then,

$$\frac{dw}{dv} \sim 1,$$

is obtained from equation (13). By integrating, we get,

$$w \sim v,$$

so that $c = 0$, which leads to a contradiction.

Summing up and from equation (13), we get,

$$w \sim 1,$$

so that,

$$\frac{dv}{ds} \sim 1, \quad s \to \infty \, .$$

By integrating, we finally obtain,

$$v \sim s,$$

as $s \to \infty$, so that,

$$y \sim \frac{1}{\ln \xi}, \quad \xi \to \infty\,.$$

That proves our claim.

Claim (Shu [48])
Let $\lambda > 0$. The negative asymptotic expansion of KdV–B can be written in the form (see Fig. 2),

$$u = -\frac{2k^2 v^2 U_\infty}{\delta} e^{\frac{-(k-1)v\zeta}{2\delta}} - \frac{2k^4 v^2}{\delta} \sum_{i=1}^{\infty} \frac{(2U_\infty)^{i+1} e^{-\frac{(i+1)(k-1)v\zeta}{2\delta}}}{\prod_{j=1}^{i} [j(k-1)+2k]\, j(k-1)}, \quad \zeta \to \infty, \tag{14}$$

where $k = \sqrt{1 + \frac{4\lambda\delta}{v^2}}$ and $U_\infty > 0$ is a constant.

To prove the latter, we first notice that since,

$$e^{-\frac{(i+1)(k-1)v\zeta}{2\delta}}$$

exists, the infinite series converges. Let,

$$u_m = -\frac{2k^2 v^2 U_\infty}{\delta} e^{\frac{-(k-1)v\zeta}{2\delta}} - \frac{2k^4 v^2}{\delta} \sum_{i=1}^{\infty} \frac{(2U_\infty)^{i+1} e^{-\frac{(i+1)(k-1)v\zeta}{2\delta}}}{\prod_{j=1}^{i} [j(k-1)+2k]\, j(k-1)}$$

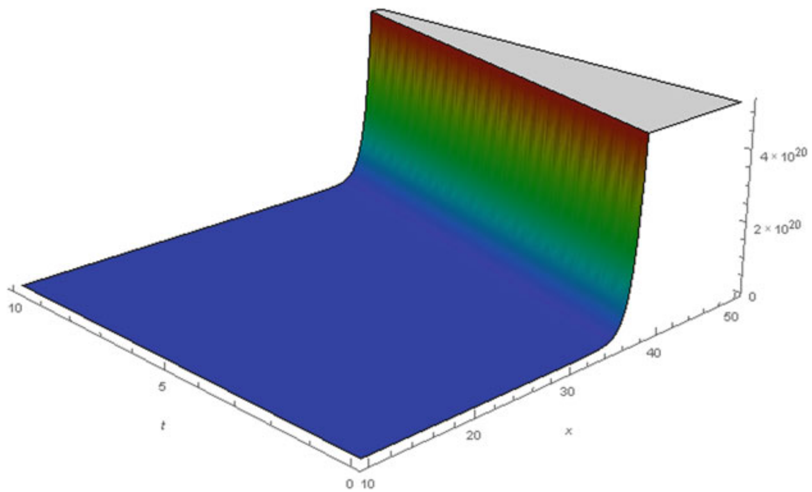

Fig. 2 The graph shows the asymptotic expansion of the KdV–B equation (see Equation (14)) for the parameters, $\alpha = 0.1$, $\beta = 0.7$, $\gamma = 1$, $\lambda = 1$, $U_\infty = 1$ and $\sigma = -2.25$. The shock wave characteristics can be observed

and

$$y_m = U_\infty + \sum_{i=1}^{m} \frac{2^{i-1} u_\infty^{1+i} \xi^{i(\sigma+2}}{\prod_{j=1}^{i} [j(\sigma+2)-1]\, j(\sigma+2)} .$$

Then,

$$y_{m+1}[1+O(1)] = U_\infty + \int_\xi^\infty \left(\int_t^\infty \tau^\sigma y_m^2 [1+O(1)]^2 d\tau \right) dt$$

can be obtained for an arbitrary integer m. Since $u_m \to u$ as $m \to \infty$, we get,

$$y_m \to y_\infty, \quad m \to \infty,$$

so that,

$$y_\infty = U_\infty + \int_\xi^\infty \left(\int_t^\infty \tau^\sigma y_\infty^2 d\tau \right) dt$$

and y_∞ is the positive asymptotic expansion of equation (7).

4 Hyperbolic Methods for Traveling Wave Solutions of KdV–B

Since the late 1980s, various methods for seeking explicit exact solutions to the KdV–B equation have been independently proposed by many mathematicians, engineers and physicists. The first analytical traveling wave solution to the Burgers-KdV equation was obtained by Xiong [56] in 1989. Two different methods for the construction of exact solutions to the KdV–B equation were proposed by Jeffrey and Mohamad [26]. Wang [52] applied the homogeneous balance method to the study of exact solutions of the compound KdV–B equation. Demiray [13] proposed a so-called "hyperbolic tangent approach" for finding the exact solution to the KdV–B equation, which is actually the Parkes and Duffy's automated method [39, 40]. Recently, Feng [16–18] introduced the first-integral method to study the exact solution of KdV–B, which is based on the ring theory of commutative algebra. The Cauchy problem for the KdV–B equation was investigated by Bona and Schonbek [7]. They proved the existence and uniqueness of bounded traveling wave solutions which tend to constant states at plus and minus infinity.

We focus on deriving traveling wave solutions for KdV–B, using the Tanh and Sech methods [21, 24, 32, 33, 53]. We start with the Tanh method, considering,

$$-\lambda u + \gamma \frac{u^2}{2} - \alpha \frac{du}{d\zeta} + \beta \frac{d^2 u}{d\zeta^2} = 0, \tag{15}$$

where λ is the wave velocity, assuming that both our solution and its spatial derivatives vanish at either plus or minus infinity.

The Tanh method uses a finite series,

$$u(x,t) = u(\mu\zeta) = s(y) = \sum_{m=0}^{M} a_m y^m, \tag{16}$$

where μ is the wave number, inversely proportional to the width of the wave, and M is a positive integer, in most cases, that will be determined. However if M is not an integer, a transformation formula is usually used. Substituting equation (16) into equation (15) yields an equation in powers of y.

To determine the parameter M, we usually balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms. With M determined, we collect all coefficients of powers of y in the resulting equation where these coefficients have to vanish.

This will give a system of algebraic equations involving the parameters a_m, $m = 0, \ldots, M$, μ and λ. Having determined these parameters, knowing that M is a positive integer in most cases, and using equation (16), we obtain an analytic solution in a closed form. We introduce,

$$y = tanh(\mu\zeta), \tag{17}$$

that leads to the change of derivatives,

$$\begin{cases} \dfrac{d}{d\zeta} = \dfrac{d}{dy}\dfrac{dy}{dz} = \mu(1-y^2)\dfrac{d}{dy}, \\ \dfrac{d^2}{d\zeta^2} = \dfrac{d}{d\zeta}\dfrac{d}{d\zeta} = \mu^2(1-y^2)\left(-2y\dfrac{d}{dy} + (1-y^2)\dfrac{d^2}{dy^2}\right). \end{cases} \tag{18}$$

Therefore, by replacing equation (16) in equation (15) and using equation (18), we derive an equation with respect to u as follows,

$$\begin{aligned} &-\lambda\left(\sum_{m=0}^{M} a_m y^m\right) + \frac{\gamma}{2}\left(\sum_{m=0}^{M} a_m y^m\right)^2 - \alpha\mu(1-y^2)\frac{d}{dy}\left(\sum_{m=0}^{M} a_m y^m\right) \\ &\quad + \beta\mu^2(1-y^2)\left(-2y\frac{d}{dy}\left(\sum_{m=0}^{M} a_m y^m\right) + (1-y^2)\frac{d^2}{dy^2}\left(\sum_{m=0}^{M} a_m y^m\right)\right) = 0\,. \end{aligned}$$

To determine M, we follow the procedure described above to get $M = 2$. This gives the solution in the form,

$$s = \sum_{m=0}^{M} a_m y^m = a_0 + a_1 y + a_2 y^2, \quad a_2 \neq 0. \tag{19}$$

Substituting equation (19) into equation (15), we get,

$$-\lambda(a_0 + a_1 y + a_2 y^2) + \frac{\gamma}{2}(a_0 + a_1 y + a_2 y^2)^2 - \alpha\mu(1 - y^2)(a_1 + 2a_2 y) + \beta\mu^2(1 - y^2) \left(-2y(a_1 + 2a_2 y) + 2a_2(1 - y^2)\right) = 0.$$

Collecting the coefficients of different powers of y, gives the following system of algebraic equations for λ μ, a_0, a_1 and a_2,

$$\begin{cases} a_2(\gamma a_2 + 12\beta\mu^2) = 0 \\ \gamma a_1 a_2 + 2\beta\mu^2 a_1 + 2\alpha\mu a_2 = 0 \\ -\lambda a_2 - 8\beta\mu^2 a_2 + \gamma a_0 a_2 + \frac{\gamma}{2} a_1^2 + \alpha\mu a_1 = 0 \\ -\lambda a_1 - 2\beta\mu^2 a_1 + \gamma a_0 a_1 - 2\alpha\mu a_2 = 0 \\ -\lambda a_0 + \frac{\gamma}{2} a_0^2 - \alpha\mu a_1 + 2\beta\mu^2 a_2 = 0 \end{cases}$$

with solution,

$$\begin{cases} \lambda = \pm\frac{6}{25}\frac{\alpha^2}{\beta}, \\ \mu = \pm\frac{\alpha}{10\beta}, \\ a_0 = \frac{\lambda}{\gamma} + 12\frac{\beta\mu^2}{\gamma}, \\ a_1 = -\frac{12}{5}\frac{\alpha\mu}{\gamma}, \\ a_2 = -12\frac{\beta\mu^2}{\gamma}. \end{cases} \tag{20}$$

Using the trigonometric identities,

$$\begin{cases} tanh^2(\theta) = 1 - sech^2(\theta) \\ tanh(-\theta) = -tanh(\theta) \end{cases}, \quad \theta \in \mathbb{R}$$

and requiring for both our solution and its spatial derivatives to vanish at plus infinity, we get the following traveling wave solution,

$$u_{1\ \infty}(\zeta) = \frac{3}{25}\frac{\alpha^2}{\beta\gamma}\left(sech^2(\mu\zeta) - 2tanh(\mu\zeta) + 2\right), \quad \mu, \lambda > 0 \tag{21}$$

Requiring for both our solution and its spatial derivatives to vanish at minus infinity, we get the following traveling wave solution,

$$u_{2\ -\infty}(\zeta) = \frac{3}{25}\frac{\alpha^2}{\beta\gamma}\left(sech^2(\mu\zeta) - 2tanh(\mu\zeta) - 2\right), \quad \mu > 0, \quad \lambda < 0 \tag{22}$$

Remark 2 A notable result is that our traveling wave solutions are expressed as a composition of a bell-profile solitary wave (KdV) and a kink-profile solitary wave (Burgers') with velocity $\lambda = \pm\frac{6}{25}\frac{\alpha^2}{\beta}$. The shock profile is dominant here. All those exact solutions, and others mentioned in literature, can be proved to be algebraically equivalent to each other [18]. That is, essentially only one explicit traveling solitary wave solution to the KdV–B equation is known which can be expressed as a composition of a bell-profile solitary wave and a kink-profile solitary wave. In other words, a feature of this solution is that is a linear combination of particular solutions of the KdV equation and the Burgers equation [18, 26].

By following similar steps as with the Tanh method for traveling wave solutions of KdV–B, the Sech method uses the variable transformation [21, 24, 53],

$$\begin{cases} y = sech(\mu\zeta), \quad \mu \neq 0, \\ \dfrac{d}{d\zeta} = -\mu y\sqrt{1-y^2}\dfrac{d}{dy}, \\ \dfrac{d^2}{d\zeta^2} = \mu^2 y[(1-2y^2)\dfrac{d}{dy} + (y-y^3)\dfrac{d^2}{dy^2}] \end{cases} \tag{23}$$

and the ansatz,

$$u(x,t) = u(\mu\zeta) = s(y) = \sum_{m=0}^{2} a_m y^m. \tag{24}$$

Then by replacing equation (24) in equation (15) and using equation (23), we derive an equation with respect to u as follows,

$$\begin{aligned} [a_2(\gamma\frac{a_2}{2} - \mu\alpha - 6\beta\mu^2)]y^4 + [a_1(\gamma a_2 - \frac{\alpha\mu}{2} - 2\beta\mu^2)]y^3 \\ + [-\lambda a_2 + \frac{\gamma}{2}a_1^2 + \gamma a_0 a_2 + 2\alpha\mu a_2 \\ + 4\beta\mu^2 a_2]y^2 + [a_1(-\lambda + \gamma a_0 + \alpha\mu + \beta\mu^2)]y \\ + \frac{\gamma}{2}a_0^2 - \lambda a_0 = 0. \end{aligned}$$

Above, we used the first order Taylor approximation,

$$\sqrt{1-y^2} \sim 1 - \frac{y^2}{2}.$$

Collecting the coefficients of different powers of y, gives the following system of algebraic equations for λ μ, a_0, a_1 and a_2,

$$\begin{cases} a_2(\gamma \dfrac{a_2}{2} - \mu\alpha - 6\beta\mu^2) = 0 \\ a_1(\gamma a_2 - \dfrac{\alpha\mu}{2} - 2\beta\mu^2) = 0 \\ -\lambda a_2 + \dfrac{\gamma}{2} a_1^2 + \gamma a_0 a_2 + 2\alpha\mu a_2 + 4\beta\mu^2 a_2 \\ a_1(-\lambda + \gamma a_0 + \alpha\mu + \beta\mu^2) = 0 \\ \dfrac{\gamma}{2} a_0^2 - \lambda a_0 = 0 \end{cases}$$

with a solution being,

$$\begin{cases} \lambda = 2\mu(\alpha + 2\beta\mu) \\ a_0 = \dfrac{4\mu}{\gamma}(\alpha + 2\beta\mu) \\ a_1 = 0 \\ a_2 = \dfrac{2\mu}{\gamma}(\alpha + 6\beta\mu) \end{cases} , \quad \mu \neq 0 \tag{25}$$

giving a solitary profile traveling wave solution,

$$\frac{2\mu}{\gamma}\left(2(\alpha + 2\beta\mu) + (\alpha + 6\beta\mu)sech^2(\mu\zeta)\right), \quad \mu \neq 0. \tag{26}$$

Remark 3 A notable result is that the Sech method can give “purely” solitary profile traveling wave solutions.
The following graph, Fig. 3, depicts the solutions studied in this section.

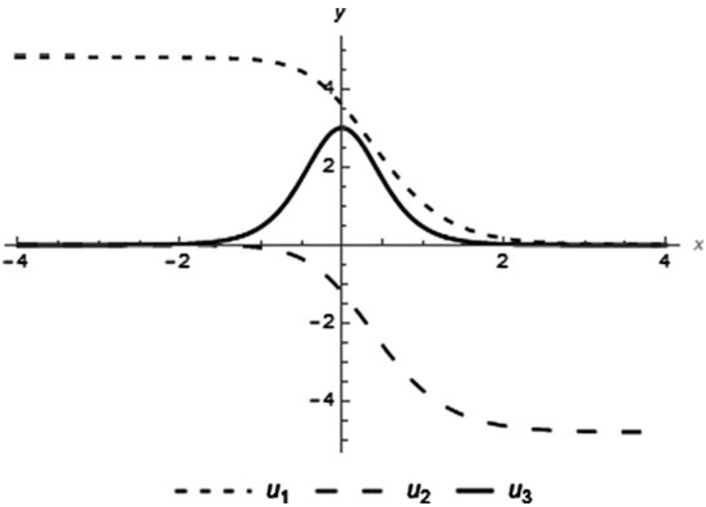

Fig. 3 u_1 and u_2 are two shock profile traveling waves of the KdV–B equation vanishing at plus and minus infinity, respectively, whereas u_3 is a solitary profile traveling wave solution of the KdV equation, for the parameters, $\alpha = 1$, $\beta = 0.1$, $\gamma = 1$, $\lambda = 1$

5 Spectral Fourier Analysis for the Numerical Solution of KdV–B

We define the Fourier and Inverse Fourier Transform of a function, say f, in the sense that the following symbols make sense, to be [2, 31],

$$\begin{cases} F[f(x)] = \hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)dx, \\ F^{-1}[\hat{f}(k)] = f(x) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k)dk. \end{cases} \tag{27}$$

It is easy to see, integrating by parts, that regarding the nth derivative of f and its Fourier Transform, the following results hold,

$$\begin{cases} \dfrac{d^n f}{dx^n} = (i^n) F^{-1} \left[k^n \hat{f}(k) \right], \\ \dfrac{d^n \hat{f}}{dx^n} = (-i)^n F \left[x^n f(x) \right]. \end{cases} \tag{28}$$

Now consider the KdV–B equation,

$$\frac{\partial u}{\partial t} + \gamma u \frac{\partial u}{\partial x} - \alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^3 u}{\partial x^3} = 0, \quad u = u(t, x). \tag{29}$$

Rearranging the terms of equation (29), we get,

$$\frac{\partial u}{\partial t} = -\gamma u \frac{\partial u}{\partial x} + \alpha \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial^3 u}{\partial x^3} . \tag{30}$$

By means of the Inverse Fourier Transform, F^{-1}, equation (30) can be written in the form [2, 50],

$$\frac{\partial u}{\partial t} = f(t, u), \tag{31}$$

where we have substituted the x-partial derivatives with,

$$\begin{cases} \dfrac{\partial u}{\partial x} = i F^{-1}(\kappa \hat{u}), \\ \dfrac{\partial^2 u}{\partial x^2} = -F^{-1}(\kappa^2 \hat{u}), \\ \dfrac{\partial^3 u}{\partial x^3} = -i F^{-1}(\kappa^3 \hat{u}). \end{cases}$$

Now, equation (31) is suitable for applying the 4th order explicit Runge–Kutta method, giving us the following,

$$\begin{cases} u_{n+1} = u_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ t_{n+1} = t_n + h \\ k_1 = f(t_n, u_n) \\ k_2 = f(t_n + \frac{h}{2}, u_n + \frac{hk_1}{2}) \\ k_3 = f(t_n + \frac{h}{2}, u_n + \frac{hk_2}{2}) \\ k_4 = f(t_n + h, u_n + hk_3) \end{cases} , \quad n = 0, 1, \ldots \tag{32}$$

For $n = 0$, we may choose either a soliton of the KdV equation or a similarity solution of the viscous Burgers equation or a traveling wave solution of KdV–B.

Below we exhibit the obtained numerical results for each case separately. In Fig. 4, the evolution of an initial solitary profile solution of the KdV equation is depicted, where diffusive effects are absent. It can be observed that the solitary waveform is retained. In Fig. 5, the evolution of a solitary profile solution of the

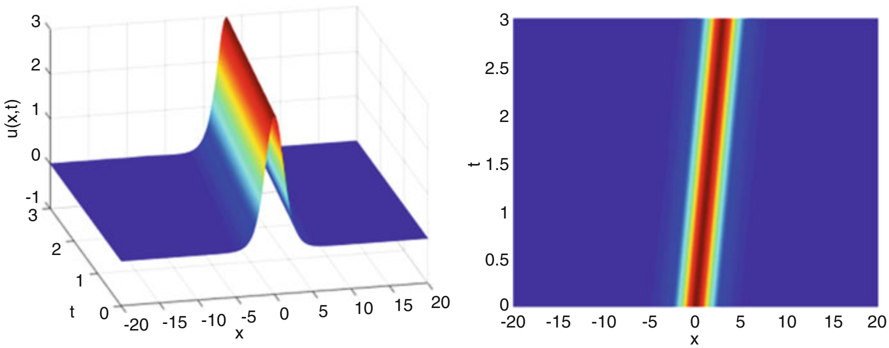

Fig. 4 A solution of the KdV–B equation, evolving a solitary profile solution of the KdV equation, where diffusive effects are absent (KdV case), for the parameters, $\lambda = 1$, $\alpha = 0$, $\beta = 0.7$, $\gamma = 1$

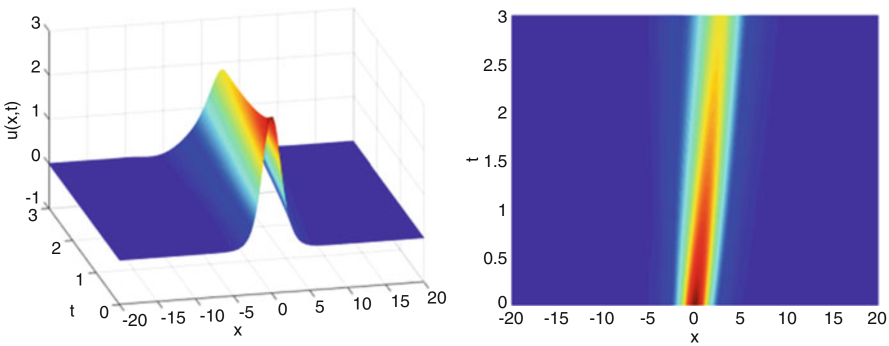

Fig. 5 A solution of the KdV–B equation, evolving a solitary profile solution of the KdV equation, where both diffusive and dispersive effects coexist, for the parameters, $\lambda = 1$, $\alpha = 0.5$, $\beta = 0.7$, $\gamma = 1$

KdV equation is presented, where both diffusive and dispersive effects coexist. It is observed that the profile loses energy and reduces in amplitude drastically. Additionally, in Fig. 6 we present the evolution of a similarity shock profile solution of the viscous Burgers equation, where both diffusive and dispersive effects coexist. In this case a wavefront can be observed revealing a shock-like behavior. Finally, in Fig. 7, the evolution of a traveling shock wave profile solution of the KdV–B equation is presented, where both diffusive and dispersive effects coexist. These numerical solutions clearly reveal both solitary and shock wave features of the KdV–B equation, revealing its connection to cardiac hemodynamics where all these phenomena, such as convection, diffusion and dispersion, can be observed.

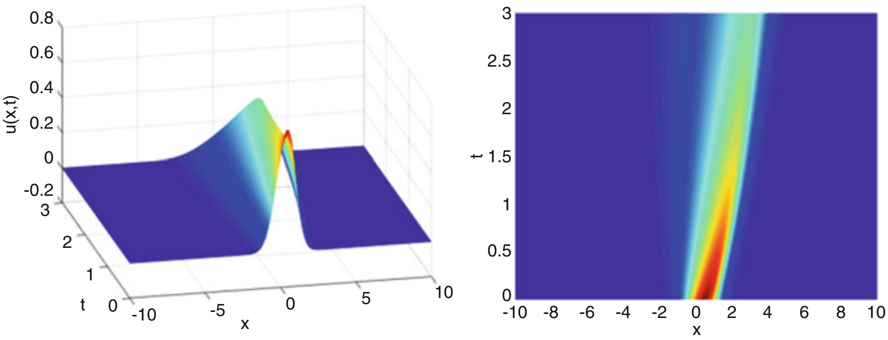

Fig. 6 A solution of the KdV–B equation, evolving a similarity shock profile solution of the viscous Burgers equation, where both diffusive and dispersive effects coexist, for the parameters, $\lambda = 1.8$, $\alpha = 0.19$, $\beta = 0.01$, $\gamma = 3.4$

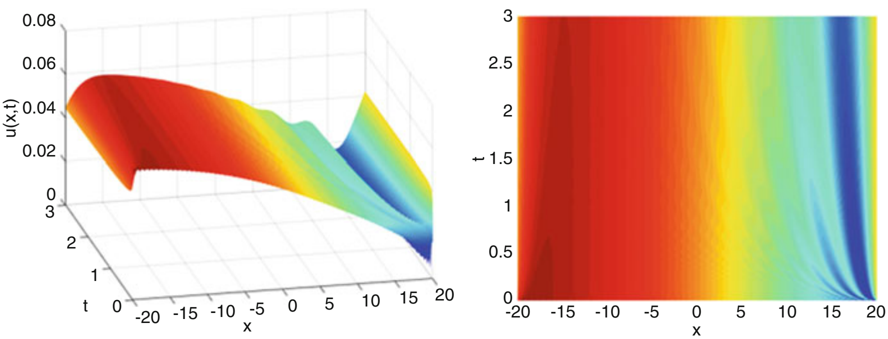

Fig. 7 A solution of the KdV–B equation, evolving a traveling wave shock profile solution of the KdV–B equation, where both diffusive and dispersive effects coexist, for the parameters, $\alpha = 0.3$, $\beta = 0.7$, $\gamma = 1$

6 Conclusions

Recent advancements concerning cardiac dynamics pose important questions about the cardiac waveform. A governing equation, namely the KdV–B equation (Korteweg–de Vries–Burgers), which is a partial differential equation can be utilized to answer several of those questions. The KdV–B equation features both solitary and shock wave characteristics due to the dispersion and dissipation terms, as also occurring in the arterial tree. This study focuses on describing cardiac dynamics with the applications of mathematics and nonlinear analysis. It is customarily difficult to solve nonlinear problems, especially by analytical techniques. Therefore, seeking suitable solving methods, such as, exact, approximate or numerical methods, is an active task in branches of applied mathematics and nonlinear analysis.

In this chapter, the phase plane of the KdV–B equation is analyzed and its qualitative behavior is derived, depicting the stability states of the equation contributing to the decisions made for further analytical and numerical consideration. The analysis reveals a saddle point (0, 0), and an additional one that could be a source point or a spiral source point or a central point depending on the equation's parameters, α, β and λ.

Furthermore, an asymptotic expansion is presented, providing a reliable basis for estimating the advantages and disadvantages when seeking and applying numerical methods to KdV–B equation. Furthermore, traveling wave solutions under both solitary and shock profiles are obtained from the hyperbolic methods, whose strength is their ease of use to find which solitary wave structures and/or shock-wave (kinks) profiles satisfy nonlinear wave and evolution equations. These techniques allow to develop algorithms for symbolic software packages, so that nonlinear partial differential equations and difference equations, can be studied automatically whether (or not) they possess traveling wave solutions. Additionally, numerical solutions are obtained for the equation, by means of the Spectral Fourier analysis. Both these solutions and the latter traveling wave solutions are evolved in time by the Runge–Kutta method. These solutions clearly depict both solitary and shock wave characteristics of the KdV–B equation. This analysis provides vital information about the equation and its connection to cardiac hemodynamics.

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