The Semi-discrete Method for the Approximation of the Solution of Stochastic Differential Equations

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Abstract We study the numerical approximation of the solution of stochastic differential equations (SDEs) that do not follow the standard smoothness assumptions. In particular, we focus on SDEs that admit solutions which take values in a certain domain; examples of these equations appear in various fields of application such as mathematical finance and natural sciences among others, where the quantity of interest may be the interest rate, which takes non-negative values, or the population dynamics which takes values between zero and one. We review the Semi-Discrete method (SD), a numerical method that has the qualitative feature of domain preservation among other desirable properties.

1 Introduction

We are interested in the numerical approximation of stochastic differential equations (SDEs) that admit solutions in a certain domain and do not satisfy the usual assumptions. Such equations appear in mathematical finance, e.g. interest rate models, but also in other fields of applications such as natural and social sciences. Generally speaking, explicit solutions of these SDEs are unknown, so numerical methods have to be used to simulate them. While numerical methods exist that converge strongly to the true solution of SDEs with non-standard coefficients, few of them are able to maintain the solution process domain. Implicit methods can in some cases succeed in that direction, but they are usually more time-consuming. Let us state the problem in mathematical terms.

Throughout, let $T > 0$ and $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ be a complete probability space, meaning that the filtration $\{\mathscr{F}_t\}_{0\leq t\leq T}$ is right continuous and $\mathscr{F}_0$ includes all $\mathbb{P}$-null sets. We are interested in the following SDE in integral form

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$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s, \quad t \in [0, T], \tag{1}$$

where $W_{t,\omega} : [0, T] \times \Omega \to \mathbb{R}^m$ is an m-dimensional Wiener process adapted to the filtration $\{\mathscr{F}_t\}_{0\leq t\leq T}$, the drift coefficient $a : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and the diffusion coefficient $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are measurable functions such that (1) has a unique strong solution and X_0 is independent of all $\{W_t\}_{0\leq t\leq T}$. SDE (1) has non-autonomous coefficients, i.e. $a(t, x), b(t, x)$ depend explicitly on t. More precisely, we assume the existence of a predictable stochastic process $X : [0, T] \times \Omega \to \mathbb{R}^d$ such that, c.f. [22, Def. 5.2.1], [24, Def. 2.1],

$$\{a(t, X_t)\} \in \mathscr{L}^1([0, T]; \mathbb{R}^d), \quad \{b(t, X_t)\} \in \mathscr{L}^2([0, T]; \mathbb{R}^{d\times m})$$

and

$$\mathbb{P}\left[X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s\right] = 1, \quad \text{for every } t \in [0, T].$$

The drift coefficient a is the infinitesimal mean of the process (X_t) and the diffusion coefficient b is the infinitesimal standard deviation of the process (X_t). SDEs like (1) have rarely explicit solutions so numerical approximations are required for path simulations of the solution process $X_t(\omega)$.

We are interested in strong approximations (mean-square) of (1), in the case of nonlinear drift and diffusion coefficients. Strongly converging numerical schemes have applications in many areas, such as simulating scenarios, filtering or visualizing stochastic dynamics (c.f [20, Sec. 4] and references therein), they are of theoretical interest (they provide basic insight into weak-sense schemes) and usually do not require simulations over long-time periods or of a significant number of trajectories. In the same time we aim for numerical methods that preserve the domain of the original process, or as we say possess an *eternal life time*.

Definition 1 (Eternal Life Time of Numerical Solution) Let $D \subseteq \mathbb{R}^d$ and consider a process (X_t) well defined on the domain $\overline{D}$, with initial condition $X_0 \in \overline{D}$ and such that

$$\mathbb{P}(\{\omega \in \Omega : X(t, \omega) \in \overline{D}\}) = 1,$$

for all $t > 0$. A numerical solution $(Y_{t_n})_{n\in\mathbb{N}}$ has an *eternal life time* if

$$\mathbb{P}(Y_{t_{n+1}} \in \overline{D} \,\big|\, Y_{t_n} \in \overline{D}) = 1.$$

Let us consider the following nonlinear model both in the drift and diffusion coefficient:

$$x_t = x_0 + \int_0^t (\alpha x_s - \beta x_s^2)ds + \int_0^t \sigma x_s^{3/2} dW_s, \quad t \in [0, T], \tag{2}$$

where x_0 is independent of $\{W_t\}_{0\leq t\leq T}$, $x_0 > 0$ a.s. and $\sigma \in \mathbb{R}$. SDE (2) is referred to as the 3/2-model [18] or the inverse square root process [1] and is used for modeling stochastic volatility. The conditions $\alpha > 0$ and $\beta > 0$ are necessary and sufficient for the stationarity of the process (x_t) and such that neither zero nor infinity is attainable in finite time [1, App. A].

A "good" numerical scheme for the approximation of the solution of an SDE that takes positive values, as (2), should preserve positivity, c.f. [2, 21]. The explicit Euler scheme does not have that property, since its increments are conditionally Gaussian and therefore there is a positive probability of producing negative values. We refer, among other papers, to [23] that considers Euler type schemes, modifications of them to overcome the above drawback, and the importance of positivity.

SDE (2) is a special case of super-linear models of the form (1) where one of the coefficients $a(\cdot), b(\cdot)$ is super-linear, i.e. when we have that

$$a(x) \geq \frac{|x|^\beta}{C},\ b(x) \leq C|x|^\alpha,\ \text{for every} \quad |x| \geq C, \tag{3}$$

or

$$b(x) \geq \frac{|x|^\beta}{C},\ a(x) \leq C|x|^\alpha,\ \text{for every} \quad |x| \geq C, \tag{4}$$

where $\beta > 1, \beta > \alpha \geq 0, C > 0$.

Another issue that arises at the numerical approximation of super-linear problems like (3) or (4), is that the moments of the scheme may explode, see [19, Th. 1]. A method that overcomes this drawback is the tamed Euler method, which reads in a general from

$$Y_{n+1}^N(\omega) := Y_n^N(\omega) + a_\Delta(Y_n^N(\omega)) \cdot \Delta + b_\Delta(Y_n^N(\omega))\Delta W_n(\omega), \tag{5}$$

for every $n \in \{0, 1, \ldots, N-1\}, N \in \mathbb{N}$ and all $\omega \in \Omega$ where $\Delta W_n(\omega) := W_{\frac{(n+1)T}{N}}(\omega) - W_{\frac{nT}{N}}(\omega)$ are the increments of the Wiener process, $Y_0^N(\omega) := x_0(\omega)$ and the control functions are such that $a_\Delta \to a$ and $b_\Delta \to b$ as $\Delta \to 0$, c.f [20, (4)], [31, Rel. (3.1)], [27], for various choices of a_Δ and b_Δ. These balanced type schemes are explicit, do not explode in finite time and converge strongly to the exact solution. Nevertheless, in general they do not preserve positivity. We should also mention here other interesting implicit methods, c.f. [26] and [25], which are unfortunately time-consuming.

We study SDEs of the general type (1) with solutions in a certain domain and our aim is to construct explicit numerical schemes which on the one hand, converge strongly to the solution process and on the other, preserve the domain of the original SDE.

The semi-discrete (SD) method, originally proposed in [7], has all the above properties and more, that is:

- it is explicit in general and therefore does not require a lot of computational time,
- it does not explode in non-linear problems, see [8, Sec. 3], [15, Sec. 4], [11]

- it strongly converges to the exact solution of the original SDE, [7, Sec. 3], [10–15, 28, 29]
- has the qualitative property of domain preservation, [7, Sec. 3.2], [10, 12–14], [15, Sec. 4], [11, 28, 29]
- preserves monotonicity, [7, Sec. 3.1]
- preserves the a.s. asymptotic stability of the underlying SDE, [16].

2 The Semi-discrete Method: Setting and General Results

We address first the scalar differential equation (1), that is the one-dimensional case $(d = 1)$, which we rewrite here

$$x_t = x_0 + \int_0^t a(s, x_s)ds + \int_0^t b(s, x_s)dW_s, \quad t \in [0, T]. \tag{6}$$

Consider the equidistant partition $0 = t_0 < t_1 < \ldots < t_N = T$ with step-size $\Delta = T/N$. We assume that there is a unique strong solution a.s. to the following SDE

$$y_t = y_{t_n} + \int_{t_n}^t f(t_n, s, y_{t_n}, y_s)ds + \int_{t_n}^t g(t_n, s, y_{t_n}, y_s)dW_s, \quad t \in (t_n, t_{n+1}], \tag{7}$$

for every $n \in \mathbb{N}, n \leq N - 1$, with $y_0 = x_0$. Here, the auxiliary functions f and g satisfy the following assumption.

Assumption 2.1 *Let $f(s, r, x, y), g(s, r, x, y) : [0, T]^2 \times \mathbb{R}^2 \to \mathbb{R}$ be such that $f(s, s, x, x) = a(s, x), g(s, s, x, x) = b(s, x)$, where f, g satisfy the following conditions:*

$$|f(s_1, r_1, x_1, y_1) - f(s_2, r_2, x_2, y_2)| \leq C_R \left(|s_1 - s_2| + |r_1 - r_2| + |x_1 - x_2| + |y_1 - y_2|\right)$$
$$|g(s_1, r_1, x_1, y_1) - g(s_2, r_2, x_2, y_2)| \leq C_R \left(|s_1 - s_2| + |r_1 - r_2| + |x_1 - x_2| + |y_1 - y_2| + \sqrt{|x_1 - x_2|}\right),$$

for any $R > 0$ such that $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq R$, where the constant C_R depends on R and $x \vee y$ denotes the maximum of x, y.

We consider the following interpolation process of the semi-discrete approximation, in a compact form,

$$y_t = y_0 + \int_0^t f(\hat{s}, s, y_{\hat{s}}, y_s)ds + \int_0^t g(\hat{s}, s, y_{\hat{s}}, y_s)dW_s, \tag{8}$$

where $\hat{s} = t_n$ when $s \in [t_n, t_{n+1})$. In that way we may compare with the exact solution x_t, which is a continuous time process. The first and third variable in f, g denote the discretized part of the original SDE. We observe from (8) that in order to solve for y_t, we have to solve, in general, an SDE and not an algebraic equation. We can reproduce the Euler scheme if we choose $f(s, r, x, y) = a(s, x)$ and $g(s, r, x, y) = b(s, x)$. The semi-discrete method (8) can be appropriately modified to produce an implicit scheme that is explicitly and easily solved if necessary (see [11, 14, 29]).

In the case of superlinear coefficients the numerical scheme (8) converges to the true solution x_t of SDE (6) and this is stated in the following, see [15, Th. 2.1].

Theorem 1 (Strong Convergence) *Suppose Assumption 2.1 holds and (7) has a unique strong solution for every $n \le N-1$, where $x_0 \in \mathscr{L}^p(\Omega, \mathbb{R})$. Let also*

$$\mathbb{E}(\sup_{0\le t\le T} |x_t|^p) \vee \mathbb{E}(\sup_{0\le t\le T} |y_t|^p) < A,$$

for some $p > 2$ and $A > 0$. Then the semi-discrete numerical scheme (8) converges to the true solution of (6) in the $\mathscr{L}^2$-sense, that is

$$\lim_{\Delta\to 0} \mathbb{E} \sup_{0\le t\le T} |y_t - x_t|^2 = 0. \tag{9}$$

Theorem 1 is an extension of [8, Th. 1] to time-dependent coefficients which covers super-linear diffusion coefficients, like for example of the form $b(t, x) = \beta(t) \cdot x^{3/2}$. In all other cases we may assume the usual local Lipschitz assumption for both f and g.

We understand by the general form of decomposition (7) that we may produce many different semi-discrete numerical schemes. In a sense the method is problem dependent, since the form of the drift and diffusion coefficients, a and b, of the original SDE suggest the way of discretization. We will see in the following Sections 3 and 4 applications of the semi-discrete method which all have in common the qualitative property of domain preservation.

Relation (9) does not reveal the order of convergence. In order to show the order of convergence, we work with a truncated version of the SD method, see [30].

We choose a strictly increasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every $s, r \le T$

$$\sup_{|x|\le u} (|f(s, r, x, y)| \vee |g(s, r, x, y)|) \le \mu(u)(1 + |y|), \qquad u \ge 1. \tag{10}$$

The inverse function of μ, denoted by μ^{-1}, maps $[\mu(1), \infty)$ to $\mathbb{R}_+$. Moreover, we choose a strictly decreasing function $h : (0, 1] \to [\mu(1), \infty)$ and a constant $\hat{h} \ge 1 \vee \mu(1)$ such that

$$\lim_{\Delta\to 0} h(\Delta) = \infty \quad \text{and} \quad \Delta^{1/6} h(\Delta) \le \hat{h} \quad \text{for every} \quad \Delta \in (0, 1]. \tag{11}$$

Let $\Delta \in (0, 1]$ and f_Δ, g_Δ defined by

$$\phi_\Delta(s, r, x, y) := \phi\left(s, r, (|x| \wedge \mu^{-1}(h(\Delta)))\frac{x}{|x|}, y\right), \tag{12}$$

for $x, y \in \mathbb{R}$ where we set $x/|x| = 0$ when $x = 0$. Using the truncated auxiliary functions f_Δ and g_Δ we may redefine SDEs (7) and (8), which now read

$$y_t^\Delta = y_{t_n}^\Delta + \int_{t_n}^t f_\Delta(t_n, s, y_{t_n}^\Delta, y_s^\Delta)ds + \int_{t_n}^t g_\Delta(t_n, s, y_{t_n}^\Delta, y_s^\Delta)dW_s, \quad t \in (t_n, t_{n+1}], \tag{13}$$

and

$$y_t^\Delta = y_0 + \int_0^t f_\Delta(\hat{s}, s, y_{\hat{s}}^\Delta, y_s^\Delta)ds + \int_0^t g_\Delta(\hat{s}, s, y_{\hat{s}}^\Delta, y_s^\Delta)dW_s. \tag{14}$$

respectively, with $y_0 = x_0$ a.s.

Assumption 2.2 *Let the truncated versions $f_\Delta(s, r, x, y), g_\Delta(s, r, x, y)$ of f, g satisfy the following condition ($\phi_\Delta \equiv f_\Delta, g_\Delta$)*

$$|\phi_\Delta(s_1, r_1, x_1, y_1) - \phi_\Delta(s_2, r_2, x_2, y_2)| \le h(\Delta)\Big(|s_1 - s_2| + |r_1 - r_2| + |x_1 - x_2| + |y_1 - y_2|\Big)$$

for all $0 < \Delta \le 1$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, where $h(\Delta)$ is as in (11).

Let us also assume that the coefficients $a(t, x), b(t, x)$ of the original SDE satisfy the Khasminskii-type condition.

Assumption 2.3 *We assume the existence of constants $p \ge 2$ and $C_K > 0$ such that $x_0 \in \mathscr{L}^p(\Omega, \mathbb{R})$ and*

$$xa(t, x) + \frac{p-1}{2}b(t, x)^2 \le C_K(1 + |x|^2)$$

for all $(t, x) \in [0, T] \times \mathbb{R}$.

Under the local Lipschitz and the Khasminskii-type condition SDE (6) has a unique solution and finite moment bounds of order p, c.f. [24], i.e. for all $T > 0$, there exists a constant $A > 0$ such that $\sup_{0 \le t \le T} \mathbb{E}|x_t|^p < A$. We rewrite the main result [30, Th. 3.1].

Theorem 2 (Order of Strong Convergence) *Suppose Assumption 2.2 and Assumption 2.3 hold and (13) has a unique strong solution for every $n \le N - 1$, where $x_0 \in \mathscr{L}^p(\Omega, \mathbb{R})$ for some $p \ge 14 + 2\gamma$. Let $\epsilon \in (0, 1/3)$ and define for $\gamma > 0$*

$$\mu(u) = \overline{C}u^{1+\gamma}, \quad u \ge 0 \quad \textit{and} \quad h(\Delta) = \overline{C} + \sqrt{\ln \Delta^{-\epsilon}}, \quad \Delta \in (0, 1],$$

where $\Delta \leq 1$ and $\hat{h}$ are such that (11) holds. Then the semi-discrete numerical scheme (14) converges to the true solution of (6) in the $\mathscr{L}^2$-sense with order arbitrarily close to $1/2$, *that is*

$$\mathbb{E} \sup_{0 \leq t \leq T} |y_t^{\Delta} - x_t|^2 \leq C \Delta^{1-\epsilon}. \tag{15}$$

3 Applications of the Semi-discrete Method: Mathematical Finance

3.1 3/2-Model

Let us first consider the more general 3/2-model (2) with super-linear drift and diffusion coefficients, see [15, Sec. 4.1],

$$x_t = x_0 + \int_0^t (k_1(s) x_s - k_2(s) x_s^2) ds + \int_0^t k_3(s) x_s^{3/2} \phi(x_s) dW_s, \quad t \in [0, T], \tag{16}$$

where $\phi(\cdot)$ is a locally Lipschitz and bounded function with locally Lipschitz constant C_R^{ϕ}, bounding constant K_{ϕ}, x_0 is independent of all $\{W_t\}_{0 \leq t \leq T}, x_0 \in \mathscr{L}^{4p}(\Omega, \mathbb{R})$ for some $2 < p$ and $x_0 > 0$ a.s., $\mathbb{E}(x_0)^{-2} < A$, $k_1(\cdot), k_2(\cdot), k_3(\cdot)$ are positive and bounded functions with $k_{2,\min} > \frac{7}{2}(K_{\phi} k_{3,\max})^2$. It holds that $x_t > 0$ a.s. The following semi-discrete numerical scheme,

$$y_t = y_0 + \int_0^t (k_1(s) - k_2(s) y_{\hat{s}}) y_s ds + \int_0^t k_3(s) \sqrt{y_{\hat{s}}} \phi(y_{\hat{s}}) y_s dW_s, \tag{17}$$

where $\hat{s} = t_n$, when $s \in [t_n, t_{n+1})$, produces a linear SDE with solution

$$y_t = x_0 \exp\left\{ \int_0^t \left(k_1(s) - k_2(s) y_{\hat{s}} - k_3^2(s) \frac{y_{\hat{s}} \phi^2(y_{\hat{s}})}{2} \right) ds + \int_0^t k_3(s) \sqrt{y_{\hat{s}}} \phi(y_{\hat{s}}) dW_s \right\}, \tag{18}$$

where $y_t = y_t(t_0, x_0)$. We call (18) an exponential semi-discrete approximation of (16). The exponential semi-discrete numerical scheme (18) converges to the true solution of (16) in the mean square sense, is positive and has finite moments $\mathbb{E}(\sup_{0 \leq t \leq T} (y_t)^p)$ for appropriate p, see [15, Sec. 4.1]. See also the very recent work [17], a combination of the Lamperti transformation with the SD method, named LSD method.

3.2 CEV Process

The following SDE

$$x_t = x_0 + \int_0^t (k_1 - k_2 x_s)ds + \int_0^t k_3 (x_s)^q dW_s, \quad t \in [0, T], \tag{19}$$

where k_1, k_2, k_3 are positive and $1/2 < q < 1$ is known as a mean-reverting CEV process. Equation (19) may represent the instantaneous volatility or the instantaneous variance of the underlying financially observable. Here the diffusion coefficient is sub-linear. Feller's test implies that there is a unique non-explosive strong solution such that $x_t > 0$ a.s. when $x_0 > 0$ a.s. c.f. [22, Prop. 5.22]. The steady-state level of x_t is k_1/k_2 and the rate of mean-reversion is k_2.

Here we examine two versions of an implicit SD scheme that are solved explicitly. In [14], we propose

$$y_t = x_0 + \int_0^t (k_1 - k_2(1-\theta)y_{\hat{s}} - k_2\theta y_{\widetilde{s}})\, ds + k_3 \int_0^t (y_{\hat{s}})^{q-\frac{1}{2}} \sqrt{y_s} dW_s$$
$$+ \int_t^{t_{n+1}} \left(k_1 - k_2(1-\theta)y_{t_n} - \frac{(k_3)^2}{4(1+k_2\theta\Delta)} (y_{t_n})^{2q-1} - k_2\theta y_t \right) ds, \tag{20}$$

for $t \in (t_n, t_{n+1}]$ where

$$\hat{s} = t_j,\ s \in (t_j, t_{j+1}],\ j=0,\ldots,n, \qquad \widetilde{s} = \begin{cases} t_{j+1}, & \text{for } s \in [t_j, t_{j+1}], \\ t, & \text{for } s \in [t_n, t], \end{cases} \quad j=0,\ldots,n-1$$

and $\theta \in [0, 1]$ represents the level of implicitness. After rearranging

$$y_t(q) = y_n + \int_{t_n}^t \frac{(k_3)^2}{4(1+k_2\theta\Delta)^2} (y_{t_n})^{2q-1} ds + \frac{k_3}{1+k_2\theta\Delta} (y_{t_n})^{q-\frac{1}{2}} \int_{t_n}^t \mathrm{sgn}(z_s)\sqrt{y_s} dW_s, \tag{21}$$

with solution

$$y_t(q) = (z_t)^2, \quad z_t := \sqrt{y_n} + \frac{k_3}{2(1+k_2\theta\Delta)} (y_{t_n})^{q-\frac{1}{2}} (W_t - W_{t_n}), \tag{22}$$

where y_n is

$$y_n := y_{t_n}\left(1 - \frac{k_2\Delta}{1+k_2\theta\Delta}\right) + \frac{k_1\Delta}{1+k_2\theta\Delta} - \frac{(k_3)^2}{4(1+k_2\theta\Delta)^2}(y_{t_n})^{2q-1}\Delta.$$

The SD method (22) is positive by construction and under some conditions on the coefficients k_i, the level of implicitness θ and the step-size Δ, it strongly converges to the solution of (19) with a logarithmic rate if also $\mathbb{E}(x_0)^p < A$ for some $p \geq 4$

and with a polynomial rate of convergence of magnitude $\frac{1}{2}(q-\frac{1}{2})$ if $x_0 \in \mathbb{R}$, see [14, Th.1 and Th.2]. The other version of the implicit SD scheme, see [12], is written in each sub-interval,

$$\widetilde{y}_t(q) = \widetilde{y}_n + \int_{t_n}^{t} \frac{q(k_3)^2}{2}(\widetilde{y}_s)^{2q-1}ds + k_3 \int_{t_n}^{t} \mathrm{sgn}(\widetilde{z}_s)(\widetilde{y}_s)^q dW_s \tag{23}$$

with solution

$$\widetilde{y}_t(q) = |\widetilde{z}_t|^{1/(1-q)}, \quad \widetilde{z}_t := (\widetilde{y}_n)^{1-q} + k_3(1-q)(W_t - W_{t_n}), \tag{24}$$

where

$$\widetilde{y}_n := \widetilde{y}_{t_n}(1 - k_2\Delta) + k_1\Delta - \frac{q(k_3)^2\Delta}{2}(\widetilde{y}_{t_n})^{2q-1}.$$

The SD method (24) is again positive by construction and under some conditions on the coefficients k_i, the level of implicitness θ and the step-size Δ, it strongly converges to the solution of (19) with a polynomial rate of convergence of magnitude $q(q-\frac{1}{2})$ if $x_0 \in \mathbb{R}$. See also how LSD performs [17].

3.3 CIR/CEV Delay Models with Jump

Here we study a general model of type (19) including delay and jump terms. In particular we consider the following stochastic delay differential equation (SDDE) with jump,

$$x_t = \begin{cases} \xi_0 + \int_0^t (k_1 - k_2 x_{s^-})ds + \int_0^t k_3 b(x_{s-\tau}) x_{s^-}^{\alpha} dW_s + \int_0^t g(x_{s^-})d\widetilde{N}_s, & t \in [0, T], \\ \xi(t), & t \in [-\tau, 0], \end{cases} \tag{25}$$

where $x_{s^-} = \lim_{r \uparrow s} x_r$, the coefficient $b \in \mathscr{C}(\mathbb{R}_+, \mathbb{R}_+)$[1] and is assumed to be γ-Hölder continuous with $\gamma > 0$, the jump coefficient $g : \mathbb{R} \mapsto \mathbb{R}$ is assumed deterministic for simplicity, the function $\xi \in \mathscr{C}([-\tau, 0], (0, \infty))$ and $\tau > 0$ is a positive constant which represents the delay. Process $\widetilde{N}(t) = N(t) - \lambda t$ a compensated Poisson process with intensity $\lambda > 0$ independent of W_t. (25) has a unique and nonnegative solution and under some conditions on $\|\xi\|$ and the step-size Δ the following scheme strongly converges to the solution of (25) with polynomial or logarithmic rate, see [29],

[1] $\mathscr{C}(A, B)$ the space of continuous functions $\phi : A \mapsto B$ with norm $\|\phi\| = \sup_{u \in A} \phi(u)$.

$$\begin{cases} y_{t_{k+1}-} = (z_{t_{k+1}})^2, \\ y_{t_{k+1}} = y_{t_{k+1}-} + g(y_{t_{k+1}-})\Delta\widetilde{N}_k, \end{cases} \tag{26}$$

where

$$z_t = \sqrt{y_{t_k}\left(1 - \frac{k_2\Delta_k}{1+k_2\theta\Delta_k}\right) + \frac{k_1\Delta_k}{1+k_2\theta\Delta_k} - \frac{(k_3)^2}{4(1+k_2\theta\Delta_k)^2}\frac{b^2(y_{t_{k-\tau}})}{(1+b(y_{t_{k-\tau}})\Delta_k^m)^2}(y_{t_k})^{2\alpha-1}\Delta_k}$$
$$+ \frac{k_3}{2(1+k_2\theta\Delta_k)}\frac{b(y_{t_{k-\tau}})}{1+b(y_{t_{k-\tau}})\Delta_k^m}(y_{t_k})^{\alpha-\frac{1}{2}}(W_t - W_{t_k})$$

$y_t = \xi(t)$ when $t \in [-\tau, 0]$ and for $k = 0, 1, \ldots, n_T - 1$, and $\Delta_k = t_{k+1} - t_k$, $\Delta\widetilde{N}_k := \widetilde{N}(t_{k+1}) - \widetilde{N}(t_k) = \Delta N_k - \lambda\Delta_k$ and $\theta \in [0, 1]$ represents the level of implicitness, with $m = 1/4$. The SD scheme (26) combines the semi-discrete idea with a taming procedure. For the case $\alpha = 1/2$, known as the CIR model, where no delay and jump terms, see also [7, 11] and the application of the LSD method [17]. For extensions of the SD method to the two-factor CIR, see [10].

3.4 Aït-Sahalia Model

Let

$$x_t = x_0 + \int_0^t (\frac{a_1}{x_s} - a_2 + a_3x_s - a_4x_s^r)ds + \sigma\int_0^t x_s^\rho dW_s, \tag{27}$$

where $x_0 > 0$, the coefficients a_i are nonnegative and $r > 1, \rho > 1$. SDE (27), known as the *Aït-Sahalia* model, is used as an interest rate model and satisfies $x_t > 0$ a.s. The approximation of (27), by a combination of the splitting step method and the semi-discrete method, is proposed in [13]. In fact the SD approximation for the transformed process $z_t = x_t^2$ takes place first with dynamics given by

$$z_t = z_0 + \int_0^t (2a_1z_s - 2a_2\sqrt{z_s} + 2a_3z_s - 2a_4z_s^{(r+1)/2} + \sigma^2z_s^\rho)ds + 2\sigma\int_0^t z_s^{(\rho+1)/2}dW_s. \tag{28}$$

Splitting (28) in each subinterval with $t \in [t_n, t_{n+1}]$ as

$$z_1(t) = z_2(t_n) + \int_{t_n}^t (\ln(4/3)z_1(s) - 2a_2\sqrt{z_1(s)})ds \tag{29}$$

$$z_2(t) = z_1(t_{n+1}) + \int_{t_n}^t (2a_1 + (2a_3 - \ln(4/3))z_2(s) - 2a_4z_2^{(r+1)/2}(s) + \sigma^2z_2^\rho(s))ds$$
$$+ 2\sigma\int_{t_n}^t z_2^{(\rho+1)/2}(s)dW_s, \tag{30}$$

where $z_2(0) = x_0$ suggests that we may take the solution of (29)

$$z_1(t) = \left(\frac{2a_2}{\ln(4/3)} + \left(\sqrt{z_2(t_n)} - \frac{2a_2}{\ln(4/3)} \right) \left(\frac{4}{3} \right)^{(t-t_n)/2} \right)^2 \tag{31}$$

and approximate (30) with

$$\begin{aligned} \widetilde{z}_2(t) &= z_1(t_{n+1}) + 2a_1 \Delta \\ &\quad + \int_{t_n}^{t} \left(2a_3 - \ln(4/3) - 2a_4 \widetilde{z}_2^{(r-1)/2}(\hat{s}) + \sigma^2 \widetilde{z}_2^{(\rho-1)/2}(\hat{s}) \right) \widetilde{z}_2(s) ds \\ &\quad + 2\sigma \int_{t_n}^{t} \widetilde{z}_2^{(\rho-1)/2}(\hat{s}) \widetilde{z}_2(s) dW_s. \end{aligned} \tag{32}$$

We end up with the following SD numerical scheme for the transformed process z_t

$$\begin{aligned} \widetilde{z}_{n+1} &= \left(2a_1 \Delta + \left(\frac{2a_2}{\ln(4/3)} + \left(\sqrt{\widetilde{z}_n} - \frac{2a_2}{\ln(4/3)} \right) \left(\frac{4}{3} \right)^{\Delta/2} \right)^2 \right) \\ &\quad \times \exp\{(2a_3 - \ln(4/3) - 2a_4 \widetilde{z}_n^{(r-1)/2} - \sigma^2 \widetilde{z}_n^{\rho-2}) \Delta + 2\sigma \widetilde{z}_n^{(\rho-1)/2} \Delta W_n\} \end{aligned} \tag{33}$$

and then take $y_n = \sqrt{\widetilde{z}_n}$ for the approximation of the original Aït-Sahalia model, which is positive, strongly convergent with finite moment bounds, when $r+1 > 2\rho$, with $\rho \geq 2$, see [13]. See also the performance of LSD [17].

4 Applications of the Semi-discrete Method: Population Dynamics and Biology

4.1 Wright-Fisher Model

The next class of SDEs appears in population dynamics to describe fluctuations in gene frequency of reproducing individuals among finite populations [5] and ion channel dynamics within cardiac and neuronal cells, (cf. [3, 4, 6] and references therein),

$$x_t = x_0 + \int_0^t (k_1 - k_2 x_s) ds + k_3 \int_0^t \sqrt{x_s(1 - x_s)} dW_s, \tag{34}$$

where $k_i > 0, i = 1, 2, 3$. If $x_0 \in (0, 1)$ and $(k_1 \wedge (k_2 - k_1)) \geq (k_3)^2/2$, then $0 < x_t < 1$ a.s. The process

$$y_t = y_{t_n} + \int_{t_n}^{t_{n+1}} \left(k_1 - \frac{(k_3)^2}{4} + y_{t_n}\left(\frac{(k_3)^2}{2} - k_2\right)\right) ds + \int_{t_n}^{t} \frac{(k_3)^2}{4}(1-2y_s)ds$$
$$+k_3 \int_{t_n}^{t} \sqrt{y_s(1-y_s)}\,\mathrm{sgn}(z_s)dW_s, \quad (35)$$

for $t \in (t_n, t_{n+1}]$, with $y_0 = x_0$ a.s. and $z_t = \sin\left(k_3 \Delta W_n^t + 2\arcsin(\sqrt{y_n})\right)$, where $y_n := y_{t_n} + \left(k_1 - \frac{(k_3)^2}{4} + y_{t_n}\left(\frac{(k_3)^2}{2} - k_2\right)\right) \cdot \Delta$ has the following solution

$$y_t = \sin^2\left(\frac{k_3}{2}\Delta W_n^t + \arcsin(\sqrt{y_n})\right), \quad (36)$$

which has the pleasant feature that $y_t \in (0,1)$ when $y_0 \in (0,1)$. Process (36) is well defined when $0 < y_n < 1$, which is achieved for appropriate Δ. To simplify conditions on the parameters and the step size d we may adopt the strategy presented in [28] considering a perturbation of order Δ in the initial condition. Here we used an additive discretization of the drift coefficient and the eternal life time SD scheme (36) strongly converges to the solution of (34), see [28]. Moreover, in [28], an application of the SD method to an extension of the Wright-Fisher model to the multidimensional case is treated, producing a strongly converging and boundary preserving scheme.

4.2 Predator-Prey Model

The following system of SDEs, c.f. [20],

$$X_t^{(1)} = X_0^{(1)} + \int_0^t (aX_s^{(1)} - bX_s^{(1)}X_s^{(2)})ds + \int_0^t k_1 X_s^{(1)} dW_s^{(1)},$$
$$X_t^{(2)} = X_0^{(2)} + \int_0^t (cX_s^{(1)}X_s^{(2)} - dX_s^{(2)})ds + \int_0^t k_2 X_s^{(2)} dW_s^{(2)},$$

where $a, b, c, d > 0$ and $k_1, k_2 \in \mathbb{R}$ with independent Brownian motions $W_t^{(1)}, W_t^{(2)}$ was studied in [9]. Under some moment bound conditions for $(X_t^{(i)}), i = 1, 2$ and when $X_0^{(1)} > 0$ and $X_0^{(2)} > 0$ then $X_t^{(1)} > 0$ and $X_t^{(2)} > 0$ a.s. Transforming the second equation $Z_t^{(2)} = \ln(X_t^{(2)})$ produces the following system

$$X_t^{(1)} = X_0^{(1)} + \int_0^t (a - be^{Z_s^{(2)}})X_s^{(1)} ds + \int_0^t k_1 X_s^{(1)} dW_s^{(1)},$$
$$Z_t^{(2)} = Z_0^{(2)} + \int_0^t (cX_s^{(1)} - d - (k_2)^2)ds + k_2 W_t^{(2)},$$

which is approximated by the following SD scheme

$$Y_t^{(1)} = X_0^{(1)} + \int_0^t (a - be^{Y_{\hat{s}}^{(2)}}) Y_s^{(1)} ds + \int_0^t k_1 Y_s^{(1)} dW_s^{(1)},$$

$$Y_t^{(2)} = Y_0^{(2)} + \int_0^t (cY_{\hat{s}}^{(1)} - d - (k_2)^2) ds + k_2 W_t^{(2)},$$

which reads

$$Y_{t_{n+1}}^{(1)} = Y_{t_n}^{(1)} \exp\{(a - be^{Y_{t_n}^{(2)}} - \frac{(k_1)^2}{2})\Delta + k_1 \Delta W_n^{(1)}\}$$

$$Y_{t_{n+1}}^{(2)} = Y_{t_n}^{(2)} + (cY_{t_n}^{(1)} - d - (k_2)^2)\Delta + k_2 \Delta W_n^{(2)}.$$

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