Canonical Systems of Partial Differential Equations

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Abstract We use critical point theory to find solutions of the nonlinear steady state Schrödinger equations arising in the study of photonic lattices.

1 Introduction

Systems of partial differential equations arise in many investigations in the physical sciences. Depending on the application and on the questions asked, different types of systems emerge. Usually, if one is interested in finding steady states solutions, the resulting system is elliptic in nature. Such systems may display severe difficulties when one tries to solve them. Most of the time they admit a trivial solution, where all of the unknown functions are identically zero. However, the physical application requires a solution which is not identically zero. In such cases, the methods of solution may be very difficult. In particular, one has to show that the solution obtained is not trivial. The system that we study is not only deceptive, but it is almost impossible to tell if one has solved the whole system or only parts of the system. I call it "canonical." I shall elaborate on this later.

Many general systems are the form

$$\mathscr{A}v = f(x, v, w), \quad x \in Q \subset \mathbb{R}^n, \tag{1}$$

$$\mathscr{B}w = g(x, v, w), \quad x \in Q \subset \mathbb{R}^n, \tag{2}$$

where $\mathscr{A}, \mathscr{B}$ are linear partial differential operators. I call this system "deceptive" if $(v, 0)$ is a solution of (1) and (2) whenever v satisfies

$$\mathscr{A}v = f(x, v, 0), \quad x \in Q \subset \mathbb{R}^n, \tag{3}$$

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or $(0, w)$ is a solution whenever w satisfies

$$\mathscr{B}w = g(x, 0, w), \quad x \in Q \subset \mathbb{R}^n. \tag{4}$$

In this case it is very difficult to determine if both components of a solution are nontrivial.

The particular system I have chosen consist of nonlinear Schrödinger equations arising in optics (cf. [16]) describing the propagation of a light wave in induced photonic lattices. They can be written in the form

$$iV_t + \Delta V = \frac{PV}{1 + |V|^2 + |W|^2}$$

$$iW_t + \Delta W = \frac{PW}{1 + |V|^2 + |W|^2}$$

for the periodic wave functions $V(x, t)$, $W(x, t)$ over a periodic bounded spacial domain $\Omega \subset \mathbb{R}^2$, where P, Q are parameters (cf. [2, 21]). To find a steady state solution, we look for solutions of the form

$$V(x, t) = e^{i\lambda t} v(x), \quad W(x, t) = e^{i\lambda t} w(x),$$

where λ is a real constant. This leads to the following system of equations over a periodic domain $\Omega \subset \mathbb{R}^2$:

$$\Delta v = \frac{Pv}{1 + v^2 + w^2} + \lambda v, \tag{5}$$

$$\Delta w = \frac{Qw}{1 + v^2 + w^2} + \lambda w, \tag{6}$$

where P, Q, λ are parameters. The solutions v, w are to be periodic in Ω with the same periods. One wishes to obtain intervals of the parameter λ for which there are nontrivial solutions. This will provide continuous energy spectrum that allows the existence of steady state solutions. This system was studied in [2], where it was shown that

1. If P, Q, λ are all positive, then the only solution is trivial.
2. If $P < 0$ and $0 < \lambda < -P$, then the system (5) and (6) has a nontrivial solution.
3. If $P, Q > 0$, there is a constant $\delta > 0$ such that the system (5) and (6) has a nontrivial solution provided $0 < -\lambda < \delta$.
4. All of these statements are true if we replace P by Q.

Wave propagation in nonlinear periodic lattices has been studied by many reseachers (cf., e.g., [1–11, 17, 20–23] and their bibliographies.)

In the present paper, we wish to cover some remaining situations not mentioned in [2] as well as extending their results to higher dimensions. We shall show that there are many intervals of the parameters in which nontrivial solutions exist. Our results are true in any dimension.

In stating our results, we shall make use of the following considerations. Let Ω be a bounded periodic domain in $\mathbb{R}^n$, $n \geq 1$. Consider the operator $-\Delta$ on functions in $L^2(\Omega)$ having the same periods as Ω. The spectrum of $-\Delta$ consists of isolated eigenvalues of finite multiplicity:

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_\ell < \cdots ,$$

with eigenfunctions in $L^\infty(\Omega)$. Let λ_ℓ, $\ell \geq 0$, be one of these eigenvalues, and define

$$N = \bigoplus_{\lambda \leq \lambda_\ell} E(\lambda), \quad M = N^\perp.$$

As noted in [2], to prove the existence of a nontrivial solution of system (5) and (6), it suffices to obtain a nontrivial solution of either

$$\Delta v = \frac{Pv}{1+v^2} + \lambda v, \tag{7}$$

or

$$\Delta w = \frac{Qw}{1+w^2} + \lambda w. \tag{8}$$

This stems from the fact that $(v, 0)$ is a solution of (5) and (6) if v is a solution of (7) and $(0, w)$ is a solution of (5) and (6) if w is a solution of (8). The author is unaware if such solutions are desirable from the physical point of view. However, we have been able to find values of P, Q, λ for which the system (5) and (6) has a solution (v, w) where $v \neq 0, w \neq 0$.

We shall prove

Theorem 1 *If* $0 < \lambda < -P$ *or* $0 < \lambda < -Q$, *then (5) and (6) has a nontrivial solution.*

Theorem 2 *If* $0 < -\lambda < P$ *or* $0 < -\lambda < Q$, *then (5) and (6) has a nontrivial solution.*

Theorem 3 *If* $P > 0, Q > 0,\ \sigma = -\lambda > 0$, *and either* $0 \leq \sigma - P < \lambda_1 < \sigma$ *or* $0 \leq \sigma - Q < \lambda_1 < \sigma$, *then (5) and (6) has a nontrivial solution.*

Theorem 4 *If* $P > 0, Q > 0,\ \sigma = -\lambda > 0$, *and either* $\lambda_\ell \leq \sigma - P < \lambda_{\ell+1} < \sigma$ *or* $\lambda_\ell \leq \sigma - Q < \lambda_{\ell+1} < \sigma$ *then (5) and (6) has a nontrivial solution.*

Theorem 5 *If* $P > 0,\ Q = 0,\ \lambda = -\lambda_\ell < 0,\ -\Delta w = \lambda_\ell w$ *and*

$$\lambda_\ell < P \int_\Omega \frac{1}{1+w^2}/|\Omega|,$$

then (5) and (6) has a nontrivial solution. If $w \neq 0$, then the solution has both components nonzero.

Theorem 6 *If $Q > 0,\ P = 0,\ \lambda = -\lambda_\ell < 0,\ -\Delta v = \lambda_\ell v$ and*

$$\lambda_\ell < Q \int_\Omega \frac{1}{1+v^2}/|\Omega|,$$

then (5) and (6) has a nontrivial solution. If $v \neq 0$, then the solution has both components nonzero.

2 Some Lemmas

In proving our results we shall make use of the following lemmas (cf., e.g., [12, 14, 15, 18]). For the definition of linking, cf. [12].

Lemma 1 *Let M, N be closed subspaces of a Hilbert space E such that one of them is finite dimensional and $E = M \oplus N$. Take $B = \partial\mathscr{B}_\delta \cap M$, and let w_0 be any element in $\partial\mathscr{B}_1 \cap M$. Take A to be the set of all u of the form*

$$u = v + sw_0, \quad v \in N,\ s \in \mathbb{R},$$

satisfying the following

(a) $\|v\|_E \leq R,\ s = 0$
(b) $\|v\|_E \leq R,\ s = 2R_0$
(c) $\|v\|_E = R,\ 0 \leq s \leq 2R_0,$

where $0 < \delta < \min(R, R_0)$. Then A and B link each other.

Lemma 2 *The sets $\|u\|_E = R > 0$ and $\{e_1, e_2\}$ link each other provided $\|e_1\|_E < R$ and $\|e_2\|_E > R$.*

Lemma 3 *If $G(u) \in C^1(E, \mathbb{R})$ satisfies*

$$\alpha = \inf_E G > -\infty, \tag{9}$$

then there is a sequence $\{u_k\}$ such that

$$G(u_k) \to \alpha, \quad (1 + \|u_k\|_E)\|G'(u_k)\| \to 0. \tag{10}$$

Lemma 4 *If A links B, and $G(u) \in C^1(E, \mathbb{R})$ satisfies*

$$a_0 = \sup_A G \leq b_0 = \inf_B G, \tag{11}$$

then there is a sequence $\{u_k\}$ *such that*

$$G(u_k) \to c \geq b_0, \quad (1 + \|u_k\|_E)\|G'(u_k)\| \to 0. \tag{12}$$

We let E be the subspace of $H^{1,2}(\Omega)$ consisting of those functions having the same periodicity as Ω with norm given by

$$\|w\|_E^2 = \|\nabla w\|^2 + \|w\|^2.$$

Assume $P \neq 0$, $Q \neq 0$, $\lambda \neq 0$. Let

$$a(u) = \frac{1}{P}[\ \|\nabla v\|^2 + \lambda\ \|v\|^2] + \frac{1}{Q}[\ \|\nabla w\|^2 + \lambda\ \|w\|^2], \quad v, w \in E \tag{13}$$

and

$$G(u) = a(u) + \int_\Omega \ln(1 + u^2)\, dx. \tag{14}$$

We have

Lemma 5 *If G(u) is given by (14), then every sequence satisfying (10) has a subsequence converging in E. Consequently, there is a* $u \in E$ *such that G(u)=c and* $G'(u) = 0$.

Proof The sequence satisfies

$$G(u_k) = \frac{1}{P}\|\nabla v_k\|^2 + \frac{\lambda}{P}\|v_k\|^2 + \frac{1}{Q}\|\nabla w_k\|^2 + \frac{\lambda}{Q}\|w_k\|^2 + \int_\Omega \ln\{1 + |u_k|^2\}\, dx \to c, \tag{15}$$

$$(G'(u_k), q)/2 = \frac{1}{P}(\nabla v_k, \nabla g) + \frac{\lambda}{P}(v_k, g) + \frac{1}{Q}(\nabla w_k, \nabla h) + \frac{\lambda}{Q}(w_k, h) + \int_\Omega \frac{u_k q}{1 + u_k^2}\, dx \to 0, \quad q = (g, h), \tag{16}$$

$$(G'(u_k), v_k)/2 = \frac{1}{P}(\nabla v_k, \nabla v_k) + \frac{\lambda}{P}(v_k, v_k) \tag{17}$$

$$+\int_\Omega \frac{u_k v_k}{1+u_k^2}\,dx \to 0.$$

and

$$(G'(u_k), w_k)/2 = \frac{1}{Q}(\nabla w_k, \nabla w_k) + \frac{\lambda}{Q}(w_k, w_k) \\ + \int_\Omega \frac{u_k w_k}{1+u_k^2}\,dx \to 0. \tag{18}$$

Thus,

$$\int_\Omega H(x, u_k)\,dx \to c, \tag{19}$$

where

$$H(x,t) = \ln(1+t^2) - \frac{t^2}{1+t^2}. \tag{20}$$

Let $\rho_k = \|u_k\|_H$, where

$$\|u\|_H^2 = \frac{1}{|P|}[\|\nabla v\|^2 + |\lambda|\ \ \|v\|^2] \\ + \frac{1}{|Q|}[\|\nabla w\|^2 + |\lambda|\ \ \|w\|^2], \quad u = (v, w) \in E. \tag{21}$$

Assume first that $\rho_k \to \infty$. Let $\tilde{u}_k = u_k/\rho_k$. Then $\|\tilde{u}_k\|_H = 1$. Hence, there is a renamed subsequence such that $\tilde{u}_k \rightharpoonup \tilde{u}$ in E, and $\tilde{u}_k \to \tilde{u}$ in $L^2(\Omega)$ and a.e. Now

$$\|u_k\|_H^2 = \frac{1}{|P|}[\|\nabla v_k\|^2 + |\lambda|\ \|v_k\|^2] + \frac{1}{|Q|}[\|\nabla w_k\|^2 + |\lambda|\ \|w_k\|^2]. \tag{22}$$

By (17) and (18),

$$\|u_k\|_H^2 \le |(G'(u_k), v_k)|/2 + |(G'(u_k), w_k)|/2 \\ + \frac{|\lambda| - \lambda}{|P|}\|v_k\|^2 + \frac{|\lambda| - \lambda}{|Q|}\|w_k\|^2 \\ + \int_\Omega \frac{u_k^2}{1+u_k^2}\,dx.$$

Hence,

$$1 = \|\tilde{u}_k\|_H^2 \leq [|(G'(u_k), v_k)|/2 + |(G'(u_k), w_k)|/2]/\rho_k^2 + C\|\tilde{u}_k\|^2. \tag{23}$$

In the limit we have,

$$1 \leq C\|\tilde{u}\|^2.$$

This shows that $\tilde{u} \not\equiv 0$. Let Ω_0 be the subset of Ω where $\tilde{u}(x) \neq 0$. Then $|\Omega_0| \neq 0$. Thus

$$\begin{aligned}\int_\Omega H(x, u_k)\, dx &= \int_{\Omega_0} H(x, u_k)\, dx + \int_{\Omega \setminus \Omega_0} H(x, u_k)\, dx \\ &\geq \int_{\Omega_0} H(x, u_k)\, dx \to \infty.\end{aligned}$$

This contradicts (19). Thus, the sequence satisfying (10) is bounded in E. Hence, there is a renamed subsequence such that $u_k \rightharpoonup u_0$ in E, and $u_k \to u_0$ in $L^2(\Omega)$ and a.e. Taking the limit in (17), we obtain

$$\begin{aligned}(G'(u_0), q)/2 =& \frac{1}{P}(\nabla v_0, \nabla g) + \frac{\lambda}{P}(v_0, g) \\ &+ \frac{1}{Q}(\nabla w_0, \nabla h) + \frac{\lambda}{Q}(w_0, h) \\ &+ \int_\Omega \frac{u_0 q}{1 + u_0^2}\, dx = 0, \quad q = (g, h),\end{aligned} \tag{24}$$

Thus, u_0 satisfies $G'(u_0) = 0$. Since $u_0 \in E$, it satisfies

$$\begin{aligned}(G'(u_0), u_0)/2 =& \frac{1}{P}(\nabla v_0, \nabla v_0) + \frac{\lambda}{P}(v_0, v_0) \\ &+ \frac{1}{Q}(\nabla w_0, \nabla w_0) + \frac{\lambda}{Q}(w_0, w_0) \\ &+ \int_\Omega \frac{u_0^2}{1 + u_0^2}\, dx = 0\end{aligned} \tag{25}$$

Also, from the limit in (17), we have

$$\begin{aligned}\lim \frac{1}{P}\|\nabla v_k\|^2 =& \lim (G'(u_k), v_k)/2 \\ &- \lim[\frac{\lambda}{P}\|v_k\|^2 + \int_\Omega \frac{v_k^2}{1 + u_k^2}\, dx]\end{aligned}$$

$$= -\left[\frac{\lambda}{P}\|v\|^2 + \int_\Omega \frac{v^2}{1+u^2}\,dx\right]$$
$$= \frac{1}{P}\|\nabla v\|^2,$$

with a similar statement for $\|\nabla w\|^2$. Consequently, $\nabla u_k \to \nabla u$ in $L^2(\Omega)$. This shows that $G(u_k) \to G(u_0)$. Hence, $G(u_0) = c$.

Lemma 6 *If $G'(u) = 0$, then (v,w) is a solution of (5) and (6).*

Proof From (24) we see that

$$|(\nabla u, \nabla q)| \le C\|q\|, \quad q \in E.$$

From the fact that the functions and Ω are periodic with the same period, it follows that $u \in H^{2,2}(\Omega)$ and satisfies (5) and (6) (cf., e.g., [13]).

Lemma 7

$$\int_\Omega \ln(1+u^2)dx/\|u\|_H^2 \to 0, \quad \|u\|_H \to \infty. \tag{26}$$

Proof Suppose $u_k \in H$ is a sequence such that $\rho_k = \|u_k\|_H \to \infty$. Let $\tilde{u}_k = u_k/\rho_k$. Then $\|\tilde{u}_k\|_H = 1$. Hence, there is a renamed subsequence such that $\tilde{u}_k \rightharpoonup \tilde{u}$ in H, and $\tilde{u}_k \to \tilde{u}$ in $L^2(\Omega)$ and a.e. Now

$$\frac{\ln(1+u_k^2)}{\rho_k^2} = \frac{\ln(1+u_k^2)}{u_k^2}\tilde{u}_k^2 \to \ 0 \ a.e.$$

and it is dominated a.e. by $\tilde{u}_k^2 \to \tilde{u}^2$ in $L^1(\Omega)$. Thus

$$\int_\Omega \frac{\ln(1+u_k^2)}{\rho_k^2}dx \to \ 0.$$

Since this is true for any sequence satisfying $\|u_k\|_H \to \infty$, we see that (26) holds.

Corollary 1 *If*

$$I(u) = \|u\|_H^2 - \int_\Omega \ln(1+u^2)\,dx,$$

then

$$I(v) \to \infty \ as \ \|v\|_H \to \infty. \tag{27}$$

Proof We have

$$I(u)/\|u\|_H^2 = 1 - \int_\Omega \ln(1+u^2)dx/\|u\|_H^2 \to 1, \quad \|u\|_H \to \infty$$

by Lemma 7. This gives (27).

Lemma 8

$$\int_\Omega [u^2 - \ln(1+u^2)]dx/\|u\|_H^2 \to 0, \quad \|u\|_H \to 0. \tag{28}$$

Proof Suppose $u_k \in H$ is a sequence such that $\rho_k = \|u_k\|_H \to 0$. In particular, there is a renamed subsequence such that $u_k \to 0$ a.e. Let $\tilde{u}_k = u_k/\rho_k$. Then $\|\tilde{u}_k\|_H = 1$. Hence, there is a renamed subsequence such that $\tilde{u}_k \rightharpoonup \tilde{u} \in H$, and $\tilde{u}_k \to \tilde{u}$ in $L^2(\Omega)$ and a.e. Now

$$\frac{u_k^2 - \ln(1+u_k^2)}{\rho_k^2} \le \frac{u_k^2}{1+u_k^2}\tilde{u}_k^2 \to \ 0 \ a.e.$$

and it is dominated a.e. by $\tilde{u}_k^2 \to \tilde{u}^2$ in $L^1(\Omega)$. Thus

$$\int_\Omega \frac{u_k^2 - \ln(1+u_k^2)}{\rho_k^2} dx \to \ 0.$$

Since this is true for any sequence satisfying $\|u_k\|_H \to 0$, we see that (28) holds.

3 Proofs of the Theorems

Proof of Theorem 1 We let E be the subspace of $H^{1,2}(\Omega)$ consisting of those functions having the same periodicity as Ω with norm given by

$$\|w\|_E^2 = \|\nabla w\|^2 + \|w\|^2.$$

Let $u = (v, w)$, where $v, w \in E$ and $u^2 = v^2 + w^2$. If $q = (g, h)$, we write $uq = vg + wh$. Define

$$\|u\|_H^2 = \frac{1}{|P|}[\|\nabla v\|^2 + |\lambda| \ \ \|v\|^2] + \frac{1}{|Q|}[\|\nabla w\|^2 + |\lambda| \ \ \|w\|^2], \quad v, w \in E. \tag{29}$$

Assume that P, Q, λ do not vanish. Then $\|u\|_H^2$ is a norm on $H = E \times E$ having a scalar product $(u, h)_H$.

Let

$$I(u) = \|u\|_H^2 - \int_\Omega \ln(1+u^2)\,dx. \tag{30}$$

Then,

$$(I'(u), q)/2 = (u,q)_H - \int_\Omega \frac{uq}{1+u^2}\,dx, \quad q \in H. \tag{31}$$

If $I'(u) = 0$, then

$$\Delta v = \frac{-|P|v}{1+|v|^2+|w|^2} + |\lambda|v, \tag{32}$$

$$\Delta w = \frac{-|Q|w}{1+|v|^2+|w|^2} + |\lambda|w. \tag{33}$$

This is equivalent to (5) and (6) if $P < 0$, $Q < 0$, $\lambda > 0$. To prove the theorem, we must show that there is a nontrivial solution of $I'(u) = 0$ when either $0 < \lambda < -P$ or $0 < \lambda < -Q$.

Assume $0 < \lambda < -P$. We show that $I(u)$ has a minimum $u \neq 0$.

Let the sequence $u_k \in H$ satisfy

$$I(u_k) \searrow \alpha = \inf_H I$$

(which may be $-\infty$). By (27), $\rho_k = \|u_k\|_H$ is bounded. Hence, there is a renamed subsequence such that $u_k \rightharpoonup u_0$ in H, and $u_k \to u_0$ in $L^2(\Omega)$ and a.e. Since

$$\|u_k\|_H^2 - 2([u_k - u_0], u_0)_H = \|u_0\|_H^2 + \|u_k - u_0\|_H^2,$$

we have

$$\begin{aligned} I(u_0) &\leq \|u_k\|_H^2 - 2([u_k - u_0], u_0)_H \\ &- \int_Q \ln(1+u_0^2)dx \\ &= I(u_k) - 2([u_k - u_0], u_0)_H \\ &- \int_Q [\ln(1+u_0^2) - \ln(1+u_k^2)]dx \\ &\to \ \alpha. \end{aligned}$$

Thus,

$$\alpha \leq I(u_0) \leq \alpha,$$

showing that α is finite and that u_0 is a minimum. Thus, $I'(u_0) = 0$ and u_0 is a solution of

$$\Delta v = \frac{-|P|v}{1 + |v|^2 + |w|^2} + \lambda v, \tag{34}$$

$$\Delta w = \frac{-|Q|w}{1 + |v|^2 + |w|^2} + \lambda w. \tag{35}$$

Next, we show that $u_0 \neq 0$. We do this by showing that $\alpha < 0$. Consider a constant function $u = (s, 0)$. Then,

$$I(u) = [\frac{\lambda}{|P|}s^2 - \ln(1 + s^2)]|\Omega|, \quad s \in \mathbb{R}.$$

This has a negative minimum if $\lambda < |P|$. Thus $I(u_0) = \alpha < 0$. Since $I(0, 0) = 0$, we see that $u_0 \neq 0$. However, u_0 satisfies (34) and (35), not (5) and (6). To rectify the situation, we merely note that the same method produces a negative minimum v_0 for $I(v, 0)$, and $(v_0, 0)$ is a nontrivial solution of (5) and (6). This completes the proof for the case $0 < \lambda < -P$. The case $0 < \lambda < -Q$ is treated similarly.

Proof of Theorem 2 Assume $0 < \sigma < P,\ 0 < \sigma < Q$, and let $a(u)$ and $G(u)$ be given by (13) and (14), respectively. Then $G'(u) = 0$ iff $u = (v, w)$ is a solution of (5) and (6). We search for a nontrivial solution.

Let $\rho_k = \|u_k\|_H$, where

$$\|u\|_H^2 = \frac{1}{|P|}[\|\nabla v\|^2 + |\lambda| \ \ \|v\|^2] + \frac{1}{|Q|}[\|\nabla w\|^2 + |\lambda| \ \ \|w\|^2], \quad u = (v, w) \in E. \tag{36}$$

Assume that $\rho_k \to 0$. Let $\tilde{u}_k = u_k/\rho_k$. Then $\|\tilde{u}_k\|_H = 1$. Hence, there is a renamed subsequence such that $\tilde{u}_k \rightharpoonup \tilde{u}$ in E, and $\tilde{u}_k \to \tilde{u}$ in $L^2(\Omega)$ and a.e. We have

$$G(u_k)/\rho_k^2 = \frac{1}{P}\|\nabla \tilde{v}_k\|^2 + \frac{\lambda + P}{P}\|\tilde{v}_k\|^2 + \frac{1}{Q}\|\nabla \tilde{w}_k\|^2 + \frac{\lambda + Q}{Q}\|\tilde{w}_k\|^2 + \int_\Omega [\ln\{1 + |u_k|^2\} - u_k^2]\, dx/\rho_k^2.$$

Since $P > \sigma$ and $Q > \sigma$, we see in view of Lemma 7 that there are positive constants ε, η such that

$$G(u)/\|u\|_H^2 \geq \varepsilon, \quad \|u\|_H \leq \eta.$$

Let A be the set of those $u \in H$ such that $\|u\|_H = \eta$. Consider a constant function $u = (s, 0)$. Then,

$$G(u)/s^2 = [\frac{\lambda}{P} + s^{-2}\ln(1+s^2)]|\Omega| \to \frac{\lambda}{P}|\Omega| < 0, \quad s \to \infty.$$

Hence, there is a $u \in H$ such that $\|u\|_H > \eta$ and $G(u) < \varepsilon\eta^2$. Since $G(0,0) = 0$, there is a $u \in H$ such that $\|u\|_H < \eta$ and $G(u) < \varepsilon\eta^2$. The theorem now follows from Lemmas 2, 4, and 5.

Proof of Theorem 3 Assume $P > 0, Q > 0, \lambda < 0$ and $\sigma = -\lambda > \max[P, \lambda_1]$. Let

$$a(u) = \frac{1}{P}[\ \|\nabla v\|^2 - \sigma\ \|v\|^2] + \frac{1}{Q}[\ \|\nabla w\|^2 - \sigma\ \|w\|^2], \quad v, w \in E \tag{37}$$

and

$$G(u) = a(u) + \int_\Omega \ln(1+u^2)\,dx. \tag{38}$$

Then $G'(u) = 0$ iff u satisfies (5) and (6).

First, we note that

$$G(u) \le 0, \quad u \in N,$$

if $\sigma \ge P,\ \sigma \ge Q$. To see this, let $u = (c, d) \in N$. Then

$$a(c,d) = -\frac{\sigma}{P}c^2|\Omega| - \frac{\sigma}{Q}d^2|\Omega|$$

and

$$\int_\Omega \ln(1+c^2+d^2)dx \le (c^2+d^2)|\Omega|.$$

Thus,

$$G(u) \le [1 - \frac{\sigma}{P}]c^2|\Omega| + [1 - \frac{\sigma}{Q}]d^2|\Omega|.$$

This means that

$$G(u) \le 0, \quad u \in N, \tag{39}$$

provided $\sigma \ge P,\ \sigma \ge Q$.

Next, let ψ be an eigenfunction of $-\Delta$ corresponding to the eigenvalue λ_1. If we take $u = (\psi + c, \psi + d)$, we have

$$a(u) = \frac{1}{P}[\ \|\nabla(\psi + c)\|^2 - \sigma\ \|(\psi + c)\|^2] + \frac{1}{Q}[\ \|\nabla(\psi + d)\|^2 - \sigma\ \|(\psi + d)\|^2],$$

and this gives

$$a(u) = \frac{1}{P}[(\lambda_1 - \sigma)\ \|\psi\|^2 - \sigma c^2] + \frac{1}{Q}[(\lambda_1 - \sigma)\ \|\psi\|^2 - \sigma d^2],$$

which will be negative if $\sigma > \lambda_1$. Moreover,

$$\int_\Omega \ln(1 + 2\psi^2 + c^2 + d^2)dx/\|u\|_H^2 \to 0, \quad \|u\|_H \to \infty.$$

This follows from the fact that

$$\int_\Omega \ln(1 + u^2)dx/\|u\|_H^2 \to 0, \quad \|u\|_H \to \infty \tag{40}$$

(Lemma 7). Consequently,

$$\limsup_{\|(\psi+c,\psi+d)\|_H \to \infty} G(\psi + c, \psi + d) < 0 \tag{41}$$

provided $\sigma > \lambda_1$.

Next, let $u = (v, w)$ be any function in M. Then $\|\nabla u\|^2 = \|\nabla v\|^2 + \|\nabla w\|^2 \geq \lambda_1\|v\|^2 + \lambda_1\|w\|^2 = \lambda_1\|u\|^2$. Then

$$a(u) + \|u\|^2 \geq \frac{1}{P}[1 - \frac{\sigma - P}{\lambda_1}]\ \|\nabla v\|^2 + \frac{1}{Q}[1 - \frac{\sigma - Q}{\lambda_1}]\ \|\nabla w\|^2.$$

Thus, there is an $\varepsilon > 0$ such that

$$a(u) + \|u\|^2 \geq 2\varepsilon\|\nabla u\|^2, \quad u \in M, \tag{42}$$

when $\sigma - \lambda_1 < \min[P, Q]$.

Now

$$\int_\Omega [u^2 - \ln(1 + u^2)]dx/\|u\|_H^2 \to 0, \quad \|u\|_H \to 0 \tag{43}$$

by Lemma 8. If we combine (42) and (43), we see that there is an $\varepsilon > 0$ such that

$$G(u) \geq \varepsilon \|\nabla u\|^2, \quad u \in M, \tag{44}$$

when $\|\nabla u\|^2$ is small and $\sigma - \lambda_1 < \min[P, Q]$.

Take $A = \partial(N \oplus \{\psi\})$, $B = \partial\mathscr{B}_\delta \cap M$. By (39), (41), and (44) one can apply Lemma 1 to obtain (10) and then Lemma 5 to conclude that (5) and (6) has a nontrivial solution. To see this, note that $a_0 = 0 < \varepsilon \leq b_0$, showing that the solution u_0 satisfies $G(u_0) \geq \varepsilon > 0$. Since $G(0) = 0$, we see that $u_0 \neq 0$. If $\max[P, \lambda_1] < -\lambda < P + \lambda_1$ is true, but $\max[Q, \lambda_1] < -\lambda < Q + \lambda_1$, is not, we can apply the argument used in the proof of Theorem 1. The same is true in the other direction. This completes the proof.

Proof of Theorem 4 First, we note that

$$G(u) \leq 0, \quad u \in N, \tag{45}$$

if $\sigma \geq \lambda_\ell + \max[P, Q]$. To see this, let $u = (v, w) \in N$. Then $\|\nabla u\|^2 = \|\nabla v\|^2 + \|\nabla w\|^2 \leq \lambda_\ell \|v\|^2 + \lambda_\ell \|w\|^2 = \lambda_\ell \|u\|^2$. Then

$$G(u) \leq \frac{1}{P}[\lambda_\ell - \sigma + P]\ \|v\|^2 + \frac{1}{Q}[\lambda_\ell - \sigma + Q]\ \|w\|^2 \leq 0.$$

Next, let g be an eigenfunction of $-\Delta$ corresponding to the eigenvalue $\lambda_{\ell+1}$. If we take $u = (g + v, g + w)$, we have

$$\begin{aligned} a(u) =& \frac{1}{P}[\ \|\nabla(g+v)\|^2 - \sigma\ \|(g+v)\|^2] \\ &+ \frac{1}{Q}[\ \|\nabla(g+w)\|^2 - \sigma\ \|(g+w)\|^2], \end{aligned}$$

and this gives

$$\begin{aligned} a(u) =& \frac{1}{P}[(\lambda_{\ell+1} - \sigma)\ \|g\|^2 + (\lambda_\ell - \sigma)\|v\|^2] \\ &+ \frac{1}{Q}[(\lambda_{\ell+1} - \sigma)\ \|g\|^2 + (\lambda_\ell - \sigma^2)\|w\|^2], \end{aligned}$$

which will be negative if $\sigma > \lambda_{\ell+1}$. Moreover, by Lemma 7,

$$\int_\Omega \ln(1 + 2g^2 + v^2 + w^2)dx/\|u\|_H^2 \to 0, \quad \|u\|_H \to \infty. \tag{46}$$

Consequently, (46) holds provided $\sigma > \lambda_{\ell+1}$.

Next, let $u = (v, w)$ be any function in M. Then $\|\nabla u\|^2 = \|\nabla v\|^2 + \|\nabla w\|^2 \geq \lambda_{\ell+1}\|v\|^2 + \lambda_{\ell+1}\|w\|^2 = \lambda_{\ell+1}\|u\|^2$. Then

$$a(u) + \|u\|^2 \geq \frac{1}{P}[1 - \frac{\sigma - P}{\lambda_{\ell+1}}] \, \|\nabla v\|^2 + \frac{1}{Q}[1 - \frac{\sigma - Q}{\lambda_{\ell+1}}] \, \|\nabla w\|^2.$$

Thus, there is an $\varepsilon > 0$ such that

$$a(u) + \|u\|^2 \geq 2\varepsilon \|\nabla u\|^2, \quad u \in M, \tag{47}$$

when $\sigma - \lambda_{\ell+1} < \min[P, Q]$.

Now by Lemma 8,

$$\int_\Omega [u^2 - \ln(1 + u^2)] dx / \|u\|_H^2 \to 0, \quad \|u\|_H \to 0. \tag{48}$$

If we combine (47) and (48), we see that there is an $\varepsilon > 0$ such that

$$G(u) \geq \varepsilon \|\nabla u\|^2, \quad u \in M, \tag{49}$$

when $\|\nabla u\|^2$ is small and $\sigma - \lambda_{\ell+1} < \min[P, Q]$.

By (45), (46), and (49) one can apply Lemma 1 to obtain (10) and then Lemma 5 to conclude that (5) and (6) has a nontrivial solution u_0 taking $A = \partial(N \oplus \{g\}),\ B = \partial\mathscr{B}_\delta \cap M$. Then $a_0 = 0 < \varepsilon \leq b_0$, showing that $G(u_0) \geq \varepsilon > 0$. Since $G(0, 0) = 0$, we see that $u_0 \neq 0$. If $\lambda_\ell < \sigma - P < \lambda_{\ell+1} < \sigma$ is true, but $\lambda_\ell < \sigma - Q < \lambda_{\ell+1} < \sigma$ is not, we can apply the argument used in the proof of Theorem 1. The same is true in the other direction. This completes the proof.

Proof of Theorem 5 If $w = 0$, this follows from Theorem 1 since $0 < \lambda_\ell < -P$. Otherwise, let

$$I_w(v) = \frac{1}{P}\|\nabla v\|^2 - \frac{\lambda_\ell}{P}\|v\|^2 + \int_\Omega \ln\{1 + v^2 + w^2\} \, dx, \quad v \in H. \tag{50}$$

Then,

$$(I_w'(v), g)/2 = \frac{1}{P}(\nabla v, \nabla g) - \frac{\lambda_\ell}{P}(v, g) + \int_\Omega \frac{vg}{1 + v^2 + w^2} \, dx. \tag{51}$$

If $I_w'(v) = 0$, then $u = (v, w)$ satisfies

$$\Delta v = \frac{Pv}{1 + v^2 + w^2} - \lambda_\ell v, \tag{52}$$

$$\Delta w = -\lambda_\ell w, \tag{53}$$

which is (5) and (6) for the case $Q = 0,\ \lambda = -\lambda_\ell$. If we can find a solution $v \neq 0$ of $I_w'(v) = 0$, then we shall have a solution $u = (v, w)$ of (5) and (6) with $v \neq 0,\ w \neq 0$. This was done in Theorem 3 of [19].

The proof of Theorem 6 is similar to that of Theorem 5 and is omitted.

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