Best Hyers–Ulam Stability Constants on a Time Scale with Discrete Core and Continuous Periphery

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Abstract Consider a time scale consisting of a discrete core with uniform step size, augmented with a continuous-interval periphery. On this time scale, we determine the best constants for the Hyers–Ulam stability of a first-order dynamic equation with complex constant coefficient, based on the placement of the complex coefficient in the complex plane, with respect to the imaginary axis and the Hilger circle. These best constants are then related to known results for the special cases of completely continuous and uniformly discrete time scales.

1 Introduction

In this paper we explore the Hyers–Ulam stability of a certain dynamic equation on a new time scale with a discrete, uniform core and continuous periphery. Ulam inaugurated this type of stability [37], followed by Hyers [22] and Rassias [34]. Since then, there has been wide-spread interest in this type of stability, including for difference equations, recurrence relations, h-difference equations, quantum equations, and dynamic equations on time scales. For early papers on difference equations, see Popa [31, 32]; more current works include Anderson and Onitsuka [5, 6], Baias and Popa [13], Brzdęk and Wójcik [16], Onitsuka [29, 30], Rasouli, Abbaszadeh, and Eshaghi [33], Xu and Brzdęk [38]. A related monograph is Brzdęk, Popa, Raşa, and Xu [17]. Quantum equations and Hyers–Ulam stability are investigated in Anderson and Onitsuka [7, 8]. For some work on matrix and nonlinear difference equations, see Jung and Nam [24, 25], and Nam [26–28]. For early papers on time scales, see András and Mészáros [12], Hua, Li, and Feng [21];

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contemporary results include Anderson [2], Anderson, Jennissen, and Montplaisir [10], Anderson and Onitsuka [3, 4, 9], Shen [35], Shen and Li [36]. For recent papers with non-constant or periodic coefficients, see Anderson [1], Anderson, Onitsuka, and Rassias [11], Baias, Blaga, and Popa [14], Buşe, Lupulescu, and O'Regan [19], Buşe, O'Regan, and Saierli [18].

This work will proceed as follows. In Section 2, we will define the time scale with discrete core and continuous periphery, introduce the basic derivative and exponential function for this time scale, and define Hyers–Ulam stability for the dynamic equation with a complex constant coefficient. In Section 3, we establish the best Hyers–Ulam stability constants in Theorem 5, based on the location of the complex coefficient with respect to the imaginary axis, and for negative real part, with respect to the left Hilger circle. If we expand the discrete core to all of $h\mathbb{Z}$, or shrink it to recover the continuum $\mathbb{R}$, we are able to relate our new results with the current literature in the field. As we do this, an interesting case arises when the real part of the complex coefficient is negative but it lies outside the Hilger circle; this case is explored in Section 4. After that, we provide a brief conclusion and future direction.

2 Time Scale with Discrete Core and Continuous Periphery

Let $\mathbb{N}_0$ denote the non-negative integers $\{0, 1, 2, \ldots\}$, let $m \in \mathbb{N}_0$, and let $h > 0$. Define the time scale with discrete core and continuous periphery via

$$\mathbb{T}_{hm} := (-\infty, -hm) \cup \{-hm, \ldots, -h, 0, h, \ldots, hm\} \cup (hm, \infty).$$

Here, $h > 0$ is the uniform step size in the discrete core, with discrete spread $m \in \mathbb{N}_0$ out to the continuous periphery. Define the graininess function $\mu : \mathbb{T}_{hm} \to \mathbb{R}$ via

$$\mu(t) = \begin{cases} 0 & : t \in (-\infty, -hm) \cup [hm, \infty), \\ h & : t \in \{-hm, \ldots, -h, 0, h, \ldots, h(m-1)\}. \end{cases}$$

As $h \to 0$, or if $m = 0$, we have $\mathbb{T}_{0,m} = \mathbb{T}_{h,0} = \mathbb{R}$, and we recover results for classical differential equations; as $m \to \infty$ for fixed $h > 0$, we have $\mathbb{T}_{h,\infty} = h\mathbb{Z}$ and we recover results for standard h-difference equations.

In this section we introduce the first-order linear homogeneous equation with constant complex-valued coefficient

$$x^{\Delta}(t) - \lambda x(t) = 0, \quad \lambda \in \mathbb{C} \setminus \left\{ \frac{-1}{h} \right\}, \quad t \in \mathbb{T}_{hm}, \tag{1}$$

where

$$x^{\Delta}(t) := \begin{cases} \frac{d}{dt}x(t) & : t \in (-\infty, -hm) \cup [hm, \infty) \\ \frac{x(t+h)-x(t)}{h} & : t \in \{-hm, \dots, -h, 0, h, \dots, h(m-1)\}. \end{cases}$$

Lemma 1 (Exponential Function) *Fix $h > 0$. For $t \in \mathbb{T}_{hm}$, define the function*

$$e_\lambda(t, 0) := \begin{cases} (1 + h\lambda)^{-m} e^{\lambda(t+hm)} & : t \in (-\infty, -hm) \\ (1 + h\lambda)^{\frac{t}{h}} & : t \in \{-hm, \dots, -h, 0, h, \dots, hm\} \\ (1 + h\lambda)^{m} e^{\lambda(t-hm)} & : t \in (hm, \infty). \end{cases} \qquad (2)$$

Then, $x(t) = x_0 e_\lambda(t, 0)$ for $e_\lambda(t, 0)$ given in (2) is the unique solution of (1) satisfying $x(0) = x_0 \in \mathbb{C}$.

3 Best Constants for First-Order Equations with Constant Complex Coefficient

In this section, we consider on $\mathbb{T}_{hm}$ the Hyers–Ulam stability of (1), defined as follows.

Definition 1 (HUS) Let $\varepsilon > 0$ be arbitrary. Equation (1) has Hyers–Ulam stability (HUS) if and only if given $\phi : \mathbb{T}_{hm} \to \mathbb{C}$ satisfying $|\phi^{\Delta}(t) - \lambda\phi(t)| \le \varepsilon$ for all $t \in \mathbb{T}_{hm}$, there exists a solution $x : \mathbb{T}_{hm} \to \mathbb{C}$ of (1) and a constant $K > 0$ such that $|\phi(t) - x(t)| \le K\varepsilon$ for all $t \in \mathbb{T}_{hm}$. Such a constant K is called an HUS constant for (1) on $\mathbb{T}_{hm}$.

Theorem 1 *Let $\lambda \in \mathbb{C}\backslash \left\{\frac{-1}{h}\right\}$ with $\mathrm{Re}(\lambda) > 0$. Let $\varepsilon > 0$ be a fixed arbitrary constant, and let ϕ be a function on $\mathbb{T}_{hm}$ satisfying the inequality*

$$\left|\phi^{\Delta}(t) - \lambda\phi(t)\right| \le \varepsilon, \qquad t \in \mathbb{T}_{hm}.$$

Then, $\lim_{t\to\infty} \frac{\phi(t)}{e_\lambda(t, 0)}$ exists, and the function x given by

$$x(t) := \left(\lim_{t\to\infty} \frac{\phi(t)}{e_\lambda(t, 0)}\right) e_\lambda(t, 0)$$

is the unique solution of (1) with

$$|\phi(t) - x(t)| \le \varepsilon \left(\frac{1}{\mathrm{Re}(\lambda)}\right)$$

for all $t \in \mathbb{T}_{hm}$.

Proof Let $\lambda \in \mathbb{C}\backslash\{\frac{-1}{h}\}$ with $\mathrm{Re}(\lambda) > 0$. Throughout this proof, as $|\phi^{\Delta}(t) - \lambda\phi(t)| \leq \varepsilon$ for all $t \in \mathbb{T}_{hm}$, there exists a function $q : \mathbb{T}_{hm} \to \mathbb{C}$ such that

$$\phi^{\Delta}(t) - \lambda\phi(t) = q(t), \quad |q(t)| \leq \varepsilon$$

for all $t \in \mathbb{T}_{hm}$. The variation of constants formula then yields

$$\phi(t) = \phi_0 e_\lambda(t,0) + e_\lambda(t,0)\int_0^t \frac{q(\tau)}{e_\lambda(\sigma(\tau),0)}\Delta\tau.$$

Since $\mathrm{Re}(\lambda) > 0$ and $|q(t)| \leq \varepsilon$, we can rewrite ϕ as

$$\phi(t) = \left(\phi_0 + \int_0^\infty \frac{q(\tau)}{e_\lambda(\sigma(\tau),0)}\Delta\tau\right) e_\lambda(t,0) - e_\lambda(t,0)\int_t^\infty \frac{q(\tau)}{e_\lambda(\sigma(\tau),0)}\Delta\tau, \tag{3}$$

where

$$x_0 := \phi_0 + \int_0^\infty \frac{q(\tau)}{e_\lambda(\sigma(\tau),0)}\Delta\tau \in \mathbb{C}$$

exists and is finite. Clearly

$$x(t) := x_0 e_\lambda(t,0), \quad t \in \mathbb{T}_{hm}$$

is a solution of (1), and

$$\lim_{t\to\infty}\frac{\phi(t)}{e_\lambda(t,0)} = \lim_{t\to\infty}\left(\phi_0 + \int_0^t \frac{q(\tau)}{e_\lambda(\sigma(\tau),0)}\Delta\tau\right) = x_0$$

exists, so

$$x(t) = \left(\lim_{t\to\infty}\frac{\phi(t)}{e_\lambda(t,0)}\right)e_\lambda(t,0).$$

We take into account three cases based on the three branches of the exponential function in (2).

(a). For $\mathrm{Re}(\lambda) > 0$ and $t \in (hm, \infty)$, using (3) we have that

$$\begin{aligned}
|\phi(t) - x(t)| &= \left|-e_\lambda(t,0)\int_t^\infty \frac{q(\tau)}{e_\lambda(\sigma(\tau),0)}\Delta\tau\right| \\
&\leq \varepsilon|e_\lambda(t,0)|\int_t^\infty \frac{\Delta\tau}{|e_\lambda(\sigma(\tau),0)|} \\
&= \varepsilon|1+h\lambda|^m e^{\mathrm{Re}(\lambda)(t-hm)}\int_t^\infty \frac{d\tau}{|1+h\lambda|^m e^{\mathrm{Re}(\lambda)(\tau-hm)}} \\
&= \frac{\varepsilon}{\mathrm{Re}(\lambda)}
\end{aligned}$$

holds for all $t \in (hm, \infty)$.

(b). For $\mathrm{Re}(\lambda) > 0$ and $t \in \{-hm, \ldots, -h, 0, h, \ldots, hm\}$, using (3) we have

$$\begin{aligned}
|\phi(t) - x(t)| &\le \varepsilon |e_\lambda(t,0)| \int_t^\infty \frac{\Delta\tau}{|e_\lambda(\sigma(\tau),0)|} \\
&= \varepsilon |1+h\lambda|^{\frac{t}{h}} \left(\int_t^{hm} + \int_{hm}^\infty \right) \frac{\Delta\tau}{|e_\lambda(\sigma(\tau),0)|} \\
&= \varepsilon |1+h\lambda|^{\frac{t}{h}} \left(\sum_{j=\frac{t}{h}}^{m-1} \frac{h}{|1+h\lambda|^{j+1}} + \int_{hm}^\infty \frac{d\tau}{|1+h\lambda|^m e^{\mathrm{Re}(\lambda)(\tau-hm)}} \right) \\
&= \varepsilon |1+h\lambda|^{\frac{t}{h}} \left(\frac{h\Big(|1+h\lambda|^{-\frac{t}{h}} - |1+h\lambda|^{-m}\Big)}{|1+h\lambda|-1} + \frac{1}{|1+h\lambda|^m \,\mathrm{Re}(\lambda)} \right) \\
&= \varepsilon \left(\frac{h}{|1+h\lambda|-1} + |1+h\lambda|^{\frac{t}{h}-m} \left(\frac{1}{\mathrm{Re}(\lambda)} - \frac{h}{|1+h\lambda|-1} \right) \right) \\
&\le \frac{\varepsilon}{\mathrm{Re}(\lambda)}
\end{aligned}$$

for all $t \in \{-hm, \ldots, -h, 0, h, \ldots, hm\}$, as $\frac{t}{h} \le m$ and $\frac{1}{\mathrm{Re}(\lambda)} \ge \frac{h}{|1+h\lambda|-1}$ for $\mathrm{Re}(\lambda) > 0$ and $h > 0$.

(c). For $\mathrm{Re}(\lambda) > 0$ and $t \in (-\infty, -hm)$, using (3) we have

$$\begin{aligned}
|\phi(t) - x(t)| &\le \varepsilon |e_\lambda(t,0)| \int_t^\infty \frac{\Delta\tau}{|e_\lambda(\sigma(\tau),0)|} \\
&= \frac{\varepsilon\, e^{\mathrm{Re}(\lambda)(t+hm)}}{|1+h\lambda|^m} \left(\int_t^{-hm} + \int_{-hm}^{hm} + \int_{hm}^\infty \right) \frac{\Delta\tau}{|e_\lambda(\sigma(\tau),0)|} \\
&= \frac{\varepsilon\, e^{\mathrm{Re}(\lambda)(t+hm)}}{|1+h\lambda|^{2m}} \left(\frac{1}{\mathrm{Re}(\lambda)} + \frac{\left(e^{-\mathrm{Re}(\lambda)(t+hm)} - 1\right)}{|1+h\lambda|^{-2m}\,\mathrm{Re}(\lambda)} \right. \\
&\qquad \left. + \frac{h\left(|1+h\lambda|^{2m} - 1\right)}{|1+h\lambda|-1} \right) \\
&= \varepsilon \left\{ \frac{1}{\mathrm{Re}(\lambda)} + e^{\mathrm{Re}(\lambda)(t+hm)} \left(\frac{1}{|1+h\lambda|^{2m}\,\mathrm{Re}(\lambda)} \right. \right. \\
&\qquad \left. \left. + \left(\frac{h}{|1+h\lambda|-1} \right) \left(1 - \frac{1}{|1+h\lambda|^{2m}} \right) - \frac{1}{\mathrm{Re}(\lambda)} \right) \right\} \\
&\le \frac{\varepsilon}{\mathrm{Re}(\lambda)}
\end{aligned}$$

for all $t \in (-\infty, -hm)$, as $t < -hm$, and the expression inside the square brackets is negative.

We next show that x is the unique solution of (1) such that $|\phi(t) - x(t)| \le K\varepsilon := \frac{1}{\mathrm{Re}(\lambda)}\varepsilon$ for all $t \in \mathbb{T}_{hm}$. Suppose $\phi : \mathbb{T}_{hm} \to \mathbb{C}$ is an approximate solution of (1) such that

$$\left|\phi^{\Delta}(t) - \lambda\phi(t)\right| \le \varepsilon \text{ for all } t \in \mathbb{T}_{hm}$$

for some $\varepsilon > 0$. Suppose further that $x_1, x_2 : \mathbb{T}_{hm} \to \mathbb{C}$ are two different solutions of (1) such that $|\phi(t) - x_j(t)| \le K\varepsilon$ for all $t \in \mathbb{T}_{hm}$, for $j = 1, 2$. Then, we have for constants $c_j \in \mathbb{C}$ that

$$x_j(t) = c_j e_\lambda(t, 0), \quad c_1 \ne c_2,$$

and

$$|c_1 - c_2| \cdot |e_\lambda(t, 0)| = |x_1(t) - x_2(t)| \le |x_1(t) - \phi(t)| + |\phi(t) - x_2(t)| \le 2K\varepsilon;$$

letting $t \to \infty$ yields $\infty < 2K\varepsilon$, a contradiction. Consequently, x is the unique solution of (1) such that $|\phi(t) - x(t)| \le \frac{\varepsilon}{\mathrm{Re}(\lambda)}$ for all $t \in \mathbb{T}_{hm}$. This completes the proof. □

Theorem 2 *Let $\lambda \in \mathbb{C}\backslash\left\{\frac{-1}{h}\right\}$ with $\mathrm{Re}(\lambda) < 0$. Let $\varepsilon > 0$ be a fixed arbitrary constant, and let ϕ be a function on $\mathbb{T}_{hm}$ satisfying the inequality*

$$\left|\phi^{\Delta}(t) - \lambda\phi(t)\right| \le \varepsilon, \qquad t \in \mathbb{T}_{hm}.$$

Then, $\displaystyle\lim_{t\to-\infty} \frac{\phi(t)}{e_\lambda(t,0)}$ exists, and the function x given by

$$x(t) := \left(\lim_{t\to-\infty} \frac{\phi(t)}{e_\lambda(t,0)}\right) e_\lambda(t, 0)$$

is the unique solution of (1) with $|\phi(t) - x(t)| \le K\varepsilon$ for all $t \in \mathbb{T}_{hm}$, where

$$K := \begin{cases} \frac{-1}{\mathrm{Re}(\lambda)} + 2hm & : |1 + h\lambda| = 1 \\ \max\left\{\frac{-1}{\mathrm{Re}(\lambda)}, \frac{h}{1-|1+h\lambda|} + |1 + h\lambda|^{2m}\left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\mathrm{Re}(\lambda)}\right)\right\} & : |1 + h\lambda| \ne 1. \end{cases} \tag{4}$$

In particular, the following holds.

(i) If $t \in (-\infty, -hm)$, then

$$|\phi(t) - x(t)| \le \varepsilon\left(\frac{-1}{\mathrm{Re}(\lambda)}\right)$$

for all $t \in (-\infty, -hm)$.

(ii) *If* $|1+h\lambda| = 1$ *and* $t \in \{-hm, \ldots, -h, 0, h, \ldots, hm\} \cup (hm, \infty)$, *then*

$$|\phi(t) - x(t)| \le \varepsilon \left(\frac{-1}{\mathrm{Re}(\lambda)} + 2hm \right)$$

for all $t \in \{-hm, \ldots, -h, 0, h, \ldots, hm\} \cup (hm, \infty)$.

(iii) *If* $0 < |1+h\lambda| < 1$ *and* $t \in \{-hm, \ldots, -h, 0, h, \ldots, hm\} \cup (hm, \infty)$, *then*

$$|\phi(t) - x(t)| \le \varepsilon \left(\frac{h}{1-|1+h\lambda|} + |1+h\lambda|^{2m} \left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\mathrm{Re}(\lambda)} \right) \right)$$

for all $t \in \{-hm, \ldots, -h, 0, h, \ldots, hm\} \cup (hm, \infty)$.

(iv) *If* $|1+h\lambda| > 1$ *and* $t \in \{-hm, \ldots, -h, 0, h, \ldots, hm\} \cup (hm, \infty)$, *then*

$$|\phi(t) - x(t)| \le \varepsilon \left(\frac{h}{1-|1+h\lambda|} + |1+h\lambda|^{2m} \left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\mathrm{Re}(\lambda)} \right) \right)$$

for all $t \in \{-hm, \ldots, -h, 0, h, \ldots, hm\} \cup (hm, \infty)$.

Proof Let $\lambda \in \mathbb{C}\backslash\{\frac{-1}{h}\}$ with $\mathrm{Re}(\lambda) < 0$. Supposing $|\phi^{\Delta}(t) - \lambda\phi(t)| \le \varepsilon$ for all $t \in \mathbb{T}_{hm}$, there exists a function $q : \mathbb{T}_{hm} \to \mathbb{C}$ such that

$$\phi^{\Delta}(t) - \lambda\phi(t) = q(t), \quad |q(t)| \le \varepsilon$$

for all $t \in \mathbb{T}_{hm}$. Then, we have

$$\phi(t) = \phi_0 e_\lambda(t,0) + e_\lambda(t,0) \int_0^t \frac{q(\tau)}{e_\lambda(\sigma(\tau),0)} \Delta\tau.$$

Since $\mathrm{Re}(\lambda) < 0$ and $|q(t)| \le \varepsilon$, we can rewrite ϕ as

$$\phi(t) = \left(\phi_0 - \int_{-\infty}^0 \frac{q(\tau)}{e_\lambda(\sigma(\tau),0)} \Delta\tau \right) e_\lambda(t,0) + e_\lambda(t,0) \int_{-\infty}^t \frac{q(\tau)}{e_\lambda(\sigma(\tau),0)} \Delta\tau, \tag{5}$$

where

$$x_0 := \phi_0 - \int_{-\infty}^0 \frac{q(\tau)}{e_\lambda(\sigma(\tau),0)} \Delta\tau \in \mathbb{C}$$

exists and is finite. As in the previous case,

$$x(t) := x_0 e_\lambda(t,0), \quad t \in \mathbb{T}_{hm}$$

is a solution of (1), and

$$\lim_{t\to-\infty} \frac{\phi(t)}{e_\lambda(t,0)} = \lim_{t\to-\infty} \left(\phi_0 - \int_t^0 \frac{q(\tau)}{e_\lambda(\sigma(\tau),0)} \Delta\tau \right) = x_0$$

exists, so

$$x(t) = \left(\lim_{t \to -\infty} \frac{\phi(t)}{e_\lambda(t,0)} \right) e_\lambda(t,0).$$

We again work our way through the three cases based on the three branches of the exponential function in (2).

(i). For $\mathrm{Re}(\lambda) < 0$ and $t \in (-\infty, -hm)$, using (5) we have

$$\begin{aligned}
|\phi(t) - x(t)| &= \left| e_\lambda(t,0) \int_{-\infty}^{t} \frac{q(\tau)}{e_\lambda(\sigma(\tau),0)} \Delta\tau \right| \\
&\le \varepsilon |e_\lambda(t,0)| \int_{-\infty}^{t} \frac{\Delta\tau}{|e_\lambda(\sigma(\tau),0)|} \\
&= \varepsilon |1+h\lambda|^{-m} e^{\mathrm{Re}(\lambda)(t+hm)} \int_{-\infty}^{t} \frac{|1+h\lambda|^m d\tau}{e^{\mathrm{Re}(\lambda)(\tau+hm)}} \\
&= -\frac{\varepsilon}{\mathrm{Re}(\lambda)}
\end{aligned}$$

holds for $\mathrm{Re}(\lambda) < 0$ and for all $t \in (-\infty, -hm)$.

(ii) (a). For $\mathrm{Re}(\lambda) < 0$ with $|1+h\lambda| = 1$ and $t \in \{-hm, \ldots, -h, 0, h, \ldots, hm\}$, using (5) we have

$$\begin{aligned}
|\phi(t) - x(t)| &\le \varepsilon |e_\lambda(t,0)| \int_{-\infty}^{t} \frac{\Delta\tau}{|e_\lambda(\sigma(\tau),0)|} \\
&= \varepsilon \left(\int_{-\infty}^{-hm} + \int_{-hm}^{t} \right) \frac{\Delta\tau}{|e_\lambda(\sigma(\tau),0)|} \\
&= \varepsilon \left(\int_{-\infty}^{-hm} \frac{d\tau}{e^{\mathrm{Re}(\lambda)(\tau+hm)}} + \sum_{j=-m}^{\frac{t-h}{h}} h \right) \\
&= \varepsilon \left(\frac{-1}{\mathrm{Re}(\lambda)} + hm + t \right) \\
&\le \varepsilon \left(\frac{-1}{\mathrm{Re}(\lambda)} + 2hm \right),
\end{aligned}$$

for $\mathrm{Re}(\lambda) < 0$ with $|1+h\lambda| = 1$ and $t \in \{-hm, \ldots, -h, 0, h, \ldots, hm\}$.

(ii) (b). For $\mathrm{Re}(\lambda) < 0$ with $|1+h\lambda| = 1$ and $t \in (hm, \infty)$, using (5) we have

$$|\phi(t) - x(t)| \le \varepsilon\, e^{\mathrm{Re}(\lambda)(t-hm)} \left(\int_{-\infty}^{-hm} + \int_{-hm}^{hm} + \int_{hm}^{t} \right) \frac{\Delta\tau}{|e_\lambda(\sigma(\tau),0)|}$$

$$= \varepsilon\, e^{\mathrm{Re}(\lambda)(t-hm)} \left(2hm - \frac{e^{-\mathrm{Re}(\lambda)(t-hm)}}{\mathrm{Re}(\lambda)}\right)$$

$$\leq \varepsilon \left(\frac{-1}{\mathrm{Re}(\lambda)} + 2hm\right),$$

as $t > hm$ and $\mathrm{Re}(\lambda) < 0$.

(iii) (a). For $\mathrm{Re}(\lambda) < 0$ with $0 < |1 + h\lambda| < 1$ and $t \in \{-hm, \ldots, -h, 0, h, \ldots, hm\}$, using (5) we have

$$|\phi(t) - x(t)| \leq \varepsilon |e_\lambda(t,0)| \int_{-\infty}^{t} \frac{\Delta\tau}{|e_\lambda(\sigma(\tau),0)|}$$

$$= \varepsilon |1 + h\lambda|^{\frac{t}{h}} \left(\int_{-\infty}^{-hm} + \int_{-hm}^{t}\right) \frac{\Delta\tau}{|e_\lambda(\sigma(\tau),0)|}$$

$$= \varepsilon |1 + h\lambda|^{\frac{t}{h}} \left(\int_{-\infty}^{-hm} \frac{|1+h\lambda|^m d\tau}{e^{\mathrm{Re}(\lambda)(\tau+hm)}} + \sum_{j=-m}^{\frac{t-h}{h}} \frac{h}{|1+h\lambda|^{j+1}}\right)$$

$$= \varepsilon |1 + h\lambda|^{\frac{t}{h}} \left(\frac{h\left(|1+h\lambda|^m - |1+h\lambda|^{-\frac{t}{h}}\right)}{|1+h\lambda| - 1} - \frac{|1+h\lambda|^m}{\mathrm{Re}(\lambda)}\right)$$

$$= \varepsilon \left(\frac{h}{1 - |1+h\lambda|} + |1+h\lambda|^{\frac{t}{h}+m} \left(\frac{h}{|1+h\lambda| - 1} - \frac{1}{\mathrm{Re}(\lambda)}\right)\right)$$

$$\leq \varepsilon \left(\frac{h}{1 - |1+h\lambda|} + |1+h\lambda|^{2m} \left(\frac{h}{|1+h\lambda| - 1} - \frac{1}{\mathrm{Re}(\lambda)}\right)\right),$$

as $\frac{t}{h} \leq m$ and $\frac{h}{|1+h\lambda|-1} - \frac{1}{\mathrm{Re}(\lambda)} \leq 0$ for $\mathrm{Re}(\lambda) < 0$ with $0 < |1 + h\lambda| < 1$ and $h > 0$.

(iii) (b). For $\mathrm{Re}(\lambda) < 0$ with $0 < |1 + h\lambda| < 1$ and $t \in (hm, \infty)$, using (5) we have

$$|\phi(t) - x(t)| \leq \frac{\varepsilon e^{\mathrm{Re}(\lambda)(t-hm)}}{|1+h\lambda|^{-m}} \left(\int_{-\infty}^{-hm} + \int_{-hm}^{hm} + \int_{hm}^{t}\right) \frac{\Delta\tau}{|e_\lambda(\sigma(\tau),0)|}$$

$$= \varepsilon e^{\mathrm{Re}(\lambda)(t-hm)} \left(\frac{1 - e^{-\mathrm{Re}(\lambda)(t-hm)} - |1+h\lambda|^{2m}}{\mathrm{Re}(\lambda)}\right.$$

$$\left. + \frac{h\left(|1+h\lambda|^{2m} - 1\right)}{|1+h\lambda| - 1}\right)$$

$$= \varepsilon\left(\frac{-1}{\operatorname{Re}(\lambda)} + e^{\operatorname{Re}(\lambda)(t-hm)}\left(\frac{|1+h\lambda|^{2m}-1}{-\operatorname{Re}(\lambda)} + \frac{h\left(|1+h\lambda|^{2m}-1\right)}{|1+h\lambda|-1}\right)\right)$$
$$\le \varepsilon\left(\frac{h}{1-|1+h\lambda|} + |1+h\lambda|^{2m}\left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\operatorname{Re}(\lambda)}\right)\right),$$

as $t > hm$, $\operatorname{Re}(\lambda) < 0$ with $0 < |1+h\lambda| < 1$, and the expression inside the square brackets is non-negative.

(iv) (a). For $\operatorname{Re}(\lambda) < 0$ with $|1+h\lambda| > 1$ and $t \in \{-hm, \dots, -h, 0, h, \dots, hm\}$, using the same calculation as in case (iii)(a), we get

$$|\phi(t) - x(t)| \le \varepsilon\left(\frac{h}{1-|1+h\lambda|} + |1+h\lambda|^{\frac{t}{h}+m}\left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\operatorname{Re}(\lambda)}\right)\right)$$
$$\le \varepsilon\left(\frac{h}{1-|1+h\lambda|} + |1+h\lambda|^{2m}\left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\operatorname{Re}(\lambda)}\right)\right),$$

as $\frac{t}{h} \le m$ and $\frac{h}{|1+h\lambda|-1} - \frac{1}{\operatorname{Re}(\lambda)} > 0$ for $\operatorname{Re}(\lambda) < 0$ with $|1+h\lambda| > 1$ and $h > 0$.

(iv) (b). For $\operatorname{Re}(\lambda) < 0$ with $|1+h\lambda| > 1$ and $t \in (hm, \infty)$, using the same calculation as in case (iii)(b), we get

$$|\phi(t) - x(t)| \le \varepsilon\left(\frac{-1}{\operatorname{Re}(\lambda)} + e^{\operatorname{Re}(\lambda)(t-hm)}\left(\frac{|1+h\lambda|^{2m}-1}{-\operatorname{Re}(\lambda)} + \frac{h\left(|1+h\lambda|^{2m}-1\right)}{|1+h\lambda|-1}\right)\right)$$
$$\le \varepsilon\left(\frac{h}{1-|1+h\lambda|} + |1+h\lambda|^{2m}\left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\operatorname{Re}(\lambda)}\right)\right),$$

as $t > hm$, $\operatorname{Re}(\lambda) < 0$ with $|1+h\lambda| > 1$, and the expression inside the square brackets is positive.

We next show that x is the unique solution of (1) such that $|\phi(t) - x(t)| \le K\varepsilon$ for all $t \in \mathbb{T}_{hm}$, where K is given by (4). Suppose $\phi : \mathbb{T}_{hm} \to \mathbb{C}$ is an approximate solution of (1) such that

$$\left|\phi^{\Delta}(t) - \lambda\phi(t)\right| \le \varepsilon \text{ forall } t \in \mathbb{T}_{hm}$$

for some $\varepsilon > 0$. Suppose further that $x_1, x_2 : \mathbb{T}_{hm} \to \mathbb{C}$ are two different solutions of (1) such that $|\phi(t) - x_j(t)| \le K\varepsilon$ for all $t \in \mathbb{T}_{hm}$, for $j = 1, 2$. Then, we have for constants $c_j \in \mathbb{C}$ that

$$x_j(t) = c_j e_\lambda(t, 0), \quad c_1 \ne c_2,$$

and

$$|c_1 - c_2| \cdot |e_\lambda(t, 0)| = |x_1(t) - x_2(t)| \le |x_1(t) - \phi(t)| + |\phi(t) - x_2(t)| \le 2K\varepsilon;$$

letting $t \to -\infty$ yields $\infty < 2K\varepsilon$, a contradiction. Consequently, x is the unique solution of (1) such that $|\phi(t) - x(t)| \le \varepsilon K$ for all $t \in \mathbb{T}_{hm}$. This completes the proof. □

Theorem 3 *Let* $\lambda \in \mathbb{C}\setminus\left\{\frac{-1}{h}\right\}$ *with* $\mathrm{Re}(\lambda) = 0$*. Then,* (1) *is not Hyers–Ulam stable on* $\mathbb{T}_{hm}$*.*

Proof Assume $\mathrm{Re}(\lambda) = 0$ for $\lambda \in \mathbb{C}$. Let arbitrary $\varepsilon > 0$ be given, and let $\lambda = i\beta$ for some $\beta \in \mathbb{R}$. Then,

$$\phi(t) := \frac{\varepsilon t e_{i\beta}(t,0)}{\left(1 + h^2\beta^2\right)^{\frac{m+1}{2}}}, \qquad t \in \mathbb{T}_{hm}$$

satisfies the inequality

$$\left|\phi^{\Delta}(t) - i\beta\phi(t)\right| = \frac{\varepsilon\left|(1 + i\beta\mu(t))e_{i\beta}(t,0)\right|}{\left(1 + h^2\beta^2\right)^{\frac{m+1}{2}}} \le \frac{\varepsilon\left|e_{i\beta}(t,0)\right|}{\left(1 + h^2\beta^2\right)^{\frac{m}{2}}} \le \varepsilon$$

for all $t \in \mathbb{T}_{hm}$. Since $x(t) = x_0 e_{i\beta}(t,0)$ is the general solution of (1) when $\lambda = i\beta$, then

$$|\phi(t) - x(t)| = \frac{\left|e_{i\beta}(t,0)\right|}{\left(1 + h^2\beta^2\right)^{\frac{m+1}{2}}}\left|\varepsilon t - x_0\left(1 + h^2\beta^2\right)^{\frac{m+1}{2}}\right| \to \infty$$

as $t \to \pm\infty$ for $t \in \mathbb{T}_{hm}$ and for any $x_0 \in \mathbb{C}$, $\beta \in \mathbb{R}$, $h > 0$. So, (1) lacks HUS on $\mathbb{T}_{hm}$ if $\lambda = i\beta$. □

Using the previous theorems, we can establish the following results.

Theorem 4 *Let* $\lambda \in \mathbb{C}\setminus\{\frac{-1}{h}\}$*. Equation* (1) *has HUS on* $\mathbb{T}_{hm}$ *if and only if* $\mathrm{Re}(\lambda) \neq 0$*.*

Proof By Theorems 1, 2 and 3, we obtain the result, immediately. □

Lemma 2 *Let* $\lambda \in \mathbb{C}\setminus\left\{\frac{-1}{h}\right\}$ *with* $\mathrm{Re}(\lambda) \neq 0$*.*

(i) If $\mathrm{Re}(\lambda) > 0$*, then the HUS constant* K *for* (1) *satisfies*

$$K \ge \frac{1}{\mathrm{Re}(\lambda)}.$$

(ii) If $\mathrm{Re}(\lambda) < 0$*, then the HUS constant* K *for* (1) *satisfies*

$$K \ge \begin{cases} \frac{-1}{\mathrm{Re}(\lambda)} + 2hm & : |1 + h\lambda| = 1 \\ \max\left\{\frac{-1}{\mathrm{Re}(\lambda)}, \frac{h}{1-|1+h\lambda|} + |1 + h\lambda|^{2m}\left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\mathrm{Re}(\lambda)}\right)\right\} & : |1 + h\lambda| \neq 1. \end{cases}$$

Proof Since $\lambda \in \mathbb{C}\backslash\left\{\frac{-1}{h}\right\}$ with $\operatorname{Re}(\lambda) \neq 0$, Equation (1) has HUS by Theorem 4. We will proceed by cases.

(i). Let $\lambda = \alpha + i\beta \in \mathbb{C}\backslash\left\{\frac{-1}{h}\right\}$, and assume $\operatorname{Re}(\lambda) = \alpha > 0$; set

$$\phi(t) := \frac{-\varepsilon e_{i\beta}(t,0)}{\alpha\left(1+h^2\beta^2\right)^{\frac{m}{2}}} + \frac{\varepsilon}{\alpha}e_\lambda(t,0).$$

It follows that

$$\left|\phi^\Delta(t) - \lambda\phi(t)\right| = \frac{\left|\varepsilon\alpha e_{i\beta}(t,0)\right|}{\alpha\left(1+h^2\beta^2\right)^{\frac{m}{2}}} = \frac{\varepsilon\left|e_{i\beta}(t,0)\right|}{\left(1+h^2\beta^2\right)^{\frac{m}{2}}} \leq \varepsilon.$$

Since $x(t) = \frac{\varepsilon}{\alpha}e_\lambda(t,0)$ is a solution of (1),

$$|\phi(t) - x(t)| = \frac{\varepsilon\left|e_{i\beta}(t,0)\right|}{\alpha\left(1+h^2\beta^2\right)^{\frac{m}{2}}} \leq \frac{\varepsilon}{\alpha},$$

with equality at $t = hm$, so the minimal HUS constant K for (1) satisfies

$$K \geq \frac{1}{\alpha} = \frac{1}{\operatorname{Re}(\lambda)}.$$

This ends the proof of case (i).

(ii) (a). Assume $\operatorname{Re}(\lambda) < 0$ with $|1 + h\lambda| = 1$. Let

$$\phi(t) = e_\lambda(t,0)\int_0^t \frac{q(\tau)}{e_\lambda(\sigma(\tau),0)}\Delta\tau, \quad q(\tau) = \frac{\varepsilon e_\lambda(\sigma(\tau),0)}{|e_\lambda(\sigma(\tau),0)|}, \tag{6}$$

for all $t \in \mathbb{T}_{hm}$. Then,

$$\phi^\Delta(t) - \lambda\phi(t) = q(t), \quad |q(t)| = \varepsilon,$$

and, employing (6), we see that ϕ takes the form

$$\phi(t) = \varepsilon\begin{cases}\left(\frac{1}{\operatorname{Re}(\lambda)} - hm\right)e_\lambda(t,0) - \frac{e^{i\operatorname{Im}(\lambda)(t+hm)}}{(1+h\lambda)^m\operatorname{Re}(\lambda)} & : t \in (-\infty, -hm)\\ te_\lambda(t,0) & : t \in \{-hm,\ldots,hm\}\\ \left(\frac{1}{\operatorname{Re}(\lambda)} + hm\right)e_\lambda(t,0) - \frac{e^{i\operatorname{Im}(\lambda)(t-hm)}}{(1+h\lambda)^{-m}\operatorname{Re}(\lambda)} & : t \in (hm, \infty).\end{cases}$$

If we take

$$x(t) := \varepsilon\left(\frac{1}{\operatorname{Re}(\lambda)} - hm\right)e_\lambda(t,0),$$

then x is a solution of (1), and

$$|\phi(t)-x(t)| = \begin{cases} \varepsilon\left|\frac{-e^{i\,\mathrm{Im}(\lambda)(t+hm)}}{(1+h\lambda)^m\,\mathrm{Re}(\lambda)}\right| = \frac{-\varepsilon}{\mathrm{Re}(\lambda)} & : t\in(-\infty,-hm)\\ \varepsilon\left|-\frac{1}{\mathrm{Re}(\lambda)}+t+hm\right| \le \varepsilon\left(\frac{-1}{\mathrm{Re}(\lambda)}+2hm\right) & : t\in\{-hm,\dots,hm\}\\ \varepsilon\left|-\frac{1}{\mathrm{Re}(\lambda)}+2hme^{\mathrm{Re}(\lambda)(t-hm)}\right| \le \varepsilon\left(\frac{-1}{\mathrm{Re}(\lambda)}+2hm\right) & : t\in(hm,\infty),\end{cases}$$

where we have equality at $t = hm$. This shows that the HUS constant K must satisfy

$$K \ge \left(\frac{-1}{\mathrm{Re}(\lambda)} + 2hm\right)$$

for $\mathrm{Re}(\lambda) < 0$ with $|1+h\lambda| = 1$. Here ends the proof of case $(ii)(a)$.

(ii)(b). Assume $\mathrm{Re}(\lambda) < 0$ with $|1+h\lambda| \neq 1$. Again, let ϕ be given by (6) for all $t \in \mathbb{T}_{hm}$. Then,

$$\phi^{\Delta}(t) - \lambda\phi(t) = q(t), \quad |q(t)| = \varepsilon,$$

and in this case ϕ takes the form

$$\phi(t) = \varepsilon\begin{cases} \left(\frac{|1+h\lambda|^m}{\mathrm{Re}(\lambda)} - \frac{h(|1+h\lambda|^m-1)}{|1+h\lambda|-1}\right)e_\lambda(t,0) - \frac{|1+h\lambda|^m e^{i\,\mathrm{Im}(\lambda)(t+hm)}}{(1+h\lambda)^m\,\mathrm{Re}(\lambda)} & : t\in(-\infty,-hm)\\ \frac{h\left(|1+h\lambda|^{\frac{t}{h}}-1\right)}{|1+h\lambda|^{\frac{t}{h}}(|1+h\lambda|-1)}e_\lambda(t,0) & : t\in\{-hm,\dots,hm\}\\ \left(\frac{|1+h\lambda|^{-m}}{\mathrm{Re}(\lambda)} + \frac{h(|1+h\lambda|^m-1)}{|1+h\lambda|^m(|1+h\lambda|-1)}\right)e_\lambda(t,0) - \frac{(1+h\lambda)^m e^{i\,\mathrm{Im}(\lambda)(t-hm)}}{|1+h\lambda|^m\,\mathrm{Re}(\lambda)} & : t\in(hm,\infty).\end{cases}$$

If we take

$$x(t) := \varepsilon\left(\frac{|1+h\lambda|^m}{\mathrm{Re}(\lambda)} - \frac{h\left(|1+h\lambda|^m-1\right)}{|1+h\lambda|-1}\right)e_\lambda(t,0), \tag{7}$$

then x is a solution of (1), and

$$|\phi(t)-x(t)| = \frac{-\varepsilon}{\mathrm{Re}(\lambda)}, \quad t\in(-\infty,-hm).$$

For $t \in \{-hm,\dots,-h,0,h,\dots,hm\}$,

$$|\phi(t)-x(t)| = \varepsilon\left(\frac{h\left(|1+h\lambda|^{m+\frac{t}{h}}-1\right)}{|1+h\lambda|-1} - \frac{|1+h\lambda|^{m+\frac{t}{h}}}{\mathrm{Re}(\lambda)}\right).$$

If $0 < |1+h\lambda| < 1$, then as in the proof of Theorem 2 $(iii)(a)$, we have

$$|\phi(t)-x(t)| = \varepsilon\left(\frac{h}{1-|1+h\lambda|} + |1+h\lambda|^{\frac{t}{h}+m}\left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\operatorname{Re}(\lambda)}\right)\right)$$
$$\le \varepsilon\left(\frac{h}{1-|1+h\lambda|} + |1+h\lambda|^{2m}\left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\operatorname{Re}(\lambda)}\right)\right),$$

as $\frac{t}{h} \le m$ and $\frac{h}{|1+h\lambda|-1} - \frac{1}{\operatorname{Re}(\lambda)} \le 0$ for $\operatorname{Re}(\lambda) < 0$ with $0 < |1+h\lambda| < 1$ and $h > 0$, with equality at $t = hm$. If $|1+h\lambda| > 1$, then as in the proof of Theorem 2 $(iv)(a)$,

$$|\phi(t)-x(t)| = \varepsilon\left(\frac{h}{1-|1+h\lambda|} + |1+h\lambda|^{\frac{t}{h}+m}\left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\operatorname{Re}(\lambda)}\right)\right)$$
$$\le \varepsilon\left(\frac{h}{1-|1+h\lambda|} + |1+h\lambda|^{2m}\left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\operatorname{Re}(\lambda)}\right)\right),$$

as $\frac{t}{h} \le m$ and $\frac{h}{|1+h\lambda|-1} - \frac{1}{\operatorname{Re}(\lambda)} > 0$ for $\operatorname{Re}(\lambda) < 0$ with $|1+h\lambda| > 1$ and $h > 0$. For $t \in (hm, \infty)$ and $0 < |1+h\lambda| < 1$, then as in the proof of Theorem 2 $(iii)(b)$,

$$|\phi(t)-x(t)| = \varepsilon\left(\frac{-1}{\operatorname{Re}(\lambda)} + e^{\operatorname{Re}(\lambda)(t-hm)}\left(\frac{|1+h\lambda|^{2m}-1}{-\operatorname{Re}(\lambda)} + \frac{h\left(|1+h\lambda|^{2m}-1\right)}{|1+h\lambda|-1}\right)\right)$$
$$\le \varepsilon\left(\frac{h}{1-|1+h\lambda|} + |1+h\lambda|^{2m}\left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\operatorname{Re}(\lambda)}\right)\right),$$

as $t > hm$, $\operatorname{Re}(\lambda) < 0$ with $0 < |1+h\lambda| < 1$, and the expression inside the square brackets is non-negative. For $t \in (hm, \infty)$ and $|1+h\lambda| > 1$, then as in the proof of Theorem 2 $(iv)(b)$, we have

$$|\phi(t)-x(t)| = \varepsilon\left(\frac{-1}{\operatorname{Re}(\lambda)} + e^{\operatorname{Re}(\lambda)(t-hm)}\left(\frac{|1+h\lambda|^{2m}-1}{-\operatorname{Re}(\lambda)} + \frac{h\left(|1+h\lambda|^{2m}-1\right)}{|1+h\lambda|-1}\right)\right)$$
$$\le \varepsilon\left(\frac{h}{1-|1+h\lambda|} + |1+h\lambda|^{2m}\left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\operatorname{Re}(\lambda)}\right)\right),$$

as $t > hm$, $\operatorname{Re}(\lambda) < 0$ with $|1+h\lambda| > 1$, and the expression inside the square brackets is positive. This ends the proof of case $(ii)(b)$, and thus the overall result holds.

□

Theorem 5 *Let* $\lambda \in \mathbb{C}\backslash\{\frac{-1}{h}\}$. *If* $\operatorname{Re}(\lambda) \neq 0$, *then* (1) *has HUS on* $\mathbb{T}_{hm}$.

(i) If $\operatorname{Re}(\lambda) > 0$, *then*

$$K = \frac{1}{\operatorname{Re}(\lambda)}$$

is the best (minimal) HUS constant.

(ii) *If* $\mathrm{Re}(\lambda) < 0$*, then*

$$K = \begin{cases} \frac{-1}{\mathrm{Re}(\lambda)} + 2hm & : |1+h\lambda| = 1 \\ \max\left\{\frac{-1}{\mathrm{Re}(\lambda)}, \frac{h}{1-|1+h\lambda|} + |1+h\lambda|^{2m}\left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\mathrm{Re}(\lambda)}\right)\right\} & : |1+h\lambda| \neq 1 \end{cases}$$

is the best (minimal) HUS constant.

Proof This result follows immediately from the definitions of HUS and HUS constant, Theorems 1–4, and Lemma 2. □

Remark 1 If $m = 0$, then $\mathbb{T}_{h,0} = \mathbb{R}$, and the results in Theorems 1 and 2 $(i) - (iv)$ match exactly the known results for $\mathbb{T} = \mathbb{R}$, namely that $x'(t) - \lambda x(t) = 0$ has HUS on $\mathbb{R}$, and

$$K = \frac{1}{|\mathrm{Re}(\lambda)|}$$

is the best possible HUS constant. If $h \to 0$, then $\mathbb{T}_{0,m} = \mathbb{R}$, and the results in Theorems 1 and 2 $(i) - (iv)$ also recover the known results for $\mathbb{T} = \mathbb{R}$, because

$$\lim_{h\to 0+}\left(\frac{-1}{\mathrm{Re}(\lambda)} + 2hm\right) = \frac{-1}{\mathrm{Re}(\lambda)}, \quad \lim_{h\to 0+}\frac{h}{|1+h\lambda|-1} = \lim_{h\to 0+}\frac{1}{\mathrm{Re}_h(\lambda)} = \frac{1}{\mathrm{Re}(\lambda)}$$

hold, where $\mathrm{Re}_h(\lambda)$ represents the Hilger real part [20] for h-difference equations.

For fixed $h > 0$, if $m \to \infty$, then $\mathbb{T}_{h,\infty} = h\mathbb{Z}$, and the results in Theorem 1 and Theorem 2 (i) and (iii) match exactly the known results for $\mathbb{T} = h\mathbb{Z}$, namely that $\Delta_h x(t) - \lambda x(t) = 0$ has HUS on $h\mathbb{Z}$, and

$$K = \frac{h}{\big|1 - |1+h\lambda|\big|} = \frac{1}{|\mathrm{Re}_h(\lambda)|}$$

is the best possible HUS constant. Theorem 2 (ii) shows an interesting connection; as $m \to \infty$, the HUS constant in (ii) goes to infinity as well. This is accurate, as $\mathrm{Re}(\lambda) < 0$ with $|1 + h\lambda| = 1$ makes the h-difference equation version of (1) Hyers–Ulam unstable on $h\mathbb{Z}$, as $\lambda \in \mathbb{C}$ is then on the left Hilger circle [5]; see [15, Chapter 2.1], [20], and [23] for more on the Hilger complex plane, and [2, 5, 10] for more on the Hilger circle and HUS. On the other hand, in case (iv) $\mathrm{Re}(\lambda) < 0$ with $|1 + h\lambda| > 1$, a result that does not match is obtained, that is,

$$\lim_{m\to\infty}\left(\frac{h}{1-|1+h\lambda|} + |1+h\lambda|^{2m}\left(\frac{h}{|1+h\lambda|-1} - \frac{1}{\mathrm{Re}(\lambda)}\right)\right) = \infty$$

when $\mathrm{Re}(\lambda) < 0$ with $|1 + h\lambda| > 1$, but, we know that $\Delta_h x(t) - \lambda x(t) = 0$ has HUS when $|1 + h\lambda| > 1$ (see [5]). Why does this logical gap occur? According to the information of Theorem 2.5 (ii) in [5], in this case, the unique solution x

is determined when $t \to \infty$. As you can see from the claim of Theorem 2, even in this case, the unique solution x is determined by the information of $t \to \infty$. Therefore, we can say that the case $\mathrm{Re}(\lambda) < 0$ with $|1 + h\lambda| > 1$ is a distinguishing characteristic of Hyers–Ulam stability on this time scale with discrete core and continuous periphery. We explore this anomaly in the next section.

4 Connection with h-Difference Equations in the Case $|1 + h\lambda| > 1$

The following result is effective for clarifying the connection with the h-difference equation $\Delta_h x(t) - \lambda x(t) = 0$ with $|1 + h\lambda| > 1$.

Theorem 6 *Let* $\lambda \in \mathbb{C} \setminus \left\{ \frac{-1}{h} \right\}$ *with* $\mathrm{Re}(\lambda) < 0$ *and* $|1 + h\lambda| > 1$. *Let* $\varepsilon > 0$ *be a fixed arbitrary constant, and let* ϕ *be a function on* $\mathbb{T}_{hm}$ *satisfying the inequality*

$$\left|\phi^{\Delta}(t) - \lambda\phi(t)\right| \le \varepsilon, \qquad t \in \mathbb{T}_{hm}.$$

Then, the function x *given by*

$$x(t) := \left(\frac{\phi(hm)}{e_\lambda(hm, 0)} \right) e_\lambda(t, 0)$$

is a solution of (1) *with*

$$|\phi(t) - x(t)| \le \varepsilon \max \left\{ \frac{h\left(1 - |1 + h\lambda|^{-2m}\right)}{|1 + h\lambda| - 1}, \frac{-1}{\mathrm{Re}(\lambda)} \right\}$$

for all $t \in \{-hm, \ldots, -h, 0, h, \ldots, hm\} \cup (hm, \infty)$. *In particular, the following holds.*

(i) If $t \in \{-hm, \ldots, -h, 0, h, \ldots, hm\}$, *then*

$$|\phi(t) - x(t)| \le \varepsilon \left(\frac{h\left(1 - |1 + h\lambda|^{-2m}\right)}{|1 + h\lambda| - 1} \right)$$

for all $t \in \{-hm, \ldots, -h, 0, h, \ldots, hm\}$.

(ii) If $t \in (hm, \infty)$, *then*

$$|\phi(t) - x(t)| \le \varepsilon \left(\frac{-1}{\mathrm{Re}(\lambda)} \right)$$

for all $t \in (hm, \infty)$.

Proof Let $\lambda \in \mathbb{C}\backslash\left\{\frac{-1}{h}\right\}$ with $\text{Re}(\lambda) < 0$ and $|1 + h\lambda| > 1$. Suppose that $|\phi^{\Delta}(t) - \lambda\phi(t)| \le \varepsilon$ for all $t \in \mathbb{T}_{hm}$, there exists a function $q : \mathbb{T}_{hm} \to \mathbb{C}$ such that

$$\phi^{\Delta}(t) - \lambda\phi(t) = q(t), \quad |q(t)| \le \varepsilon$$

for all $t \in \mathbb{T}_{hm}$. Then, we have

$$\phi(t) = \phi_0 e_\lambda(t, 0) + e_\lambda(t, 0) \int_0^t \frac{q(\tau)}{e_\lambda(\sigma(\tau), 0)} \Delta\tau.$$

Let $x(t) = x_0 e_\lambda(t, 0)$ be the solution of (1) with

$$x_0 := \phi_0 + \int_0^{hm} \frac{q(\tau)}{e_\lambda(\sigma(\tau), 0)} \Delta\tau \in \mathbb{C}.$$

It follows that

$$\phi(t) - x(t) = -e_\lambda(t, 0) \left(\int_t^{hm} \frac{q(\tau)}{e_\lambda(\sigma(\tau), 0)} \Delta\tau \right). \tag{8}$$

(a) For $\text{Re}(\lambda) < 0$ with $|1 + h\lambda| > 1$ and $t \in \{-hm, \ldots, -h, 0, h, \ldots, hm\}$, using (8) we have

$$|\phi(t) - x(t)| \le \varepsilon |e_\lambda(t, 0)| \int_t^{hm} \frac{\Delta\tau}{|e_\lambda(\sigma(\tau), 0)|} = \varepsilon |1 + h\lambda|^{\frac{t}{h}} \sum_{j=\frac{t}{h}}^{m-1} \frac{h}{|1 + h\lambda|^{j+1}}$$

$$= \varepsilon \left(\frac{h\left(1 - |1 + h\lambda|^{-m+\frac{t}{h}}\right)}{|1 + h\lambda| - 1} \right) \le \varepsilon \left(\frac{h\left(1 - |1 + h\lambda|^{-2m}\right)}{|1 + h\lambda| - 1} \right),$$

as $\frac{t}{h} \le m$, $h > 0$, and $\text{Re}(\lambda) < 0$ with $|1 + h\lambda| > 1$.

(b) For $\text{Re}(\lambda) < 0$ with $|1 + h\lambda| > 1$ and $t \in [hm, \infty)$, using (8) we have

$$\begin{aligned} |\phi(t) - x(t)| &= \left| e_\lambda(t, 0) \left(\int_{hm}^t \frac{q(\tau)}{e_\lambda(\sigma(\tau), 0)} \Delta\tau \right) \right| \\ &\le \varepsilon |1 + h\lambda|^m e^{\text{Re}(\lambda)(t-hm)} \int_{hm}^t \frac{\Delta\tau}{|e_\lambda(\sigma(\tau), 0)|} \\ &= \varepsilon \left(\frac{-1 + e^{\text{Re}(\lambda)(t-hm)}}{\text{Re}(\lambda)} \right) \le \varepsilon \left(\frac{-1}{\text{Re}(\lambda)} \right), \end{aligned}$$

as $t \ge hm$ and $\text{Re}(\lambda) < 0$ with $|1 + h\lambda| > 1$. This completes the proof.

□

Remark 2 For fixed $h > 0$, if $m \to \infty$, then $\mathbb{T}_{h,\infty} = h\mathbb{Z}$, and the result in Theorem 6 (i) reproduces exactly the known result for $\mathbb{T} = h\mathbb{Z}$ (see, Theorem 2.5 (ii) in [5]). Actually, we will explain this fact. Since

$$\left|\int_0^{hm} \frac{q(\tau)}{e_\lambda(\sigma(\tau),0)}\Delta\tau\right| \le \varepsilon\int_0^{hm}\frac{\Delta\tau}{|e_\lambda(\sigma(\tau),0)|} \le \varepsilon\sum_{j=0}^{m-1}\frac{h}{|1+h\lambda|^{j+1}} < \frac{h}{|1+h\lambda|-1}$$

holds, we see that

$$\lim_{m\to\infty}\frac{\phi(hm)}{e_\lambda(hm,0)} = \lim_{t\to\infty}\frac{\phi(t)}{e_\lambda(t,0)}$$

exists. In addition, an HUS constant is

$$\lim_{m\to\infty}\frac{h\left(1-|1+h\lambda|^{-2m}\right)}{|1+h\lambda|-1} = \frac{h}{|1+h\lambda|-1}.$$

Theorem 6 (i) says that the function x given by

$$x(t) := \left(\lim_{t\to\infty}\frac{\phi(t)}{e_\lambda(t,0)}\right)e_\lambda(t,0)$$

is a solution of (1) with

$$|\phi(t)-x(t)| \le \frac{h\varepsilon}{|1+h\lambda|-1}$$

for all $t \in h\mathbb{Z}$. As $m \to \infty$, our exponential function $e_\lambda(t,0)$ corresponds to $(1+h\lambda)^{\frac{t}{h}}$ for all $t \in h\mathbb{Z}$. In this case, we can prove the uniqueness of the solution. Let $\varepsilon > 0$, and let $\phi : \{-hm,\dots,-h,0,h,\dots,hm\} \to \mathbb{C}$ satisfy

$$\left|\phi^\Delta(t)-\lambda\phi(t)\right| \le \varepsilon \text{ for all } t \in \{-hm,\dots,-h,0,h,\dots,hm\}.$$

Suppose that $x_1, x_2 : \{-hm,\dots,-h,0,h,\dots,hm\} \to \mathbb{C}$ are two different solutions of (1) such that $|\phi(t)-x_j(t)| \le K\varepsilon := \frac{h}{|1+h\lambda|-1}\varepsilon$ for all $t \in \{-hm,\dots,-h,0,h,\dots,hm\}$, for $j = 1,2$. Then, we have for constants $c_j \in \mathbb{C}$ that

$$x_j(t) = c_j(1+h\lambda)^{\frac{t}{h}}, \quad c_1 \ne c_2,$$

and

$$|c_1-c_2||1+h\lambda|^{\frac{t}{h}} = |x_1(t)-x_2(t)| \le |x_1(t)-\phi(t)|+|\phi(t)-x_2(t)| \le 2K\varepsilon;$$

letting $m \to \infty$ and $t \to \infty$ yields $\infty < 2K\varepsilon$, a contradiction. Consequently, x is the unique solution of $\Delta_h x(t) - \lambda x(t) = 0$ such that $|\phi(t) - x(t)| \le \varepsilon K$ for all $t \in h\mathbb{Z}$.

5 Conclusion and Future Directions

In this paper we determined the best Hyers–Ulam stability constants for a first-order complex constant coefficient dynamic equation on a time scale with a discrete core and continuous periphery. In the future, we will study a time scale with a discrete periphery and continuous core, whose exponential function for $\lambda \in \mathbb{C}\backslash\{-\frac{1}{h}\}$ is

$$e_\lambda(t,0) := \begin{cases} (1+h\lambda)^{\frac{t}{h}+m} e^{-hm\lambda} & : t \in \{\dots, -h(m+2), -h(m+1)\} \\ e^{\lambda t} & : t \in [-hm, hm] \\ (1+h\lambda)^{\frac{t}{h}-m} e^{hm\lambda} & : t \in \{h(m+1), h(m+2), \dots\}. \end{cases}$$

on $\mathbb{T}_{hm} = \{\dots, -h(m+2), -h(m+1)\} \cup [-hm, hm] \cup \{h(m+1), h(m+2), \dots\}$, where $h > 0$ is the discrete step size and m is a non-negative integer.

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