Factorization and Solution of Linear and Nonlinear Second Order Differential Equations with Variable Coefficients and Mixed Conditions

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Abstract This chapter deals with the factorization and solution of initial and boundary value problems for a class of linear and nonlinear second order differential equations with variable coefficients subject to mixed conditions. The technique for nonlinear differential equations is based on their decomposition into linear components of the same or lower order and the factorization of the associated second order linear differential operators. The implementation and efficiency of the procedure is shown by solving several examples.

1 Introduction

One of the most important categories of ordinary differential equations is the second order differential equations with variable coefficients. Many problems from engineering and science are within this large class of differential equations. These equations, in addition to their natural significance, have also been used as a vehicle for the study of other higher order differential equations. Both exact and numerical methods have been developed for the their solution [2]. Most of the explicit techniques rely on the knowledge of fundamental solutions. The factorization method does not require any fundamental solution of the given second order differential equation, but its applicability is limited to certain problems. For a review of the factorization of differential operators the interested reader can look at the selected articles [1, 3–9, 14–16].

Following the work in [10–13] and [17], this paper is concerned with the exact solution of a class of linear and nonlinear differential equations of second order with variable coefficients subject to nonlocal boundary conditions by direct factorization of the differential equation as well as the boundary conditions. Specifically, in Section 2, we recall some basic results and consider linear first order problems with

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Th. M. Rassias (ed.), *Nonlinear Analysis, Differential Equations, and Applications*, Springer Optimization and Its Applications 173,
https://doi.org/10.1007/978-3-030-72563-1_18

a mixed boundary condition. In Section 3, we present the operator factorization method for solving, under certain conditions, the linear second order differential equation

$$u''(x) + p(x)u(x) + q(x)u(x) = f(x), \quad x \in (a, b), \tag{1}$$

where the coefficients $p(x),\ q(x) \in C[a, b]$ and the forcing function $f(x) \in C[a, b]$, subject to general boundary conditions

$$\begin{aligned} \mu_{11}u(a) + \mu_{12}u(b) &= \beta_1, \\ \mu_{21}u'(a) + \mu_{22}u'(b) + \mu_{23}u(a) + \mu_{24}u(b) &= \beta_2, \end{aligned} \tag{2}$$

where $\mu_{ij},\ \beta_i \in \mathbb{R},\ i = 1, 2,\ j = 1, 2, 3, 4$. In Section 4, we deal with the construction of explicit solutions to two kinds of nonlinear differential equations of second order, which can be decomposed initially into linear second order differential equations. First, we consider the equation of the form

$$u''(x)u'(x) + \left[q(x)u'(x) + g(x)u''(x)\right]u(x) + q(x)g(x)u^2(x) = 0, \tag{3}$$

for $x \in (a, b)$ and $q(x),\ g(x) \in C[a, b]$, along with the general boundary conditions (2). Also, we consider the nonlinear differential equation of the type

$$F\left(\frac{u''(x)}{u(x)}, x\right) = F\left(w(x), x\right) = w^2(x) + a(x)w(x) + b(x) = 0, \quad x \in (a, b), \tag{4}$$

where the nonlinear function F is a second degree polynomial of $w(x) = u''(x)/u(x)$ and the coefficients $a(x),\ b(x) \in C[a, b]$, subject to general boundary conditions (2). Finally, some conclusions are quoted in Section 5.

2 Preliminaries

We first recall some basic results. A linear operator $P : C[a, b] \to C[a, b]$ is said to be *correct* if P is injective, $R(P) = C[a, b]$ and its inverse P^{-1} is bounded on $C[a, b]$. Let $A : C[a, b] \to C[a, b]$ be the linear first order operator

$$Ay(x) = y'(x) + a(x)y(x), \quad D(A) = C^1[a, b], \tag{5}$$

where $a(x) \in C[a, b]$, and $\widehat{A}$ be its restriction on

$$D(\widehat{A}) = \{y(x) \in D(A) : \ y(x_0) = y_0\}, \tag{6}$$

where $x_0 \in [a, b]$ and y_0 is an arbitrary real initial value. Then the following fundamental theorem holds.

Theorem 1 *The linear operator $\widehat{A}$ in (5) and (6) is correct and the unique solution of the initial value problem*

$$\widehat{A}y(x) = f(x), \quad \forall f(x) \in C[a, b], \tag{7}$$

is given by

$$y(x) = \widehat{A}^{-1} f(x) = e^{-\int_{x_0}^{x} a(t)dt} \left(y_0 + \int_{x_0}^{x} f(t) e^{\int_{t_0}^{t} a(\tau)d\tau} dt \right). \tag{8}$$

Accordingly, let $B : C[a, b] \to C[a, b]$ be the linear second order operator

$$By(x) = y''(x) + b_1(x)y'(x) + b_2(x)y(x), \quad D(B) = C^2[a, b], \tag{9}$$

where $b_1(x),\ b_2(x) \in C[a, b]$, and $\widehat{B}$ be its restriction on

$$D(\widehat{B}) = \left\{ y(x) \in D(B) :\ y(x_0) = y_0,\ y'(x_0) = y_0' \right\}, \tag{10}$$

where $x_0 \in [a, b]$ and $y_0,\ y_0'$ is a couple of given real numbers. Then we have the next fundamental theorem.

Theorem 2 *The linear operator $\widehat{B}$ in (9) and (10) is correct and the initial value problem*

$$\widehat{B}y(x) = f(x), \quad \forall f(x) \in C[a, b], \tag{11}$$

has exactly one solution $y(x) = \widehat{B}^{-1} f(x)$.

We now consider a problem for a first order differential equation and a nonlocal boundary condition, which we will encounter below. For this, we prove the next theorem.

Theorem 3 *Let the general linear first order problem with a nonlocal boundary condition*

$$\begin{aligned} \widehat{Q}y(x) &= y'(x) + q(x)y(x) = f(x), \\ D(\widehat{Q}) &= \left\{ y(x) \in C^1[a, b] :\ \mu_1 y(a) + \mu_2 y(b) = \beta \right\}, \end{aligned} \tag{12}$$

where the operator $\widehat{Q} : C[a, b] \to C[a, b]$, the given functions $q(x),\ f(x) \in C[a, b]$, and the constants $\mu_1, \mu_2, \beta \in \mathbb{R}$. If

$$\mu_1 + \mu_2 e^{-\int_a^b q(t)dt} \neq 0, \tag{13}$$

then the operator $\widehat{Q}$ is correct and the unique solution of problem (12) is given by

$$y(x) = \widehat{Q}^{-1} f(x) = e^{-\int_a^x q(t)dt} \left(C + \int_a^x f(t) e^{\int_a^t q(\tau)d\tau} dt \right), \tag{14}$$

where

$$C = \left(\mu_1 + \mu_2 e^{-\int_a^b q(t)dt}\right)^{-1} \left(\beta - \mu_2 e^{-\int_a^b q(t)dt} \int_a^b f(t) e^{\int_a^t q(\tau)d\tau} dt \right).$$

Proof It is known that the general solution of the first order differential equation in (12) is

$$y(x) = e^{-\int_{x_0}^x q(t)dt} \left(C + \int_{x_0}^x f(t) e^{\int_{t_0}^t q(\tau)d\tau} dt \right), \tag{15}$$

where $x_0 \in [a, b]$. For $x_0 = a$, we have

$$y(a) = C, \quad y(b) = e^{-\int_a^b q(t)dt} \left(C + \int_a^b f(t) e^{\int_a^t q(\tau)d\tau} dt \right).$$

Substituting these values into the boundary condition in (12), we obtain

$$\left(\mu_1 + \mu_2 e^{-\int_a^b q(t)dt}\right) C = \beta - \mu_2 e^{-\int_a^b q(t)dt} \int_a^b f(t) e^{\int_a^t q(\tau)d\tau} dt.$$

If relation (13) holds, then

$$C = \left(\mu_1 + \mu_2 e^{-\int_a^b q(t)dt}\right)^{-1} \left(\beta - \mu_2 e^{-\int_a^b q(t)dt} \int_a^b f(t) e^{\int_a^t q(\tau)d\tau} dt \right). \tag{16}$$

From (15) and (16) it is implied (14). □

3 Factorization Method for Linear Differential Equations

Let the linear differential operators of first order $L_1 : C[a, b] \to C[a, b]$ and $L_2 : C[a, b] \to C[a, b]$ be defined by

$$L_1 u(x) = [D + r(x)]\, u(x), \quad D(L_1) = C^1[a, b], \tag{17}$$

$$L_2 u(x) = [D + s(x)]\, u(x), \quad D(L_2) = C^1[a, b], \tag{18}$$

respectively, where $D = \frac{d}{dx}$, and the coefficients $r(x) \in C[a, b]$ and $s(x) \in C^1[a, b]$. Consider the composition,

$$
\begin{aligned}
L_1 L_2 u(x) &= L_1 (L_2 u(x)) \\
&= [D + r(x)] ([D + s(x)] u(x)) \\
&= \Big[D^2 + (r(x) + s(x)) D + (s'(x) + r(x)s(x)) \Big] u(x). \qquad (19)
\end{aligned}
$$

This gives rise to the following proposition.

Proposition 1 *Let the linear differential operator of second order $L : C[a, b] \to C[a, b]$ be defined by*

$$
Lu(x) = \Big[D^2 + p(x)D + q(x) \Big] u(x), \quad D(L) = C^2[a, b], \qquad (20)
$$

where the coefficients $p(x)$, $q(x) \in C[a, b]$. If there exist two functions $r(x) \in C[a, b]$ and $s(x) \in C^1[a, b]$ satisfying the relations

$$
r(x) + s(x) = p(x), \qquad (21)
$$

$$
s'(x) + r(x)s(x) = q(x), \qquad (22)
$$

then the operator L can be factorized into a product of the two linear differential operators of first order L_1, L_2 in (17) and (18), respectively, such that

$$
Lu(x) = L_1 L_2 u(x). \qquad (23)
$$

Remark 1 By solving equation (21) with respect to $r(x)$ and then substituting into (22), we get

$$
r(x) = p(x) - s(x), \qquad (24)
$$

$$
s'(x) + p(x)s(x) - s^2(x) = q(x), \qquad (25)
$$

where (25) is the nonlinear Riccati equation.

Consider the linear second order initial value problem

$$
Lu(x) = f(x), \quad u(x_0) = \beta_1, \quad u'(x_0) = \beta_2, \qquad (26)
$$

where $f(x) \in C[a, b]$ is a forcing function, x_0 is a point in $[a, b]$, $\beta_i \in \mathbb{R}$, $i = 1, 2$, and $u(x) \in C^2[a, b]$ is the unknown function describing the response of the system modeled by (26). If (21) and (22) hold true, then this problem can be factorized and solved in closed form as it is shown in the next theorem.

Theorem 4 *Let L be the linear differential operator of second order in (20) and* $\widehat{L}$ *be its restriction on*

$$D(\widehat{L}) = \{u(x) \in D(L) : u(x_0) = \beta_1, \ u'(x_0) = \beta_2\}, \tag{27}$$

where $x_0 \in [a, b]$ *and* $\beta_1, \beta_2 \in \mathbb{R}$. *If the prerequisites (21) and (22) are fulfilled then:*

(i) The operator $\widehat{L}$ *can be factorized as*

$$\widehat{L}u(x) = \widehat{L}_1\widehat{L}_2u(x), \tag{28}$$

where $\widehat{L}_1, \widehat{L}_2$ *are correct restrictions of the linear first order differential operators* L_1, L_2, *defined in (17) and (18), on*

$$D(\widehat{L}_1) = \{z(x) \in D(L_1) : z(x_0) = \beta_2 + s(x_0)\beta_1\}, \tag{29}$$

$$D(\widehat{L}_2) = \{u(x) \in D(L_2) : u(x_0) = \beta_1\}, \tag{30}$$

respectively.

(ii) The operator $\widehat{L}$ *is correct and the unique solution of the initial value problem*

$$\widehat{L}u(x) = f(x), \quad \forall f(x) \in C[a, b], \tag{31}$$

is given in closed form by

$$\begin{aligned} u(x) &= \widehat{L}^{-1}u(x) = \widehat{L}_2^{-1}\widehat{L}_1^{-1}f(x) = \widehat{L}_2^{-1}z(x) \\ &= e^{-\int_{x_0}^{x} s(t)dt}\left(\beta_1 + \int_{x_0}^{x} z(t)e^{\int_{t_0}^{t} s(\tau)d\tau}dt\right), \end{aligned} \tag{32}$$

where

$$z(x) = \widehat{L}_1^{-1}f(x) = e^{-\int_{x_0}^{x} r(t)dt}\left(\beta_2 + s(x_0)\beta_1 + \int_{x_0}^{x} f(t)e^{\int_{t_0}^{t} r(\tau)d\tau}dt\right). \tag{33}$$

Proof

(i) From the definition of $\widehat{L}$ and Proposition 1, we have

$$Lu(x) = L_1L_2u(x) = f(x), \quad u(x_0) = \beta_1, \quad u'(x_0) = \beta_2. \tag{34}$$

By setting

$$L_2u(x) = u'(x) + s(x)u(x) = z(x), \tag{35}$$

we get

$$L_1 z(x) = z'(x) + r(x)z(x) = f(x). \tag{36}$$

From (35) it is implied that

$$u'(x_0) = z(x_0) - s(x_0)u(x_0),$$

which when is substituted into the second condition in (34) yields

$$z(x_0) = \beta_2 + s(x_0)\beta_1.$$

Whence, we have the two linear first order initial value problems

$$L_1 z(x) = f(x), \quad z(x_0) = \beta_2 + s(x_0)\beta_1, \tag{37}$$

$$L_2 u(x) = z(x), \quad u(x_0) = \beta_1. \tag{38}$$

That is $\widehat{L}u(x) = \widehat{L}_1\widehat{L}_2 u(x)$. It remains to show that $D(\widehat{L}) = D(\widehat{L}_1\widehat{L}_2)$. By using (29) and (30), we obtain

$$\begin{aligned} D(\widehat{L}_1\widehat{L}_2) &= \{u(x) \in D(\widehat{L}_2) : \widehat{L}_2 u(x) \in D(\widehat{L}_1)\} \\ &= \{u(x) \in D(L_2) : u(x_0) = \beta_1,\ u'(x) + s(x)u(x) \in D(\widehat{L}_1)\} \\ &= \left\{u(x) \in C^2[a,b] : u(x_0) = \beta_1,\ u'(x_0) + s(x_0)u(x_0) = \beta_2 + s(x_0)\beta_1\right\} \\ &= \left\{u(x) \in C^2[a,b] : u(x_0) = \beta_1,\ u'(x_0) = \beta_2\right\}. \end{aligned} \tag{39}$$

(ii) The linear first order initial value problem (37) possesses exactly one solution $z(x)$, which can be found by using the standard means, such as the method of integrating factors [2], and is given in (33). Having obtained $z(x)$, we can solve the linear first order initial value problem (38) in like manner to obtain the solution $u(x)$ in (32), which is the solution of the linear second order initial value problem (31). The operator $\widehat{L} = \widehat{L}_1\widehat{L}_2$ is correct because $\widehat{L}_1$ and $\widehat{L}_2$ are correct.

□

The factorization method also applies to some types of boundary value problems, although it is more complicated. Let the linear second order differential equation,

$$Lu(x) = u''(x) + p(x)u(x) + q(x)u(x) = f(x), \quad x \in (a,b), \tag{40}$$

where the coefficients $p(x),\ q(x) \in C[a,b]$ and $f(x) \in C[a,b]$, and assume that the operator $L : C[a,b] \to C[a,b]$ is factorable, i.e. there exist $r(x) \in C[a,b]$ and

$s(x) \in C^1[a, b]$ such that $r(x) + s(x) = p(x)$ and $s'(x) + r(x)s(x) = q(x)$. Let also the two boundary conditions

$$\mu_{11}u(a) + \mu_{12}u(b) = \beta_1,$$
$$\mu_{21}[u'(a) + s(a)u(a)] + \mu_{22}[u'(b) + s(b)u(b)] = \beta_2, \tag{41}$$

where $\mu_{ij}, \ \beta_i \in \mathbb{R}, i = 1, 2, \ j = 1, 2$. Notice that (41) are the boundary conditions as in (2) when

$$\mu_{23} = s(a)\mu_{21}, \quad \mu_{24} = s(b)\mu_{22}. \tag{42}$$

We claim that the boundary value problem for the differential equation (40) and the boundary conditions (41) can be factorized and solved explicitly. We prove the following theorem.

Theorem 5 *Let L be the linear second order differential operator in (40) and assume that there exist two functions $r(x) \in C[a, b]$ and $s(x) \in C^1[a, b]$ which satisfy (21) and (22). Let $\bar{L}$ be a restriction of L on*

$$D(\bar{L}) = \{u(x) : u(x) \in D(L), \ \mu_{11}u(a) + \mu_{12}u(b) = \beta_1,$$
$$\mu_{21}[u'(a) + s(a)u(a)] + \mu_{22}[u'(b) + s(b)u(b)] = \beta_2\}, \tag{43}$$

where $\mu_{ij}, \ \beta_i \in \mathbb{R}, i = 1, 2, \ j = 1, 2$. Then:

(i) The operator $\bar{L}$ can be factorized as follows

$$\bar{L}u(x) = \bar{L}_1\bar{L}_2u(x), \tag{44}$$

where $\bar{L}_1, \bar{L}_2$ are restrictions of the two first order linear differential operators L_1, L_2, defined in (17) and (18), on

$$D(\bar{L}_1) = \left\{ z(x) \in C^1[a, b] : \ \mu_{21}z(a) + \mu_{22}z(b) = \beta_2 \right\}, \tag{45}$$

$$D(\bar{L}_2) = \left\{ u(x) \in C^1[a, b] : \ \mu_{11}u(a) + \mu_{12}u(b) = \beta_1 \right\}, \tag{46}$$

respectively.

(ii) If

$$\mu_{21} + \mu_{22}e^{-\int_a^b r(t)dt} \neq 0, \quad \mu_{11} + \mu_{12}e^{-\int_a^b s(t)dt} \neq 0, \tag{47}$$

then the operator $\bar{L}$ is correct and the unique solution of the boundary value problem

$$\bar{L}u(x) = f(x), \quad \forall f(x) \in C[a, b], \tag{48}$$

is given by

$$u(x) = \bar{L}^{-1}f(x) = \bar{L}_2^{-1}\bar{L}_1^{-1}f(x) = \bar{L}_2^{-1}z(x)$$
$$= e^{-\int_a^x s(t)dt}\left(C_2 + \int_a^x z(t)e^{\int_a^t s(\tau)d\tau}dt\right), \tag{49}$$

where

$$z(x) = \bar{L}_1^{-1}f(x) = e^{-\int_a^x r(t)dt}\left(C_1 + \int_a^x f(t)e^{\int_a^t r(\tau)d\tau}dt\right), \tag{50}$$

and

$$C_1 = \left(\mu_{21} + \mu_{22}e^{-\int_a^b r(t)dt}\right)^{-1}\left(\beta_2 - \mu_{22}e^{-\int_a^b r(t)dt}\int_a^b f(t)e^{\int_a^t r(\tau)d\tau}dt\right),$$
$$C_2 = \left(\mu_{11} + \mu_{12}e^{-\int_a^b s(t)dt}\right)^{-1}\left(\beta_1 - \mu_{12}e^{-\int_a^b s(t)dt}\int_a^b z(t)e^{\int_a^t s(\tau)d\tau}dt\right).$$

Proof

(i) From the definition of $\bar{L}$ and Proposition 1, we have

$$Lu(x) = L_1L_2u(x) = f(x), \tag{51}$$

and

$$\mu_{11}u(a) + \mu_{12}u(b) = \beta_1,$$
$$\mu_{21}[u'(a) + s(a)u(a)] + \mu_{22}[u'(b) + s(b)u(b)] = \beta_2. \tag{52}$$

Let

$$L_2u(x) = u'(x) + s(x)u(x) = z(x). \tag{53}$$

It follows that

$$u'(a) + s(a)u(a) = z(a), \quad u'(b) + s(b)z(b) = z(b),$$

and upon substitution into the second boundary condition in (52), we get

$$\mu_{21}z(a) + \mu_{22}z(b) = \beta_2.$$

Thus, we have

$$L_1z(x) = z'(x) + r(x)z(x) = f(x), \quad \mu_{21}z(a) + \mu_{22}z(b) = \beta_2, \tag{54}$$

$$L_2u(x) = u'(x) + s(x)u(x) = z(x), \quad \mu_{11}u(a) + \mu_{12}u(b) = \beta_1. \tag{55}$$

That is $\bar{L}u(x) = \bar{L}_1\bar{L}_2u(x)$. It remains to show that $D(\bar{L}) = D(\bar{L}_1\bar{L}_2)$. By using the definition of $D(\bar{L}_1\bar{L}_2)$ we obtain

$$\begin{aligned} D(\bar{L}_1\bar{L}_2) &= \{u(x) \in D(\bar{L}_2) : \bar{L}_2u(x) \in D(\bar{L}_1)\} \\ &= \Big\{u(x) \in C^1[a,b] : \mu_{11}u(a) + \mu_{12}u(b) = \beta_1, \\ &\quad u'(x) + s(x)u(x) \in D(\bar{L}_1)\Big\} \\ &= \Big\{u(x) \in C^1[a,b] : \mu_{11}u(a) + \mu_{12}u(b) = \beta_1, \\ &\quad z(x) = u'(x) + s(x)u(x) \in C^1[a,b], \\ &\quad \mu_{21}[u'(a) + s(a)u(a)] + \mu_{22}[u'(b) + s(b)u(b)] = \beta_2\Big\} \\ &= \Big\{u(x) \in C^2[a,b] : \mu_{11}u(a) + \mu_{12}u(b) = \beta_1, \\ &\quad \mu_{21}[u'(a) + s(a)u(a)] + \mu_{22}[u'(b) + s(b)u(b)] = \beta_2\Big\} \\ &= D(\bar{L}). \end{aligned} \tag{56}$$

(ii) Application of Theorem 3 to solve boundary value problem (54) yields (50). Substituting this unique solution $z(x) = \bar{L}_1^{-1}f(x)$ into (55) and applying Theorem 3 once more, we obtain (49), which is the solution to boundary value problem (48). The correctness of $\bar{L} = \bar{L}_1\bar{L}_2$ follows from the correctness of $\bar{L}_1$ and $\bar{L}_2$.

□

To elucidate the implementation of the above procedure, we solve the following example problem.

Example 1 Let the boundary value problem

$$\begin{gathered} u''(x) - \tfrac{x+2}{x+1}u'(x) + \tfrac{1}{x+1}u(x) = 3(x+1), \quad 0 < x < 1, \\ u(0) - 5u(1) = 0, \\ 3u'(0) - 4u'(1) - 3u(0) + 4u(1) = 2. \end{gathered} \tag{57}$$

We take

$$p(x) = -\frac{x+2}{x+1}, \quad \text{and} \quad q(x) = \frac{1}{x+1},$$

which are continuous on [0, 1]. Notice that equations (21) and (22) are satisfied by

$$r(x) = -\frac{1}{x+1}, \quad \text{and} \quad s(x) = -1,$$

which are continuous on [0, 1] and $s'(x) = 0$. Lastly, the second of the boundary conditions (57) can be put in the form

$$3[u'(0) + (-1)u(0)] - 4[u'(1) + (-1)u(1)] = 2.$$

Thus (57) is carried to

$$\begin{aligned} \bar{L}u(x) &= u''(x) - \frac{x+2}{x+1}u'(x) + \frac{1}{x+1}u(x) = f(x), \\ D(\bar{L}) &= \Big\{u(x) : u(x) \in C^2[0, 1],\ u(0) - 5u(1) = 0, \\ &\qquad 3[u'(0) + s(0)u(0)] - 4[u'(1) + s(1)u(1)] = 2\Big\}, \end{aligned} \tag{58}$$

where $f(x) = 3(x+1)$. By Theorem 5, the operator $\bar{L}$ can be factorized as $\bar{L}u(x) = \bar{L}_1\bar{L}_2u(x)$, where

$$\bar{L}_1z(x) = z'(x) - \frac{1}{x+1}z(x), \quad D(\bar{L}_1) = \Big\{z(x) \in C^1[0, 1] :\ 3z(0) - 4z(1) = 2\Big\},$$

$$\bar{L}_2u(x) = u'(x) - u(x), \quad D(\bar{L}_2) = \Big\{u(x) \in C^1[0, 1] :\ u(0) - 5u(1) = 0\Big\}.$$

Furthermore,

$$\mu_{21} + \mu_{22}e^{-\int_0^1 r(t)dt} = -5 \neq 0, \quad \mu_{11} + \mu_{12}e^{-\int_0^1 s(t)dt} = 1 - 5e \neq 0, \tag{59}$$

and therefore (58) has only one solution. To construct the solution, we first solve the problem $\bar{L}_1u(x) = f(x)$ by means of (50), which yields

$$z(x) = (x+1)(3x - \frac{26}{5}). \tag{60}$$

Then by utilizing (60) and solving $\bar{L}_2u(x) = z(x)$ by (49), we get

$$u(x) = \frac{142e^x}{5(5e-1)} - \frac{15x^2 + 19x - 7}{5}. \tag{61}$$

This is the unique solution of the given boundary value problem (57).

4 Factorization Method for Nonlinear Differential Equations

In this section, we deal with the solution of a class of nonlinear boundary value problems for second order differential equations. Let the nonlinear differential equation of the form

$$u''(x)u'(x) + \left[q(x)u'(x) + g(x)u''(x)\right]u(x) + q(x)g(x)(u(x))^2 = 0, \tag{62}$$

for $x \in (a, b)$, and where $q(x),\ g(x) \in C[a, b]$, together with the boundary conditions

$$\begin{aligned} \mu_{11}u(a) + \mu_{12}u(b) &= \beta_1, \\ \mu_{21}u'(a) + \mu_{22}u'(b) + \mu_{23}u(a) + \mu_{24}u(b) &= \beta_2, \end{aligned} \tag{63}$$

where $\mu_{ij},\ \beta_i \in \mathbb{R},\ i = 1, 2,\ j = 1, 2, 3, 4.$

The nonlinear equation (62) can be decomposed as the product

$$\left[u''(x) + q(x)u(x)\right]\left[u'(x) + g(x)u(x)\right] = 0,$$

and hence, either

$$u''(x) + q(x)u(x) = 0, \tag{64}$$

or

$$u'(x) + g(x)u(x) = 0. \tag{65}$$

As a consequence, the solutions of the nonlinear boundary value problem (62) and (63) may be obtained by solving the linear second order problem (64) and (63) and the linear first order problem (65) and (63).

For the solution of the linear second order problem (64) and (63), we may employ Theorem 5 provided that prerequisites (21) and (22) are satisfied, i.e. there exist $r(x) \in C[a, b]$ and $s(x) \in C^1[a, b]$ such that

$$r(x) = -s(x), \quad s'(x) - (s(x))^2 = q(x), \tag{66}$$

and if

$$\mu_{23} = s(a)\mu_{21}, \quad \mu_{24} = s(b)\mu_{22}. \tag{67}$$

In this case problem (64), (63) can be put in the form

$$\bar{L}u(x) = u''(x) + q(x)u(x) = 0,$$

$$D(\bar{L}) = \Big\{u(x) \in C^2[a,b]:\ \mu_{11}u(a) + \mu_{12}u(b) = \beta_1,$$
$$\mu_{21}[u'(a) + s(a)u(a)] + \mu_{22}[u'(b) + s(b)u(b)] = \beta_2\Big\}. \tag{68}$$

Problem (68) can be now solved by means of Theorem 5.

The linear first order problem (65) and (63) is subjected to more conditions than the order of the differential equation and it is most likely to possess no solution. Nevertheless, we can proceed as follows. By utilizing (65) evaluate $u'(a)$ and $u'(b)$ and substitute into the second of the boundary conditions in (63). Taking into account (67), we get

$$\mu_{21}[s(a) - g(a)]u(a) + \mu_{22}[s(b) - g(b)]u(b) = \beta_2. \tag{69}$$

Thus, problem (65) and (63) may be formulated as

$$Tu(x) = u'(x) + g(x)u(x) = 0,$$
$$D(T) = \{u(x) \in C^1[0,1]:\ \mu_{11}u(a) + \mu_{12}u(b) = \beta_1,$$
$$\mu_{21}[s(a) - g(a)]u(a) + \mu_{22}[s(b) - g(b)]u(b) = \beta_2\}. \tag{70}$$

By employing Theorem 3, we find the unique solution of the problem

$$T_0u(x) = u'(x) + g(x)u(x) = 0,$$
$$D(T_0) = \{u(x) \in C^1[0,1]:\ \mu_{11}u(a) + \mu_{12}u(b) = \beta_1\}. \tag{71}$$

If the solution $u(x)$ of this problem satisfies the second boundary condition in (70), then $u(x)$ is a solution of (70); otherwise (70) has no solution.

Example 2 Let us find the solutions of the nonlinear second order boundary value problem

$$u''(x)u'(x) - \left[\tfrac{2}{(x+1)^2}u'(x) + \tfrac{1}{x+3}u''(x)\right]u(x) + \tfrac{2}{(x+3)(x+1)^2}(u(x))^2 = 0,$$
$$u(0) + 5u(1) = 0,$$
$$-u'(0) + 6u'(1) - u(0) + 3u(1) = 4, \tag{72}$$

where $x \in [0,1]$ and $u(x) \in C^2[0,1]$.

The nonlinear second order differential equation (72) is of the type (62) with

$$q(x) = -\frac{2}{(x+1)^2}, \quad g(x) = -\frac{1}{x+3},$$

and it can be decomposed as

$$\left[u''(x) - \frac{2}{(x+1)^2}u(x)\right]\left[u'(x) - \frac{1}{x+3}u(x)\right] = 0.$$

Thus, we get the following two linear problems

$$\begin{aligned}
\bar{L}u(x) &= u''(x) - \frac{2}{(x+1)^2}u(x) = 0,\\
D(\bar{L}) &= \{u(x) \in C^2[0,1] : u(0) + 5u(1) = 0,\\
&\qquad -u'(0) + 6u'(1) - u(0) + 3u(1) = 4\},
\end{aligned} \tag{73}$$

and

$$\begin{aligned}
Tu(x) &= u'(x) - \frac{1}{x+3}u(x) = 0,\\
D(T) &= \{u(x) \in C^2[0,1] : u(0) + 5u(1) = 0,\\
&\qquad -u'(0) + 6u'(1) - u(0) + 3u(1) = 4\}.
\end{aligned} \tag{74}$$

In solving the boundary value problem (73), notice that the functions

$$r(x) = -\frac{1}{x+1}, \quad s(x) = \frac{1}{x+1},$$

obey (66), $r(x) \in C[0,1]$, $s(x) \in C^1[0,1]$ and $s(0) = 1$, $s(1) = \frac{1}{2}$, and that the preconditions (67) are met. Hence, problem (73) may be written in the form (68), namely

$$\begin{aligned}
\bar{L}u(x) &= u''(x) - \frac{2}{(x+1)^2}u(x) = 0,\\
D(\bar{L}) &= \{u(x) \in C^2[0,1] : u(0) + 5u(1) = 0,\\
&\qquad -[u'(0) + s(0)u(0)] + 6[u'(1) + s(1)u(1)] = 4\}.
\end{aligned} \tag{75}$$

By Theorem 5, the boundary value problem (75) is factorized into the following two first order problems

$$\begin{aligned}
\bar{L}_1 z(x) &= z'(x) - \frac{1}{x+1}z(x) = 0,\\
D(\bar{L}_1) &= \left\{z(x) \in C^1[0,1] :\ -z(0) + 6z(1) = 4\right\},
\end{aligned} \tag{76}$$

and

$$\bar{L}_2 u(x) = u'(x) + \frac{1}{x+1} u(x) = z(x),$$
$$D(\bar{L}_2) = \left\{ u(x) \in C^1[0,1] : \, u(0) + 5u(1) = 0 \right\}. \tag{77}$$

The first of the uniqueness requirements (47) is fulfilled, viz.

$$\mu_{21} + \mu_{22} e^{-\int_0^1 r(t)dt} = -1 + 6 \left(e^{\int_0^1 \frac{1}{t+1} dt} \right) = 11 \neq 0, \tag{78}$$

and therefore the operator $\bar{L}_1$ is correct and the unique solution of (76) is derived through (50), which is

$$z(x) = \frac{4}{11}(x+1). \tag{79}$$

By substituting (79) into (77) and verifying that the second of the uniqueness conditions (47) is also satisfied, viz.

$$\mu_{11} + \mu_{12} e^{-\int_0^1 s(t)dt} = 1 + 5 \left(e^{-\int_0^1 \frac{1}{t+1} dt} \right) = \frac{7}{2} \neq 0, \tag{80}$$

it follows that the operator $\bar{L}_2$ is correct and the unique solution of (77), obtained via (49), is

$$u(x) = \frac{4(x^3 + 3x^2 + 3x - 5)}{33(x+1)}. \tag{81}$$

The function $u(x)$ in (81) is a solution to nonlinear second order boundary value problem (72).

We now examine the existence of a solution of the linear first order problem (74). By applying Theorem 3, we find that the problem

$$T_1 u(x) = u'(x) - \frac{1}{x+3} u(x) = 0,$$
$$D(T_1) = \{u(x) \in C^2[0,1] : \, u(0) + 5u(1) = 0\} \tag{82}$$

has no solution except the trivial $u(x) = 0$, which however does not satisfy the second of the boundary conditions in (74).

Summing up, the nonlinear second order boundary value problem (72) admits only the solution (81).

The technique presented above and explained in Example 2 can be extended to solve and other types of nonlinear boundary value problems. For example, consider the nonlinear differential equation of second order of the form

$$(u''(x))^2 + a(x)u''(x)u(x) + b(x)(u(x))^2 = 0, \quad x \in (a, b), \tag{83}$$

subject to two general boundary constraints

$$\begin{aligned} \mu_{11}u(a) + \mu_{12}u(b) &= \beta_1, \\ \mu_{21}u'(a) + \mu_{22}u'(b) + \mu_{23}u(a) + \mu_{24}u(b) &= \beta_2, \end{aligned} \tag{84}$$

where $a(x),\ b(x) \in C[a, b]$ and $\mu_{ij},\ \beta_i \in \mathbb{R}, i = 1, 2,\ j = 1, 2, 3, 4$.

The differential equation (83) can be put in the form

$$F\left(\frac{u''(x)}{u(x)}, x\right) = F\left(w(x), x\right) = (w(x))^2 + a(x)w(x) + b(x) = 0,$$

where $w(x) = u''(x)/u(x)$ and the nonlinear function F is a second degree polynomial of $w(x)$. Hence, it can be decomposed as

$$\left[w(x) + q^-(x)\right]\left[w(x) + q^+(x)\right] = 0,$$

where $q^-(x),\ q^+(x) \in C[a, b]$ and $a(x) = q^-(x) + q^+(x),\ b(x) = q^-(x)q^+(x)$. By substituting back $w(x) = u''(x)/u(x)$, we get

$$\left[u''(x) + q^-(x)u(x)\right]\left[u''(x) + q^+(x)u(x)\right] = 0, \tag{85}$$

from where follows that, either

$$u''(x) + q^-(x)u(x) = 0, \quad x \in (a, b), \tag{86}$$

or

$$u''(x) + q^+(x)u(x) = 0, \quad x \in (a, b). \tag{87}$$

Thus, the solution of the nonlinear boundary value problem (83) and (84) is reduced to the solution of the two linear second order boundary value problems (86), (84), and (87), (84). Whenever the conditions (21), (22) and (42) are met, Theorem 5 may be applied to acquire the solutions in closed form.

5 Conclusions

A practical technique has been presented for factorizing and solving linear initial and boundary value problems for second order differential equations with nonlocal boundary conditions. Two types of nonlinear boundary value problems for second order differential equations have also been considered where the factorization method was used to construct their solutions in closed form. The main advantage of the factorization method is that no fundamental or particular solutions are required. Its main disadvantage is that it cannot be applied to all boundary value problems except to those where certain conditions are satisfied. The efficiency of the method encourages the pursuit of further research for the extension of the method to problems with fully mixed boundary conditions and multipoint conditions.

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