Piecewise Polynomial Inversion of the Radon Transform in Three Space Dimensions via Plane Integration and Applications in Positron Emission Tomography

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Abstract The inversion of the celebrated Radon transform in three dimensions involves two-dimensional plane integration. This inversion provides the mathematical foundation of the important field of medical imaging, known as three-dimensional positron emission tomography (3D PET). In this chapter, we present an analytical expression for the inversion of the three-dimensional Radon transform, as well as a novel numerical implementation of this formula, based on piecewise polynomials of the third degree.

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1 Introduction

The celebrated Radon transform of a two-dimensional function is defined as the set of all its line integrals [1, 2]. The transition to three space dimensions yields a certain generalization of line integration, namely *plane integration*. Indeed, the Radon transform of a three-dimensional function is defined as the set of all its plane integrals.[1] The inversion of the three-dimensional Radon transform provides the mathematical foundation of the important field of medical imaging, known as 3D positron emission tomography (3D PET). The 3D Radon transform gives rise to an associated *inverse problem*, namely to “reconstruct” a function from its plane integrals. The main task in 3D PET imaging is the numerical implementation of the inversion of the 3D Radon transform.

In 3D PET, contrary to the conventional 2D PET, there is a certain generalization of the notion of image reconstruction: in the 2D case, the integration occurs in planes instead of lines. The difficulties arising in the 2D cases and their generalizations [3] are overcome in the 3D case. The inversion of the 3D Radon transform seems more straightforward than the one of the conventional Radon transform [4]. There are several numerical implementation methods in the literature, including: (i) the introduction of the concept of three-dimensional image reconstruction from “complete” projections [5]; (ii) the formulation of the 3D Radon transform for discrete 3D images (volumes), based on the summation over planes with small absolute slopes [6]; and (iii) the reconstruction of conductivities in the context of electric impedance tomography (EIT) [7]. The differences between 2D and 3D Radon transform inversion are emphasized in [8], and [9], where an analytic filter-backprojection method is introduced based on the spatially invariant detector point spread function. The authors of [10] proposed a spline-based inversion of the Radon transform in two and three dimensions; also the PET image reconstruction algorithms proposed in [11] show that analytic algorithms in 3D are linear and therefore allow easier control of the spatial resolution and noise correlations than in the case of the 2D reconstructions.

In this chapter, we present a novel formula for the inversion of the 3D Radon transform, as well as a novel numerical implementation of this formula, based on piecewise polynomial interpolation. We expect that our novel numerical implementation will enhance three-dimensional medical image reconstruction, especially in the case of 3D PET.

[1] Plane integrals are special cases of surface integrals, where the surface of integration is a plane.

2 The Radon Transform in Two Space Dimensions

In order to elucidate the properties of the three-dimensional Radon transform, it is essential to review the corresponding properties of the two-dimensional Radon transform.

A line L on the plane can be specified by the signed distance from the origin ρ, with $-\infty < \rho < \infty$, and the angle with the x_1-axis θ, with $0 \leq \theta < 2\pi$, as in Figure 1. We denote the corresponding unit vectors perpendicular and parallel to L by $\mathbf{n}$ and $\mathbf{p}$, respectively. These unit vectors are given by

$$\mathbf{n} = (-\sin\theta, \cos\theta)^{\mathrm{T}} \quad \text{and} \quad \mathbf{p} = (\cos\theta, \sin\theta)^{\mathrm{T}}, \tag{1}$$

with

$$\mathbf{n} \cdot \mathbf{p} = 0. \tag{2}$$

Every point $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$ lying on the line L in Cartesian coordinates can be expressed in terms of the so-called *local coordinates* (ρ, τ) via

$$\mathbf{x} = \rho\,\mathbf{n} + \tau\mathbf{p},$$

where τ denotes the arc length. Therefore, we parameterize each point $\mathbf{x}$ on the line L in the following manner:

$$\mathbf{x} := \mathbf{x}(\rho, \tau; \theta) = \begin{bmatrix} x_1(\rho, \tau; \theta) \\ x_2(\rho, \tau; \theta) \end{bmatrix} = \begin{bmatrix} \tau\cos\theta - \rho\sin\theta \\ \tau\sin\theta + \rho\cos\theta \end{bmatrix} \tag{3}$$

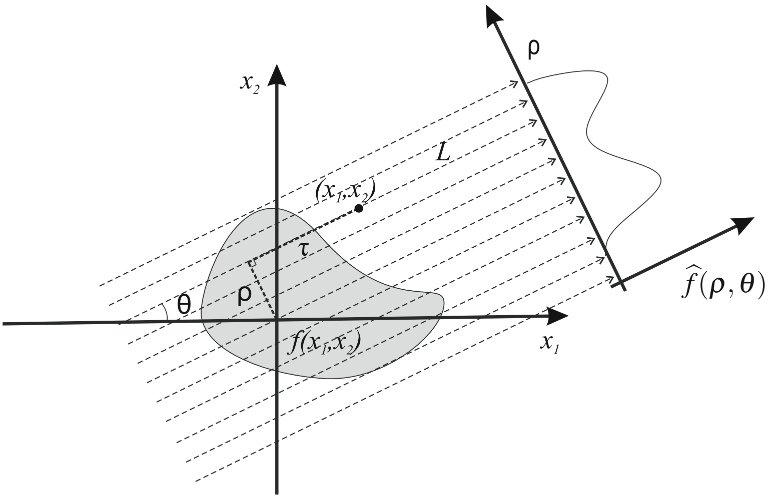

Fig. 1 A two-dimensional function $f(x_1, x_2)$ expressed in Cartesian coordinates, and its projections $\widehat{f}(\rho, \theta)$, expressed in local coordinates

Through Equation (3), we can express the local coordinates (ρ, τ) in terms of Cartesian coordinates (x_1, x_2) and the associated angle θ:

$$\begin{bmatrix}\rho\\ \tau\end{bmatrix} := \begin{bmatrix}\rho(x_1, x_2; \theta)\\ \tau(x_1, x_2; \theta)\end{bmatrix} = \begin{bmatrix}x_2\cos\theta - x_1\sin\theta\\ x_2\sin\theta + x_1\cos\theta\end{bmatrix} \tag{4}$$

We define the line integral over all lines L, defined in Equation (1), of a two-dimensional Schwartz function $f : \mathbb{R}^2 \to \mathbb{R}$, $f \in S(\mathbb{R}^2)$, as its *two-dimensional Radon transform*, $\mathscr{R}_2 f$. In the context of 2D PET, the 2D Radon transform of the function f is usually stored in the form of the so-called *sinogram*, denoted by $\widehat{f}(\rho, \theta)$

$$\mathscr{R}_2 f = \widehat{f}(\rho, \theta) = \int_L f \,\mathrm{d}s, \tag{5}$$

where ds denotes an arc length differential, and $S(\mathbb{R}^2)$ denotes the space of Schwartz functions in $\mathbb{R}^2$,

$$S(\mathbb{R}^2) = \left\{ f \in C^\infty(\mathbb{R}^2) : ||f||_{\alpha,\beta} < \infty \right\} \subset C^\infty(\mathbb{R}^2), \tag{6}$$

and

$$||f||_{\alpha,\beta} = \sup_{x\in\mathbb{R}^2} |x^\alpha D^\beta f(x)|, \quad \forall\, \text{multi} - \text{index}\ \alpha, \beta, \quad |x^\alpha D^\beta f(x)| \to 0, \quad \text{as}\ |x| \to \infty. \tag{7}$$

Equation (5) may be rewritten via a parameterization $\mathbf{x} := \mathbf{x}(\tau)$ of the line L, with $\mathbf{x} : \mathbb{R}^2 \to L$, as follows:

$$\widehat{f}(\rho, \theta) = \int_{-\infty}^{\infty} f\left(\mathbf{x}(\tau)\right) \left|\left|\mathbf{x}'(\tau)\right|\right|_2 \mathrm{d}\tau, \tag{8}$$

where $||\cdot||_2$ denotes the L^2-norm in $\mathbb{R}^2$. The parameterization provided by (3) will be proven to be very convenient and easy-to-manipulate, especially for the description of parallel lines. In this case, it is worth noting that

$$\left|\left|\mathbf{x}'(\tau)\right|\right|_2 = \sqrt{\left(\frac{\mathrm{d}x_1}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}x_2}{\mathrm{d}\tau}\right)^2} = \cos^2\theta + \sin^2\theta = 1. \tag{9}$$

Hence, Equation (3) is a natural parameterization of the set of parallel lines L. Therefore, the 2D Radon transform defined in Equation (5) may be expressed as follows:

$$\mathscr{R}_2 f = \widehat{f}(\rho, \theta) = \int_{-\infty}^{\infty} f(\tau\cos\theta - \rho\sin\theta, \tau\sin\theta + \rho\cos\theta)\mathrm{d}\tau, \tag{10}$$

with $0 \leq \theta < 2\pi$ and $-\infty < \rho < \infty$. If we use a Dirac delta function, or a line impulse, then the Radon transform, denoted by $\mathscr{R}_{2\mathrm{D}}$ defined in (10) may be rewritten in the form:

$$\mathscr{R}_2 f = \widehat{f}(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2)\delta(\rho + x_1\sin\theta - x_2\cos\theta)\mathrm{d}x_1\mathrm{d}x_2, \tag{11}$$

taking into account Equation (4).

The 2D Radon transform (10) gives rise to one of the most significant inverse problems in emission tomography. This specific inverse problem implies the “reconstruction” of the function $f(x_1, x_2)$, from its two-dimensional Radon transform, i.e. the function $\widehat{f}(\rho,\theta)$.

3 The Radon Transform in Three Space Dimensions

In the two-dimensional case, the Radon transform is considered on sets of parallel lines. This consideration implies the involvement of line integrals. However, in the three-dimensional case, the Radon transform is restricted on two-dimensional planes. In this direction, the transition to three space dimensions yields a certain generalization of line integration, namely *plane integration*.[2]

Therefore, we define the surface integral over all planes P of a three-dimensional Schwartz function $f : \mathbb{R}^3 \to \mathbb{R}$, $f \in S(\mathbb{R}^3)$, as its *three-dimensional Radon transform*, $\mathscr{R}_3 f$. In the context of 3D PET, the 3D Radon transform of the function f is usually stored in the form of the so-called *3D sinogram*, denoted by $\widehat{f}(\rho,\theta,\phi)$:

$$\mathscr{R}_3 f = \widehat{f}(\rho,\theta,\phi) = \iint_P f\,\mathrm{d}s, \tag{12}$$

where ds denotes an area differential and $S(\mathbb{R}^3)$ denotes the space of Schwartz functions in $\mathbb{R}^3$:

$$S(\mathbb{R}^3) = \left\{ f \in C^\infty(\mathbb{R}^3) : ||f||_{\alpha,\beta} < \infty \right\} \subset C^\infty(\mathbb{R}^3), \tag{13}$$

and

$$||f||_{\alpha,\beta} = \sup_{x\in\mathbb{R}^3} |x^\alpha D^\beta f(x)|, \quad \forall\, \text{multi}-\text{index}\ \alpha,\beta, \quad |x^\alpha D^\beta f(x)| \to 0, \quad \text{as } |x| \to \infty. \tag{14}$$

Equation (12) may be rewritten via a parameterization $\mathbf{x} := \mathbf{x}(u, v)$ of the plane P, with $\mathbf{x} : \mathbb{R}^3 \to P$, as follows:

[2] Plane integrals are special cases of surface integrals, where the surface of integration is a plane.

$$\widehat{f}(\rho,\theta,\phi) = \int_{-\infty}^{\infty} f\left(\mathbf{x}(u,v)\right) \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \mathrm{d}u\mathrm{d}v, \tag{15}$$

where, for the area differential we employed

$$\mathrm{d}s = \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \mathrm{d}u\mathrm{d}v. \tag{16}$$

In the three-dimensional setting, for convenience, we characterize each two-dimensional plane P by a vector and a scalar, namely:

(i) the unit normal vector $\mathbf{n}$

$$\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}, \quad \text{with} \quad \sqrt{n_1^2 + n_2^2 + n_3^2} = 1, \quad \text{and} \tag{17}$$

(ii) the signed distance from the origin ρ.

The normal from the origin to the plane intersects the plane at the point $\rho\mathbf{n}$. Thus, if $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}$ is a point on the plane under investigation, then

$$(\rho\mathbf{n} - \mathbf{x}) \cdot \mathbf{n} = 0 \tag{18}$$

The above implies

$$\begin{bmatrix} \rho n_1 - x_1 \\ \rho n_2 - x_2 \\ \rho n_3 - x_3 \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0,$$

or

$$\rho(n_1^2 + n_2^2 + n_3^2) - (n_1 x_1 + n_2 x_2 + n_3 x_3) = 0.$$

Hence, the equation of the plane is

$$\rho - \mathbf{n} \cdot \mathbf{x} = 0, \quad \forall\, \mathbf{x} \in P. \tag{19}$$

We suppose that the plane of integration P, specified by its signed distance from the origin ρ and its unit normal vector $\mathbf{n}$, intersects the x_1x_2-plane in an angle θ, and the x_2x_3-plane in an angle ϕ (spherical angles). In this connection, the unit normal vector $\mathbf{n}$ is uniquely specified by the two spherical angles, i.e. $\mathbf{n} := \mathbf{n}(\theta, \phi)$. Thus $f(\rho, \mathbf{n})$ involves three variables, namely $f(\rho, \mathbf{n}) = f(\rho, \theta, \phi)$. In this direction, we characterize $\mathbf{n}$ in terms of spherical angles, as follows:

$$\mathbf{n}(\theta, \phi) = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}, \quad 0 \le \theta \le \pi \quad 0 \le \phi \le 2\pi. \tag{20}$$

If a point $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}$ lies on the plane of integration P, i.e. $\mathbf{x} \in P$, then Equation (19) implies

$$\rho - x_1 \sin\theta \cos\phi - x_2 \sin\theta \sin\phi - x_3 \cos\theta = 0. \tag{21}$$

Equation (21) provides a convenient parameterization of the plane under investigation via x_1 and x_2, treated hereafter as independent variables. In this setting, x_3 will be considered as a dependent variable. Taking into account the parameterization induced by Equation (21), we rewrite the equation of a point $\mathbf{x}$ lying on the plane of integration P as $\mathbf{x} = \mathbf{x}(x_1, x_2)$. More specifically, if $\mathbf{x} \in P$, then:

$$\mathbf{x} := \mathbf{x}(x_1, x_2; \rho, \theta, \phi) = (x_1, x_2, \csc\theta\,(\rho - x_1 \sin\theta\cos\phi - x_2 \sin\theta \sin\phi))^{\mathrm{T}}, \tag{22}$$

where $\csc\theta$ denotes the cosecant of the angle θ, i.e.,

$$\csc\theta = \frac{1}{\cos\theta}. \tag{23}$$

Hence, the area differential ds is given by

$$\mathrm{d}s = \left| \frac{\partial \mathbf{x}(x_1, x_2)}{\partial x_1} \times \frac{\partial \mathbf{x}(x_1, x_2)}{\partial x_2} \right| \mathrm{d}x_1 \mathrm{d}x_2, \tag{24}$$

where

$$\frac{\partial \mathbf{x}}{\partial x_1} \times \frac{\partial \mathbf{x}}{\partial x_2} = \begin{vmatrix} \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 & \hat{\mathbf{x}}_3 \\ \frac{\partial x_1}{\partial x_1} & \frac{\partial x_2}{\partial x_1} & \frac{\partial x_3}{\partial x_1} \\ \frac{\partial x_1}{\partial x_2} & \frac{\partial x_2}{\partial x_2} & \frac{\partial x_3}{\partial x_2} \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 & \hat{\mathbf{x}}_3 \\ 1 & 0 & -\tan\theta\cos\phi \\ 0 & 1 & \tan\theta\cos\phi \end{vmatrix} = \begin{bmatrix} \tan\theta\cos\phi \\ \tan\theta\cos\phi \\ 1 \end{bmatrix}, \tag{25}$$

and $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$ are the corresponding unit vectors in the x_1, x_2, and x_3 directions, respectively. Taking into account Equation (25), the magnitude of the above "Jacobian" vector is,

$$\left| \frac{\partial \mathbf{x}}{\partial x_1} \times \frac{\partial \mathbf{x}}{\partial x_2} \right| = \sqrt{\tan^2\theta + 1} = \csc\theta. \tag{26}$$

Hence Equation (24) yields

$$\mathrm{d}s = \csc\theta \mathrm{d}x_1 \mathrm{d}x_2. \tag{27}$$

Thus, Equation (15) becomes

$$\widehat{f}(\rho, \theta, \phi) = \int_{-\infty}^{\infty} \mathrm{d}x_1 \int_{-\infty}^{\infty} \csc\theta f\,(x_1, x_2, \csc\theta(\rho - x_1 \sin\theta\cos\phi - x_2 \sin\theta\sin\phi))\,\mathrm{d}x_2. \tag{28}$$

An alternative way to express the three-dimensional Radon transform of a function $f : \mathbb{R}^3 \to \mathbb{R}$ involves a Dirac delta, or "plane impulse", namely

$$\mathscr{R}_3 f = \widehat{f}(\rho, \theta, \phi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3) \\ \times \delta(\rho - x_1 \sin\theta\cos\phi - x_2 \sin\theta\sin\phi - x_3\cos\theta)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3. \qquad (29)$$

The alternative definition given by Equation (29) will be proven very useful (see Theorem 1) for the inversion and the numerical implementation of the three-dimensional Radon transform, as discussed in Sections 4 and 5.

4 The Inversion of the Radon Transform in Three Space Dimensions via Plane Integration

For the analytical inversion of the Radon transform in three space dimensions defined in Equation (28) we shall employ plane integration. In this direction, we will make use of the so-called *central slice theorem* (CST). This specific theorem, applied in the three-dimensional case, provides a fundamental tool for the Fourier-based inversion of the 3D Radon transform.

Theorem 1 (Central Slice Theorem in 3D) *The three-dimensional Fourier transform $\mathscr{F}_3$ of a function $f(x_1, x_2, x_3)$, usually denoted by $\widetilde{f} = \mathscr{F}_3 f$, equals the one-dimensional Fourier transform with respect to the signed distance from the origin $\mathscr{F}_1^{(\rho)}$ of the three-dimensional Radon transform $\mathscr{R}_3$ of the same function $\widehat{f} = \mathscr{R}_3 f$, i.e.*

$$\mathscr{F}_3 f = \mathscr{F}_1^{(\rho)}\mathscr{R}_3 f, \quad \text{or} \quad \widetilde{f} = \mathscr{F}_1^{(\rho)}\widehat{f}, \qquad (30)$$

where

$$\widetilde{f}(k_1, k_2, k_3) := (\mathscr{F}_3 f)(k_1, k_2, k_3) \\ = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)e^{-2\pi\mathrm{i}(k_1x_1 + k_2x_2 + k_3x_3)}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3, \qquad (31)$$

and

$$\mathscr{F}_1^{(\rho)}\widehat{f} = \int_{-\infty}^{\infty} \widehat{f}(\rho, \theta, \phi)e^{-2\pi\mathrm{i}k\rho}\mathrm{d}\rho. \qquad (32)$$

Proof For the proof of the central slice theorem in three dimensions, it is convenient to employ the alternative definition of the three-dimensional Radon transform as provided via a delta function in Equation (29). In this case, we expand Equation (32) as follows:

$$
\begin{aligned}
\mathscr{F}_1^{(\rho)}\widehat{f} &= \int_{-\infty}^{\infty} \widehat{f}(\rho,\theta,\phi)e^{-2\pi \mathrm{i}k\rho}\mathrm{d}\rho \qquad (33)\\
&= \int_{-\infty}^{\infty}\bigg(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2,x_3)\\
&\quad \times \delta(\rho - x_1\sin\theta\cos\phi - x_2\sin\theta\sin\phi - x_3\cos\theta)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\bigg)e^{-2\pi \mathrm{i}k\rho}\mathrm{d}\rho\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2,x_3)\bigg(\int_{-\infty}^{\infty} e^{-2\pi \mathrm{i}k\rho}\\
&\quad \times \delta(\rho - x_1\sin\theta\cos\phi - x_2\sin\theta\sin\phi - x_3\cos\theta)\mathrm{d}\rho\bigg)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2,x_3)e^{-2\pi \mathrm{i}k(x_1\sin\theta\cos\phi + x_2\sin\theta\sin\phi + x_3\cos\theta)}\\
&\quad \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2,x_3)e^{-2\pi \mathrm{i}[(k\sin\theta\cos\phi)x_1 + (k\sin\theta\sin\phi)x_2 + (k\cos\theta)x_3]}\\
&\quad \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2,x_3)e^{-2\pi \mathrm{i}(k_1x_1 + k_2x_2 + k_3x_3)}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\
&= \widetilde{f},
\end{aligned}
$$

where we introduced the new k-variables in the Fourier space vector $\mathbf{k} := (k_1, k_2, k_3)^{\mathrm{T}}$ of spatial frequencies as follows:

$$k_1 = k\sin\theta\cos\phi, \quad k_2 = k\sin\theta\sin\phi, \quad k_3 = k\cos\theta, \qquad (34)$$

as the new k-variables in the Fourier space. □

Hence, the Fourier transform with respect to ρ of the "data" equals the three-dimensional Fourier transform of the function under investigation evaluated at the new set of variables. The inversion of Equation (30) yields

$$f = \mathscr{F}_3^{-1}\widetilde{f}. \qquad (35)$$

For the inversion of the 3D Radon transform we will utilize the following corollary.

Corollary 1 (One-Dimensional Fourier Transform of the Second Derivative) *For any twice differentiable function g with respect to* ρ*, the one-dimensional Fourier transform of the second derivative of g,* $\mathscr{F}_1^{(\rho)} g''$*, is related with the one-dimensional Fourier transform of g,* $\mathscr{F}_1^{(\rho)} g$*, by the following expression:*

$$\left(\mathscr{F}_1^{(\rho)} g''\right)(\xi) = -4\pi^2\xi^2 G(\xi), \tag{36}$$

where $\mathscr{F}_1^{(\rho)}$ *denotes the one-dimensional Fourier transform with respect to* ρ *defined in Equation (32),* g'' *denotes the second derivative of g with respect to* ρ*, i.e.,*

$$g'' = \frac{\partial^2 g}{\partial \rho^2}, \tag{37}$$

and G denotes the one-dimensional Fourier transform of g,

$$G = \mathscr{F}_1^{(\rho)} g. \tag{38}$$

Proof Inverting Equation (32) and employing g instead of $\widetilde{f}$ yields

$$g = \left\{\mathscr{F}_1^{(\rho)}\right\}^{-1} G = \int_{-\infty}^{\infty} G(\xi) e^{2\pi \mathrm{i}\rho\xi} \mathrm{d}\xi, \tag{39}$$

where G is defined in Equation (38). As in [12], we take the second derivative of both sides of Equation (39):

$$\begin{aligned} g'' := \frac{\partial^2 g}{\partial \rho^2} &= \frac{\partial^2}{\partial \rho^2}\left(\int_{-\infty}^{\infty} G(\xi) e^{2\pi \mathrm{i}\rho\xi} \mathrm{d}\xi\right) \\ &= \int_{-\infty}^{\infty} G(\xi)\left[\frac{\partial^2}{\partial \rho^2}\left(e^{2\pi \mathrm{i}\rho\xi}\right)\right] \mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} G(\xi)\left[(2\pi \mathrm{i}\xi)^2\, e^{2\pi \mathrm{i}\rho\xi}\right] \mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} \left[-4\pi^2\xi^2 G(\xi)\right] e^{2\pi \mathrm{i}\rho\xi} \mathrm{d}\xi. \end{aligned} \tag{40}$$

From the above it is clear that the functions g'' and $-4\pi^2\xi^2 G(\xi)$ form a Fourier transform pair. Hence, Equation (40), combined with Equation (39), imply Equation (36). □

Theorem 2 (Inversion of the Three-Dimensional Fourier Transform) *The inverse of the three dimensional Radon transform* $\widetilde{f} = \mathscr{R}_3 f$*, defined in Equations (28) and (29), of a Schwartz function* $f \in \mathscr{S}(\mathbb{R}^3)$ *is given by*

$$f(x_1, x_2, x_3) = -\frac{1}{4\pi^2} \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^{2\pi} \widetilde{f}''(\rho^*, \theta, \phi)\mathrm{d}\phi, \tag{41}$$

where, as in Equation (37), prime denotes differentiation with respect to ρ*, i.e.*

$$\widetilde{f}''(\rho^*, \theta, \phi) = \frac{\partial^2}{\partial\rho^2} \widehat{f}(\rho, \theta, \phi)\bigg|_{\rho=\rho^*}, \tag{42}$$

and ρ^* *is given by*

$$\rho^* = x_1 \sin\theta \cos\phi + x_2 \sin\theta \sin\phi + x_3 \cos\theta. \tag{43}$$

Proof Equation (35) implies

$$\left(\mathscr{F}_3^{-1} g\right)(x_1, x_2, x_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1, x_2, x_3) e^{2\pi \mathrm{i}(k_1x_1+k_2x_2+k_3x_3)}\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3. \tag{44}$$

In this direction, the inversion of the three-dimensional Fourier transform will reveal the unknown function f in the sense that:

$$f(x_1, x_2, x_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{f}(k_1, k_2, k_3) e^{2\pi \mathrm{i}(k_1x_1+k_2x_2+k_3x_3)}\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3. \tag{45}$$

We proceed by making a change of variables from (k_1, k_2, k_3) to (k, θ, ϕ) defined by Equation (34). The corresponding Jacobian is given by

$$J(k, \theta, \phi) = \begin{vmatrix} \frac{\partial k_1}{\partial k} & \frac{\partial k_1}{\partial \theta} & \frac{\partial k_1}{\partial \phi} \\ \frac{\partial k_2}{\partial k} & \frac{\partial k_2}{\partial \theta} & \frac{\partial k_2}{\partial \phi} \\ \frac{\partial k_3}{\partial k} & \frac{\partial k_3}{\partial \theta} & \frac{\partial k_3}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & k\cos\theta\cos\phi & -k\sin\theta\sin\phi \\ \sin\theta\sin\phi & k\cos\theta\sin\phi & k\sin\theta\cos\phi \\ \cos\theta & -k\sin\theta & 0 \end{vmatrix} = k^2\sin\theta. \tag{46}$$

Thus we modify Equation (45) in the following manner:

$$f(x_1, x_2, x_3) = \int_0^{\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(k\sin\theta\cos\phi, k\sin\theta\sin\phi, k\cos\theta) \times e^{2\pi \mathrm{i}k(\sin\theta\cos\phi x_1+\sin\theta\sin\phi x_2+\cos\theta x_3)} J(k, \theta, \phi)\mathrm{d}k\mathrm{d}\theta\mathrm{d}\phi. \tag{47}$$

However, a point $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}$ lying on the plane of integration P ($\mathbf{x} \in P$), according to Equation (21), satisfies

$$\rho^* - x_1\sin\theta\cos\phi - x_2\sin\theta\sin\phi - x_3\cos\theta = 0, \tag{48}$$

where ρ^* is the signed distance of the plane P from the origin, see Equation (43). Hence, taking into account Equations (46), (48), and (47) may be rewritten as

$$f(x_1, x_2, x_3) = \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^{2\pi} \left[\int_{-\infty}^{\infty} \widetilde{f}(k\sin\theta\cos\phi, k\sin\theta\sin\phi, k\cos\theta) \times e^{2\pi \mathrm{i} k\rho^*} k^2 \mathrm{d}k \right] \mathrm{d}\phi. \tag{49}$$

We combine Equations (30) with (32) and (34) to obtain

$$\widetilde{f}(k\sin\theta\cos\phi, k\sin\theta\sin\phi, k\cos\theta) = \int_{-\infty}^{\infty} \widehat{f}(\rho, \theta, \phi) e^{-2\pi \mathrm{i} k\rho} \mathrm{d}\rho. \tag{50}$$

In Equation (49), we replace $\widetilde{f}$ by the right-hand side of Equation (50)

$$f(x_1, x_2, x_3) = \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^{2\pi} \left[\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \widehat{f}(\rho, \theta, \phi) e^{-2\pi \mathrm{i} k\rho} \mathrm{d}\rho \right) e^{2\pi \mathrm{i} k\rho^*} k^2 \mathrm{d}k \right] \mathrm{d}\phi. \tag{51}$$

We denote the one-dimensional Fourier transform of the three-dimensional Radon transform of f by $\widehat{F}$,

$$\widehat{F}(k) := \int_{-\infty}^{\infty} \widehat{f}(\rho, \theta, \phi) e^{-2\pi \mathrm{i} k\rho} \mathrm{d}\rho, \tag{52}$$

and insert Equation (52) into Equation (51):

$$f(x_1, x_2, x_3) = \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^{2\pi} \left[\int_{-\infty}^{\infty} k^2 \widehat{F}(k) e^{2\pi \mathrm{i} k\rho^*} \mathrm{d}k \right] \mathrm{d}\phi. \tag{53}$$

The final step involves the rewriting of Equation (53) in the following manner:

$$f(x_1, x_2, x_3) = -\frac{1}{4\pi^2} \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^{2\pi} \left[\int_{-\infty}^{\infty} \left(-4\pi^2 k^2 \widehat{F}(k) \right) e^{2\pi \mathrm{i} k\rho^*} \mathrm{d}k \right] \mathrm{d}\phi. \tag{54}$$

We employ Corollary 1, and replace the integral inside the brackets on the left-hand side of Equation (54), by the left-hand side of the first line of Equation (40) to obtain

$$f(x_1, x_2, x_3) = -\frac{1}{(4\pi)^2} \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^{2\pi} \left. \frac{\partial^2 \widetilde{f}(\rho, \theta, \phi)}{\partial \rho^2} \right|_{\rho=\rho^*} \mathrm{d}\phi, \tag{55}$$

which, via Equation (42), is Equation (41). □

5 Numerical Implementation of the Inversion of the Radon Transform in Three Space Dimensions via Piecewise Cubic Polynomials

For the numerical implementation of the inversion of the Radon transform in three space dimensions we will employ piecewise continuous cubic polynomials, namely cubic splines. It is important to note that all integrals involving the second derivative with respect to ρ of the 3D Radon transform will be evaluated at $\rho = \rho^*$, namely at

$$\rho^* = x_1 \sin\theta \cos\phi + x_2 \sin\theta \sin\phi + x_3 \cos\theta. \tag{56}$$

As shown in the previous section, the 3D inverse Radon transform can be expressed as

$$f(x_1, x_2, x_3) = -\frac{1}{4\pi^2} \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^{2\pi} \widehat{f}''(\rho, \theta, \phi)\bigg|_{\rho=\rho^*} \mathrm{d}\phi \tag{57}$$

where ρ^*, $\widehat{f}$ and $\widehat{f}''$ are defined in Equations (56), (28), and (42), respectively.

We assume that the three-dimensional Radon transform, $\widehat{f}$, is given for every θ and every ϕ at the n knots $\{\rho_i\}_1^n$. We denote the value of $\widehat{f}$ at ρ_i by $\widehat{f_i}$, namely

$$\widehat{f_i} = \widehat{f}(\rho_i, \theta, \phi), \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi], \quad i = 1, \ldots, n-1. \tag{58}$$

We also assume that both $\widehat{f}(\rho, \theta, \phi)$ and $\widehat{f}'(\rho, \theta, \phi)$, where

$$\widehat{f}'(\rho, \theta, \phi) = \frac{\partial \widehat{f}(\rho, \theta, \phi)}{\partial \rho}, \tag{59}$$

vanish at the endpoints $\rho_1 = -1$ and $\rho_n = 1$, i.e.

$$\widehat{f}(\rho_1, \theta, \phi) = \widehat{f}(\rho_n, \theta, \phi) = 0, \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi], \tag{60}$$

and

$$\frac{\partial}{\partial \rho}\widehat{f}(\rho_1, \theta, \phi) = \frac{\partial}{\partial \rho}\widehat{f}(\rho_n, \theta, \phi) = 0, \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi]. \tag{61}$$

In each interval $[\rho_i, \rho_{i+1}]$, $i = 1, \ldots, n-1$, we approximate $\widehat{f}(\rho, \theta, \phi)$ by the third-degree spline $S_i^{(3)}$, namely

$$\widehat{f}(\rho, \theta, \phi) \sim S_i^{(3)}(\rho, \theta, \phi), \quad \rho \in [\rho_i, \rho_{i+1}] \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi]. \tag{62}$$

The cubic spline $S_i^{(3)}$ interpolates $\widetilde{f}$ at the points $\{\rho_i\}_1^n$:

$$S_i^{(3)}(\rho_i, \theta, \phi) = f_i, \quad i = 1, \ldots, n. \tag{63}$$

Therefore, for $\rho \in [\rho_i, \rho_{i+1}]$

$$S_i^{(3)}(\rho, \theta, \phi) = a_i(\theta, \phi) + b_i(\theta, \phi)\rho + c_i(\theta, \phi)\rho^2 + d_i(\theta, \phi)\rho^3. \tag{64}$$

Then, following Equation (62),

$$\frac{\partial}{\partial \rho}\widehat{f}(\rho, \theta, \phi) \sim \frac{\partial}{\partial \rho} S_i^{(3)}(\rho, \theta, \phi) =: S_i^{(2)}(\rho, \theta, \phi), \tag{65}$$

where

$$S_i^{(2)}(\rho, \theta, \phi) = b_i(\theta, \phi) + 2c_i(\theta, \phi)\rho + 3d_i(\theta, \phi)\rho^2. \tag{66}$$

Similarly,

$$\frac{\partial^2}{\partial \rho^2}\widehat{f}(\rho, \theta, \phi) \sim \frac{\partial^2}{\partial \rho^2} S_i^{(3)}(\rho, \theta, \phi) = \frac{\partial}{\partial \rho} S_i^{(2)}(\rho, \theta, \phi) =: S_i^{(1)}(\rho, \theta, \phi), \tag{67}$$

where

$$S_i^{(1)}(\rho, \theta, \phi) = 2c_i(\theta, \phi) + 6d_i(\theta, \phi)\rho \tag{68}$$

Hence, Equation (57) becomes

$$f(x, y, z) = -\frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi} [2c_i(\theta, \phi) + 6d_i(\theta, \phi)\rho] \sin\theta \, \mathrm{d}\phi \mathrm{d}\theta \tag{69}$$

The constants $c_i(\theta, \phi)$ and $d_i(\theta, \phi)$ involved in the above inversion integral, are given by the following expressions, see [13]:

$$c_i(\theta) = \frac{1}{2\Delta_i}(\rho_{i+1}\widehat{f}_i'' - \rho_i \widehat{f}_{i+1}''), \tag{70a}$$

$$d_i(\theta) = \frac{\widehat{f}_{i+1}'' - \widehat{f}_i''}{6\varDelta_i}, \tag{70b}$$

where

$$\varDelta_i = \rho_{i+1} - \rho i, \tag{70c}$$

and

$$\widehat{f}_i'' := \frac{\partial^2}{\partial \rho^2} \widehat{f}(\rho_i, \theta, \phi). \tag{70d}$$

It is worth noting that the inversion formula (57) involves the known constants $\{\widehat{f}\}_1^n$ and the unknown constants $\{\widehat{f}''\}_1^n$. For the computation of $\{\widehat{f}''\}_1^n$, we employ the continuity of the first derivative of the cubic spline, i.e.

$$S_i^{(2)}(\rho_{i+1}, \theta, \phi) = S_{i+1}^{(2)}(\rho_i, \theta, \phi), \quad i = 1, 2, \ldots, n-2, \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi], \tag{71a}$$

and

$$S_1^{(2)}(\rho_1, \theta, \phi) = S_{n-1}^{(2)}(\rho_n, \theta, \phi) = 0, \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi]. \tag{71b}$$

The above consists of a system of n unknowns and of n equations, namely $n - 2$ equations arising from Equation (71a) for $i = 1, 2, \ldots, n - 2$), and 2 equations arising from Equation (71b). The continuity of the cubic spline itself, i.e.

$$S_i^{(3)}(\rho_{i+1}, \theta, \phi) = S_{i+1}^{(3)}(\rho_i, \theta, \phi) = 0, \quad i = 1, 2, \ldots, n-2, \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi] \tag{72a}$$

and

$$S_1^{(2)}(\rho_1, \theta, \phi) = S_{n-1}^{(2)}(\rho_n, \theta, \phi) = 0, \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi]. \tag{72b}$$

The continuity of the cubic spline, $S_i^{(3)}(\rho, \theta, \phi)$, as expressed in Equations (72), implies that the knots $\{\rho_i\}_1^n$ are *removable* logarithmic singularities.

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References

1. J. Radon, Über die bestimmung von funktionen durch ihre integralwerte längs gewisser mannigfaltigkeiten. Akad. Wiss. **69**, 262–277 (1917)
2. P. Kuchment, The Radon transform and medical imaging, in *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (2014)
3. N.E. Protonotarios, G.A. Kastis, A.S. Fokas, *A New Approach for the Inversion of the Attenuated Radon Transform* (Springer, Cham, 2019), pp. 433–457. https://doi.org/10.1007/978-3-030-31339-5_16
4. M.-Y. Chiu, H.H. Barrett, R.G. Simpson, Three-dimensional reconstruction from planar projections. JOSA **70**(7), 755–762 (1980). https://doi.org/10.1364/JOSA.70.000755
5. M. Defrise, D. Townsend, R. Clack, Three-dimensional image reconstruction from complete projections. Phys. Med. Biol. **34**(5), 573 (1989). https://doi.org/10.1088/0031-9155/34/5/002

6. A. Averbuch, Y. Shkolnisky, 3d fourier based discrete radon transform. Appl. Comput. Harmon. Anal. **15**(1), 33–69 (2003). https://doi.org/10.1016/S1063-5203(03)00030-7
7. J. Bikowski, K. Knudsen, J.L. Mueller, Direct numerical reconstruction of conductivities in three dimensions using scattering transforms. Inverse Probl. **27**(1), 015002 (2010). https://doi.org/10.1088/0266-5611/27/1/015002
8. F.H. Fahey, Data acquisition in PET imaging. J. Nucl. Med. Technol. **30**(2), 39–49 (2002)
9. P. Kinahan, J. Rogers, Analytic 3D image reconstruction using all detected events. IEEE Trans. Nucl. Sci. **36**, 964–968 (1989). https://doi.org/10.1109/23.34585
10. P. La Rivière, X. Pan, Spline-based inverse radon transform in two and three dimensions. IEEE Trans. Nucl. Sci. **45**(4), 2224–2231 (1998). https://doi.org/10.1109/23.708352
11. M. Defrise, P.E. Kinahan, C.J. Michel, Image reconstruction algorithms in PET, in *Positron Emission Tomography* (Springer, Berlin, 2005), pp. 63–91. https://doi.org/10.1007/1-84628-007-9_4
12. J. Gaskill, J.W. Sons, *Linear Systems, Fourier Transforms, and Optics*. *Wiley Series in Pure and Applied Optics* (Wiley, New York, 1978)
13. W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery, *Numerical Recipes: The Art of Scientific Computing*, 3rd edn. (Cambridge University, New York, 2007)