Non-radial Solutions of a Supercritical Equation in Expanding Domains: The Limit Case

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Abstract In this article, we introduce a new method to prove the existence of an infinite sequence of distinct non-radial but symmetric nodal (i.e. sign changing) solutions for supercritical nonlinear elliptic problems defined in the whole Euclidean space. By 'symmetric' we mean that both the domain and the solution remain invariant under the action of a compact subgroup G of the isometry group $O(n)$, without finite subgroup. The key ingredient of the method is a process through which an open symmetric domain of the n-dimensional space can be extended in an appropriate manner to 'fill' eventually the entire space 'almost everywhere', remaining symmetric, and giving a sequence of domains where in each of them subsequently we solve an appropriate auxiliary problem. Passing to the limit we obtain the solution of the problem as a limit of the sequence formed by the solutions of the corresponding to the domains sequence of equations.

The base model problem of interest is stated bellow:

$$\text{(P)} \quad \begin{cases} \Delta_p u = |u|^a u, \quad u \in C^2(\mathbb{R}^n), \quad n \geqslant 3 \\ 1 < p < n-k, \quad 0 \leq a \leq p^*(k) = \frac{(n-k)p}{n-k-p}, \end{cases}$$

where p_G^* is the critical exponent of the embedding

$$H_{0,G}^{1,p}(\Omega) \hookrightarrow L^{p_G^*}(\Omega)$$

(being the critical of the supercritical one) and k is the minimum orbit dimension in G. However, we will focus on the critical of the supercritical case $a = p^*(k)$, since on the one hand it is the most important and on the other hand it covers all the other cases.

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By $H^{1,p}_{0,G}(\Omega)$ is denoted the closure of the subspace $C^{\infty}_{0,G}(\Omega)$ consisting of all G-invariant functions in $C^{\infty}_{0}(\Omega)$.

1 Introduction

In this article, the main objective is to prove the existence of non-radial nodal (sign-changing) solutions of the above problem (P), in the case where the exponent a is the critical of supercritical exponent, since the rest of the cases have been studied. Thus, the problem (P) is set out in detail as follows:

$$(P) \qquad \begin{cases} \Delta_p u = |u|^{p^*(k)-2}u \quad \text{in } \mathbb{R}^n, \quad n \geq 3 \\ 1 < p < n-k, \quad p^*(k) = \frac{(n-k)p}{n-k-p}, \end{cases}$$

Here, G is a group of symmetries that acts on the domains and the functions defined on them together, k is the minimum dimension orbit of all orbits of G,

$$\Delta_p u = -\text{div}\left(|\nabla u|^{p-2}\nabla u\right), \ 1 < p \neq 2$$

is the p-Laplacian operator (note that if $p = 2$, is the Laplace–Beltrami operator) and $p^*(k)$ is the critical exponent of the Sobolev embedding

$$H^{1,p}_G(\Omega) \hookrightarrow L^p(\Omega).$$

By $H^{1,p}_G(\Omega)$ is denoted the subspace of all G-invariant functions in $H^{1,p}(\Omega)$.

In problem (P) the solutions obtained are such that

$$\int_{\mathbb{R}^n} |\nabla u|^p dx \to \infty.$$

We study both the cases, $p = 2$ and $p \in (1, 2) \cup (2, n-k)$, however, to avoid any confusion we note that throughout the article we denote by

$$\Delta_p u = -\text{div}\left(|\nabla u|^{p-2}\nabla u\right), \ 1 < p < n-k$$

the p-Laplacian as well as the Laplace–Beltrami operator but when we refer to other articles the Laplace–Beltrami operator is denoted as in the referred articles, i.e. without the minus conversion.

For the problem (P), we prove the existence and find both the type and the number of the solutions to the problem (P). For this aim we use *the method of expanding domains* which was successfully introduced for the first time in [42]. In that article this method was used firstly to prove the existence of a solution and secondly to determine the type and the number of the solutions to critical nonlinear elliptic problem:

$$(\mathrm{P}_0) \qquad -\Delta u = |u|^{\frac{4}{n-2}} u, \quad u \in C^2(\mathbb{R}^n), \quad n \geq 3.$$

Concerning the method itself it seems to have particular value because it can be used and in other types of partial differential equations.

Both cases, i.e. $p = 2$ and $p \neq 2$, are extremely interesting and that is why for several decades now many researchers have been paying attention to them.

Problem (P_0) consists a special case of (P) for $p = 2$ and it owns its origin in many astrophysical and physical contexts and more precisely in the Lane-Emden-Fowler problem,

$$(\mathrm{P}_0') \qquad \begin{cases} -\Delta u = u^q \\ u > 0 \ \text{in} \ \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where Ω is a domain with smooth boundary in $\mathbb{R}^N$ and $p > 1$. But its greatest interest lies in its relation to the Yamabe problem (see in [5, 57, 64, 68]) and for a complete and detailed study we refer to [6], nevertheless it has an autonomous presence holding an important place among the most famous nonlinear partial differential equations). We refer, also, to the classical papers [20, 30, 47], which are some of the large number of very good papers that are devoted to the study of this problem.

Gidas, Ni, and Nirenberg, in their celebrated paper [30], proved symmetry and some related properties of positive solutions of a larger class of second order elliptic equations. Concerning the equation

$$-\Delta u = |u|^{\frac{4}{n-2}} u, \quad u \in C^2(\mathbb{R}^n), \quad n \geq 3,$$

they proved that any positive solution of this, which has finite energy, namely

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx < +\infty,$$

is necessarily of the form

$$u(x) = \left(\frac{\lambda \sqrt{n(n-2)}}{\lambda^2 + |x - x_0|^2} \right)^{\frac{n-2}{2}}, \ \lambda > 0, x_0 \in \mathbb{R}^n.$$

These solutions yield the well-known one-instanton solutions in a regular gauge of the Yang–Mills equation. In addition, since this equation is invariant under the conformal transformations of $\mathbb{R}^n$, if u(x) is a solution, then

$$\lambda^{\frac{n-2}{2}} u(\frac{x - x_0}{\lambda}), \ \forall \ \lambda > 0 \text{ and } x_0 \in \mathbb{R}^n$$

is also a solution. Moreover, all solutions obtained in this way have the same energy and we will say that these solutions are equivalent. In particular, all these solutions are equivalent.

Ding in his also celebrated article [20] using Ambrosetti and Rabinowitz analysis (see in [2]) proved that this problem has infinite distinct solutions $u_k \in C^2(\mathbb{R}^n)$, $k = 1, 2, \cdots$, which changes sign and such that

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} |\nabla u_k|^2 dx \to \infty.$$

Ding showed that it is possible to solve the equation in the whole Euclidean space, reduced the problem to an equivalent problem on $\mathbb{S}^n$, the Euclidean n-sphere throughout a conformal deformation. However, this method cannot be used in the case of the p-Laplacian operator, because this operator is not a conformal invariant operator.

Mazzeo and Smale in their also celebrated article [47] proved that if Ω is an open set in $\mathbb{R}^n$ and u is a positive C^2 function on Ω such that the metric $g = u^{\frac{4}{n-2}} g_0$ on Ω has scalar curvature $R(g) = n(n - l)$, then u must satisfy the equation

$$\Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0, \ u > 0$$

on Ω, where g_0 is the Euclidean metric on $\mathbb{R}^n$.

Caffarelli, Gidas, and Spruck in their classical paper [11] studied non-negative smooth solutions of the conformally invariant equation

$$-\Delta u = u^{\frac{n+2}{n-2}}, \quad u \geq 0, \quad n \geq 3,$$

in a punctured ball $B_1(0) \backslash \{0\} \subseteq \mathbb{R}^n$, with an isolated singularity at the origin. In this paper, the authors introduced a heuristic idea of asymptotic symmetry technique which can roughly be described as follows: After an inversion, the function u becomes defined in the complement of B_1, is strictly positive of ∂B_1, and in some sense 'goes to zero' at infinity. If the function u can be extended to B_1 as a super solution of our problem, then the reflection process at infinity can start and move all the way to ∂B_1. This would imply asymptotic radial symmetry at infinity. With this comprehensive report on this issue we would like, on the one hand, to emphasize the important contribution of this great article of Caffarelli, Gidas, and Spruck on the study on the direction of finding the radial solutions of our problem and on the other hand, we wish to make clear that in our procedural paper we do not care about the radial solutions but we do care about the existence of non-radial solutions.

Schoen in [57] built solutions of (P) with prescribed isolated singularities. Schoen, also, in [58], have used the geometrical meaning of problem (P) in order to derive, through ideas of conformal geometry, the existence of weak solutions having a singular set whose Hausdorff dimension is less than or equal to $\frac{n-2}{2}$. Let us notice that in this paper the authors explain how to build solutions of (P) with a singular set whose Hausdorff dimension is not necessarily an integer.

Bartsch and Schneider in [8] proved that for $N > 2m$ the equation

$$(-\Delta)^m = |u|^{\frac{4m}{N-2m}} u$$

on $\mathbb{R}^N$ has a sequence of nodal, finite energy solutions which is unbounded in $\mathscr{D}^{m,2}(\mathbb{R}^N)$, the completion of $\mathscr{D}(\mathbb{R}^N)$ with respect to the scalar product:

$$(u, v) = \begin{cases} \int_{\mathbb{R}^N} \Delta^{\frac{m}{2}} u \cdot \Delta^{\frac{m}{2}} v, & m \text{ even} \\ \int_{\mathbb{R}^N} \nabla \Delta^{\frac{m-1}{2}} u \cdot \nabla \Delta^{\frac{m-1}{2}} v, & m \text{ odd.} \end{cases}$$

This result generalizes the result of Ding for $m = 1$, and provides interesting information concerning the number and the kind of the solutions of the equation.

Wang in [66] studied the following nonlinear Neumann elliptic problem:

$$(\mathrm{P_N}) \qquad \begin{cases} -\Delta u = u^{\frac{N+2}{N-2}}, & u > 0 \text{ in } \mathbb{R}^N \backslash \Omega, \\ u(x) \to 0 & \text{as } |x| \to +\infty, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where n denotes interior unit normal vector and Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 4$. In this paper, it is proved that if $N \geq 4$, (Wang believes that the results will also hold in the case of $N = 3$), and Ω is a smooth and bounded domain then the problem $(\mathrm{P_N})$ has infinity many non-radial positive solutions, whose energy can be made arbitrarily large when Ω is convex, as seen from inside (with some symmetries). We refer to the Wang's problem $(\mathrm{P_N})$ due to its close relationship with our problem and as we will see later, if we choose suitable Ω we can have a result on this problem in almost all the space. In particular, in both problems we have to solve the same non-linear differential equation with critical exponent with boundary conditions Dirichlet and Neumann, respectively. In addition, in both cases the domain Ω presents some symmetries. However, a subsequent process in each case is completely different from that of another. In our case, our goal is to solve the problem *almost in the whole space*, starting from an open symmetric domain Ω of n-dimensional space and we extend the Ω so that it remains symmetrical to fill almost all the space. In the other case is considered the corresponding Neumann problem in $\mathbb{R}^N \backslash \Omega$ where Ω is convex as seen from inside with some symmetries. If we choose appropriate a such Ω with a small volume as much as we can say that the solutions of Wang satisfy the conditions of the problem *almost in the whole space*. Finally, in both problems we take an infinity number of non-radial solutions, whose energy can be made arbitrary large, however in the first problem we find nodal solutions while in the second are founded positive solutions.

Concerning to the progress of the study of the problem (P) for $p = 2$ a number of important articles are available (cf. [1, 3, 4, 9–11, 20, 21, 23, 27, 29, 30, 32, 37, 42, 43, 45, 47, 48, 52, 57, 58, 63, 66]).

The p-Laplace operator (or p-harmonic operator) occupies a similar position to the standard Laplace operator when it comes to nonlinear phenomena. In fact, many of the things that apply to the usual Laplace operator and consequently to the equations that relate to it also apply to the p-Laplace as well as his equations, except that the Principle of Superposition which is of course lost. A very detailed and complete study is provided by Lindqvist [44]. Also, a Morse theoretic study of a very general class of homogeneous operators that includes the p-Laplacian as a special case is presented by Perera, Agarwal, and O'Regan in [53]. However, the p-Laplacian operator also appears in many areas of physics, such as non-Newtonian fluid flows, turbulent filtration in porous media, plasticity theory, rheology, glaciology, radiation of heat (cf. [24, 35, 49]).

The p-Laplace operator is a particularly interesting and remarkable case and this fact is confirmed not only by the large number of articles dedicated to it but also by the multifaceted study of the problems related to it (cf. [13, 18, 22, 26, 28, 31, 36, 40, 46, 54, 55, 60, 67, 69]).

In the problem (P), considered for any $1 < p < n - k$, a main difficulty comes from the double lack of compactness. By lack of compactness, we mean that the functional that we consider do not satisfy the Palais-Smale condition (cf. [50, 51, 59, 61, 62, 70]), (i.e. there exists a sequence along which the functional remains bounded, its gradient goes to zero, and does not converge). However, for $p \neq 2$, a second difficulty arises from the fact that the p-Laplace operator is not conformal invariant operator so the methods used in the case of the Laplace operator cannot be applied.

Concerning the lack of compactness, the first difficulty comes from the fact that the exponent

$$p^*(k) = \frac{(n-k)p}{n-k-p}$$

is supercritical (in fact the critical of the supercritical), and the second one is some extra difficulty because of the lack of compactness in unbounded domains. But, it is well known (see in [15, 16, 25, 32]) that the symmetry property of the domain allows us to improve the Sobolev embedding in higher L^p spaces and we overcome the obstruction of the exponent. Regarding the problem of lack of compactness in unbounded domains we avoid solving problems in such domains by remaining in bounded domains and then we pass to unbounded with limit procedures. In addition, this ensures us the ability to overcome the problems due to the non-conformality of the p-Laplace operation.

To overcome all the above obstacles we consider the following corresponding problem

$$
(\mathrm{P}_\varepsilon) \qquad \begin{cases} \Delta_p u_\varepsilon + \varepsilon\, a(x)|u_\varepsilon|^{p-2}u_\varepsilon = f(x)|u_\varepsilon|^{p^*(k)-2}u_\varepsilon \\ \qquad u_\varepsilon \not\equiv 0 \ \text{ in } \ \Omega_\varepsilon, \ u_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon \\ \qquad 1 < p < n-k, \ p^*(k) = \frac{(n-k)p}{n-k-p}, \end{cases}
$$

where Ω_ε, $\varepsilon > 0$, is an expanding domain in $\mathbb{R}^n$, $n \geq 3$, invariant under the action of a subgroup G of the isometry group $O(n)$ and $a, f \in C^\infty(\overline{\Omega_\varepsilon})$ are two smooth G-invariant functions on $\overline{\Omega_\varepsilon}$.

The problem (P_ε) has been studied by many authors. We refer to [3, 4, 10, 20, 23, 27, 29, 32] and the references therein for a further discussion of both the problem itself and several variants of it. Some special cases have been also studied. For example, no solution can exist if Ω is starshaped, as a consequence of the Pohozaev identity (see in [56]). Furthermore, if Ω is an annulus, there are infinite solutions (see in [43]). Also, a general result of Bahri and Coron guarantees the existence of positive solutions in domains Ω having nontrivial topology (i.e. certain homology groups of Ω are non trivial) (see in [7]). The existence and multiplicity of positive or nodal solutions of critical equations on bounded domains or in some contractible domains have been determined by other authors (see for example in [21, 27, 32, 52, 63]). Some more nonexistence results in this case are available, (see in [1, 4, 12, 37]).

Our proof is via approximation by an infinite sequence of problems defined on a sequence of expanding symmetric bounded domains. Firstly, we solve the problem (P_ε). (see in [14, 42] for the case of the Laplacian and for the case of the p-Laplacian see in [15, 16], for $n = 3$, and for $n \geq 3$, respectively). Then we consider a sequence of problems $(\mathrm{P}_{\varepsilon_j})$, $j = 1, 2, \cdots$, defined in a sequence of expanding domains Ω_{ε_j}, $j = 1, 2, \cdots$, and henceforth, sending $\varepsilon \to 0$, we obtain the solution of the limit problem (P) as the limit of the sequence of the solutions of the problems $(\mathrm{P}_{\varepsilon_j})$. This method is a generalization of the method we used in [42] and thus a uniform treatment of both cases $p = 2$ and $p \in (1, 2) \cup (2, n-k)$ is achieved. In addition, the used method is a different from previous ones and can be used to solve poly-harmonic equations with supercritical exponent and even in the critical of supercritical case, as in our case, providing an alternative way of utilizing the best constants of the appearing Sobolev inequalities. Furthermore, this method enables us to determine the kind and the number of solutions of the problem in both cases, i.e. for $p = 2$ and for $p \in (1, 2) \cup (2, n-k)$.

This article is organized as follows: Section 2 is devoted to notations and in some necessary background material. In Section 3, we introduce our main tool, meaning the process through which an open symmetric domain of n-dimensional space can be extended in an appropriate manner to *'fill'* eventually the entire space *'almost everywhere'*, remaining symmetric, and subsequently we solve the auxiliary problem (P_ε). Section 4 is devoted to some basic definitions and to the proof of the main theorem.

2 Notations and Some Background Material

As referred in the beginning of this article, our main objective is to prove the existence of an infinite sequence of distinct non-radial nodal G-invariant solutions defined in 'almost the whole' Euclidean space for the supercritical nonlinear elliptic problem (P). However, before dealing with problem (P) let us consider the following basic problem which will play an important role in solving the problem (P).

$$(\mathrm{P}_0) \qquad \begin{cases} \Delta_p u + a(x)|u|^{p-2}u = f(x)|u|^{q-2}u \\ u \not\equiv 0 \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded, smooth, domain of $\mathbb{R}^n$, $n \geq 3$.

If we consider

$$p^* = \frac{np}{n-p}$$

it is well known by Sobolev's embedding theorem (cf. [6]) the embedding

$$H_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

is compact for any $p \in [1,\ p^*)$ but the embedding

$$H_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

is only continuous.

We say that the exponent

$$p^* = \frac{np}{n-p}$$

for the Sobolev embedding

$$H_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

is the *critical exponent* for this embedding and that the problem (P) is *supercritical*, *critical* or *supercritical* if $q-1 < p^*$, $q-1 = p^*$ or $q-1 > p^*$ respectively. If $p > n$ the problem (P) is always sub-critical.

In order to make this article self-contained we will open at this point a parenthesis where we will introduce some useful background material from the geometry. (More details see in [9] or [38]).

Consider a group G acting on a set X. The *orbit* of a point x in X is the set of elements of X to which x can be moved by the elements of G. (Just as gravity moves a planet around in its orbit, the group action moves an element around in its orbit.)

The $G-orbit$ of x is denoted by

$$O_G(x) = \{\tau(x), \tau \in G\},$$

and for any $Y \subseteq X$, we write

$$G(Y) = \{\tau(y) : y \in Y \text{ and } \tau \in G\}.$$

If for some subset $Y \subseteq X$ is valid

$$G(Y) = Y,$$

then, we say that Y is *invariant* under the action of G and in this case we denote it by Y_G.

For every $x \in X$, we define the *stabilizer subgroup* of G with respect to x (also called the *isotropy group*) as the set of all elements in G that fix x:

$$S_G(x) = \{\tau \in G : \tau(x) = x\}.$$

Moreover, if the set X is equipped with a metric, then the *isometry group* of this metric space is the set of all isometries (i.e. distance-preserving maps) from the metric space onto itself, with the function composition as group operation. Its identity element is the identity function (i.e. the isometry group of a two-dimensional sphere is the orthogonal group $O(3)$).

Given (M, g) a Riemannian manifold (complete or not, but connected), we define by $I(M, g)$ its group of isometries. It is well known (see for instance [38]) that $I(M, g)$ is a Lie group with respect to the compact open topology, and that $I(M, g)$ acts differentiably on M. Since (this is actually due to E. Cartan) any closed subgroup of a compact Lie group is a Lie group, we get that any compact subgroup of $I(M, g)$ is a sub-Lie group of $I(M, g)$. It is now classical (see [9] and [19]), that for any $x \in M$, $O_G(x)$ is a smooth compact sub-manifold of M.

We denote by $|O_G(x)|$ the volume of $O_G(x)$ for the Riemannian metric induced on $O_G(x)$. In the special case where $O_G(x)$ has finite cardinal, then,

$$|O_G(x)| = \text{card } O_G(x).$$

Let G be a closed subgroup of $I(M, g)$. Assume that for any $x \in M$,

$$\text{card } O_G(x) = +\infty,$$

and set

$$k = \min_{x \in M} \dim O_G(x).$$

Then $k \geq 1$ (see [32]), and is called *minimum orbit dimension*.

We consider a bounded, smooth domain Ω of $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, $k \geq 2, n-k \geq 1$ such that

$$\overline{\Omega} \subset \left(\mathbb{R}^k \backslash \{0\}\right) \times \mathbb{R}^{n-k}.$$

Suppose that $\overline{\Omega}$ is invariant under the action of $G_{k,n-k}$, that is

$$\tau(\overline{\Omega}) = \overline{\Omega}, \quad \text{for all} \quad \tau \in G_{k,n-k},$$

where $G_{k,n-k} = O(k) \times Id_{n-k}$ (then denoted by G), is the subgroup of the isometry group $O(n)$ of the type

$$(x_1, x_2) \longrightarrow (\sigma(x_1), x_2), \quad \sigma \in O(k), \quad x_1 \in \mathbb{R}^k, \quad x_2 \in \mathbb{R}^{n-k}.$$

For example, a such Ω in $\mathbb{R}^3$ is the solid torus

$$\overline{T} = \left\{ (x, y, z) \in \mathbb{R}^3 : \left(\sqrt{x^2 + y^2} - R \right)^2 + z^2 \leq r^2, \; R > r > 0 \right\}.$$

Also, as such Ω we can see the part of the n-dimensional ball B_n from which we have removed a part of it in such a way that the rest is invariant under the action of the group G and its cover belongs to $\overline{B}_n \subset \left(\mathbb{R}^k \backslash \{0\}\right) \times \mathbb{R}^{n-k}$. This is because the balls enjoy a large number of symmetries in addition to the radial symmetry.

We define

$$C_G^{\infty}(\Omega) = \left\{ u \in C^{\infty}(\Omega) : u \circ \tau = u \, , \; \forall \, \tau \in G \right\},$$

and

$$C_{0,G}^{\infty}(\Omega) = \left\{ u \in C_0^{\infty}(\Omega) : u \circ \tau = u \, , \; \forall \, \tau \in G \right\},$$

where $C^{\infty}(\Omega)$ denotes the space of smooth functions on Ω and where $C_0^{\infty}(\Omega)$ denotes the space of smooth functions with compact support on Ω.

We define, also, the Sobolev space $H^{1,p}(\Omega)$ as the completion of $C^{\infty}(\Omega)$ with respect to the norm

$$\|u\|_{H^{1,p}(\Omega)} = \left(\|\nabla u\|_{L^p(\Omega)}^p + \|u\|_{L^p(\Omega)}^p \right)^{1/p}, \; p \geq 1,$$

and the Sobolev space $H_0^{1,p}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ in $H^{1,p}(\Omega)$.

Finally, we denote by $H_G^{1,p}(\Omega)$ and $H_{0,G}^{1,p}(\Omega)$ the subspaces of $H^{1,p}(\Omega)$ and $H^{1,p}(\Omega)$, respectively, of all G-invariant functions defined on Ω.

It is well known that the symmetry property of the domain allows us to improve the Sobolev embedding in higher L^p spaces. More precisely, let us consider a smooth compact n-dimensional, $n \geq 3$, Riemannian manifold (M, g) invariant under the action of an arbitrary compact subgroup G of $\mathrm{Isom}_g(M)$. Let us also assume that

$$\mathrm{Card}\, O_G^x = +\infty$$

for any orbit O_G^x of G and $k \geq 1$. It is well known that the Sobolev embedding

$$H_G^{1,p}(M) \hookrightarrow L^q(M)$$

is compact for any

$$1 \leq q < \frac{(n-k)p}{n-k-p}$$

but if

$$1 \leq q \leq \frac{(n-k)p}{n-k-p}$$

is only continuous (cf. in [14, 16, 20, 25, 33, 34]).

3 Preliminary Results

Let Ω be a domain such that $\overline{\Omega} \subset \left(\mathbb{R}^k \backslash \{0\}\right) \times \mathbb{R}^{n-k}$ and also invariant under the action of the group G defined above. For any small $\varepsilon > 0$ and some $m > 0$ (which will be determined later) we consider the family of expanding domains

$$\Omega_\varepsilon = \varepsilon^{-m}\Omega = \{\varepsilon^{-m}x : x \in \Omega\}$$

Then, it is very simple to be confirmed that Ω_εs inherit the symmetry properties of Ω for any ε.

At this point we need to comment on the term 'almost the whole' space and specify the impact of this term on solutions to the problem. To do this we must describe the process by which we 'fill' the space by properly expanding the domain Ω and then see how the method of solving the problem works. In fact we consider a sequence consisting of Ω_{ε_j}, where the sequence of ε_js (for the time being) is a sequence that tends to 0 in such a way that Ω_{ε_j}s extend continuously and as $\varepsilon \to 0$ they cover "almost everywhere" the entire space. This is because this extension also entails the inside boundary of the Ω_{ε_j}s (i.e. the one that is on the zero side) and of course increases the volume of the orbit with the minimum dimension. This does not pose a problem for us in the solution because it depends only on the volume of this orbit (apart from the other parameters) (see Theorem 3) so we can extend these Ω_{ε_j}s to the inside as much as we want by extending with zero values the functions defined in them. The outer boundary of Ω_{ε_j}s does not impose any restrictions and this is because any orbits close to it do not play any role since as mentioned above only the orbit with the minimum volume affects the solutions and it is on the opposite side, the side of zero. Finally, the fact that the domain is also expanding does not affect either the Sobolev inequalities associated with the problem or the solutions because we can, for example, normalize the functions u_j i.e. so that their norms are equal to 1.

We consider now the transformation

$$\phi : \Omega \to \Omega_\varepsilon \ : \ X = \varepsilon^{-m} x, \ x \in \Omega, \ X \in \Omega_\varepsilon \tag{1}$$

and for $\ell > 0$ we set

$$u_\varepsilon(X) = \varepsilon^{-\ell} u\left(\varepsilon^m X\right).$$

In particular we obtain

$$|\nabla u| = \varepsilon^{-m} |\nabla u_\varepsilon| \tag{2}$$

and

$$\Delta_p u = -\varepsilon^{-mp} \operatorname{div}\left(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon\right). \tag{3}$$

Note the equality (3) remains valid for $p = 2$, i.e. for $\Delta_2 = \Delta$, the Laplace–Beltrami operator.

In the following, we will suppose that $p \neq 2$, since the case where $p = 2$ was studied in [42].

Applying the transformation (1) in the equation of the problem (P_0), because of (2) and (3), we obtain the following equation

$$\Delta_p u_\varepsilon + \varepsilon^{mp+\ell(2-p)} a(x) |u_\varepsilon|^{p-2} u_\varepsilon = \varepsilon^{mp+\ell(2-q)} f(x) |u_\varepsilon|^{q-2} u_\varepsilon.$$

Since ℓ is an arbitrary positive real, we can choose $\ell = \frac{mp}{q-2}$ and thus we obtain the following equation:

$$\Delta_p u_\varepsilon + \varepsilon^{mp+mp(2-p)/(q-2)} a(x) |u_\varepsilon|^{p-2} u_\varepsilon = f(x) |u_\varepsilon|^{q-2} u_\varepsilon. \tag{4}$$

Finally, replacing the $\varepsilon^{mp+mp(2-p)/(q-2)}$ by ε, we can write the equation (4) in the following form

$$\Delta_p u_\varepsilon + \varepsilon a(x) |u_\varepsilon|^{p-2} u_\varepsilon = f(x) |u_\varepsilon|^{q-2} u_\varepsilon. \tag{5}$$

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, G-invariant and k be the minimum of the dimensions of all orbits of G with infinite cardinal. Let, also, Ω_ε as defined above. A such Ω is the above defined solid torus T and a such Ω_ε, in this case, is an expanding torus T_ε.

Now for any $\varepsilon > 0$ consider the following auxiliary problem:

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} \Delta_p u_\varepsilon + \varepsilon\, a(x)|u_\varepsilon|^{p-2}u_\varepsilon = f(x)|u_\varepsilon|^{p^*(k)-2}u_\varepsilon \\ u_\varepsilon \not\equiv 0 \ \text{ in } \ \Omega_\varepsilon, \ u_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon \\ 1 < p < n-k, \ p^*(k) = \frac{(n-k)p}{n-k-p}, \end{cases}$$

where a, f are two smooth $\mathscr{H}_\sigma^p$-invariant functions (defined bellow).

Before, we solve the problem (P_ε), we must compute the best constant $K_G^p(\Omega_\varepsilon)$ in the following Sobolev inequality, which appears in this problem:

$$\left(\int_{\Omega_\varepsilon} |u|^{p^*(k)}\, dx\right)^{\frac{p}{p^*(k)}} \leqslant \left(K_G^p(\Omega_\varepsilon) + \epsilon\right) \int_{\Omega_\varepsilon} |\nabla u|^p\, dx + B_\epsilon \int_{\Omega_\varepsilon} |u|^p\, dx, \tag{6}$$

where ϵ is a positive constant no matter how small, but it cannot disappear and B_ϵ a positive constant.

In fact we will express the best constant $K_G(\Omega_\varepsilon)$ of inequality (6) as a function of the optimal constant the best constant $K_G^p(\Omega)$ and ε.

Concerning this best constant the following theorem holds:

Theorem 1

$$K_G(\Omega_\varepsilon) = \varepsilon^m K_G(\Omega) = \frac{K\,(n-k,\, p)}{\varepsilon^{-m}\mathscr{V}^{\frac{1}{n-k}}}$$

where $K(n-k, p)$ is the best constant in the classical Sobolev inequality of $\mathbb{R}^{n-k}$ and $\mathscr{V}$ denotes the minimum of the volume of the k-dimensional orbits in Ω.

Proof According to the Theorem 2.1 in [16], (see, also, Theorem 3.1 in [15]) we have

$$K_G^p(\Omega_\varepsilon) = \frac{K(n-k, p)}{\mathscr{V}_\varepsilon^{\frac{1}{n-k}}}$$

and since

$$\mathscr{V}_\varepsilon = \varepsilon^{-m(n-k)}\mathscr{V}$$

we obtain

$$K_G^p(\Omega_\varepsilon) = \frac{K(n-k,p)}{\varepsilon^{-m\mathscr{V}\frac{1}{n-k}}}$$

□

Now, for the problem (P_ε), consider the functional

$$J(u_\varepsilon) = \int_{\Omega_\varepsilon} \Big(|\nabla u_\varepsilon|^p + \varepsilon a(x)|u_\varepsilon|^p\Big)dx$$

and suppose that the operator

$$L_p(u_\varepsilon) = \Delta_p u_\varepsilon + \varepsilon\, a(x)|u_\varepsilon|^{p-2}u_\varepsilon$$

is coercive.

Denote

$$\mathscr{H}^p = \left\{ u_\varepsilon \in H_{0,G}^{1,p}(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} f(x)|u_\varepsilon|^q dx = 1 \right\},$$

$$\mu_\varepsilon = \inf J(u_\varepsilon),$$

for all $u_\varepsilon \in \mathscr{H}^p$, and suppose that exists an isometry σ such that $\sigma(\Omega_\varepsilon) = \Omega_\varepsilon$. Moreover we suppose that the functions $a(x)$ and $f(x)$ are invariant under the action of σ, and

$$\mathscr{H}_\sigma^p = \mathscr{H}^p \cap \Big\{u_\varepsilon \in H_{0,G}^{1,p}(\Omega_\varepsilon) : u_\varepsilon \circ \sigma = -u_\varepsilon\Big\} \neq \emptyset.$$

Then, we have the following theorems.

Theorem 2 *For $p = 2$ and $n \geq 3$, the problem* (P_ε)*, always, has a non-radial nodal solution u. Moreover, if $f(x) > 0$ for all $x \in \overline{\Omega}_\varepsilon$,* ($\mathrm{P}_0$) *has an infinity sequence $\{u_{\varepsilon_i}\}$ of non-radial nodal solutions, such that*

$$\lim_{i\to\infty} \int_{\Omega_\varepsilon} (|\nabla u_{\varepsilon_i}|^2 + u_{\varepsilon_i}^2)dx = +\infty.$$

In addition, u and $\{u_{\varepsilon_i}\}_{i=1,2,\ldots}$ are G-invariant and σ-antisymmetrical.

Theorem 3 *Let a and f be two smooth functions, $\mathscr{H}_\sigma^p$-invariant and p, q be two real numbers defined as in* (P_ε)*. Suppose that $sup_{x\in\Omega_\varepsilon} f(x) > 0$ and the operator L_p is coercive. Then the problem* (P_ε) *has a non-radial nodal $\mathscr{H}_\sigma^p$-invariant solution, that belongs to $C^{1,\alpha}(\Omega_\varepsilon)$ for some $\alpha \in (0,1)$, if*

$$\mu_\varepsilon < K_G^p(\Omega_\varepsilon)^{-p}\left(\sup_{x\in\Omega_\varepsilon} f(x)\right)^{-p/q}.$$

The proofs of Theorems 2 and 3 use standard variational methods, under the assumptions of Lemma 3.6 in [16], (cf. [14, 15, 17, 25]).

4 Solution of the Problem (P)

We return to our main problem

$$\text{(P)} \quad \begin{cases} \Delta_p u = |u|^{p^*(k)} u, \quad u \in C^2(\mathbb{R}^n), \quad n \geqslant 3 \\ \quad 1 < p < n-k, \quad p^*(k) = \frac{(n-k)p}{n-k-p}. \end{cases}$$

In the problem (P) direct variational methods are not applicable because of the double lack of compactness. To overcome this problem we will use an approximate method. That is, we consider a sequence of expanding Ω_{ε_j} (where $\varepsilon_j \to 0$ as $j \to \infty$) as well as the sequence of problems

$$(\mathrm{P}_{\varepsilon_\mathrm{j}}) \quad \begin{cases} \Delta_p u_{\varepsilon_j} + \varepsilon_j a(x)\left|u_{\varepsilon_j}\right|^{p-2} u_{\varepsilon_j} = f(x)\left|u_{\varepsilon_j}\right|^{p^*(k)-2} u_{\varepsilon_j} \\ \qquad u_{\varepsilon_j} \not\equiv 0 \ \text{ in } \Omega, \quad u_{\varepsilon_j} = 0 \ \text{ on } \partial\Omega \\ \qquad 1 < p < n-k, \quad p^*(k) = \frac{(n-k)p}{n-k=p}, \end{cases}$$

where a, f are as in the problem $(\mathrm{P}_{\varepsilon_\mathrm{j}})$.

According to the Theorems 2 and 3, every problem $(\mathrm{P}_{\varepsilon_\mathrm{j}})$ has a non-radial nodal $\mathscr{H}_\sigma^p$-invariant solution. Thus, a solution to the problem (P) may be then obtained by the limit procedure as $\varepsilon_j \to 0$.

Before we will approximate the solutions in $\mathbb{R}^n$ by solutions in bounded domains $\Omega_{\varepsilon_j} \in \mathbb{R}^n$, we note that, in the generalized setting of the problems in Ω_{ε_j}s, the Dirichlet condition $u_{\varepsilon_j}(x) = 0$ on $\partial\Omega_{\varepsilon_j}$ may actually be included in the condition $u_{\varepsilon_j} \in H_{0,G}^{1,p}(\Omega_{\varepsilon_j})$.

Moreover, since any function $u_{\varepsilon_j} \in H_{0,G}^{1,p}(\Omega_{\varepsilon_j})$ can be extended onto $\mathbb{R}^n$ by

$$\tilde{u}_\varepsilon(x) = \begin{cases} u_{\varepsilon_j}(x), & x \in \Omega_{\varepsilon_j} \\ 0, & x \in \mathbb{R}^n \backslash \Omega_{\varepsilon_j}, \end{cases}$$

generalized solutions may be defined in Ω_{ε_j}s analogously to the case in $\mathbb{R}^n$.

We need now the following two definitions:

Definition 1 A function $u_{\varepsilon_j} \in H^{1,p}_{0,G}(\Omega_{\varepsilon_j})$ is a *generalized solution* of ($\mathrm{P}_{\varepsilon_j}$) if the function

$$g(x, u_{\varepsilon_j}) = \varepsilon_j a(x) u_{\varepsilon_j} - f(x)|u_{\varepsilon_j}|^{p^*(k)-2} u_{\varepsilon_j}$$

is locally integrable and for all $\varphi \in C_0^\infty(\Omega_{\varepsilon_j})$, the following holds:

$$\int_{\Omega_{\varepsilon_j}} |\nabla u_{\varepsilon_j}|^{p-2}(\nabla u_{\varepsilon_j}, \nabla \varphi) dx + \int_{\Omega_{\varepsilon_j}} f(x, u_{\varepsilon_j}) \varphi dx = 0.$$

Definition 2 A function $u_\varepsilon \in C^2(\Omega_\varepsilon) \cap C(\overline{\Omega}_\varepsilon)$ is a *classical solution* to (P_ε) if after substituting it into equation of (P_ε), this equation becomes the identity at each $x \in \Omega_\varepsilon$ and $u_\varepsilon(x) = 0$ provided $x \in \partial \Omega_\varepsilon$.

Provided that all the conditions of the Theorem 3 are satisfied, we apply it to the sequence of the problems ($\mathrm{P}_{\varepsilon_j}$) and denote by $\{u_j\}_{j=1}^\infty$ the sequence of the corresponding solutions.

Under the above considerations to following theorem holds.

Theorem 4 *The problem*

$$\Delta_p u = f(x)|u|^{p^*(k)-2} u \quad \text{in } \mathbb{R}^n, \quad n \geq 3$$

has a generalized non-radial nodal $\mathscr{H}_\sigma^p$-invariant solution u and there is a subsequence of $\{u_j\}$ (again denoted by $\{u_j\}$) such that

$$u_j \rightharpoonup u \quad \text{in} \quad H^{1,p}_{0,G} \quad \text{as} \quad j \to \infty.$$

In addition

$$\lim_{j \to \infty} \int_{\mathbb{R}^n} |\nabla u_j|^p dx = +\infty.$$

Proof The case $p = 2$ is presented in [42], thus, we will prove the case $p \in (1, 2) \cup (2, n-k)$. However, we present a unified proof for both cases. For the proof we borrow ideas from [42] and carried out in 5 steps.

Step 1. According to the above Theorem 3, every problem ($\mathrm{P}_{\varepsilon_j}$) has at least one non-radial nodal G-invariant and σ antisymmetrical solution u_j. Let u_j, $j = 1, 2, \ldots$, an arbitrary sequence of such solutions. Since the problem ($\mathrm{P}_{\varepsilon_j}$) has a nontrivial solution belonging to one of the spaces considered earlier, then for any $\lambda > 0$ the function

$$v_j = \lambda^{\frac{1}{p^*(k)-p}} u_j \in H^{1,p}_{0,G}\left(\Omega_{\varepsilon_j}\right)$$

is a non trivial solution to the problem:

$$(\mathrm{P}_{\varepsilon_j}^{\lambda}) \quad \begin{cases} \Delta_p v_j + \varepsilon_j a(x) \left|v_j\right|^{p-2} v_j = \lambda f(x) \left|v_j\right|^{p^*(k)-2} v_j \\ v_j \not\equiv 0 \ \text{ in } \Omega, \ \ v_j = 0 \ \text{ on } \partial\Omega \\ 1 < p < n-k, \quad p^*(k) = \frac{(n-k)p}{n-k=p}. \end{cases}$$

In this first step of the proof we prove that there exists a sub-sequence of the sequence of the solutions to the problems $\left(P_{\varepsilon_j}^{\lambda}\right)$ which converges weakly in $H_0^{1,p}(\mathbb{R}^n)$.
For

$$\lambda = \left\|u_j\right\|_{H^{1,p}\left(\Omega_{\varepsilon_j}\right)}^{-(p^*(k)-p)}$$

we obtain that

$$v_j = \frac{u_j}{\left\|u_j\right\|_{H^{1,p}\left(\Omega_{\varepsilon_j}\right)}},$$

which means that the sequence $\{\upsilon_{\varepsilon_j}\}$ is bounded in $H^{1,p}(\Omega_{\varepsilon_j})$ for all $j = 1, 2, \ldots$.
Therefore, there is a positive constant C not dependent on j and such that:

$$\|\upsilon_j\|_{H^{1,p}(\Omega_{\varepsilon_j})} \leq C, \ \ \forall \ j = 1, 2, \cdots . \tag{7}$$

Because of the reflexivity of $H_0^{1,p}(\mathbb{R}^n)$ and condition (7) we may choose a subsequence of $\{\upsilon_j\}$ (again denoted by $\{\upsilon_j\}$) such that:

$$\upsilon_j \rightharpoonup \upsilon \ \text{ in } \ H_0^{1,p}(\mathbb{R}^n) \ \text{ as } \ j \to +\infty. \tag{8}$$

Step 2. In this step we prove that the function υ is a nontrivial G-invariant generalized solution of the limit problem obtained from the sequence of problems $(\mathrm{P}_{\varepsilon_j}^{\lambda})$ as $j \to \infty$.
We choose an arbitrary $\varphi \in C_0^{\infty}(\mathbb{R}^n)$. Then, according to the definition of $C_0^{\infty}(\mathbb{R}^n)$, the support of φ is bounded in $\mathbb{R}^n$, which means that there is an Ω_{ε_0} such that supp$\varphi \subset \Omega_{\varepsilon_0}$. Since, by definition, the Ω_{ε_j}s constitute a family of expanding domains, we can choose the Ω_{ε_0} such that $\Omega_{\varepsilon_0} \subset \Omega_{\varepsilon_1}$ and so $\Omega_{\varepsilon_0} \subset \Omega_{\varepsilon_j}$ for all $j = 1, 2, \ldots$.
Let

$$g(x, v_j) = \varepsilon_j a(x) v_j - \lambda f(x) |v_j|^{p^*(k)-2} v_j.$$

Then, because the υ_j is a generalized solution to $(\mathrm{P}^{\lambda}_{\varepsilon_j})$, it holds

$$\int\limits_{\mathbb{R}^n} |\nabla \upsilon_j|^{p-2}(\nabla \upsilon_j, \nabla\varphi)dx = -\int\limits_{\Omega_{\varepsilon_j}} g(x, \upsilon_j)\varphi dx = -\int\limits_{\Omega_{\varepsilon_0}} g(x, \upsilon_j)\varphi dx. \tag{9}$$

for all Ω_{ε_j}.

By the weak convergence (8), we obtain the following limit relation for the left-hand side of (9):

$$\lim_{j\to\infty} \int\limits_{\mathbb{R}^n} |\nabla \upsilon_j|^{p-2}(\nabla \upsilon_j, \nabla\varphi)dx = \int\limits_{\mathbb{R}^n} |\nabla \upsilon|^{p-2}(\nabla \upsilon, \nabla\varphi)dx. \tag{10}$$

In addition, the critical exponent of the embedding

$$H^{1,p}_G(\Omega_{\varepsilon_0}) \hookrightarrow L^p(\Omega_{\varepsilon_0})$$

is equal to

$$p^*(k) = \frac{(n-k)p}{n-k-k} > \frac{np}{n-p} = p^*.$$

Let some p_0 such that

$$p^* < p_0 < p^*(k).$$

Then the embedding is compact and thus from the Sobolev and Kondrashov theorems together and (9) arises that

$$\upsilon_j \to \upsilon \text{ in } L^{p_0-1}(\Omega_{\varepsilon_0}), \text{ as } j \to +\infty. \tag{11}$$

Furthermore, by definition of $a(x)$ and $f(x)$, there exists a positive constant C such that:

$$|g(x,t)| \le C(|t| + |t|^{p_0-1}), \quad p^* < p_0 < p^*(k),$$

for almost all $x \in \Omega_{\varepsilon_j}$, $j = 1, 2, \ldots$ and for all $t \in \mathbb{R}$.

Therefore, by Vainberg-Krasnoselskii Theorem (cf. [39, 65] or [41]) gives that:

$$\varphi g(\cdot, \upsilon_j(\cdot)) \to \varphi g(\cdot, \upsilon(\cdot)) \text{ in } L^{\frac{p_0}{p^*}}(\Omega_{\varepsilon_0}) \text{ as } j \to +\infty \tag{12}$$

and the Hölder inequality from (12) follows that:

$$\varphi g(\cdot, \upsilon_j(\cdot)) \to \varphi g(\cdot, \upsilon(\cdot)) \text{ in } L^{r}(\Omega_{\varepsilon_0}) \text{ as } j \to +\infty, \tag{13}$$

for all $1 \le r \le \frac{p_0}{p^*}$.
By (13) the limit relation from the right hand-side of (10) yields:

$$\lim_{j\to\infty} \int\limits_{\Omega_{\varepsilon_0}} g(x, \upsilon_j)\varphi dx = \int\limits_{\Omega_{\varepsilon_0}} g(x, \upsilon)\varphi dx. \tag{14}$$

Finally, passing to the limit in (9) because of (8) and (14), we obtain:

$$\int\limits_{\mathbb{R}^n} |\nabla\upsilon|^{p-2}(\nabla\upsilon, \nabla\varphi)dx = -\int\limits_{\Omega_{\varepsilon_0}} g(x, \upsilon)\varphi dx = -\int\limits_{\mathbb{R}^n} g(x, \upsilon)\varphi dx,$$

which corresponds to the definition of a weak solution. This is a generalized solution by the force of (9) and since the function f is regular enough it is a classical solution, (see §§ 1.2 and 3.1 in [41]). As convergence in L^p spaces implies a.e. convergence by (11) follows that the function υ will be G-invariant.

Step 3. In this step we prove that the solution v is non trivial, that is $\upsilon \not\equiv 0$. Suppose, by contradiction, that $\upsilon \equiv 0$. Then, for any $\varepsilon > 0$ we have

$$|\upsilon| < \frac{\varepsilon}{2}. \tag{15}$$

On the other hand, from (13) arises that

$$\upsilon_j \to \upsilon \ \text{ in } \ L^1(\Omega_{\varepsilon_0}),$$

which means that for any $\varepsilon > 0$ there exists a positive integer j_0 such that:

$$|\upsilon_j - \upsilon| < \frac{\varepsilon}{2} \ \text{ for all } \ j > j_0. \tag{16}$$

Therefore, by the standard inequality

$$|\upsilon_j| \le |\upsilon_j - \upsilon| + |\upsilon|$$

due to (15) and (16) we obtain that:

$$|\upsilon_j| < \varepsilon \ \text{ for any } \ j \ge j_0. \tag{17}$$

We recall now that every solution to the problem $(\mathrm{P}_{\varepsilon_j})$ belongs to the set

$$\mathscr{H}_\varepsilon^\sigma = \left\{ u_\varepsilon \in H_{0,G}^{1,p}(\Omega_{\varepsilon_j}) : \ u_{\varepsilon_j} \circ \sigma = -u_{\varepsilon_j} \ \text{ and } \int\limits_{\Omega_{\varepsilon_j}} f(x)|u_{\varepsilon_j}|^{p^*(k)}dx = 1 \right\}.$$

Since every υ_j corresponds to an $u_{\varepsilon_j} \in \mathscr{H}_\varepsilon^\sigma$, and $\upsilon_{\varepsilon_j} = \lambda^{\frac{1}{p^*(k)-p}} u_{\varepsilon_j}$, by definition, we have the following:

$$1 = \int_{\Omega_{\varepsilon_j}} f(x)\lambda^{-\frac{p^*(k)}{p^*(k)-p}} |\upsilon_j|^{p^*(k)} dx < \int_{\Omega_{\varepsilon_j}} f(x)\lambda^{-\frac{p^*(k)}{p^*(k)-p}} \varepsilon^{p^*(k)} dx,$$

which is false due to (17) as the $\varepsilon > 0$ can be chosen as small as we want.

Step 4. We have proved that the limit problem

$$(\mathrm{P}^\lambda) \qquad \Delta_p \upsilon = \lambda f(x)|\upsilon|^{p^*(k)-2}\upsilon \quad \text{in } \mathbb{R}^n, \quad n \geq 3$$

has a generalized non-radial nodal G-invariant and σ-anti-symmetrical solution υ, which means that the function $u = \lambda^{\frac{1}{p^*(k)-p}} \upsilon$ is a generalized non-radial nodal G-invariant and σ-anti-symmetrical solution to the limit problem:

$$(\mathrm{P}) \qquad \Delta_p u = f(x)|u|^{p^*(k)-2}u \quad \text{in } \mathbb{R}^n, \quad n \geq 3.$$

Step 5. It remains to prove that

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} |\nabla u_j|^p dx = +\infty.$$

The Sobolev inequality (6) after a normalization of the sequence u_js so that $\|u_j\|_{L^{p^*(k)}(\Omega_{\varepsilon_j})=1}$ and provided that the constants B_ϵ are positive give us that

$$1 \leqslant \left(K_G^p\left(\Omega_{\varepsilon_j}\right) + \epsilon\right) \int_{\Omega_{\varepsilon_j}} \left|\nabla u_j\right|^p dx. \tag{18}$$

From (18) after a replacement of the constant $K_G^p(\Omega_{\varepsilon_j})$ from the one calculated in Theorem 1 we obtain the inequality

$$\frac{1}{\varepsilon_j^m V^{-\frac{1}{n-k}} K\,(n-k,\ p) + \epsilon} < \int_{\Omega_{\varepsilon_j}} \left|\nabla u_j\right|^p dx. \tag{19}$$

By inequality (19) taking the limits for $j \to \infty$ we have that $\varepsilon_j \to 0$ and then

$$\int_{\mathbb{R}^n} \left|\nabla u_j\right|^p dx \to \infty.$$

This completes the proof of the theorem. □

Corollary 1 *The problem:*

$$(\mathrm{P}) \quad \begin{cases} \Delta_p u = |u|^{p^*(k)} u, \quad u \in C^2(\mathbb{R}^n), \quad n \geqslant 3 \\ 1 < p < n-k, \quad p^*(k) = \frac{(n-k)p}{n-k-p}, \end{cases}$$

has a sequence $\{u_j\}$ *of non-radial nodal* G*-invariant and* σ*-anti-symmetrical solutions, such that:*

$$\lim_{j\to+\infty} \int_{\mathbb{R}^n} |\nabla u_j|^p 0 dx = +\infty.$$

Proof The result is obtained if we put

$$f(x) = \frac{1}{|\Omega_{\varepsilon_j}|} - \varepsilon_j |x|^{\alpha}, \ \alpha > -n$$

and follows the spirit of the approach in Theorem 4. □

Remark 1 The number of the sequences of non-radial nodal G-invariant and σ-anti-symmetrical solutions to the problem (P), depends on the number of all subgroups of $O(n)$ of which the cardinal of orbits with minimum volume is infinite, that are on the dimension n of the domain.

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