Chapter 14
Growth and Cycles as a Struggle: Lotka–Volterra, Goodwin and Phillips

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14.1 Introduction

In the early 1960s, the Phillips work generated many empirical studies of the relationship between the inflation rate and unemployment. In that work, other explanatory variables, not only unemployment, were used to model either wages or price dynamics. However, many papers published in those years did not pay much attention to the evidence suggesting that the Phillips curve was not stable over time.

The breakdown of the empirical Phillips relationship began in the late 1960s with the theoretical works by Phelps [25] and Friedman [14]. According to these authors, workers are rational and take into account the expected price increases. For this reason, Friedman argued that the expectation-augmented Phillips curve would shift in such a way that, in the long run, a higher rate of inflation would not result in any change in unemployment. The price stability is consistent only with a rate of unemployment named by Friedman "natural rate of unemployment." This rate is determined by the real factors, which affect the amount of frictional and structural unemployment in the economy. On the Keynesian side, inflationary expectations either adjust to past wages and prices or, according to the "new-Keynesian" models of price stickiness, motivate forward-looking inflation expectations [6, 12, 40]. In a series of tests, Rudd and Whelan challenged the validity of those models: e. g. the

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ability of the labor share in a neo-Keynesian version of the Phillips curve to model inflation [31]; the specification of delayed and future inflation ("hybrid" inflation) [29]; the explanatory power of rational sticky price expectations models [30].

The link between the growth, cycles and the Phillips curve was introduced by Goodwin [15] who transformed the conventional labour share model (for a review, see Foley et al. [13]) into a dynamic struggle between capitalists and workers. In fact, while the share between labour and capital could be assumed constant in the long run, it fluctuates in actual economics. The Goodwin model has become a powerful framework able to accommodate extensions in many directions, from the inclusion among the endogenous variables of technical changes [34, 39] to the coupling of Goodwin's model with the financial instability hypothesis (FIH) by Minsky [16, 23, 36] and from an open economy where the long-term output growth rate is constrained by the balance of payments [9] to the incorporation of elements by Kalecki (investment function independent of savings and mark-up pricing in oligopolistic goods markets) and Marx (the reserve army) [33].

14.2 The Phillips Curve

The Phillips curve is a statistical relationship between unemployment and the rate of change of the money wage rate studied by Alban W. Phillips, a New Zealand economist, at the London School of Economics. Published in Economica in 1958 [26], the study showed that there was a nonlinear inverse relationship between the annual average percentage rate of unemployment and the annual rate of change of money wage rate:

$$\frac{\dot{w}}{w} = f(U) \quad \text{s.t.} \quad f' < 0, \tag{14.1}$$

where $\dot{w}/w$ is the rate of change of the money wage rate and U the unemployment. The curve is similar to a hyperbola with horizontal asymptote in the fourth quadrant.

The data used by Phillips were those of UK in 1861–1957. He displayed this relationship by fitting a curve to this data. Since the observations for 1948–1957 lay quite close to the curve fitted for the years 1861–1913, the relationship was thought to be stable and persistent over a long period of time. This was the reason why, in the following years, the Phillips curve played a central role in economic policy decisions to support employment. The use of the curve as an instrument of policy was made possible because, as suggested by Lipsey [19], the curve could be moved from a relationship between $\dot{w}/w$ and unemployment to one between the rate of change of the price level and unemployment. This is possible both when the markets are assumed perfectly competitive or monopolistic. In the Keynesian framework, the Phillips curve meant that inflation would erode real wages and, thus, boost labour demand.

Figure 14.1 shows the relation between unemployment rate [2] and inflation [1] in the United Kingdom for the whole period studied by Phillips while Fig. 14.2 displays the said relation for the years 1861–1913 and 1948–1947 separately.

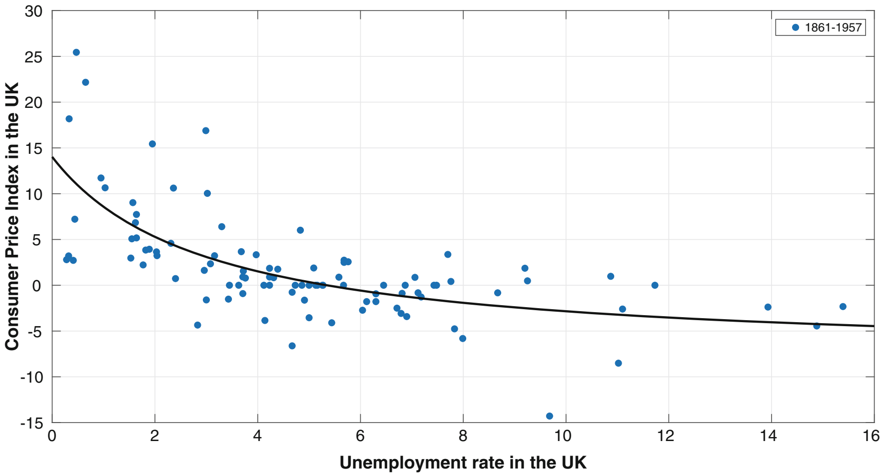

Fig. 14.1 Relation between unemployment rate [2] and inflation [1] in the United Kingdom, 1861-1957.

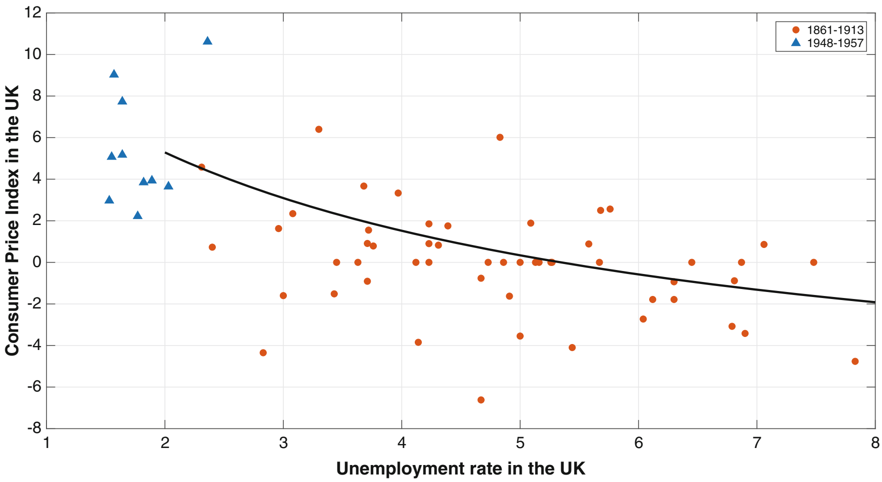

Fig. 14.2 Relation between unemployment rate [2] and inflation [1] in the United Kingdom, 1861–1913 (orange dots) and 1948–1947 (blue diamonds).

However, while there might be a relationship between employment and inflation in the short run, Phelps [25] and Friedman [14] argued that such relationship is hard to find in the long run. In particular, Friedman, by giving credit to Samuelson and Solow [32], explained that in the long run, workers and employers negotiate wages by taking into account inflation, so that pay rises increase at rates near anticipated inflation. Given a natural level of employment determined by the characteristics of the economy, an increase of inflation determines a temporary increase of employment. Agents' expectations play a role in restoring unemployment back to its previous level (how quickly it depends on the context). This process could lead to stagflation characterized by high inflation and unemployment as experienced in developed economies in 1970s. To prevent stagflation, Friedman suggested that central banks should not set unemployment targets below the natural rate.

A more radical critique to the foundations of Keynesian was made by the rational expectations school led by Robert Lucas and Thomas Sargent which challenged the idea that monetary policy could systematically affect output even in the short run. To those critics, new Keynesian models incorporate rational expectations and assume some price rigidity, i.e., sticky prices. In that context, markets do not clear instantaneously: aggregate output may be below the potential level, and an increase in liquidly can produce a short-run increase in consumption thus boosting output without inflationary consequences. Among others, we mention the paper by Chen et al. [8] in which it is possible to find a baseline disequilibrium AS-AD model empirically calibrated on quarterly time series data of the US economy 1965.1-2001.1. The model exhibits a Phillips curve, a dynamic IS curve and a Taylor interest rate rule. The outcome is "that monetary policy should allow for sufficient steady state inflation in order to avoid stability problems in areas of the phase space where wages are not flexible in a downward direction" [8].

14.2.1 Perfectly Competitive Markets

We assume that, in the economy as a whole, labour is the only variable productive factor in the short run. Given the production function $y = F(L)$ with L as input, the profit maximization problem of the firm is

$$\max_L \Pi = \max_L (pF(L) - wL), \tag{14.2}$$

where Π is the profit, p the market price of output y and wL the labour cost. The first-order condition $pF' = w$ requires that the value of the marginal productivity of labour F' must be equal to his price w/p. As the marginal productivity is decreasing $F'' < 0$, the second-order conditions are satisfied. By setting $F' = l_m$, the logarithmic differentiation of the first-order condition with respect to time yields

$$\frac{\dot{p}}{p} = \frac{\dot{w}}{w} - \frac{\dot{l_m}}{l_m}. \tag{14.3}$$

Therefore, by substitution of the Phillips curve, Eq. (14.1), we get

$$\frac{\dot{p}}{p} = f(U) - \frac{\dot{l_m}}{l_m}. \tag{14.4}$$

This means that, in competitive markets, if wages change according to the long-run changes of the marginal productivity of labour, then the average labour cost of produced goods remains unchanged and there will be no price increase in the system.

14.2.2 Monopolistic Markets

In this case, the assumption is that firms define the price by means of markup over the average cost of labour

$$p = m\frac{wL}{y} = m\frac{w}{l_a}, \tag{14.5}$$

where m is the unit markup and $l_a = y/L$ the average productivity of labour. After the logarithmic differentiation of Eq. (14.5) with respect to time, we still get

$$\frac{\dot{p}}{p} = \frac{\dot{w}}{w} - \frac{\dot{l}_a}{l_a} = f(U) - \frac{\dot{l}_a}{l_a}, \tag{14.6}$$

if the markup is assumed constant. Like the case of competitive markets, the productivity of labour (either marginal or average) plays a role in the price dynamics. Nevertheless, when the markets are not competitive, the market power of the firms cannot be neglected.

14.2.3 Calvo Model and New Keynesian Economics

As mentioned in the introduction, new Keynesian economics relies on the reinterpretation of the Phillips curve in terms of forward looking expectations and is based on sticky prices. Among the most influential contributors, we recall Fischer [12], Taylor [40] and Calvo [6].

Because of its simplicity, we use Calvo framework that deals with natural expectations and sticky prices of the new Keynesian economics.

We adopt following notation:

- z_t is the log price at time t,
- μ is the markup over the marginal cost mc_t,
- p^*_{t+k} is the log of the optimal price that the firm would set in period $t+k$ in absence of price rigidity,
- $(1-\theta)^{t+k}$ is the probability for a firm to set its price p^*_{t+k} in period $t+k$,
- $E_t\left(z_t - p^*_{t+k}\right)$ is the expected loss at time t for a firm that is not able to set the price at p^*_{t+k},
- $0 \leq \beta \leq 1$ is a discount rate,
- $\pi_t = p_t - p_{t-1}$ is the inflation rate.

The loss function for a firm is

$$L(z_t) = \sum_{k=0}^{\infty}(\theta\beta)^k[E_t\,(z_t - p_{t+k})]^2\,, \tag{14.7}$$

which implies that all future losses are considered, each one weighted by the discount rate β and the probability θ.

Equation (14.7) is minimized by differentiating with respect to the price z_t:

$$L^{'}(z_t) = 2\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left(z_t - p^*_{t+k}\right) = 0\,, \tag{14.8}$$

so that

$$\sum_{k=0}^{\infty}(\theta\beta)^k z_t = \sum_{k=0}^{\infty}(\theta\beta)^k E_t(p^*_{t+k})\,. \tag{14.9}$$

The left-hand side of Eq. (14.9) is

$$\sum_{k=0}^{\infty}(\theta\beta)^k z_t = \frac{z_t}{1-\theta\beta}\,, \tag{14.10}$$

and thus

$$\frac{z_t}{1-\theta\beta} = \sum_{k=0}^{\infty}(\theta\beta)^k E_t(p^*_{t+k})\,, \tag{14.11}$$

so that

$$z_t = (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_t(p^*_{t+k})\,. \tag{14.12}$$

Note that the second order condition for the minimum is

$$L^{''}(z_t) = 2\sum_{k=0}^{\infty}(\theta\beta)^k = \frac{2}{(1-\theta\beta)} > 0 \tag{14.13}$$

which is satisfied because θ and $\beta \in (0, 1)$. Thus, Eq. (14.12) states that the optimal solution for the firm, in presence of sticky prices, is a weighted average of expected future prices.

Given that firms should set the price as a markup over marginal cost, we may assume that

$$p^*_{t+k} = \mu + mc_{t+k}\,, \tag{14.14}$$

so the reset price in Eq. (14.12) can be rewritten as

$$z_t = (1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_t(\mu + mc_{t+k}). \tag{14.15}$$

In general, a first-order stochastic difference equation of type

$$y_t = aE_t(y_{t+1}) + bx_t \tag{14.16}$$

has the following solution:

$$y_t = b\sum_{k=0}^{\infty} a^k E_t(x_{t+k})\,. \tag{14.17}$$

Eq. 14.15 says that z_t is the solution of

$$z_t = \theta\beta E_t(z_{t+1}) + (1-\theta\beta)(\mu + mc_t), \tag{14.18}$$

where $y_t = z_t$, $x_t = \mu + mc_t$, $a = \theta\beta$ and $b = 1-\theta\beta$.

At the aggregate level, prices are a weighted average of previous prices and current reset prices

$$p_t = \theta p_{t-1} + (1-\theta)z_t\,, \tag{14.19}$$

which rearranged is

$$z_t = \frac{1}{1-\theta}(p_t - \theta p_{t-1})\,, \tag{14.20}$$

or equivalently

$$\begin{aligned} z_t &= \frac{1}{1-\theta}\Big((1-\theta)p_t + \theta p_t - \theta p_{t-1}\Big) \\ &= p_t + \frac{\theta}{1-\theta}(p_t - p_{t-1}) = p_t + \frac{\theta}{1-\theta}\pi_t \\ z_t &= \frac{1}{1-\theta}\Big(p_t - p_{t-1} + (1-\theta)p_{t-1}\Big) \\ &= \frac{1}{1-\theta}(p_t - p_{t-1}) + p_{t-1} = \frac{1}{1-\theta}\pi_t + p_{t-1}\,. \end{aligned} \tag{14.21}$$

$$p_t + \frac{\theta}{1-\theta}\pi_t = \theta\beta E_t\Big(\frac{1}{1-\theta}\pi_{t+1} + p_t\Big) + (1-\theta\beta)(\mu + mc_t)$$

$$p_t + \frac{\theta}{1-\theta}\pi_t = \frac{\theta\beta}{1-\theta} E_t(\pi_{t+1}) + \theta\beta p_t + (1-\theta\beta)(\mu + mc_t)$$

$$\frac{\theta}{1-\theta}\pi_t = \frac{\theta\beta}{1-\theta} E_t(\pi_{t+1}) + (1-\theta\beta)(\mu + mc_t - p_t) .$$

So that, by rearranging, we arrive at the *New-Keynesian Phillips curve*

$$\pi_t = \beta E_t(\pi_{t+1}) + \frac{(1-\theta)(1-\beta\theta)}{\theta}(\mu + mc_t - p_t) . \tag{14.22}$$

Equation (14.22) states that current prices depend on next period expected inflation rate $E_t(\pi_{t+1})$ and real marginal costs $mc_t - p_t$. As the latter is not observed nor recorded in national accounts, this relationship is hard to test empirically.

14.3 Lotka–Volterra Model

The Lotka–Volterra 'predator–prey' model describes the interaction between two species: the predator and the prey. This model was initially proposed by Alfred J. Lotka [20], who borrowed from Verhulst the logistic map [43]. Independently, Vito Volterra developed the same equations to explain the dynamics of the fish catches in the Adriatic Sea [44] (cf. for further details Kinoshita [17]).

The assumptions of the model are as follows:

(a) Preys have access to unlimited food.
(b) Preys are the unique source of food for predators which, in turn, have limitless appetite.
(c) The rate of change of both populations is proportional to the size.
(d) Genetic adaptation and environment changes are not considered.

And the model equations read

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta x y, \\ \frac{dy}{dt} &= \delta x y - \gamma y, \end{aligned} \tag{14.23}$$

where

- x is the number of preys,
- y is the number of predators,
- α is the natural growth rate of preys in the absence of predation,
- β is the death rate of preys due to predation,

- δxy is the natural growth rate of predators or efficiency rate of turning preys into predators,
- γ is the natural death rate of predators in the absence of preys.

By dividing the second equation by the first in (14.24), we get

$$\frac{dy}{dx} = -\frac{y}{x}\frac{\delta x - \gamma}{\beta y - \alpha} \tag{14.24}$$

from which integration yields

$$\frac{\beta y - \alpha}{y}\,dy + \frac{\delta x - \gamma}{x}\,dx = 0,$$

i.e.,

$$\delta x - \gamma \ln x + \beta y - \alpha \ln y = A,$$

where A is constant.

14.4 The Goodwin Model

Richard M. Goodwin, was one of the first economists to develop a nonlinear model of the business cycle and one of the first pioneers of chaotic dynamics in economics. In his model on the growth cycle [15], Goodwin founds his assumptions on the Harrod intuition that a capitalist economy grows until it arrives near full employment, after which it collapses. To formalize this intuition, that is, the coexistence of growth and cycle in the same model, Goodwin suggests an economic adaptation of the Lotka–Volterra predator–prey system. In contrast to the mainstream approach [7, 11, 35] in which cycles were caused by exogenous shocks, this model had the advantage to explain endogenously output fluctuations together with the ones of employment and wages.

In the framework, we are discussing that the economy produces a single good, workers consume all their wage and capitalists save and invest all their profits. Economic growth rate is positively related to both saving rate and capital share. In fact, as workers do not save, a decrease in the profit share reduces investments and, as a consequence, future output. Thus, during a recession, the lower labour demand brings salaries down and restores the profit share of capitalists (who will again start investing more).

In the Goodwin model and its extensions, when the economy expands, higher labour demand generates wage inflation, so that real wages increase more than labour productivity. This in turn implies that the wage share increases as production increases. So when the economy is expanding, the rigidity of the labor market can increase wages more than productivity, thus reducing investment and growth.

14.4.1 Assumptions of Goodwin's Model

The key assumptions of Goodwin, as described in his original work, are

(a) steady technical progress (disembodied),
(b) steady growth in the labour force,
(c) only two factors of production, labour and "capital" (plant and equipment), both homogeneous and non-specific,
(d) all quantities real and net,
(e) all wages consumed, all profits saved and invested,
(f) a constant capital-output ratio,
(g) a real wage rate that rises in the neighbourhood of full employment.

Caveats in this list are in assumption (e), which could be changed into constant proportional savings without altering the logic of the model, and in assumption (f), which could be softened at the cost of overcomplicating the system. Assumptions (f) and (g) are empirical and disputable.
In the following, we list the symbols that are used in Sect. 14.4.2 and are consistent with Goodwin's original paper:

(i) q output,
(ii) k capital,
(iii) w wage,
(iv) $a = a_0\, e^{\alpha t}$ labour productivity, where α is the growth parameter,
(v) $s = q/k = 1/\sigma$ capital productivity,
(vi) $k/q = \sigma$ capital-output ratio,
(vii) $u = w/a$ workers' share of product,
(viii) $(1 - w/a)$ capitalists' share of product,
(ix) $(1 - w/a)q = \dot{k}$ surplus = profit = savings = investments,
(x) $\dot{k}/k = \dot{q}/q = (1 - w/a)/\sigma$ profit rate,
(xi) $n = n_0\, e^{\beta t}$ labour supply, where β is the growth parameter,
(xii) $l = q/a$ employment,
(xiii) $v = l/n$ employment rate.

14.4.2 Dynamics of Goodwin's Model

The logarithmic differentiation of the employment rate and of the workers' share of product yields, respectively:

$$\begin{aligned} \frac{\dot{v}}{v} &= \frac{\dot{l}}{l} - \frac{\dot{n}}{n} = \frac{\dot{q}}{q} - \alpha - \beta \\ \frac{\dot{u}}{u} &= \frac{\dot{w}}{w} - \alpha, \end{aligned} \tag{14.25}$$

where $\frac{\dot{w}}{w} = \rho v - \gamma$ is the linearized Phillips curve.
By virtue of notation (x) the system in Eq. (14.25) becomes

$$\begin{aligned}\dot{v} &= \left[\frac{1-u}{\sigma} - (\alpha+\beta)\right] v \\ \dot{u} &= [-(\gamma+\alpha) + \rho v]\, u.\end{aligned} \tag{14.26}$$

At the equilibrium, it must be $\dot{v} = \dot{u} = 0$, so the solutions are: $v = u = 0$ and (v^*, u^*), such that

$$\begin{aligned}v^* &= \frac{\gamma+\alpha}{\rho}\,, \\ u^* &= [1 - (\beta+\alpha)\sigma]\ .\end{aligned}$$

To have economic meaning, Goodwin imposes that $u^* > 0$, i.e. $\dfrac{1}{\sigma} > (\alpha+\beta)$.

By means of the linear approximation method near the two equilibria, we get the following Jacobian matrices, respectively:

$$J(0,0) = \begin{bmatrix} \frac{1}{\sigma} - (\alpha+\beta) & 0 \\ 0 & -(\gamma+\alpha) \end{bmatrix} \quad \text{and} \quad J(v^*, u^*) = \begin{bmatrix} 0 & -\frac{1}{\sigma} v^* \\ \rho u^* & 0 \end{bmatrix}.$$

As $J_0 = J(0,0)$ is a diagonal matrix, the eigenvalues are real and of opposite sign, the origin is a saddle point. At the equilibrium point (v^*, u^*), the eigenvalues are purely imaginary. Therefore, the fixed point is neutrally stable (or equivalently structurally unstable), and the trajectories are closed orbits. The specific closed orbit where the system will be located depends on the initial condition. We are bound to remind the reader that the system (14.26) is a rare example of integrable system of nonlinear differential equations. The procedure is as follows.

Similarly to the Lotka-Volterra model, let us rewrite the system (14.26) as

$$\begin{aligned}\frac{dv}{dt} &= [s - (\alpha+\beta) - u\, s]\, v\,, \\ \frac{du}{dt} &= [-(\gamma+\alpha) + \rho v]\, u\,,\end{aligned} \tag{14.27}$$

and divide the second equation by the first, so that

$$\frac{du}{dv} = \frac{\rho v - \gamma - \alpha}{s - (\alpha+\beta) - u\, s}\frac{u}{v}$$

or equivalently

$$[s - (\alpha+\beta) - u\, s]\, v\, du + [(\gamma+\alpha) - \rho v]\, u\, dv = 0.$$

As the variables are separable, through the division by uv, we get

$$\left[\frac{s-(\alpha+\beta)}{u}-s\right]du+\left[\frac{(\gamma+\alpha)}{v}-\rho\right]dv=0$$

and integrating

$$\int\left[\frac{s-(\alpha+\beta)}{u}-s\right]du+\int\left[\frac{(\gamma+\alpha)}{v}-\rho\right]dv=0$$
$$\text{i.e. } [s-(\alpha+\beta)]\log u-s\,u+(\gamma+\alpha)\log v-\rho\,v=A.$$

It follows that

$$u^{s-(\alpha+\beta)}\,e^{-s\,u}\,v^{\gamma+\alpha}\,e^{-\rho v}=e^{A}. \tag{14.28}$$

By setting

$$U=u^{s-(\alpha+\beta)}\,e^{-s\,u},\quad V=v^{-(\gamma+\alpha)}\,e^{\rho\,v}\text{ and }B=e^{A}, \tag{14.29}$$

(14.28) can be rewritten as

$$U(u)=BV(v). \tag{14.30}$$

This equality allows us to obtain the integral curves in the plane (v,u). In fact, to each value of the arbitrary constant B, there is a corresponding integral curve. To draw the integral curves, we have to investigate the shape of the curves U and V.

Hence, the curve

- U has a maximum in u^* because

$$\left.\frac{dU}{du}\right|_{u=u^*}=u^{*,s-(\alpha+\beta)}\,e^{-su^*}\left[\frac{s-(\alpha+\beta)}{u^*}-s\right]=0$$

  and

$$\left.\frac{d^2U}{du^2}\right|_{u=u^*}=-U\left[\frac{s-(\alpha+\beta)}{u^{*,2}}\right]<0.$$

- V has a minimum in v^* because

$$\frac{dV}{dv}=v^{-(\gamma+\alpha)}\,e^{\rho\,v}\left[\frac{-(\alpha+\gamma)}{v}+\rho\right]=0$$

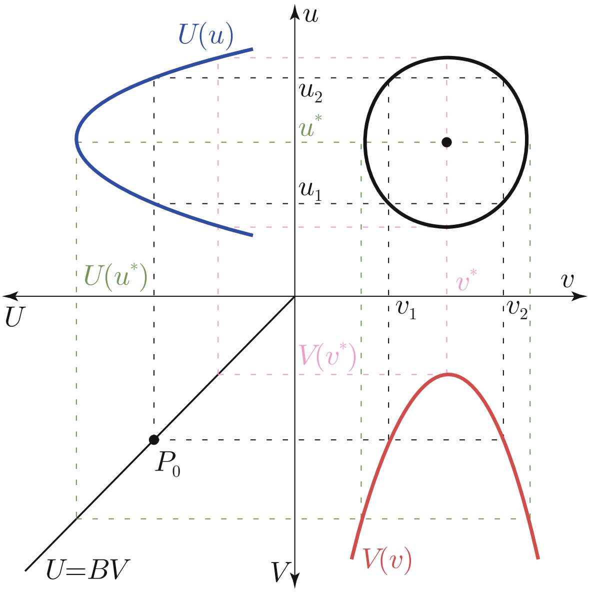

Fig. 14.3 Graphical proof that system (14.27) displays infinite closed orbits

and

$$\left.\frac{d^2V}{dv^2}\right|_{v=v^*} = V\left[\frac{\alpha+\gamma}{v^{*,2}}\right] > 0.$$

Now, we are in position to display the integral curves in Fig. 14.3. In the second and fourth quadrants, we qualitatively report the curves $U(u)$ (blue) and $V(v)$ (red) as well as their optima u^* and v^*.

Let us now consider a point P_0 satisfying (14.30). This corresponds to choosing a point on the straight line $U = BV$ in the third quadrant. P_0 can be projected in the second quadrant through the inverse of $U(u)$ so that points u_1 and u_2 correspond to $U^{-1}(P_0)$. Similarly, the inverse mapping $V^{-1}(P_0)$ identifies points v_1 and v_2 in the fourth quadrant. The projections of the four points in the first quadrant (i.e., the (u, v) plane) identify the coordinates (u_1, v_1), (u_1, v_2), (u_2, v_1) and (u_2, v_2) that satisfy (14.30). Thus, by iterating the process, we can state that the system has a periodic closed orbit corresponding, graphically, to the curve drawn in the (u, v) plane (first quadrant). Note that, as the choice of the parameter B is arbitrary, the system has infinitely many periodic closed orbits, around the equilibrium point (u^*, v^*).

14.5 Kolmogorov Prey–Predator Model

Although the Goodwin model is able to describe persistent oscillations of an economic system, it cannot display structural stability.[1] Many attempts to add this feature were made in the 1970s and the 1980s (e.g., Desai [10] and van der Ploeg [27, 28]). However, despite the use of additional hypotheses, the Goodwin non-trivial equilibrium point remained a centre; if not, it became a stable node or a focus. This type of result is a direct consequence of the structural instability: any small perturbation (as an effect of additional hypotheses) leads to the loss of the cycle. Kolmogorov [18] was the first to raise the problem of structural instability, suggesting a more general version of the predator–prey system as follows:

$$\begin{aligned} \frac{\dot{N}_1}{N_1} &= K_1(N_1, N_2) \\ \frac{\dot{N}_2}{N_2} &= K_2(N_1, N_2), \end{aligned} \tag{14.31}$$

where $K_1(0, 0) = 0$, $K_2(0, 0) = 0$, K_1 and K_2 are continuous functions with continuous first derivatives for all N_1 (the preys) and N_2 (the predators). By imposing some appropriate conditions on K_1 and K_2 the integral curves of system 14.31 are the coordinates displayed in Fig. 14.4.

The isoclines $K_1 = 0$ and $K_2 = 0$ divide the first quadrant into four parts (see Fig. 14.4) and the singular points are the origin $(0, 0)$, $Z = (N_1^*, N_2^*)$ obtained by the intersection of the isoclines $K_1 = K_2 = 0$ and B corresponding to $N_2 = 0$ and $K_1 = 0$. Kolmogorov provided the functions K_1 and K_2 with well-founded assumptions in biological theory, and showed that system (14.31) may generate limit cycles when the equilibrium point (N_1^*, N_2^*) is unstable. As Kolmogorov [18] affirmed, "no integral curve starting in the domain $N_1 > 0$, $N_2 > 0$ can move asymptotically toward the coordinate axes. In other words, if initially both $N_1 > 0$ and $N_2 > 0$, neither species can completely disappear". In the classical model, either there is a globally stable equilibrium or there is a globally stable cycle. Other modifications, such as intraspecific competition among prey and predators (e.g., see [24]), confirm the results obtained by Kolmogorov and show that the transition from the globally stable equilibrium to the stable cycle is obtained through non-catastrophic Hopf bifurcations. Contrarily, Tyutyunov et al. [41], by adopting the Patlak–Keller–Segel taxis model for the predator and "assuming that movement velocities of predators are proportional to the gradients of specific cues emitted by prey," showed that the stationary regime of the model becomes unstable with respect to small perturbations.

[1] For a detailed treatment of the subject, see Veneziani et al. [42].

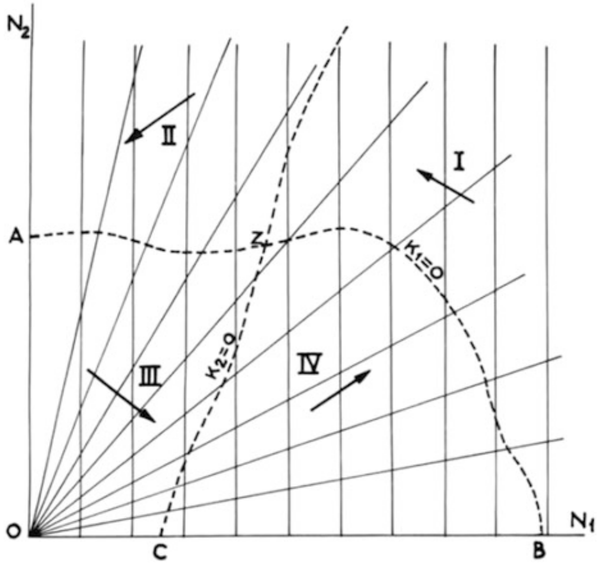

Fig. 14.4 First quadrant describing the dynamics of prey-predators: in part I (above $K_1 = 0$ and to the right of $K_2 = 0$) preys decrease and predators increase, in part II (above $K_1 = 0$ and to the left of $K_2 = 0$) both preys and predators decrease, in part III (below $K_1 = 0$ and to the left of $K_2 = 0$) preys increase and predators decrease, in part IV (below $K_1 = 0$ and to the right of $K_2 = 0$) both preys and predators increase. Source[18].

In summary, from a practical standpoint, Kolmogorov's approach has the merit of emphasizing the analytical properties that a predator–prey system must satisfy, in order to ensure structural stability. Since then, this approach has become a basic landmark for specific predator–prey models in biology and economics. In such models, the existence of stable limit cycles is proved either by the Poincare–Bendixon theorem or by the Hopf bifurcation theorem.

Among others, see the models by May [21] and Tanner [38] in mathematical biology and Medio [22] and Sportelli [37] for applications to economics. Last but not least, according to the Goodwin model, the wage share should lag behind the employment rate. Moreover, this is not always true in reality, see Fig. 14.5, where wage share ratio (blue line) [3, 5] is not close at all to employment (red line) [4].

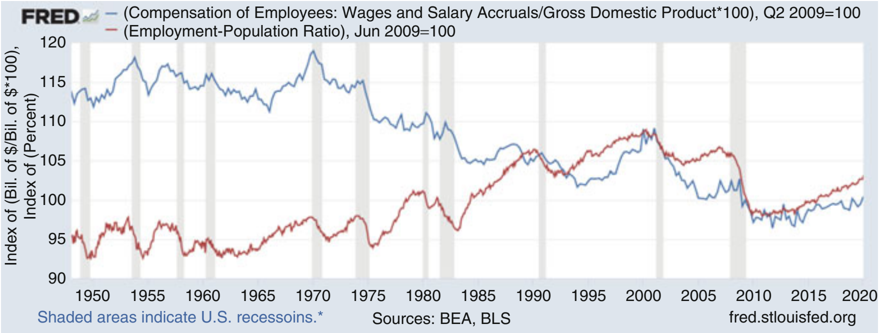

Fig. 14.5 Blue: compensation of employees: wages and salary accruals/USA GDP *100), quarterly, seasonally adjusted annual rate. Red: employment-population ratio (percentage), monthly, seasonally adjusted

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