On a Cournot Dynamic Game with Cost Uncertainty and Relative Profit Maximization

Georges Sarafopoulos and Kosmas Papadopoulos

Abstract In this paper, a Cournot duopoly model with homogeneous goods is examined with uncertain cost function. A random linear cost function is introduced in this model for the first player. The case of homogeneous expectations is studied. The existence and uniqueness of the equilibrium are obtained. The asymptotic behavior of the equilibrium point is also investigated. Complete stability and bifurcation analysis are provided. The obtained theoretical results are verified by numerical simulations.

Keywords Cournot duopoly game · Cost uncertainty · Relative profit maximization · Discrete dynamical system · Nash equilibrium · Stability · Bifurcation diagrams · Lyapunov numbers · Strange attractors · Chaotic behavior

1 Introduction

An Oligopoly is a market structure between monopoly and perfect competition, where there are only a few number of firms in the market producing homogeneous products. The dynamic of an oligopoly game is more complex because firms must consider not only the behaviors of the consumers, but also the reactions of the competitors i.e. they form expectations concerning how their rivals will act. Cournot, in 1838 has introduced the first formal theory of oligopoly. In 1883 another French mathematician Joseph Louis Francois Bertrand modified Cournot game suggesting that firms actually choose prices rather than quantities. Originally Cournot and Bertrand models were based on the premise that all players follow naive expectations, so that in every step, each player (firm) assumes the last values that were taken by the competitors without estimation of their future reactions. However, in real market conditions

G. Sarafopoulos (✉) · K. Papadopoulos
Department of Economics, Democritus University of Thrace, University Campus, 69100 Komotini, Greece
e-mail: gsarafop@econ.duth.gr

K. Papadopoulos
e-mail: kpapa@econ.duth.gr

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such an assumption is very unlikely since not all players share naive beliefs. Therefore, different approaches to firm behavior were proposed. Some authors considered duopolies with homogeneous expectations and found a variety of complex dynamics in their games, such as appearance of strange attractors (Agiza [1]; Agiza et al. [4]; Agliari et al. [5, 6]; Bischi and Kopel [11]; Kopel [18]; Puu [23]; Sarafopoulos [24, 25]; Sarafopoulos et al. [28]; Zhang et al. [32]). Also models with heterogeneous agents were studied (Agiza and Elsadany [2, 3]; Den Haan [12]; Fanti and Gori [15]; Hommes [17]; Sarafopoulos et al. [26, 27, 29]; Tramontana [30]; Zhang et al. [31]).

In the real market producers do not know the entire demand function, though it is possible that they have a perfect knowledge of technology, represented by the cost function. Hence, it is more likely that firms employ some local estimate of the demand. This issue has been previously analyzed by Baumol and Quandt [9]; Puu [22]; Naimzada and Ricchiuti [20]; Askar [7]; Askar [8]. Bounded rational players (firms) update their strategies based on discrete time periods and by using a local estimate of the marginal profit. With such local adjustment mechanism, the players are not requested to have a complete knowledge of the demand and the cost functions (Agiza and Elsadany [2]; Naimzada and Sbragia [21]; Zhang et al. [32]; Askar [8]; Bischi et al. [10, 11]).

In this paper we study the dynamics of a Cournot-type duopoly with homogeneous goods where each firm behaves with homogeneous expectations. We show that the model gives more complex chaotic and unpredictable trajectories as a consequence of change in the speed of players' adjustment. The paper is organized as follows: In Sect. 2, the dynamics of the duopoly game with homogeneous expectations, linear demand and cost functions and relative profit functions for two players are analyzed. A cost uncertainty is introduced into first player's utility function. We set both players as bounded rational players. The existence and local stability of the equilibrium points are also analyzed. In Sect. 3 numerical simulations are used to verify the algebraic results of Sect. 2 plotting the bifurcation diagrams of the game's system and to show the complex dynamics via computing Lyapunov numbers, and sensitive dependence on initial conditions.

2 The Game

2.1 The Construction of the Game

In this study we assume that in the two companies there is a separation between ownership and management, so there is a possibility that the managers who make decisions for the company to decide at the expense of their company trying to increase the profits of the competitor. Also, we consider homogeneous players and more specifically, we consider that both firms choose the quantity of their productions in a rational way, following an adjustment mechanism (bounded rational players). We consider a simple Cournot-type duopoly market where firms (players) produce the same good and offer it at discrete-time periods on a common market. Production decisions are taken at discrete time periods t = 0, 1, 2,… At each period t, every firm must form an expectation of the rival's strategy in the next time period in order to determine the corresponding profit-maximizing prices for period t + 1. We suppose that q_1, q_2 are the production quantities of each firm. Also, we consider that the preferences of consumers represented by the equation:

$$U(q_1, q_2) = \alpha(q_1 + q_2) - \frac{1}{2}\left(q_1^2 + q_2^2 + 2dq_1q_2\right) \tag{1}$$

where α is a positive parameter (α > 0), which expresses the market size and $d \in [-1, 1]$ is the parameter that reveals the differentiation degree of products [13]. For example, if $d = 0$ then both products are independently and each firm participates in a monopoly. But, if $d = 1$ then one product is a substitute for the other, since the products are homogeneous. It is understood that for positive values of the parameter d the larger the value, the less diversification we have in both products. On the other hand negative values of the parameter d are described that the two products are complementary and when $d = -1$ then we have the phenomenon of full competition between the two companies. The inverse demand functions (as functions of quantities) coming from the maximizing of (1) are given by the following equations (assuming d = 1):

$$\mathrm{p}_1(q_1, q_2) = \alpha - q_1 - q_2 \text{ and } \mathrm{p}_2(q_1, q_2) = \alpha - q_2 - q_1, (d = 1) \tag{2}$$

In this work we suppose that the first player's cost function contains an uncertainty by which the marginal cost (linear cost function) is equal to the combination between the parameters:$c_1, c_2 > 0$, which is described by the following equation:

$$C_1(q_1) = [p \cdot c_1 + (1 - p) \cdot c_2] \cdot q_1 \tag{3}$$

where $p \in [0, 1]$, is the positive uncertainty cost parameter.

On the other hand the second player uses a simple linear cost function that its marginal cost is equal to $c_1 > 0$ and it is described by the equation:

$$C_2(q_2) = c_1 \cdot q_2 \tag{4}$$

With these assumptions the profits of the firms are given by:

$$\Pi_1(q_1, q_2) = p_1 \cdot q_1 - C_1(q_1) = [\alpha - q_1 - q_2 - p \cdot c_1 - (1 - p) \cdot c_2] \cdot q_1 \tag{5}$$

and

$$\Pi_2(q_1, q_2) = p_2 \cdot q_2 - C_2(q_2) = [\alpha - c_1 - q_1 - q_2]q_2 \tag{6}$$

Then the marginal profits at the point of the strategy space are given by:

$$\frac{\partial \Pi_1}{\partial q_1} = \alpha - p \cdot c_1 - (1 - p) \cdot c_2 - 2q_1 - q_2, \frac{\partial \Pi_1}{\partial q_2} = -q_1 \tag{7}$$

and

$$\frac{\partial \Pi_2}{\partial q_2} = \alpha - c_1 - q_1 - 2q_2, \frac{\partial \Pi_2}{\partial q_1} = -q_2 \tag{8}$$

As it is noticed both managers care about the maximization of a utility function that contains a percentage of opponent company's profits (generalized relative profit function), which is given by:

$$U_i = (1 - \mu_i) \cdot \Pi_i + \mu_i \cdot (\Pi_i - \Pi_j) = \Pi_i - \mu_i \cdot \Pi_j \tag{9}$$

where $\mu \in [0, 1]$ is the percentage that the player i takes into account the opponent company's prifots. So, the marginal utility of the player i is given by the following equation:

$$\frac{\partial U_i}{\partial q_i} = \frac{\partial \Pi_i}{\partial q_i} - \mu_i \cdot \frac{\partial \Pi_j}{\partial q_i} \tag{10}$$

and the marginal utilities for each player are:

$$\frac{\partial U_1}{\partial q_1} = \alpha - p \cdot c_1 - (1 - p) \cdot c_2 - 2q_1 - (1 - \mu)q_2 \tag{11}$$

and

$$\frac{\partial U_2}{\partial q_2} = \alpha - c_1 - (1 - \mu)q_1 - 2q_2 \tag{12}$$

Both players are characterized as bounded rational players. According to the existing literature it means that they decide their productions following a mechanism that is described by the equation:

$$\frac{q_i(t+1) - q_i(t)}{q_i(t)} = k \cdot \frac{\partial U_i}{\partial q_i}, k \leq 0 \tag{13}$$

Through this mechanism the player increases his level of adaptation when his marginal utility is positive or decreases his level when his marginal utility is negative, where k is the speed of adjustment of player, it is a positive parameter (k > 0), which gives the extend variation of production quantity of the each company, following a given utility signal.

The dynamical system of the players is described by:

$$\begin{cases} q_1(t+1) = q_1(t) + k \cdot q_1(t) \cdot \dfrac{\partial U_1}{\partial q_1} \\ q_2(t+1) = q_2(t) + k \cdot q_2(t) \cdot \dfrac{\partial U_2}{\partial q_2} \end{cases} \tag{14}$$

We will focus on the dynamics of this system to the parameter k.

2.2 Dynamical Analysis

The dynamical analysis of the discrete dynamical system involves finding equilibrium positions and studying them for stability. The ultimate goal of this algebraic study is to formulate a proposition that will be the stability condition of the Nash Equilibrium position. Finally, these algebraic results are verified and visualized doing some numerical simulations using the program of Mathematica.

2.2.1 The Equilibrium Positions

The equilibriums of the dynamical system (14) are obtained as the nonnegative solutions of the algebraic system:

$$\begin{cases} q_1^* \cdot \dfrac{\partial U_1}{\partial q_1} = 0 \\ q_2^* \cdot \dfrac{\partial U_2}{\partial q_2} = 0 \end{cases} \tag{15}$$

which is obtained by setting: $q_1(t+1) = q_1(t) = q_1^*$ and $q_2(t+1) = q_2(t) = q_2^*$.

- If $q_1^* = q_2^* = 0$ then the boundary equilibrium position is the point:

$$E_0 = (0, 0) \tag{16}$$

- If $q_1^* = 0$ and $\frac{\partial U_2}{\partial q_2} = 0$ then: $q_2^* = \frac{\alpha - c_1}{2}$ and the equilibrium position is the point:

$$E_1 = \left(0, \frac{\alpha - c_1}{2}\right) \tag{17}$$

- If $q_2^* = 0$ and $\frac{\partial U_1}{\partial q_1} = 0$ then: $q_1^* = \frac{\alpha - p \cdot c_1 - (1-p) \cdot c_2}{2}$ and the equilibrium position is the point:

$$E_2 = \left(\frac{\alpha - p \cdot c_1 - (1 - p) \cdot c_2}{2}, 0 \right) \tag{18}$$

- If $\frac{\partial U_1}{\partial q_1} = \frac{\partial U_2}{\partial q_2} = 0$ then the following system is obtained:

$$\begin{cases} \alpha - p \cdot c_1 - (1 - p) \cdot c_2 - 2q_1^* - (1 - \mu) \cdot q_2^* = 0 \\ \alpha - c_1 - (1 - \mu) \cdot q_1^* - 2q_2^* = 0 \end{cases} \tag{19}$$

and the nonnegative solution of this algebraic system will give the Nash Equilibrium position $E_* = \left(q_1^*, q_2^*\right)$ where:

$$q_1^* = \frac{\alpha(1 + \mu) + (1 - \mu - 2p) \cdot c_1 - 2(1 - p) \cdot c_2}{4 - (1 - \mu)^2} \tag{20}$$

and

$$q_2^* = \frac{\alpha(1 + \mu) - (2 - p + p \cdot \mu) \cdot c_1 + (1 - p) \cdot (1 - \mu) \cdot c_2}{4 - (1 - \mu)^2} \tag{21}$$

This means that:

$$\alpha(1 + \mu) + (1 - \mu - 2p) \cdot c_1 - 2(1 - p) \cdot c_2 > 0 \tag{22}$$

and

$$\alpha(1 + \mu) - (2 - p + p \cdot \mu) \cdot c_1 + (1 - p) \cdot (1 - \mu) \cdot c_2 > 0. \tag{23}$$

2.2.2 Stability of Equilibrium Points

To study the stability of the equilibrium positions we need the Jacobian matrix of the dynamical system Eq. (15) which is the matrix:

$$J\left(q_1^*, q_2^*\right) = \begin{bmatrix} f_{q_1} & f_{q_2} \\ g_{q_1} & g_{q_2} \end{bmatrix} \tag{24}$$

where:

$$\begin{aligned} f(q_1, q_2) &= q_1 + k \cdot q_1 \cdot \frac{\partial U_1}{\partial q_1} \\ g(q_1, q_2) &= q_2 + k \cdot q_2 \cdot \frac{\partial U_2}{\partial q_2} \end{aligned} \tag{25}$$

and as a result the Jacobian matrix of game's discrete dynamical system Eq. (14) is the following matrix:

$$J\left(q_1^*, q_2^*\right) = \begin{bmatrix} 1 + k \cdot \left(\frac{\partial U_1}{\partial q_1} + q_1^* \cdot \frac{\partial^2 U_1}{\partial q_1^2}\right) & k \cdot q_1^* \cdot \frac{\partial^2 U_1}{\partial q_1 \partial q_2} \\ k \cdot q_2^* \cdot \frac{\partial^2 U_2}{\partial q_2 \partial q_1} & 1 + k \cdot \left(\frac{\partial U_2}{\partial q_2} + q_2^* \cdot \frac{\partial^2 U_2}{\partial q_2^2}\right) \end{bmatrix} \tag{26}$$

For the E_0 the Jacobian matrix becomes as:

$$J(E_0) = \begin{bmatrix} 1 + k \cdot \frac{\partial U_1}{\partial q_1} & 0 \\ 0 & 1 + k \cdot \frac{\partial U_2}{\partial q_2} \end{bmatrix} \overset{A=1+k\cdot\frac{\partial U_1}{\partial q_1}}{\underset{B=1+k\cdot\frac{\partial U_2}{\partial q_2}}{=}} \begin{bmatrix} \mathrm{A} & 0 \\ 0 & B \end{bmatrix} \tag{27}$$

with $Tr = \mathrm{A} + B$ and $Det = A \cdot B$.

From the characteristic equation of $J(E_0)$, we find the nonnegative eigenvalues:

$$r_1 = A = 1 + k \cdot \frac{\partial U_1}{\partial q_1} \text{ and } r_2 = B = 1 + k \cdot \frac{\partial U_2}{\partial q_2} \tag{28}$$

it's clearly seems that $|r_1|, |r_2| > 1$ and the E_0 equilibrium is unstable.

For the E_1 the Jacobian matrix becomes as:

$$J(E_1) = \begin{bmatrix} 1 + k \cdot \frac{\partial U_1}{\partial q_1} & 0 \\ -k \cdot (1-\mu) q_2^* & 1 - 2k \cdot q_2^* \end{bmatrix} \overset{C=1+k\cdot\frac{\partial U_1}{\partial q_1}}{\underset{E=1-2k\cdot q_2^*}{=}} \begin{bmatrix} C & 0 \\ D & E \end{bmatrix} \tag{29}$$

with $Tr = C + E$ and $Det = C \cdot E$.

From the characteristic equation of $J(E_1)$, we find the nonnegative eigenvalue:

$$r_1 = C = 1 + k \cdot \frac{\alpha(1+\mu) + (1-\mu-2p) \cdot c_1 - (1-p) \cdot c_2}{2} \tag{30}$$

it's clearly seems that $|r_1| > 1$, because:

$\alpha(1+\mu) + (1-\mu-2p) \cdot c_1 - 2(1-p) \cdot c_2 > 0$ Eq. (22) and the E_1 equilibrium is unstable.

For the E_2 the Jacobian matrix becomes as:

$$J(E_2) = \begin{bmatrix} 1 - 2k \cdot q_1^* & -k \cdot (1-\mu) q_1^* \\ 0 & 1 + k \cdot \frac{\partial U_2}{\partial q_2} \end{bmatrix} \overset{F=1-2k\cdot q_1^*}{\underset{H=1+k\cdot\frac{\partial U_2}{\partial q_2}}{=}} \begin{bmatrix} F & G \\ 0 & H \end{bmatrix} \tag{31}$$

with $Tr = F + H$ and $Det = F \cdot H$.

From the characteristic equation of $J(E_2)$, we find the nonnegative eigenvalue:

$$r_2 = H = 1 + k \cdot \frac{\alpha(1+\mu) - (2-p+p\cdot\mu) \cdot c_1 + (1-p) \cdot (1-\mu) \cdot c_2}{2} \tag{32}$$

it's clearly seems that $|r_2| > 1$, because:
$\alpha(1+\mu) - (2 - p + p \cdot \mu) \cdot c_1 + (1-p) \cdot (1-\mu) \cdot c_2 > 0$ Eq. (23) and the E_2 equilibrium is unstable.
For the E_* the Jacobian matrix becomes as:

$$J(E_*) = \begin{bmatrix} 1 + k \cdot q_1^* \cdot \frac{\partial^2 U_1}{\partial q_1^2} & k \cdot q_1^* \cdot \frac{\partial^2 U_1}{\partial q_1 \partial q_2} \\ k \cdot q_2^* \cdot \frac{\partial^2 U_2}{\partial q_2 \partial q_1} & 1 + k \cdot q_2^* \cdot \frac{\partial^2 U_2}{\partial q_2^2} \end{bmatrix} \tag{33}$$

with

$$Tr = 2 - 2k \cdot q_1^* - 2k \cdot q_2^* \tag{34}$$

and

$$Det = 1 - 2k \cdot q_1^* - 2k \cdot q_2^* + \left[4 - (1-\mu)^2\right] \cdot k^2 \cdot q_1^* \cdot q_2^* \tag{35}$$

To study the stability of Nash equilibrium we use three conditions that the equilibrium position is locally asymptotically stable when they are satisfied simultaneously [14, 16, 19]:

$$\begin{array}{ll} (i) & 1 - Det > 0 \\ (ii) & 1 - Tr + Det > 0 \\ (iii) & 1 + Tr + Det > 0 \end{array} \tag{36}$$

The condition (i) gives:

$$1 - Det > 0 \Leftrightarrow 2k\left(q_1^* + q_2^*\right) - \left[4 - (1-\mu)^2\right] \cdot k^2 \cdot q_1^* \cdot q_2^* > 0 \tag{37}$$

It's easy to find that the first condition (i) is always satisfied:

$$1 - Tr + Det > 0 \Leftrightarrow \left[4 - (1-\mu)^2\right] \cdot k^2 \cdot q_1^* \cdot q_2^* > 0 > 0 \tag{38}$$

because: $\left[4 - (1-\mu)^2\right] > 0$.
Finally, the condition (iii) becomes as:

$$1 + Tr + Det > 0 \Leftrightarrow \left[4 - (1-\mu)^2\right] \cdot q_1^* \cdot q_2^* \cdot k^2 - 4\left(q_1^* + q_2^*\right) \cdot k + 4 > 0 \tag{39}$$

Proposition: The Nash equilibrium of the discrete dynamical system Eq. (15) is locally asymptotically stable if:

$$2k\left(q_1^* + q_2^*\right) - \left[4 - (1-\mu)^2\right] \cdot k^2 \cdot q_1^* \cdot q_2^* > 0$$

and

$$\left[4-(1-\mu)^2\right]\cdot q_1^*\cdot q_2^*\cdot k^2-4\left(q_1^*+q_2^*\right)\cdot k+4>0.$$

3 Numerical Simulations Focusing on the Parameter k

From the condition (i) focusing on the parameter k we take the following inequality:

$$0<k<\frac{2\left(q_1^*+q_2^*\right)}{\left[4-(1-\mu)^2\right]\cdot q_1^*\cdot q_2^*} \tag{40}$$

The condition (iii) is the following:

$$\left[4-(1-\mu)^2\right]\cdot q_1^*\cdot q_2^*\cdot k^2-4\left(q_1^*+q_2^*\right)\cdot k+4>0$$

And its discriminant is positive:

$$\Delta=16\left[\left(q_1^*-q_2^*\right)^2+(1-\mu)^2\right]>0 \tag{41}$$

so the condition (iii) is satisfied if:

$$k\in(0,k_1)\cup(k_2,+\infty) \tag{42}$$

where:

$$k_{1,2}=\frac{4\left(q_1^*+q_2^*\right)\pm\sqrt{\Delta}}{2\left[4-(1-\mu)^2\right]\cdot q_1^*\cdot q_2^*} \tag{43}$$

are its two positive roots.

To provide some numerical evidence for the chaotic behavior of the system Eq. (14), as a consequence of change in the parameter k (the speed of adjustment), we present various numerical results here to show the chaoticity, including its bifurcations diagrams, strange attractors, Lyapunov numbers and sensitive dependence on initial conditions.

In order to study the local stability properties of the equilibrium points, it is convenient to take specific values for the other parameters: $\alpha = 5$, c1 = 1, c2 = 0.5 and p = μ = 0.5. So, as a result we find that $q_1^* \simeq 1.73$ and $q_2^* \simeq 1.57$ and the stability condition becomes as:

$$0<k<0.48 \tag{44}$$

This algebraic result is verified by the bifurcation diagrams of q_1^* (Fig. 1) and q_2^* (Fig. 2) with respect to the parameter k. As it seems there is a locally asymptotically stable orbit until the value of 0.48 for the parameter k and after this value doubling period bifurcations are appeared and finally, for higher values of the parameter k the system's behavior becomes chaotic and unpredictable (Fig. 3).

Fig. 1 Bifurcation diagram with respect to the parameter d against the variable q_1^* with 400 iterations of the map Eq. (15) for $\alpha = 5$, $c_1 = 1$, $c_2 = 0.50$, $p = 0.50$ and $\mu = 0.50$

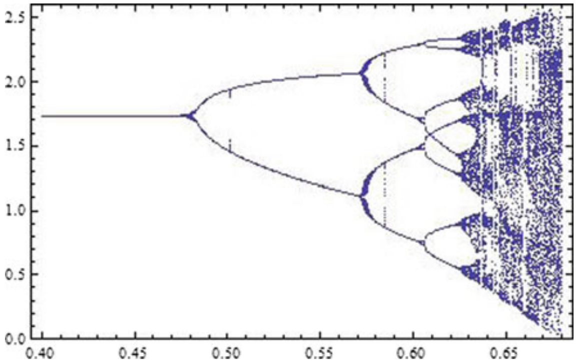


Fig. 2 Bifurcation diagram with respect to the parameter d against the variable q_2^* with 400 iterations of the map Eq. (15) for $\alpha = 5$, $c_1 = 1$, $c_2 = 0.50$, $p = 0.50$ and $\mu = 0.50$

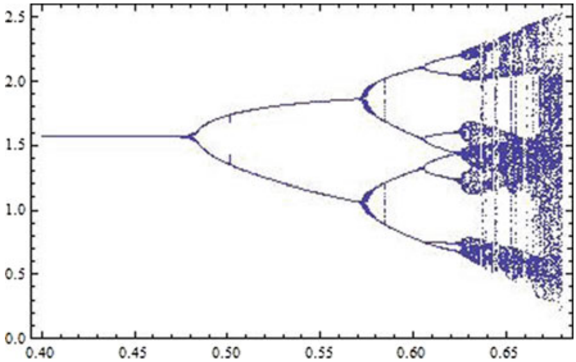
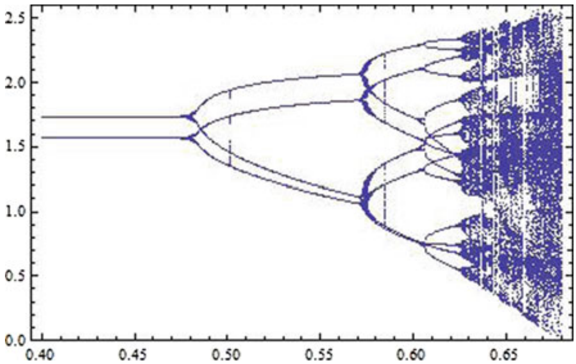


Fig.3 The two previous bifurcation diagrams of Figs. 1 and 2 in one



This chaotic trajectory can create strange attractors (Fig. 4) for a higher value of the parameter k like 0.675, outside the stability space. Also, computing the Lyapunov numbers (Fig. 5) for this value of the parameter k and setting the same fixed values for the other parameters α, c_1, c_2, p and μ it seems that they are getting over the value of 1 as an evidence for the chaotic trajectory.

This chaotic trajectory makes the system sensitive on initial conditions, which means that only a small change on a coordinate may change completely the system's behavior. For example, choosing two different initial conditions (0.1,0.1) (Fig. 6) and (0.101,0.1) (Fig. 7) with a small change at the q_1^*-coordinate and plotting the time series of system it seems that at the beginning the time series are indistinguishable, but after a number of iterations, the difference between them builds up rapidly.

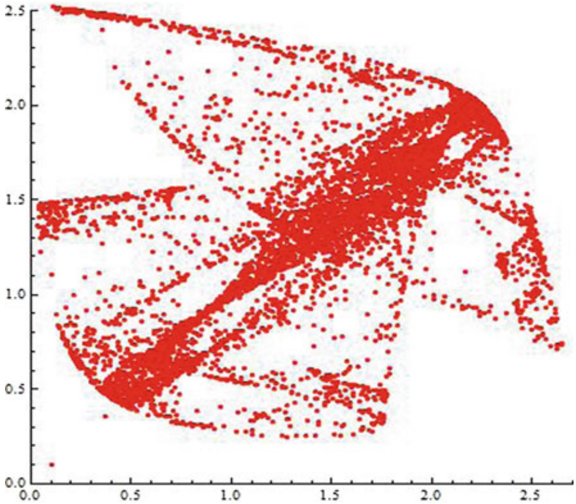


Fig. 4 Phase portrait (strange attractor) of the orbit of (0.1,0.1) with 8000 iterations of the map Eq. (15) for $\alpha = 5$, $c_1 = 1$, $c_2 = 0.50$, $p = 0.50$, $\mu = 0.50$ and $k = 0.675$

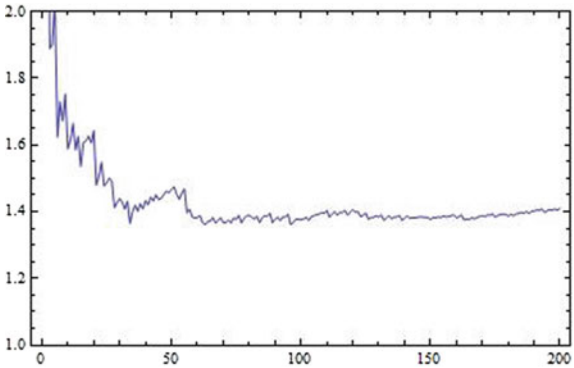


Fig. 5 Lyapunov numbers of the orbit of (0.1,0.1) with 8000 iterations of the map Eq. (15) for $\alpha = 5$, $c_1 = 1$, $c_2 = 0.50$, $p = 0.50$, $\mu = 0.50$ and $k = 0.675$

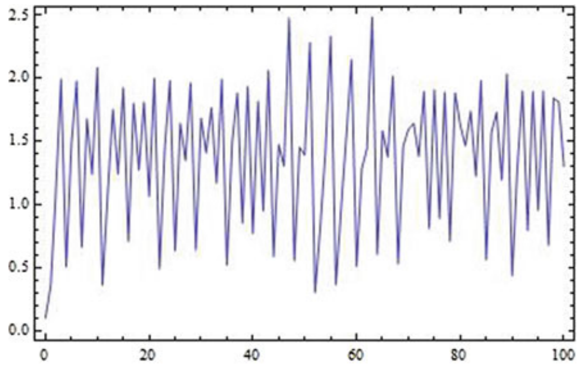


Fig. 6 Sensitive dependence on initial conditions for q_1^*-coordinate plotted against the time: the orbit of (0.1,0.1) of the system Eq. (15) for $\alpha = 5$, $c_1 = 1$, $c_2 = 0.50$, $p = 0.50$, $\mu = 0.50$ and $k = 0.675$

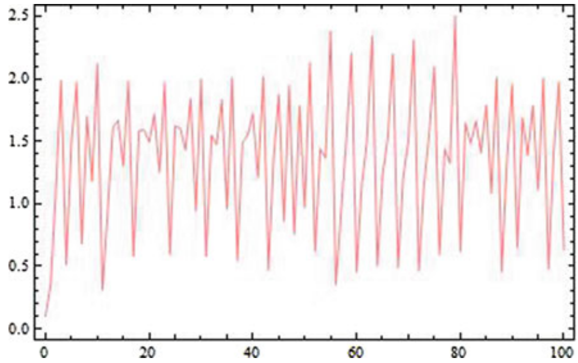


Fig. 7 Sensitive dependence on initial conditions for q_1^*-coordinate plotted against the time: the orbit of (0.101,0.1) of the system Eq. (15) for $\alpha = 5$, $c_1 = 1$, $c_2 = 0.50$, $p = 0.50$, $\mu = 0.50$ and $k = 0.675$

4 Conclusions

In this paper we analyzed the dynamics of a differentiated Cournot duopoly with homogeneous expectations, linear demand and cost functions. An uncertainty of the first firm's cost function is introduced. By assuming that at each time period each firm maximizes its expected relative profit under the same expectations, a discrete dynamical system was obtained. Existence and stability of equilibrium of this system are studied. We showed numerically that the model gives chaotic and unpredictable trajectories. The main result is that higher values of the speed of adjustment may destabilize the Cournot–Nash equilibrium.

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