Alternative Quantum Formulations and Systems at the Classical-Quantum Border

R. Vilela Mendes

Abstract There are, at least, three completely equivalent formulations of quantum mechanics: the Hilbert space approach, the phase-space deformation approach and the tomographic one. The Hilbert space approach is the most widely used to describe dynamics at the microscopic level. However, with the recent emergence of "quantum technology" it became important to have appropriate models for systems with behavior at the classical-quantum border. For these systems, the deformation and tomographic approaches turn out to be more convenient than the Hilbert space one. This paper presents a short review of the alternative quantum formalisms as well as some applications, one of them discussed at the Nice conference.

Keywords Phase-space deformation · Quantum tomography · Quantum kinetics · Quantum complexity

1 Introduction

There are, at least, three completely equivalent formulations of quantum mechanics: the Hilbert space approach, the phase-space deformation approach and the tomographic one.[1] In the Hilbert space approach quantum states are mapped to rays in Hilbert space and physical measurable quantities are obtained from expectation values of the self-adjoint operators that represent observables. This is the most widely used formalism to describe dynamics at the microscopic level. However, with the recent developments in quantum technology it became important to have suitable models for systems with a behavior at the classical-quantum border. For these systems, it would be convenient to have a "smooth" transition between the phase space

[1]For other formulations of quantum mechanics see D. F. Styer et al. Am. J. Phys. 70 (2002) 288–297. Here however I have emphasized those that seem most useful to describe systems at the classical-quantum border.

R. V. Mendes (✉)
CMAFCIO, University of Lisbon, Lisbon, Portugal
e-mail: rvilela.mendes@gmail.com; rvmendes@fc.ul.pt

C. Bernardin et al. (eds.), *From Particle Systems to Partial Differential Equations*,
Springer Proceedings in Mathematics & Statistics 352,
https://doi.org/10.1007/978-3-030-69784-6_18

description of classical dynamics and the quantum phenomena. This is achieved by a phase space formulation of quantum mechanics, the quantum effects corresponding to a replacement of the classical commutative algebra of velocity and momenta by a non-commutative algebra. Likewise the non-commutative nature of quantum observables is circumvented in the tomographic approach by dealing with linear combinations of the noncommuting observables, the full scope of quantum dynamics being obtained by varying the coefficients in the linear combination. The Hilbert space approach being very familiar, this paper only presents a short review of the other two quantum formalisms as well as some applications. Finally it is pointed out that the alternative approaches corresponding to a replacement of the operators by functions (operator symbols) with a non-commutative algebra, it is, in principle, possible to develop many other equivalent formalisms for quantum mechanics. This is formalized in a quantizer-dequantizer framework.

2 Quantum Mechanics and Deformation Theory

The phase space of classical mechanics is a symplectic manifold $W = (T^*M, \boldsymbol{\omega})$ where T^*M is the cotangent bundle over the configuration space M and $\boldsymbol{\omega}$ is a symplectic form. In local (Darboux) coordinates (p_i, q_i) the symplectic form is

$$\boldsymbol{\omega} = \sum dp_i \wedge dq_i \tag{1}$$

The Poisson bracket gives a Lie algebra structure to the C^∞-functions on W, namely

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \tag{2}$$

in local coordinates.

The transition to quantum mechanics is now regarded as a deformation of this Poisson algebra [1]. Let for example $T^*M = \boldsymbol{R}^{2n}$. Then,

$$\boldsymbol{\omega} = \sum_{1 \le i, j \le n} \omega_{ij} dx^i \wedge dx^j = \sum_{1 \le i \le n} dx^i \wedge dx^{i+n} \tag{3}$$

Consider the following bidifferential operator

$$P^r(f, g) = \sum_{\substack{i_1 \cdots i_r \\ j_1 \cdots j_r}} \boldsymbol{\omega}^{i_1 j_1} \cdots \boldsymbol{\omega}^{i_r j_r}\, \partial_{i_1} \cdots \partial_{i_r}(f) \partial_{j_1} \cdots \partial_{j_r}(g) \tag{4}$$

$P^1(f, g)$ is the Poisson bracket. $P^3(f, g)$ is a non-trivial 2-cocycle and, barring obstructions, one expects the existence of non-trivial deformations of the Poisson algebra. Existence of non-trivial deformations have indeed been proved in a very general context [2–5]. They always exist if W is finite-dimensional and for a flat Poisson manifold they are all equivalent to the Moyal [6] bracket

$$[f, g]_M = \frac{2}{\hbar} sin\left(\frac{\hbar}{2}P\right)(f, g) = \{f, g\} - \frac{\hbar^2}{4.3!}P^3(f, g) + \cdots \tag{5}$$

Moreover $[f, g]_M = \frac{1}{i\hbar}(f *_\hbar g - g *_\hbar f)$ where $f *_\hbar g$ is an associative star-product

$$f *_\hbar g = \exp\left(i\frac{\hbar}{2}P(f, g)\right) \tag{6}$$

Correspondence with quantum mechanics formulated in Hilbert space is obtained by the Weyl quantization prescription. Let $f(p, q)$ be a function in phase space and $\widetilde{f}$ its Fourier transform. Then, if to the function f we associate the Hilbert space operator

$$\boldsymbol{\Omega}(f) = \int \widetilde{f}(x_i, y_i) e^{-\frac{i}{\hbar}\sum x_i Q_i + y_i P_i} dx_i dy_i \tag{7}$$

where $Q_i \boldsymbol{\psi} = \boldsymbol{x}_i \boldsymbol{\psi}$ and $P_i = -i\hbar \frac{\partial}{\partial x_i}\boldsymbol{\psi}$, one finds

$$[\boldsymbol{\Omega}(f), \boldsymbol{\Omega}(g)] = i\hbar\boldsymbol{\Omega}([f, g]_M) \tag{8}$$

with, in the left-hand side, the usual commutator for Hilbert space operators and in right hand side the Moyal bracket. Therefore quantum mechanics may be described either by associating self-adjoint operators in Hilbert space to the observables or, equivalently, staying in the classical setting of phase-space functions but deforming their product to a $*_\hbar$- product and their Poisson brackets to Moyal brackets.

Time evolution of the observables is described by the Moyal equation

$$\frac{\partial}{\partial t} f = [f, H]_M = \frac{1}{i\hbar}(f *_\hbar H - H *_\hbar f) \tag{9}$$

Somewhat related to quantization by deformation is the geometric quantization theory. Geometric quantization [7] is a very nice and profound theory. Starting from a classical phase-space it aims to construct, in a consistent manner, a Hilbert space representing the corresponding quantum theory. The final product being a Hilbert space, a setting quite different from the classical phase-space, geometric quantization is probably not so useful to study systems at the classical-quantum border. I might be wrong.

Here is a brief sketch of the geometric quantization scheme: One starts from a manifold M with a symplectic structure ω and construct a Hermitean line bundle L with a connection of curvature $-i\omega$. L is the prequantization line bundle and the Hilbert space H_0 of square-integrable sections of L is the prequantum Hilbert space. Smooth functions on M are mapped to operators on H_0 taking Poisson brackets to commutators. H_0 is in general too big a space. H_0 is then cut down by polarization, which picks out a subspace P_x of the complexified tangent space at $x \in M$. The quantum Hilbert space H is then defined to be the space of square-integrable sections of L that yield zero when we take their covariant derivative at any point x in the direction of any vector in P_x.

3 Quantum Mechanics in the Tomography Approach

The tomography approach may be used both for classical and quantum mechanics, which makes the classical-quantum transition quite easy. One therefore starts by describing the tomography formulation of classical statistical mechanics. States in classical statistical mechanics are described by a function $\rho(x, p)$, which is the probability distribution in phase space,

$$\rho(x, p) \geq 0, \qquad \int \rho(x, p)\, dp = P(x), \qquad \int \rho(x, p)\, dx = \widetilde{P}(p), \tag{10}$$

$P(x)$ and $\widetilde{P}(p)$ being the (marginal) probability distributions for position and momentum.

The density function $\rho(x, p)$ is normalized

$$\int \rho(x, p)\, dx\, dp = 1 .$$

Consider now a parametrized observable, a linear function on the phase space of the system,

$$X(x, p) = \mu x + \nu p , \tag{11}$$

The variable $X(x, p)$ can be considered as the position of the system when measured in a rotated and rescaled reference frame in the classical phase space. All the position and momentum features of the system are obtained by varying the μ and ν parameters. The tomography map is defined as

$$M(X, \mu, \nu) = \frac{1}{2\pi} \int e^{-ik(X - \mu x - \nu p)} \rho(x, p)\, dx\, dp\, dk . \tag{12}$$

which is an homogeneous function,

$$M(\lambda X, \lambda\mu, \lambda\nu) = |\lambda|^{-1} M(X, \mu, \nu) , \tag{13}$$

and the Eq. (12) can be inverted,

$$\rho\,(x,\ p) = \frac{1}{4\pi^2}\int M\,(X,\ \mu,\ \nu)\ \exp\left[-i\,(\mu x + \nu p - X)\right]\,dX\,d\mu\,d\nu\,. \tag{14}$$

Therefore the classical system may be equivalently described by the phase space density $\rho\,(x,\ p)$ or by the tomography map. The tomography map cannot be an arbitrary function, it must be such that the corresponding $\rho\,(x,\ p)$ in (12) is a nonnegative function. As seen from (12) the classical tomography map is the Fourier transform of a characteristic function

$$M\,(X) = \frac{1}{2\pi}\int \langle e^{ikX}\rangle e^{-ikX}\,dk \tag{15}$$

which is a real nonnegative function. Furthermore

$$M\,(X) = \int \rho\,(x,\ p)\ \delta\,(X(x,\ p) - X)\ dx\,dp\,. \tag{16}$$

and

$$\int M\,(X)\ dX = \int \rho\,(x,\ p)\ dx\,dp = 1\,. \tag{17}$$

The evolution equation for the classical phase space density of a particle with mass $m = 1$ and potential $V(x)$,

$$\frac{\partial\rho\,(x,\ p,\ t)}{\partial t} + p\,\frac{\partial\rho\,(x,\ p,\ t)}{\partial x} - \frac{\partial V(x)}{\partial x}\,\frac{\partial\rho\,(x,\ p,\ t)}{\partial p} = 0 \tag{18}$$

can be rewritten in terms of the tomography map $M\,(X,\ \mu,\ \nu,\ t)$

$$\dot{M} - \mu\frac{\partial}{\partial\nu}M - \frac{\partial V}{\partial x}\,(\widetilde{q})\left[\nu\frac{\partial}{\partial X}M\right] = 0\,, \tag{19}$$

with the argument of the function $\partial V/\partial x$ being replaced by the operator

$$\widetilde{q} = -\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}\,. \tag{20}$$

For the mean value of position in classical statistical mechanics, one has

$$\langle x\rangle = \int \rho\,(x,\ p)\,x\,dx\,dp = i\int M\,(X,\ \mu,\ \nu)\,e^{iX}\,\delta'\,(\mu)\ \delta\,(\nu)\ dX\,d\mu\,d\nu\,. \tag{21}$$

In quantum mechanics one considers the observable ($\hbar = 1$)

$$\widehat{X} = \mu\hat{q} + \nu\hat{p}\,, \tag{22}$$

$\hat{q}$ and $\hat{p}$ being the quantum position and momentum. The quantum tomography map may be defined directly from the wave function or the density matrix. However it was originally defined [8–12] in terms of the Wigner function $W(q, p)$ as follows:

$$M(X, \mu, \nu) = \int \exp\left[-ik(X - \mu q - \nu p)\right] W(q, p) \frac{dk\, dq\, dp}{(2\pi)^2}. \tag{23}$$

One sees that the formula (23) is identical to (12) of the classical case. For a pure state, with wave function $\Psi(y)$, the quantum tomography map has the form [13]

$$M(X, \mu, \nu) = \frac{1}{2\pi|\nu|} \left| \int \Psi(y) \exp\left(\frac{i\mu y^2}{2\nu} - \frac{iyX}{\nu} \right) dy \right|^2. \tag{24}$$

From Eq. (24) one sees that the tomography map is the amplitude squared of a projection of the quantum state on the eigenvectors of the operator $\widehat{X}$ in (22).

The formula (23) can be inverted and, as in the classical case, the Wigner function can be expressed in terms of the tomography map [8],

$$W(q, p) = \frac{1}{2\pi} \int M(X, \mu, \nu) \exp\left[-i(\mu q + \nu p - X)\right] d\mu\, d\nu\, dX. \tag{25}$$

As was shown in [10], for systems with the Hamiltonian

$$H = \frac{\hat{p}^2}{2} + V(\hat{q}), \tag{26}$$

the tomography map satisfies a quantum time-evolution equation

$$\begin{aligned} \dot{M} - \mu \frac{\partial}{\partial \nu} M - i \Bigg[V\left(-\frac{1}{\partial/\partial X} \frac{\partial}{\partial \mu} - i \frac{\nu}{2} \frac{\partial}{\partial X} \right) \\ - V\left(-\frac{1}{\partial/\partial X} \frac{\partial}{\partial \mu} + i \frac{\nu}{2} \frac{\partial}{\partial X} \right) \Bigg] M = 0, \end{aligned} \tag{27}$$

which is an alternative to the Schrödinger equation.

The evolution Eq. (27) can also be presented in the form

$$\dot{w} - \mu \frac{\partial w}{\partial \nu} - \frac{\partial V}{\partial x}(\widetilde{q})\, \nu \frac{\partial}{\partial X} w + 2 \sum_{n=1}^{\infty} \frac{V^{2n+1}(\widetilde{q})}{(2n+1)!} \left(\frac{\nu}{2} \frac{\partial}{\partial X} \right)^{2n+1} (-1)^{n+1} w = 0, \tag{28}$$

where $\widetilde{q}$ is given by (20) and

$$V^{2n+1}(\widetilde{q}) \equiv \frac{d^{2n+1} V}{dq^{2n+1}}(\widetilde{q}).$$

This evolution equation is in fact Moyal equation (9) in the tomography representation.

The tomography approach has another interesting application in another classical, but non-commutative, context. In signal processing one deals with time and frequency which, as $\hat{q}$ and $\hat{p}$, are also non-commutative variables. Then, the tomography map, being a positive quantity with a probability interpretation, provides a robust and unambiguous tool for feature extraction in signal processing [13–18].

4 Applications

In this section one illustrates the use of the alternative quantum formulations in two situations where the classical-quantum border is quite apparent. The deformation approach is quite appropriate to obtain the quantum formulation, or quantum corrections, to the kinetic equations, because the natural setting of such equations is the phase space of positions and momenta. On the other hand, another notion that is very useful in classical mechanics is the notion of sensitive dependence to initial conditions or chaotic behavior. In classical mechanics this notion finds a rigorous formulation through the Lyapunov exponents of the dynamics. However, it is not obvious how to correctly carry the notion of Lyapunov exponent to quantum mechanics in the Hilbert space formulation. By first defining classical Lyapunov exponents in a tomographic formulation it becomes an easy matter to carry them to quantum mechanics and then, if needed, to carry the definition to Hilbert space. Of course this is possible because all the alternative formulations are equivalent. To use one or another is a question of computational and conceptual convenience. Another situation of current interest at the classical-quantum border is the cooling of levitated nanoparticles [19].

4.1 Kinetic Equations and Quantum Corrections

A kinetic equation deals with the evolution of a probability density $f(t, x, p)$ of particles in phase space. The typical form is

$$\frac{\partial}{\partial t} f + \frac{p}{m} \cdot \nabla_x f + F_{ext} \cdot \nabla_p f = S(f) \tag{29}$$

the left hand side being a drift term defining the characteristics along which the particles move between collisions and the right hand side a collision term. It is therefore an equation involving a probability distribution in the (x, p) phase space. In quantum mechanics $f(x, p)$ cannot be a classical probability distribution because x and p are non-commuting variables. However $f(x, p)$ may be interpreted as a functional of elements in an algebra with a deformed product and, as discussed before, this leads to the correct quantum results.

It is therefore tempting, to obtain the quantum corrections to Eq. (29), by simply replacing all products by deformed products. However, recalling that at the basis of the deformation interpretation of quantum mechanics is the deformation of a Poisson algebra, it is more appropriate to deform the kinetic equation when their (canonical or non-canonical) Hamiltonian structure is exhibited. This is the approach that will be followed.

4.1.1 The Poisson–Vlasov Equation

The Poisson–Vlasov equation describing a collisionless plasma with purely electrostatic interactions is

$$\frac{\partial f}{\partial t}+\frac{p}{m}\cdot\frac{\partial f}{\partial x}-e\frac{\partial \phi}{\partial x}\cdot\frac{\partial f}{\partial p}=0 \tag{30}$$

with

$$\triangle\phi=-e\int dpf\left(x,p,t\right) \tag{31}$$

It is a non-canonical Hamiltonian system [20], with Hamiltonian,

$$H_{PV}=\frac{1}{2}\int\left|\frac{p}{2m}\right|^{2}f\left(x,p,t\right)dxdp+e\int dx\phi\left(x\right)\int f\left(x,p,t\right)dp \tag{32}$$

the time evolution of arbitrary phase-space functions given by

$$\frac{dF}{dt}=\left[F,H_{PV}\right] \tag{33}$$

the Poisson structure $[\cdot,\cdot]$ being

$$\left[F,G\right]=\int f\left\{\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right\}dxdp \tag{34}$$

where $\{\cdot,\cdot\}$ stands for the usual Poisson bracket for functions of x and p

$$\left\{A,B\right\}=\sum_{i}\left(\frac{\partial A}{\partial x_{i}}\frac{\partial B}{\partial p_{i}}-\frac{\partial A}{\partial p_{i}}\frac{\partial B}{\partial x_{i}}\right) \tag{35}$$

and the functional derivative $\frac{\delta F}{\delta f}$ being related to the Fréchet derivative by

$$\left(D_{f}F\right)\cdot f^{'}=\int\frac{\delta F}{\delta f}f^{'}dxdv \tag{36}$$

Taking into attention that

$$\frac{\delta f(y,\mu,t)}{\delta f(x,p,t)} = \delta^3(y-x)\,\delta^3(\mu-p)$$
$$\frac{\delta H_{PV}}{\delta f(x,p,t)} = \frac{1}{2m}|p|^2 + e\phi(x) \tag{37}$$

and using Eq. (34) one obtains the classical Poisson–Vlasov equation

$$\frac{df}{dt} = [f, H_{PV}] = -\frac{p}{m}\cdot\nabla_x f + e\nabla_x\phi\cdot\nabla_p f \tag{38}$$

For the quantum version all one has to do is to replace in Eq. (34) the Poisson bracket (35) by the Moyal bracket (5).

$$\frac{df}{dt} = \int d^3x d^3p f(x,p,t)\,\frac{2}{\hbar}\sin\left(\frac{\hbar}{2}P\right)\left(\frac{\delta f}{\delta f}, \frac{\delta H_{PV}}{\delta f}\right) \tag{39}$$

P being the bidifferential operator in (4).

Of special interest is the leading quantum correction. The 6-dimensional ω matrix in the symplectic form (3) has $\omega_{i,i+3} = -\omega_{i+3,i} = 1$ with all the other elements being zero. Because $\frac{\delta H_{PV}}{\delta f(x,p,t)}$ is quadratic in p, all terms in $\omega_{i,i+3}\omega_{j,j+3}\omega_{k,k+3}$ vanish. Finally one obtains in leading $\hbar^2$ order,

$$\frac{df}{dt} = [f, H_{PV}]_M = -\frac{p}{m}\cdot\nabla_x f + e\nabla_x\phi\cdot\nabla_p f - e\frac{\hbar^2}{24}\sum_{i,j,k=1}^{3}\frac{\partial^3 f}{\partial p_i\partial p_j\partial p_k}\frac{\partial^3\phi}{\partial x_i\partial x_j\partial x_k} + O\left(\hbar^4\right) \tag{40}$$

4.1.2 The Maxwell–Vlasov Equation

The Maxwell–Vlasov equation,

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f + \frac{e}{m}\left(E + \frac{v\times B}{c}\right)\cdot\nabla_v f = 0 \tag{41}$$

describing a classical collisionless plasma in an electromagnetic field, is also a non-canonical Hamiltonian system. There are several variational formulations of the Maxwell–Vlasov system, the most complete one being probably the one by Marsden and Weinstein [21]. However, in their formulation, part of the dynamics is coded on the Poisson structure rather than on the Hamiltonian and to apply the deformation theory for the transition to quantum mechanics, one would also need to handle the deformation of the electromagnetic field dynamics, not just the replacement of the Poisson bracket involving position and momentum of the particles. Hence, because here one only wants to obtain the quantum corrections to the f dynamics, it is more convenient to use the Low [22] Hamiltonian,

$$H_{MV} = \int d^3xd^3p \left\{ \frac{1}{2m} \left(p - \frac{e}{c}A \right)^2 + e\phi(x) \right\} f(x, p, t) + \int d^3x \left(E^2 + B^2 \right) \tag{42}$$

where $E = -\nabla_x \phi - \frac{1}{c}\frac{\partial A}{\partial t}$, $B = \nabla \times A$ in terms of the independent variables (ϕ, A).

The Poisson structure is the same as in (34) for the f dynamics. With this Hamiltonian

$$\frac{\delta H_{MV}}{\delta f(x, p, t)} = \frac{1}{2m} \left(p^2 - \frac{e}{c}(p \cdot A + A \cdot p) + \frac{e^2}{c^2}A^2 \right) + e\phi(x) \tag{43}$$

Then, using (34) and (35) one obtains for the classical equation

$$\begin{aligned} \frac{\partial f}{\partial t} &= -\frac{1}{m}\left(p - \frac{e}{c}A \right) \cdot \nabla_x f + \left(\frac{e}{m}\nabla_x \phi - \frac{e}{mc} p \cdot \nabla_x A + \frac{e^2}{2mc^2}\nabla_x A^2 \right) \cdot \nabla_p f \\ &= -\frac{1}{m}\left(p - \frac{e}{c}A \right) \cdot \nabla_x f + \left(-eE - \frac{e}{c} v \times B - \frac{e}{c}\frac{dA}{dt} \right) \cdot \nabla_p f \end{aligned} \tag{44}$$

with

$$\begin{aligned} \frac{dA}{dt} &= \frac{\partial A}{\partial t} + v \cdot \nabla_x A \\ v &= \frac{1}{m}\left(p - \frac{e}{c}A \right) \end{aligned} \tag{45}$$

Equation (44) is the same as (41) written in the variables (x, p) instead of (x, v). The first set is the most convenient one because the Moyal bracket deformation acts on these variables. Then, the quantum Maxwell–Vlasov equation becomes[2]

$$\frac{\partial f}{\partial t} = \int d^3xd^3pf(x, p, t) \frac{2}{\hbar} \sin\left(\frac{\hbar}{2}P \right) \left(\frac{\delta f}{\delta f}, \frac{\delta H_{MV}}{\delta f} \right) \tag{46}$$

and, computing the leading quantum corrections, one obtains

$$\begin{aligned} \frac{\partial f}{\partial t} &== -\frac{1}{m}\left(p - \frac{e}{c}A \right) \cdot \nabla_x f + \left(-eE - \frac{e}{c} v \times B - \frac{e}{c}\frac{dA}{dt} \right) \cdot \nabla_p f \\ &- \frac{e\hbar^2}{24} \sum_{i,j,k=1}^{3} \frac{\partial^3 f}{\partial p_i \partial p_j \partial p_k} \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} \left(\phi + \frac{e}{2mc^2}A^2 \right) \\ &+ \frac{e\hbar^2}{24mc} \sum_{i,j,k=1}^{3} \left(\frac{\partial^3 f}{\partial p_i \partial p_j \partial p_k} p \cdot \frac{\partial^3 A}{\partial x_i \partial x_j \partial x_k} - 3 \frac{\partial^3 f}{\partial x_i \partial p_j \partial p_k} \frac{\partial^2 A_k}{\partial x_j \partial x_k} \right) + O\left(\hbar^4 \right) \end{aligned} \tag{47}$$

[2]Notice that in the Hamiltonian, products should also be replaced by $*$-products. However $p * A + A * p = 2p \cdot A$.

4.2 A Quantum Lyapunov Exponent

Bounded classical systems that are chaotic, display exponential growth of initial perturbations and other interesting long-time asymptotics, like exponential decay of correlations. In contrast, quantum Hamiltonians of bounded systems with time-independent potentials, having discrete spectrum, their wave functions are almost periodic functions. For this reason the work on "quantum chaos" has shifted from consideration of long-time properties to the statistics of energy levels of quantum systems with a chaotic classical counterpart.

However, quantum systems with bounded configuration space but time-dependent interactions (for example particles in an accelerator subjected to electromagnetic kicks or the systems used in quantum control) may have continuous spectrum. Therefore the estimation and control, of the rate of growth of the perturbed matrix elements of observables, becomes an issue of both theoretical and practical concern.

In classical mechanics the most important asymptotic indicator of chaotic behavior is the Lyapunov exponent (an ergodic invariant). Therefore a natural first step to discuss rates of growth in quantum mechanics seems to be the construction of a quantum Lyapunov exponent.

The tomography approach, because of the similarity of its structure in the classical and the quantum cases, seems to be an appropriate setting to carry out this construction. As a precondition it is necessary to carry the definition of Lyapunov exponent, usually defined in terms of orbits and tangent maps, to a definition in terms of densities. This was carried out in [23]. Given an initial density $\rho\left(x,\ p,\ t=0\right)\equiv\rho\left(x,\ p\right)$ for a classical particle, let it have a general time evolution defined by a smooth kernel

$$\rho\left(x,p,t\right)=\int\mathscr{K}\left(x,p,x',p',t\right)\rho\left(x',p'\right)\,dx'\,dp'. \tag{48}$$

The evolution of the distribution $\rho\left(x,p,t\right)$, described by Eq. (48), is equivalent to the action of the Frobenius–Perron operator used in [23, 24]. Consider now a small perturbation in the initial condition

$$\widetilde{\rho}\left(x,p\right)=\rho\left(x,p\right)+\varepsilon\left(\mathbf{n}\nabla\right)\delta(q)\,\delta(p)\,, \tag{49}$$

where

$$\mathbf{n}=\left(n_1,n_2\right),\qquad \mathbf{n}^2=1\,,\qquad \nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial p}\right),$$

and

$$\left(\mathbf{n}\nabla\right)=n_1\frac{\partial}{\partial x}+n_2\frac{\partial}{\partial p}\,.$$

The perturbed initial density evolves like

$$\rho\left(x,p,t\right)=\int\mathscr{K}\left(x,p,x',p',t\right)\left[\rho\left(x',p'\right)+\varepsilon\left(\mathbf{n}\nabla'\right)\delta(x')\,\delta(p')\right]\,dx'\,dp',\tag{50}$$

where

$$\nabla'=\left(\frac{\partial}{\partial x'},\frac{\partial}{\partial p'}\right).$$

Let us now compare the expectation values, for example, of the position of the perturbed and unperturbed initial densities at time t

$$\Delta x(t)=\int x\left[\widetilde{\rho}\left(x,p,t\right)-\rho\left(x,p,t\right)\right]\,dx\,dp\,,\tag{51}$$

which equals

$$\Delta x(t)=\varepsilon\int x\mathscr{K}\left(x,p,x',p',t\right)\left(\nabla'\mathbf{n}\right)\delta(x')\delta(p')\,dx'\,dp'.\tag{52}$$

In order to obtain the Lyapunov exponent one computes

$$\lambda=\lim_{t\to\infty}\frac{1}{t}\log\left|\frac{\Delta x(t)}{\Delta x(0)}\right|.\tag{53}$$

leading to

$$\lambda=\lim_{t\to\infty}\frac{1}{t}\left\{\log\mid\varepsilon\int x\mathscr{K}\left(x,p,x',p',t\right)\right.$$
$$\left.\times\left(\nabla'\mathbf{n}\right)\delta(x')\delta(p')\,dx'\,dp'-\log\mid\Delta x(0)\mid\right\}.\tag{54}$$

To translate this procedure to the tomography framework of classical mechanics, the initial probability density is transformed to an initial tomography map

$$\rho\left(x,p\right)\to M\left(X,\mu,\nu,t=0\right)\equiv M\left(X,\mu,\nu\right).\tag{55}$$

The density $\delta\left(x-x_0\right)\delta\left(p-p_0\right)$ is mapped to the tomography map

$$M_\delta\left(X,\mu,\nu\right)=\delta\left(X-\mu q_0-\nu p_0\right),\tag{56}$$

and the perturbed term

$$\widetilde{\rho}\left(x,p\right)-\rho\left(x,p\right)=\left(\mathbf{n}\nabla\right)\delta\left(x-x_0\right)\,\delta\left(p-p_0\right)\tag{57}$$

is mapped to the tomographic perturbation

$$M_\eta\left(X,\mu,\nu\right)=\left(n_1\mu+n_2\nu\right)\delta'\left(X-\mu q_0-\nu p_0\right),\tag{58}$$

The unperturbed and perturbed initial tomography maps evolve with the classical propagator $\Pi_{\text{cl}}\left(X,\ \mu,\ \nu,\ X',\ \mu',\ \nu',\ t_2,\ t_1\right)$ that connects the maps at times t_1 and t_2 $(t_2 > t_1)$

$$M\left(X, \mu, \nu, t_2\right) = \int \Pi_{\text{cl}}\left(X, \mu, \nu, X', \mu', \nu', t_2, t_1\right) M\left(X', \mu', \nu', t_1\right)\, dX'\, d\mu'\, d\nu'. \tag{59}$$

the propagator satisfying the equation

$$\begin{aligned} &\frac{\partial \Pi_{\text{cl}}}{\partial t_2} - \mu\, \frac{\partial}{\partial \nu} \Pi_{\text{cl}} - \frac{\partial V}{\partial x}\left(\widetilde{q}\right) \nu\, \frac{\partial}{\partial X} \Pi_{\text{cl}} \\ &= \delta\left(t_2 - t_1\right) \delta\left(X - X'\right) \delta\left(\mu - \mu'\right) \delta\left(\nu - \nu'\right). \end{aligned}$$

hence,

$$\begin{aligned} M_\eta\left(X, \mu, \nu, t\right) = \int & \Pi_{\text{cl}}\left(X, \mu, \nu, X', \mu', \nu', t\right) \\ & \times \left(n_1 \mu' + n_2 \nu'\right) \delta'\left(X' - \mu' q_0 - \nu' p_0\right) dX'\, d\mu'\, d\nu'. \end{aligned} \tag{60}$$

The position perturbations at time zero and time t are

$$\begin{aligned} \Delta x(0) &= \varepsilon \int M_\eta\left(X, \mu, \nu\right) e^{iX} \delta'(\mu)\, \delta(\nu)\, dX\, d\mu\, d\nu \\ \Delta x(t) &= \varepsilon \int M_\eta\left(X, \mu, \nu, t\right) e^{iX} \delta'(\mu)\, \delta(\nu)\, dX\, d\mu\, d\nu \end{aligned}$$

and by replacement in (53) the classical Lyapunov exponent is expressed as a function of the tomography maps.

For the quantum Lyapunov exponent all one has to do is to obtain the quantum values of $\Delta x(0)$ and $\Delta x(t)$. This is obtained by replacing, in the Eqs. (59) and (60), the classical by the quantum propagator which satisfies the equation

$$\begin{aligned} &\frac{\partial \Pi}{\partial t_2} - \mu\, \frac{\partial \Pi}{\partial \nu} - \frac{\partial V}{\partial x}\left(\widetilde{q}\right) \nu\, \frac{\partial \Pi}{\partial X} + 2 \sum_{n=1}^{\infty} \frac{V^{2n+1}\left(\widetilde{q}\right)}{(2n+1)!} \left(\frac{\nu}{2} \frac{\partial}{\partial X}\right)^{2n+1} (-1)^{n+1} \Pi \\ &= \delta\left(t_2 - t_1\right) \delta\left(X - X'\right) \delta\left(\mu - \mu'\right) \delta\left(\nu - \nu'\right), \end{aligned} \tag{61}$$

the quantum Lyapunov exponent being also obtained from (53).

After some algebra one arrives at the Lyapunov exponent expression in the quantum-mechanical case,

$$\begin{aligned} \lambda = \lim_{t \to \infty} \frac{1}{t} \log | & \int dX\, d\mu'\, d\nu'\, X \left(n_1 \mu' + n^2 \nu'\right) \\ & \times \frac{\partial \Pi}{\partial X'} \left(X, 1, 0, X' \to \mu' q_0 + \nu' p_0, \mu', \nu', t\right) | \ . \end{aligned} \tag{62}$$

A satisfactory construction was thus achieved [25] in the sense that the phase-space observables that are used are exactly the same in classical and quantum mechanics. The only difference between the classical and the quantum exponent lies in the time evolution dynamics.

It is of some interest to express this results in the Hilbert space framework of quantum mechanics [26]. The tomographic maps being related to traces of operators, it turns out that the quantum Lyapunov exponent measures the rate of growth of the trace of position and momentum observables starting from a singular initial density matrix. A positive Lyapunov exponent would correspond to exponential growth of these traces. However, the same quantities may serve to characterize other types of growth, leading to a generalized notion of quantum sensitive dependence.

There are examples where exponential rates of growth (as in classical chaos) are also found in quantum systems [25]. However, in many other cases, quantum mechanics seems to have a definite taming effect on classical chaos. Therefore, a generalized notion of quantum sensitive dependence, corresponding to rates of growth milder than exponential, might be of interest to classify different types of quantum complexity or to characterize the degree of accuracy achievable in quantum control.

As a first step rewrite the result for the quantum Lyapunov exponent along the phase-space vector $v = (v_1 v_2)$

$$\lambda_v = \lim_{t\to\infty} \frac{1}{t} \log \left\| \int d^n X \, d^n \mu \, d^n \nu e^{iX\bullet\mathbf{1}} \left(\begin{pmatrix} \nabla_\mu \\ \nabla_\nu \end{pmatrix} \delta^n(\mu) \delta^n(\nu) \right) M_t(X,\mu,\nu) \right\|$$
$$M_t(X,\mu,\nu) = \int \Pi\left(X,\ \mu,\ \nu,\ X',\ \mu',\ \nu',\ t,\ 0\right) M_0(X,\mu,\nu)\, dX'^n d\mu'^n d\nu'^n$$
$$M_0\left(X', \mu', \nu'\right) = \left(\left(v_1 \circledast \mu' + v_2 \circledast \nu'\right) \bullet \nabla_{X'}\right) \delta^n\left(X' - \mu' q_0 - \nu' p_0\right) \tag{63}$$

with $(a \circledast b)_i = a_i b_i$.

For a system with Hamiltonian

$$H = \frac{p^2}{2} + V(q)\,, \tag{64}$$

the evolution equation for the quantum propagator of the tomographic densities is

$$\begin{aligned} &\tfrac{\partial \Pi}{\partial t} - \mu \bullet \nabla_\nu \Pi - \nabla_x V\left(\widetilde{q}\right) \bullet \left(\nu \circledast \nabla_X \Pi\right) \\ &+ \tfrac{2}{\hbar} \textstyle\sum_{n=1}^{\infty} (-1)^{n+1} \left(\tfrac{\hbar}{2}\right)^{2n+1} \tfrac{\nabla_{i_1 \cdots i_{2n+1}} V(\widetilde{q})}{(2n+1)!} \left(\nu \circledast \nabla_X\right)_{i_1} \cdots \left(\nu \circledast \nabla_X\right)_{i_{2n+1}} \Pi \\ &= 0 \end{aligned} \tag{65}$$

with initial condition

$$\lim_{t\to t_0} \Pi\left(X,\mu,\nu,X',\mu',\nu',t,t_0\right) = \delta^n\left(X-X'\right) \delta^n\left(\mu-\mu'\right) \delta^n\left(\nu-\nu'\right) \tag{66}$$

reducing for $\hbar = 0$ to the classical evolution equation.

In the tomographic formulation, classical and quantum mechanics are both described by a set of positive probability distributions $M_t\left(X,\ \mu,\ \nu\right)$, the $\hbar$-deformation appearing only in the time-evolution. It is this fact that allows the notion of Lyapunov exponent to be carried over without ambiguity from classical to quantum mechanics. However, to relate the Lyapunov exponent to the behavior of operator matrix elements and the spectral properties of the Hamiltonian, it is more convenient to rewrite it as a functional of the density matrix $\rho\left(x, x^{'}\right)$. The first step is to consider the Fourier transform $G_t\left(\mu, \mu\right)$ of the tomographic density $M_t\left(X,\ \mu,\ \nu\right)$

$$G_t\left(\mu, \nu\right) \doteq G_t\left(1, \mu, \nu\right) = \int d^n X\, e^{iX\bullet\mathbf{1}} M_t\left(X, \mu, \nu\right) \tag{67}$$

and perform the integrals in (63) to obtain

$$\lambda_{\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}} = \lim_{t\to\infty} \frac{1}{t} \log \left\| \begin{matrix} \nabla_\mu G_t\left(\mu, \nu\right)|_{\mu=\nu=0} \\ \nabla_\nu G_t\left(\mu, \nu\right)|_{\mu=\nu=0} \end{matrix} \right\| \tag{68}$$

Now, using the relation between the tomographic densities and the density matrix, namely

$$G_t\left(\mu, \nu\right) = \left(\frac{1}{2\pi}\right)^n \int d^n X\, d^n p d^n x d^n x^{'}\, e^{i\left(X\bullet\mathbf{1}-p\bullet\left(x-x^{'}\right)\right)} \rho_t\left(x, x^{'}\right) \\ \delta^n\left(X - \mu \circledast \left(\frac{x+x^{'}}{2}\right) + \nu \circledast p\right) \tag{69}$$

one easily obtains

$$\lambda_{\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}} = \lim_{t\to\infty} \frac{1}{t} \log \left\| \begin{matrix} Tr\left\{\rho_t x\right\} \\ Tr\left\{\rho_t p\right\} \end{matrix} \right\| \tag{70}$$

the density matrix at time zero (corresponding to $M_0\left(X^{'}, \mu^{'}, \nu^{'}\right)$ in Eq. (63)) being

$$\rho_0\left(x, x^{'}\right) = -e^{ip_0\bullet\left(x-x^{'}\right)} \left\{\left(v_1 \bullet \nabla\right)\delta^n\left(q_0 - \frac{x+x^{'}}{2}\right) + iv_2 \bullet \left(x-x^{'}\right)\delta^n\left(q_0 - \frac{x+x^{'}}{2}\right)\right\} \tag{71}$$

Equation (70) means that the quantum Lyapunov exponent measures the exponential rate of growth of the expectation values of position and momentum, starting from the initial singular perturbation ρ_0. This is a rather appealing and intuitive form for the Lyapunov exponent.

Using the time-dependent operators in the Heisenberg picture

$$\begin{aligned} x_H\left(t\right) &= U^\dagger x U \\ p_H\left(t\right) &= U^\dagger p U \end{aligned} \tag{72}$$

one has an equivalent form for $\lambda_{\vec{v}}$

$$\lambda_{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}} = \lim_{t\to\infty} \frac{1}{t} \log \left\| \begin{matrix} Tr' \{\rho_0 x_H (t)\} \\ Tr' \{\rho_0 p_H (t)\} \end{matrix} \right\| \tag{73}$$

where we have also defined

$$Tr' \{\rho_0 x_H (t)\} = Tr \{\rho_0 x_H (t)\} / Tr \{\rho_0 x_H (0)\}$$

Whenever $\rho_0 x_H (t)$ is a trace class operator, the term corresponding to $\mathrm{Tr}\{\rho_0 x_H (0)\}$ has no contribution in the $t \to \infty$ limit. On the other hand, by taking the appropriate cut-off and a limiting procedure, the above expression may also make mathematical sense even in some non-trace class cases.

5 Quantizers and Dequantizers: An Unified View of Alternative Quantum Formulations

Tomography maps may be framed not only as amplitudes of projections on a complete basis of eigenvectors of a family of operators, as in (24), but also as operator symbols [27]. That is, as a map of operators to a space of functions where the operators non-commutativity is replaced by a modification of the usual product to a star-product.

Let $\hat{A}$ be an operator in Hilbert space $\mathscr{H}$ and $\hat{U}(\mathbf{x})$, $\hat{D}(\mathbf{x})$ two families of operators called *dequantizers* and *quantizers*, respectively, such that

$$\mathrm{Tr}\left\{ \hat{U}(\mathbf{x})\hat{D}(\mathbf{x}') \right\} = \delta(\mathbf{x} - \mathbf{x}') \tag{74}$$

The labels $\mathbf{x}$ (with components $x_1, x_2, \ldots x_n$) are coordinates in a linear space V where the functions (operator symbols) are defined. Some of the coordinates may take discrete values, then the delta function in (74) should be understood as a Kronecker delta. Provided the property (74) is satisfied, one defines the *symbol of the operator* $\hat{A}$ by the formula

$$f_A(\mathbf{x}) = \mathrm{Tr}\left\{ \hat{U}(\mathbf{x})\hat{A} \right\}, \tag{75}$$

assuming the trace to exist. In view of (74), one has the reconstruction formula

$$\hat{A} = \int f_A(x)\hat{D}(\mathbf{x})\, d\mathbf{x} \tag{76}$$

The role of quantizers and dequantizers may be exchanged. Then

$$f_A^d(\mathbf{x}) = \mathrm{Tr}\left\{\hat{D}(\mathbf{x})\,\hat{A}\right\} \tag{77}$$

is called the dual symbol of $f_A(\mathbf{x})$ and the reconstruction formula is

$$\hat{A} = \int f_A^d(x)\hat{U}(\mathbf{x})\,d\mathbf{x} \tag{78}$$

Symbols of operators can be multiplied using the star-product kernel as follows

$$f_A(\mathbf{x}) \star f_B(\mathbf{x}) = \int f_A(\mathbf{y}) f_B(\mathbf{z}) K(\mathbf{y}, \mathbf{z}, \mathbf{x})\,d\mathbf{y}\,d\mathbf{z} \tag{79}$$

the kernel being

$$K(\mathbf{y}, \mathbf{z}, \mathbf{x}) = \mathrm{Tr}\left\{\hat{D}(\mathbf{y})\hat{D}(\mathbf{z})\hat{U}(\mathbf{x})\right\} \tag{80}$$

The star-product is associative,

$$(f_A(\mathbf{x}) \star f_B(\mathbf{x})) \star f_C(\mathbf{x}) = f_A(\mathbf{x}) \star (f_B(\mathbf{x}) \star f_C(\mathbf{x})) \tag{81}$$

this property corresponding to the associativity of the product of operators in Hilbert space.

With the dual symbols the trace of an operator may be written in integral form

$$\mathrm{Tr}\left\{\hat{A}\hat{B}\right\} = \int f_A^d(\mathbf{x}) f_B(\mathbf{x})\,d\mathbf{x} = \int f_B^d(\mathbf{x}) f_A(\mathbf{x})\,d\mathbf{x}. \tag{82}$$

For two different symbols $f_A(\mathbf{x})$ and $f_A(\mathbf{y})$ corresponding, respectively, to the pairs $(\hat{U}(\mathbf{x}), \hat{D}(\mathbf{x}))$ and $(\hat{U}_1(\mathbf{y}), \hat{D}_1(\mathbf{y}))$, one has the relation

$$f_A(\mathbf{x}) = \int f_A(\mathbf{y}) K(\mathbf{x}, \mathbf{y})\,d\mathbf{y}, \tag{83}$$

with intertwining kernel

$$K(\mathbf{x}, \mathbf{y}) = \mathrm{Tr}\left\{\hat{D}_1(\mathbf{y})\hat{U}(\mathbf{x})\right\} \tag{84}$$

Let now a wave function be identified with the projection operator Π_ψ on the function $\psi\,(t)$, denoted by

$$\Pi_\psi = |\psi\rangle\,\langle\psi| \tag{85}$$

Then the tomography maps (tomograms), and also other transforms, are symbols of the projection operators for several choices of quantizers and dequantizers.

Some examples:

Denote position and momentum by q and p (for signal processing the corresponding set of non-commuting variable would be t and ω).

The *Wigner–Ville function*: is the symbol of $\mid \psi\rangle\langle\psi \mid$ corresponding to the dequantizer

$$\hat{U}(\mathbf{x}) = 2\hat{\mathscr{D}}(2\alpha)\hat{P}, \qquad \alpha = \frac{q+ip}{\sqrt{2}} \tag{86}$$

where $\hat{P}$ is the inversion operator

$$\hat{P}\psi(q) = \psi(-q) \tag{87}$$

and $\hat{\mathscr{D}}(\gamma)$ is a "displacement" operator

$$\hat{\mathscr{D}}(\gamma) = \exp\left[\frac{1}{\sqrt{2}}\gamma\left(q - \frac{\partial}{\partial q}\right) - \frac{1}{\sqrt{2}}\gamma^*\left(q + \frac{\partial}{\partial q}\right)\right] \tag{88}$$

The quantizer operator is

$$\hat{D}(\mathbf{x}) := \hat{D}(q,p) = \frac{1}{2\pi}\hat{U}(q,p), \tag{89}$$

The Wigner function is

$$W(q,p) = 2\mathrm{Tr}\left\{\mid \psi\rangle\langle\psi \mid \hat{D}(2\alpha)\hat{D}\right\} \tag{90}$$

or, in integral form

$$W(q,p) = 2\int \psi^*(q)\hat{\mathscr{D}}(2\alpha)\psi(-q)\,dq \tag{91}$$

The *symplectic tomogram* (position-momentum or time-frequency in signal processing) tomogram of $\mid \psi\rangle\langle\psi \mid$ corresponds to the dequantizer

$$\hat{U}(\mathbf{x}) := \hat{U}(X,\mu,\nu) = \delta\left(X\hat{1} - \mu\hat{q} - \nu\hat{p}\right), \tag{92}$$

Here the notation $\delta\left(X\hat{1} - \mu\hat{q} - \nu\hat{p}\right)$ stands for the projector on the eigenvector of $\mu\hat{q} + \nu\hat{p}$ corresponding to the eigenvalue X and

$$\hat{q}\psi(q) = q\psi(q), \qquad \hat{p}\psi(t) = -i\frac{\partial}{\partial q}\psi(q) \tag{93}$$

and $X, \mu, \nu \in R$. The quantizer of the symplectic tomogram is

$$\hat{D}(\mathbf{x}) := \hat{D}(X, \mu, \nu) = \frac{1}{2\pi} \exp\left[i\left(X\hat{1} - \mu\hat{q} - \nu\hat{p}\right)\right] \tag{94}$$

The *optical tomogram* is the same as above for the case

$$\mu = \cos\theta, \qquad \nu = \sin\theta. \tag{95}$$

Thus the optical tomogram is

$$\begin{aligned} M(X,\theta) &= \mathrm{Tr}\left\{\mid \psi\rangle\langle\psi \mid \delta\left(X\hat{1} - \mu\hat{q} - \nu\hat{p}\right)\right\} \\ &= \frac{1}{2\pi}\int \psi^*(q)e^{ikX}\exp\left[ik\left(X - q\cos\theta + i\frac{\partial}{\partial q}\sin\theta\right)\right]\psi(q)\,dq\,dk \\ &= \frac{1}{2\pi|\sin\theta|}\left|\int \psi(q)\exp\left[i\left(\frac{\cot\theta}{2}q^2 - \frac{Xq}{\sin\theta}\right)\right]dq\right|^2. \end{aligned} \tag{96}$$

One important feature of the formulation of tomograms as operator symbols is that one may work with deterministic functions $\psi\ (q)$ as easily as with probabilistic ones. In this latter case the projector in (85) would be replaced by

$$\Pi_p = \int p_\mu \left|\psi_\mu\right\rangle\left\langle\psi_\mu\right| d\mu \tag{97}$$

with $\int p_\mu d\mu = 1$, the tomogram being the symbol of this new operator.

This also provides a framework for an algebraic formulation of signal processing, perhaps more general than the one described in [28]. There, a signal model is a triple $(\mathscr{A}, \mathscr{M}, \Phi)$ $\mathscr{A}$ being an algebra of linear filters, $\mathscr{M}$ a $\mathscr{A}$-module and Φ a map from the vector space of signals to the module. With the operator symbol interpretation, both deterministic or random signals and linear or nonlinear transformations on signals are operators. By the application of the dequantizer (Eq. 75) they are mapped to functions, the filter operations becoming star-products.

References

1. F. Bayen, M. Flato, C. Fronsdal, C. Lichnerowicz, D. Sternheimer, Deformation theory and quantization. Ann. Phys (NY) **111**, 61–151 (1978)
2. J. Vey, Déformation du crochet de Poisson sur une variété symplectique. Comment. Math. Helv. **50**, 421–454 (1975)
3. M. De Wilde, P. Lecomte, Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds. Lett. Math. Phys. **7**, 487–496 (1983)
4. O.M. Neroslavsky, A.T. Vlassov, Sur les deformations de l'algebre des fonctions d'une variété symplectique. C. R. Acad. Sci. Paris **292**(I), 71–73 (1981)
5. M. Cahen, S. Gutt, Regular star-representations of Lie algebras. Lett. Math. Phys. **6**, 395–404 (1982)

6. J. Moyal, Quantum mechanics as a statistical theory. Proc. Cambridge Phil. Soc. **45**, 99–124 (1949)
7. S. Bates, A. Weinstein, *Lectures on the Geometry of Quantization, Berkeley Mathematics Lecture Notes 8* (American Mathematical Society, Providence, 1997)
8. S. Mancini, V.I. Man'ko, P. Tombesi, Wigner function and probability distribution for shifted and squeezed quadratures. Quantum Semiclass. Opt. **7**, 615–623 (1995)
9. G.M. D'Ariano, S. Mancini, V.I. Man'ko, P. Tombesi, Reconstructing the density operator by using generalized field quadratures. Quantum Semiclass. Opt. **8**, 1017–1027 (1996)
10. S. Mancini, V.I. Man'ko, P. Tombesi, Symplectic tomography as classical approach to quantum systems. Phys. Lett. A **213**, 1–6 (1996)
11. S. Mancini, V.I. Man'ko, P. Tombesi, Classical-like description of quantum dynamics by means of symplectic tomography. Found. Phys. **27**, 801–824 (1997)
12. O. Man'ko, V.I. Man'ko, Quantum states in probability representation and tomography. J. Russ. Laser Res. **18**, 407–444 (1997)
13. V.I. Man'ko, R. Vilela Mendes, Noncommutative time–frequency tomography. Phys. Lett. A **263**, 53–61 (1999)
14. M.A. Man'ko, V.I. Man'ko, R. Vilela Mendes, Tomograms and other transforms: a unified view. J. Phys. A: Math. Gen. **34**, 8321–8332 (2001)
15. F. Clairet, F. Briolle, B. Ricaud, S. Heuraux, New signal processing technique for density profile reflectometry on Tore Supra. Rev. Sci. Instrum. **82**, 083502 (2011)
16. C. Aguirre, R. Vilela Mendes, Signal recognition and adapted filtering by non-commutative tomography. IET Signal Proc. 8, 67–75 (2014)
17. R. Vilela Mendes, H.C. Mendes, T. Araújo, Signals on graphs: Transforms and tomograms. Phys. A **450**, 1–17 (2016)
18. R. Vilela Mendes, Non-commutative tomography and signal processing. Phys. Scr. **90**, 07402 (2015)
19. R. Vilela Mendes, J. Leitão, V.I. Man'ko, The classical-quantum transition in the cooling of levitated nanoparticles, in preparation
20. P.J. Morrison, Hamiltonian and action principle formulations of plasma physics. Phys. Plasmas **12**, 058102 (2005)
21. J.E. Marsden, A. Weinstein, The Hamiltonian structure of the Maxwell-Vlasov equations. Phys. D **4**, 394–406 (1982)
22. F.E. Low, A Lagrangian formulation of the Boltzmann-Vlasov equation for plasmas. Proc. R. Soc. Lond. Ser. A **248**, 282–287 (1958)
23. R. Vilela Mendes, Sensitive dependence in quantum systems: some examples and results. Phys. Lett. A **171**, 253–258 (1992)
24. R. Vilela Mendes, R. Coutinho, On the computation of quantum characteristic exponents. Phys. Lett. A **239**, 239–245 (1998)
25. V.I. Man'ko, R. Vilela Mendes, Lyapunov exponent in quantum mechanics. A phase-space approach. Phys. D **145**, 330–348 (2000)
26. V.I. Man'ko, R. Vilela Mendes, Quantum sensitive dependence. Phys. Lett. A **300**, 353–360 (2002)
27. F. Briolle, V.I. Man'ko, B. Ricaud, R. Vilela Mendes, Noncommutative tomography: A tool for data analysis and signal processing. J. Russian Laser Res. **33**, 103–121 (2012)
28. M. Puschel, J.F. Moura, Algebraic signal processing theory: 1-D space. IEEE Trans. Signal Proc. **56**, 3586–3599 (2008)