Chapter 4
Mental and Neural Foundations of Numerical Magnitude

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Abstract Number symbols allow learners to communicate and operate on quantitative information. For more than 50 years, research in mathematics education, cognitive psychology, and neuroscience has investigated the mechanisms underlying the understanding and processing of numerical magnitude, an essential property of numbers. From single-digit and multidigit natural numbers to rational numbers and beyond, learning numerical magnitude presents learners with challenges of increasing complexity across the schooling process. Here, we review interdisciplinary research that has contributed to understanding how people's minds and brains process numerical magnitude through diverse number systems. We focus on main issues during the development of this research, which has mostly relied on number comparison tasks, but it has also incorporated quantity estimation and number line positioning. We include key themes across number systems, such as the emergence of number-space associations and whether perceiving a numerical symbol automatically activates its numerical magnitude.

Keywords Numerical magnitude · Numerical cognition · Mental calculation · Educational neuroscience · Mathematics education

4.1 Introduction

The study of the understanding of quantity and number, particularly by humans, has garnered increasing interest in the last 50 years both worldwide (LeFevre 2016) and in the Latin American context (Haase et al. 2020), giving rise to the field of *numerical cognition*. From its very beginnings, this field of research has played a key role in understanding human mental processing and development. Mehler and Bever (1967) took an empirical approach to assess the well-known Piagetian observation

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that 4-year-old children fail to distinguish between the spatial extent occupied by a set of objects and the quantity of its elements. This lack of distinction was commonly assumed to imply that children of this age have not yet learned that the spatial rearrangement of a set's elements conserves their quantity. Piaget's demonstration consisted of presenting children with sets of objects and asking them whether both sets were *the same* or if one of them *had more*. Most 4-year-old children failed to indicate a set of six objects as *having more* than one of four objects when both sets are displayed in a linear array and the smaller set is more spatially spread than the larger set. Mehler and Bever examined Piaget's findings and conclusions by conducting two similar experiments, each of them with a specific methodological variation.

First, they presented the same task not only to 4- and 5-year-olds, but also to children younger than 4 years old. According to Piaget's theory, younger children are in an earlier stage of development than 4-year-olds, so if the latter fail in the task, the former should fail as well. Contrary to this prediction, Mehler and Bever (1967) found that even children younger than 3 years old answered the task in a manner consistent with quantity conservation.[1]

The second variation introduced by these authors was the use of candy sets instead of sets of arbitrary objects. In this version of the task, children were asked not which set *had more*, but rather, to choose one of the sets and eat those candies. The results revealed that most of the children across all the age ranges tested (2.5–4.5 years, in total) displayed knowledge of quantity conservation.

More than 50 years after Mehler and Bever's (1967) study, the field of numerical cognition has grown significantly. In this period, it has actively and continuously contributed to the understanding of children's learning and development (Alcock et al. 2016; Beller et al. 2018).

4.2 Number Symbols and the Problem of Numerical Magnitude

Many species have been shown to perceive and use quantitative information from their surroundings, such as monkeys (Cantlon and Brannon 2006), dogs (Bonanni et al. 2011), crows (Ditz and Nieder 2016), fish (Agrillo et al. 2008), and even insects (Pahl et al. 2013).

However, a critical point where the human journey to mathematics departs from the quantitative capacities of other animals is the association of arbitrary symbols to quantities of objects. In this sense, the most basic symbols, widely shared across cultures, are linguistic objects such as words denoting approximate

[1] That is to say, 2- and 4.5-year-old children demonstrated knowledge of quantity conservation, but children in between these ages did not. Mehler and Bever (1967) interpreted this peculiar outcome as stemming from a difference between implicit and explicit understandings of quantity conservation (see Dehaene 2011, Ch. 2, for an explanation based on children's developing theory of mind).

quantities—*one*, *some*, *many*—and words denoting exact quantities (Núñez 2017). It is worth noting that extracting the quantity of items in a collection and associating it to a word is a task that requires a high level of abstraction, since quantity is not a property of any *particular* object but rather of a *collection* of objects. For example, in the phrase, *three red cars*, the property *red* can be recognized as true of each car, but the property *three* refers to no individual car.

Once quantities have been associated with symbols, these symbols are used to represent the quantities and make inferences such as that a set of five toys has more elements than a set of two toys, even if one cannot directly experience either set. Most importantly, in principle, symbols are arbitrary: any symbol could be used to represent any quantity, meaning that there are no systematic cues in the symbol allowing someone to infer how large a represented quantity is (without knowledge of the symbol-quantity mapping).

A pervasive question in cognition relates to the semantics of symbols. In the case of quantitative or numerical symbols, their semantics directly relate to the magnitude of their associated quantities. These symbols may then inherit at least *some* of the properties of the quantities they stand for. In this chapter, we focus on the issue of *numerical magnitude*: When an educated person perceives or mentally works with a numerical symbol, is the magnitude of the associated quantity necessarily activated in their mind? We begin with a review of seminal studies in the field of numerical cognition and present evidence for the hypothesis that the educated human mind processes numbers in a manner akin to a mental number line. We then review studies relevant to the processing of numerical magnitude across different number systems: natural numbers, rational numbers, integer numbers, and irrational numbers.

4.2.1 Seminal Studies

Among the many studies focused on the processing and learning of numerical symbols, three of them can be considered as foundational because they led to the view that our mental representations of numbers behave in a way akin to a mental number line.

The first study in this list was conducted by Moyer and Landauer (1967), who tested a sample of young adults by showing them pairs of single-digit numbers and asking them to decide—as quickly as possible—which digit in each pair was numerically larger. They recorded not only the correctness of each answer, but also precise measurements of how long participants took to answer each item. These authors observed that participants' error rates and response times increased with the numerical distance between the digits (a *distance effect*). In other words, participants were more accurate and faster when comparing numerically distant digits (e.g., 2 vs. 7) rather than numerically close digits (e.g., 6 vs. 7). This result reveals that the mental comparison of numerical symbols works in an analogical manner, in sharp contrast with the numerical processing of a computer, for example.

Inspired by psychophysical research and modeling, these authors also observed that their participants' response times could be modeled as a function of the ratio between the digits' values: if one defines the ratio Q as the smaller value divided by the larger value, then higher values of Q were associated with longer response times. The appearance of this ratio between numerical values has deep implications for our understanding of the mental organization of numerical representations, as it implies that the absolute difference between the digits' numerical values is not the best description of the distance between their mental representations. This implies, for instance, that comparing 2 vs. 5 ($Q = 0.4$) is just as difficult as comparing 10 vs. 25 ($Q = 0.4$), and less difficult than comparing 12 vs. 15 ($Q = 0.8$). This pattern of results would come as no surprise if we were referring to the comparison of sets of objects according to their quantity. Imagine two boxes containing toys: Box A contains 10 red toys and 25 green toys, while and Box B contains 12 red toys and 15 green toys. If you are asked to determine whether each box contains more red toys or green toys by only taking a quick look at their contents, you will naturally anticipate experiencing more difficulty with Box B because the amount of toys of each color is somehow more balanced than in Box A. Moyer and Landauer's (1967) work demonstrated that this intuitive pattern also applies to the comparison of numerical symbols or, in other words, that the comparison of numerical symbols according to their magnitude behaves consistently with a model in which numerical symbols are mentally associated with the magnitude of their corresponding quantities before comparison.

The ratio effect is also consistent with *Weber's Law*, which summarizes a regularity about people's perception of stimuli that differ in a continuous magnitude, such as brightness, loudness, and weight. Weber's Law states that whenever a person has to determine if two stimuli are different or not, the smallest discriminable difference is proportional to the reference magnitude being used (Nutler 2010): a person might successfully discriminate between a weight of 100 g and another of 103 g (3% difference) and between a weight of 1000 g and another of 1030 g (3% difference), but be unable to distinguish between a weight of 1000 g and another of 1003 g (0.3% difference). This observation has led researchers to believe that numerical symbols are mentally processed in a manner similar to physical magnitudes such as those mentioned above.

The task used by Moyer and Landauer (1967) asked participants to focus explicitly on the numerical magnitude of the digits presented. Nevertheless, this is not the only possible criterion to compare number symbols, because any visual symbol must be drawn with a given physical size. This allows people to compare digits in terms of their numerical size or their physical size (e.g., 2 vs. 4). Later studies looked further into the contrast between comparing digits' numerical sizes and physical sizes.

Henik and Tzelgov (1982) asked adults to compare single-digit numbers in terms of each of these dimensions. They presented pairs of digits varying in both physical and numerical size, and asked participants to select the largest one in terms of either physical size or numerical size, while disregarding the other dimension. In this task, the two size dimensions could lead to *congruent* or *incongruent* answers. Comparing

2 vs. 4 is an example of a congruent stimulus because the latter has a larger numerical size and a larger physical size. In contrast, comparing 2 vs. 4 is an example of an incongruent stimulus, as the latter has a larger numerical size but a smaller physical size. With this design, these authors were able not only to measure performance when comparing stimuli in terms of the dimension that participants were asked to attend, but also to examine how this performance is modulated by the congruency between the attended and the unattended dimensions. Whereas physical size is a perceptual property of the stimuli, and it is therefore accessed immediately upon stimulation, numerical size depends on the meaning of each symbol and must be retrieved from memory. Henik and Tzelgov's (1982) results showed that comparing digits' physical sizes was more difficult (higher error rates, longer response times) in incongruent pairs than in congruent pairs, which demonstrated that numerical size information interferes with physical size judgments. Their findings suggested that the retrieval of a digit's numerical magnitude is not an optional process that people use only when needed, but one that occurs automatically (i.e., even when the task requires one to ignore numerical magnitude).

A third seminal study is that of Dehaene et al. (1993). These authors observed a different kind of automatic processing of numerical magnitude in people's performance in a task where magnitude is irrelevant, a parity judgment task. Participants were instructed to press keys with their left or right hand to indicate if a single-digit number presented on a screen was even or odd. As it was expected that participants would respond more quickly with their dominant hand, the authors took care that in half of the experimental session the *even* response key was pressed with the left hand and in the other half with the right hand. While numerical magnitude was completely irrelevant to this task, the authors observed that participants responded more quickly to numerically larger digits when the answer (*even* or *odd*) was given with their right hand, and to numerically smaller digits when the answer was given with their left hand. This difference, named *spatial-numerical association of response codes* (SNARC) effect, provided further support for the automaticity of processing numerical magnitude. It also indicates that numerical mental representations are intimately associated with space.

Many studies aiming to assess the nature of the mental representations of more advanced number symbols and, in particular, if their numerical magnitude is processed automatically, have resorted to variants of one of these tasks. Comparison tasks based on numerical magnitude may lead to the observation of distance effects; comparison tasks based on symbols' physical sizes may lead to the observation of interference or congruency effects; and tasks involving magnitude-related or magnitude-unrelated judgments about numbers may reveal SNARC effects.

4.2.2 A Mental Number Line

Altogether, the presence of distance effects in number comparison, the automaticity of the activation of single digits' numerical magnitude, and the association between numbers and space (with smaller numbers associated to the left side and larger numbers associated with the right side) have been interpreted by many researchers as an indication that people's mental representation of single-digit numbers is similar to a mental number line oriented from left to right. To account for the variability in participants' responses to multiple presentations of a given numerical stimulus, this number line is assumed to work analogically: numbers are represented in an approximate manner rather than in an exact one.

While the mental number line interpretation has not been unanimous (e.g., Fischer 2006), it has received support from the study of the behavior of patients who have suffered a brain injury leading to a condition known as hemispatial neglect (Zorzi et al. 2002, 2006; Umiltà et al. 2008). These patients often fail to be aware of objects—sometimes even of people—located in the side of space opposite to their lesion: a right-hemisphere brain lesion might lead to *left neglect* and an inability to acknowledge objects located on the left side of the patient's extrapersonal space (Husain 2008). When asked to point to the middle of a visually presented line, left neglect patients tend to point to a position that is shifted to the right with respect to the real midpoint. More surprisingly, a similar pattern holds when they are asked to indicate the midpoint of a numerical interval. For example, they may state 17 as the midpoint of the interval 11–19 (Zorzi et al. 2002). Further evidence from nonhuman animals and young human infants suggests that the association between number symbols and space might be a reflection of an association between quantities and space that is present early in development (De Hevia et al. 2012).

The scaling of the mental number line has also been a matter of debate. One widely known model assumes linear scaling (Gallistel and Gelman 1992), where pairs of numbers with a fixed absolute difference are located in equidistant positions. For instance, in such a number line, the mental representations of 2, 4, 6, and 8 form a uniformly spaced sequence. This model further states that larger numbers are represented in a less precise manner than smaller numbers.[2] Another well-known model poses logarithmic scaling (Dehaene and Mehler 1992), where pairs of numbers with a fixed ratio are located in equidistant positions. In contrast to the linear model, in this number line the mental representations of larger numbers are compressed: 4 and 6 are closer than 2 and 4, and 6 and 8 are closer than 4 and 6. An example of a uniformly spaced sequence in a logarithmic number line is 2, 4, 8, and 16, where each element is obtained by doubling the previous one. In the logarithmic

[2] A more precise statement requires conceptualizing the representation of each number as a random position on the number line. The standard deviation of the associated probability distribution can be taken as an inverse measure of how *exact* the representation is. The linear model states that the standard deviation corresponding to a number representation grows proportionally to the value of the number (e.g., the standard deviation of the mental representation of 8 is twice as large as that of the mental representation of 4).

model, it is assumed that all numbers are mentally represented with a similar degree of precision.[3]

Both models successfully reproduce the distance effect, since in both of them the number representations are ordered and, therefore, increasing numerical distance between numbers translates into numerical representations that are farther away on the number line. Both models also account for the size effect: in the linear model, the numerical representations of a pair of numbers become less precise with increasing numerical quantity, leading to a more difficult discrimination. In the logarithmic model, instead, two numbers with a fixed difference become closer with increasing numerical magnitude, leading again to higher difficulty.

Dehaene (2001) suggested that both models led to essentially indistinguishable predictions in some settings and conjectured that animal behavior data alone would be unable to choose a model as better than the other. In recent years, however, novel modeling approaches that take into consideration not only participants' error rates and response times, but also the interplay between them, have succeeded in differentiating between these models (e.g., Ratcliff and McKoon 2018). This suggests that we are closer to understanding if one of these models should be preferred, and under which conditions.

4.2.3 Neuronal and Neural Data

Researchers studying number sense have also analyzed neuronal and neural[4] data. Nieder and Miller (2003) investigated the neuronal activity of rhesus monkeys while they watched sets of dots and decided if pairs of these sets depicted the same or a different quantity. Neurons in the monkeys' prefrontal brain cortex showed a pattern of performance similar to that predicted by the logarithmic model: noisy responses that, once plotted in a logarithmic scale, could be described as random distributions with the same degree of precision.

A later study by the same authors (Nieder and Miller 2004) revealed that neurons located in the posterior parietal cortex also responded in a selective manner to the quantity of a set of objects. Moreover, they observed that the response of parietal neurons occurred closer in time to the presentation of the sets of objects than the response of prefrontal neurons, suggesting that the processing of numerical information in the brain flows from the parietal to the prefrontal cortex.

The type of recordings that Nieder and Miller (2003, 2004) used to evaluate neurons' responses in monkeys cannot be usually used to study humans. Researchers

[3] In terms of random positions and probability distributions, the logarithmic model states that all number representations share the same standard deviation.

[4] Although these terms are sometimes exchangeable, *neuronal* refers to neurons whereas *neural* refers to nerves. In the context of this review, neuronal activity is used for a single neuron or a small group of neurons, while neural activity is used for large populations of neurons where single units cannot be distinguished.

wanting to investigate brain activity in response to numerical information in humans have thus resorted to other techniques, such as positron emission tomography (PET) and, more recently, functional magnetic resonance imaging (fMRI).

Piazza et al. (2004) conducted an fMRI study to search for evidence of neural activity responsive to numerical magnitude in human adults. They asked participants to passively observe a sequence of dot sets representing different quantities. Most of the time the sets in these sequences represented a fixed quantity such as 16, with occasional changes to quantities which were numerically close (e.g., 14 or 18) or far (e.g., 8 or 32) from the fixed standard. These researchers reasoned that, if there are brain regions that respond specifically to the quantity of objects in a set, these regions should get *habituated* after the frequent repetition of the fixed standard quantity and thus reduce their activity. Moreover, after habituation, these regions should reactivate in response to the presentation of novel quantities, and this reactivation should be larger for novel quantities that are far away from the standard. Neural data revealed the presence of a brain region that showed precisely this pattern of responses, located within the parietal brain lobe: the *intraparietal sulcus*. Results from previous studies have demonstrated that this brain area plays a role in several numerical tasks, such as number comparison, subtraction, and detection of number symbols (Eger et al. 2003; Lee 2000; Pesenti et al. 2000), suggesting that it plays a role in the processing of magnitudes associated to both numerical symbols and quantities. This interpretation received further support from another study by Piazza et al. (2007), who observed that the pattern of neural habituation and reactivation in the intraparietal sulcus occurred even if the presented stimuli switched between formats (number symbols and quantities), as well as from other works (e.g., Venkatraman et al. 2005).

4.2.4 Debates

Although the view described above has reached wide agreement in the numerical cognition community, several points are still under debate. One of these points refers to whether the numerical representations hosted by the parietal cortex are actually *abstract*, namely, if they encode numbers' magnitudes regardless of the sensory means by which these numbers are presented to a person (sets of objects, number symbols, or even spoken number words). While many studies, including those mentioned above, have favored the belief in a truly abstract number representation in the parietal cortex, several researchers have challenged this view (e.g., Ansari 2016; Cohen Kadosh and Walsh 2009; Wilkey and Ansari 2020). Cohen Kadosh and Walsh (2009) argued that several pieces of evidence for non-abstractness have been reported, but they have been ignored or explained by alternative means. They also noted that most of the support for abstractness has arisen from an actual lack of evidence for differences in processing numbers presented in diverse formats, which could be ascribed to small sample sizes and low statistical power. Wilkey and Ansari (2020) also indicated that the brain's involvement in numerical tasks is not restricted to the intraparietal sulcus, but it is configured as a widely distributed network that has not yet been fully understood. Moreover, they noted that many common tasks to measure numerical processing require not only numerical capacities, but also other cognitive functions (e.g., executive functions) that need to be disentangled before drawing conclusions.

Other debates relate to the extent to which the behavioral and neural results obtained so far reflect innate dispositions or uniquely human capacities. In contrast to basic quantitative capacities, which are shared across many species, number symbols seem to be uniquely human and culturally learned. Therefore, should people's mental representation of symbolic numbers be akin to a left-to-right oriented number line, this representation must be acquired through experience (Núñez 2011). Furthermore, even though not all human cultures possess number symbols as commonly understood, they may still utilize quantitative symbols such as the verbal quantifiers *a few* and *a lot* (Pica et al. 2004). These verbal quantifiers seem to share some properties with quantities and numbers in terms of human perception (Pezzelle et al. 2018). Núñez (2017) called for a clearer distinction between quantitative and numerical cognitive capacities, with quantitative capacities being biologically evolved and shared across species, and numerical capacities being culturally mediated.

As Wilkey and Ansari (2020) argue, the link between cognitive capacities and brain activity is difficult to assess. The brain is a system whose complexity is well beyond our current understanding, and its mechanisms resist simplistic explanations. This becomes a challenge for interdisciplinary dialogues such as those between neuroscience and mathematics education, two disciplines with different histories, aims, and focus of interest.

4.3 Numerical Magnitude Across Different Number Systems

In this section, we review research on numerical magnitude in specific number systems: natural numbers, rational numbers, integers, and irrational numbers.

Following the lead of Moyer and Landauer (1967), most studies in numerical cognition dealing with magnitude have used number comparison tasks. Here, participants are typically asked to determine which one of two visually presented number symbols represent a number greater than the other. The usual empirical outcomes measured are response accuracy and/or response time. Other tasks ask participants to locate a given number on a visually presented number line (e.g., Siegler and Opfer 2003) and estimate the number of objects in a collection (e.g., Izard and Dehaene 2008). In the following sections, we review findings on the mental and neural processing of numerical magnitude, considering these different tasks as well as number systems from natural to irrational numbers.

4.3.1 Natural Numbers

Many major number systems use a place value codification, where arbitrarily large numbers are built from a finite set of symbols—called *digits*—that are arranged in space so that a symbol's value depends on its relative location within the number. For instance, the digit 1 in 145 means a hundred, but in 514 it means ten. Consequently, the study of the numerical magnitude of natural numbers can look into at least two levels. First, how the numerical magnitude of isolated, visually presented digits is processed; and second, how the numerical magnitude of compound, multidigit numbers is processed.

Many of the earliest investigations of numerical magnitude, such as that of Moyer and Landauer (1967) presented above, focused on the processing of single digits. As we have described, the study of single-digit numbers has reached a broad—though not unanimous—agreement on the idea that these numbers are mentally represented in a manner akin to a number line, in many cases oriented from left to right so that numerically small digits are associated with the left side of space and numerically large digits are associated with the right side of space.

Regarding multidigit numbers, a natural first step was to examine if a similar pattern of results to that of single digits holds. Hinrichs et al. (1981) and later Dehaene et al. (1990) extended Moyer and Landauer's (1967) work to investigate if a numerical distance effect emerges in the comparison of two-digit numbers. In both studies, researchers asked participants to compare a given two-digit number (e.g., 39) against a given reference (e.g., 55), while recording the accuracy and timing of their responses. If participants processed these numbers purely according to their magnitude, their responses would be expected to show a distance effect similar to that of comparing single digits (*holistic* model). Alternatively, responses could reflect strategies similar to those taught in school: participants might examine the tens digits in order to compare the numbers, and resort to the units digits only when the tens were equal (*lexicographic* model). Yet another option was possible: that participants would process two-digit numbers as independent single digits and only later integrate them into a representation of the magnitude of the two-digit number (*interference* model). Results showed a mixed pattern: while there was an overall effect of numerical distance, response times showed discontinuities depending on the tens digits at least when using some reference values (Dehaene et al. 1990). The researchers interpreted these findings as supporting the holistic model instead of the other two. Still, even then it was known that there is a limit to the capacity of processing multidigit numbers in a holistic manner: when presented with numbers comprising four or six digits, a pattern of results consistent with the lexicographic model emerges clearly (e.g., Poltrock and Schwartz 1984).

Later studies showed that the interference model should not be quickly discarded. Nuerk et al. (2001) made a slight modification to the previous comparison tasks, in which numbers were compared against a fixed reference, through a design in which participants compared pairs of numbers. Crucially, they chose number pairs in which the tens and units digits varied more systematically, leading the researchers to introduce the concept of *unit-decade-compatibility*: a number pair such as 42 and 57 was named *compatible* because the larger number is composed by both the larger tens digit and the larger units digit. In contrast, a number pair such as 47 and 62 was named *incompatible* because one of the numbers had the larger tens digit but the smaller units digit. The authors reasoned that these pairs provided a stringent test for the interference model against the holistic and the lexicographic models. Since both number pairs share the same numerical distance between numbers, the holistic model predicted similar performance in both of them. Moreover, both numbers have different tens digits, so the lexicographic model can also predict similar performances or, at most, better performance when comparing the incompatible pair because of a larger difference in the tens digits with respect to the compatible pair. Only the interference model provided an opposite prediction: that the incompatible item would prove more difficult than the compatible one. This prediction follows from the fact that in the compatible item, both the comparison of units and the comparison of tens lead to the same answer; however, in the incompatible one, the comparison of these digits leads to conflicting answers. The data supported Nuerk et al.'s (2001) intuition, showing that participants were quicker and more accurate when comparing compatible pairs. Further studies have provided additional support to the idea that multidigit number processing differs in many aspects from single-digit processing (e.g., Macizo and Herrera 2008). Nuerk et al. (2011) reviewed a number of phenomena that are proper to the processing of multidigit numbers, which a simple holistic model cannot successfully explain. Moeller et al. (2010) conducted a computational modeling study and suggested that the distance effects that have been typically interpreted as supporting the holistic model could also emerge from a non-holistic representation. In other words, this means that the human mind could have no specific representation of a multidigit number's magnitude but still produce patterns of behavior consistent with it.

The work by Nuerk et al. (2001, 2011, among others) suggested that the identities and magnitudes of each of the digits composing a two-digit number are processed automatically, but that it is not so for the magnitude of the full number. An intermediate step would be to ask if the place value of each digit is automatically processed. Kallai and Tzelgov (2012a) tackled this question by presenting adult participants with strings of digits and asking them to decide which string contained the larger non-zero digit. For instance, when presented with 030 and 005, participants were expected to select the latter. The authors reasoned that if the mind accesses not only the digits' identity, but also their place value, such as the 3 in 030 being interpreted as *3 tens*, a conflict would occur when a smaller digit has a larger place value. This led them to measure people's performance in a *congruent* (e.g., 050 vs. 003) and an *incongruent* (e.g., 030 vs. 005) condition. Their results showed that participants' accuracy rates and response times were indeed affected by congruency—more specifically, by an interference of place value information in digit magnitude judgments— suggesting that participants processed the digits' place value despite its being irrelevant for the task at hand. They also applied an adaptation of the physical comparison task described above (Henik and Tzelgov 1982), where participants were presented with digit strings of different physical sizes and asked which string was physically larger than the other. In this second step, they used items with the same digits but different place values (e.g., 0300 vs. 0003) to isolate the place value component. Although the identity and positions of the non-zero digits were irrelevant to the task, Kallai and Tzelgov (2012a) observed an interference effect of place value information in participants' physical size judgments: these judgments were easier when the physically larger digit string corresponded to the number with larger magnitude. These findings demonstrate that the numerical magnitude of digits within multidigit numbers can also be automatically activated; however, it is still unclear if the magnitude of complex multidigit numbers is the main driving force behind people's mental numerical judgments. Whereas behavioral data show similar signatures to the processing of single digits (e.g., distance effects), it cannot be taken for granted that they rely on the same underlying mechanism such as a mental number line.

While most studies of natural number magnitude have used comparison tasks, there have been other interesting tasks that have contributed to understanding how the human mind processes numerical magnitude.

Izard and Dehaene (2008) used an estimation task from psychophysics research (e.g., Indow and Ida 1977), in which adult participants were presented with pictures of sets of 1–100 dots and asked to estimate their quantity. The authors observed that participants' responses were consistent in the sense that larger quantities led to larger estimates, although in general participants underestimated the target quantities (but see Crollen et al. 2011). Furthermore, the authors probed participants' ability to adjust their responses based on calibrating information: the presentation of a given picture said to contain 30 dots. Participants successfully adjusted their estimates to this information, even when in some cases the calibrating picture did not actually contain the stated quantity of dots. The estimation task taps directly into people's ability to connect number symbols and nonsymbolic quantities (Mundy and Gilmore 2009), sparking interest in the study of the mental representations of magnitude in children and adults with developmental dyscalculia (Castro Cañizares and Reigosa-Crespo 2011; Mejias et al. 2012).

Siegler and Opfer (2003) asked school children and adults to convert natural numbers from a symbolic representation to a position on a 0–1000 number line and vice versa. Adults' responses, as expected, showed proportionality between number size and number line positioning, which was evident from a strong linear relation between their responses and the target positions and numbers. Sixth-grade children showed a pattern of responses very similar to adults, but second- and fourth-graders differed from them: the relation between the numbers presented and the locations they selected on the number line showed not a linear shape but a logarithmic one, meaning that small numbers were more spaced than large numbers. For example, the median responses of second- and fourth-graders located numbers in the range 200–250 close to the midpoint of the 0–1000 number line. These authors also showed that the numerical context was relevant to second graders' responses, because in spite of showing a logarithmic pattern in locating numbers on a 0–1000 number line, they positioned numbers on a 0–100 line proportionally to their magnitudes (i.e., with linear rather than logarithmic spacing). Further studies have investigated the development of children's mapping of numbers on the number line (e.g., Berteletti et al. 2010; Siegler and Booth 2004) and its relation to mental calculation (e.g., Domínguez Suraña and Aguilar Villagrán 2017). Although it is debated whether children actually switch over development between two mental models of numerical mapping or there is just one model that gets tuned with experience (Barth and Paladino 2011; Opfer et al. 2011, 2016), the overall behavioral pattern is uncontested. This task has been used to measure children's mathematical knowledge with several aims, such as assessing children's learning in response to interventions (Navarrete et al. 2018; Siegler and Ramani 2008).

4.3.2 Rational Numbers

Rational numbers represent ratios between natural numbers. They constitute one of the very first instances of non-natural numbers that children encounter in school, challenging learners to find novel ways to understand number, numerical magnitude, and arithmetic (Charalambous and Pitta-Pantazi 2007; Van Dooren et al. 2015). Rational numbers are often presented as fractions and are visually displayed as a couple of natural numbers—numerator and denominator—separated by a line. Children may struggle to understand that fractions are not merely two numbers put together but single numbers by themselves (Stafylidou and Vosniadou 2004). As such, fractions have a numerical magnitude of their own, related but somewhat independent of the numerical magnitudes of the natural numbers composing them: these natural numbers can change without affecting the magnitude of the fraction (e.g., 5/6, 10/12, and 4440/5328 are all the same rational number).

The study of the cognitive basis of people's processing of fractions is much more recent than that of natural numbers. Bonato et al. (2007) investigated the nature of our mental representation of fractions, asking if these are represented as pairs of natural numbers or as integrated entities. In the first case, termed a *componential representation*, the only numerical magnitudes immediately available to a person would be those of the numerator and denominator, whereas the magnitude of the fraction would be inferred from these. In the second case, named a *holistic representation*, the numerical magnitude of the fraction would be automatically available to the person. Following the results obtained from the study of natural numbers, Bonato et al. (2007) looked for a distance effect and a SNARC effect in a task where individual *unit fractions* (1/n) had to be compared against 1/5 to decide if they were numerically smaller or larger. Although their results confirmed the presence of a distance effect, they also found a reversed SNARC effect: responses to fractions larger than 1/5 were quicker with the left hand, while responses to fractions smaller than 1/5 were quicker with the right hand. This pattern of results was interpreted as an indication that participants were not answering the task based on the fractions' numerical magnitudes, but on the numerical magnitudes of their denominators.

It must be noted that an absence of evidence in favor of a holistic representation is not evidence for the absence of such a representation, so several researchers challenged Bonato et al.'s (2007) results on methodological grounds. Kallai and Tzelgov (2009) argued that the reported results were driven by the authors' choice to use unit fractions and other fractions that varied only in one component at a time. They investigated whether adults would automatically process the magnitudes of fractions by using a physical size comparison task, obtaining results that did not support the hypothesis that fractions are automatically associated with their numerical magnitude. Schneider and Siegler (2010) used a richer set of fractions than Bonato et al. (2007) and reported a distance effect of a similar shape to that attested with natural numbers. These authors inferred that participants had accessed a mental representation of the fractions' magnitudes similar to the mental number line for naturals. Their results notwithstanding, in two of their three experiments participants took extremely long to respond to many items (showing median response times of 5–30 s), preventing any claims about the automaticity of participants' cognitive processing. Later, Toomarian and Hubbard (2018) revisited this issue, showing that the same task used by Bonato et al. (2007) could lead to a reversed SNARC (as attested by Bonato and colleagues) or to a regular SNARC effect depending on the specific set of fractions used, which suggests that participants adapted their responses to the fraction sets.

Ischebeck et al. (2009) investigated the neural representation of fractions using fMRI and focusing on the aforementioned componential-holistic debate. They devised a set of fraction comparison items that pitted the magnitudes of the fractions and of their components against one another (similar to Nuerk et al.'s 2001 design with natural numbers). For instance, in items like 2/5 vs. 4/5 and 1/3 vs. 4/7, the larger fraction is the one with the larger natural number components, whereas in items like 1/3 vs. 1/4 and 2/3 vs. 4/9, the larger fraction is the one with the smaller components. Their results showed that, although brain activity was significantly modulated by the congruency relation between fractions' and components' magnitudes, the intraparietal sulcus responded significantly to the numerical distance between fractions (Ischebeck et al. 2010), involving the same brain area associated with processing the magnitude of natural numbers (see also Jacob and Nieder 2009).

Meert et al.'s (2009, 2010) studies marked the beginning of a departure from a rigid debate about a single mental representation for fractions, claiming that these representations may be hybrid: a holistic approach would be used when a rational number and its magnitude are well known (e.g., by means of extensive practice), whereas fractions that are less familiar or more complex would elicit componential processing. The field slowly moved from an initial focus on discovering the *nature* of fractions' mental representations to investigating the *conditions* in which one type of processing or the other is preferred. Gabriel et al. (2013) asked adult participants to decide whether pairs of fractions were *numerically* equal or *visually* equal. For example, 1/2 and 1/2 are numerically and visually equal, 1/2 and 2/4 are numerically equal but visually different, and 1/2 an 3/5 are different numerically and visually. These researchers reported distance effects for the comparison of fractions according to their numerical magnitude, but not according to their visual appearance, suggesting that the numerical magnitude of fractions is not automatically accessed but only accessed when required by the task.

The evidence against automatic processing of fractions was accumulating, but no explanations were given. Kallai and Tzelgov (2012b) carried out a study where adult participants were trained to associate arbitrary figures with unit fractions. After training, they asked participants to perform a physical comparison task with these fractions, using the traditional notation and the newly learned symbols in separate moments. Their results confirmed that standard fractions were not automatically processed according to their holistic magnitude; in contrast, the newly learned symbols elicited automatic processing. These results highlight a possible role of the usual mathematical notation in people's struggle with fractions, that is, the high familiarity and automaticity of processing of natural numbers impairs the development of a similar type of processing for the fractions' magnitudes.

Another line of research has focused on the development of rational numbers' concept of magnitude. It has been suggested that, for children to learn fractions and rational numbers, a conceptual reorganization is needed (Stafylidou and Vosniadou 2004; Vamvakoussi et al. 2013). The school mathematics curriculum usually provides extensive practice with natural numbers before introducing rational numbers, and hence the intuitions developed by children in the context of natural numbers might interfere with their later learning of rationals. Ni and Zhou (2005) introduced this problem as a *whole number bias* or *natural number bias* affecting children's reasoning in the initial stages of learning (e.g., Gómez and Dartnell 2019; Stafylidou and Vosniadou 2004; Van Hoof et al. 2013). Later studies suggested that this bias persists into adulthood and might affect even mathematics experts' intuitive responses (Obersteiner et al. 2013; but see Morales et al. 2020). This natural number bias would be manifest in many aspects of rational number knowledge, such as numerical magnitude, arithmetic operations, representations, and density (Van Dooren et al. 2015; Van Hoof et al. 2015).

With respect to numerical magnitude, rational numbers behave according to principles different from those of natural numbers. A fraction composed of large natural numbers (e.g., 4440/5328) does not necessarily have a large magnitude in itself. Many children fail to understand fraction magnitude. A classic example was documented by Reys et al. (1982): when asked to estimate the value of the outcome of 12/13 + 7/8, a large share of children selected 19 or 21 as the answer, indicating that they reasoned about the fractions by naively adding their components. In opposition to focusing on numerical magnitude, children and adults may deploy a variety of strategies, such as component-wise comparison, benchmarking, and gap thinking to solve problems about fractions (Clarke and Roche 2009; Gómez and Dartnell 2019; Obersteiner and Tumpek 2016; Obersteiner et al. 2020). All these strategies involve fraction magnitude to different extents.

Several studies have looked into the predictive power of rational number knowledge in mathematics achievement (Booth and Newton 2012; Siegler et al. 2012; Torbeyns et al. 2015). Overall, their results point to the crucial role of understanding fraction magnitude for future achievement in advanced mathematical topics such as algebra.

Although our focus in this section has been on fractions, rational numbers can also be represented as decimal numbers. Given the large body of literature that has addressed learners' difficulties with understanding and processing decimal numbers' magnitude (e.g., Resnick et al. 1989, 2019), we cannot delve into this due to space constraints. In addition, it should be noted that there is an emerging line of research on the perception and processing of ratios or proportional relations, understood as being to rational numbers what nonsymbolic quantities are to natural numbers. Lewis et al. (2016) and Jacob et al. (2012) argue that the mind and brain have specialized mechanisms to process ratios akin to those for processing natural numbers. Such mechanisms would challenge long-held views in the field that assume that humans lack basic intuitions on which rational numbers could be grounded (e.g., Gallistel and Gelman 1992; Ni and Zhou 2005).

4.3.3 Numerical Magnitude in Other Number Sets

In the previous sections, we have presented an overview of studies dealing with the numerical magnitude of natural and rational numbers. Although other number sets have been studied as well, most efforts have concentrated in these previous ones. In this last section, we review research on the numerical magnitude of integer (negative) numbers as well as irrational numbers.

4.3.3.1 Integers

The set of natural numbers is not closed under subtraction (e.g., no natural number is the result of an operation such as 4–7), and integers fill this gap by extending the set of natural numbers so that subtraction becomes a closed operation. The set of integers comprises natural numbers, zero, and the negative natural numbers, extending the natural number line to the left. It is important to note that negative numbers challenge our definition of numerical magnitude in a novel manner. Within the domains of natural numbers and (positive) rational numbers, large numbers are always located in the number line to the right side of small numbers. Mathematically, an integer's magnitude is defined as its distance to zero in the number line—its absolute value. Therefore, a negative number can have a larger magnitude than a positive number despite being located to its left (e.g., −5 vs. 2) or, in more general terms, numerical *magnitude* and numerical *ordering* become dissociated. This novel context requires reassessing several concepts. For instance, how should the SNARC effect be hypothesized to extend to integers? The spatial dimension might become associated with numerical magnitude or with numerical ordering, leading to opposite conclusions.

Fischer (2003), Shaki and Petrusic (2005), and Fischer and Rottmann (2005) made early attempts at studying negative numbers and how they are mentally processed. Fischer (2003), as well as Shaki and Petrusic (2005), investigated whether negative numbers were represented in the same mental number line as natural numbers by asking adults to compare pairs of integers and looking for a SNARC effect.

When the task presented negative and positive numbers in a mixed manner, responses were quicker for negative numbers with the left hand and for positive numbers with the right hand, implying a standard SNARC effect with respect to number line ordering. Shaki and Petrusic (2005) also included a condition in which participants answered only to pairs of negative numbers, and in this case, they observed a different pattern: negative numbers closer to zero were associated with quicker left-hand responses, and negative numbers farther from zero were associated with quicker right-hand responses. This outcome can be interpreted in two ways: as a *reverse* SNARC effect in terms of number line ordering, since negative numbers closer to zero are located on the number line to the right side of negative numbers farther from zero; or as a *standard* SNARC effect with respect to the magnitude of negative numbers, since negative numbers closer to zero have a smaller magnitude than negative numbers farther from zero. This result, therefore, suggests that the mental representation of negative numbers is not automatically associated with the left-side portion of the mental number line, which implies that the mental processing of negative numbers depends on the experimental task and items (similarly to the research on rational numbers).

Fischer and Rottmann (2005) further investigated the processing of negative numbers, focusing specifically on automaticity. Following the lead of the original SNARC study (Dehaene et al. 1993), they asked participants to classify positive and negative integers as even or odd. Results supported the idea that numerical magnitude (absolute value) was automatically activated, but that it was not so for numerical ordering. This suggests a componential mental representation for negative numbers: while their magnitude is automatically accessed, their sign (or polarity) must be later integrated with magnitude. Tzelgov et al. (2009) investigated if this component-based processing was due to the presentation of magnitude and sign via independent symbols by replacing the negative sign with color. Their results indicated that, still, negative numbers were processed with disintegrated magnitude and sign. The researchers carried out an additional test, replacing the number symbols with Japanese letters (unfamiliar to the participants) so that the magnitude and sign information were not associated to different portions of the symbol but rather to each symbol as a whole. Even in this new setting, participants did not display automatic processing of numerical ordering, suggesting that the componential nature of integer representations is not due to the specific standard notation used for them. Negative numbers seem to be initially processed based on their magnitude, and their sign is later integrated when relevant.

There is also research on children's learning of integer number ordering, specifically using number line tasks (Young and Booth 2015; Brez et al. 2015). Overall, it has been observed that response times and accuracy for negative numbers are lower, indicating that children probably process negative numbers via strategies different from those for positive numbers.

4.3.3.2 Irrational Numbers

Irrational numbers challenge learners from their very definition. Unlike the previous number sets, irrational numbers are defined by what they are *not*: they are all real numbers that are not rational, that is to say, all real numbers that cannot be expressed as the ratio of two integers. Whereas some irrational numbers can be interpreted in terms of measurement (e.g., $\sqrt{2}$ as the length of the diagonal of a unit square, and π as the length of the contour of a circle of radius 1/2), many others are defined in algebraic terms (e.g., ϕ is the larger solution of $x^2 = x + 1$), and others even lack an algebraic characterization (e.g., e cannot be expressed as a root of a polynomial with integer coefficients). The only common ground across irrational numbers seems to be the fact that, when written in decimal notation, they are infinitely long and non-periodic (e.g., Chapernowne's number, obtained by concatenating all natural numbers as 0.123456789101112131415…). Hence, the mathematical interest in irrational numbers stems more from their algebraic and analytical properties than from their magnitude.

Interpreting the ability to estimate the magnitude of number symbols as crucial for number sense, Obersteiner and Hofreiter (2017) asked if skilled mathematicians would be able to automatically access a mental representation of irrational numbers' magnitudes. In a comparison task, participants were asked to select the largest of two positive irrational numbers as quickly as possible. The task included number pairs such as $\sqrt[3]{21}$ and $\sqrt[3]{14}$. The results showed that mathematicians struggled to utilize numerical magnitude, resorting instead to strategies specific to the number pairs. Another study about irrational numbers is the one by Patel and Varma (2018). They asked undergraduate students to compare pairs of square roots such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, and $\sqrt{9}$. Additionally, participants completed a number line estimation task in four blocks: natural numbers, roots of single-digit numbers, roots of perfect square numbers, and roots of two-digit numbers, and finally, they completed a knowledge test about irrational numbers. Their results showed, similarly to Obersteiner and Hofreiter's (2017) work, that students tended to compare these radical expressions by focusing on their subradical components (e.g., $\sqrt{3} > \sqrt{2}$ because $3 > 2$). For the number line task, students also used strategies such as comparison against a perfect square. As pointed out by Patel and Varma (2018), it seems likely that the relevance of numerical magnitude representations diminishes as the abstraction of numbers increases.

4.4 Final Remarks

This chapter aimed at providing a general introduction to the processing of numerical magnitude to early researchers, as it is an entry point to the field of numerical cognition. Because of space constraints, several important topics have been left out, such as the effect of individual differences in task performance on mathematics achievement.

All the research described herein investigated number sets contained within the set of real numbers, where numerical magnitude is intimately related to ordinality (although the link is not perfect, as evidenced by negative numbers). Nonetheless, ordinality is not necessary for a well-defined concept of numerical magnitude, as it is the case of complex numbers. There is a lack of research of numerical magnitude in such settings, which needs researchers and mathematics educators to work together to define interesting research questions. In contrast, a line of research on the mental mechanisms underlying number ordinality has emerged in recent years (see Lyons et al. 2016, for a review), revealing interesting differences with numerical magnitude, such as a reversal of the distance effect (i.e., the numerically closer two numbers are, the easier it is to judge if they are in ascending order).

Numerical magnitude has been proposed as a crucial understanding in the study of numbers at school (Fazio et al. 2014; Siegler et al. 2012). While that certainly seems to be the case for natural and rational numbers, it is unclear if this statement holds for more advanced number sets such as negative, irrational, and complex numbers, where the focus shifts toward more algebraic properties of numbers. In this sense, the development of intuitions for progressively more advanced number systems might partly account for the relations observed between proficiency in rational number knowledge and later achievement in algebra (Booth and Newton 2012; Siegler et al. 2012; Torbeyns et al. 2015). The shift from a concrete, measure-based conception of number toward an algebraic one has traditionally challenged school children and, now, also researchers in numerical cognition.

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