Chapter 6
A Hybrid High-Order Method for Multiple-Network Poroelasticity

Lorenzo Botti, Michele Botti, and Daniele A. Di Pietro

Abstract We develop Hybrid High-Order methods for multiple-network poroelasticity, modelling seepage through deformable fissured porous media. The proposed methods are designed to support general polygonal and polyhedral elements. This is a crucial feature in geological modelling, where the need for general elements arises, e.g., due to the presence of fracture and faults, to the onset of degenerate elements to account for compaction or erosion, or when nonconforming mesh adaptation is performed. We use as a starting point a mixed weak formulation where an additional total pressure variable is added, that ensures the fulfilment of a discrete inf-sup condition. A complete theoretical analysis is performed, and the results are demonstrated on a panel of numerical tests.

Keywords Hybrid High-Order methods · Discontinuous Galerkin methods · Polytopal methods · Multi-network poroelasticity · Barenblatt-Biot equations

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L. Botti
Department of Engineering and Applied Sciences, University of Bergamo, Bergamo, Italy
e-mail: lorenzo.botti@unibg.it

M. Botti
MOX-Laboratory for Modeling and Scientific Computing, Dipartimento di Matematica, Politecnico di Milano, Milan, Italy
e-mail: michele.botti@polimi.it

D. A. Di Pietro (✉)
IMAG, University of Montpellier, CNRS, Montpellier, France
e-mail: daniele.di-pietro@umontpellier.fr

D. A. Di Pietro et al. (eds.), *Polyhedral Methods in Geosciences*,
SEMA SIMAI Springer Series 27,
https://doi.org/10.1007/978-3-030-69363-3_6

6.1 Introduction

In this work, we develop and analyse Hybrid High-Order (HHO) methods for the multiple-network poroelastic problem.

In the standard quasi-static poroelasticity theory [18], the medium is modelled as a continuous superposition of solid and fluid phases. The corresponding set of equations, named after Biot in recognition of his pioneering contributions [7, 8], result from the balances of force and mass. Specifically, mechanical equilibrium is assumed, with the total stress tensor decomposed into one contribution due to the strain of the porous matrix and one due to the pore pressure; see [32]. A standard description of the flow, on the other hand, is obtained combining the mass balance with the Darcy law. This simplified description can fail to capture physically relevant phenomena in fissured media. A modification of the Darcy model accounting for the simultaneous presence of pore and fissure networks was originally proposed by Barenblatt et al. in [4] for the rigid case. Plugging this description into the Biot model gives raise to the so-called Barenblatt–Biot equations. These ideas can be naturally extended to M porous networks, finding applications, e.g., in the modelling of the interactions between biological fluids and tissue; see, e.g, [33]. A different extension of the Biot model is considered in Chap. 4, where thermal effects are incorporated into a single network model.

In the context of computational geosciences, the use of discretisation methods that support general polytopal meshes and, possibly, high-order has been recently advocated by several authors; see, e.g., [2, 3, 6, 15–17, 27, 31] and references therein. The support of polyhedral meshes enables, e.g., a seamless treatment of degenerate elements which may arise due to erosion or compaction in corner-point descriptions of petroleum basins, of non-matching interfaces across fractures or faults, and of non-conforming mesh refinement or agglomeration [5]. High-order methods, on the other hand, typically lead to a better usage of computational resources than low-order methods whenever the solution exhibits sufficient (local) regularity or mesh adaptation is available.

Our focus is here on a specific family of polytopal discretisations, HHO methods. Originally introduced in [23] in the context of linear elasticity, HHO methods rely on two key ingredients: *local reconstructions* obtained by solving small, embarrassingly parallel problems inside each element and *stabilisation terms* that penalise, inside each element, residuals designed so as to preserve optimal approximation properties. A general and up-to-date overview of HHO methods can be found in the recent monograph [22]. Hybrid High-Order methods are linked to the hybridized version of the Mixed Virtual Element methods considered in Chaps. 7 and 8; see [1, 24] and also [22, Sects. 5.4 and 5.5]. Concerning their application to poroelasticity, we can cite, in particular: the HHO-Discontinuous Galerkin method for the Biot problem proposed and analysed in [9], based in turn on the methods of [23] for the mechanics and [25] for the flow; its extension to nonlinear elastic laws proposed in [14], where the mechanical term is discretised according to [13]; its application to the treatment of stochastic coefficients considered in [12] in conjunction with Polynomial Chaos techniques. An abstract analysis framework covering general schemes for the linear Biot problem in fully discrete formulation (cf. [20]) has been recently proposed in [10] including, in particular, a variation of the method of [9] where also the flow equation is discretised in the HHO spirit. Other applications of HHO methods to problems in geosciences include flows in fractured porous media [16, 17] and miscible fluid flows in porous media [2].

The method proposed in the present work uses as a starting point a mixed formulation inspired by [30], where an additional total pressure variable is introduced that accounts for the pore and mechanical pressures. Given an integer polynomial degree $k \geq 0$, the discretisation of the mechanical term in the equilibrium equation follows [13] if $k \geq 1$ and [12] if $k = 0$. This choice induces a natural discretisation for the total pressure in the space of broken polynomials of total degree $\leq k$, which ensures inf-sup stability. As it has been done in [10], we consider two different discretisations of the Darcy term in the mass balance equations (enforcing mass conservation in each pore network). The first scheme is based on the HHO method of [26], so the discrete unknowns for the pore pressures are located both at elements and faces. The second scheme is obtained by using the Discontinuous Galerkin (DG) method of [25]. In both cases, the linear exchange terms as well as the porosity are discretised using element unknowns only. The resulting methods have several appealing features: they support general polytopal meshes and high-order; they can be applied to an arbitrary number $M \geq 1$ of pore networks; they are well-behaved for quasi-incompressible porous matrices; they deliver an L^2-error estimate for the total pressure robust in the entire range of geophysical parameters.

From the practical standpoint, a relevant difference between the two schemes is that the HHO-HHO version can benefit from static condensation, leading to linear systems where the only globally coupled unknowns are displacement and pore pressure at faces, and global pressures at elements. On typical meshes, this results in fewer unknowns compared to the HHO-DG scheme and better computational efficiency, particularly in three space dimensions; see, e.g., the numerical tests on meshes with planar faces in [11]. On the other hand, the HHO-DG scheme may be easier to implement, as it does not require the introduction of pore pressures at faces, nor the computation of local pore pressure reconstructions or static condensation. From the theoretical point of view, the analysis of the HHO-DG scheme requires elliptic regularity (in Theorem 6.2, the convexity of the domain is assumed) to achieve optimal orders of convergence. As pointed out in [10], this is not the case for the HHO-HHO scheme. In this paper, we focus on the HHO-DG scheme for the numerical tests of Sect. 6.5, and postpone a comparison with the HHO-HHO scheme to a future work.

The rest of this paper is organised as follows. In Sect. 6.2 we establish the continuous setting and state the multiple-network poroelasticity problem in weak formulation. Section 6.3 describes the discrete setting and contains the statement of the discrete problem. The analysis of the method is carried out in Sect. 6.4 focusing, for the sake of simplicity, on the HHO-HHO variant. The pivotal result is here an a priori estimate for an abstract problem whose purpose is twofold: when applied to the HHO scheme, it yields its well-posedness; when applied to the error equations, it establishes a basic error estimate. Finally, Sect. 6.5 contains a thorough numerical validation of the method.

6.2 Continuous Setting

In what follows, given an open bounded set $X \subset \mathbb{R}^d$, we denote by $(\cdot, \cdot)_X$ the usual scalar product of $L^2(X; \mathbb{R})$, $L^2(X; \mathbb{R}^d)$, or $L^2(X; \mathbb{R}^{d\times d})$, according to the context. When $X = \Omega$, the subscript is omitted. Given a vector space V and two real numbers $a < b$, we additionally denote by $C^0([a, b]; V)$ the spaces of continuous V-valued functions of time on $[a, b]$ and by $H^m(a, b; V)$ the space of V-valued functions that are square-integrable along with their derivatives up to the m-th on (a, b), equipped with the usual norms.

We consider the evolution over a finite time $t_{\rm F} > 0$ of a porous medium which, in its reference configuration, occupies a fixed region of space $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, and hosts $M \geq 1$ pore networks. For the sake of simplicity, we assume that Ω is a polygon or a polyhedron, so that it can be covered exactly by a spatial mesh made of polygonal or polyhedral elements. Denote by $\mu > 0$ and $\lambda \geq 0$ the Lamé parameters of the matrix and, for any $i \in [\![1, M]\!]$, by $C_i \geq 0$, $\alpha_i \in (0, 1]$, and $K_i > 0$, respectively, the constrained specific storage, Biot–Willis, and permeability coefficients of each network. We additionally denote by $\boldsymbol{f} \in H^1(0, t_{\rm F}; L^2(\Omega; \mathbb{R}^d))$ a volumetric force and, for any $i \in [\![1, M]\!]$, by $g_i \in C^0([0, t_{\rm F}]; L^2(\Omega; \mathbb{R}))$ a source term for the ith pore network. The above physical parameters and forcing terms will be collectively referred to as the *problem data*.

Let $\boldsymbol{U} := H^1_0(\Omega; \mathbb{R}^d)$, $P_0 := \left\{q \in L^2(\Omega; \mathbb{R}) \ : \ \int_\Omega q = 0\right\}$, and, for all $i \in [\![1, M]\!]$, $P_i := H^1_0(\Omega; \mathbb{R})$. We also set, for the sake of brevity, $\boldsymbol{\alpha} := (1, \alpha_1, \ldots, \alpha_M) \in \mathbb{R}^{M+1}$ and, denoting by p_0 the total pressure field and, for any $i \in [\![1, M]\!]$, by p_i the pressure field in the ith porous network, $\boldsymbol{p} := (p_0, p_1, \ldots, p_M)$. We consider a weak formulation inspired by (but not coincident with) the one considered in [30]: Find the displacement $\boldsymbol{u} \in C^0([0, t_{\rm F}]; \boldsymbol{U})$, the total pressure $p_0 \in H^1(0, t_{\rm F}; P_0)$ and, for all $i \in [\![1, M]\!]$, the ith pore network pressure $p_i \in C^0([0, t_{\rm F}]; P_i) \cap H^1(0, t_{\rm F}; L^2(\Omega; \mathbb{R}))$ such that it holds, for almost every $t \in (0, t_{\rm F}]$, all $\boldsymbol{v} \in \boldsymbol{U}$, all $q_0 \in P_0$, and all $q_i \in P_i$, $i \in [\![1, M]\!]$,

$$
2\mu\, a(\boldsymbol{u}(t), \boldsymbol{v}) + b(\boldsymbol{v}, p_0(t)) = (\boldsymbol{f}(t), \boldsymbol{v}) \tag{6.1a}
$$

$$
b(\boldsymbol{u}(t), q_0) - \lambda^{-1}(\boldsymbol{\alpha}\cdot\boldsymbol{p}, q_0) = 0, \tag{6.1b}
$$

$$
(\mathrm{d}_t\psi_i(\boldsymbol{p}(t)), q_i) + (S_i(\boldsymbol{p}(t)), q_i) + K_i\, c(p_i, q_i) = (g_i(t), q_i) \quad \forall i \in [\![1, M]\!], \tag{6.1c}
$$

where we have set, for all $i \in [\![1, M]\!]$ and all $\boldsymbol{q} \in \mathbb{R}^{M+1}$,

$$
\psi_i(\boldsymbol{q}) := C_i q_i + \alpha_i \lambda^{-1} \boldsymbol{\alpha}\cdot\boldsymbol{q}, \tag{6.2}
$$

and we have introduced the bilinear forms $a : \boldsymbol{U} \times \boldsymbol{U} \to \mathbb{R}$, $b : \boldsymbol{U} \times P_0 \to \mathbb{R}$, and $c : H^1(\Omega; \mathbb{R}) \times H^1(\Omega; \mathbb{R}) \to \mathbb{R}$ such that, for all $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{U}$, all $q_0 \in P_0$, and all $r, q \in H^1(\Omega; \mathbb{R})$,

$$a(\boldsymbol{w}, \boldsymbol{v}) := (\nabla_{\mathrm{s}}\boldsymbol{w}, \nabla_{\mathrm{s}}\boldsymbol{v}), \qquad b(\boldsymbol{v}, q_0) := (\nabla\cdot\boldsymbol{v}, q_0), \qquad c(r, q) := (\nabla r, \nabla q). \tag{6.3}$$

In the expression of the bilinear form a, ∇_{s} denotes the symmetric part of the gradient applied to vector fields. In (6.1b), the exchange term is expressed by the function $S_i : \mathbb{R}^{M+1} \to \mathbb{R}$ such that, for any $\boldsymbol{q} \in \mathbb{R}^{M+1}$,

$$S_i(\boldsymbol{q}) := \sum_{j=1}^{M} \xi_{i\leftarrow j}(q_i - q_j),$$

where $\{\xi_{i\leftarrow j} \ : \ i, j \in [\![1, M]\!]\}$ is a family of nonnegative real numbers such that $\xi_{i\leftarrow j} = \xi_{j\leftarrow i}$ for all $i, j \in [\![1, M]\!]$. We assume that the initial pressures $p_i^0 \in P_i$, $i \in [\![0, M]\!]$, are given, so that an initial equilibrium displacement $\boldsymbol{u}^0 \in \boldsymbol{U}$ can be computed from (6.1a).

6.3 Discrete Setting

6.3.1 Space and Time Meshes

We consider spatial meshes corresponding to couples $\mathcal{M}_h := (\mathcal{T}_h, \mathcal{F}_h)$, where $\mathcal{T}_h$ is a finite collection of polyhedral elements such that $h := \max_{T\in\mathcal{T}_h} h_T > 0$ with h_T denoting the diameter of T, while $\mathcal{F}_h$ is a finite collection of planar faces. It is assumed henceforth that the mesh $\mathcal{M}_h$ matches the geometrical requirements detailed in [22, Definition 1.4]. This covers, essentially, any reasonable partition of Ω into polyhedral sets, not necessarily convex.

For every mesh element $T \in \mathcal{T}_h$, we denote by $\mathcal{F}_T$ the subset of $\mathcal{F}_h$ containing the faces that lie on the boundary ∂T of T. For any mesh element $T \in \mathcal{T}_h$ and each face $F \in \mathcal{F}_T$, $\boldsymbol{n}_{TF}$ is the constant unit vector normal to F pointing out of T. Boundary faces lying on $\partial\Omega$ and internal faces contained in Ω are collected in the sets $\mathcal{F}_h^{\mathrm{b}}$ and $\mathcal{F}_h^{\mathrm{i}}$, respectively. For any $F \in \mathcal{F}_h^{\mathrm{i}}$, we denote by T_1 and T_2 the elements of $\mathcal{T}_h$ such that $F \subset \partial T_1 \cap \partial T_2$. The numbering of T_1 and T_2 is arbitrary but fixed once and for all, and we set $\boldsymbol{n}_F := \boldsymbol{n}_{T_1 F}$.

Our focus being on the h-convergence analysis, we consider a sequence of refined polygonal or polyhedral meshes that is regular in the sense of [22, Definition 1.9]. This implies, in particular, that the diameter h_T of a mesh element $T \in \mathcal{T}_h$ is comparable to the diameter h_F of each face $F \in \mathcal{F}_T$ uniformly in h, and that the number of faces in $\mathcal{F}_T$ is bounded above by an integer N_∂ independent of h; see [22, Lemma 1.12]. In order to have the stability of the bilinear form discretising the mechanical term when discrete unknowns are polynomials of degree $k \geq 1$, we will further assume that every element $T \in \mathcal{T}_h$ is star-shaped with respect to every point of a ball of diameter uniformly comparable to h_T. This assumption ensures, in particular, that uniform local Korn inequalities hold inside each element; cf. the Appendix of [11] and also [22, Chap. 7].

The time mesh is obtained subdividing $[0, t_{\rm F}]$ into $N \in \mathbb{N}^*$ uniform subintervals. We introduce the timestep $\tau := t_{\rm F}/N$ and the discrete times $t^n := n\tau$, $n \in [\![0, N]\!]$.

For all $n \in [\![1, N]\!]$ and all $\varphi \in C^0([0, t_{\rm F}]; V)$ we let, for the sake of brevity,

$$\varphi^n := \varphi(t^n)$$

and define the discrete backward time derivative operator $\delta_t^n : C^0([0, t_{\rm F}]; V) \to V$ at time n as

$$\delta_t^n \varphi := \frac{\varphi^n - \varphi^{n-1}}{\tau}. \tag{6.4}$$

Denoting by $(\cdot, \cdot)_V$ an inner product in V with associated norm $\|\cdot\|_V$, and letting $\varphi \in H^1(0, t_{\rm F}; V)$, it holds

$$\sum_{n=1}^{N} \tau \|\delta_t^n \varphi\|_V^2 \leq \|\varphi\|_{H^1(0,t_{\rm F};V)}^2. \tag{6.5}$$

6.3.2 Local and Broken Spaces and Projectors

Let a polynomial degree $l \geq 0$ be fixed. For all $X \in \mathcal{T}_h \cup \mathcal{F}_h$, denote by $\mathbb{P}^l(X; \mathbb{R})$ the space spanned by the restriction to X of d-variate polynomials of total degree $\leq l$, and let $\pi_X^l : L^1(X; \mathbb{R}) \to \mathbb{P}^l(X; \mathbb{R})$ be the corresponding L^2-orthogonal projector such that, for any $v \in L^1(X; \mathbb{R})$,

$$(\pi_X^l v - v, w)_X = 0 \qquad \forall w \in \mathbb{P}^l(X; \mathbb{R}).$$

Denoting by $m \geq 1$ an integer, the vector version $\boldsymbol{\pi}_X^l : L^1(X; \mathbb{R}^m) \to \mathbb{P}^l(X; \mathbb{R}^m)$, is obtained applying π_X^l component-wise. We will also need, in what follows, the space of $d \times d$ symmetric matrix-valued fields with polynomial entries, denoted by $\mathbb{P}^l(T; \mathbb{R}^{d\times d}_{\rm sym})$.

At the global level, we introduce the broken polynomial space

$$\mathbb{P}^l(\mathcal{T}_h; \mathbb{R}) := \left\{ v \in L^1(\Omega; \mathbb{R}) \ : \ v_{|T} \in \mathbb{P}^l(T; \mathbb{R}) \quad \forall T \in \mathcal{T}_h \right\},$$

the corresponding vector version $\mathbb{P}^l(\mathcal{T}_h; \mathbb{R}^d)$, and the space $\mathbb{P}^l(\mathcal{T}_h; \mathbb{R}^{d\times d}_{\rm sym})$ of $d \times d$ symmetric matrix-valued fields with broken polynomial entries. The L^2-orthogonal projector on $\mathbb{P}^l(\mathcal{T}_h; \mathbb{R})$ is $\pi_h^l : L^1(\Omega; \mathbb{R}) \to \mathbb{P}^l(\mathcal{T}_h; \mathbb{R})$ such that, for all $v \in L^1(\Omega; \mathbb{R})$,

$$(\pi_h^l v)_{|T} = \pi_T^l v_{|T} \qquad \forall T \in \mathcal{T}_h. \tag{6.6}$$

Broken polynomial spaces constitute special instances of the broken Sobolev spaces $H^m(\mathcal{T}_h; \mathbb{R}) := \{v \in L^2(\Omega; \mathbb{R}) \ : \ v_{|T} \in H^m(T; \mathbb{R}) \quad \forall T \in \mathcal{T}_h\}$, which will be used, along with their vector-valued counterparts, to express the regularity requirements on the exact solution in the error estimate of Theorems 6.1 and 6.2. For any function $v \in H^1(\mathcal{T}_h; \mathbb{R})$ we define, for all $F \in \mathcal{F}_h^{\mathrm{i}}$, the jump operator such that

$$[v]_F := v_{|T_1} - v_{|T_2},$$

where we remind the reader that T_1 and T_2 are the mesh elements that share F as a face, taken in an arbitrary but fixed order. On boundary faces, the jump operator simply returns the trace of its argument on $\partial\Omega$.

6.3.3 *Discrete Spaces and Reconstructions*

To formulate the discrete problem, we need scalar and vector HHO spaces. From this point on, we let an integer $k \geq 0$ be fixed, corresponding to the polynomial degrees of the discrete unknowns.

6.3.3.1 Scalar HHO Space and Pressure Reconstruction

The scalar HHO space, that will be used to discretise network pressures in the HHO-HHO scheme (6.23), is

$$\underline{Q}_h^k := \Big\{ \underline{q}_h = ((q_T)_{T\in\mathcal{T}_h}, (q_F)_{F\in\mathcal{F}_h}) \ : \\ q_T \in \mathbb{P}^k(T; \mathbb{R}) \text{ for all } T \in \mathcal{T}_h \text{ and } q_F \in \mathbb{P}^k(F; \mathbb{R}) \text{ for all } F \in \mathcal{F}_h \Big\}.$$

The interpolator $\underline{I}_h^k : H^1(\Omega; \mathbb{R}) \to \underline{Q}_h^k$ is defined setting, for all $q \in H^1(\Omega; \mathbb{R})$,

$$\underline{I}_h^k q := \big((\pi_T^k q_{|T})_{T\in\mathcal{T}_h}, (\pi_F^k q_{|F})_{F\in\mathcal{F}_h}\big).$$

For all $\underline{q}_h \in \underline{Q}_h^k$, we define the broken polynomial function $q_h \in \mathbb{P}^k(\mathcal{T}_h; \mathbb{R})$ obtained patching element unknowns, that is,

$$(q_h)_{|T} := q_T \qquad \forall T \in \mathcal{T}_h.$$

For any element $T \in \mathcal{T}_h$, we denote by $\underline{Q}_T^k$ the restriction of $\underline{Q}_h^k$ to T, and we introduce the pressure reconstruction $\mathrm{r}_T^{k+1} : \underline{q}_T^k \to \mathbb{P}^{k+1}(T; \mathbb{R})$ such that, for all $\underline{q}_T \in \underline{Q}_T^k$,

$$(\nabla \mathsf{r}_T^{k+1} \underline{q}_T, \nabla w)_T = -(q_T, \Delta w)_T + \sum_{F \in \mathcal{F}_T} (q_F, \nabla w \cdot \boldsymbol{n}_{TF})_F \quad \forall w \in \mathbb{P}^{k+1}(T; \mathbb{R}),$$
$$\int_T \mathsf{r}_T^{k+1} \underline{q}_T = \int_T q_T.$$

The global pressure reconstruction operator $\mathsf{r}_h^{k+1} : \underline{Q}_h^k \to \mathbb{P}^{k+1}(\mathcal{T}_h; \mathbb{R})$ is obtained patching the local ones: For all $\underline{q}_h \in \underline{Q}_h^k$,

$$(\mathsf{r}_h^{k+1} \underline{q}_h)_{|T} := \mathsf{r}_T^{k+1} \underline{q}_T \qquad \forall T \in \mathcal{T}_h.$$

6.3.3.2 Vector HHO Space, Strain, and Displacement Reconstructions

The vector HHO space, that will be used to discretise the displacement, is

$$\underline{\boldsymbol{V}}_h^k := \Big\{ \underline{\boldsymbol{v}}_h = ((\boldsymbol{v}_T)_{T \in \mathcal{T}_h}, (\boldsymbol{v}_F)_{F \in \mathcal{F}_h}) \ : \\ \boldsymbol{v}_T \in \mathbb{P}^k(T; \mathbb{R}^d) \text{ for all } T \in \mathcal{T}_h \text{ and } \boldsymbol{v}_F \in \mathbb{P}^k(F; \mathbb{R}^d) \text{ for all } F \in \mathcal{F}_h \Big\}.$$

For all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{V}}_h^k$, we let $\boldsymbol{v}_h \in \mathbb{P}^k(\mathcal{T}_h; \mathbb{R}^d)$ be such that

$$(\boldsymbol{v}_h)_{|T} := \boldsymbol{v}_T \qquad \forall T \in \mathcal{T}_h.$$

The interpolator $\underline{\boldsymbol{I}}_h^k : H^1(\Omega; \mathbb{R}^d) \to \underline{\boldsymbol{V}}_h^k$ is such that, for any $\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d)$,

$$\underline{\boldsymbol{I}}_h^k \boldsymbol{v} := \big((\boldsymbol{\pi}_T^k \boldsymbol{v}_{|T})_{T \in \mathcal{T}_h}, (\boldsymbol{\pi}_F^k \boldsymbol{v}_{|F})_{F \in \mathcal{F}_h} \big).$$

For any element $T \in \mathcal{T}_h$, we denote by $\underline{\boldsymbol{V}}_T^k$ the restriction of $\underline{\boldsymbol{V}}_h^k$ to T and we introduce the strain reconstruction $\mathbf{E}_T^k : \underline{\boldsymbol{V}}_T^k \to \mathbb{P}^k(T; \mathbb{R}^{d \times d}_{\text{sym}})$ such that, for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{V}}_T^k$,

$$(\mathbf{E}_T^k \underline{\boldsymbol{v}}_T, \boldsymbol{\tau})_T = -(\boldsymbol{v}_T, \nabla \cdot \boldsymbol{\tau})_T + \sum_{F \in \mathcal{F}_T} (\boldsymbol{v}_F, \boldsymbol{\tau} \boldsymbol{n}_{TF})_F \qquad \forall \boldsymbol{\tau} \in \mathbb{P}^k(T; \mathbb{R}^{d \times d}_{\text{sym}}).$$

For any $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{V}}_T^k$, we reconstruct from $\mathbf{E}_T^k \underline{\boldsymbol{v}}_T$ a high-order displacement $\mathbf{r}_T^{k+1} \underline{\boldsymbol{v}}_T \in \mathbb{P}^{k+1}(T; \mathbb{R}^d)$ enforcing the following conditions:

$$(\nabla_{\text{s}} \mathbf{r}_T^{k+1} \underline{\boldsymbol{v}}_T - \mathbf{E}_T^k \underline{\boldsymbol{v}}_T, \nabla_{\text{s}} \boldsymbol{w})_T = 0 \qquad \forall \boldsymbol{w} \in \mathbb{P}^{k+1}(T; \mathbb{R}^d),$$
$$\int_T \mathbf{r}_T^{k+1} \underline{\boldsymbol{v}}_T = \int_T \boldsymbol{v}_T, \text{ and } \int_T \nabla_{\text{ss}} \mathbf{r}_T^{k+1} \underline{\boldsymbol{v}}_T = \frac{1}{2} \sum_{F \in \mathcal{F}_T} \int_F (\boldsymbol{v}_F \otimes \boldsymbol{n}_{TF} - \boldsymbol{n}_{TF} \otimes \boldsymbol{v}_F),$$

where ∇_{ss} denotes the skew-symmetric part of the gradient applied to vector fields. The global strain and displacement reconstructions $\mathbf{E}_h^k : \underline{\boldsymbol{V}}_h^k \to \mathbb{P}^k(\mathcal{T}_h; \mathbb{R}^{d \times d}_{\text{sym}})$ and

$\mathbf{r}_h^{k+1} : \underline{\boldsymbol{V}}_h^k \to \mathbb{P}^{k+1}(\mathcal{T}_h; \mathbb{R}^d)$ are obtained setting, for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{V}}_h^k$,

$$(\mathbf{E}_h^k \underline{\boldsymbol{v}}_h)_{|T} := \mathbf{E}_T^k \underline{\boldsymbol{v}}_T \text{ and } (\mathbf{r}_h^{k+1} \underline{\boldsymbol{v}}_h)_{|T} := \mathbf{r}_T^{k+1} \underline{\boldsymbol{v}}_T \text{ for all } T \in \mathcal{T}_h.$$

We also define a global divergence reconstruction $\mathsf{D}_h^k : \underline{\boldsymbol{V}}_h^k \to \mathbb{P}^k(\mathcal{T}_h; \mathbb{R})$ as the trace of $\mathbf{E}_h^k$, that is, for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{V}}_h^k$,

$$\mathsf{D}_h^k \underline{\boldsymbol{v}}_h := \operatorname{tr}(\mathbf{E}_h^k \underline{\boldsymbol{v}}_h).$$

6.3.3.3 Displacement and Pressure Spaces

The discrete spaces for the displacement including the strongly enforced homogeneous boundary conditions and for the total pressure including the zero-average condition are, respectively:

$$\underline{\boldsymbol{U}}_h^k := \left\{ \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{V}}_h^k \ : \ \boldsymbol{v}_F = \boldsymbol{0} \text{ for all } F \in \mathcal{F}_h^{\mathrm{b}} \right\} \text{ and } P_{h,0}^k := \mathbb{P}^k(\mathcal{T}_h; \mathbb{R}) \cap P_0,$$

with P_0 defined in Sect. 6.2. When using the HHO method for the discretisation of the flow equations, for any $i \in [\![1, M]\!]$, the space for the ith network pressure is

$$\underline{P}_{h,i}^k := \underline{Q}_{h,\mathrm{D}}^k \text{ with } \underline{Q}_{h,\mathrm{D}}^k := \left\{ \underline{q}_h \in \underline{Q}_h^k \ : \ q_F = 0 \text{ for all } F \in \mathcal{F}_h^{\mathrm{b}} \right\},$$

while, when using the DG method, we use instead

$$P_{h,i}^k := \mathbb{P}^k(\mathcal{T}_h; \mathbb{R}).$$

6.3.4 Discrete Bilinear Forms

We discuss in this section the approximation of the continuous bilinear forms defined in (6.3). In order to alleviate the exposition, from this point on we use the abridged notation $a \lesssim b$ for the inequality $a \leq Cb$ with real number $C > 0$ independent of the meshsize, the time step and, for local inequalities, on the mesh element or face. Further dependencies of the hidden constant will be specified when appropriate.

6.3.4.1 Mechanical Term

The discrete counterpart of the continuous bilinear form a is $\mathrm{a}_h : \underline{\boldsymbol{V}}_h^k \times \underline{\boldsymbol{V}}_h^k \to \mathbb{R}$ such that, for all $\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{V}}_h^k$,

$$\mathrm{a}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) := \begin{cases} (\mathbf{E}_h^k \underline{\boldsymbol{w}}_h, \mathbf{E}_h^k \underline{\boldsymbol{v}}_h) + \mathrm{s}_{\mathrm{a},h}(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) & \text{if } k \geq 1, \\ (\mathbf{E}_h^0 \underline{\boldsymbol{w}}_h, \mathbf{E}_h^0 \underline{\boldsymbol{v}}_h) + \mathrm{s}_{\mathrm{a},h}(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) + \mathrm{j}_h(\mathbf{r}_h^1 \underline{\boldsymbol{w}}_h, \mathbf{r}_h^1 \underline{\boldsymbol{v}}_h) & \text{if } k = 0, \end{cases}$$

with stabilising bilinear form $\mathrm{s}_{\mathrm{a},h} : \underline{\boldsymbol{V}}_h^k \times \underline{\boldsymbol{V}}_h^k \to \mathbb{R}$ and jump penalisation bilinear form $\mathrm{j}_h : H^1(\mathcal{T}_h; \mathbb{R}^d) \times H^1(\mathcal{T}_h; \mathbb{R}^d) \to \mathbb{R}$ such that

$$\mathrm{s}_{\mathrm{a},h}(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) := \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F^{-1} (\boldsymbol{\delta}_{TF}^k \underline{\boldsymbol{w}}_T, \boldsymbol{\delta}_{TF}^k \underline{\boldsymbol{v}}_T)_F \qquad \forall \underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{V}}_h^k,$$

$$\mathrm{j}_h(\boldsymbol{w}, \boldsymbol{v}) := \sum_{F \in \mathcal{F}_h} h_F^{-1} ([\boldsymbol{w}]_F, [\boldsymbol{v}]_F)_F \qquad \forall \boldsymbol{w}, \boldsymbol{v} \in H^1(\mathcal{T}_h; \mathbb{R}^d),$$

where, for all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$, $\boldsymbol{\delta}_{TF}^k \underline{\boldsymbol{v}}_T := \boldsymbol{\pi}_F^k (\mathbf{r}_T^{k+1} \underline{\boldsymbol{v}}_T - \boldsymbol{v}_F) - \boldsymbol{\pi}_T^k (\mathbf{r}_T^{k+1} \underline{\boldsymbol{v}}_T - \boldsymbol{v}_T)$. A discussion on the case $k = 0$, including a justification of the term involving the bilinear form j_h, can be found in [12]; see also [22, Sect. 7.6].

Following [22, Chap. 7], the bilinear form a_h defines an inner product on $\underline{\boldsymbol{U}}_h^k$, and we denote by $\|\cdot\|_{\mathrm{a},h}$ the induced norm. The corresponding dual norm $\|\cdot\|_{\mathrm{a},h,*}$ is defined such that, for any linear form $\ell_h : \underline{\boldsymbol{U}}_h^k \to \mathbb{R}$,

$$\|\ell_h\|_{\mathrm{a},h,*} := \sup_{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k \setminus \{\underline{\boldsymbol{0}}\}} \frac{\ell_h(\underline{\boldsymbol{v}}_h)}{\|\underline{\boldsymbol{v}}_h\|_{\mathrm{a},h}}. \tag{6.7}$$

The following consistency property holds: For all $\boldsymbol{w} \in \boldsymbol{U} \cap H^{k+2}(\mathcal{T}_h; \mathbb{R}^d)$,

$$\|\mathcal{E}_{\mathrm{a},h}(\boldsymbol{w}; \cdot)\|_{\mathrm{a},h,*} \lesssim h^{k+1} |\boldsymbol{w}|_{H^{k+2}(\mathcal{T}_h; \mathbb{R}^d)}, \tag{6.8}$$

where the hidden constant is independent of both h and $\boldsymbol{w}$ and the consistency error linear form $\mathcal{E}_{\mathrm{a},h}(\boldsymbol{w}; \cdot) : \underline{\boldsymbol{U}}_h^k \to \mathbb{R}$ is such that, for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$,

$$\mathcal{E}_{\mathrm{a},h}(\boldsymbol{w}; \underline{\boldsymbol{v}}_h) := -(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{w}, \boldsymbol{v}_h) - \mathrm{a}_h(\underline{\boldsymbol{I}}_h^k \boldsymbol{w}, \underline{\boldsymbol{v}}_h). \tag{6.9}$$

We additionally have the following discrete Korn–Poincaré inequality:

$$\|\boldsymbol{v}_h\|_{L^2(\Omega; \mathbb{R}^d)} \leq C_{\mathrm{K}} \|\underline{\boldsymbol{v}}_h\|_{\mathrm{a},h} \qquad \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k, \tag{6.10}$$

where the real number $C_{\mathrm{K}} > 0$ is independent of h, but possibly depends on Ω, d, k, and the mesh regularity parameter. In the case $k \geq 1$, this inequality results from [22, Eq. (7.75) with $2\mu = 1$ and $\lambda = 0$ together with Remark 7.26] whereas, in the case $k = 0$, it is a consequence of [22, Eq. (7.109) with $\lambda = 0$ and Remark 7.26].

6.3.4.2 Pressure–Displacement Coupling

The coupling between the total pressure and the displacement is realised by means of the bilinear form $\mathrm{b}_h : \underline{\boldsymbol{V}}_h^k \times \mathbb{P}^k(\mathcal{T}_h;\mathbb{R})$ such that, for all $(\underline{\boldsymbol{v}}_h, q_h) \in \underline{\boldsymbol{V}}_h^k \times \mathbb{P}^k(\mathcal{T}_h;\mathbb{R})$,

$$\mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h) := (\mathsf{D}_h^k \underline{\boldsymbol{v}}_h, q_h).$$

The following inf-sup condition holds: There is a real number $\beta > 0$ independent of h, but possibly depending on Ω, d, k, and the mesh regularity parameter, such that

$$\beta \|q_h\|_{L^2(\Omega;\mathbb{R})} \leq \|\mathrm{b}_h(\cdot, q_h)\|_{\mathrm{a},h,*} \qquad \forall q_h \in P_{h,0}^k. \tag{6.11}$$

Moreover, we have the following consistency properties: For all $\boldsymbol{v} \in \boldsymbol{U}$,

$$\mathrm{b}_h(\underline{\boldsymbol{I}}_h^k \boldsymbol{v}, q_h) = b(\boldsymbol{v}, q_h) \qquad \forall q_h \in P_{h,0}^k \tag{6.12}$$

and, for all $q \in H^1(\Omega;\mathbb{R}) \cap H^{k+1}(\mathcal{T}_h;\mathbb{R})$,

$$\|\mathcal{E}_{\mathrm{b},h}(q;\cdot)\|_{\mathrm{a},h,*} \lesssim h^{k+1} |q|_{H^{k+1}(\mathcal{T}_h;\mathbb{R})}, \tag{6.13}$$

where the hidden constant is independent of both h and q and the consistency error linear form $\mathcal{E}_{\mathrm{b},h}(q;\cdot) : \underline{\boldsymbol{U}}_h^k \to \mathbb{R}$ is such that, for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$,

$$\mathcal{E}_{\mathrm{b},h}(q;\underline{\boldsymbol{v}}_h) := -(\nabla q, \boldsymbol{v}_h) - \mathrm{b}_h(\underline{\boldsymbol{v}}_h, \pi_h^k q). \tag{6.14}$$

6.3.4.3 HHO Discretisaton of the Darcy Term

Denote by ∇_h the broken gradient acting element-wise. The Darcy bilinear form c is approximated by $\mathrm{c}_h^{\mathrm{hho}} : \underline{Q}_h^k \times \underline{Q}_h^k \to \mathbb{R}$ such that, for all $\underline{r}_h, \underline{q}_h \in \underline{Q}_h^k$,

$$\mathrm{c}_h^{\mathrm{hho}}(\underline{r}_h, \underline{q}_h) := (\nabla_h \mathrm{r}_h^{k+1} \underline{r}_h, \nabla_h \mathrm{r}_h^{k+1} \underline{q}_h) + \mathrm{s}_{\mathrm{c},h}(\underline{r}_h, \underline{q}_h),$$

with stabilising bilinear form

$$\mathrm{s}_{\mathrm{c},h}(\underline{r}_h, \underline{q}_h) := \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} h_F^{-1} (\delta_{TF}^k \underline{r}_T, \delta_{TF}^k \underline{q}_T)_F,$$

where, for all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$, $\delta_{TF}^k \underline{q}_T := \pi_F^k(\mathrm{r}_T^{k+1}\underline{q}_T - q_F) - \pi_T^k(\mathrm{r}_T^{k+1}\underline{q}_T - q_T)$. The bilinear form $\mathrm{c}_h^{\mathrm{hho}}$ defines an inner product on $\underline{Q}_{h,\mathrm{D}}^k$ as a consequence of [22, Eq. (2.41) and Corollary 2.16], and we denote by $\|\cdot\|_{\mathrm{c},h,\mathrm{hho}}$ the induced norm. The corresponding dual norm is such that, for any linear form $\ell_h : \underline{Q}_{h,\mathrm{D}}^k \to \mathbb{R}$,

$$\|\ell_h\|_{\mathrm{c},h,*} := \sup_{\underline{q}_h \in \underline{Q}_{h,\mathrm{D}}^k \setminus \{\underline{0}\}} \frac{\ell_h(\underline{q}_h)}{\|\underline{q}_h\|_{\mathrm{c},h,\mathrm{hho}}}. \tag{6.15}$$

It follows from [22, Eq. (2.42)] that, for all $r \in H_0^1(\Omega; \mathbb{R}) \cap H^{k+2}(\mathcal{T}_h; \mathbb{R})$ such that $\Delta r \in L^2(\Omega; \mathbb{R})$,

$$\|\mathcal{E}_{\mathrm{c},h}^{\mathrm{hho}}(r; \cdot)\|_{\mathrm{c},h,*} \lesssim h^{k+1} |r|_{H^{k+2}(\mathcal{T}_h;\mathbb{R})}, \tag{6.16}$$

where the hidden constant is independent of both h and r, and the consistency error linear form $\mathcal{E}_{\mathrm{c},h}^{\mathrm{hho}}(r; \cdot) : \underline{Q}_{h,\mathrm{D}}^k \to \mathbb{R}$ is such that, for all $\underline{q}_h \in \underline{Q}_{h,\mathrm{D}}^k$,

$$\mathcal{E}_{\mathrm{c},h}^{\mathrm{hho}}(r; \underline{q}_h) := -(\Delta r, q_h) - \mathrm{c}_h^{\mathrm{hho}}(\underline{I}_h^k r, \underline{q}_h). \tag{6.17}$$

The following discrete Poincaré inequality results combining [22, Lemma 2.15 and Eq. (2.41)]: For all $\underline{q}_h \in \underline{Q}_{h,\mathrm{D}}^k$,

$$\|q_h\|_{L^2(\Omega;\mathbb{R})} \le C_{\mathrm{P}} \|\underline{q}_h\|_{\mathrm{c},h,\mathrm{hho}}, \tag{6.18}$$

with real number $C_{\mathrm{P}} > 0$ independent of h and $\underline{q}_h$, but possibly depending on Ω, d, k, and the mesh regularity parameter.

6.3.4.4 DG Discretisation of the Darcy Term

For the DG approximation of the Darcy operator we need to assume $k \ge 1$ to have consistency. Let the normal trace average operator be defined such that, for all $\boldsymbol{\psi} \in H^1(\mathcal{T}_h; \mathbb{R}^d)$ and all $F \in \mathcal{F}_h^{\mathrm{i}}$ shared by the mesh elements T_1 and T_2,

$$\{\boldsymbol{\psi} \cdot \boldsymbol{n}\}_F := \frac{1}{2} \left(\boldsymbol{\psi}_{|T_1} + \boldsymbol{\psi}_{|T_2}\right)_{|F} \cdot \boldsymbol{n}_F.$$

The DG method hinges on the bilinear form $\mathrm{c}_h^{\mathrm{dg}} : \mathbb{P}^k(\mathcal{T}_h; \mathbb{R}) \times \mathbb{P}^k(\mathcal{T}_h; \mathbb{R}) \to \mathbb{R}$ such that, for all $r_h, q_h \in \mathbb{P}^k(\mathcal{T}_h; \mathbb{R})$,

$$\begin{aligned} \mathrm{c}_h^{\mathrm{dg}}(r_h, q_h) := (\boldsymbol{\nabla}_h r_h, \boldsymbol{\nabla}_h q_h) + \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} ([r_h]_F, [q_h]_F)_F \\ - \sum_{F \in \mathcal{F}_h} \left[([r_h]_F, \{\boldsymbol{\nabla}_h q_h \cdot \boldsymbol{n}\}_F)_F + (\{\boldsymbol{\nabla}_h r_h \cdot \boldsymbol{n}\}_F, [q_h]_F)_F \right], \end{aligned} \tag{6.19}$$

where the stabilisation parameter $\eta > 0$ is chosen large enough to ensure coercivity with respect to the norm $\|\cdot\|_{\mathrm{c},h,\mathrm{dg}}$ defined such that, for all $q_h \in \mathbb{P}^k(\mathcal{T}_h; \mathbb{R})$,

$$\|q_h\|_{\mathrm{c},h,\mathrm{dg}} := \left(\|\nabla_h q_h\|^2_{L^2(\Omega;\mathbb{R}^d)} + \sum_{F\in\mathcal{F}_h} h_F^{-1} \|[q_h]_F\|^2_{L^2(F)} \right)^{\frac{1}{2}}.$$

Let $r \in H_0^1(\Omega, \mathbb{R})$ be such that $\Delta r \in L^2(\Omega, \mathbb{R})$, and consider the elliptic projection problem that consists in finding $r_h \in \mathbb{P}^k(\mathcal{T}_h; \mathbb{R})$ such that

$$\mathrm{c}_h^{\mathrm{dg}}(r_h, q_h) = -(\Delta r, q_h)_{L^2(\Omega)} \qquad \forall q_h \in \mathbb{P}^k(\mathcal{T}_h, \mathbb{R}). \tag{6.20}$$

It is inferred from [21, Appendix A] that, if Ω is convex and $r \in H^{m+1}(\mathcal{T}_h, \mathbb{R})$ for some $m \in \{0, \ldots, k\}$, it holds

$$\|r_h - r\|_{L^2(\Omega)} + h\|r_h - r\|_{\mathrm{c},h,\mathrm{dg}} \lesssim h^{m+1}|r|_{H^{m+1}(\mathcal{T}_h)}, \tag{6.21}$$

with hidden constant independent of h and r.

6.3.5 Discrete Problems

Assume the initial pressures given, and denote by $\boldsymbol{u}^0 \in \boldsymbol{U}$ the corresponding initial equilibrum displacement. Enforce the initial condition by setting

$$\underline{\boldsymbol{u}}_h^0 := \underline{\boldsymbol{I}}_h^k \boldsymbol{u}^0, \qquad p_{h,i}^0 := \pi_h^k p_i^0 \quad \forall i \in [\![0, M]\!]. \tag{6.22}$$

The discrete problem with HHO discretisation of the Darcy term (HHO-HHO scheme) reads:

Problem 6.1 (HHO-HHO scheme) *For $n = 1, \ldots, N$, find $\underline{\boldsymbol{u}}_h^n \in \underline{\boldsymbol{U}}_h^k$, $p_{h,0}^n \in P_{h,0}^k$ and, for all $i \in [\![1, M]\!]$, $\underline{p}_{h,i}^n \in \underline{P}_{h,i}^k$ such that, for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$, all $q_{h,0} \in P_{h,0}^k$, and all $\underline{q}_{h,i} \in \underline{P}_{h,i}^k$, $i \in [\![1, M]\!]$,*

$$2\mu\, \mathrm{a}_h(\underline{\boldsymbol{u}}_h^n, \underline{\boldsymbol{v}}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, p_{h,0}^n) = (\boldsymbol{f}^n, \boldsymbol{v}_h), \tag{6.23a}$$

$$\mathrm{b}_h(\underline{\boldsymbol{u}}_h^n, q_{h,0}) - \lambda^{-1}(\boldsymbol{\alpha}\cdot\boldsymbol{p}_h^n, q_{h,0}) = 0, \tag{6.23b}$$

$$(\delta_t^n \psi_i(\boldsymbol{p}_h), q_{h,i}) + (S_i(\boldsymbol{p}_h^n), q_{h,i}) + K_i \mathrm{c}_h^{\mathrm{hho}}(\underline{p}_{h,i}^n, \underline{q}_{h,i}) = (g_i^n, q_{h,i}) \quad \forall i \in [\![1, M]\!], \tag{6.23c}$$

where we have set, for any $n \in [\![0, N]\!]$, $\boldsymbol{p}_h^n := (p_{h,0}^n, p_{h,1}^n, \ldots, p_{h,M}^n)$ and we remind the reader that ψ_i is defined by (6.2).

The problem resulting from the DG approximation of the flow operator (HHO-DG scheme) reads:

Problem 6.2 (HHO-DG scheme) *For $n = 1, \ldots, N$, find $\underline{\boldsymbol{u}}_h^n \in \underline{\boldsymbol{U}}_h^k$ and $p_{h,0}^n \in P_{h,0}^k$ such that (6.23a)–(6.23b) hold for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$ and all $q_{h,0} \in P_{h,0}^k$, respectively, and,* *for all* $i \in [\![1, M]\!]$, $p_{h,i}^n \in P_{h,i}^k$ *such that, for all* $q_{h,i} \in P_{h,i}^k$, $i \in [\![1, M]\!]$,

$$(\delta_t^n \psi_i(\boldsymbol{p}_h), q_{h,i}) + (S_i(\boldsymbol{p}_h^n), q_{h,i}) + K_i \mathrm{c}_h^{\mathrm{dg}}(p_{h,i}^n, q_{h,i}) = (g_i^n, q_{h,i}) \quad \forall i \in [\![1, M]\!]. \tag{6.24}$$

6.4 Convergence Analysis

We carry out a convergence analysis for the methods formulated in Sect. 6.3.5. For the sake of conciseness, the focus is on the HHO-HHO scheme (6.23). The modifications needed to adapt the results to the HHO-DG scheme are discussed in Sect. 6.4.4. A unified analysis covering both HHO-HHO and HHO-DG methods for the single-network Biot problem can be found in [10].

6.4.1 An Abstract A Priori Estimate

We derive an a priori estimate for an auxiliary problem analogous to (6.23), but with modified right-hand side. Applied to the discrete problem (6.23), this estimate can be used to infer its well-posendess. Applied to the error equations (6.50) below, it gives a basic error estimate.

Problem 6.3 (HHO-HHO scheme with abstract right-hand side) *Let the families of linear forms* $(\ell_1^n : \underline{\boldsymbol{U}}_h^k \to \mathbb{R})_{n\in[\![0,N]\!]}$, *and, for all* $i \in [\![1, M]\!]$, $(\ell_{2,i}^n : \underline{P}_{h,i}^k \to \mathbb{R})_{n\in[\![1,N]\!]}$, *be given. Assume* $\underline{\boldsymbol{w}}_h^0 \in \underline{\boldsymbol{U}}_h^k$, $r_{h,0}^0 \in P_{h,0}^k$, *and, for all* $i \in [\![1, M]\!]$, $\underline{r}_{h,i}^0 \in \underline{P}_{h,i}^k$ *also given. For* $n = 1, \ldots, N$, $\underline{\boldsymbol{w}}_h^n \in \underline{\boldsymbol{U}}_h^k$, $r_{h,0}^n \in P_{h,0}^k$ *and, for all* $i \in [\![1, M]\!]$, $\underline{r}_{h,i}^n \in \underline{P}_{h,i}^k$ *are such that, for all* $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$, *all* $q_h \in P_{h,0}^k$, *and all* $\underline{q}_{h,i} \in \underline{P}_{h,i}^k$, $i \in [\![1, M]\!]$,

$$2\mu\, \mathrm{a}_h(\underline{\boldsymbol{w}}_h^n, \underline{\boldsymbol{v}}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, r_{h,0}^n) = \ell_1^n(\underline{\boldsymbol{v}}_h), \tag{6.25a}$$

$$\mathrm{b}_h(\underline{\boldsymbol{w}}_h^n, q_{h,0}) - \lambda^{-1}(\boldsymbol{\alpha}\cdot\boldsymbol{r}_h^n, q_{h,0}) = 0, \tag{6.25b}$$

$$(\delta_t^n \psi_i(\boldsymbol{r}_h), q_{h,i}) + (S_i(\boldsymbol{r}_h^n), q_{h,i}) + K_i\, \mathrm{c}_h^{\mathrm{hho}}(\underline{r}_{h,i}^n, \underline{q}_{h,i}) = \ell_{2,i}^n(\underline{q}_{h,i}) \quad \forall i \in [\![1, M]\!], \tag{6.25c}$$

where, for any $n \in [\![0, N]\!]$, $\boldsymbol{r}_h^n := (r_{h,0}^n, r_{h,1}^n, \ldots, r_{h,M}^n)$.

Applying discrete time derivation to (6.25b) we obtain, for all $n \in [\![1, N]\!]$,

$$\mathrm{b}_h(\delta_t^n \underline{\boldsymbol{w}}_h, q_{h,0}) - \lambda^{-1}(\boldsymbol{\alpha}\cdot\delta_t^n \boldsymbol{r}_h, q_{h,0}) = 0 \qquad \forall q_{h,0} \in P_{h,0}^k. \tag{6.26}$$

Lemma 6.1 (Abstract a priori estimate) *Assuming* τ *small enough (with threshold independent of* h*), the solution to (6.25) satisfies the following a priori estimate:*

$$\max_{n\in[\![1,N]\!]}\left(\mu\|\underline{\boldsymbol{w}}_h^n\|_{\mathrm{a},h}^2+\lambda^{-1}\|\boldsymbol{\alpha}\cdot\boldsymbol{r}_h^n\|_{L^2(\Omega;\mathbb{R})}^2+\sum_{i=1}^M C_i\|r_{h,i}^n\|_{L^2(\Omega;\mathbb{R})}^2\right)$$
$$+\sum_{n=1}^N\tau\|\boldsymbol{r}_h^n\|_{\boldsymbol{\xi}}^2+\sum_{i=1}^M\sum_{n=1}^N\tau K_i\|\underline{r}_{h,i}^n\|_{\mathrm{c},h,\mathrm{hho}}^2\le\exp\left(\frac{t_{\mathrm{F}}}{1-\tau}\right)(\mathcal{N}_\ell+\mathcal{N}_0),\quad (6.27)$$

where we have introduced the exchange norm

$$\|\boldsymbol{r}_h^n\|_{\boldsymbol{\xi}}^2:=\sum_{i=1}^M\sum_{j=1}^M\|\xi_{i\leftarrow j}(r_{h,i}^n-r_{h,j}^n)\|_{L^2(\Omega;\mathbb{R})}^2$$

and we have set

$$\mathcal{N}_\ell:=\frac{1}{2\mu}\max_{n\in[\![1,N]\!]}\|\ell_1^n\|_{\mathrm{a},h,*}^2+\frac{1}{\mu}\sum_{n=1}^N\tau\|\delta_t^n\ell_1\|_{\mathrm{a},h,*}^2+\sum_{i=1}^M\sum_{n=1}^N\tau K_i^{-1}\|\ell_{2,i}^n\|_{\mathrm{c},h,*}^2,\quad (6.28a)$$

$$\mathcal{N}_0:=2\|\ell_1^0\|_{\mathrm{a},h,*}\|\underline{\boldsymbol{w}}_h^0\|_{\mathrm{a},h}+2\mu\|\underline{\boldsymbol{w}}_h^0\|_{\mathrm{a},h}^2+\frac{1}{\lambda}\|\boldsymbol{\alpha}\cdot\boldsymbol{r}_h^0\|_{L^2(\Omega;\mathbb{R})}^2+\sum_{i=1}^M C_i\|r_{h,i}^0\|_{L^2(\Omega;\mathbb{R})}^2.\quad (6.28b)$$

Moreover, it holds

$$\frac{\beta^2}{\mu}\max_{n\in[\![1,N]\!]}\|r_{h,0}^n\|_{L^2(\Omega;\mathbb{R})}^2\le\frac{2}{\mu}\max_{n\in[\![1,N]\!]}\|\ell_1^n\|_{\mathrm{a},h,*}^2+4\beta^2\exp\left(\frac{t_{\mathrm{F}}}{1-\tau}\right)(\mathcal{N}_\ell+\mathcal{N}_0).\quad (6.29)$$

Proof We start by deriving a basic energy estimate and then, leveraging the discrete inf-sup condition (6.11), deduce from the latter the estimate on the total pressure.

(i) *Basic energy estimate*. Let $\mathsf{N}\in[\![1,N]\!]$ and $n\in[\![1,\mathsf{N}]\!]$. Taking $\underline{\boldsymbol{v}}_h=\delta_t^n\underline{\boldsymbol{w}}_h$ in (6.25a), $q_{h,0}=-r_{h,0}^n$ in (6.26), and, for all $i\in[\![1,M]\!]$, $\underline{q}_{h,i}=\underline{r}_{h,i}^n$ in (6.25c), and summing the resulting equations we obtain, after expanding $\delta_t^n\psi_i(\boldsymbol{r}_h)$ according to its definition,

$$2\mu\,\mathrm{a}_h(\underline{\boldsymbol{w}}_h^n,\delta_t^n\underline{\boldsymbol{w}}_h^n)+\lambda^{-1}\,(\boldsymbol{\alpha}\cdot\delta_t^n\boldsymbol{r}_h^n,\boldsymbol{\alpha}\cdot\boldsymbol{r}_h^n)+\sum_{i=1}^M C_i\,(\delta_t^n r_{h,i},r_{h,i}^n)$$
$$+\sum_{i=1}^M(S_i(\boldsymbol{r}_h^n),r_{h,i}^n)+\sum_{i=1}^M K_i\,\mathrm{c}_h^{\mathrm{hho}}(\underline{r}_{h,i}^n,\underline{r}_{h,i}^n)=\ell_1^n(\delta_t^n\underline{\boldsymbol{w}}_h)+\sum_{i=1}^M\ell_{2,i}(\underline{r}_{h,i}^n).\quad (6.30)$$

Denote by $\mathcal{L}^n=\mathcal{L}_1^n+\cdots\mathcal{L}_5^n$ and $\mathcal{R}^n=\mathcal{R}_1^n+\mathcal{R}_2^n$, respectively, the left- and right-hand side of the above expression, and set $\mathcal{L}:=\sum_{n=1}^{\mathsf{N}}\tau\mathcal{L}^n$ and, for $i\in\{1,2\}$,

$\mathcal{R}_i := \sum_{n=1}^{\mathsf{N}} \tau \mathcal{R}_i^n$.

(i.A) *Lower bound for $\mathcal{L}$*. Recalling the definition (6.4) of the discrete time derivative and using multiple times the formula

$$x(x-y) = \frac{1}{2}\left(x^2 + (x-y)^2 - y^2\right) \tag{6.31}$$

with $x = \bullet^n$ and $y = \bullet^{n-1}$, we can write for the first three terms in $\mathcal{L}^n$

$$\begin{aligned}
\mathcal{L}_1^n &= \frac{\mu}{\tau}\left(\|\underline{\boldsymbol{w}}_h^n\|_{\mathrm{a},h}^2 + \|\underline{\boldsymbol{w}}_h^n - \underline{\boldsymbol{w}}_h^{n-1}\|_{\mathrm{a},h}^2 - \|\underline{\boldsymbol{w}}_h^{n-1}\|_{\mathrm{a},h}^2\right), \\
\mathcal{L}_2^n &= \frac{1}{2\lambda\tau}\left(\|\boldsymbol{\alpha}\cdot\boldsymbol{r}_h^n\|_{L^2(\Omega;\mathbb{R})}^2 + \|\boldsymbol{\alpha}\cdot(\boldsymbol{r}_h^n - \boldsymbol{r}_h^{n-1})\|_{L^2(\Omega;\mathbb{R})}^2 - \|\boldsymbol{\alpha}\cdot\boldsymbol{r}_h^{n-1}\|_{L^2(\Omega;\mathbb{R})}^2\right), \\
\mathcal{L}_3^n &= \sum_{i=1}^M \frac{C_i}{2\tau}\left(\|r_{h,i}^n\|_{L^2(\Omega;\mathbb{R})}^2 + \|r_{h,i}^n - r_{h,i}^{n-1}\|_{L^2(\Omega;\mathbb{R})}^2 - \|r_{h,i}^{n-1}\|_{L^2(\Omega;\mathbb{R})}^2\right).
\end{aligned} \tag{6.32}$$

For the fourth term, using again (6.31) this time with $x = r_{h,i}^n$ and $y = r_{h,j}^n$ along with $\xi_{i\leftarrow j} = \xi_{j\leftarrow i}$, we get

$$\begin{aligned}
\mathcal{L}_4^n &= \sum_{i=1}^M \sum_{j=1}^M (\xi_{i\leftarrow j}(r_{h,i}^n - r_{h,j}^n), r_{h,i}^n) \\
&= \frac{1}{2}\sum_{i=1}^M \sum_{j=1}^M \left(\|\xi_{i\leftarrow j}^{\frac{1}{2}} r_{h,i}^n\|_{L^2(\Omega;\mathbb{R})}^2 + \|\xi_{i\leftarrow j}^{\frac{1}{2}}(r_{h,i}^n - r_{h,j}^n)\|_{L^2(\Omega;\mathbb{R})}^2 - \|\xi_{j\leftarrow i}^{\frac{1}{2}} r_{h,j}^n\|_{L^2(\Omega;\mathbb{R})}^2\right) \\
&= \frac{1}{2}\sum_{i=1}^M \sum_{j=1}^M \|\xi_{i\leftarrow j}^{\frac{1}{2}}(r_{h,i}^n - r_{h,j}^n)\|_{L^2(\Omega;\mathbb{R})}^2 = \frac{1}{2}\|\boldsymbol{r}_h^n\|_{\boldsymbol{\xi}}^2.
\end{aligned} \tag{6.33}$$

Multiplying (6.30) by τ, summing over $n \in [\![1, \mathsf{N}]\!]$, using (6.32) and (6.33), and telescoping out the appropriate summands, we get

$$\begin{aligned}
&\mu\|\underline{\boldsymbol{w}}_h^{\mathsf{N}}\|_{\mathrm{a},h}^2 + \frac{1}{2\lambda}\|\boldsymbol{\alpha}\cdot\boldsymbol{r}_h^{\mathsf{N}}\|_{L^2(\Omega;\mathbb{R})}^2 + \sum_{i=1}^M \frac{C_i}{2}\|r_{h,i}^{\mathsf{N}}\|_{L^2(\Omega;\mathbb{R})}^2 + \frac{1}{2}\sum_{n=1}^{\mathsf{N}} \tau\|\boldsymbol{r}_h^n\|_{\boldsymbol{\xi}}^2 + \sum_{i=1}^M \sum_{n=1}^{\mathsf{N}} \tau K_i \|\underline{r}_{h,i}^n\|_{\mathrm{c},h,\mathrm{hho}}^2 \\
&\quad \le \mathcal{R} + \mu\|\underline{\boldsymbol{w}}_h^0\|_{\mathrm{a},h}^2 + \frac{1}{2\lambda}\|\boldsymbol{\alpha}\cdot\boldsymbol{r}_h^0\|_{L^2(\Omega;\mathbb{R})}^2 + \sum_{i=1}^M \frac{C_i}{2}\|r_{h,i}^0\|_{L^2(\Omega;\mathbb{R})}^2.
\end{aligned} \tag{6.34}$$

(i.B) *Upper bound for $\mathcal{R}$*. A discrete integration by parts in time gives for the first term

$$
\begin{aligned}
\mathcal{R}_1 &= \ell_1^{\mathsf{N}}(\underline{\boldsymbol{w}}_h^{\mathsf{N}}) - \ell_1^0(\underline{\boldsymbol{w}}_h^0) - \sum_{i=1}^{\mathsf{N}} \tau(\delta_t^n \ell_1)(\underline{\boldsymbol{w}}_h^{n-1}) \\
&\leq \|\ell_1^{\mathsf{N}}\|_{\mathrm{a},h,*}\|\underline{\boldsymbol{w}}_h^{\mathsf{N}}\|_{\mathrm{a},h} + \|\ell_1^0\|_{\mathrm{a},h,*}\|\underline{\boldsymbol{w}}_h^0\|_{\mathrm{a},h} + \sum_{n=1}^{\mathsf{N}} \tau\mu^{-\frac{1}{2}}\|\delta_t^n \ell_1\|_{\mathrm{a},h,*}\mu^{\frac{1}{2}}\|\underline{\boldsymbol{w}}_h^{n-1}\|_{\mathrm{a},h} \\
&\leq \frac{1}{4\mu}\|\ell_1^{\mathsf{N}}\|_{\mathrm{a},h,*}^2 + \frac{\mu}{2}\|\underline{\boldsymbol{w}}_h^{\mathsf{N}}\|_{\mathrm{a},h}^2 + \|\ell_1^0\|_{\mathrm{a},h,*}\|\underline{\boldsymbol{w}}_h^0\|_{\mathrm{a},h} \\
&\quad + \frac{1}{2\mu}\sum_{n=1}^{\mathsf{N}} \tau\|\delta_t^n \ell_1\|_{\mathrm{a},h,*}^2 + \frac{\mu}{2}\sum_{n=0}^{\mathsf{N}} \tau\|\underline{\boldsymbol{w}}_h^n\|_{\mathrm{a},h}^2,
\end{aligned}
\tag{6.35}
$$

where we have used multiple times the definition of dual norm (6.7) to pass to the second line and we have concluded invoking the standard and generalised Young inequalities and rearranging.

Moving to the second term, we use the definition (6.15) of the dual norm and the Young inequality to write, for all $i \in [\![1, M]\!]$,

$$
\begin{aligned}
\sum_{n=1}^{\mathsf{N}} \tau\ell_{2,i}^n(\underline{r}_{h,i}^n) &\leq \sum_{n=1}^{\mathsf{N}} \tau K_i^{-\frac{1}{2}}\|\ell_{2,i}^n\|_{\mathrm{c},h,*}\, K_i^{\frac{1}{2}}\|\underline{r}_{h,i}^n\|_{\mathrm{c},h,\mathrm{hho}} \\
&\leq \frac{1}{2}\sum_{n=1}^{\mathsf{N}} \tau K_i^{-1}\|\ell_{2,i}^n\|_{\mathrm{c},h,*}^2 + \frac{1}{2}\sum_{n=1}^{\mathsf{N}} \tau K_i\|\underline{r}_{h,i}^n\|_{\mathrm{c},h,\mathrm{hho}}^2.
\end{aligned}
$$

Hence, summing over $i \in [\![1, M]\!]$,

$$
\mathcal{R}_2 \leq \frac{1}{2}\sum_{i=1}^{M}\sum_{n=1}^{\mathsf{N}} \tau K_i^{-1}\|\ell_{2,i}^n\|_{\mathrm{c},h,*}^2 + \frac{1}{2}\sum_{i=1}^{M}\sum_{n=1}^{\mathsf{N}} \tau K_i\|\underline{r}_{h,i}^n\|_{\mathrm{c},h,\mathrm{hho}}^2. \tag{6.36}
$$

Gathering (6.35) and (6.36) and rearranging, we arrive at

$$
\begin{aligned}
\mathcal{R} &\leq \frac{\mu}{2}\|\underline{\boldsymbol{w}}_h^{\mathsf{N}}\|_{\mathrm{a},h}^2 + \frac{1}{2}\sum_{i=1}^{M}\sum_{n=1}^{\mathsf{N}} \tau K_i\|\underline{r}_{h,i}^n\|_{\mathrm{c},h,\mathrm{hho}}^2 + \frac{\mu}{2}\sum_{n=0}^{\mathsf{N}} \tau\|\underline{\boldsymbol{w}}_h^n\|_{\mathrm{a},h}^2 \\
&\quad + \frac{1}{4\mu}\|\ell_1^{\mathsf{N}}\|_{\mathrm{a},h,*} + \frac{1}{2\mu}\sum_{n=1}^{\mathsf{N}} \tau\|\delta_t^n \ell_1\|_{\mathrm{a},h,*}^2 + \frac{1}{2}\sum_{i=1}^{M}\sum_{n=1}^{\mathsf{N}} \tau K_i^{-1}\|\ell_{2,i}^n\|_{\mathrm{c},h,*}^2 \\
&\quad + \|\ell_1^0\|_{\mathrm{a},h,*}\|\underline{\boldsymbol{w}}_h^0\|_{\mathrm{a},h}.
\end{aligned}
\tag{6.37}
$$

(i.C) *Basic estimate*. Combining (6.34) and (6.37) and multiplying by 2, we arrive at

$$\mu\|\underline{\boldsymbol{w}}_h^{\mathsf{N}}\|_{\mathrm{a},h}^2 + \lambda^{-1}\|\boldsymbol{\alpha}\cdot\boldsymbol{r}_h^{\mathsf{N}}\|_{L^2(\Omega;\mathbb{R})}^2 + \sum_{i=1}^{M} C_i\|r_{h,i}^{\mathsf{N}}\|_{L^2(\Omega;\mathbb{R})}^2$$
$$+ \sum_{n=1}^{\mathsf{N}} \tau\|\boldsymbol{r}_h^n\|_{\boldsymbol{\xi}}^2 + \sum_{i=1}^{M}\sum_{n=1}^{\mathsf{N}} \tau K_i\|\underline{r}_{h,i}^n\|_{\mathrm{c},h,\mathrm{hho}}^2 \le \mu\sum_{n=0}^{\mathsf{N}}\tau\|\underline{\boldsymbol{w}}_h^n\|_{\mathrm{a},h}^2 + \mathcal{N}_\ell + \mathcal{N}_0. \tag{6.38}$$

The estimate (6.27) follows from the discrete Gronwall inquality of [28, Lemma 5.1].

(ii) *Estimate on the total pressure*. For all $n \in [\![1, \mathsf{N}]\!]$, using the inf-sup stability (6.11) of the pressure-displacement coupling, we can write

$$\begin{aligned} \beta\|r_{h,0}^n\|_{L^2(\Omega;\mathbb{R})} &\le \sup_{\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{U}}_h^k\setminus\{\underline{\boldsymbol{0}}\}} \frac{\mathrm{b}_h(\underline{\boldsymbol{v}}_h, r_{h,0}^n)}{\|\underline{\boldsymbol{v}}_h\|_{\mathrm{a},h}} \\ &\le \sup_{\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{U}}_h^k\setminus\{\underline{\boldsymbol{0}}\}} \frac{\ell_1^n(\underline{\boldsymbol{v}}_h) - 2\mu\,\mathrm{a}_h(\underline{\boldsymbol{w}}_h^n, \underline{\boldsymbol{v}}_h)}{\|\underline{\boldsymbol{v}}_h\|_{\mathrm{a},h}} \\ &\le \|\ell_1^n\|_{\mathrm{a},h,*} + 2\mu\,\|\underline{\boldsymbol{w}}_h^n\|_{\mathrm{a},h}, \end{aligned} \tag{6.39}$$

where we have used (6.25a) in the second line and we have concluded using the definition (6.7) of dual norm for the first term and a Cauchy–Schwarz inequality on the symmetric positive definite bilinear form a_h for the second. Squaring, dividing both sides by μ, passing to the maximum over $n \in [\![1, N]\!]$, and using (6.27) to estimate the second term in the right-hand side, (6.41) follows.

6.4.2 A Priori Estimate for the HHO-HHO Scheme

The following lemma contains an a priori estimate on the discrete solution, from which the well posedness of problem (6.23) can be inferred.

Lemma 6.2 (A priori estimate on the discrete solution) *Assuming τ small enough, any solution $\left(\underline{\boldsymbol{u}}_h^n, p_{h,0}^n, (p_{h,i})_{1\le i\le M}\right)_{1\le n\le N}$ to the discrete problem (6.23) satisfies the following a priori bound:*

$$\max_{n\in[\![1,N]\!]}\left(\mu\|\underline{\boldsymbol{u}}_h^n\|_{\mathrm{a},h}^2 + \lambda^{-1}\|\boldsymbol{\alpha}\cdot\boldsymbol{p}\|_{L^2(\Omega;\mathbb{R})}^2 + \sum_{i=1}^{M} C_i\|p_{h,i}^n\|_{L^2(\Omega;\mathbb{R})}^2\right)$$
$$+ \sum_{n=1}^{N}\tau\|\boldsymbol{r}_h^n\|_{\boldsymbol{\xi}}^2 + \sum_{i=1}^{M}\sum_{n=1}^{N}\tau K_i\|\underline{p}_{h,i}^n\|_{\mathrm{c},h,\mathrm{hho}}^2 \le \exp\left(\frac{t_{\mathrm{F}}}{1-\tau}\right)(\mathcal{A}+\mathcal{B}), \tag{6.40}$$

where

$$\begin{aligned}\mathcal{A} &:= \frac{C_{\mathrm{K}}^2}{2\mu}\|\boldsymbol{f}\|^2_{C^0([0,t_{\mathrm{F}}];L^2(\Omega;\mathbb{R}^d))} + \frac{1}{\mu}\|\boldsymbol{f}\|^2_{H^1(0,t_{\mathrm{F}};L^2(\Omega;\mathbb{R}^d))} \\ &\quad + C_{\mathrm{P}} t_{\mathrm{F}} \sum_{i=1}^{M} \frac{1}{K_i}\|g_i\|^2_{C^0([0,t_{\mathrm{F}}];L^2(\Omega;\mathbb{R}))} \\ \mathcal{B} &:= 2C_{\mathrm{K}}\|\boldsymbol{f}^0\|_{L^2(\Omega;\mathbb{R}^d)}\|\underline{\boldsymbol{u}}_h^0\|_{\mathrm{a},h} + 2\mu\|\underline{\boldsymbol{u}}_h^0\|^2_{\mathrm{a},h} + \lambda^{-1}\|\boldsymbol{\alpha}\cdot\boldsymbol{p}_h^0\|^2_{L^2(\Omega;\mathbb{R})} \\ &\quad + \sum_{i=1}^{M} C_i\|p_{h,i}^0\|^2_{L^2(\Omega;\mathbb{R})}.\end{aligned}$$

Moreover, it holds

$$\frac{\beta^2}{\mu}\max_{n\in[\![1,N]\!]}\|p_{h,0}^n\|^2_{L^2(\Omega;\mathbb{R})} \le \frac{2C_{\mathrm{K}}^2}{\mu}\|\boldsymbol{f}\|^2_{C^0([0,t_{\mathrm{F}}];L^2(\Omega;\mathbb{R}^d))} + 4\beta^2\exp\left(\frac{t_{\mathrm{F}}}{1-\tau}\right)(\mathcal{A}+\mathcal{B}). \tag{6.41}$$

Proof We apply Lemma 6.1 with $\ell_1^n = \left(\underline{\boldsymbol{U}}_h^k \ni \underline{\boldsymbol{v}}_h \mapsto (\boldsymbol{f}, \boldsymbol{v}_h) \in \mathbb{R}\right)$ for all $n \in [\![0, N]\!]$ and $\ell_2^n = \left(\underline{P}_{h,i}^n \ni \underline{q}_{h,i} \mapsto (g_i, q_{h,i}) \in \mathbb{R}\right)$ for all $n \in [\![1, N]\!]$ and all $i \in [\![1, M]\!]$, and show that

$$\mathcal{N}_\ell \le \mathcal{A} \text{ and } \mathcal{N}_0 \le \mathcal{B}. \tag{6.42}$$

Let us prove the first bound in (6.42). Denote by $\mathcal{N}_{\ell,i}$, $i \in [\![1, 3]\!]$, the terms in the right-hand side of (6.28a). We start by noticing that, for all $n \in [\![0, N]\!]$,

$$\begin{aligned}\|\ell_1^n\|_{\mathrm{a},h,*} &= \sup_{\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{U}}_h^k\setminus\{\underline{\boldsymbol{0}}\}} \frac{\ell_1^n(\underline{\boldsymbol{v}}_h)}{\|\underline{\boldsymbol{v}}_h\|_{\mathrm{a},h}} \\ &= \sup_{\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{U}}_h^k\setminus\{\underline{\boldsymbol{0}}\}} \frac{\|\boldsymbol{f}^n\|_{L^2(\Omega;\mathbb{R}^d)}\|\boldsymbol{v}_h\|_{L^2(\Omega;\mathbb{R}^d)}}{\|\underline{\boldsymbol{v}}_h\|_{\mathrm{a},h}} \\ &= \sup_{\underline{\boldsymbol{v}}_h\in\underline{\boldsymbol{U}}_h^k\setminus\{\underline{\boldsymbol{0}}\}} \frac{C_{\mathrm{K}}\|\boldsymbol{f}^n\|_{L^2(\Omega;\mathbb{R}^d)}\|\underline{\boldsymbol{v}}_h\|_{\mathrm{a},h}}{\|\underline{\boldsymbol{v}}_h\|_{\mathrm{a},h}} \le C_{\mathrm{K}}\|\boldsymbol{f}^n\|_{L^2(\Omega;\mathbb{R}^d)},\end{aligned} \tag{6.43}$$

where we have used the definition (6.7) of the dual norm in the first line, a Cauchy–Schwarz inequality to pass to the the second line, and the discrete Korn inequality (6.10) to pass to the third line. As a consequence,

$$\mathcal{N}_{\ell,1} \le \frac{C_{\mathrm{K}}^2}{2\mu}\max_{n\in[\![1,N]\!]}\|\boldsymbol{f}^n\|^2_{L^2(\Omega;\mathbb{R}^d)} = \frac{C_{\mathrm{K}}^2}{2\mu}\|\boldsymbol{f}\|^2_{C^0([0,t_{\mathrm{F}}];L^2(\Omega;\mathbb{R}^d))}. \tag{6.44}$$

Proceeding similarly for the second term and invoking the boundedness (6.5) of the discrete time derivative with $V = L^2(\Omega; \mathbb{R}^d)$ and $\varphi = \boldsymbol{f}$, we get

$$\mathcal{N}_{\ell,2} \le \frac{C_{\rm K}^2}{2\mu}\sum_{n=1}^{n}\tau\|\delta_t^n \boldsymbol{f}\|_{L^2(\Omega;\mathbb{R}^d)}^2 \le \frac{C_{\rm K}^2}{2\mu}\|\boldsymbol{f}\|_{H^1(0,t_{\rm F};L^2(\Omega;\mathbb{R}^d))}^2. \tag{6.45}$$

To bound the third term, we observe that, using the definition (6.15) of the dual norm and the Poincaré inequality in a similar manner as above, it holds, for all $n \in [\![1, N]\!]$ and all $i \in [\![1, M]\!]$, $\|\ell_{2,i}^n\|_{{\rm c},h,*} \le K_i^{-1} C_{\rm P}\|g_i^n\|_{L^2(\Omega;\mathbb{R})}$, hence

$$\begin{aligned}\mathcal{N}_{\ell,3} &\le C_{\rm P}\sum_{i=1}^{M}\frac{1}{K_i}\sum_{n=1}^{N}\tau\|g_i^n\|_{L^2(\Omega;\mathbb{R})}^2\\ &\le C_{\rm P}t_{\rm F}\sum_{i=1}^{M}\frac{1}{K_i}\max_{n\in[\![1,N]\!]}\|g_i^n\|_{L^2(\Omega;\mathbb{R})}^2 = C_{\rm P}t_{\rm F}\sum_{i=1}^{M}\frac{1}{K_i}\|g_i\|_{C^0([0,t_{\rm F}];L^2(\Omega;\mathbb{R}))}^2.\end{aligned} \tag{6.46}$$

Gathering (6.44)–(6.46), the first bound in (6.30) follows. The second bound in (6.30) is an immediate after invoking (6.43) with $n = 0$. This concludes the proof.

6.4.3 *Error Estimate for the HHO-HHO Scheme*

Following the general ideas of [20], we estimate the error such that, for all $n \in [\![0, N]\!]$,

$$\underline{\boldsymbol{e}}_h^n := \underline{\boldsymbol{u}}_h^n - \underline{\hat{\boldsymbol{u}}}_h^n, \qquad \epsilon_{h,0}^n := p_{h,0}^n - \hat{p}_{h,0}^n, \qquad \underline{\epsilon}_{h,i}^n := \underline{p}_{h,i}^n - \underline{\hat{p}}_{h,i}^n \quad \forall i \in [\![1, M]\!], \tag{6.47}$$

where the interpolate of the continuous solution is obtained setting, for all $n \in [\![0, N]\!]$,

$$\underline{\hat{\boldsymbol{u}}}_h^n := \underline{\boldsymbol{I}}_h^k \boldsymbol{u}^n, \qquad \hat{p}_{h,0}^n := \pi_h^k p_0^n, \qquad \underline{\hat{p}}_{h,i}^n := \underline{I}_h^k p_i^n \quad \forall i \in [\![1, M]\!]. \tag{6.48}$$

The starting point for the error analysis is the following proposition, which establishes that the errors solve the auxiliary problem (6.25) for a suitable choice of the right-hand sides ℓ_1 and $\ell_{2,i}$, $i \in [\![1, M]\!]$.

Proposition 6.1 (Error equations) *We have that*

$$\underline{\boldsymbol{e}}_h^0 = \underline{\boldsymbol{0}}, \qquad \epsilon_{h,0}^0 = 0, \qquad \underline{\epsilon}_{h,i}^0 = \underline{0} \quad \forall i \in [\![1, M]\!]. \tag{6.49}$$

Additionally, for $n = 1, \ldots, N$, it holds, for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$, all $q_{h,0} \in P_{h,0}^k$,

$$2\mu\, {\rm a}_h(\underline{\boldsymbol{e}}_h^n, \underline{\boldsymbol{v}}_h) + {\rm b}_h(\underline{\boldsymbol{v}}_h, \epsilon_{h,0}^n) = 2\mu\mathcal{E}_{{\rm a},h}(\boldsymbol{u}^n; \underline{\boldsymbol{v}}_h) + \mathcal{E}_{{\rm b},h}(p_0^n; \underline{\boldsymbol{v}}_h), \tag{6.50a}$$

$${\rm b}_h(\underline{\boldsymbol{e}}_h^n, q_{h,0}) - \lambda^{-1}(\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}_h^n, q_{h,0}) = 0, \tag{6.50b}$$

and, for all $i \in [\![1, M]\!]$ and all $\underline{q}_{h,i} \in \underline{P}_{h,i}^k$,

$$
\begin{aligned}
&(\delta_t^n \psi_i(\boldsymbol{\epsilon}_h), q_{h,i}) + (S_i(\boldsymbol{\epsilon}_h^n), q_{h,i}) + K_i\ \mathrm{c}_h^{\mathrm{hho}}(\underline{\epsilon}_{h,i}^n, \underline{q}_{h,i}) \\
&\qquad = (\mathrm{d}_t^n \psi_i(\boldsymbol{p}) - \delta_t^n \psi_i(\boldsymbol{p}), q_{h,i}) + \mathcal{E}_{\mathrm{c},h}^{\mathrm{hho}}(p_i^n; \underline{q}_{h,i}),
\end{aligned} \tag{6.50c}
$$

where we have set, for all $n \in [\![0, N]\!]$, $\boldsymbol{\epsilon}_h^n := (\epsilon_{h,0}^n, \epsilon_{h,1}^n, \ldots, \epsilon_{h,M}^n)$ *and, given a function of time* φ *smooth enough, we have introduced the abridged notation* $\mathrm{d}_t^n \varphi := \mathrm{d}_t \varphi(t^n)$.

Proof Equation (6.49) is an immediate consequence of the definition (6.47) of the errors along with the discrete initial condition (6.22).

Let now $n \in [\![1, N]\!]$. To prove (6.50a), it suffices to subtract from both sides of (6.23a) the quantity $2\mu\ \mathrm{a}_h(\underline{\hat{\boldsymbol{u}}}_h^n, \underline{\boldsymbol{v}}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, \hat{p}_{h,0}^n)$, observe that $\boldsymbol{f}^n = -2\mu \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u}^n) - \boldsymbol{\nabla} p_0^n$ almost everywhere in Ω, and recall the definitions (6.9) and (6.14) of the consistency error linear forms associated with a_h and b_h.

Moving to (6.50b), we observe that, for all $q_{h,0} \in P_{h,0}^k$,

$$
\begin{aligned}
\mathrm{b}_h(\underline{\hat{\boldsymbol{u}}}_h^n, q_{h,0}) - \lambda^{-1}(\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}_h^n, q_{h,0}) &= \mathrm{b}_h(\underline{\boldsymbol{I}}_h^k \boldsymbol{u}^n, q_{h,0}) - \lambda^{-1}(\boldsymbol{\alpha}\cdot\boldsymbol{\pi}_h^k \boldsymbol{p}^n, q_{h,0}) \\
&= b(\boldsymbol{u}, q_{h,0}) - \lambda^{-1}(\boldsymbol{\alpha}\cdot\boldsymbol{p}^n, q_{h,0}) = 0,
\end{aligned} \tag{6.51}
$$

where, to pass to the second line, we have used the consistency property (6.12) of b_h together with the definition (6.6) of the global L^2-orthogonal projector and $q_{h,0} \in \mathbb{P}^k(\mathcal{T}_h; \mathbb{R})$ to remove it from the second term, while the conclusion follows from (6.1a) after observing that $P_{h,0}^k \subset P_0$. The error equation (6.50b) then follows subtracting (6.51) from (6.23b) and using the linearity of the bilinear forms in the left-hand side.

Finally, to prove (6.50c) for a given $i \in [\![1, M]\!]$ and $\underline{q}_{h,i} \in \underline{P}_{h,i}^k$, we subtract from both sides the quantity $(\delta_t^n \psi_i(\hat{\boldsymbol{p}}_h), q_{h,i}) + (S_i(\hat{\boldsymbol{p}}_h^n), q_{h,i}) + K_i\ \mathrm{c}_h^{\mathrm{hho}}(\underline{\hat{p}}_{h,i}^n, \underline{q}_{h,i})$ and observe that

$$
\begin{aligned}
(g_i^n, q_{h,i}) &= (\mathrm{d}_t^n \psi_i(\boldsymbol{p}), q_{h,i}) + (S_i(\boldsymbol{p}^n), q_{h,i}) - (K_i \Delta p_i^n, q_{h,i}) \\
&= (\mathrm{d}_t^n \psi_i(\boldsymbol{p}) - \delta_t^n \psi_i(\boldsymbol{p}), q_{h,i}) + \mathcal{E}_{\mathrm{c},h}^{\mathrm{hho}}(p_i^n; \underline{q}_{h,i}) \\
&\quad + (\delta_t^n \psi_i(\hat{\boldsymbol{p}}_h), q_{h,i}) + (S_i(\hat{\boldsymbol{p}}_h^n), q_{h,i}) + K_i\ \mathrm{c}_h^{\mathrm{hho}}(\underline{\hat{p}}_{h,i}^n, \underline{q}_{h,i}),
\end{aligned}
$$

where, to pass to the second line, we have added and subtracted $(\delta_t^n \psi_i(\hat{\boldsymbol{p}}_h), q_{h,i}) + \mathrm{c}_h^{\mathrm{hho}}(\underline{\hat{p}}_{h,i}^n, \underline{q}_{h,i})$, used the fact that $q_{h,i} \in \mathbb{P}^k(\mathcal{T}_h; \mathbb{R})$ along with the linearity of ψ and the definition (6.6) of the global L^2-orthogonal projector to write $(\delta_t^n \psi_i(\hat{\boldsymbol{p}}_h), q_{h,i}) = (\delta_t^n \psi_i(\boldsymbol{p}), q_{h,i})$, and recalled the definition (6.17) of the consistency error associated with the bilinear form $\mathrm{c}_h^{\mathrm{hho}}$.

Theorem 6.1 (Error estimate for the HHO-HHO scheme) *Assume the additional regularity*

$$\begin{aligned}
\boldsymbol{u} &\in H^1(0,t_{\mathrm{F}};H^{k+2}(\mathcal{T}_h;\mathbb{R}^d)),\\
p_0 &\in H^1(0,t_{\mathrm{F}};H^{k+1}(\mathcal{T}_h;\mathbb{R})),\\
\forall i \in [\![1,M]\!],\quad p_i &\in C^0([0,t_{\mathrm{F}}];H^{k+2}(\mathcal{T}_h;\mathbb{R})),\\
\forall i \in [\![1,M]\!],\quad \psi_i(\boldsymbol{p}) &\in H^2(0,t_{\mathrm{F}};L^2(\Omega;\mathbb{R})).
\end{aligned}$$

Then, for a time step τ small enough (with threshold independent of h), it holds that

$$\max_{n\in[\![1,N]\!]}\left(\mu\|\underline{\boldsymbol{e}}_h^n\|_{\mathrm{a},h}^2+\lambda^{-1}\|\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}_h^n\|_{L^2(\Omega;\mathbb{R})}^2+\sum_{i=1}^M C_i\ \|\epsilon_{h,i}^n\|_{L^2(\Omega;\mathbb{R})}^2+\frac{\beta^2}{\mu}\|\epsilon_{h,0}^n\|_{L^2(\Omega;\mathbb{R})}^2\right)$$
$$+\sum_{n=1}^N\tau\|\boldsymbol{\epsilon}_h^n\|_{\boldsymbol{\xi}}^2+\sum_{i=1}^M\sum_{n=1}^N\tau K_i\ \|\underline{\epsilon}_{h,i}^n\|_{\mathrm{c},h,\mathrm{hho}}^2\lesssim h^{2(k+1)}\mathcal{A}+\tau^2\mathcal{B},\tag{6.52}$$

where the hidden constant is independent of h, τ, of the problem data, of $\boldsymbol{u}$, and of p_i, $i\in[\![0,M]\!]$, but possibly depends on Ω, t_{F}, the mesh regularity parameter, and k, and we have set

$$\begin{aligned}
\mathcal{A} &:= \|\boldsymbol{u}\|_{H^1(0,t_{\mathrm{F}};H^{k+2}(\mathcal{T}_h;\mathbb{R}^d))}^2+\mu^{-1}\|p_0\|_{H^1(0,t_{\mathrm{F}};H^{k+1}(\mathcal{T}_h;\mathbb{R}^d))}^2\\
&\quad+\sum_{i=1}^M K_i^{-1}\|p_i\|_{C^0([0,t_{\mathrm{F}}];H^{k+2}(\mathcal{T}_h;\mathbb{R}))}^2,\\
\mathcal{B} &:= \sum_{i=1}^M K_i^{-1}\ \|\psi_i(\boldsymbol{p})\|_{H^2(0,t_{\mathrm{F}};L^2(\Omega;\mathbb{R}))}^2.
\end{aligned}$$

Proof For the sake of brevity, denote by $\mathcal{E}_{h\tau}$ the left-hand side of (6.52). Applying Lemma 6.1 with, for all $n\in[\![1,N]\!]$,

$$\begin{aligned}
\ell_1^n &= 2\mu\mathcal{E}_{\mathrm{a},h}(\boldsymbol{u}^n;\cdot)+\mathcal{E}_{\mathrm{b},h}(p_0^n;\cdot),\\
\ell_{2,i}^n &= (\mathrm{d}_t^n\psi_i(\boldsymbol{p})-\delta_t^n\psi_i(\boldsymbol{p}),\cdot)+\mathcal{E}_{\mathrm{c},h}^{\mathrm{hho}}(p_i;\cdot)\qquad\forall i\in[\![1,M]\!],
\end{aligned}$$

using multiple times the triangle inequality, and rearranging the terms, we arrive at

$$\begin{aligned}
\mathcal{E}_{h\tau} \lesssim \mu^{-1}&\max_{n\in[\![1,N]\!]}\|2\mu\mathcal{E}_{\mathrm{a},h}(\boldsymbol{u}^n;\cdot)+\mathcal{E}_{\mathrm{b},h}(p_0^n;\cdot)\|_{\mathrm{a},h,*}^2\\
&+\mu^{-1}\sum_{n=1}^N\tau\|\delta_t^n\left(2\mu\mathcal{E}_{\mathrm{a},h}(\boldsymbol{u};\cdot)+\mathcal{E}_{\mathrm{b},h}(p_0;\cdot)\right)\|_{\mathrm{a},h,*}^2\\
&+\sum_{i=1}^M\sum_{n=1}^N\tau K_i^{-1}\|\mathcal{E}_{\mathrm{c},h}^{\mathrm{hho}}(p_i^n;\cdot)\|_{\mathrm{c},h,*}^2
\end{aligned}$$

$$+\sum_{i=1}^{M}\sum_{n=1}^{N}\tau K_i^{-1}\|(\mathrm{d}_t^n\psi_i(\boldsymbol{p})-\delta_t^n\psi_i(\boldsymbol{p}),\cdot)\|_{\mathrm{c},h,*}^2 =: \mathfrak{T}_1+\cdots+\mathfrak{T}_4. \tag{6.53}$$

We proceed to bound the terms in the right-hand side of the above expression. For the first term, we write

$$\begin{aligned}
\mathfrak{T}_1 &\lesssim \mu^{-1}\left(\max_{n\in[\![1,N]\!]}\|2\mu\mathcal{E}_{\mathrm{a},h}(\boldsymbol{u}^n;\cdot)\|_{\mathrm{a},h,*}^2+\max_{n\in[\![1,N]\!]}\|\mathcal{E}_{\mathrm{b},h}(p_0^n;\cdot)\|_{\mathrm{a},h,*}^2\right)\\
&\lesssim h^{2(k+1)}\,\mu^{-1}\max_{n\in[\![1,N]\!]}\left(2\mu|\boldsymbol{u}^n|_{H^{k+2}(\mathcal{T}_h;\mathbb{R}^d))}^2+|p_0^n|_{H^{k+1}(\mathcal{T}_h;\mathbb{R}))}^2\right)\\
&\le h^{2(k+1)}\,\left(2\|\boldsymbol{u}\|_{C^0([0,t_\mathrm{F}];H^{k+2}(\mathcal{T}_h;\mathbb{R}^d))}^2+\mu^{-1}\|p_0\|_{C^0([0,t_\mathrm{F}];H^{k+1}(\mathcal{T}_h;\mathbb{R}))}^2\right)\\
&\lesssim h^{2(k+1)}\mathcal{A},
\end{aligned} \tag{6.54}$$

where, to pass to the second line, we have used the consistency properties (6.8) of a_h and (6.13) of b_h, while the conclusion follows from the embedding $H^1(0,t_\mathrm{F};V)\hookrightarrow C^0([0,t_\mathrm{F}];V)$ valid in dimension 1.

For the second term, we write

$$\begin{aligned}
\mathfrak{T}_2 &\lesssim \mu^{-1}\sum_{n=1}^{N}\tau\left(\|2\mu\mathcal{E}_{\mathrm{a},h}(\delta_t^n\boldsymbol{u};\cdot)\|_{\mathrm{a},h,*}^2+\|\mathcal{E}_{\mathrm{b},h}(\delta_t^n p_0;\cdot)\|_{\mathrm{a},h,*}^2\right)\\
&\lesssim h^{2(k+1)}\,\mu^{-1}\sum_{n=1}^{N}\tau\left(2\mu|\delta_t^n\boldsymbol{u}|_{H^{k+2}(\mathcal{T}_h;\mathbb{R}^d)}^2+|\delta_t^n p_0|_{H^{k+1}(\mathcal{T}_h;\mathbb{R})}^2\right)\\
&\lesssim h^{2(k+1)}\,\left(\|\boldsymbol{u}\|_{H^1(0,t_\mathrm{F};H^{k+2}(\mathcal{T}_h;\mathbb{R}^d))}^2+\mu^{-1}\|p_0\|_{H^1(0,t_\mathrm{F};H^{k+1}(\mathcal{T}_h;\mathbb{R}))}^2\right)\\
&\lesssim h^{2(k+1)}\mathcal{A},
\end{aligned} \tag{6.55}$$

where, in the first line, we have used the fact that $\delta_t^n\left(2\mu\mathcal{E}_{\mathrm{a},h}(\boldsymbol{u};\cdot)+\mathcal{E}_{\mathrm{b},h}(p_0;\cdot)\right)=2\mu\mathcal{E}_{\mathrm{a},h}(\delta_t^n\boldsymbol{u};\cdot)+\mathcal{E}_{\mathrm{b},h}(\delta_t^n p_0;\cdot)$ followed by a triangle inequality, we have invoked the consistency (6.8) of a_h and (6.13) of b_h to pass to the second line, and the boundedness (6.5) of the backward time derivative operator to pass to the third line.

For the third term, the consistency properties (6.16) of $\mathrm{c}_h^{\mathrm{hho}}$ readily give

$$\begin{aligned}
\mathfrak{T}_3 &\le h^{2(k+1)}\sum_{i=1}^{M}\sum_{n=1}^{N}\tau K_i^{-1}|p_i^n|_{H^{k+2}(\mathcal{T}_h;\mathbb{R})}^2\\
&\lesssim h^{2(k+1)}t_\mathrm{F}\sum_{i=1}^{M}K_i^{-1}\|p_i\|_{C^0([0,t_\mathrm{F}];H^{k+2}(\mathcal{T}_h;\mathbb{R}))}^2\lesssim h^{2(k+1)}\mathcal{A}.
\end{aligned} \tag{6.56}$$

Let us now move to the fourth term. For the sake of conciseness, we let, for all $i\in[\![1,M]\!]$, $\psi_i:=\psi_i(\boldsymbol{p})$, regarded as an element $H^1(0,t_\mathrm{F};L^2(\Omega;\mathbb{R}))$, and we conventionally denote $\psi(\boldsymbol{x}, t) := \psi(t)(\boldsymbol{x})$ for all $t \in [0, t_{\mathrm{F}}]$ and almost every $\boldsymbol{x} \in \Omega$. Let $i \in [\![1, M]\!]$. It holds, for all $n \in [\![1, N]\!]$,

$$\begin{aligned} d_t^n \psi_i - \delta_t^n \psi_i &= d_t^n \psi_i - \frac{1}{\tau} \int_{t^{n-1}}^{t^n} d_t \psi_i(t)\,\mathrm{d}t \\ &= d_t^n \psi_i - \frac{1}{\tau} \int_{t^{n-1}}^{t^n} \left(d_t^n \psi_i - \int_t^{t^n} d_t^2 \psi_i(s)\,\mathrm{d}s \right) \mathrm{d}t \\ &= \frac{1}{\tau} \int_{t^{n-1}}^{t^n} \int_t^{t^n} d_t^2 \psi_i(s)\,\mathrm{d}s\,\mathrm{d}t \le \int_{t^{n-1}}^{t^n} |d_t^2 \psi_i(t)|\,\mathrm{d}t. \end{aligned}$$

Combining this result with the Jensen inequality, we infer

$$\begin{aligned} \|d_t^n \psi_i - \delta_t^n \psi_i\|_{L^2(\Omega;\mathbb{R})}^2 &\le \int_\Omega \left| \int_{t^{n-1}}^{t^n} |d_t^2 \psi_i(\boldsymbol{x}, t)|\,\mathrm{d}t \right|^2 \mathrm{d}\boldsymbol{x} \\ &\le \tau \int_{t^{n-1}}^{t^n} \|d_t^2 \psi_i(t)\|_{L^2(\Omega;\mathbb{R})}^2\,\mathrm{d}t \\ &\le \tau \|\psi_i\|_{H^2(t^{n-1},t^n;L^2(\Omega;\mathbb{R}))}^2. \end{aligned} \tag{6.57}$$

We next write, for all $n \in [\![1, N]\!]$, all $i \in [\![1, M]\!]$, and all $\underline{q}_{h,i} \in \underline{P}_{h,i}^k$,

$$\begin{aligned} \left|(\mathrm{d}_t^n \psi_i - \delta_t^n \psi_i, q_{h,i})\right| &\le \|\mathrm{d}_t^n \psi_i - \delta_t^n \psi_i\|_{L^2(\Omega;\mathbb{R})}\ \|q_{h,i}\|_{L^2(\Omega;\mathbb{R})} \\ &\le \tau^{\frac{1}{2}} \|\psi_i\|_{H^2(t^{n-1},t^n;L^2(\Omega;\mathbb{R}))}\ \|q_{h,i}\|_{L^2(\Omega;\mathbb{R})} \\ &\lesssim \tau^{\frac{1}{2}} \|\psi_i\|_{H^2(t^{n-1},t^n;L^2(\Omega;\mathbb{R}))}\ \|\underline{q}_{h,i}\|_{\mathrm{c},h,\mathrm{hho}}, \end{aligned}$$

where we have used a Cauchy–Schwarz inequality in the first line, the bound (6.57) in the second line, and a discrete global Poincaré inequality in HHO spaces (resulting from a combination of [19, Proposition 5.4] and [26, Lemma 4]) to conclude. Using the above estimate in conjunction with the definition (6.15) of the dual norm, we have that

$$\|(\mathrm{d}_t^n \psi_i(\boldsymbol{p}) - \delta_t^n \psi_i(\boldsymbol{p}), \cdot)\|_{\mathrm{c},h,*}^2 \lesssim \tau \|\psi_i(\boldsymbol{p})\|_{H^2(t^{n-1},t^n;L^2(\Omega;\mathbb{R}))}^2.$$

Using this bound, we obtain

$$\begin{aligned} \mathfrak{T}_4 &\lesssim \sum_{i=1}^M \sum_{n=1}^N \tau^2 K_i^{-1} \|\psi_i(\boldsymbol{p})\|_{H^2(t^{n-1},t^n;L^2(\Omega;\mathbb{R}))}^2 \\ &= \tau^2 \sum_{i=1}^N K_i^{-1} \|\psi_i(\boldsymbol{p})\|_{H^2(0,t_{\mathrm{F}};L^2(\Omega;\mathbb{R}))}^2 = \tau^2 \mathcal{B}. \end{aligned} \tag{6.58}$$

Plugging (6.54)–(6.58) into (6.53) yields (6.52).

6.4.4 Error Estimate for the HHO-DG Scheme

The proof of the error estimate for the HHO-DG scheme follows by adapting the arguments used in Theorem 6.1 to a different choice of the interpolates of the continuous pressures in (6.48). For all $n \in [\![0, N]\!]$ and all $i \in [\![1, M]\!]$, we set

$$\epsilon_{h,i}^n := p_{h,i}^n - \hat{p}_{h,i}^n,$$

where $\hat{p}_{h,i}^0 := \pi_h^k p_i^0$ and, for $n \geq 1$, $\hat{p}_{h,i}^n$ is the solution of problem (6.20) with $r = p_i^n$.

Theorem 6.2 (Error estimate for the HHO-DG scheme) *Assume $k \geq 1$, Ω convex, and the additional regularity*

$$\begin{aligned}
\boldsymbol{u} &\in H^1(0, t_{\mathrm{F}}; H^{k+2}(\mathcal{T}_h; \mathbb{R}^d)),\\
p_0 &\in H^1(0, t_{\mathrm{F}}; H^{k+1}(\mathcal{T}_h; \mathbb{R})),\\
\psi_0(\boldsymbol{p}) &\in H^1(0, t_{\mathrm{F}}; H^{k+1}(\mathcal{T}_h; \mathbb{R}))\\
\forall i \in [\![1, M]\!], \quad S_i(\boldsymbol{p}) &\in C^0([0, t_{\mathrm{F}}]; H^{k+1}(\mathcal{T}_h; \mathbb{R})),\\
\forall i \in [\![1, M]\!], \quad \psi_i(\boldsymbol{p}) &\in H^2(0, t_{\mathrm{F}}; L^2(\Omega; \mathbb{R})) \cap H^1(0, t_{\mathrm{F}}; H^{k+1}(\mathcal{T}_h; \mathbb{R})),
\end{aligned}$$

with $\psi_0(\boldsymbol{p}) := \lambda^{-1}(\boldsymbol{\alpha}\cdot\boldsymbol{p} - p_0)$. Then, for a time step τ small enough (with threshold independent of h), it holds that

$$\max_{n\in[\![1,N]\!]} \left(\mu \|\underline{\boldsymbol{e}}_h^n\|_{\mathrm{a},h}^2 + \lambda^{-1} \|\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}_h^n\|_{L^2(\Omega;\mathbb{R})}^2 + \sum_{i=1}^M C_i \ \|\epsilon_{h,i}^n\|_{L^2(\Omega;\mathbb{R})}^2 + \frac{\beta^2}{\mu} \|\epsilon_{h,0}^n\|_{L^2(\Omega;\mathbb{R})}^2 \right)$$
$$+ \sum_{n=1}^N \tau \|\boldsymbol{\epsilon}_h^n\|_{\boldsymbol{\xi}}^2 + \sum_{i=1}^M \sum_{n=1}^N \tau K_i \ \|\epsilon_{h,i}^n\|_{\mathrm{c},h,\mathrm{dg}}^2 \lesssim h^{2(k+1)} \mathcal{A}^{\mathrm{dg}} + \tau^2 \mathcal{B}^{\mathrm{dg}}, \tag{6.59}$$

where the hidden constant is independent of h, τ, of the problem data, of $\boldsymbol{u}$, and of p_i, $i \in [\![0, M]\!]$, but possibly depends on Ω, t_{F}, k, and we have set

$$\begin{aligned}
\mathcal{A}^{\mathrm{dg}} &:= \|\boldsymbol{u}\|_{H^1(0,t_{\mathrm{F}};H^{k+2}(\mathcal{T}_h;\mathbb{R}^d))}^2 + \mu^{-1} \|p_0\|_{H^1(0,t_{\mathrm{F}};H^{k+1}(\mathcal{T}_h;\mathbb{R}^d))}^2\\
&\quad + \sum_{i=0}^M \lambda\alpha_i^{-2} \|\psi_i(\boldsymbol{p})\|_{H^1(0,t_{\mathrm{F}};H^{k+1}(\mathcal{T}_h;\mathbb{R}))}^2 + \sum_{i=1}^M \lambda\alpha_i^{-2} \|S_i(\boldsymbol{p})\|_{L^2(0,t_{\mathrm{F}};H^{k+1}(\mathcal{T}_h;\mathbb{R}))}^2,\\
\mathcal{B}^{\mathrm{dg}} &:= \sum_{i=1}^M \lambda\alpha_i^{-2} \ \|\psi_i(\boldsymbol{p})\|_{H^2(0,t_{\mathrm{F}};L^2(\Omega;\mathbb{R}))}^2.
\end{aligned}$$

Proof Proceeding as in the proof of Proposition 6.1 and recalling the definition of the elliptic projection in (6.20), it is readily inferred that

$$\underline{\boldsymbol{e}}_h^0 = \underline{\mathbf{0}}, \qquad \epsilon_{h,i}^0 = 0, \quad \forall i \in [\![0, M]\!] \tag{6.60a}$$

and, for $n \in [\![1, N]\!]$, it holds, for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$, all $q_{h,0} \in P_{h,0}^k$,

$$2\mu\ \mathrm{a}_h(\underline{\boldsymbol{e}}_h^n, \underline{\boldsymbol{v}}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, \epsilon_{h,0}^n) = 2\mu\mathcal{E}_{\mathrm{a},h}(\boldsymbol{u}^n; \underline{\boldsymbol{v}}_h) + \mathcal{E}_{\mathrm{b},h}(p_0^n; \underline{\boldsymbol{v}}_h), \tag{6.60b}$$

$$\mathrm{b}_h(\delta_t^n \underline{\boldsymbol{e}}_h, q_{h,0}) - \lambda^{-1}(\delta_t^n(\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}_h), q_{h,0}) = -(\delta_t^n(\psi_0(\boldsymbol{p} - \hat{\boldsymbol{p}}_h), q_{h,0}), \tag{6.60c}$$

and, for all $i \in [\![1, M]\!]$ and $q_{h,i} \in P_{h,i}^k$,

$$\begin{aligned}(\delta_t^n \psi_i(\boldsymbol{\epsilon}_h), q_{h,i}) &+ (S_i(\boldsymbol{\epsilon}_h^n), q_{h,i}) + K_i\ \mathrm{c}_h^{\mathrm{dg}}(\epsilon_{h,i}^n, q_{h,i}) \\ &= (S_i(\boldsymbol{p}^n - \hat{\boldsymbol{p}}_h^n), q_{h,i}) + (\mathrm{d}_t^n \psi_i(\boldsymbol{p}) - \delta_t^n \psi_i(\boldsymbol{p}), q_{h,i}) + (\delta_t^n \psi_i(\boldsymbol{p} - \hat{\boldsymbol{p}}_h), q_{h,i}),\end{aligned} \tag{6.60d}$$

where, in (6.60c), we have applied discrete time derivation and introduced the linear function ψ_0 defined such that, for all $\boldsymbol{q} \in \mathbb{R}^{M+1}$, $\psi_0(\boldsymbol{q}) := \lambda^{-1}(\boldsymbol{\alpha}\cdot\boldsymbol{q} - q_0)$. Then, following the first two step of the proof of Lemma 6.1 we obtain an estimate similar to (6.34), namely, for an arbitrary $\mathsf{N} \in [\![1, N]\!]$ it holds

$$\begin{aligned}&\mu\|\underline{\boldsymbol{e}}_h^{\mathsf{N}}\|_{\mathrm{a},h}^2 + \frac{\|\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}_h^{\mathsf{N}}\|_{L^2(\Omega;\mathbb{R})}^2}{2\lambda} + \sum_{i=1}^{M} \frac{C_i}{2}\|\epsilon_{h,i}^{\mathsf{N}}\|_{L^2(\Omega;\mathbb{R})}^2 + \sum_{n=1}^{\mathsf{N}} \frac{\tau}{2}\|\boldsymbol{\epsilon}_h^n\|_{\boldsymbol{\xi}}^2 + \sum_{i=1}^{M}\sum_{n=1}^{\mathsf{N}} \tau K_i \|\epsilon_{h,i}^n\|_{\mathrm{c},h,\mathrm{dg}}^2 \\ &\le \sum_{n=1}^{\mathsf{N}} \tau\left(2\mu\mathcal{E}_{\mathrm{a},h}(\boldsymbol{u}^n; \delta_t^n \underline{\boldsymbol{e}}_h) + \mathcal{E}_{\mathrm{b},h}(p_0^n; \delta_t^n \underline{\boldsymbol{e}}_h)\right) + \sum_{i=0}^{M}\sum_{n=1}^{\mathsf{N}} \tau(\mathcal{E}_{i,h}^n(\boldsymbol{p}), \epsilon_{h,i}^n),\end{aligned} \tag{6.61}$$

with $\mathcal{E}_{0,h}^n(\boldsymbol{p}) := \delta_t^n \psi_0(\boldsymbol{p} - \hat{\boldsymbol{p}}_h)$ and, for all $i \in [\![1, M]\!]$,

$$\mathcal{E}_{i,h}^n(\boldsymbol{p}) := (\mathrm{d}_t^n \psi_i(\boldsymbol{p}) - \delta_t^n \psi_i(\boldsymbol{p})) + S_i(\boldsymbol{p}^n - \hat{\boldsymbol{p}}_h^n) + \delta_t^n \psi_i(\boldsymbol{p} - \hat{\boldsymbol{p}}_h).$$

The first term in the right-hand side of (6.61) can be bounded as in (6.35). We bound the second term by using the Cauchy–Schwarz and Young inequality to write

$$\sum_{i=0}^{M}\sum_{n=1}^{\mathsf{N}} \tau(\mathcal{E}_{i,h}^n(\boldsymbol{p}), \epsilon_{h,i}^n) \le \sum_{i=0}^{M}\sum_{n=1}^{\mathsf{N}} \frac{\tau\lambda}{2\alpha_i^2}\|\mathcal{E}_{i,h}^n(\boldsymbol{p})\|_{L^2(\Omega,\mathbb{R})}^2 + \sum_{n=1}^{\mathsf{N}} \frac{\tau}{2\lambda}\|\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}_h^n\|_{L^2(\Omega;\mathbb{R})}^2.$$

Therefore, proceeding as in steps (i.C) and (ii) of Lemma 6.1, yields

$$\max_{n\in[\![1,N]\!]}\left(\mu\|\underline{\boldsymbol{e}}_h^n\|_{\mathrm{a},h}^2+\lambda^{-1}\|\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}_h^n\|_{L^2(\Omega;\mathbb{R})}^2+\sum_{i=1}^M C_i\ \|\epsilon_{h,i}^n\|_{L^2(\Omega;\mathbb{R})}^2+\frac{\beta^2}{\mu}\|\epsilon_{h,0}^n\|_{L^2(\Omega;\mathbb{R})}^2\right)$$
$$+\sum_{n=1}^N\tau\|\boldsymbol{\epsilon}_h^n\|_{\boldsymbol{\xi}}^2+2\sum_{i=1}^M\sum_{n=1}^N\tau K_i\ \|\epsilon_{h,i}^n\|_{\mathrm{c},h,\mathrm{dg}}^2\lesssim\exp\left(\frac{t_\mathrm{F}}{1-\tau}\right)\left(\mathfrak{T}_1+\mathfrak{T}_2+\mathfrak{T}_3^{\mathrm{dg}}+\mathfrak{T}_4^{\mathrm{dg}}\right),\tag{6.62}$$

where

$$\mathfrak{T}_3^{\mathrm{dg}}:=\sum_{n=1}^N\tau\left(\sum_{i=0}^M\lambda\alpha_i^{-2}\|\delta_t^n\psi_i(\boldsymbol{p}-\hat{\boldsymbol{p}}_h)\|_{L^2(\Omega;\mathbb{R})}^2+\sum_{i=1}^M\lambda\alpha_i^{-2}\|S_i(\boldsymbol{p}^n-\hat{\boldsymbol{p}}_h^n)\|_{L^2(\Omega;\mathbb{R})}^2\right),$$
$$\mathfrak{T}_4^{\mathrm{dg}}:=\sum_{i=1}^M\sum_{n=1}^N\tau\lambda\alpha_i^{-2}\|\mathrm{d}_t^n\psi_i(\boldsymbol{p})-\delta_t^n\psi_i(\boldsymbol{p})\|_{L^2(\Omega;\mathbb{R})}^2,$$

and the terms $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are defined in (6.53) and bounded in (6.54) and (6.55), respectively. The term $\mathfrak{T}_4^{\mathrm{dg}}$ can be bounded using (6.57) and (6.58) to obtain $\mathfrak{T}_4^{\mathrm{dg}}\lesssim\tau^2\mathcal{B}^{\mathrm{dg}}$. Hence, it only remains to bound $\mathfrak{T}_3^{\mathrm{dg}}$. Owing to the linearity of the backward time derivative δ_t^n and the functions ψ_i and S_i for all $i\in[\![1,M]\!]$, the approximation property (6.21) of the elliptic projection, and the boundedness property (6.5), we infer

$$\mathfrak{T}_3^{\mathrm{dg}}\lesssim h^{2(k+1)}\sum_{n=1}^N\tau\left(\sum_{i=0}^M\lambda\alpha_i^{-2}\|\delta_t^n\psi_i(\boldsymbol{p})\|_{H^{k+1}(\mathcal{T}_h;\mathbb{R})}^2+\sum_{i=1}^M\lambda\alpha_i^{-2}\|S_i(\boldsymbol{p}^n)\|_{H^{k+1}(\mathcal{T}_h;\mathbb{R})}^2\right)$$
$$\lesssim h^{2(k+1)}\mathcal{A}^{\mathrm{dg}}.$$

Combining the previous bounds with (6.62) leads to the conclusion.

Table 6.1 Model parameters

Parameter	Unit	Set *i*	Set *ii*	Set *iii*	Set *iv*
μ	MPa	4.2	4.2	4.2	4.2
λ	MPa	2.4	$2.4\cdot10^5$	2.4	2.4
α_1	–	0.95	0.95	0.95	0.95
α_2	–	0.12	0.12	0.12	0.12
C_1	MPa^{-1}	0.054	0.054	0.0	0.054
C_2	MPa^{-1}	0.014	0.014	0.0	0.014
K_1	$\mathrm{m}^2\,\mathrm{MPa}^{-1}\,\mathrm{s}^{-1}$	$6.18\cdot10^{-6}$	$6.18\cdot10^{-6}$	$6.18\cdot10^{-6}$	10^{-12}
K_2	$\mathrm{m}^2\,\mathrm{MPa}^{-1}\,\mathrm{s}^{-1}$	$2.72\cdot10^{-5}$	$2.72\cdot10^{-5}$	$2.72\cdot10^{-5}$	10^{-11}
$\xi_{1\leftarrow2}$	$\mathrm{MPa}^{-1}\,\mathrm{s}^{-1}$	0.01	0.01	0.01	0.01

Table 6.2 Convergence rates for the HHO-DG discretisation with polynomial degree $k = 1$ based on manufactured solutions of the Barenblatt–Biot problem, see text for details

Set	$\|\underline{\boldsymbol{e}}_{h\tau}\|_{\infty,1}$	EOC	$\|\epsilon_{0,h\tau}\|_{\infty,0}$	EOC	$\|\epsilon_{1,h\tau}\|_{\infty,0}$	EOC	$\|\epsilon_{2,h\tau}\|_{\infty,0}$	EOC
i	2.39e−01	–	5.60e−01	–	4.78e−01	–	2.48e−01	–
	6.23e−02	1.94	1.11e−01	2.24	9.31e−02	2.36	4.80e−02	2.37
	1.51e−02	2.05	2.28e−02	2.28	1.88e−02	2.31	1.01e−02	2.24
	3.73e−03	2.01	4.92e−03	2.21	3.83e−03	2.29	2.52e−03	2.01
	9.39e−04	1.99	1.08e−03	2.19	7.55e−04	2.34	6.28e−04	2.00
ii	2.43e−01	–	8.25e−01	–	1.43e−01	–	1.32e−01	–
	6.26e−02	1.95	1.55e−01	2.41	3.76e−02	1.92	3.86e−02	1.77
	1.51e−02	2.05	3.09e−02	2.33	9.16e−03	2.04	9.52e−03	2.02
	3.73e−03	2.02	6.84e−03	2.18	2.34e−03	1.97	2.49e−03	1.93
	9.35e−04	2.00	1.71e−03	2.00	6.04e−04	1.95	6.27e−04	1.99
iii	2.39e−01	–	5.67e−01	–	4.79e−01	–	3.08e−01	–
	6.23e−02	1.94	1.14e−01	2.31	9.43e−02	2.34	6.48e−02	2.25
	1.51e−02	2.05	2.40e−02	2.24	1.97e−02	2.26	1.40e−02	2.21
	3.73e−03	2.01	5.50e−03	2.13	4.45e−03	2.15	3.27e−03	2.10
	9.35e−04	2.00	1.38e−03	1.99	1.12e−03	1.99	8.19e−04	2.00
iv	2.42e−01	–	8.00e−01	–	7.78e−01	–	4.14e−01	–
	6.25e−02	1.95	1.46e−01	2.46	1.41e−01	2.47	6.28e−02	2.72
	1.51e−02	2.05	2.79e−02	2.39	2.62e−02	2.43	1.11e−02	2.50
	3.73e−03	2.01	5.58e−03	2.32	4.88e−03	2.42	2.61e−03	2.09
	9.39e−04	1.99	1.12e−03	2.31	8.43e−04	2.53	6.40e−04	2.03

6.5 Numerical Tests

In this section, we present some numerical examples to illustrate the theoretical results. In order to confirm the convergence rates predicted in Theorem 6.2, we rely on a manufactured smooth solution of a two-network poroelasticity problem (i.e. the Barenblatt–Biot problem) on the unit square domain $\Omega = (0, 1)^2$ and time interval $[0, t_F = 1)$. The exact displacement $\boldsymbol{u}$ and exact pressures p_1 and p_2 are given by,

$$
\begin{aligned}
\boldsymbol{u}(\boldsymbol{x}, t) &= \sin(\pi t) \begin{pmatrix} -\cos(\pi x_1)\cos(\pi x_2) \\ \sin(\pi x_1)\sin(\pi x_2) \end{pmatrix}, \\
p_1(\boldsymbol{x}, t) &= \pi \sin(\pi t)\big[\sin(\pi x_1)\cos(\pi x_2) + \cos(\pi x_1)\sin(\pi x_2)\big], \\
p_2(\boldsymbol{x}, t) &= \pi \sin(\pi t)\big[\sin(\pi x_1)\cos(\pi x_2) - \cos(\pi x_1)\sin(\pi x_2)\big].
\end{aligned}
$$

The total pressure p_0, volumetric load $\boldsymbol{f}$, and source terms g_1 and g_2 are inferred from the exact solution. In order to assess the robustness with respect to the model coefficients, we consider the four sets of parameters depicted in Table 6.1. The first set of model parameters is taken from [29]. The second, third, and fourth sets are meant

Table 6.3 Convergence rates for the HHO-DG discretisation with polynomial degree $k = 2$ based on manufactured solutions of the Barenblatt–Biot problem, see text for details

Set	$\|\underline{e}_{h\tau}\|_{\infty,1}$	EOC	$\|\epsilon_{0,h\tau}\|_{\infty,0}$	EOC	$\|\epsilon_{1,h\tau}\|_{\infty,0}$	EOC	$\|\epsilon_{2,h\tau}\|_{\infty,0}$	EOC
i	3.29e−02	–	8.38e−02	–	7.16e−02	–	3.31e−02	–
	4.05e−03	3.02	7.36e−03	3.51	6.15e−03	3.54	2.65e−03	3.64
	5.40e−04	2.91	8.04e−04	3.19	6.15e−04	3.32	3.48e−04	2.93
	6.93e−05	2.96	8.58e−05	3.23	5.70e−05	3.43	4.55e−05	2.93
	8.68e−06	3.00	9.43e−06	3.19	5.68e−06	3.33	5.68e−06	3.00
ii	3.36e−02	–	1.22e−01	–	1.69e−02	–	2.16e−02	–
	4.05e−03	3.05	9.88e−03	3.63	2.33e−03	2.86	2.46e−03	3.13
	5.37e−04	2.91	1.17e−03	3.08	3.21e−04	2.86	3.47e−04	2.83
	6.82e−05	2.98	1.46e−04	3.00	4.20e−05	2.94	4.56e−05	2.93
	8.52e−06	3.00	1.81e−05	3.01	5.52e−06	2.93	5.69e−06	3.00
iii	3.29e−02	–	8.61e−02	–	7.23e−02	–	4.77e−02	–
	4.04e−03	3.02	7.84e−03	3.46	6.56e−03	3.46	4.39e−03	3.44
	5.38e−04	2.91	9.51e−04	3.04	7.90e−04	3.05	5.39e−04	3.02
	6.83e−05	2.98	1.20e−04	2.99	9.90e−05	3.00	6.81e−05	2.99
	8.54e−06	3.00	1.49e−05	3.01	1.23e−05	3.01	8.45e−06	3.01
iv	3.35e−02	–	1.14e−01	–	1.12e−01	–	4.67e−02	–
	4.05e−03	3.05	8.78e−03	3.71	8.36e−03	3.75	2.90e−03	4.01
	5.40e−04	2.91	8.94e−04	3.30	7.69e−04	3.44	3.61e−04	3.01
	6.93e−05	2.96	8.97e−05	3.32	6.56e−05	3.55	4.59e−05	2.98
	8.68e−06	3.00	9.45e−06	3.25	5.80e−06	3.50	5.69e−06	3.01

to check the robustness of the method in the nearly incompressible case (i.e. large values of λ), in the vanishing storage coefficients case, and in the small permeabilities case, respectively. We remark that the value of μ and λ considered in the second test corresponds to a Poisson ratio $\nu = 0.49999$.

We consider the HHO method described in Sect. 6.3 with DG discretisation of the Darcy term for polynomial degrees $k \in \{1, 2, 3\}$ over a trapezoidal elements mesh sequence $(\mathcal{T}_h)_j$ with 2^{2+2j} elements, for $j \in [\![1, 5]\!]$. The time discretisation is based on Backward Differentiation Formulas (BDF) of order $(k + 1)$ with a fixed time step $\tau = 10^{-3}$. The boundary conditions are inferred from the exact solution. On the bottom edge $\{\boldsymbol{x} \in \partial\Omega \ : \ x_2 = 0\}$, we enforce Dirichlet conditions for the displacement and Neumann conditions for both the network pressures p_1 and p_2. On the rest of the domain boundary we set Neumann conditions for the displacement and Dirichlet for the two pressures. Initial conditions are specified by means of L^2-projections over mesh elements according to (6.22). Initialisation is performed at several time points $(t_i = -\tau\, i,\ \ i = 1, \ldots, k + 1)$, in agreement with the BDF order.

In Tables 6.2, 6.3 and 6.4 we report the convergence rates for the four set of model parameters indicated in Table 6.1. We use the following shorthand notations for the error measures:

Table 6.4 Convergence rates for the HHO-DG discretisation with polynomial degree $k = 3$ based on manufactured solutions of the Barenblatt–Biot problem, see text for details

Set	$\|\underline{\boldsymbol{e}}_{h\tau}\|_{\infty,1}$	EOC	$\|\epsilon_{0,h\tau}\|_{\infty,0}$	EOC	$\|\epsilon_{1,h\tau}\|_{\infty,0}$	EOC	$\|\epsilon_{2,h\tau}\|_{\infty,0}$	EOC
i	3.30e−03	–	8.57e−03	–	7.41e−03	–	2.77e−03	–
	2.42e−04	3.77	5.34e−04	4.00	4.48e−04	4.05	1.66e−04	4.06
	1.42e−05	4.09	2.64e−05	4.34	2.03e−05	4.46	9.44e−06	4.14
	9.26e−07	3.94	1.41e−06	4.23	8.87e−07	4.52	6.29e−07	3.91
	5.79e−08	4.00	7.49e−08	4.24	3.89e−08	4.51	3.89e−08	4.02
ii	3.36e−03	–	1.19e−02	–	1.94e−03	–	1.83e−03	–
	2.43e−04	3.79	7.14e−04	4.06	1.42e−04	3.77	1.57e−04	3.54
	1.42e−05	4.10	3.83e−05	4.22	8.91e−06	4.00	9.39e−06	4.07
	9.14e−07	3.96	2.37e−06	4.01	5.94e−07	3.91	6.28e−07	3.90
	5.66e−08	4.01	1.45e−07	4.03	3.83e−08	3.96	3.89e−08	4.01
iii	3.31e−03	–	8.94e−03	–	7.62e−03	–	4.80e−03	–
	2.42e−04	3.77	5.78e−04	3.95	4.88e−04	3.97	3.17e−04	3.92
	1.42e−05	4.10	3.15e−05	4.20	2.65e−05	4.20	1.73e−05	4.19
	9.14e−07	3.95	1.99e−06	3.99	1.67e−06	3.99	1.10e−06	3.98
	5.67e−08	4.01	1.22e−07	4.02	1.03e−07	4.02	6.78e−08	4.02
iv	3.34e−03	–	1.09e−02	–	1.08e−02	–	3.25e−03	–
	2.42e−04	3.78	6.23e−04	4.13	5.95e−04	4.18	1.78e−04	4.19
	1.42e−05	4.09	2.91e−05	4.42	2.53e−05	4.56	9.62e−06	4.21
	9.27e−07	3.94	1.45e−06	4.33	9.93e−07	4.67	6.31e−07	3.93
	5.79e−08	4.00	7.47e−08	4.28	3.94e−08	4.66	3.89e−08	4.02

$$\|\underline{\boldsymbol{e}}_{h\tau}\|_{\infty,1} := \max_{n\in[\![1,N]\!]} \|\underline{\boldsymbol{u}}_h^n - \underline{\boldsymbol{I}}_h^k \boldsymbol{u}^n\|_{\mathrm{a},h},$$
$$\|\epsilon_{i,h\tau}\|_{\infty,0} := \max_{n\in[\![1,N]\!]} \|p_{i,h}^n - \pi_h^k p_i^n\|_{L^2(\Omega;\mathbb{R})}, \quad \forall i \in [\![0,2]\!].$$

Each error measure is accompanied by the corresponding estimated order of convergence (EOC). The observed convergence rates are in agreement with the error estimate of Theorem 6.2. We remark that the performance is not affected by the different choices of the model parameters. Hence, the method is robust in all the limit cases of vanishing storage, nearly incompressible, and poorly permeable media.

Acknowledgements M. Botti acknowledges funding from the European Commission through the H2020-MSCA-IF-EF project PDGeoFF, Polyhedral Discretisation Methods for Geomechanical Simulation of Faults and Fractures in Poroelastic Media (Grant no. 896616).

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