The Pearcey Equation: From the Salpeter Relativistic Equation to Quasiparticles

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Abstract This work presents the Pearcey equation, a quasi-relativistic wave equation for spinless particles with non-zero rest mass. This equation was introduced as a mathematical tool to address the problem of nonlocality concerning the pseudo-differential operator in the Hamiltonian of the Salpeter equation. The Pearcey equation can be considered as a *way* to relativity since it embeds the peculiar features of the relativistic evolution even if it looks very similar to the Schrödinger equation. In light of the catastrophe theory, the Pearcey equation acquires a deeper physical meaning as a candidate for describing quasiparticles.

Keywords Pearcey equation · Quasiparticles · Relativistic equation · Catastrophe theory

1 The Salpeter Equation: An Historical Review from Classical to Quantum Mechanics

At the end of the 19th century, physicists believed that Newton's laws of mechanics and Maxwell's electromagnetic theory provided the necessary foundations for the understanding of almost all physical phenomena. It was well known that Newton's laws explained the dynamics of matter from heavenly bodies down to falling apples while Maxwell's four equations correctly described the character of radiation not only by the unification of electrical and magnetic phenomena but also by laying the foundations of the study of light [1–4].

Distinguished examples of the effectiveness of the "classical" interpretative model based on Newton's laws and Maxwell's equations, were the discovery of the planet Neptune, made in 1846 by the astronomer Galle using the calculations of Leverrier,

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and the discovery of electromagnetic waves by Hertz in 1886 supporting the theoretical predictions by Maxwell in 1873. These outstanding experimental confirmations led physicists to believe that they had reached the *end of physics*.

In the wake of this *optimism*, they began to erect Newton's laws and Maxwell's equations as pillars of Hercules in order to mark the confines of Physics. In fact, they thought that the few remaining unanswered questions could be solved in the well-understood framework of that time.

The words of Albert A. Michelson [5] could summarize the optimistic spirit of the physicists at the end of the XIX century:
"*The more important fundamental laws and facts of physical science have all been discovered, and these are now so firmly established that the possibility of their ever being supplanted in consequence of new discoveries is exceedingly remote*".

The revolution, which upset Physics at the beginning of the XX century, was carried by the theory of special relativity and by quantum wave mechanics.

In 1926, Schrödinger introduced a partial differential equation [6], the so-called free-Schrödinger equation, which describes the de Broglie's "matter waves" [7] inaugurating the quantum wave mechanics.

As Felix Bloch recollected, Erwin Schrödinger claimed with satisfaction: "*My colleague Debye suggested that one should have a wave equation; well I have found one!*" [8] whose expression in $(1+1)$ dimensions reads

$$ i\hbar\partial_t\Psi(x,t) = -\frac{\hbar^2}{2m}\partial_x^2\Psi(x,t). \tag{1} $$

In the above equation, the initial condition is denoted with $\Psi(x,0) = \Psi_0(x)$, $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant and m is the (rest) mass of the particle.

The Schrödinger equation was very successful in describing the known energy levels of the hydrogen atom. It has been highly successful in describing the absorption or emission of radiation where an atom undergoes the transition from one energy state to another. The frequency of the emitted radiation follows the Bohr radiation condition $\hbar\nu = E_f - E_i$. But contrary to what Bohr did, Schrödinger did not have to impose quantization because this flowed naturally from the boundary conditions imposed on the solutions of his equation.

In general, the solution to the Schrödinger equation describes the dynamical behaviour of the particle in quantum mechanics, in a similar way as Newton's equation describes the dynamics of a particle in classical physics. However, there is an important difference: the wave function Ψ does not give the trajectory of a particle as Newton's law does. Therefore physicists asked themselves what type of information Ψ gives. The answer was given by Max Born: the square-modulus of Ψ gives the probability to find the particle in a region of space at a given time. The probabilistic statement replaces the deterministic statement of classical physics and from then on, our concept of physical reality has changed.

Einstein was among many who objected to this vision, suggesting that quantum theory was incomplete. He explained his point of view in a letter to P. S. Epstein: "*I incline to the opinion that the wave function does not (completely) describe what is real, but only a (to us) empirically accessible maximal knowledge regarding that which really exists [...]. That is what I mean when I advance the view that quantum mechanics gives an incomplete description of the real state of affairs...*" [9].

Einstein's opposition was overcome thanks to the great power of prediction that the established theory of quantum mechanics had shown in explaining experiments conducted during the last century.

These experimental confirmations pushed scientists to accept the principles and postulates of quantum mechanics, although the question of what there is beyond the experiments, remains an open question. Bohr offered consolation stating that: "*There is no quantum world. There is only an abstract quantum physical description. It is wrong to think that the task of physics is to find out how Nature is. Physics concerns what we can say about Nature*" [10].

In the attempt to unify special relativity and quantum wave mechanics the spinless Salpeter equation was introduced to generalize the Schrödinger equation in the context of relativistic quantum mechanics [11–13].

Without loss of generality, limiting ourselves to consider the initial value problem in (1+1) dimensions, the spinless Salpeter equation is

$$
i\hbar\partial_t \ \psi(x,t) = \sqrt{m^2c^4 - c^2\hbar^2 \frac{\partial^2}{\partial x^2}} \ \psi(x,t),
$$
$$
\psi(x,0) = \psi_0(x),
$$
(2)

where $\psi(x,0) = \psi_0(x)$ denotes the initial condition, c is the speed of light in vacuum and m the mass of the particle.

As shown in [14–19], the solution of the initial value problem of the spinless Salpeter equation in coordinate space is given by

$$
\psi(x,t) = \int_{-\infty}^{+\infty} S(x-x',t)\psi_0(x')dx'
$$
$$
= \frac{mc^2t}{\pi\hbar} \int_{-\infty}^{+\infty} \frac{K_1\left(\frac{imc}{\hbar}\sqrt{c^2t^2 - (x-x')^2}\right)}{\sqrt{c^2t^2 - (x-x')^2}}\psi_0(x')dx',
$$
(3)

where K_1 is the modified Bessel function of the second kind of first order, also well known as McDonald function [20].

The spinless Salpeter equation (2) is different from the Klein-Gordon [21] equation, since it is a first-order in the time derivative, which would make it more similar to the Dirac equation [22]. The difference between the two is that the spinless Salpeter equation preserves the scalar nature of the wave function and it does not present

problems of probabilistic interpretation at quantum level: it only possesses positive energy solutions.

Moreover, the spinless Salpeter equation, differently from the Dirac equation, has a well-defined classical relativistic counterpart.

Although the Salpeter equation is a relativistic version of the Schrödinger equation, it has only recently stimulated the interest of the scientific community [14–19, 23–28]. This is definitely because of the mathematical complexity concerning its non local nature carried by the pseudo-differential Hamiltonian operator in (3).

However, the non locality does not disturb the light cone structure as it was analytically and numerically proven in [14–16] but makes it difficult to obtain rigorous analytical statements about the time-dependent and stationary solutions of the equation. Another difficulty arises from the fact that the equation is not a covariant equation as it should be according to the special relativity principles.

However, it has been proven that the Hilbert space of its solutions is invariant under the Lorentz-group of transformations [29].

To conclude, the Salpeter equation makes it possible to have a quantum relativistic description which does not conflict with the principles of the special relativity, even if it is a non-local and non-covariant equation: it is widely used in the phenomenological description of the quark-antiquark-gluon systems as a hadron model [30, 31] and it is as good as the Klein-Gordon equation in describing the experimental spectrum of mesonic atoms [32].

From the analysis of the Salpeter equation, a new wave-like equation, the Pearcey equation has been introduced in [14–19] in order to probe the onset of the relativistic features. This equation was introduced as an alternative way to deal with the problem of nonlocality in relativistic quantum mechanics.

It allowed to test the correctness of the results obtained for the Salpeter equation in the analysis of its solutions and the Lie-point symmetries establishing a link between two theories: the classical quantum wave mechanics and the relativistic quantum wave mechanics.

The aim of this work is to present the Pearcey equation in the context of quasi-relativistic physics. The parallelism with optics and with the catastrophe theory gives a deeper physical meaning to the Pearcey equation and its solutions. In particular, the Pearcey equation can be a candidate to describe quasiparticles.

The rest of the paper is organized as follows. Section 1 introduces the Pearcey equation, a new partial evolution equation first order in time.

Section 3 analyzes the behaviour of a Lorentzian wave-packet in order to add another example of evolution ruled by the Pearcey equation with respect to the ones presented in the previous papers on the subject [14–19]. Section 4 is devoted to the role of catastrophe theory in the light-cone structure of the solution of the Pearcey equation and of the Salpeter equation.

2 The Pearcey Equation

In the Introduction, we emphasized the important role played by the Schrödinger equation in non-relativistic quantum mechanics [1–3] and the corresponding remarkable role played by the Salpeter equation as a counterpart of the Schrödinger equation within the framework of the relativistic quantum mechanics.

Since the specific features of the solutions of the Schrödinger equation are definitely different from those of the Salpeter equation, as emerged from the numerical and asymptotic analysis developed in [14–16], it is within reason to ponder about the existence of a third theory that could bridge the two approaches, emerging as an asymptotic limit between the classical and the relativistic quantum mechanics. This third possibility can be named quasi-relativistic.

In order to define the dynamic evolution of a quasi-relativistic quantum system, the first step to consider is the series expansion of the relativistic energy-momentum relation for a freely moving particle

$$E = \sqrt{m^2c^4 + p^2c^2}, \tag{4}$$

where m and p denote respectively the rest mass and the momentum of the particle and c is the speed of light in vacuo.

The first correction of the Schrödinger equation is given by stopping the series expansion of (4) with respect to $\frac{p}{mc}$ at the fourth order, namely

$$E = mc^2\Big[1 + \frac{1}{2}\Big(\frac{p}{mc}\Big)^2 - \frac{1}{8}\Big(\frac{p}{mc}\Big)^4\Big]. \tag{5}$$

By the standard quantization rules

$$E \rightarrow i\hbar\frac{\partial}{\partial t} \qquad\qquad x \rightarrow x \qquad\qquad p \rightarrow -i\hbar\frac{\partial}{\partial x}$$

consistent with the Newton-Wigner localization scheme, we get the (1+1)D Pearcey equation

$$i\hbar\frac{\partial\tilde{\psi}(p,t)}{\partial t} = mc^2\Big[1 + \frac{1}{2}\Big(\frac{p}{mc}\Big)^2 - \frac{1}{8}\Big(\frac{p}{mc}\Big)^4\Big]\tilde{\psi}(p,t), \tag{6}$$

which in coordinate representation reads

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = mc^2\Big[1 - \frac{\hbar^2}{2m^2c^2}\frac{\partial^2}{\partial x^2} - \frac{\hbar^4}{8m^4c^4}\frac{\partial^4}{\partial x^4}\Big]\psi(x,t). \tag{7}$$

The Pearcey equation (7) is an evolution equation in time and thus, it requires the knowledge of the initial condition $\psi(x,0) = \psi_0(x)$ to fix the dynamics of the system.

In order to define the solution of the Pearcey equation (7), we resort to one of the most applied techniques: the Fourier transform method.

Following the Fourier-transformed procedure, the general solution of (7) is

$$\psi(x,t) = \frac{e^{-\frac{it}{\hbar}mc^2}}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{\frac{i}{\hbar}mc^2\left(\frac{t}{8}p^4 - \frac{t}{2}p^2 + p\cdot x\right)} \tilde{\psi}_0(p)dp. \tag{8}$$

To simplify the analysis of the equation and to facilitate the parallelism between optics and quantum mechanics, we introduce the dimensionless variables ξ and τ, expressed in term of the reduced Compton wavelength λ_C:

$$\xi = \frac{x}{\lambda_C}, \quad \tau = \frac{ct}{\lambda_C}, \quad \kappa = \frac{p}{mc} \tag{9}$$

so that (6) becomes:

$$i\frac{\partial\tilde{\psi}(\kappa,\tau)}{\partial\tau} = \left[1 + \frac{1}{2}\kappa^2 - \frac{1}{8}\kappa^4\right]\tilde{\psi}(\kappa,\tau). \tag{10}$$

Fixing the initial condition equals to a Dirac delta function, $\psi_0(\xi) = \delta(\xi)$, the above equation yields the response of the system to an impulse, i.e.

$$S(\xi,\tau) = \frac{e^{-i\tau}}{2\pi} \int_{-\infty}^{+\infty} e^{i\left(\frac{\tau}{8}\kappa^4 - \frac{\tau}{2}\kappa^2 + \kappa\cdot\xi\right)}d\kappa. \tag{11}$$

$S(\xi,\tau)$ represents the fundamental solution of (10), and hence also the kernel of the transformation which encodes also the reason behind the choice of the name: Pearcey equation. The integral $S(\xi,\tau)$ is known in optics and in particular in catastrophe theory as the Pearcey function (see Sect. 4 and reference therein for further details). The Pearcey function is defined in [33–35] by

$$P(x,y) = \int_{-\infty}^{+\infty} e^{i(s^4 + xs^2 + ys)}ds, \tag{12}$$

where x, y may be in general complex numbers and s is the integration variable.

The analogy can be clearly appreciated by making a comparison between the contourplots of the Pearcey function in Fig. 1a and the fundamental solution of the Pearcey equation in Fig. 1b where the peculiar caustic-like structure conveyed by "isolated spots" can be still recognized.

The "isolated spots" in Fig. 1 seem to resemble the uniformly distributed *granularity*, which is also present in the contourplot of the fundamental solution of the Salpeter equation [14, 15], the distinctive pattern of the relativistic behaviour.

The symmetric distribution of the spots positioned at the intersections of the parallels to the edges of the light cone is due to the homogeneity of the spacetime and, when the first and the second order coupling interactions are involved [15], it is resembling
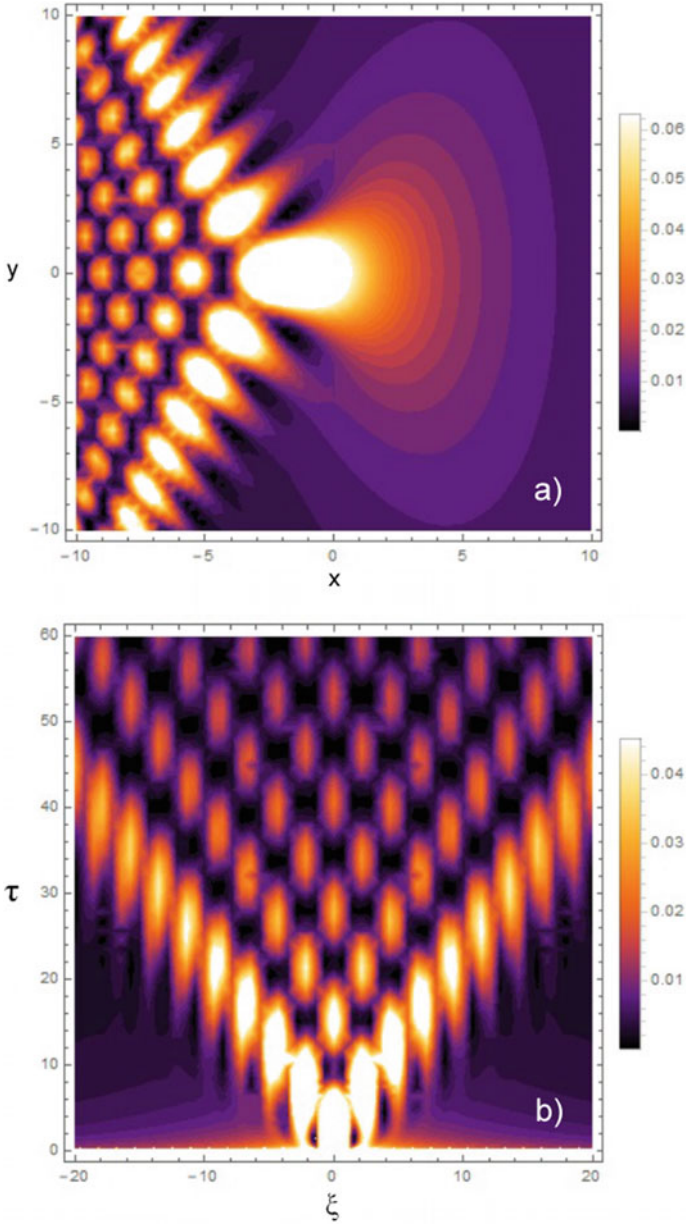
Fig. 1 A comparison between the (ξ, τ)-contourplots of the squared modulus of the Pearcey function $|P(\xi, \tau)|^2$ (**a**) and the squared modulus of the fundamental solution of the Pearcey equation $|\psi(\xi, \tau)|^2$ (**b**)

the discrete-like diffraction patterns arising from the impulse response (i.e. single state excitation) in periodic photonic lattices.

3   Evolution of the Lorentzian Wave Packet

The wave function arising from an initial Lorentzian function manifests an interesting behavior from the optical point of view, i.e. the Lorentz beams [36].
The Lorentz beam, based on experimental observations, is suitable to model the radiation emitted by single-mode diode laser [37–39].

So, as a further example in the collection of results presented for the Pearcey equation in [14–18], we can consider as initial wave function a Lorentzian defined as

$$\psi_0^L(\xi) = \frac{1}{\pi} \frac{w}{(\xi^2 + w^2)}, \qquad\qquad w > 0, \qquad\qquad (13)$$

and normalized such that $\lim_{\xi \to 0} \psi_0^L(\xi) = \delta(\xi)$.
The evolution of the Lorentzian initial input (13) ruled by the Peaecey equation reads

$$\psi^L(\xi, \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{\frac{i\tau}{8}\kappa^4 - i\frac{\tau}{2}\kappa^2 + i\kappa\xi} e^{-w|\kappa|} d\kappa =$$
$$= \frac{1}{2\pi} \left\{ \int_0^{+\infty} e^{i\frac{\tau}{8}\kappa^4 - i\frac{\tau}{2}\kappa^2 + i\kappa(\xi + iw)} d\kappa + \int_0^{+\infty} e^{i\frac{\tau}{8}\kappa^4 - i\frac{\tau}{2}\kappa^2 - i\kappa(\xi - iw)} d\kappa \right\}.$$

Here, the parameter w determines the "height" of the curve, being $\psi_0^L(0) = \frac{1}{w\pi}$, the width at the half maximum is $\psi_0^L(\pm w) = \frac{1}{2\pi w} = \frac{1}{2}\psi_0^L(0)$, and the variances of the function in both the spaces are $\sigma_\xi^2 = w^2$ and $\sigma_\kappa^2 = \frac{1}{2w^2}$, yielding $\sigma_\xi \sigma_\kappa = \frac{1}{\sqrt{2}}$.
Just like $\psi^L(\xi, \tau)$, the integrals entering the above expression can be put in relation with the Pearcey function but with the integral extending from 0 to ∞, i.e. with what is reported in the literature as the half-Pearcey function $P_{\frac{1}{2}}(x, y)$ [40, 41].
The (ξ, τ)-contour plots of the squared modulus of ψ^L are shown in the Fig. 2 for some value of w which rules the width of the Lorentzian.

As for the Gaussian initial input, the isolated spots of the fundamental function, roughly comprised within a V-like contour, dominates the evolution for small w-values.
The wave function behaves much like the fundamental solution $S(\xi, \tau)$. With increasing w, as it is evident from Fig. 2d, the spots tend to be "absorbed" in a more compact structure, as observed in Fig. 3 for the Lorentzian input under the Salpeter evolution. As w further increases, the solutions displays a hybrid relativistic-non relativistic behaviour.
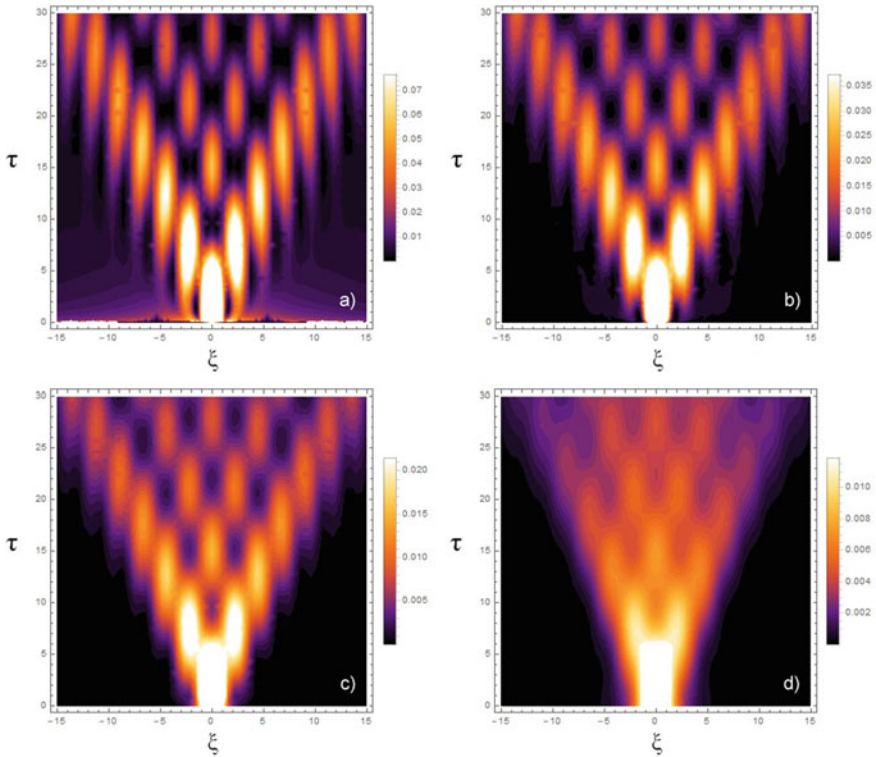
Fig. 2 (ξ, τ)-contourplot of the squared modulus of the Pearcey function $|\psi^L(\xi, \tau)|^2$ for a Lorentzian initial input with $w = 0.01$, $w = 0.5$, $w = 1$ and $w = 2$ in **a**, **b**, **c** and **d** respectively

4 Quantum Caustics and Light-Cone Structure: The Onset of Granularity in the Nature of the Light-Cone Structure

The catastrophe theory is a framework originated by the French mathematician René Thom in the 1960s [42]; it deals with the modeling of the so-called *catastrophes* which are discontinuous transitions and even sudden changes caused by smooth variations of the control parameters (or variables) involved in the system. Among all the catastrophes, there are seven types called *elementary* since they have a dependency only on a few parameters.

This classification shows a hierarchical structure where the elementary catastrophes of higher order contain the lower order ones. The lowest catastrophe is called fold and it occurs when, given one control parameter and a single state variable, the gradient of the mapping vanishes. The next higher order catastrophe is the cusp and it is originated exactly in the point where two fold lines meet [43]. Other higher order catastrophes can be generated in the same hierarchical way: the intersection of two cusp lines results in a swallowtail, and so on.
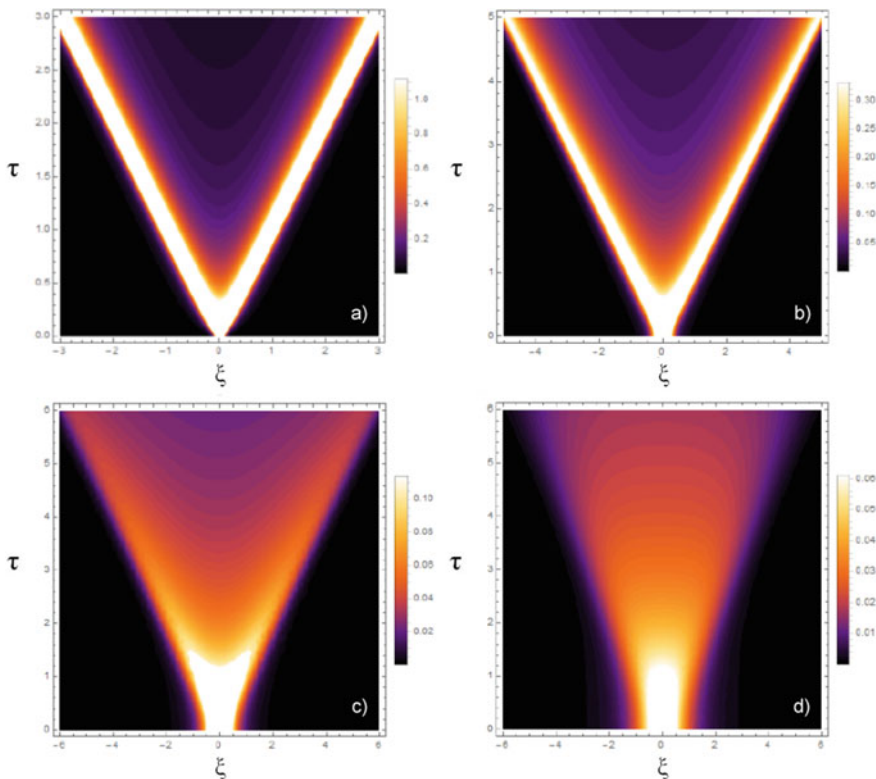
Fig. 3 (ξ, τ)-contourplot of the squared modulus of the fundamental solution of the Salpeter equation $|\psi_{Salpeter}^{L}(\xi, \tau)|^2$ for a Lorentzian initial input with $w = 0.01$, $w = 0.5$, $w = 1$ and $w = 2$ in **a**, **b**, **c** and **d** respectively

Table 1 collects the descriptions and formulas of all the seven elementary catastrophes as listed in [42].

In catastrophe theory, the wave function ϕ, also known as diffraction integral or caustic beam, is an essential tool in the description of a variety of diffraction phenomena. These seven catastrophes share the same general formula for their diffraction integrals

$$\phi(\mathbf{v}) = \int_{-\infty}^{+\infty} e^{iF(\mathbf{s},\mathbf{v})} d\mathbf{s}, \tag{14}$$

where the function F is one of the generating functions listed in Table 1 depending on the state variable vector $\mathbf{s} = (s, t)$ and on the vector $\mathbf{v} = (x, y, z, w)$ whose components are the control parameters. Each generating function defines a specific diffraction integral.

Replacing the cusp generating function (see Table 1) in (14), we get the Pearcey function (12) as defined in Sect. 2, displayed below for the reader's convenience:

Table 1 Classification of the Seven Elementary Catastrophes. The co-rank is the number of the state variables involved: s and t, whereas the control variables x, y, z and w are independent parameters that influence the state of the system. The codimension is a measure of the function's degeneracy removed by the generating functions. The generating function gives the universal unfolding of the corresponding singularity

Catastrophe	Co-rank	Control variables	Codimension	Potential functions
Fold	1	1	1	$s^3 + xs$
Cusp	1	2	2	$s^4 + xs^2 + ys$
Swallowtail	1	3	3	$s^5 + xs^3 + ys^2 + zs$
Butterfly	1	4	4	$s^6 + xs^4 + ys^3 + zs^2 + ws$
Hyperbolic umbilic	2	3	3	$s^3 + t^3 + xst + ys + zt$
Elliptic umbilic	2	3	3	$s^3 - st^2 + x(s^2 + t^2) + ys + zt$
Parabolic umbilic	2	4	4	$s^2t + t^4 + xs^2 + yt^2 + zs + wt$

$$P(x, y) = \int_{-\infty}^{+\infty} e^{i(s^4 + xs^2 + ys)} ds. \tag{15}$$

Considering the similarity between Fig. 1a, b, there should be also a mathematical correspondence between the Pearcey function and the fundamental solution of the Pearcey equation. This correspondence can be unveiled by changing the variable in the fundamental solution of the Pearcey equation (11) according to the following formula

$$\kappa = s\left(\frac{8}{\tau}\right)^{\frac{1}{4}}. \tag{16}$$

After changing the variable (16), the fundamental solution of the Pearcey equation can be written in terms of the Pearcey function:

$$\begin{aligned} S(\xi, \tau) &= \frac{1}{\pi(2\tau)^{\frac{1}{4}}} \int_{-\infty}^{+\infty} e^{i\left[s^4 - \sqrt{2\tau}s^2 + \left(\frac{8}{\tau}\right)^{\frac{1}{4}}\xi s\right]} ds \\ &= \frac{e^{-i\tau}}{\pi(2\tau)^{\frac{1}{4}}} P\left(-\sqrt{2\tau}, \left(\frac{8}{\tau}\right)^{\frac{1}{4}}\xi\right), \end{aligned} \tag{17}$$

allowing to interpret the fundamental solution as a *caustic*-like beam.

Equation (17) extends the analogy between optics and classical mechanics which characterizes the solutions of the Schrödinger equation and the paraxial wave equa-

tion to a quasi-relativistic framework [18].

The term *quasi-relativistic* denotes what happens in the gray region between the classical and the relativistic quantum theories. It is an appropriate definition since the solutions of the Pearcey equation recall the relativistic ones ruled by the Salpeter, although the former contains only one more term compared to the Schrödinger equation, that is the first relativistic correction to the kinetic energy.

Among all the peculiar features, the most relevant one is the presence of a light-cone structure. The more terms are considered in the series expansion of the relativistic energy-momentum relation (4), the more the kernel of the corresponding evolution equation tends to the Salpeter one and consequently the corresponding light-cone structure starts to visibly emerge.

Adding more terms in the Hamiltonian of the Pearcey function—namely, considering higher orders of the series expansion of the Salpeter Hamiltonian—other equations can be introduced in quasi-relativistic physics as for example the butterfly equation whose partial differential equation in the momentum space reads

$$i\hbar\frac{\partial\tilde{\psi}(p,t)}{\partial t} = mc^2\Big[1 + \frac{1}{2}\Big(\frac{p}{mc}\Big)^2 - \frac{1}{8}\Big(\frac{p}{mc}\Big)^4 + \frac{1}{16}\Big(\frac{p}{mc}\Big)^6\Big]\tilde{\psi}(p,t), \qquad (18)$$

and in coordinate representation is

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = mc^2\Big[1 - \frac{\hbar^2}{2m^2c^2}\frac{\partial^2}{\partial x^2} - \frac{\hbar^4}{8m^4c^4}\frac{\partial^4}{\partial x^4} - \frac{\hbar^6}{16m^6c^6}\frac{\partial^6}{\partial x^6}\Big]\psi(x,t). \quad (19)$$

This process for building up quasi-relativistic wave equations, based on the series expansion of the square root operator in the Hamiltonian of the Salpeter equation, allows to probe the relativistic onset in the solutions, overcoming the difficulties related to the presence of the square root. The Hamiltonians in the evolution equations generated by the series expansion seem to show a hierarchical structure quite similar to the one illustrated for the diffraction generating functions in the catastrophe theory [44].

Each higher order Hamiltonian embeds the previous smaller order one as it can be easily deduced comparing the Pearcey equation (7) with the butterfly equation (19). This hierarchical organization can be extended asymptotically until it naturally reaches the fundamental solution of the Salpeter equation which can be interpreted as the highest order diffraction integral function or caustic beam embedding the entire hierarchy of the light-cones structures, that can now be considered as caustics. This mathematical consideration has a relevant physical consequence: the Pearcey equation can be a candidate to describe quasiparticles.

Quasiparticle is a key concept that provides an intuitive understanding of complex phenomena in many-body physics [45]: it is a collective state of many particles, an elementary excitation or even a bound state of a pair of particles that has an energy-momentum relationship like a particle. An example borrowed from superconductors is the Cooper pair [46]. In a Cooper pair, the electrons can form a bound state with

opposite momenta and opposite spin. So, they can be defined in terms of a standard s-wave or a spin-0 object.

To support this idea, the light-cone structure should be related to an intrinsic velocity limit. The observed light-cone hides the presence of such a limit on the velocity of the particles and for this the Pearcey equation is a candidate for describing particles in quasi-relativistic framework. It is important to verify the caustic nature of the light-cone since this condition is a key notion for introducing the limit on the velocity for the particles.

Previously, it has been shown that the fundamental solution of the Pearcey equation is a caustic beam and all the probability density can be embedded by a couple of caustics forming a first attempt of a light-cone. In order to have further confirmations, let us consider the result shown in [44, 47] where the caustic nature of the light-cone can be mathematically appreciated considering the following couple of equations

$$
\begin{aligned}
\frac{dF}{ds} &= 0 \\
\frac{d^2 F}{ds^2} &= 0,
\end{aligned}
\tag{20}
$$

defining the necessary condition to have caustics as the edges of the light-cone.

In [44], the conditions (20) can be replaced by the so-called Lieb-Robinson (LR) maximal group velocity. The LR velocity [48] is a theoretical upper limit on the speed at which information can propagate: information cannot travel instantaneously in quantum theory, even when the relativity limits of the speed of light do not play a central role.

This velocity is finally associated with quasiparticles that were previously excited by a quench and then freely propagated in the sample [44, 49]:

$$
v_{LR} = \max_k |\frac{d\epsilon_k}{dk}|,
\tag{21}
$$

where ϵ_k is the dispersion relation for a quasiparticle in function of the quasimomentum k.

This result confirms that the Pearcey equation can bridge the Schrödinger equation and the relativistic spinless Salpeter equation, since it shows the rising of the light-cone structure and therefore it unveils the intrinsic velocity limit. In this scenario, the connection between optics and quantum mechanics, remarked by the formal analogy between the Schrödinger equation and the paraxial wave equation, is strengthened and it is extended to the quasi-relativistic physics.

Another interesting feature emerged in the analysis of the fundamental solution of the Pearcey equation (but also in the Salpeter equation) is the granularity of the edges in the light-cone structure which is still present whereas the spot-like pattern in the central part of the light-cone tends to disappear when higher order terms in the Hamiltonian are considered.

This nests the idea that spots along the edges have a different origin from the spots inside the light-cone in the fundamental solution of the Pearcey equation. The strategy consists into focusing our attention not on the "colored" spots, i.e. where the probability to find the particle or the intensity is greater, but on the darkest regions in Fig. 1, which are usually called vortices in the optical catastrophe theory [44].

An optical vortex corresponds to a zero in the optical field, so it explains why it can be found in the darkest region between the spots in the contourplot of Fig. 1. The appearance of these vortices in the fundamental solution of the Pearcey equation can be explained considering the relation with the Pearcey caustic beam, whose complex nature is at the origin of the vortices.

As observed in [44], it is possible to distinguish the darkest regions or vortices in two "groups" forming a fine structure in the diffraction integrals. The reason behind the vortices classification is their positions in the light-cone structure. In fact there is a network of vortex-antivortex pair inside it and single rows of vortices lining the outer edges.

The procedure to find these vortices in a continuum approach, consists in covering the plane with loops around which we integrate the phase of the Pearcey function. This interpretation is intriguing and it perfectly matches the result obtained in [14–19] concerning the fundamental solution of the Pearcey and the Salpeter equations. The singularity on the edges of the light-cone structure shows a discontinuity by $e^{i\pi}$ which generates the peculiar granularity in the relativistic behaviour.

In conclusion, the Pearcey equation, introduced from the series expansion of the Hamiltonian in the spinless Salpeter equation, seems to be a perfect candidate for describing spinless quasiparticles in the quasi-relativistic framework.

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