Minimal Surfaces Under Constrained Willmore Transformation

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Abstract The class of constrained Willmore (CW) surfaces in space-forms constitutes a Möbius invariant class of surfaces with strong links to the theory of integrable systems, with a *spectral deformation* [8], defined by the action of a loop of flat metric connections, and *Bäcklund transformations* [9], defined by a dressing action by simple factors. Constant mean curvature (CMC) surfaces in 3-dimensional space-forms are [25] examples of CW surfaces, characterized by the existence of some *polynomial conserved quantity* [21, 22, 24]. Both CW spectral deformation and CW Bäcklund transformation preserve [21, 22, 24] the existence of such a conserved quantity, defining, in particular, transformations within the class of CMC surfaces in 3-dimensional space-forms, with, furthermore [21, 22, 24], preservation of both the space-form and the mean curvature, in the latter case. A classical result by Thomsen [28] characterizes, on the other hand, isothermic Willmore surfaces in 3-space as minimal surfaces in some 3-dimensional space-form. CW transformation preserves [8, 9] the class of Willmore surfaces, as well as the isothermic condition, in the particular case of spectral deformation [8]. We define, in this way, a CW spectral deformation and CW Bäcklund transformations of minimal surfaces in 3-dimensional space-forms into new ones, with preservation of the space-form in the latter case. This paper is dedicated to a reader-friendly overview of the topic.

Keywords Willmore energy · Constrained Willmore surfaces · Constant mean curvature surfaces · Minimal surfaces · Isothermic surfaces · Bäcklund transformations · Spectral deformation · Polynomial conserved quantities

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1 Introduction

A central theme in Mathematics is that of the search for the optimal representative within a certain class of objects, often driven by the minimization of some energy, reflecting what occurs in many physical processes. From the early 1960s, Thomas Willmore devoted particular attention to the quest for the optimal immersion of a given closed surface in Euclidean 3-space, regarding the minimization of some natural energy, motivated by questions on the elasticity of certain biological membranes and the energetic cost associated with membrane bending deformations.

We can characterize how much a membrane is bent at a particular point on the membrane by means of the curvature of the osculating circles of the planar curves obtained as perpendicular cross sections through the point. The curvature of these circles consists of the inverse of their radii, with a positive or negative sign depending on whether the membrane curves upwards or downwards, respectively. The minimal and maximal values of the radii of the osculating circles associated with a particular point on the membrane define the principal curvatures, k_1 and k_2, and, from these, the mean curvature, $H = (k_1 + k_2)/2$ and the Gaussian curvature, $K = k_1 k_2$, at the point.

In modern literature on the elasticity of membranes, a weighed sum

$$a \int H + b \int H^2 + c \int K,$$

of the total mean curvature, the total squared mean curvature and the total Gaussian curvature, is considered to be the elastic bending energy of a membrane. By physical considerations, the total mean curvature is neglected. On the other hand, from the perspective of critical points of energy, in deformations conserving the topological type, the total Gaussian curvature can be ignored, according to Gauss–Bonnet theorem. What's left is what Willmore considered to be the *elastic bending energy* of a compact, oriented Riemannian surface, without boundary, isometrically immersed in $\mathbb{R}^3$, nowadays known as the *Willmore energy*.

The Willmore energy had already made its appearance early in the nineteenth century, through the works of Marie-Sophie Germain [15, 16] and Siméon Poisson [20] and their pioneering studies on elasticity and vibrating properties of thin plates, with the claim that the elastic force of a thin plate is proportional to its mean curvature. Since then, the mean curvature has remained a key concept in the theory of elasticity. The Willmore energy appeared again in the 1920s, in the works of Wilhelm Blaschke [1] and Gerhard Thomsen [28], but their findings were forgotten and only brought to light after the increased interest on the subject motivated by the work of Thomas Willmore.

Willmore surfaces are the critical points of the Willmore energy functional. Minimal surfaces, in their turn, are defined variationally as the stationary configurations for the area functional, amongst all those spanning a given boundary.

Minimal surfaces were first considered by Joseph-Louis Lagrange [17], in 1762, who raised the question of existence of surfaces of least area among all those spanning a given closed curve in Euclidean 3-space as boundary. Earlier, Leonhard Euler [14] had already discussed minimizing properties of the surface now known as the catenoid, although he only considered variations within a certain class of surfaces. The problem raised by Lagrange became known as the Plateau's Problem, referring to Joseph Plateau [19], who first experimented with soap films.

A physical model of a minimal surface can be obtained by dipping a wire loop into a soap solution. The resulting soap film is minimal in the sense that it always tries to organize itself so that its surface area is as small as possible whilst spanning the wire contour. This minimal surface area is, naturally, reached for the flat position, which happens to be a position of vanishing mean curvature. This does not come as a particular feature of this rather simple example of minimal surface. In fact, the Euler–Lagrange equation of the variational problem underlying minimal surfaces turns out to be precisely the zero mean curvature equation, as discovered by Jean Baptiste Meusnier [18]. The flat position of the soap film is also the position in which the membrane is the most relaxed. These surfaces are elastic energy minimals and, in this way, examples of Willmore surfaces.

Unlike flat soap films, soap bubbles exist under a certain surface tension, in an equilibrium where slightly greater pressure inside the bubble is balanced by the area-minimizing forces of the bubble itself. With their spherical shape, soap bubbles are examples of area-minimizing surfaces under the constraint of volume enclosed. These are surfaces of (non-zero) constant mean curvature and examples of *constrained Willmore surfaces*, the generalization of Willmore surfaces that arises when we consider critical points of the Willmore functional only with respect to infinitesimally conformal variations.

A very interesting fact about the Willmore energy is that it is scale-invariant: if one dilates the surface by any factor, the Willmore energy remains the same. Think of a round sphere in $\mathbb{R}^3$ as an example: if one increases the radius, the surface becomes flatter and its squared mean curvature decreases, but, at the same time, the surface area gets larger, which increases the value of the total squared mean curvature over the surface. One can show that these two phenomena counterbalance each other on any surface. In fact, the Willmore energy has the remarkable property of being invariant under any conformal transformation of $\mathbb{R}^3$, as established in a paper by James White [30] and, actually, already known to Blaschke [1] and Thomsen [28].

The class of constrained Willmore surfaces in space-forms constitutes a Möbius invariant class of surfaces with strong links to the theory of integrable systems, with a *spectral deformation*, defined by Fran Burstall, Franz Pedit and Ulrich Pinkall [8], by the action of a loop of flat metric connections, and *Bäcklund transformations* [9], defined by a dressing action by simple factors.

Constant mean curvature surfaces in 3-dimensional space-forms are examples of constrained Willmore surfaces, as established by Jörg Richter [25], characterized by the existence of some *polynomial conserved quantity* [21, 22, 24]. Both constrained Willmore spectral deformation and constrained Willmore Bäcklund transformation preserve [21, 22, 24] the existence of such a conserved quantity, for special choices of parameters, defining, in particular, transformations within the class of constant mean curvature surfaces in 3-spaces, with, furthermore [21, 22, 24], preservation of both the space-form and the mean curvature, in the latter case.

A classical result by Thomsen [28] characterizes, on the other hand, isothermic Willmore surfaces in 3-space as minimal surfaces in some 3-dimensional space-form. Constrained Willmore transformation preserves [8, 9] the class of Willmore surfaces, as well as, in the particular case of spectral deformation [8], the isothermic condition.

We define, in this way, a constrained Willmore spectral deformation and constrained Willmore Bäcklund transformations of minimal surfaces in 3-dimensional space-forms into new ones, with preservation of the space-form, in the latter case. This paper is dedicated to a reader-friendly overview of the topic. A detailed account of elementary computations can be found in [22, 23].

Along this text, we shall make no explicit distinction between a bundle and its complexification, and move from real tensors to complex tensors by complex multilinear extension, preserving notation. Our theory is local and, throughout this text, restriction to a suitable non-empty open set shall be underlying. Underlying throughout will be, as well, the identification

$$\wedge^2\mathbb{R}^{n+1,1} \cong o(\mathbb{R}^{n+1,1})$$

of the exterior power $\wedge^2\mathbb{R}^{n+1,1}$ with the orthogonal algebra $o(\mathbb{R}^{n+1,1})$ via

$$u \wedge v(w) := (u, w)v - (v, w)u,$$

for $u, v, w \in \mathbb{R}^{n+1,1}$.

2 The Willmore Energy

Among the classes of Riemannian submanifolds, there is that of *Willmore surfaces*, named after Willmore [31], in the 1960s, although the topic was mentioned by Blaschke [1] and Thomsen [28], in the 1920s, as a variational problem of optimal realization of a given closed surface in Euclidean 3-surface, regarding the minimization of some natural energy, motivated by questions on the elasticity of certain biological membranes and vesicles.

In modern literature on the elasticity of membranes, a weighed sum of the total squared mean curvature and the total Gaussian curvature, is considered to be the elastic energy of a membrane. From the perspective of critical points of energy, in deformations conserving the topological type, the total Gaussian curvature can be ignored, according to Gauss–Bonnet theorem. What's left is what is defined as the *Willmore energy*,

$$\mathcal{W} = \int_M H^2 dA,$$

of a compact, oriented (Riemannian) surface M, without boundary, (isometrically) immersed in $\mathbb{R}^3$.

From the perspective of energy extremals, the Willmore functional can be extended to compact, oriented (Riemannian) surfaces (isometrically) immersed in a general Riemannian manifold $\hat{M}$ with constant sectional curvature, or *space-form*, by means of

$$\mathcal{W} = \int_M |\Pi_0|^2 dA,$$

the total squared norm of the trace-free part Π_0 of the second fundamental form: by the Gauss equation, relating the curvature tensors of M and $\hat{M}$, we have

$$|\Pi_0|^2 = 2(|\mathcal{H}|^2 - K + \hat{K}),$$

for $\mathcal{H}$ the mean curvature vector and K and $\hat{K}$ the sectional curvatures of M and $\hat{M}$, respectively, so that, in the particular case of surfaces in $\mathbb{R}^3$,

$$|\Pi_0|^2 = 2(H^2 - K),$$

and, therefore, the two functionals share critical points. *Willmore surfaces* are the critical points of the Willmore functional.

3 Conformal Invariance and the Central Sphere Congruence

It is well-known that the Levi-Civita connection is not a conformal invariant (see, for example, [32, Sect. 3.12]). Although the second fundamental form is not conformally invariant, under a conformal change of the metric, its trace-free part remains invariant (see [23, Sect. 2.1]), so the respective squared norm and the area element change in inverse ways, leaving the Willmore energy unchanged and establishing the class of Willmore surfaces as a conformally invariant class. There is then no reason for carrying a distinguished metric—instead, we consider a conformal class of metrics.

Our study is one of surfaces in n-dimensional space-forms, with $n \geq 3$, from a conformally invariant point of view. So let S^n be the conformal n-sphere, in which, by stereographic projection, we find, in particular, the Euclidean n-space, as well as two copies of hyperbolic n-space. Our surfaces are immersions

$$\Lambda : M \to S^n$$

of a compact, oriented surface M, which we provide with the conformal structure $\mathcal{C}_\Lambda$ induced by Λ and with the canonical complex structure (that is, 90° rotation in the positive direction in tangent spaces, a notion that is, obviously, invariant under conformal changes of the metric). We find a convenient setting in Darboux's light-cone model of the conformal n-sphere [11]. We follow the modern account presented in [3]. So consider the Lorentzian space $\mathbb{R}^{n+1,1}$ and its light-cone $\mathcal{L}$, and fix a unit time-like vector t_0. We identify $v \in S^n \subseteq \mathbb{R}^{n+1}$ with the light-line through $v + t_0$, identifying, in this way, S^n with the projectivized light-cone,

$$S^n \cong \mathbb{P}(\mathcal{L}).$$

For us, a surface is, in this way, a null line subbundle $\Lambda = \langle\sigma\rangle$ of the trivial bundle $\underline{\mathbb{R}}^{n+1,1} = M \times \mathbb{R}^{n+1,1}$, with $\sigma : M \to \mathcal{L}$ a never-zero section of Λ. For further reference, set

$$\Lambda^{1,0} := \Lambda \oplus d\sigma(T^{1,0}M), \quad \Lambda^{0,1} := \Lambda \oplus d\sigma(T^{0,1}M),$$

independently of the choice of a never-zero $\sigma \in \Gamma(\Lambda)$, and then

$$\Lambda^{(1)} := \Lambda^{1,0} + \Lambda^{0,1}.$$

A fundamental construction in conformal geometry of surfaces is the mean curvature sphere congruence, or central sphere congruence, the bundle of 2-spheres tangent to the surface and sharing with it mean curvature vector at each point (although the mean curvature vector is not conformally invariant, under a conformal change of the metric, it changes in the same way for the surface and the osculating 2-sphere). In the light-cone picture, 2-spheres correspond to (3, 1)-planes in $\mathbb{R}^{n+1,1}$ and, in this way, the central sphere congruence defines a map

$$S : M \to \mathrm{Gr}_{(3,1)}(\mathbb{R}^{n+1,1}),$$

into the Grassmannian $\mathcal{G} := \mathrm{Gr}_{(3,1)}(\mathbb{R}^{n+1,1})$ of (3, 1)-planes in $\mathbb{R}^{n+1,1}$. We have, therefore, a decomposition

$$\mathbb{R}^{n+1,1} = S \oplus S^\perp$$

and then a decomposition of the trivial flat connection d as

$$d = \mathcal{D} + \mathcal{N},$$

for $\mathcal{D}$ the connection given by the sum of the connections induced by d on S and $S^\perp$, respectively, through orthogonal projection.

Given $\mu, \eta \in \Omega^1(S^*T\mathcal{G})$, let $(\mu \wedge \eta)$ be the 2-form defined from the metric on $S^*T\mathcal{G}$:

$$(\mu \wedge \eta)_{(X,Y)} = (\mu_X, \eta_Y) - (\mu_Y, \eta_X),$$

for all $X, Y \in \Gamma(TM)$. Next we present a manifestly conformally invariant formulation of the Willmore energy. It follows the definition presented in [7], in the quaternionic setting, for the particular case of $n = 4$. The intervention of the conformal structure restricts to the Hodge $*$-operator, which is conformally invariant on 1-forms over a surface.

Theorem 1 ([7])

$$\mathcal{W}(\Lambda) = \frac{1}{2}\int_M (dS \wedge *dS).$$

Note that

$$(dS \wedge *dS) = -(*dS \wedge dS) = (dS, dS)dA,$$

$(dS \wedge *dS)$ is a conformally invariant way of writing $(dS, dS)_g dA_g$, for $g \in \mathcal{C}_\Lambda$, with dA_g denoting the area element of (M, g) and $(\, ,\,)_g$ denoting the Hilbert–Schmidt metric on $L((TM, g), S^*T\mathcal{G})$. It follows that the Willmore energy of Λ coincides with the Dirichlet energy of S with respect to any of the metrics in the conformal class $\mathcal{C}_\Lambda$,

$$\mathcal{W}(\Lambda) = E(S).$$

4 Constrained Willmore Surfaces and Harmonicity

Harmonic maps are the critical points of the Dirichlet energy functional. Willmore surfaces are closely related to harmonic maps via the central sphere congruence, in a key result established by Blaschke [1], for $n = 3$, and, independently, Ejiri [13] and Rigoli [26], for general n:

Theorem 2 ([1, 13, 26]) *Λ is a Willmore surface if and only if its central sphere congruence S is a harmonic map.*

The well-developed theory of harmonic maps into Grassmannians now applies. First of all, it provides a zero-curvature characterization of Willmore surfaces: for a map into a Grassmannian, the harmonicity amounts to the flatness of a certain[1] family of connections, as established by Uhlenbeck [29], and so does then the Willmore surface condition:

Theorem 3 *Λ is a Willmore surface if and only if $d^\lambda := \mathcal{D} + \lambda\mathcal{N}^{1,0} + \lambda^{-1}\mathcal{N}^{0,1}$ is a flat connection, for all $\lambda \in S^1$.*

A larger class of surfaces arises when one imposes the weaker requirement that a surface extremize the Willmore functional only with respect to infinitesimally conformal variations: these are the *constrained Willmore surfaces*. The introduction of a constraint in the variational problem equips surfaces Λ with *Lagrange multipliers*,

[1]In the literature, the associated family of flat connections corresponding to a different choice of orientation in M can also be found.

as first proven by Burstall–Pedit–Pinkall [8] and then given the following manifestly conformally invariant formulation by Burstall–Calderbank [4]:

Theorem 4 ([4, 8]) *Λ is a constrained Willmore surface if and only if there exists a real form $q \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ such that*

$$d_q^\lambda := \mathcal{D} + \lambda\mathcal{N}^{1,0} + \lambda^{-1}\mathcal{N}^{0,1} + (\lambda^2 - 1)q^{1,0} + (\lambda^{-2} - 1)q^{0,1}$$

is a flat connection, for all $\lambda \in S^1$. Such a form q is said to be a (Lagrange) multiplier *for Λ and Λ is said to be a q-constrained Willmore surface. At times, it will be convenient to make an explicit reference to the central sphere congruence of Λ, writing $d_S^{\lambda,q}$ for d_q^λ.*

Willmore surfaces are the constrained Willmore surfaces admitting the zero multiplier. This is not necessarily the only multiplier, as we shall see.

5 Isothermic Constrained Willmore Surfaces

Isothermic surfaces are classically defined by the existence of conformal curvature line coordinates. Although the second fundamental form is not conformally invariant, conformal curvature line coordinates are preserved under conformal changes of the metric and, therefore, so is the isothermic surface condition. The next result presents a manifestly conformally invariant formulation of the isothermic surface condition, established by Burstall–Donaldson–Pedit–Pinkall [6].

Proposition 1 ([6]) *Λ is an isothermic surface if and only if there exists a non-zero closed real 1-form $\eta \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$. In this case, we say that Λ is a η-isothermic surface.*

If $q_1 \neq q_2$ are multipliers for Λ, then Λ is a $*(q_1 - q_2)$-isothermic surface, and, reciprocally, if Λ is a η-isothermic q-constrained Willmore surface, then the set of multipliers for Λ is the affine space $q + \langle *\eta\rangle_{\mathbb{R}}$. Hence:

Proposition 2 ([9]) *A constrained Willmore surface Λ admits a unique multiplier if and only if Λ is not an isothermic surface.*

A classical result by Thomsen [28] characterizes isothermic Willmore surfaces in 3-space as minimal surfaces in some 3-dimensional space-form. (In contrast to constrained Willmore surfaces, constant mean curvature surfaces are not conformally invariant objects, requiring a distinguished space-form to be considered.)

Theorem 5 ([28]) *Λ is a minimal surface in some 3-dimensional space-form if and only if Λ is an isothermic Willmore surface in 3-space.*

Constant mean curvature surfaces in 3-dimensional space-forms are examples of isothermic constrained Willmore surfaces, as proven by Richter [25]. However, isothermic constrained Willmore surfaces in 3-space are not necessarily constant mean curvature surfaces in some space-form, as established by an example due to Burstall, presented in [2], of a constrained Willmore cylinder that does not have constant mean curvature in any space-form.

6 Transformations of Constrained Willmore Surfaces

The zero-curvature characterization of constrained Willmore surfaces presented above allows one to deduce two types of symmetry.

Suppose that Λ is a q-constrained Willmore surface. The two types of transformations that we describe next apply to any choice of the multiplier q (when there is a choice to be made) and depend on it. In the particular case that Λ is a Willmore surface, we consider q to be the zero multiplier, without further reference.

6.1 Spectral Deformation

The simplest transformation of Λ into new constrained Willmore surfaces arises from exploiting a scaling freedom in the spectral parameter, as follows.

For each $\lambda \in S^1$, the flatness of the metric connection d_q^λ establishes, at least locally, the existence of an isometry of bundles

$$\phi_\lambda : (\underline{\mathbb{R}}^{n+1,1}, d_q^\lambda) \to (\underline{\mathbb{R}}^{n+1,1}, d),$$

preserving connections, defined on a simply connected component of M and unique up to a Möbius transformation. We define a *spectral deformation* of $\Lambda = \langle \sigma \rangle$ by setting, for each $\lambda \in S^1$,

$$\Lambda_\lambda := \phi_\lambda \Lambda = \langle \phi_\lambda \sigma \rangle.$$

For each $\lambda \in S^1$, set

$$q_\lambda := \phi_\lambda \circ (\lambda^2 q^{1,0} + \lambda^{-2} q^{0,1}) \circ (\phi_\lambda)^{-1}.$$

The central sphere congruence of Λ_λ is $\phi_\lambda S$ and, given $\mu \in S^1$, we have

$$d_{\phi_\lambda S}^{\mu, q_\lambda} = \phi_\lambda \circ d_S^{\mu\lambda, q} \circ (\phi_\lambda)^{-1},$$

establishing the flatness of $d_{\phi_\lambda S}^{\mu, q_\lambda}$ from the one of $d_S^{\mu\lambda, q}$ (note that $\mu\lambda \in S^1$). It follows that:

Theorem 6 *Λ_λ is a q_λ-constrained Willmore surface, for all $\lambda \in S^1$.*

In particular, this spectral deformation preserves the zero multiplier.

Corollary 1 *If Λ is a Willmore surface, then so is Λ_λ, for all $\lambda \in S^1$.*

This spectral deformation coincides, up to reparameterization, with the one presented in [8], in terms of the *Hopf differential* and the *Schwarzian derivative* (see [22, Sect. 6.4.1]).

The isothermic surface condition is known [8] to be preserved under constrained Willmore spectral deformation. In our setting, one can verify (see [23, Sect. 2.3.5]) that, if Λ is also a η-isothermic surface, then Λ_λ is a η_λ-isothermic surface, for

$$\eta_\lambda := \phi_\lambda \circ (\lambda\eta^{1,0} + \lambda^{-1}\eta^{0,1}) \circ (\phi_\lambda)^{-1}.$$

Proposition 3 ([8]) *If Λ is an isothermic surface, then so is Λ_λ, for all $\lambda \in S^1$.*

From Theorem 5, it follows that:

Corollary 2 *If Λ is a minimal surface in some 3-dimensional space-form, then so is Λ_λ, for each $\lambda \in S^1$ (although not necessarily with preservation of the space-form).*

As we shall see later in this text, this spectral deformation preserves, as well, the class of constant mean curvature surfaces in 3-dimensional space-forms, for special choices of the spectral parameter.

6.2 Bäcklund Transformation

Having exploited the equivalence of $d_S^{\lambda,q}$ to the trivial flat connection, as flat metric connections, by means of

$$d_S^{\lambda,q} = (\phi_\lambda)^{-1} \circ d \circ \phi_\lambda,$$

we now explore equivalences starting from $d_S^{\lambda,q}$, i.e., equivalences given by

$$d_{S^*}^{\lambda,q^*} = r(\lambda) \circ d_S^{\lambda,q} \circ r(\lambda)^{-1},$$

for some q^* and some S^*, with $r(\lambda) \in \Gamma(O(\underline{\mathbb{R}}^{n+1,1}))$, so that the flatness of $d_S^{\lambda,q}$ establishes that of $d_{S^*}^{\lambda,q^*}$. The difficulties involved are of two different orders, namely, the preservation of the algebraic shape of $d_S^{\lambda,q}$, together with ensuring that S^* is the central sphere congruence of some surface, so that the family of flat connections $d_{S^*}^{\lambda,q^*}$ is the associated family to some constrained Willmore surface. A version of the Terng–Uhlenbeck [27] dressing action by simple factors proves to offer a simple construction, out of two parameters, a complex number α and a null line bundle L, parallel with respect to $d_S^{\alpha,q}$, from which we define, respectively, the eigenvalues and the eigenspaces of two different types of linear fractional transformations, out of which we define $r(\lambda)$, as follows.

Let ρ denote reflection across S,

$$\rho = \pi_S - \pi_{S^\perp},$$

for π_S and $\pi_{S^\perp}$ the orthogonal projections of $\underline{\mathbb{R}}^{n+1,1}$ onto S and $S^\perp$, respectively. Given $\alpha \in \mathbb{C}$ and L a null line subbundle of $\underline{\mathbb{R}}^{n+1,1}$ such that $\rho L \cap L^\perp = 0$, set

$$p_{\alpha,L}(\lambda) := I \begin{cases} \frac{\alpha-\lambda}{\alpha+\lambda} & \text{on } L \\ 1 & \text{on } (L \oplus \rho L)^\perp \\ \frac{\alpha+\lambda}{\alpha-\lambda} & \text{on } \rho L \end{cases},$$

for $\lambda \in \mathbb{C}\backslash\{\pm\alpha\}$ and $I \in \Gamma(O(\underline{\mathbb{R}}^{n+1,1}))$ the identity map of $\underline{\mathbb{R}}^{n+1,1}$. Let $q_{\alpha,L}$ denote the map obtained from $p_{\alpha,L}$ by considering the additive inverses of the eigenvalues associated to the eigenspaces L and ρL, respectively. Define $p_{\alpha,L}(\infty)$ and $q_{\alpha,L}(\infty)$ by holomorphic extension of

$$p_{\alpha,L}, q_{\alpha,L} : \mathbb{C}\backslash\{\pm\alpha\} \to \Gamma(O(\underline{\mathbb{R}}^{n+1,1}))$$

respectively.

Now consider $\alpha \in \mathbb{C}\backslash(S^1 \cup \{0\})$ and L a $d_S^{\alpha,q}$-parallel null line subbundle of $\underline{\mathbb{R}}^{n+1,1}$ such that $\rho L \cap L^\perp = 0$ (whose existence is established in [9]). Set $\alpha^* := \overline{\alpha}^{-1}$, $L' := p_{\alpha,L}(\alpha^*)\overline{L}$ and, for each $\lambda \in \mathbb{P}^1\backslash\{\pm\alpha\}$,

$$r(\lambda) := q_{\alpha^*,L'}(\lambda)\, p_{\alpha,L}(\lambda).$$

Set, furthermore,

$$\Lambda^* := (r(1)^{-1} r(0)\Lambda^{1,0}) \cap (r(1)^{-1} r(\infty)\Lambda^{0,1}).$$

Theorem 7 ([9]) *Λ^* is a q^*-contrained Willmore surface, for*

$$q^* := r(1)^{-1} \circ (r(\infty) \circ q^{1,0} \circ r(\infty)^{-1} + r(0) \circ q^{0,1} \circ r(0)^{-1}) \circ r(1),$$

with central sphere congruence

$$S^* := r(1)^{-1} S;$$

said to be the Bäcklund transform of Λ of parameters α, L.

In particular, Bäcklund transformation preserves the zero multiplier.

Corollary 3 *If Λ is a Willmore surface, then so is Λ^*.*

It is not clear that if Λ is an isothermic surface, then so is Λ^*. So far, it is not clear either that Bäcklund transformation preserves the class of minimal surfaces in 3-dimensional space-forms. However, as we shall see later, that proves to be the case. We shall see, furthermore, that Bäcklund transformation preserves the class of constant mean curvature surfaces in 3-dimensional space-forms, for special choices of parameters, with preservation of both the mean curvature and the curvature of space.

7 Polynomial Conserved Quantities for Constrained Willmore Surfaces

The isothermic surface condition amounts [6], just as well, to the flatness of a certain family ∇^t of connections, indexed in $\mathbb{R}$. In [10], the classical notion of *special isothermic surface*, introduced by Darboux [12], is given a simple explanation in terms of the integrable systems approach to isothermic surfaces. They are realized as a particular case of a hierarchy of classes of isothermic surfaces filtered by an integer d. Here is the basic idea: The theory of ordinary differential equations ensures that one can find ∇^t-parallel sections depending smoothly on the spectral parameter t. The existence of such sections with polynomial dependence of degree d on t is of particular geometric significance, as first observed by Burstall–Calderbank [5], and gave rise to the notion of *polynomial conserved quantity of type* d, developed in [10], in the isothermic context, where the notion of *special isothermic surface of type* d is introduced, having the classical notion as a particular case ($d = 2$).

We are in this way led to the notion of *special constrained Willmore surface of type* d, presented in [24]:

Definition 1 Let Λ be a q-constrained Willmore surface and $d \in \mathbb{N}_0$. A Laurent polynomial

$$p(\lambda) = \overline{p_{_d}}\,\lambda^{-d} + \cdots + \overline{p_{_1}}\,\lambda^{-1} + p_{_0} + p_{_1}\,\lambda + \cdots + p_{_d}\,\lambda^d$$

with

$$p_{_d} \in \Gamma(S^\perp)$$

and $p_{_k} \in \Gamma(S^\perp)$ if and only if k and d have the same parity, or, otherwise, $p_{_k} \in \Gamma(S)$; is said to be a *polynomial conserved quantity of type* d *of* Λ if

$$p(1) \neq 0$$

and

$$d_q^\lambda\, p(\lambda) = 0,$$

for all $\lambda \in S^1$. We say that Λ is a *special constrained Willmore surface of type* d if it admits a polynomial conserved quantity of type d.

The case $d = 1$ recovers the notion of conserved quantity presented in [21, 22], an idea by Burstall–Calderbank [5].

The fact that $p(\lambda)$ is a polynomial conserved quantity of type d of Λ establishes, in particular, that $p(1)$ is real and constant, that is, $p(1) \in \mathbb{R}^{n+1,1}$. As we shall see, $p(1)$ carries very important information regarding both the curvature of space in which, under some conditions, Λ proves to have constant mean curvature, and the mean curvature of the surface Λ in such a space.

In the isothermic context, type 1 characterizes [5] *H-generalised surfaces*, surfaces admitting a parallel unit normal vector field which has constant inner product with the mean curvature vector (see also [10]). In the constrained Willmore context, type 1 with parallel top term characterizes surfaces with parallel mean curvature vector:

Theorem 8 ([24]) *Λ is a special constrained Willmore surface of type* 1*, admitting a polynomial conserved quantity with parallel top term, if and only if Λ has parallel mean curvature vector in some space-form.*

In codimension 1, the condition of parallelism of the top term of a polynomial conserved quantity of type 1 proves [24] to be vacuous. It follows that, in codimension 1, type 1 characterizes constant mean curvature surfaces, in both contexts, recovering, in particular, a result established in [21, 22]:

Theorem 9 ([21, 22, 24]) *Suppose that $\Lambda \subset S^3$. Then Λ is a special constrained Willmore surface of type* 1 *if and only if Λ has constant mean curvature in some space-form.*

Furthermore:

Theorem 10 ([21, 22, 24]) *Suppose that $\Lambda \subset S^3$. If $p(\lambda)$ is a polynomial conserved quantity of type* 1 *of Λ, then Λ has constant mean curvature H, with*

$$H^2 = |\pi_{S^\perp}(p(1))|^2, \tag{1}$$

in a space-form with sectional curvature

$$K = -(p(1), p(1)). \tag{2}$$

Reciprocally, if Λ has constant mean curvature H in some space-form with sectional curvature K, then Λ admits a polynomial conserved quantity $p(\lambda)$, of type 1*, satisfying (1) and (2).*

Corollary 4 *Suppose that $\Lambda \subset S^3$. Then Λ is a minimal surface in some space-form if and only if Λ is a constrained Willmore surface admitting a polynomial conserved quantity $p(\lambda)$ of type* 1 *with*

$$p(1) \in \Gamma(S).$$

8 Transformations of Special Constrained Willmore Surfaces

The class of special constrained Willmore surfaces of any given type d is preserved under both spectral deformation and Bäcklund transformation, defining, in particular, for special choices of parameters, as established in [21, 22] ($d = 1$) and [24] (general d), as follows.

Let Λ be a q-constrained Willmore surface.

Theorem 11 ([21, 22, 24]) *Let λ be in S^1 and $\phi_\lambda : (\underline{\mathbb{R}}^{n+1,1}, d_q^\lambda) \to (\underline{\mathbb{R}}^{n+1,1}, d)$ be an isometry of bundles, preserving connections. Suppose that $p(\mu)$ is a polynomial conserved quantity of type d of Λ, with $p(\lambda)$ non-zero. Then*

$$p_\lambda(\mu) := \phi_\lambda\, p(\lambda\mu)$$

is a polynomial conserved quantity of type d of the spectral deformation $\phi_\lambda\Lambda$, of parameter λ, of Λ.

As for Bäcklund transformation of special constrained Willmore surfaces:

Theorem 12 ([21, 22, 24]) *Suppose that $p(\lambda)$ is a polynomial conserved quantity of type d of Λ. Suppose that α, L are Bäcklund transformation parameters for Λ with*

$$p(\alpha) \perp \overline{L}.$$

Then

$$p^*(\lambda) := r(1)^{-1} r(\overline{\lambda}^{-1})\, p(\lambda)$$

is a polynomial conserved quantity of type d of the Bäcklund transform Λ^* of Λ, of parameters α, L.

Note that

$$p^*(1) = p(1),$$

establishing the preservation of the curvature of space, when carrying a distinguished one.

9 Constant Mean Curvature Surfaces Under Constrained Willmore Transformation

From Theorems 9 and 11, we conclude that the class of constant mean curvature surfaces in 3-dimensional space-forms is preserved under spectral deformation, for special choices of the spectral parameter. Recall, furthermore, that, for each $\lambda \in S^1$, the central sphere congruence of Λ_λ is $\phi_\lambda S$. According to Theorem 10, it follows that:

Corollary 5 *If Λ is a constant mean curvature surface in some 3-dimensional space-form, then so is Λ_λ (although not necessarily with preservation of the space-form), for special choices of the parameter $\lambda \in S^1$. Furthermore: if $p(\mu)$ is a polynomial conserved quantity of type 1 of Λ, with $p(\lambda)$ non-zero, then Λ_λ has constant mean curvature H_λ with*

$$H_\lambda^2 = |\pi_{S^\perp}(p(\lambda))|^2$$

in a space-form with sectional curvature

$$K_\lambda = -(p(\lambda), p(\lambda)).$$

Both constrained Willmore spectral deformation and Bäcklund transformation prove [24] to preserve, furthermore, the parallelism of the top term of a polynomial conserved quantity, for special choices of parameters. Hence:

Theorem 13 ([24]) *The class of parallel mean curvature surfaces in space-forms is preserved under both spectral deformation and Bäcklund transformation, for special choices of parameters, with preservation of the space-form, in the latter case.*

In particular, the class of constant mean curvature surfaces in 3-dimensional space-forms is preserved under Bäcklund transformation, for special choices of parameters, with preservation of the space-form. Furthermore, recovering a result established in [21, 22]:

Theorem 14 ([21, 22, 24]) *If Λ is a constant mean curvature surface in some 3-dimensional space-form, then so is Λ^*, for special choices of parameters, with preservation of both the space-form and the mean curvature.*

Corollary 6 *If Λ is a minimal surface in some 3-dimensional space-form, then so is Λ^*, for special choices of parameters, with preservation of the space-form.*

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