Strong Pseudoconvexity and Strong Quasiconvexity of Non-differentiable Functions

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Abstract In this chapter, we introduce the concept of strong pseudomonotonicity and strong quasimonotonicity of set-valued maps of higher order. Non-differentiable strong pseudoconvex/quasiconvex functions of higher order are characterized by the strong pseudomonotonicity/quasimonotonicity of their corresponding set-valued maps. As a by-product, we solve the open problem (converse part of Proposition 6.2) of Karamardian and Schaible (J. Optim. Theory Appl. 66:37–46, 1990) for the more general case as strong pseudoconvexity for non-smooth, locally Lipschitz continuous functions.

Keywords Generalized convexity · Generalized monotonicity · Clarke generalized subdifferential mappings

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1 Introduction

The concept of monotone maps was introduced by Minty [10] in 1964. Karamardian [5] extended the concept of monotonicity to strict and strongly monotone maps and also established the relationship between the strongly convex functions and strongly monotone maps. Furthermore, Karamardian and Schaible [6] discussed about seven kinds of monotone maps and established their relationships with corresponding convex functions.

Besides some penalty results for nonlinear programs, Lin and Fukushima [8] introduced the concept of strongly convex functions of order $\sigma > 0$ and established their relationship with strongly monotone maps of order $\sigma > 0$.

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It is very natural to see that a function is convex, and then its generalized subgradients are monotone (see [12]). The class of non-differentiable functions plays a very crucial role in the study of generalized convexity and generalized monotonicity. The theory of generalized gradients of non-smooth functions was given by Clarke [1], Rockaffelar [13], and Hiriart-Urruty [4].

Komlósi [7] proposed the relationship of quasi (pseudo, strict pseudo) convexity of lower semicontinuous bifunctions and multifunctions with quasi (pseudo, strict pseudo) monotonicity of its generalized derivatives. In 2003, Fan et al. [3] established the relationships between (strict, strong) convexity and quasiconvexity of non-differentiable functions and (strict, strong) monotonicity and quasimonotonicity of set-valued mappings. In addition to that, Fan et al. [3] also investigated the relationships between (strict, strong, and sharp) pseudoconvexity of non-smooth functions and (strict, strong, and sharp) pseudomonotonicity of set-valued mappings. Recently, Singh et al. [14] presented the first-order characterizations of strong pseudoconvex/quasiconvex functions of higher order. In addition to that, Mishra et al. [11] established the relationships between generalized convex functions and generalized monotone maps in case of semidifferentiability.

Motivated by the work of Karamardian and Schaible [6], Lin and Fukushima [8], and Fan et al. [3], we generalize the concepts of strong convexity/pseudoconvexity/quasiconvexity to strong convexity/pseudoconvexity/ quasiconvexity of order $\sigma > 0$ for non-differentiable, locally Lipschitz continuous functions and establish their relationships with strong monotonicity/pseudomonotonicity/quasimonotonicity of order $\sigma > 0$ of set-valued mappings.

2 Preliminaries

Let X be a real Banach space with a norm $\|.\|$ and X^* be its dual space with a norm $\|.\|^*$. Let U be a non-empty open convex subset of X, $F : X \to 2^{X^*}$ be a set-valued mapping from a real Banach space to the family of non-empty subsets of X^*, and $f : X \to \mathbb{R}$ be a non-differentiable real-valued function.

Definition 2.1 ([1, 9]) Let f be locally Lipschitz continuous at a given point $x \in X$ and v be any other vector in X. The Clarke generalized directional derivative of f at x in the direction of v, denoted by $f^0(x; v)$, is defined by

$$f^0(x; v) = \limsup_{y \to x,\ t \downarrow 0} \frac{f(y + tv) - f(y)}{t}.$$

Definition 2.2 ([1, 9]) Let f be locally Lipschitz continuous at a given point $x \in X$ and v be any other vector in X. The Clarke generalized subdifferential of f at x, denoted by $\partial^c f(x)$, is defined by

$$\partial^c f(x) = \{\xi \in X^* : f^0(x; v) \geq \langle \xi, v \rangle, \forall v \in X\}.$$

Lemma 2.1 ([1, 9]) *Let f be locally Lipschitz continuous with a constant L at $x \in X$. Then,*

(a) $\partial^c f(x)$ *is a non-empty convex weak*-compact subset of* X^* *and* $\|\xi\|_* \leq L$ *for every* $\xi \in \partial^c f(x)$.
(b) For every $v \in X$, $f^0(x; v) = max\{\langle \xi, v\rangle : \xi \in \partial^c f(x)\}$.

Lemma 2.2 ([1, 9]) *If f is convex on X and locally Lipschitz continuous at $x \in X$, then $\partial^c f(x)$ coincides with the subdifferential $\partial f(x)$ of f at x in the sense of convex analysis and $f^0(x; v)$ coincides with the directional derivative $f'(x; v)$ for each $v \in X$, where*

$$\partial f(x) = \{\xi \in X^* : f(y) - f(x) \geq \langle \xi, y - x\rangle, \forall y \in X\},$$

$$f'(x; v) = \lim_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

Lemma 2.3 ([1] (Mean Value Theorem)) *Let x and y be points in X, and suppose that f is Lipschitz on an open set X containing the line segment $[x, y]$. Then, $\exists$ a point $u \in (x, y)$ such that*

$$f(x) - f(y) \in \langle \partial^c f(u), x - y\rangle.$$

Definition 2.3 ([2]) A function f is quasiconvex on a convex set X of $\mathbb{R}^n$ if $\forall x, y \in X, \lambda \in [0, 1]$, we have

$$f(x) \leq f(y) \Rightarrow f(\lambda x + (1 - \lambda)y) \leq f(y).$$

Proposition 2.1 ([2]) *Let f be a locally Lipschitz continuous function on X. Then, f is said to be quasiconvex if and only if for any $x, y \in X$ and any $\eta \in \partial^c f(y)$, we have*

$$f(x) \leq f(y) \Rightarrow \langle \eta, x - y\rangle \leq 0.$$

3 Strong Convexity and Monotonicity of Order σ

We collect some definitions related to strong convexity and strong monotonicity of order σ, where $\sigma > 0$ be any positive integer, that is, strong convexity and strong monotonicity of integer order $\sigma \geq 1$ [8].

Definition 3.1 ([8]) A function $f : X \to \mathbb{R}$ is said to be strongly convex of order $\sigma > 0$ on a non-empty open convex subset $X \subseteq \mathbb{R}^n$ if $\exists\ c > 0$ such that for any $x, y \in X$ and any $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) - c\lambda(1-\lambda)\|x-y\|^{\sigma}.$$

Definition 3.2 ([8]) F is said to be strongly monotone of order $\sigma > 0$ on X if $\exists$ a constant $\alpha > 0$ such that for any $x, y \in X$ and any $u \in F(x)$, $v \in F(y)$, we have

$$\langle u - v, x - y\rangle \geq \alpha\|x-y\|^{\sigma}.$$

Proposition 3.1 *Let f be a locally Lipschitz continuous function on an open convex subset X. Then, f is strongly convex of order $\sigma > 0$ on X if and only if $\exists\, c > 0$ and $\eta \in \partial^c f(y)$ such that*

$$f(x) - f(y) \geq \langle \eta, x - y\rangle + c\|x-y\|^{\sigma}.$$

Proof Let f be strongly convex function of order $\sigma > 0$ on X. Then, for any $x, y \in X$ and any $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) - c\lambda(1-\lambda)\|x-y\|^{\sigma}, \tag{1}$$

$$\frac{f(y + \lambda(x-y)) - f(y)}{\lambda} \leq f(x) - f(y) - c(1-\lambda)\|x-y\|^{\sigma}.$$

Taking $\limsup\limits_{\lambda\downarrow 0}$, we have

$$f^0(y, x-y) \leq f(x) - f(y) - c(1-\lambda)\|x-y\|^{\sigma}. \tag{2}$$

Again, $\exists\, \eta \in \partial^c f(y)$ such that $\langle \eta, x-y\rangle \leq f^0(y, x-y)$, and then

$$\langle \eta, x - y\rangle \leq f(x) - f(y) - c(1-\lambda)\|x-y\|^{\sigma},$$

$$f(x) - f(y) \geq \langle \eta, x-y\rangle + c'\|x-y\|^{\sigma}, \qquad c' = c(1-\lambda).$$

Conversely, suppose that $f(x) - f(y) \geq \langle \eta, x-y\rangle + c\|x-y\|^{\sigma}$.

Let $x \neq y \in X$, $\lambda \in [0, 1]$, $x_\lambda = y + \lambda(x-y) \in X$ as X is convex.

In particular, $\exists\, \eta_0 \in \partial^c f(x_\lambda)$ such that

$$f(x) - f(x_\lambda) \geq \langle \eta_0, x - x_\lambda\rangle + c\|x - x_\lambda\|^{\sigma}, \tag{3}$$

and

$$f(y) - f(x_\lambda) \geq \langle \eta_0, y - x_\lambda\rangle + c\|y - x_\lambda\|^{\sigma}. \tag{4}$$

Multiplying inequality (3) by λ and (4) by $(1-\lambda)$ and adding them, we obtain

$$\lambda f(x) + (1-\lambda)f(y) - f(x_\lambda) \geq c\lambda(1-\lambda)^{\sigma}\|x-y\|^{\sigma} + c\lambda^{\sigma}(1-\lambda)\|x-y\|^{\sigma}.$$

Consider $[(1-\lambda)^{\sigma-1}+\lambda^{\sigma-1}]$ for $0 < \lambda \leq 2$, $[(1-\lambda)^{\sigma-1}+\lambda^{\sigma-1}] \geq (1-\lambda)+\lambda = 1$, and for $\lambda > 2$, since the real function $\phi(\lambda) = \lambda^{\sigma-1}$ is convex on (0,1), then $[(1-\lambda)^{\sigma-1}+\lambda^{\sigma-1}] \geq (\frac{1}{2})^{\sigma-2}$.

It follows from the above argument that $\exists$ some constant $c^{'} > 0$ independent of x, y, and λ such that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) - c^{'}\lambda(1-\lambda)\|x-y\|^{\sigma}.$$

Therefore, f is strongly convex of order $\sigma > 0$ on X. □

Theorem 3.1 *Let f be a locally Lipschitz continuous function on X. Then, f is strongly convex of order $\sigma > 0$ on X if and only if $\partial^c f$ is strongly monotone of order $\sigma > 0$ on X.*

Proof Let f be strongly convex of order $\sigma > 0$, then for any $x, y \in X$ and $\eta \in \partial^c f(y)$, we have

$$f(x) - f(y) \geq \langle \eta, x-y \rangle + c\|x-y\|^{\sigma}. \tag{5}$$

Interchanging the role of x and y and for any $\xi \in \partial^c f(x)$, we have

$$f(y) - f(x) \geq \langle \xi, y-x \rangle + c\|y-x\|^{\sigma}. \tag{6}$$

Adding inequalities (5) and (6), we get

$$0 \geq \langle \eta - \xi, x-y \rangle + 2c\|x-y\|^{\sigma},$$

$$\langle \xi - \eta, x-y \rangle \geq \beta\|x-y\|^{\sigma}.$$

Therefore, $\partial^c f$ is strongly monotone of order σ on X.

Conversely, suppose that $\partial^c f$ is strongly monotone of order $\sigma > 0$ on X; that is, for any $x, y \in X$, $\exists\, \xi \in \partial^c f(x)$ and $\eta \in \partial^c f(y)$ such that

$$\langle \xi - \eta, x-y \rangle \geq \alpha\|x-y\|^{\sigma}.$$

By the mean value theorem, for any $x \neq y \in X$, $\exists\, z = \lambda x + (1-\lambda)y$ for some $\lambda \in (0,1)$ and $\exists\, \eta_0 \in \partial^c f(z)$ such that

$$f(x) - f(y) = \langle \eta_0, x-y \rangle = \frac{1}{\lambda}\langle \eta_0, z-y \rangle. \tag{7}$$

Since $\partial^c f$ is strongly monotone of order $\sigma > 0$ on X,

$$\langle \eta_0 - \eta, z-y \rangle \geq \alpha\|z-y\|^{\sigma},$$

for any $z \neq y \in X$.

$$\langle \eta_0, z - y\rangle \geq \langle \eta, z - y\rangle + \alpha\|z - y\|^{\sigma}. \tag{8}$$

Using inequality (8) in inequality (7), we have

$$f(x) - f(y) \geq \frac{1}{\lambda}[\langle \eta, z - y\rangle + \alpha\|z - y\|^{\sigma}],$$

$$f(x) - f(y) \geq \langle \eta, x - y\rangle + \alpha\lambda^{\sigma-1}\|x - y\|^{\sigma}.$$

Therefore,

$$f(x) - f(y) \geq \langle \eta, x - y\rangle + c\|x - y\|^{\sigma}.$$

Hence, f is strongly convex of order $\sigma > 0$. □

Remark 3.1 Proposition 3.1 and Theorem 3.1 generalize Proposition 3.1 and Theorem 3.4 of Fan et al. [3], respectively, which was given for $\sigma = 2$.

4 Strong Pseudoconvexity and Pseudomonotonicity of Order σ

We introduce the concept of strongly pseudoconvex functions of order $\sigma > 0$ for non-smooth locally Lipschitz continuous functions.

Definition 4.1 Let f be a locally Lipschitz continuous function on X. Then, f is said to be strongly pseudoconvex of order $\sigma > 0$ on X if for any $x, y \in X$ and for any $\eta \in \partial^c f(y)$ $\exists$ $\alpha > 0$, we have

$$\langle \eta, x - y\rangle + \alpha\|x - y\|^{\sigma} \geq 0 \Rightarrow f(x) - f(y) \geq 0.$$

Remark 4.1 For $\sigma = 2$, the definition was given by Fan et al. [3].

We introduce the concept of strongly pseudomonotone of set-valued mappings of order $\sigma > 0$ for non-smooth locally Lipschitz continuous functions.

Definition 4.2 F is said to be strongly pseudomonotone of order $\sigma > 0$ on X if for any $x, y \in X$ and any $u \in F(x), v \in F(y),$ $\exists$ a constant $\alpha > 0$, and we have

$$\langle v, x - y\rangle + \alpha\|x - y\|^{\sigma} \geq 0 \Rightarrow \langle u, x - y\rangle \geq 0.$$

Remark 4.2 For $\sigma = 2$, the definition was given by Karamardian and Schaible [6] for real-valued mappings.

We establish the relationship between strong pseudoconvexity of locally Lipschitz continuous functions and strong pseudomonotonicity of set-valued mappings of order $\sigma > 0$, which is the natural generalization of the locally Lipschitz strong pseudoconvex functions given by Fan et al. [3].

Remark 4.3 Fan et al. [3] have left an open problem as the converse of Theorem 4.3, and we prove necessary and sufficient both part for more general class as locally Lipschitz strong pseudoconvex functions of order $\sigma > 0$.

Theorem 4.1 *Let f be a locally Lipschitz continuous function on X. Then, f is strongly pseudoconvex of order $\sigma > 0$ on X if and only if $\partial^c f$ is strongly pseudomonotone of order $\sigma > 0$ on X.*

Proof Let f be strongly pseudoconvex of order $\sigma > 0$ on X, then for any $x, y \in X$ and $\eta \in \partial^c f(y)$ $\exists$ a constant $\alpha > 0$, such that

$$\langle \eta, x - y\rangle + \alpha\|x - y\|^{\sigma} \geq 0 \Rightarrow f(x) \geq f(y).$$

Since we know that every strongly pseudoconvex function of order $\sigma > 0$ is quasiconvex,

$$f(\lambda x + (1 - \lambda)y) \leq f(x). \tag{9}$$

Also, by the definition of non-smooth quasiconvex function if for any $x, y \in X$ and any $\xi \in \partial^c f(x)$, we have

$$f(\lambda x + (1 - \lambda)y) \leq f(x) \Rightarrow \langle \xi, (\lambda x + (1 - \lambda)y) - x\rangle \leq 0,$$

$$\Rightarrow \langle \xi, x - y\rangle \geq 0.$$

Therefore, we have

$$\langle \eta, x - y\rangle + \alpha\|x - y\|^{\sigma} \geq 0 \Rightarrow \langle \xi, x - y\rangle \geq 0.$$

Thus, $\partial^c f$ is strongly pseudomonotone of order σ on X.

Conversely, suppose that $\partial^c f$ is strongly pseudomonotone of order $\sigma > 0$, then for any $x, y \in X$ and $\xi \in \partial^c f(x), \eta \in \partial^c f(y)$, $\exists$ a constant $\beta > 0$, such that

$$\langle \eta, x - y\rangle + \beta\|x - y\|^{\sigma} \geq 0 \Rightarrow \langle \xi, x - y\rangle \geq 0.$$

Equivalently,

$$\langle \xi, x - y\rangle < 0 \Rightarrow \langle \eta, x - y\rangle + \beta\|x - y\|^{\sigma} < 0. \tag{10}$$

We want to show that f is strongly pseudoconvex of order $\sigma > 0$; that is, for any $x, y \in X$ and $\eta \in \partial^c f(y)$, $\exists$ a constant $\alpha > 0$, and we have

$$\langle \eta, x - y\rangle + \alpha\|x - y\|^{\sigma} \geq 0 \Rightarrow f(x) \geq f(y). \tag{11}$$

Suppose, on contrary, $f(x) < f(y)$.

By the mean value theorem, $\exists\ z = \lambda x + (1 - \lambda)y$ for some $\lambda \in (0, 1)$ and $\eta_0 \in \partial^c f(z)$, such that

$$f(x) - f(y) = \langle \eta_0, x - y\rangle = \frac{1}{\lambda}\langle \eta_0, z - y\rangle < 0.$$

Since $\partial^c f$ is strongly pseudomonotone of order σ,

$$\langle \eta_0, z - y\rangle < 0 \Rightarrow \langle \eta, z - y\rangle + \beta\|z - y\|^{\sigma} < 0,$$

$$\langle \eta_0, z - y\rangle < 0 \Rightarrow \langle \eta, x - y\rangle + \beta\lambda^{\sigma-1}\|x - y\|^{\sigma} < 0,$$

which contradicts to the left-side inequality of (11).

Hence, $f(x) \geq f(y)$, and f is strongly pseudoconvex of order $\sigma > 0$. □

Remark 4.4 Every strongly monotone map of order $\sigma > 0$ is strongly pseudomonotone of order $\sigma > 0$, but the converse is not necessarily true.

Example 4.1 Let $F : X \to \mathbb{R}$, where $X = [0, 4]$ defined by

$$F(x) = \begin{cases} 2 - x & \text{for } 0 \leq x < 1, \\ 1 & \text{for } 1 \leq x \leq 4. \end{cases}$$

This is an example of strongly pseudomonotone map of order $\sigma > 0$, but not strongly monotone map of order $\sigma > 0$.

5 Strong Quasiconvexity and Quasimonotonicity of Order σ

Definition 5.1 Let f be a locally Lipschitz continuous function on an open convex subset X. Then, f is said to be strongly quasiconvex of order $\sigma > 0$ on X if for any $x, y \in X$ and any $\eta \in \partial^c f(y)$ $\exists\ \alpha > 0$, we have

$$f(x) \leq f(y) \Rightarrow \langle \eta, x - y\rangle + \alpha\|x - y\|^{\sigma} \leq 0.$$

Definition 5.2 F is said to be strongly quasimonotone of order $\sigma > 0$ on X if for any $x, y \in X$ and any $u \in F(x)$, $v \in F(y)$ $\exists\ \beta > 0$, we have

$$\langle v, x - y\rangle > 0 \Rightarrow \langle u, x - y\rangle \geq \beta\|x - y\|^{\sigma}.$$

Theorem 5.1 *Let f be a locally Lipschitz continuous function on X. Then, f is strongly quasiconvex of order $\sigma > 0$ on X if and only if $\partial^c f$ is strongly quasimonotone of order $\sigma > 0$ on X.*

Proof Let f be strongly quasiconvex of order $\sigma > 0$ on X, then for any $x \neq y \in X$ and $\eta \in \partial^c f(y)$, $\exists$ a constant $\alpha > 0$, such that

$$f(x) \leq f(y) \Rightarrow \langle \eta, x - y \rangle + \alpha \|x - y\|^\sigma \leq 0. \tag{12}$$

We have to show that $\partial^c f$ is strongly quasimonotone on X; that is, for any $\xi \in \partial^c f(x)$ and $\eta \in \partial^c f(y)$, $\exists$ a constant $\beta > 0$, such that

$$\langle \eta, x - y \rangle > 0 \Rightarrow \langle \xi, x - y \rangle \geq \beta \|x - y\|^\sigma.$$

As f is strongly quasiconvex, then it is also quasiconvex; that is, for any $\eta \in \partial^c f(y)$, we have

$$\langle \eta, x - y \rangle > 0 \Rightarrow f(x) > f(y).$$

By the definition of strongly quasiconvex function of order $\sigma > 0$, we have

$$f(y) < f(x) \Rightarrow \langle \xi, y - x \rangle + \alpha \|y - x\|^\sigma \leq 0,$$

$$f(y) < f(x) \Rightarrow \langle \xi, x - y \rangle \geq \alpha \|x - y\|^\sigma.$$

Therefore, we have $\langle \eta, x - y \rangle > 0 \Rightarrow \langle \xi, x - y \rangle \geq \alpha \|x - y\|^\sigma$.

Thus, $\partial^c f$ is strongly quasimonotone of order σ.

Conversely, suppose that $\partial^c f$ is strongly quasimonotone of order $\sigma > 0$, then for any $\xi \in \partial^c f(x)$ and $\eta \in \partial^c f(y)$, $\exists$ a constant $\beta > 0$, such that

$$\langle \eta, x - y \rangle > 0 \Rightarrow \langle \xi, x - y \rangle \geq \beta \|x - y\|^\sigma.$$

We want to show that f is strongly quasiconvex of order $\sigma > 0$; that is, $f(x) \leq f(y) \Rightarrow \langle \eta, x - y \rangle + \alpha \|x - y\|^\sigma \leq 0$.

Suppose that $f(x) \leq f(y)$.

By the mean value theorem, $\exists\ z = \lambda x + (1 - \lambda) y$ for some $\lambda \in (0, 1)$ and $\eta_0 \in \partial^c f(z)$, such that

$$f(x) - f(y) = \langle \eta_0, x - y \rangle = \frac{1}{\lambda} \langle \eta_0, z - y \rangle \leq 0.$$

By the use of strongly quasimonotone map, we have

$$\langle \eta_0, y - z \rangle > 0 \Rightarrow \langle \eta, y - z \rangle \geq \beta \|y - z\|^\sigma,$$

$$\langle \eta_0, y - z \rangle > 0 \Rightarrow \langle \eta, y - x \rangle \geq \beta \lambda^{\sigma - 1} \|y - x\|^\sigma,$$

$$\langle \eta_0, y - z\rangle > 0 \Rightarrow \langle \eta, x - y\rangle + \alpha\|x - y\|^{\sigma} \le 0,$$

Hence, f is strongly quasiconvex of order $\sigma > 0$. □

Remark 5.1 Every strongly quasiconvex function of order $\sigma > 0$ is quasiconvex, but the converse is not always true.

Remark 5.2 The class of quasi-functions is the largest class, so every strongly pseudomonotone map of order $\sigma > 0$ is strongly quasimonotone of order $\sigma > 0$, but it is not always true in the converse case.

Example 5.1 Let $F : X \to \mathbb{R}$, where $X = [-2, 2]$ defined by

$$F(x) = \begin{cases} 0 & \text{for } -2 \le x < 0, \\ x & \text{for } 0 \le x < 1, \\ 2x - 1 & \text{for } 1 \le x \le 2. \end{cases}$$

This is an example of strongly quasimonotone map of order $\sigma > 0$, but not strongly pseudomonotone map of order $\sigma > 0$.

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