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Beyond Shanghai and PISA

Cognitive and Non-cognitive Competencies of Chinese Students in Mathematics

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Editors

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Preface

As the study of mathematics competencies becomes an international hot topic in mathematics education research, we are fortunate to start a research project to assess the mathematical competencies of Chinese students. As the research progresses, we gradually have greater interactions with international and domestic scholars. When *Mathematics curriculum standards for secondary education (2017 version)* was promulgated by the Ministry of Education of the People's Republic of China, we were turning our research findings into publication. We firmly believe that the conclusions drawn from the rigorous research will promote the development of mathematics competency–oriented high school mathematics curriculum in China.

With the team members' joint efforts, the manuscript presents the main findings from the mathematics competencies project. We hope that this achievement opens a window for relevant parts of the world to the understanding Chinese students' mathematics competencies, and also provides richer data and information to those who are interested in Shanghai PISA.

This book consists of 20 chapters. The first chapter aims to explain the cultural foundation of Chinese mathematics education by the characteristics of Chinese mathematics teaching. Chapter 2 reviews PISA assessment framework followed by exploring Chinese students' mathematics performance in this serial large-scale international comparison studies. Chapter 3, based on a theoretical analysis, develops a mathematics competency framework which reflects the characteristics of Chinese mathematics education and also illustrates six components of mathematics competency: mathematical problem posing, mathematical problem solving, mathematical representation and transformation, mathematical reasoning and argumentation, mathematical modeling, and mathematical communication. Based on this competency framework, Chaps. 5, 7, 9, 11, 12, and 14 further develop component-specific assessment frameworks followed by assessing and analyzing Chinese eighth graders' corresponding mathematics competencies. In order to assess students' performance, the ways how the national mathematics curriculum documents (e.g., mathematics teaching syllabus or mathematics curriculum standards) describe and stipulate different components of mathematics competencies were first classified, counted, analyzed and illuminated from a perspective of the historical development of Chinese mathematics curriculum (see Chaps. 4, 6, 8, 10 and 13) before reporting the results of students' performance in the assessments. As mathematical modeling is a relatively new topic in the development of high school mathematics curriculum in China, the analysis of the historical development of the notation and students' performance were combined in one chapter (see Chap. 12).

Besides the cognitive aspects of students' mathematics competencies, the Chinese mathematics curriculum also pays great attention to students' non-cognitive performance related to mathematics' core competencies. Therefore, Chaps. 15 and 16 mainly report Chinese students' self-efficacy, self-evaluation, and anxiety related to their core competencies using the assessment data.

If Shanghai PISA opens a window for the world to understand Chinese mathematics education, this book hopes to open a door for international and domestic scholars. We welcome everyone to come to get know and have discussions. In order to stimulate the discussions of Chinese students' mathematics competencies, this book invites three experts, Professor Emeritus Kaye Stacey from the University of Melbourne, Professor Frederick K.S. Leung from the University of Hong Kong, and Professor Linyuan Gu from the Shanghai Academy of Educational Science, to provide commentaries on the various conclusions of this project based on their own experiences and perspectives.

This book is the final product of a major project of the Humanities and Social Science Key Research Base of the Ministry of Education, "Research on Chinese Students' Mathematical Competencies Assessment" (Project Number: 16JJD880023). The working team firstly appreciates the high trust from Prof. Yunhuo Cui, who encouraged us to host the project while providing adequate financial support. Thanks also go to all students and relevant staff who participated in this investigation. For the book writing, the first thanks should be addressed to Dr. Jinfa Cai, the editor-in-chief of the book series, for the initiation of the project as well as the design of the book. The next thanks goes to Prof. Yong Zhou, whose professional advice on historical content analysis of curriculum standards inspired us greatly. Our thanks shall also extend to Julie-Ann Edwards, Xuhui Li, Wenjuan Li, and Xueying Ji who provided their constructive remarks on our early draft. Last but not the least, we are indebted to Springer for their editorial help, keen insight, and ongoing support along the way.

Shanghai, China

Binyan Xu
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Chapter 1
From 'Two Basics', to 'Four Basics' to 'Core Mathematics Competencies' in Mainland China

Jing Cheng, Jiansheng Bao, and Dianzhou Zhang

Abstract This chapter introduces 'core mathematics competencies' emphasised in mathematics curriculum standards in mainland China and the 'Four Basics' developed from a tradition of mathematics teaching in mainland China known as 'Two Basics'. 'Core mathematics competencies' include mathematical abstraction, logical reasoning, mathematical modelling, intuitive imagination, mathematical operation and data analysis. The 'Four Basics' advocates attaching importance to basic knowledge, basic skills, basic thoughts and basic activity experiences in mathematics; and it seeks the overall development of students. 'Four Basics' can include a variety of cognitive modules where knowledge, skills, thoughts and activities are closely linked and cannot be separated. 'Four Basics' is to core mathematics competencies what cells are to human organs. Chinese students' outstanding performance in international mathematics assessment may attributed to 'Four Basics'. Some questions for further study are raised at the end of the chapter.

Keywords Mathematics curriculum · Curriculum reform · Curriculum standards · Mathematics competencies · Two Basics · Four Basics · Mathematical knowledge · Mathematical skills · Mathematical thoughts · Mathematical activity experience · Mainland China

Nowadays, China is implementing a mathematics curriculum reform oriented by subject competency. As intended curricula, the mathematics curriculum standards of mainland China have clearly explained mathematics competency. In *Mathematics Curriculum Standards for Compulsory Education (2011 version)* (Ministry of Education of the People's Republic of China [MOE], 2012) and the latest edition *Mathematics Curriculum Standards for Senior Secondary Schools (2017 version)* (MOE, 2018), the corresponding requirements and suggestions on 'core

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mathematics competencies' are successively put forward. The formation of these curriculum goals can be partly attributed to the global environment of the development of mathematics education, and its more important source of attribution is the historical development and inheritance of the Chinese mathematics curriculum.

This chapter will first outline the 'core mathematics competencies' highlighted in the Chinese mathematics curriculum, and it then will focus on introducing 'Four Basics', which is developed from the 'Two Basics' of mathematics education in mainland China (Fan, Huang, Cai, & Li, 2004). 'Four Basics' is to core mathematics competencies what cells are to human organs.

1.1 Core Mathematics Competencies in Mainland China

Mathematics Curriculum Standards for Compulsory Education (2011 version) of mainland China points out that a mathematics curriculum should focus on the development of students' number sense, symbol sense, spatial concept, geometric intuition, data analysis concept, operation ability, reasoning ability and model thinking (MOE, 2012). *Mathematics Curriculum Standards for Senior Secondary Schools (2017 version)* clearly points out that mathematics teaching should help students acquire core mathematics competencies, namely, mathematical abstraction, logical reasoning, mathematical modelling, intuitive imagination, mathematical operation and data analysis (MOE, 2018).

1.1.1 Mathematics Competencies

The word 'competency' (素养) in Chinese is composed of two Chinese characters, in which '素' means 'daily' and '养' represents 'accomplishment', i.e. a certain level of thought, theory, knowledge and art. Accordingly, 'competency' usually refers to daily accomplishment. Mathematical competency is 'a comprehensive reflection of the basic characteristics of mathematics in terms of thinking quality, key abilities, emotional attitudes and values' (MOE, 2018, p. 4). Many words in English, such as literacy, ability, competence, proficiency, etc., have similar meanings to the term '素养' used in Chinese, but not necessarily the same meaning.

For example, Programme for International Student Assessment (PISA), a large-scale international comparative research project, focuses on students' mathematical literacy, which is interpreted as follows:

> *Mathematical literacy* is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognize the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens. (OECD, 2013, p. 25)

As another example, the National Assessment of Educational Progress (NAEP) used mathematical abilities and mathematical power to construct its research framework from 1996 to 2003:

> The first domain, *mathematical abilities*, describes three types of knowledge or processes required for a student to successfully respond to a question: conceptual understanding; procedural knowledge; and problem solving, the ability to synthesize several processes when confronting a mathematical situation. The second domain, *mathematical power*, reflects the three processes stressed as major goals of the mathematics curriculum: the ability to reason, to communicate, and to make connections between concepts and skills either across the mathematics content areas, or from mathematics to other curricular areas. (Braswell, Dion, Daane, & Jin, 2005, p. 3)

The Denmark KOM project, Competencies and the Learning of Mathematics, focuses on 'mathematical competence', which includes modelling, reasoning, representation, communication, etc.

> It indicated that '*Mathematical competence* means the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role. Necessary, but certainly not sufficient, prerequisites for mathematical competence are lots of factual knowledge and technical skills, in the same way as vocabulary, orthography, and grammar are necessary but not sufficient prerequisites for literacy' (Niss, 2003, pp. 6–7).

For the convenience of expression, this book uses 'mathematical competency', which is commonly used in international mathematics education research, to refer to 'mathematical literacy' or 'critical competence' which is concerned in the mathematics curriculum of mainland China.

1.1.2 Core Mathematics Competencies in the Intended Curricula

As intended curricula, the curriculum standards at both compulsory education and secondary education stages in mainland China propose requirements on mathematical competency, such as mathematical abstraction, logical reasoning, mathematical modelling, intuitive imagination, mathematical operation and data analysis, while providing explanations to their meanings.

Mathematical abstraction refers to the thinking process of 'getting rid of all the physical attributes of things to reach the object of mathematical study' (MOE, 2018, p. 4). It mainly includes abstracting mathematical concepts and the relationship between concepts from quantity and quantitative relationship, graphics and graphical relationship, abstracting general rules and structures from the concrete background, and using mathematical symbols or mathematical terms to represent them. At the stage of compulsory education, 'number sense' and 'symbol sense' are emphasised. Number sense mainly refers to the sense of numbers and quantity, quantitative relationship and the estimation of computation results. Symbol sense refers to the ability to understand and use symbols to represent numbers, quantitative relationship and the law of changes; symbols can be used for computation, reasoning, and drawing general conclusions (MOE, 2012, pp. 5–6).

Logical reasoning refers to the thinking process of deducing a proposition from some facts and propositions according to logical rules. There are two main types: one is inference from propositions held in a small range to those held in a larger range, the reasoning forms of which are mainly induction and analogy; and the other is inference from propositions held in a large range to those held in a smaller range, the reasoning form of which is mainly deductive reasoning (MOE, 2018, p. 5). The reasoning ability emphasised at the compulsory education stage generally includes plausible reasoning and deductive reasoning. Plausible reasoning infers certain results by induction and analogy based on existing facts, experience and intuition, while deductive reasoning proves based on the principles of logical reasoning from existing facts (including definitions, axioms, theorems, etc.) and determined rules (including definition, rules, sequence of operations, etc.). The two are 'mutually reinforcing' in the process of solving problems (MOE, 2012, pp. 6–7).

Mathematical modelling is the process of 'abstracting real problems, representing and solving them in mathematical language. Specific performance is, in the real situation, from the perspective of mathematics, to put forward problems, analyze problems, represent problems, build models, draw conclusions, verify results, improve models, and ultimately get realistic results' (MOE, 2018, pp. 5–6). Similarly, the process of establishing and solving models emphasised at the compulsory education stage includes abstracting mathematical problems from real-life or specific situations; using mathematical symbols to establish equations, inequalities, functions, etc. to express quantitative relationship and the law of changes in mathematical problems; finding results; and discussing the significance of results (MOE, 2012, p. 7).

Intuitive imagination refers to 'perceiving the shape and the change of things by means of spatial imagination, understanding and solving mathematical problems by means of geometric figures. It mainly includes using graphics to describe mathematical problems, establishing the relationship between graphics and symbols, building an intuitive model of mathematical problems, and exploring solutions to problems' (MOE, 2018, p. 6). At the compulsory education stage, emphasis is placed on developing students' spatial concept and geometric intuition. Spatial concept refers to 'abstracting geometric figures according to the characteristics of objects, imagining the actual objects described by geometric figures; imagining the orientation of objects and the position relationship between them; describing the movement and change of figures; drawing graphics according to the description of language'. Geometric intuition refers to 'using graphics to describe and analyze problems' (MOE, 2012, p. 6).

Mathematical operation refers to 'solving mathematical problems according to operation rules on the basis of clarifying operation objects'. It mainly includes understanding operation objects, mastering operation rules, exploring operation direction, selecting operation methods, designing operation procedures and obtaining operation results' (MOE, 2018, p. 7). At the compulsory education stage, the emphasis on operational ability mainly refers to the ability to correctly operate according to rules and properties. (MOE, 2012, p. 6).

Data analysis refers to 'the process of obtaining useful information from data and forming knowledge. It mainly includes collecting data and extracting information, displaying data with charts, constructing a model to analyze data and explaining the conclusions contained in data' (MOE, 2018, p. 7). At the compulsory education stage, the formation of the data analysis concept is emphasised: 'To know that there are many problems in real life, we should first do investigation and research, collect data, make judgments through analysis, and experience that data contains information; to know that there are many methods available for analyzing the same data, and we need to choose appropriate methods according to the context of the problem, to experience randomness through data analysis. On one hand, the data collected for the same thing may be different at each time; on the other hand, as long as there is enough data, we may identify rules from it' (MOE, 2012, p. 6).

1.2 Cells of Core Mathematics Competencies: Four Basics

In the process of doing, learning and using mathematics, students gradually form and develop their mathematical competencies. Therefore, daily mathematics teaching plays a significant part in achieving the curriculum goal of mathematical competencies. As the core idea of mathematics education in mainland China, 'Four Basics' plays an indispensable role in the development of core mathematics competencies, and its status is comparable to that of cells in human organs. Both *Mathematics Curriculum Standards for Compulsory Education (2011 version)* and *Mathematics Curriculum Standards for Senior Secondary Schools (2017 version)* list it as the necessary basis for students to further study and adapt to future development (MOE, 2012, 2018).

1.2.1 What Are the Four Basics?

'Four Basics' is the abbreviation of basic knowledge, basic skills, basic thoughts and basic activity experience (Zhang & Zheng, 2011). Among them, basic knowledge emphasises understanding concepts and principles, basic skills focus on the proficiency of procedural knowledge and basic thoughts reflect the methods and strategies adopted to solve various problems in the process of mathematics development. In addition, mathematics teaching is an activity, and the basic activity experience reflects the concern about the activity process.

Four Basics attaches great importance to the foundation, but it does not neglect the development. On the contrary, the purpose of laying a good foundation is to enable students to seek more practical and efficient comprehensive development on the basis of the Four Basics of mathematics.

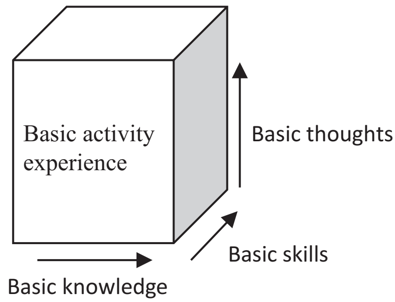

Fig. 1.1 Module diagram of four basics (Zhang & Zheng, 2011)

1.2.2 The Form of Four Basics

Mathematical Four Basics is presented in the form of a cognitive module, which is the cube shown in Fig. 1.1 (Zhang & Zheng, 2011).

The first dimension is the accumulation process of basic mathematical knowledge, the second dimension is the practice process of basic mathematical skills and the third dimension is the formation process of basic mathematical thoughts.

Mathematical basic activity experiences do not constitute a single dimension, but play an adhesive role to bond the other three basics. In fact, students' experiences gained through the ubiquitous basic mathematics activities are interwoven with the basic knowledge, skills and thoughts of mathematics, and they permeate the whole process of mathematics learning.

In the teaching process of a mathematics class, knowledge acquisition, skill training and the refinement of thoughts and methods permeate to each other. There is no pure knowledge and also no pure skill without knowledge. The mathematical thoughts and methods are built on knowledge and skills, while having their own independent value. Students' mathematical activity experiences run through the whole learning process with the above three 'basics' as the carrier.

Mathematical Four Basics cannot be separated. For example, to solve a real-world problem with an equation, the basic skills of solving equations and the mathematical thoughts of modelling are necessary, which come from students' active participation in mathematical activities. Emphasis that 'Four Basics' is a cognitive module shows that the Four Basics are closely linked and cannot be separated.

1.2.3 From Four Basics to Core Mathematics Competencies

Core mathematics competencies is the goal pursued when teaching the Four Basics of mathematics. Each core mathematics competency, which consists of certain 'knowledge, skills, thoughts and mathematical activity experience', is formed gradually during a variety of mathematics lessons.

Human cells are the basis of every organ. Like brain cells and heart cells have different organ functions, the Four Basics modules in different mathematics lessons have various characteristics. Each 'Four Basics' module has its own distinct characteristics, serving to form certain core mathematics competencies.

For example, a simple Four Basics module, such as recognising natural numbers less than 20, involves both mathematical abstraction (from concrete objects to abstract number) and mathematical modelling (counting in real-world problems) among core mathematics competencies. Conversely, the core competency of mathematical abstraction is developed from a variety of Four Basics modules with abstract characteristics, such as recognising numbers and manipulating symbols.

The long-term and comprehensive effect of Four Basics is the core mathematics competencies that the school mathematics curriculum hopes students can form.

1.3 Formation of Four Basics

The formation of Four Basics is rooted in the unique social and cultural background of China, and it comes from the long-term research, practice and reflection of mathematics educators. Its early form in mathematics curriculum is Two Basics and Three Abilities, which have gradually developed into Four Basics and Six Core Competencies in recent years.

1.3.1 Historical Development of Mathematics Curriculum in Mainland China

Mathematics curriculum in Mainland China is influenced by different social backgrounds and international environments in different historical periods, showing different characteristics which are reflected in the representative curriculum programmatic documents of the corresponding period (Curriculum and Teaching Materials Research Institute (CTMRI), 2001).

During the period of the Republic of China (before the founding of the People's Republic of China), western mathematics education was introduced, and the ancient imperial examination system changed to the modern teaching system. The representative programmatic documents of the curriculum included the promulgation of *Middle School Rules Made by Emperor Order* in 1902 and *New Education System Curriculum Standards* in 1923.

After the founding of the People's Republic of China in 1949, the development of the Chinese mathematics curriculum entered its second stage. In the early days of its founding, Chinese mathematics education was fully learned from the Soviet Union's system. In June 1950, *Simplified Mathematics Syllabus for Junior High Schools (Draft)* was promulgated. In December of the same year, the People's

Education Publishing House was established to implement the national unified supply system for primary and secondary school textbooks, compiling a unified national mathematics curriculum. In 1958, an educational reform took place, where mathematics courses were compiled according to the needs of industrial and agricultural production, seriously undermining the systematic nature of mathematics courses. Reflecting on the previous lessons, in May 1963, the Ministry of Education promulgated *Mathematics Syllabus for Full-Time High Schools (Draft)*, which took basic knowledge of mathematics and Three Abilities (correct and rapid computing ability, logical reasoning ability and spatial imagination ability) as the goal of mathematics teaching in secondary schools, a germination of Two Basics mathematics teaching. The Cultural Revolution of 1966 undermined the normal teaching order, and the mathematics curriculum also entered anarchy.

Along with the end of the Cultural Revolution in 1976, China entered the historical period of mind emancipation and reform and opening up, and the mathematics curriculum was continuously developed and innovated. In February 1978, the *Mathematics Syllabus for Full-time Ten-year High Schools (trial version)* was promulgated, and for the first time 'cultivating students' ability to analyze problems and solve problems' (CTMRI, 2001, p. 453) was included in the 'objectives of teaching', where this ability was seen as the result of computing ability, logical thinking ability and spatial imagination ability. It was also indicated that the arrangement of mathematics teaching contents should be conducive to students' 'learning basic knowledge and mastering basic skills'.

In 1986, *the Compulsory Education Law of the People's Republic of China* stipulated that the nation should implement nine-year compulsory education, and primary and junior secondary education belong to this stage. In December of the same year, the *Mathematics Syllabus for Full-time Secondary Schools* was promulgated, clearly stating that the purpose of secondary school mathematics teaching is to enable students to learn 'basic knowledge and basic skills', to cultivate students' three major abilities and to gradually form the abilities of analysing and solving problems.

At the beginning of the second century, the programmatic documents of the Chinese mathematics curriculum changed from the previous 'syllabus' to 'curriculum standard', emphasising the process goals of mathematics teaching while paying attention to the outcome goals, and putting forward new suggestions for teachers' teaching as well as student evaluation. In 2012, in the *Mathematics Curriculum Standards for Compulsory Education (2011 version)*, the Four Basics were proposed as the overall goal of the mathematics curriculum.

1.3.2 From Two Basics to Four Basics

The proposal of the Four Basics of mathematics in mainland China has its unique cultural background and educational tradition (Zhang, 2006). China's farming culture emphasises intensive cultivation and non-violation of agricultural time. Confucian culture advocates diligence and hard work, and it believes in classic literature. The imperial examination culture requires learners to be familiar with the classic way of writing and to unify the style. Folklore's educational maxims, such as 'practice makes perfect', 'one minute on stage, ten years off stage', largely reflect the values of diligence (Fan et al., 2004).

China's modern mathematics education was originally influenced by other countries. Since the Revolution of 1911, the school system and mathematics curriculum have largely referred to the United Kingdom and the United States. Meanwhile, the educational concept has been greatly influenced by Dewey's progressive teaching. In the early days of the founding of the People's Republic of China, the local mathematics education was influenced by the Soviet's school and focused on the rigour of mathematics. However, Chinese mathematics education has always adopted an eclectic approach to various educational concepts and gradually formed its own characteristics (Zhang, 2014). The *Mathematics Syllabus for Full-Time High Schools (Draft)* in 1963 summarised the positive and negative aspects of mathematics education experience and lessons since the founding of the People's Republic of China, and made a timely proposition to emphasise the basic knowledge of mathematics and the teaching of basic skills (CTMRI, 2001), where the Two Basics began to form. Although experiencing the Cultural Revolution from 1966 to 1976, after the reform and opening up, the wrong tendency of neglecting the basics was corrected, and the Two Basics in mathematics teaching were re-emphasized. In the *Mathematics Syllabus for Full-time Secondary Schools* in 1986, 'basic knowledge' and 'basic skills' were clearly listed as the purpose of mathematics teaching (CTMRI, 2001), which marked the formal establishment of the Two Basics mathematical teaching concept in the syllabus (Zhu & Bao, 2017).

In the 1980s, the study of 'mathematical thoughts and methods' initiated by Xu Lizhi gradually formed the concept of 'basic thoughts', focusing on important thoughts in mathematics rather than specific operable methods (Xu, 2000). Since the beginning of the new century, due to the attention to active participation of students, the concept of 'basic activity experience' in mathematics has been proposed. These two concepts, together with Two Basics, constitute the Four Basics (Zhang & Zheng, 2011), which was formally proposed in 2012 in the *Mathematics Curriculum Standards for Compulsory Education (2011 version)* (MOE, 2012).

It can be seen that the Four Basics is not only a product of the historical development of Chinese mathematics curriculum, but also a profound reflection on the 'China Road' of mathematics education from a global perspective (Zhang & Yu, 2013).

1.3.3 Comprehensive Development Is Inseparable from a Solid Foundation

The history of Chinese and foreign mathematics education has repeatedly shown that once the foundation is neglected and the necessary balance is lost, school education has to pay a heavy price. In the developed countries in Europe and North America, a tendency to ignore the 'foundation' has long existed in school education.

After the Second World War, because the Soviet satellite took the lead, there have been 'new mathematics movements' in some developed countries in Europe and North America. At that time, in the reform of mathematics education, in order to improve the theoretical level of mathematics curriculum, a large amount of abstract and incomprehensible contents, such as 'terminology of set', 'binary', and 'mathematical structures like group, ring and field', were introduced in the primary school mathematics curriculum. The result of the trial was a separation from students' reality and ended in failure. Consequently, in the 1970s, the slogan 'Back to Basics' was implemented as a retrospective, but there was no careful distinction between returning to the basics and simply returning to the past. By the end of the 1970s, people began to realise that 'Back to Basics' was another failure. Mechanical exercises kept students' ability to think and solve problems at a low level, and basic skills were not developed (Nie, Zheng, Sun, & Cai, 2010). In other words, although the abstract content of 'formalism' was abandoned in mathematics teaching in primary and secondary schools, it has embarked on 'reducing the difficulty without restriction' and over-promoting the improper action of 'mathematics in daily life'. Some mathematics educators who were obsessed with 'progressive education' re-implemented the educational concept of 'child centrism', regarding students' interest and happiness as the starting point, and 'design tasks with children's daily life' as the only goal of mathematics teaching. However, having a lot of basic mathematics knowledge and skill does not stimulate children's interests, such as the recitation of multiplication table, division of large numbers, operations of integers, fractions and irrational numbers, etc. Without these key points, many mathematical concepts cannot be understood, mathematical operations cannot be done, the mathematical foundation cannot be laid and thus the development of mathematics cannot be reached (Chen & Huang, 2016). Some developed countries in Europe and North America have performed poorly in international mathematics tests, and this may be part of the reason.

In contrast, the outstanding performance of Chinese students may be inseparable from the Four Basics of mathematics. From the International Assessment of Educational Progress (IAEP) of 1990–1991 (Fan et al., 2004) to the late PISA test (Gao, 2015), Chinese students have consistently ranked among the top in almost all international tests of mathematics. These tests not only reflect the students' mastery of basic knowledge and basic skills, but also examine students' ability to use mathematics knowledge to solve practical problems to a certain extent. Generally speaking, the advantage of Chinese mathematics education lies in students' solid foundation and mathematics competencies developed on the foundation.

American mathematics education had the term 'Math War' in the 1990s, and one of its debating focuses was on whether the mathematical foundation of American students was stable. In 2008, the National Mathematical Teachers' Committee (NCTM) published a document titled *Curriculum Focal Points for Pre-Kindergarten Through Grade 8 Mathematics: A Quest for Coherence*, which identified the key mathematical skills and knowledge that students must master, namely 'Curriculum Focal Points', the purpose of which was to help schools focus their mathematics on basic skills (Tong & Song, 2007). In 2006, President Bush appointed a National Mathematics Advisory Panel to help the president and the minister of education build the best American mathematics education based on scientific research. The final report of the committee in 2008 was titled *Foundations for Success*, where the success of future reforms was still relied on the foundation (National Mathematics Advisory Panel, 2008). Why repeatedly emphasized foundation? It was because foundation had long been ignored by the mainstream education concept, which should be concerned now.

The importance of foundation can also be seen in the event that the British Ministry of Education planned to learn from Shanghai, China's system. What the United Kingdom has to do is to allow students to do more exercises and improve the speed of computation, just like what is done for Chinese primary school students; they should also introduce the student practice book *One Lesson One Exercise,* published by East China Normal University Press, as a template. In fact, British education authorities hope that British primary school students simply gain a solid mathematical foundation (Fan, Ni, & Xu, 2018).

The British understanding of the 'Chinese mathematics teaching model' is that there is an importance on the foundation. However, it is one-sided to think that the 'Chinese teaching model' does not talk about development. With the continuous exchange of Chinese and foreign mathematics education experiences, the Four Basics of Chinese mathematics are also constantly developing. The foundation does not mean repeated practice, nor does it hope to build a 'thatched house' on the basis of granite. Instead, it seeks development on a solid foundation and closely combines 'laying the foundation' with 'seeking development'.

1.4 Questions Worthy of Further Study

Why do Chinese students have a solid foundation? How does this happen? The existing research provides some answers, among which the importance of the 'Four Basics' in mathematics education may be one answer. Excellent mathematics education and research results at home and abroad are constantly enriching the Four Basics in mathematics teaching. However, there is still a lack of systematic research on the connotation, classification and psychological basis of the Four Basics.

With regard to the Four Basics in mathematics teaching, there are a series of questions waiting to be answered: how does one deal with the relationship between memorising algorithms and understanding arithmetic? How is the speed of computation related to higher-order thinking? How does one realise mathematical thoughts and methods through problem solving? How do the exercises with variation avoid dull repetition? How do mathematics activities serve the formation of the other three Basics? What role do teaching methods such as teaching, teacher-student interaction, open-ended question and creation of context play in the Four Basics teaching? What is the relationship between foundation and innovation? All of these questions need further research and reflection.

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Chapter 2
What Can PISA Tell Us About Students' Mathematics Learning in Shanghai, China?

Yan Zhu

Abstract This chapter starts with a review of the development of the concept of mathematical literacy proposed by the Programme for International Student Assessment (PISA) followed by an analysis of the changes in the mathematics assessment framework across cycles. Along with devoting more attention to noncognitive skills in the field of education, PISA developed mathematics-related measures on attitudes and emotions that contribute to students using and developing their mathematical capacities. The participation of China Mainland in PISA began with Shanghai in 2009 and 2012. During these two appearances, the performance of Shanghai topped the mathematics achievement ladder, which produces a global 'PISA-shock'. However, PISA found that Shanghai students' self-concept and mathematics intentions are relatively low, while both self-belief and dispositions to mathematics show significantly positive impacts on mathematics performance. Regarding the rich information from the PISA studies, both education officials and the public in China Mainland gave generally reflective, measured and self-critical responses.

Keywords PISA · Mathematical literacy · Mathematics performance · Mathematical contents · Mathematical processes · Situations and contexts · Noncognitive skills · Self-related cognition · Dispositions towards mathematics · Subjective norms in mathematics · Intrinsic motivation · Instrumental motivation · Shanghai · East Asia · Structural equation model

2.1 About PISA Mathematics

The Programme for International Student Assessment (PISA) is a triennial international survey study by the Organization for Economic Cooperation and Development (OECD) in member and non-member nations/economies. The programme evaluates

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educational systems worldwide by measuring 15-year-old students' skills and knowledge as they approach the end of their compulsory education (OECD, 2019). In particular, it assesses how students apply what they learn in school to real-life situations. PISA has been in place since 1997 and was first implemented in 2000; it has been repeated every 3 years. To date, there have been seven cycles. In the most recent one (i.e. PISA 2018), over half a million 15-year-olds from 80 countries and economies participated.

While reading, mathematics and science are the three major competence fields of PISA, the main focus alternates between the three domains. One reason for this practice is the infeasibility of testing all students in all fields in every cycle. Consequently, mathematical literacy became the 'major' domain of PISA 2003 and PISA 2012. In the other cycles, mathematics was assessed as a 'minor' domain. Two-thirds of the testing time is devoted to 'major' competence domains, while the 'minor' domains provide a summary profile of skills.

The concept of mathematical literacy was first proposed in PISA 2000, though reading was the major competence field that year. In contrast to many other attempts to make international comparisons, PISA mathematics is concerned with students' capacity to draw upon their mathematical competencies to meet future challenges (Wong, 2003). In particular, PISA 2000 defines mathematical literacy as 'an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgements and to engage in mathematics, in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen' (OECD, 1999, p. 41). This suggests that the emphasis in PISA mathematics is 'on mathematical knowledge put into functional use in a multitude of different situations and contexts' (Blum, 2002, p. 151) to allow one to participate in societal activity (OECD, 1999). The same definition was used until PISA 2009.

In PISA 2012, mathematics again returned as the major testing field. Its assessment framework was fully revised to introduce three new mathematical processes in which students engaged as active problem solvers. In particular, the definition of mathematical literacy was renewed with explicit reference to the component processes of mathematical modelling (i.e. formulating real-world problems mathematically, employing mathematics to solve the mathematically formulated problem, and interpreting and evaluating the mathematical results in real-world terms; Stacey, 2015). Stacey remarked that the intention of revising the definition was to clarify the ideas underpinning mathematical literacy so that they can be operationalized more transparently (Stacey, 2015). As a result, PISA 2012 defined mathematical literacy as

> an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognize the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens. (OECD, 2013, p. 25)

This new definition was also used in the PISA 2015 and 2018 assessments (OECD, 2017a, 2019).

Compared to the one-time change in the definition of mathematical literacy, the mathematics framework was updated and revised on several occasions. In the first PISA assessment, two major aspects (i.e. mathematical competencies and mathematical big ideas) and two minor aspects (i.e. mathematical curricular strands and situations and contexts) were used in the assessment of mathematics. That is, the major aspects were used for the purpose of describing the scope of the assessment and the learners' proficiency, while the minor aspects were used to ensure adequate coverage of the domain and balance in the range of assessment tasks selected (OECD, 1999).

The first major aspect, mathematical competencies, includes eight general mathematics skills in a non-hierarchical order (e.g. problem posing and solving skills; symbolic, formal and technical skills; and modelling skills), which are further organized into three larger classes of competency to facilitate the operationalization. The three classes are reproduction, definitions and computations (class 1); connections and integration for problem solving (class 2); and mathematical thinking, generalization and insight (class 3). The other major aspect, mathematical big ideas, represents clusters of relevant, connected mathematical concepts that appear in real situations and contexts. While a large number of big ideas can be identified in the subject of mathematics, six of them were selected as the focus on the PISA mathematical literacy (e.g. chance, change and growth, and dependency and relationships).

The first minor aspect, mathematical curricular strands, represents the content of school mathematics as implemented in many school curricula. Nine strands are consequently identified in PISA to ensure a balance in the items and a reasonable spread of content in relation to the school mathematics curriculum (e.g. number, measurement and estimation). The other minor aspect, situations and contexts, is about the settings in which mathematics tasks are presented. Five situations are identified in PISA as being at a certain 'distance' from the learners: personal, educational, occupational, public and scientific.

Mainly because mathematical literacy was one minor assessment domain in PISA 2000, a less fully articulated framework was developed (OECD, 2009a). In 2003, the mathematics framework was updated and fully developed to guide a comprehensive assessment of mathematics as a major domain. The *PISA 2006 Technical Report* summarized five key changes to the mathematics framework between PISA 2000 and PISA 2003, including (1) expanding the rationale of the PISA emphasis on the use of mathematical knowledge and skills to solve problems, (2) restructuring and expanding domain content (i.e. expanding from two broad content areas to four; removing all references to mathematics curricular strands as a separate content categorization), (3) a more elaborate rational for the existing balance between 'realistic mathematics' and more traditional context-free items, (4) a redeveloped discussion of the relevant mathematical processes, and (5) considerable elaboration through the addition of examples (OECD, 2009a). This mathematics framework was unchanged in the next two PISA cycles (OECD, 2009a, 2009b).

When mathematics returned as the major test domain in PISA 2012, the existing framework received a review and subsequent development work. Besides revising the definition of mathematical literacy, several other changes were highlighted in the *PISA 2012 Technical Report*, including (1) the ways in which mathematical content was conceptualized and described, (2) the definition and description of how mathematical processes received substantial changes (e.g. processes are used for the first time in PISA 2012 as a primary reporting dimension), and (3) the contexts within which opportunities for students to express their levels of mathematics literacy would be provided were revised (OECD, 2014). While mathematics was assessed as a minor domain in the next two cycles (PISA 2015 and PISA 2018), the corresponding frameworks continued the description and illustration of the one set out in PISA 2012 (OECD, 2017a, 2019).

During the first few cycles of PISA, students' mathematical literacy was assessed in the classic paper-and-pencil mode. The computer-based assessment of mathematics (CBAM) was first offered in PISA 2012, while the paper-based tests remained as the main assessment mode (OECD, 2013). Computer-based tests became the main mode of assessment in PISA 2015 and PISA 2018, and paper-based alternatives were still used in countries and economies that did not have the resources available for computer-based testing in schools (OECD, 2017b). In either assessment mode, students are asked to respond to approximately 2 hours of test questions in reading, mathematics and science, with test items being a mixture of multiple-choice items and questions that require students to construct answers.

As PISA alternates its emphasis on reading, mathematics and science literacy in different cycles, a combined set of major- and minor-domain item clusters are compiled into a set of booklets in a balanced incomplete block (BIB) design. The use of the BIB design is to ensure wide coverage of content without burdening individual students (Weeks, von Davier, & Yamamoto, 2014). Figure 2.1 illustrates the representation of the PISA content area item clusters from 2000 to 2018. It is clear that the change in content representation associated with the major/minor domain design, while the representation of the minor domain content areas is inconsistent over time.

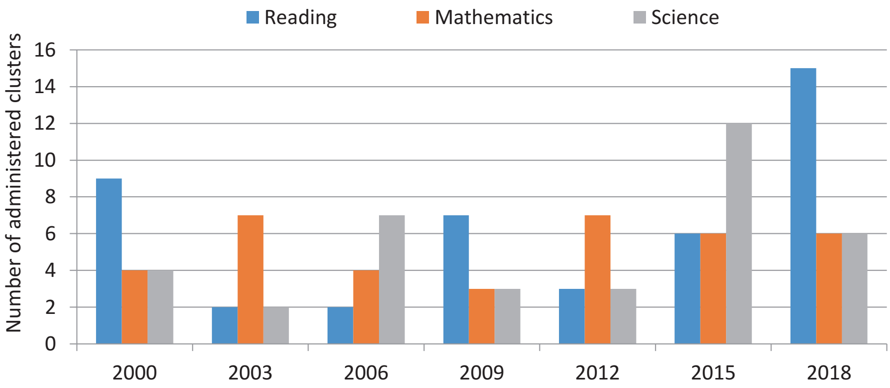

Fig. 2.1 PISA content representation

Table 2.1 Assessment design for mathematics literacy in PISA 2000 to PISA 2015

	No. of mathematics clusters	No. of mathematics items	Corresponding minutes of mathematics material	No. of booklets with mathematics items (total no. of booklets)
2000	4	32	60 mins	5 (9)
2003	7	85	210 mins	13 (13)
2006	4	48	120 mins	10 (13)
2009	3	34	90 mins	9 (13)
2012	7	110	270 mins	13 (13)
2015	6	81	180 mins	36 (66)

Note. PISA 2018 includes six mathematics clusters representing approximately 180 minutes of testing time

With the BIB design, 'major domain' items occur in all booklets, with 'minor domain' items in some. This implies that more time is devoted to 'major domain' items than 'minor domain' ones. Consequently, all sampled students responded to mathematics items in PISA 2003 and PISA 2012, while in the remaining cycles, only some students responded to mathematics items, depending on which particular booklet they were randomly assigned from the rotation design (details shown in Table 2.1).

2.2 Noncognitive Factors in PISA Mathematics

In addition to assessments, PISA includes Student and School Questionnaires to collect data that can be used in constructing indicators pointing to social, cultural, economic and education factors that are associated with student achievement (OECD, 2002). The student questionnaire is administered after the literacy assessment, which takes students about 30–35 minutes to complete. An overarching design is used in the development of the student context questionnaire, including both general variables (for all PISA cycles) and domain-specific variables (for major domains only, included every 9 years). As a result, when mathematics was the major survey domain, the PISA 2003 and PISA 2012 frameworks identified mathematics-related aspects of the assessment of attitudes that contributed to students using and further developing their mathematical capacities (Stacey & Turner, 2015). As the *PISA 2003 Assessment Framework* (OECD, 2003) highlighted, 'mathematics related attitudes and emotions such as self-confidence, curiosity, feelings of interest and relevance, and the desire to do or understand things, . . . are important contributors' (p. 26). Similarly, the *PISA 2012 Assessment and Analytical Framework* (OECD, 2013) specified important aspects of the affective domain as 'information about students' experience with mathematics in and out of school . . ., motivation, interest in mathematics and engagement with mathematics' (p.182).

Student engagement with mathematics is one important measure in the PISA 2003 survey, which refers to 'students' active involvement in learning, . . . students' beliefs about their own ability to succeed in a subject, motivation to learn a subjective and emotional relationship with a subject, as well as their choice of learning strategies for a subject' (OECD, 2005, p. 38). Five constructs are developed in this theme: self-efficacy, self-concept, anxiety, interest and enjoyment and instrumental motivation. The elaboration in PISA 2003 is listed as follows:

- Mathematics self-efficacy, MATHEFF, derived from students' responses about their perceived ability to solve a range of pure and applied mathematics problems (8 items)
- Mathematics self-concept, SCMAT, derived from students' responses about their perceived competence in mathematics (5 items)
- Mathematics anxiety, ANXMAT, derived from students' responses about feelings of stress and helplessness when dealing with mathematics (5 items)
- Interest in and enjoyment of mathematics, INTMAT, derived from students' responses about whether or not they enjoy mathematics (4 items)
- Instructional motivation to learn mathematics, INSTMOT, derived from students' responses about whether they believe mathematics is important for their future studies and careers (4 items)

Similarly, PISA 2012 also considers issues of teaching and learning mathematics. Its questionnaire design focuses on non-cognitive outcomes, explanation of students' intentions and behaviours related to mathematics and classroom teaching (OECD, 2014). In total, nine noncognitive indices are constructed in PISA 2012 (see Table 2.2).

After a careful re-evaluation of item psychometric qualities, all five constructs in PISA 2003 were retained unchanged in the PISA 2012 student questionnaire, except the index of interest in and enjoyment of mathematics, which was renamed as the index of intrinsic motivation to learn mathematics. As listed in Table 2.2, the other four were newly created in PISA 2012:

- Subjective norms in mathematics, SUBNORM, derived from students' responses about whether or not their parents and peers enjoy and value mathematics (6 items)

Table 2.2 Mathematics-related themes and constructs in PISA 2003 and PISA 2012 (student level)

Themes	Constructs	2003	2012
Self-related cognitions in mathematics	Mathematics self-efficacy	✓	✓
	Mathematics self-concept	✓	✓
	Mathematics anxiety	✓	✓
	Interest in and enjoyment of mathematics	✓	✓
	Instrumental motivation to learn mathematics	✓	✓
Dispositions towards mathematics	Subjective norms in mathematics		✓
	Mathematics intentions		✓
	Attributions to failure in mathematics		✓
	Mathematics work ethic		✓

- Mathematics intentions, MATINTFC, derived from students' responses about whether or not they intend to use mathematics in their future (5 items)
- Attributions to failure in mathematics, FAILMAT, derived from students' responses about their perceived self-responsibility for failure in mathematics (6 items)
- Mathematics work ethic, MATWKETH, derived from students' responses about their ability to dedicate time, hard work and persistence to attain mathematics competency (9 items)

2.3 Shanghai and PISA

Since the first implementation in 2000, PISA has received greater attention and uptake than many, if not all, international assessment programs (Klinger, DeLuca, & Merchant, 2016). In fact, the number of participating countries/economies has grown from 43 in the first cycle to 79 in the seventh cycle in 2018 (see Table 2.3). Among east Asian countries/economies, Hong Kong SAR, Japan and Korea have participated since the first cycle, Macau SAR since 2003, Chinese Taipei since 2006, and Shanghai-China and Singapore since 2009 (Ho, 2017). Another three Chinese regions—Beijing, Jiangsu Province and Guangdong Province—joined Shanghai in participating in PISA since 2015.

The desired base PISA target population in each country and economy consists of 15-year-old students attending educational institutions in grade 7 and higher. In all but the Russian Federation, the PISA assessment adopted a two-stage stratified sample design; that is, a sample of schools is selected systematically with probabilities proportional to the school size (PPS); then, the required number of 15-year-old students within each selected school is selected at random (35 students in schools that had 35 or more eligible students, and all students in schools with a number lower than 35; OECD, 2010).

While a minimum of 150 schools were selected in each country and economy (where this number existed), participating countries and economies further introduced stratification into their individual school sampling plan to achieve systematic

Table 2.3 Participation in PISA 2000 to PISA 2018

	No. of participating OECD countries	No. of partner countries/economies	No. of participating students	Size of population
2000[a]	29	3	250,000	17 million
2003	30	11	250,000	30 million
2006	30	27	400,000	20 million
2009[b]	34	31	470,000	26 million
2012	34	31	510,000	28 million
2015	35	37	540,000	29 million
2018	37	42	710,000	31 million

Note. [a]An additional 11 countries/economies completed the PISA 2000 assessment in 2002
[b]An additional 10 countries/economies completed the PISA 2009 assessment in 2010

distribution of specific school parameters in the sample (Ho, 2017). Stratification consists of classifying schools into *like* groups according to the selected variables (i.e. stratification variables), and two types of them are used (OECD, 2012). In particular, explicit stratification divides the schools into different strata and draws independent sample schools within each stratum (e.g. states or regions), while implicit stratification sorts the schools uniquely within each explicit stratum according to a set of criteria (e.g. type of school and urbanity). In PISA 2009, Shanghai used seven explicit strata based on four variables with three implicit stratification variables. A similar stratification was used in PISA 2012 with some modifications made to stratification variables and their corresponding levels (see Table 2.4).

As a result, 5115 Shanghai 15-year-olds from 152 schools participated in PISA 2009, and 5177 students from 155 schools participated in PISA 2012 (see Table 2.5). While a total of 9841 Chinese students in 268 schools from four regions of China participated in PISA 2015, the exact information for Shanghai is difficult to identify.

Table 2.4 Stratification variables used in PISA 2009 and PISA 2012 for the Shanghai case

	2009	2012
Explicit stratification variables	School level (3)	ISCED level (4)
	Programme (2)	ISCED programme orientation (2)
	Selectivity (2)	Selectivity (3)
	Certain selections (1)	Certain selections (1)
Number of explicit strata	7	6
Implicit stratification variables	Track (2)	Vocational school type (4)
	Funding (2)	Funding (2)
	Location (2)	Urbanity (2)

Note. For further details, see the PISA technical reports in 2009 and 2012

Table 2.5 Number of schools and number of students in PISA 2009 and PISA 2012 Shanghai sample

	2009[a]		2012[b]	
Type of school	No. of schools	No. of students	No. of schools	No. of students
Junior secondary school	52	1716	60	1899
Mixed senior secondary school	27	923	23	779
General senior secondary school	20 (key)	689	21 (model or experimental)	1412
	20 (non-key)	695	19 (ordinary)	
Vocational secondary school	29 (key)	975	32	1087
	4 (non-key)	117		
Total	152	5115	155	5177[c]

Note. [a]From Ning, van Damme, Liu, Vanlaar, and Gielen (2013)
[b]From Liang, Kidwai, and Zhang (2016)
[c]Since students taking the CBAM are a sub-sample of students sampled in the PBA, no extra student sample is needed for the CBAM, and a total of 2409 students took the CBAM in Shanghai (Ho, 2017)

2.4 Performance of Shanghai Students in Mathematical Literacy in PISA

On its first appearance in PISA, the performance of Shanghai students topped the achievement ladder, including in mathematics (see Table 2.6). The students in this Chinese city continued their impressive performance in PISA 2012, again taking first place in mathematics as well as the other two core subjects. In fact, Shanghai students are in a class of their own in mathematics, hugely outperforming second-placed Singapore (Moore, 2010). A comparison of the mathematics performance between the two cycles shows a significant improvement of 13 points in Shanghai, while the OECD average dropped by 2 points.

In PISA, students' scores are also described in terms of proficiency levels with the aim of providing insights into what students at different levels of ability can do (Shiel, Perkins, Close, & Oldham, 2007). Proficiency levels are constructed in such a way that all students performing at a particular level are expected to answer correctly at least half of the items at that level (and less than half of the items at a higher level).[1] There are six mathematics proficiency levels defined in both the PISA 2009 and PISA 2012 assessments, which is the same as the corresponding levels of the PISA 2003 scale, with some descriptions in 2012 updated to reflect new mathematical process categories in the PISA 2012 framework (OECD, 2013). Table 2.7 gives a brief description of the six levels of proficiency in mathematics.

On average, across OECD countries, 3.1% of students attained Level 6 in mathematics in PISA 2009 and 3.3% in PISA 2012. In the two cycles, Shanghai has had the highest proportion of students reaching this level in the mathematics assessment (26.6% and 30.8%, respectively), 10% higher than the second-highest economy, Singapore (15.6% and 19.0%, respectively). Moreover, more than half of the Shanghai students are proficient at Level 5 or 6 (regarded as top performers) in both PISA 2009 (50.4%) and PISA 2012 (55.4%). In both cycles, about 15% less

Table 2.6 Ranks and mean scores in Mathematics Literacy of Top-Ranking East Asian Countries and Economies in Mathematics in PISA 2009 and PISA 2012

	PISA 2009		PISA 2012	
Shanghai, China	1	600	1	613
Singapore	2	562	2	573
Hong Kong SAR	3	555	3	561
South Korea	4	546	5	554
Chinese Taipei	5	543	4	560
Japan	9	529	7	536
Macau SAR	12	525	6	538
OECD average	(20)	496	(26)	494

[1] More details can be referred to *Proficiency Levels in Assessments of Reading and Mathematics*, from http://www.erc.ie/wp-content/uploads/2017/05/PISA-NAERM-Proficiency-levels.pdf

Table 2.7 Brief descriptions of the six levels of proficiency in mathematics

Level	Lower score limit	What students can typically do
6	669	Students can conceptualize, generalize and utilize information based on their investigations and modelling of complex problem situations
5	607	Students can develop and work with models for complex situations, identifying constraints and specifying assumptions
4	545	Students can work effectively with explicit models for complex concrete situations that may involve constraints or call for making assumptions
3	482	Students can execute clearly described procedures, including those that require sequential decisions
2	420	Students can interpret and recognize situations in contexts that require no more than direct inference
1	358	Students can answer questions involving familiar contexts where all relevant information is present and the questions are clearly defined

students in the second-place system (i.e. Singapore) performed at these levels, and the OECD average percentages are no more than 13%.

In the two cycles, the share of students scoring at Level 1 or below (regarded as the lowest performers) in Shanghai is slightly below 5% (4.8% and 3.7%, respectively), while about one quarter of the OECD students performed at these levels (26% and 24.8%, respectively). In another three east Asian economies—Singapore, Hong Kong SAR and South Korea—the proportion of the students performing at Level 1 or below is between 8% and 10%. Though Shanghai students demonstrated overall high achievement levels in the PISA mathematics assessments, there were still 1.4% in 2009 and 0.8% in 2012 who scored below Level 1.

Besides students' overall mathematics scores, PISA 2012 further investigated students' performance on subscales representing different aspects of mathematics. In particular, there are three process categories (formulating, employing and interpreting) and four content categories (change and relationships, space and shape, quantity and uncertainty and data). As a result, Shanghai achieved the highest scores on all the subscales, with the scores on interpreting, quantity and uncertainty and data being lower than the overall score for mathematics (see Fig. 2.2).

Across all participating countries and economies, the average difference between the highest and lowest performance in mathematics processes is around 14 points (OECD, 2014). The corresponding difference is largest in Shanghai (46 points), which is about 16 points larger than Chinese Taipei, which has the second-highest difference. It is interesting to see that all the top-performing east Asian countries/economies perform best in formulating, while the OECD average score is highest on the interpreting subscale.

On the content subscales, Shanghai again shows the largest difference (58 points) between its strongest category (space and shape) and its weakest (quantity), followed by Chinese Taipei with a 49-point difference. Table 2.8 displays that all the top-performing east Asian systems perform the best on space and shape, while quantity and uncertainty and data are their weaknesses. In contrast, at the OECD average level, the best performance occurs on topics related to quantity and space and shape is the weakest area.

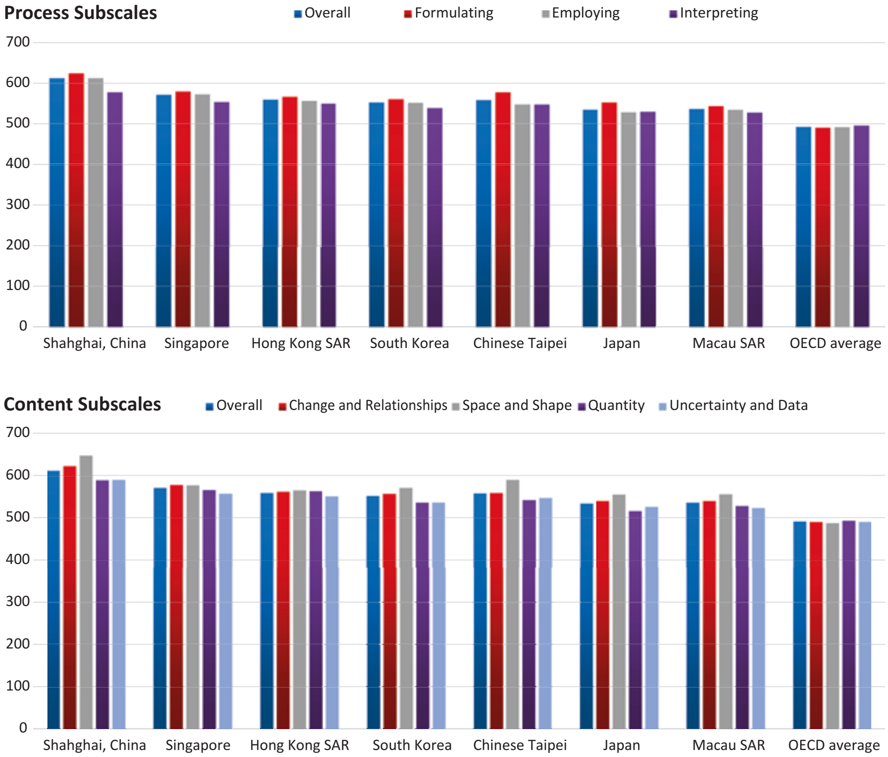

Fig. 2.2 Students' performance on mathematics subscales in PISA 2012

Table 2.8 Correlations among indices of self-related cognition ($N = 5177$)

	Mathematics self-efficacy	Mathematics self-concept	Mathematics interest	Instrumental motivation	Mathematics anxiety
Mathematics self-efficacy	1	0.40***	0.31***	0.26***	−0.37***
Mathematics self-concept		1	0.62***	0.43***	−0.72***
Mathematics interest			1	0.66***	−0.50***
Instrumental motivation				1	−0.32***
Mathematics anxiety					1

Note. All correlations are based on weighted data; *** $p < 0.001$

2.5 Shanghai Students' Self-Related Cognition in Mathematics

A total of 26 items are incorporated into the PISA 2012 student questionnaire measuring students' self-related cognition in mathematics. Five indices, as listed in Table 2.2, are constructed based on these items and then standardized, with the OECD average of zero and a standard deviation of one. A positive value on an index means that scores obtained by Shanghai students are higher than the OECD average, which indicates that students in Shanghai employ a particular self-related cognition generally more often than those from other OECD systems.

Figure 2.3 shows that while Shanghai students' mathematical self-efficacy is well above the OECD average, their mathematical self-concept does not reach the OECD average level. In fact, Shanghai students' mathematical self-efficacy ($M = 0.94$, SD = 1.10) is nearly half a standard deviation higher than that in the second highest system (Singapore: $M = 0.47$, SD = 1.02). This result suggests that Shanghai students are more confident when they are facing a mathematics task, while having overall low confidence in mathematics. A similar phenomenon is observed in the other three Chinese communities (Hong Kong SAR: 0.22 vs. −0.16, Macau SAR: 0.18 vs. −0.19, Chinese Taipei: 0.18 vs. −0.45). Singapore is the only top-performing east Asian system receiving positive scores on both the indices (0.47 vs. 0.22), while Japan (−0.36 vs. −0.38) and South Korea (−0.41 vs. −0.52) both have negative scores. In all these east Asian systems, students consistently have a higher level of self-efficacy than self-concept. Ho (2007) related such a relatively low level of self-concept to the potential impact of classroom context, parental expectations and teacher feedback.

Shanghai students show a higher level of intrinsic motivation to learn mathematics ($M = 0.43$, SD = 0.92) than instrumental motivation ($M = 0.01$, SD = 0.90). Such

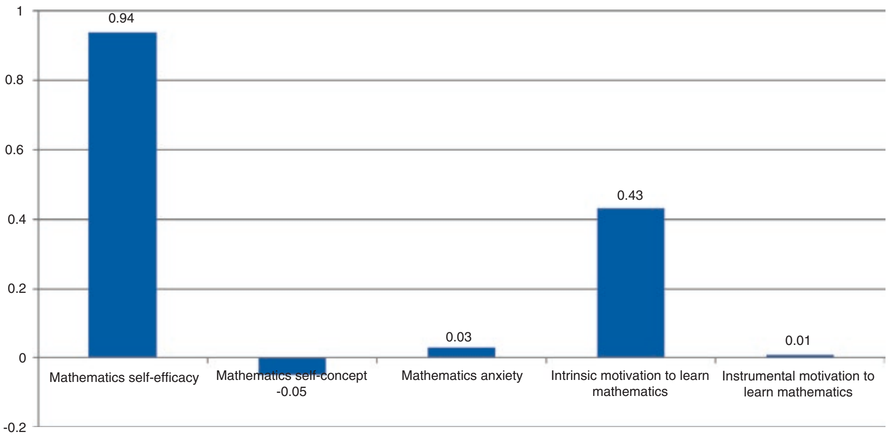

Fig. 2.3 Shanghai students' self-related cognition in PISA 2012

a pattern can also be observed in all the other top-performing east Asian systems. While Shanghai and Singapore both are above the OECD average level on the two types of motivations, Singapore scores nearly 0.40 SD higher than Shanghai on both indices (INTMAT: 0.84 vs. INSTMOT: 0.40). In contrast, both motivation types in Japan and South Korea are below the OECD average.

On the index of mathematics anxiety, all these east Asian systems receive an above-OECD-average score, with the highest in Japan (*M* = 0.36, *SD* = 1.01) and the lowest in Shanghai (*M* = 0.03, *SD* = 0.94). This suggests that Shanghai students are less anxious towards mathematics learning than their east Asian peers, while they are still more anxious than 24 participating systems in PISA 2012.

The relationship between each of the five self-related cognition indices shows a generally moderate to high level (see Table 2.8). The strongest relationship is observed between mathematics self-concept and mathematics anxiety ($r = -0.72$), which indicates that the more confident about one's own overall ability in mathematics the more anxious about mathematics. In fact, mathematics anxiety is significantly correlated with the other four self-related cognition indices in a negative way, ranging from −0.72 to −0.32. This suggests that students' anxiety hinders self-belief and the motivation to learn. The relationship between mathematics self-efficacy and the other four indices is weaker overall. In addition, the two types of motivation are highly correlated ($r = 0.66$).

2.6 Shanghai Students' Disposition Towards Mathematics

The PISA 2012 student questionnaire contains 26 items measuring students' disposition towards mathematics. Based on these items, four indices are constructed and standardized, with the OECD average of zero and standard deviation of one. Similarly, a positive value on an index indicates that students in Shanghai have a more positive disposition towards mathematics than those from other OECD systems.

As a result, on all indices related to students' dispositions towards mathematics except attributions to failure to mathematics, Shanghai's level is above the OECD average (see Fig. 2.4). In particular, Shanghai receives the second-highest level on mathematics work ethic among the top-performing east Asian systems (*M* = 0.32, SD = 0.02) and it scores about one-third standard deviation higher than the OECD average. This suggests that Shanghai students have a relatively high ability to dedicate time, hard work and persistence to attain mathematics competency.

Shanghai students' negative score on attributions to failure to mathematics indicates that the students there are more likely to attribute their failure in mathematics to themselves rather than to external factors (e.g. bad luck, bad guess or the teacher). The difference from the OECD average is close to one-half standard deviation.

On the index of subjective norms in mathematics, the average score in Shanghai is slightly above the OECD average (*M* = 0.11, SD = 1.03). This suggests that the social environment in Shanghai is characterized by a general promotion of

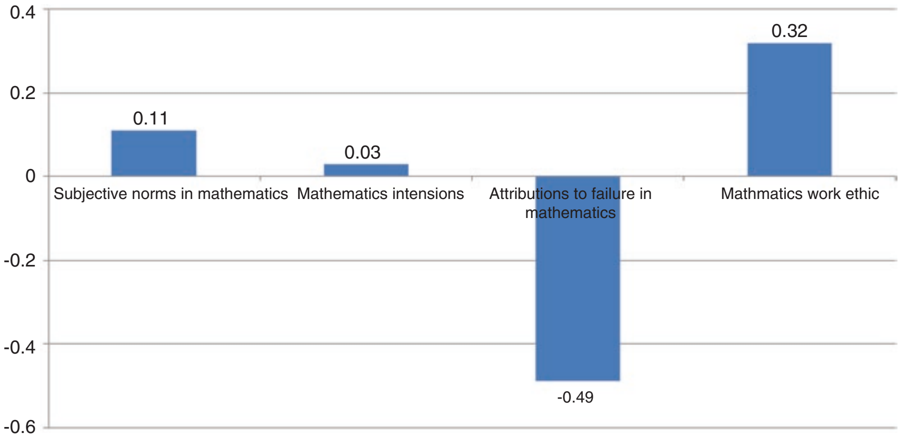

Fig. 2.4 Shanghai students' dispositions towards mathematics in PISA 2012

Table 2.9 Correlations among indices of dispositions to mathematics ($N = 5177$)

	Subjective norms in mathematics	Mathematics intentions	Mathematics work ethic	Attributions to failure in mathematics
Subjective norms in mathematics	1	0.21***	0.43***	−0.22***
Mathematics intentions		1	0.34***	−0.31***
Mathematics work ethic			1	−0.33***
Attributions to failure in mathematics				1

Note. All correlations are based on weighted data; *** $p < 0.001$

mathematics and its study. In this aspect, Singapore has a much higher score ($M = 0.80$, SD = 1.01), while the corresponding scores in the remaining top-performing east Asian systems are all below the OECD average, ranging from −0.58 (Japan) to −0.02 (Hong Kong SAR). As shown in Fig. 2.4, the extent to which Shanghai students intend to use mathematics in their future studies and careers is slightly above the OECD average. In fact, the east Asian systems generally score low on this measure. It seems that students in this region do not have strong short-term or long-term mathematics intentions.

The four disposition indices have relatively low correlations with each other (see Table 2.9). The strongest relationship occurs between subject norms in mathematics and mathematics work ethic ($r = 0.43$, $p < 0.001$), which suggests that a positive social environment for students' mathematics learning benefits the development of mathematics work ethic in Shanghai. Moreover, the index attributions to failure in mathematics is negatively correlated with all other disposition indices, ranging from

−0.22 to −0.33. This suggests that attributing their failure in mathematics to internal factors helps cultivate a positive disposition to mathematics and its learning.

2.7 Self-Related Cognition, Dispositions to Mathematics, and Mathematics Performance

In Shanghai, students' mathematics self-efficacy shows the strongest correlation with mathematics performance ($r = 0.56$) followed by mathematics self-concept ($r = 0.32$). This result is expected, as both the indices capture students' confidence in their mathematics abilities, with the former related to the specific mathematics tasks and the latter in a more general sense.

The weakest correlation is observed with subjective norms in mathematics, while it is positive ($r = 0.03$). This indicates that a social environment promoting mathematics and its learning is beneficial, though subtle, for students' mathematics performance. Two types of motivations are also found to be positively correlated with students' mathematics performance, and the strength with intrinsic motivation ($r = 0.16$) is nearly twice that with instrumental motivation ($r = 0.08$).

Among the nine noncognitive indices, mathematics anxiety and attributions to failure in mathematics show a negative correlation with students' mathematics performance. The negative correlation with mathematics anxiety ($r = -0.30$) suggests that a higher anxiety level brings lower performance in mathematics. In contrast, the fewer students attributing their failure in mathematics to external factors, the better mathematics performance they can achieve ($r = -0.23$).

Given the moderate-to-high correlations between the noncognitive indices, structural equation modelling (SEM) is used to explore the impact of students' noncognitive abilities on their mathematics performance. As a result, the model explain 63.5% of Shanghai students' performance in the PISA 2012 mathematics assessment, with an overall acceptable goodness-of-fit (RMSEA = 0.08, CFI = 0.96, TLI = 0.95; see Fig. 2.5).

It can be seen that students' self-belief about their mathematics learning, motivation towards mathematics learning and disposition to mathematics have significant contributions to students' mathematics achievement in the PISA test. In particular, students' self-belief and disposition show a significantly positive impact on their performance in mathematics ($\beta = 0.85$, $p < 0.001$ and $\beta = 0.62$, $p < 0.001$, respectively).

In contrast, the model reveals that students' motivation has a negative direct impact on mathematics achievement ($\beta = -0.68$, $p < 0.001$). Meanwhile, motivation shows a positive indirect contribution through the other two scales. In particular, it positively correlates with self-belief ($r = 0.14$, $p < 0.001$) and highly correlates with disposition ($r = 0.92$, $p < 0.001$). As a result, the total indirect effect of students' motivation on mathematics performance through the two mediator scales is 0.68 (i.e. 0.11 + 0.57).

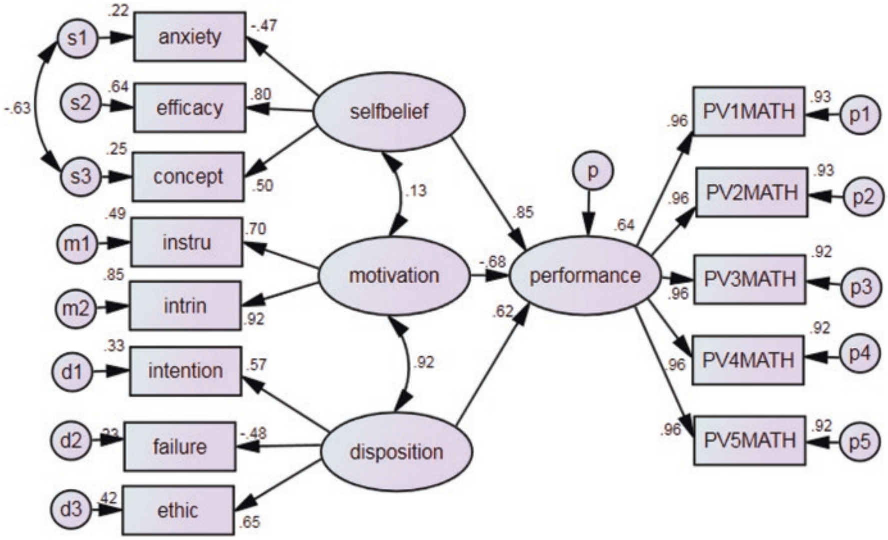

Fig. 2.5 Effects of noncognitive factors on students' mathematics achievement. *Note*. As subjective norms in mathematics refer to others' perspectives about the importance of mathematics and its learning rather than the perspectives of the students themselves, this index is excluded from the SEM model

Consistent with the correlation results, the SEM model also clearly verifies the negative influence of mathematics anxiety and attributing failure to external factors on students' mathematics performance in Shanghai. In addition, students' mathematics anxiety and their mathematics self-concept show a significant reciprocal relationship ($r = -0.63$).

2.8 Concluding Remarks

First in 2009 and then in 2012, Shanghai's 15-year-old students emerged at the top in mathematics, reading and science in the PISA assessments. Their performance is equivalent to almost 3 years of schooling over most other countries assessed. As Moore (2010) remarked, Shanghai students' mathematics performance is in a class of their own, largely outperforming their top-performing east Asian peers. Moreover, such astounding success produces a global 'PISA-shock', which has repositioned Shanghai as a significant new 'reference society' and shifted the global gaze in education away from Finland (Sellar & Lingard, 2013). Meanwhile, it also engenders some dispute over the extent to which the success is the result of sampling (Dronkers, 2015; Tan, 2017).

Whilst students in Shanghai were top-ranked internationally in the mathematics assessments, much less is known about Shanghai students' attitudes towards mathematics (Ding, Pepin, & Keith, 2015). The PISA 2012 study gives specific attention to students' noncognitive abilities in mathematics learning. As a result, Shanghai students reported a relatively low level of self-concept, instrumental motivation and mathematics intentions, and a relatively high level of self-efficacy, intrinsic motivation and mathematics work ethic, by international standards. Moreover, Shanghai students tend to attribute the failure in mathematics to their personal reasons rather than external reasons.

On the whole, Shanghai students' self-belief about their mathematics learning and dispositions to mathematics have significant positive impacts on their performance in the mathematics assessment in a direct way. Meanwhile, students' motivation shows a positive indirect influence through self-belief and dispositions, though its direct influence seems negative.

Regarding the rich information from the PISA studies, both the education officials and the public in China Mainland tend to give their own interpretations and understanding. With an analysis of newspaper articles, official documents and education essays published in China, Mainland Tan (2017) concluded that the Chinese responses are generally reflective, measured and self-critical with three board views: (a) Shanghai's PISA performance exposes educational weaknesses rather than successes, (b) Shanghai's PISA performance is only a minor success, which does not reflect all aspects of holistic education, and (c) there are still areas for improvement for Shanghai education. Tan further argues that Confucian knowledge traditions and structures in China shape such interpretations, which leads the public to downgrade Shanghai's success.

Similarly, Chinese education officials also turn to PISA data to highlight existing problems for the purpose of validating the need for local reforms (Tan, 2019). Two recent educational initiatives in Shanghai, *Green Indices for the Academic Quality of Primary and Secondary Students in Shanghai* and the *New High Quality School* project, have been identified as containing elements borrowed from PISA. Further, information from PISA is used to garner support for ongoing reform initiatives to refine the aims and nature of education in Shanghai. As Tan (2019) suggests, with the continual global influence of PISA, Chinese officials will likely persist in utilizing information from PISA to validate and sustain the reform initiatives in Shanghai and further the self-determined realities.

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Chapter 3
Towards a Framework of Mathematical Competencies in China

Binyan Xu, Yan Zhu, Jiansheng Bao, and Qiping Kong

Abstract This study intends to establish a mathematical competence model and assessment framework for compulsory education. With reference to international research and in consideration of talent development goals in China and the country's mathematics education characteristics, the study takes into account three phases of mathematical activities, namely, mathematisation of real situations, logical organisation of mathematical materials, and application of mathematical theories. The study argues that such mathematical activities are closely related to a number of mathematics competencies, including mathematical problem posing, solving problems mathematically, mathematical representation and transformation, mathematical reasoning and argumentation, mathematical modelling, and mathematical communication. This study divides mathematical core competencies into three levels—reproduction, connection, and reflection—and considers specific performance indicators at each of the three levels based on ability. Therefore, we include three dimensions in the core mathematics competencies framework: mathematical content, core competencies, and ability levels.

Keywords Mathematical core competencies · Mathematical problem posing · Solving problems mathematically · Mathematical representation and transformation · Mathematical reasoning and argumentation · Mathematical modelling · Mathematical communication · Mathematical activities · Mathematical content · Ability level · Reproduction · Connection · Reflection · Behavioural performance · Assessment framework

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3.1 Background

3.1.1 Studies on Education Quality in China

The reform and development of China's primary and secondary education entered a new stage beginning in 2000. Critical tasks of this stage include achieving educational equity, improving the quality of education, and promoting the development of educational awareness. The Ministry of Education of the People's Republic of China (hereinafter referred to as MOE) emphasises that education administrative departments at all levels should gradually establish a standardised, scientific, and institutionalised system for assessing teaching quality and instructional guidance in compulsory education, and actively explore a comprehensive evaluation system with academic level tests and student comprehension as the main indicators (MOE, 2005). In 2007, the National Assessment Center for Education Quality was established under MOE to monitor the quality of study and physical and mental health of students in primary and secondary education as well as related factors affecting students' development.

3.1.1.1 Evaluation of Regional Education Quality

One of the most important ways to establish and improve an educational quality assurance system is to develop standards which are both consistent with an international perspective and conform to China's educational circumstances (Yang, 2012). All provinces and cities in China have been actively exploring and implementing education quality monitoring in recent years, driven by comprehensive quality evaluation on the part of the national government (Xu, 2012). For example, Shanghai has implemented a comprehensive evaluation reform of the green index of academic quality, shifting the emphasis from academic knowledge to the overall development of the students. The green index of academic quality in Shanghai includes ten key aspects: students' academic level, learning motivation, learning burden, teacher-student relationship, teacher's teaching methods, principal's curriculum leadership, the influence of students' social and economic background on academic achievement, students' moral behaviour, physical and mental health, and progress between two school years. The former research group of the National Institute of Education Sciences has developed an index system and evaluation tool for the evaluation of academic achievement in each of the four core subjects—Chinese, mathematics, science, and morality and society—for grade-six students. These tools, which analyse students' academic achievement, can provide feedback for and promote the improvement of subject teaching and student learning (Research Group of Learning Achievement on Primary and Middle School Students in the National Institute of Education Sciences, 2011). Researchers have also focused on the evaluation of academic quality and student comprehension, by developing a framework and tools for evaluating students' academic achievement in various disciplines in agreement with both international goals and Chinese traditions. Some studies also absorb the latest achievements of the taxonomy of educational objectives in the cognitive domain, educational measurement, and evaluation theory to sort out evaluation dimensions and design integrated evaluation frameworks (Zhang & Cui, 2012).

3.1.1.2 Evaluations of Regional Academic Achievement in Mathematics

In the study of academic evaluation, research on the evaluation of mathematics academic achievement is particularly prominent. Researchers take quality education as the goal and the national mathematics curriculum standard as the basis to compile test questions that measure academic achievement in mathematics and to inform the test investigation. Some studies focus on the two basic dimensions of mathematical content and mathematical cognitive ability to evaluate the degree to which the three curriculum objectives related to mathematics knowledge and skill, mathematics process and method, and attitudes toward mathematics have been achieved (Shen, Yang, & Song, 2009). In this study, in order to ensure the scientificity and effectiveness of the mathematics academic evaluation test questions, test questions were compiled according to the following process: a comparative analysis of teaching materials, the establishment of evaluation criteria, the formulation of the propositional bi-directional list, compilation of test questions, and a sampling experiment.

Some researchers have also studied the status and influence factors of the mathematics academic level of grade-eight students in three regions of China, and found that these students have a middle standard rate; the students' standard rate of 'applying' ability was lower than that of 'knowing, understanding and mastering' abilities (Qi, Zhang, & Wang, 2015). In addition, the research found that teaching methods, knowledge representation, learning evaluation, attention to students, and learning habits have an impact on students' academic performance.

In summary, particular attention has been paid in recent years to education quality and academic quality in China, from the government level to the local level and on to the research level, which is in line with international trends in primary and secondary education reform and quality assurance.

3.1.2 International Comparison Studies

The construction of disciplinary competency models has become central to the development of academic quality standards internationally.

3.1.2.1 International Projects

The Trends in International Mathematics and Science Study (TIMSS), an international scale test sponsored by the International Association for the Evaluation of Educational Achievement (IEA) and held every 4 years since 1995, mainly measures the academic proficiency in mathematics and science of grade-four and grade-eight students and factors that influence their performance. So far, this project has been the most influential and popular mathematics education evaluation project in the world (Lai, 2008).

For example, the TIMSS 2007 mathematics assessment framework consisted of two major dimensions: mathematical content and cognitive ability. In terms of mathematical content, the framework asserts that fourth graders need to learn numbers, geometric shapes and measures, and data display, while eighth graders need to study numbers, algebra, geometry, and data and chance. Each of the content areas contains several themes, which are further refined into different sets of objectives by many participating countries. Cognitive ability was divided into three levels in both grades: knowing, applying, and reasoning. *Knowing* includes the facts, procedures, and concepts that students need to know, involving recall, recognition, computation, retrieval, measurement, and classification/ordering. *Applying* indicates students' ability to solve or answer questions by applying the knowledge and concepts they have learned, using selection, representation, modelling, implementation, and routine problem solving. *Reasoning* refers to the shift from tackling conventional problems to tackling unfamiliar, complex, and multi-step problems involving analysis, generalisation, synthesis/integration, justification, and non-routine problem solving (Mullis et al., 2005).

The Programme for International Student Assessment (PISA), an international test sponsored by the Organisation for Economic Co-operation and Development (OECD), has been held every 3 years since 2000 to assess the competency of 15-year-old students in the fields of reading, mathematics, and science, focusing on one competency at a time. The evaluations in 2003 and 2012 focused on mathematics competency.

The PISA defines mathematics competency as 'an individual's capacity to recognise the role that mathematics plays in the world and make well-founded mathematical judgements, and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen' (OECD, 2010, p. 122). The PISA assesses three main aspects of mathematics competency: mathematical content (including four main concepts—quantity, space and shape, change and relationship, and uncertainty—followed by specific content, such as numbers, algebra, and geometry); mathematical processes (including eight kinds of mathematical abilities—thinking and reasoning, argument, communication, modelling, problem presentation and solution, expression, symbols and standard language usage, auxiliary tools usage—and three kinds of ability groups, namely, reproducing groups, connecting and integrating groups, and reflecting groups); and mathematical situations (including five situations: personal situation, educational situation, occupational situation, public situation, and scientific situation) (OECD, 2013).

3.1.2.2 Research Projects of Different Countries/Regions

In addition to the above two cross-country and cross-region mathematics evaluation projects, many countries have their own set of relatively mature evaluation systems for assessing mathematical ability. The National Assessment of Educational

Progress of the United States (NAEP), the only continuous and long-term primary and secondary student achievement measure in the United States, assesses fourth-, eighth-, and twelfth-grade students with reading and mathematics as compulsory subjects (National Assessment Governing Board, 2007). Other such projects include: the National Curriculum Test of the UK (NCT), aiming at grades two, six, and nine with English and mathematics as compulsory subjects; the General Certificate of Secondary Education (GCSE), aiming at 16-year-old students whose compulsory education includes math; International Competitions and Assessment for Schools of Australia (ICAS), comprehensively assessing school systems with mathematics included as one of the subjects; and others (Huang, Wang, & Xu, 2004).

With the aim of improving and assuring the quality of the Germen educational system, the Institute for Education Quality Improvement (IQB) in Germany develops its test question bank based on the educational standards of mathematics adopted by the Standing Conference of the Ministers of Education and Cultural Affairs of the Land in the Federal Republic of Germany, conducts nationwide tests on all grade-three and grade-eight students, and statistically analyses their mathematical ability, thus detecting the degree to which the national standard of mathematics education has been reached (Granzer, Koeller, & Bremerich-Vos, 2009). It also provides teachers with test results and open test banks, enabling them to diagnose students' mathematical ability level, and provides resources on how to improve classroom teaching (Xu, 2011).

3.1.3 Mathematical Ability in Mathematics Curriculum Standards

In the United States, the idea of core mathematics competencies has always accompanied mathematics education reforms. *The Principles and Standards of School Mathematics Education in the United States*, published by NCTM in 2000, attached great importance to the interrelation between mathematical understanding and mathematics capability, and put forward 10 equally weighted standards for mathematical content and ability, five of which are mathematical communication, problem solving, reasoning, connection, and representation (NCTM, 2000).

The mathematics education standards for grade-10 students promulgated by Germany in 2003 also represent a typical competence-oriented approach. The standards address three dimensions, namely, mathematical process, content, and competence level. The 'process' dimension describes six major mathematical competencies, including mathematical argumentation, solving problems mathematically, mathematical modelling, using mathematical representation, the mastery of mathematical symbols, formulas, and techniques, and mathematical communication (Kultusministerkonferenz, 2004).

Since 2000, Singapore has issued mathematical syllabi centred on the development of students' ability to solve mathematical problems and has outlined mathematical process skills, such as thinking skills and mathematical reasoning, communication, and relations (Ministry of Education Singapore, 2011). The course of study for mathematics for junior high schools in Japan established in 2009 abandoned the curriculum targets focused on learning with pleasure and relaxation that had a negative effect on mathematics courses, and raised the content of teaching and the number of teaching hours to facilitate students' assimilation of basic knowledge and skills. On this basis, the revised principles emphasise the cultivation of three abilities: thinking, judgement, and expression (Chen, 2010).

The concepts of 'mathematical thinking' and 'problem solving' were first used in the *Mathematics Curriculum Standards for Full-time Compulsory Education (Experimental Version)* (MOE, 2001) in China, which extended the three traditional major mathematical abilities (calculation ability, spatial imagination ability, logical thinking ability), and the newly promulgated *Mathematics Curriculum Standards for Compulsory Education (2011 version)* (MOE, 2012) retained these concepts. The descriptions of core competencies in mathematics curricula or education standards in different countries show that the reform of mathematics education not only stresses the cultivation of mathematical ability in the strict mathematical sense, but also emphasises the development of the mathematical abilities of process and application.

Research in China and all over the world shows that the evaluation of core competence in various disciplines is key to capturing students' mathematics achievements, and thus provides a reference for improving teaching and student learning.

3.2 Theoretical Perspective

Research on mathematical competence has become an important topic in international mathematics education research. Niss (2011) believes that 'to master mathematics means to process mathematical competence', and that mathematical competence 'means the ability to understand, judge, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role' (pp. 6–7). Eight competencies that are recognised to be constituents of mathematical competence have been widely cited in recent literature: (1) mathematical thinking, (2) problem posing and solving, (3) mathematical modelling, (4) mathematical reasoning, (5) mathematical representation, (6) mathematical symbolisation and formalisation, (7) mathematical communication, and (8) tools usage. Yu emphasises that the core competence of mathematics should include the individual's ability to apply mathematical knowledge in practice. Yu, therefore, proposes that individual ability includes mathematical communication, mathematisation, mathematical representation, mathematical reasoning and argumentation, mathematical strategic thinking, and the use of symbols, formulas, and technical language (Yu, 2010, pp. 316–318).

In order to provide a meaningful and operable reference for the evaluation of mathematics education in China, this study intends to establish a mathematical competence model which takes into account not only the essential characteristics of mathematics as an academic subject, but also the new requirements for the mathematics education brought about by social development. Such a model involves the following two theoretical perspectives.

3.2.1 The Process of Mathematisation and Mathematical Competencies

Although mathematics, as a science which studies the relation of quality and the form of space in reality, has demonstrated a strong deductive system, Stoliar points out that, like any other human knowledge system, when mathematics is developing, we may discover the theorem before we can prove it; and we should guess the way to prove it before we succeed in proving it. In this sense, to reflect the creation process of mathematics in mathematical teaching, we must not only teach students to 'prove', but also teach students to 'guess' (Stoliar, 1984).

Freudenthal (1994) believed that mathematics is rooted in such common sense that could be organised by people through their own practice and reflection, systematising mathematics horizontally and longitudinally. Therefore, he regarded mathematical learning as an activity of 'recreation' or 'mathematisation', an experience which must be had by learners themselves rather than anyone else. In mathematics education, special attention should be paid to this process of mathematisation, and to cultivating students' attitude toward acquiring and building their own mathematics. A very important aspect of mathematisation is self-reflection. Cao (1990) advocated that mathematical competence is a personal psychological feature which is necessary for the successful completion of mathematics activities and has a direct impact on activities' efficiency, and that mathematical competence is also a relatively stable psychological feature which is shaped, developed, and demonstrated in mathematics activities.

Thus, we can conclude that mathematical competencies should be developed through the exploration and creation of mathematical knowledge in mathematical activities. The teaching of mathematics should be the instruction of mathematical activities, where not only the acquisition of basic mathematical knowledge, skills, and thinking methods in the strict mathematical sense are achieved, but also the accumulation of experience in mathematical activities, such as exploring, inventing, creating, and communicating, as the newly issued Compulsory Education Mathematics Curriculum Standards has advocated (MOE, 2012). Therefore, mathematical competencies are closely associated with the essence of mathematical activities, which is the focus of our study. The requirements of modern society for mathematical activities will be considered in our study as well.

3.2.2 The Nature of Mathematical Activities and Mathematical Competencies

Previous studies show that mathematical activities can be basically divided into three phases: mathematisation of real situations, logical organisation of mathematical materials, and application of mathematical theories, which also reflect the formation and development of mathematics (Kruchevskii, 1983). As a mathematical activity, the teaching of mathematics is not a matter of conveying ready-made content printed in textbooks, but rather the chance for students to detect what has been discovered in science on their own, to logically organise the mathematical materials gained from experience, and in the end to apply mathematical theories to various real situations.

3.2.2.1 Organising Empirical Materials Mathematically

In the teaching of mathematics, students will encounter plenty of empirical materials, including various situations or problems from their daily life experience, objects that they own or encounter, and relationships with other subject domains (such as physics, chemistry, biology, geography, etc.), in addition to objects prepared purposely for instruction (such as teaching materials, teaching aids, etc.) and mathematical materials (objects) that need further generalisation and abstraction.

Students need to process the empirical materials by means of observation, experiment, induction, analogy, and summarisation to find factual bases or information which can easily be understood from a mathematical point of view. For example, with the mathematical material 'the sum of degrees of interior angles of a triangle is 180', students can conduct observations and experiments such as protractor measurement or clipping to obtain a better understanding of it. Although these activities do not constitute proof, they help students accumulate experience for finding the proof. In mathematical activities, familiar daily experience can be selected to promote discussion. For example, as there are many lines from the classroom to the canteen on large campuses, it is useful to ask students to discuss optimal routes and the reasons behind them from a mathematical point of view. Learning mathematics through activity is quite beneficial to students as it helps them develop the abilities of problem posing from a mathematical perspective, mathematical communication, mathematical representation, mathematical modelling, and so on.

3.2.2.2 Organising Mathematical Materials Logically

After organising or accumulating empirical materials from a mathematical perspective, students need to extract the original concepts and axiomatic system which can be further used as a base for formulating theories in a deductive way. The deductive structure of a theory is an important feature in the mathematical concept system. In the course of teaching, teachers can and should set up instruction situations that help reveal this characteristic to students.

Take the following illustration as an example. A square is a diamond with a right angle; a diamond is a parallelogram with equal adjacent edges; a parallelogram is a quadrangle with two pairs of parallel sides; a quadrangle is a polygon with four edges; a polygon is a figure enclosed by closed fold lines; a figure is a set of points. By leading from one concept to another, we can finally reach the original concepts of 'set' and 'point'. Logical organisation also includes propositions which are proved and formulated via induction and presented in a hypothetical form. In this process, special attention should be paid to the role of induction and general plausible reasoning in mathematics activities, including what to prove, where to prove it, and how to prove it. Hence, supported by the process of mathematics teaching, students' various abilities can be cultivated.

3.2.2.3 The Application of Mathematical Theory

No matter how abstract modern mathematics is, its root is always deeply grounded in practices, which is true from land measurement and commercial trades in the past to modern physics, biology, and economics, etc. Mathematical methods are often valuable in solving problems arising in the fields of science, technology, sociology, or even history. To solve problems in non-mathematical fields, translation into a mathematical language is the first step, followed by transforming these problems into abstract mathematical ones which can be solved in the strict mathematical world. Students' capacity to abstract mathematical problems from specific contents via problem observation and active thinking is emphasised in this stage; this ability is fostered and consolidated through long-term practice, and which in turn contributes to enhancement of problem-solving ability, mathematical communication ability, mathematical reasoning ability, mathematical modelling ability, and so on.

Based on the above analysis, mathematical activities are closely related to a number of mathematics competencies, including mathematical problem posing, mathematical representation and transformation, mathematical reasoning and argumentation, solving problems mathematically, mathematical communication, mathematical modelling, and so on. Thus, mathematics instruction of this type will help students form and develop these six core competencies. Figure 3.1 shows the relationship between the three mathematical activity stages and mathematical competencies (Xu, 2013).

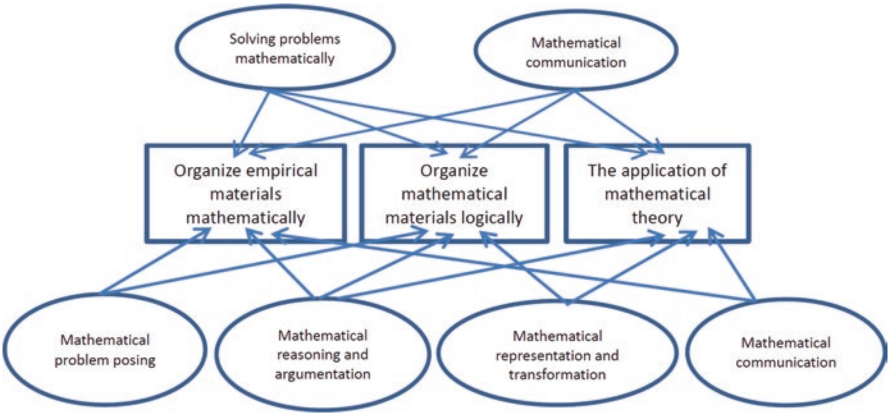

Fig. 3.1 The relationships between mathematical activity stages and core mathematics competencies

3.3 The Meaning of Core Mathematical Competencies

The composition of students' core mathematical competencies is determined by the nature of mathematical activities. In mathematics instruction which focuses on mathematical activities, students will form and develop these abilities. The following is a detailed analysis of these abilities.

3.3.1 Mathematical Problem Posing

Researchers explore the meaning of problem posing ability from different perspectives. Silver identifies two kinds of problem posing: (1) problem formulation or reformulation occurring within the process of problem solving and (2) recreation of a given problem in different ways to make it more accessible for solution (Silver, 1994). Moreover, posing can occur before, during, or after the solution of a problem, but this does not override the importance of reformulating a problem as one attempts to solve it (English, 1997). In problem posing contexts, students are stimulated to make observations, experiment by varying some of the data and analysing the results, and devise their own new problems that can be solved by using similar or different patterns (Singer, Ellerton, Cai, & Leung, 2011). Based on the above analysis, this study defines the ability to pose problems from a mathematical perspective as: to be able to propose new mathematical problems based on a certain contexts or problems, or come up with new sub-problems during or after the process of problem solving, and use mathematical language to present these proposed, creative, and independent new mathematical problems.

3.3.2 Solving Problems Mathematically

There is currently no unified definition in the field for the ability to solve problems mathematically. For example, in the standards promulgated by NCTM in 2000, the United States described solving problems mathematically as: to build new mathematical knowledge through problem solving; to solve problems that arise in mathematics and in other contexts; to apply and adapt a variety of appropriate strategies to solve problems; and to monitor and reflect on the process of mathematical problem solving (National Council of Teachers of Mathematics, 2000). In the mathematics curriculum standards promulgated in 2003, Germany defined solving problems mathematically as: to have appropriate mathematical strategies to find and rethink the ideas or methods for solving problems (KMK, 2004). Chinese mathematics education has always attached great importance to the ability of solving problems mathematically, and the Compulsory Education Mathematics Curriculum Standards promulgated in 2011 give a detailed explanation of mathematical problem solving, emphasising that middle school students should learn how to solve mathematics problems in their mathematics courses (MOE, 2011). Through text analysis, this study defines the ability of mathematically solving problems as: to be able to solve problems occurring in mathematics or other situations by using various kinds of appropriate mathematical knowledge, methods, and strategies, and to verify and reflect on the process of mathematical problem solving.

3.3.3 Mathematical Representation and Transformation

The above analysis of the existing research shows that mathematical representation and transformation are central to mathematics education reform in various countries. As is illustrated in related researches, mathematical representation refers to expressing mathematical concepts or relations in some form and contributes to students' understanding of concepts, relationships, correlations, and the mathematical knowledge used in the process of problem solving (Cai, Frank, & Lester, 2005). If a learner wants to understand a mathematical problem, it is necessary to establish a mapping between this mathematical problem and a more comprehensible mathematical problem, and representation is the mapping process. Comparing existing research results, we define the ability of mathematical representation as: to be able to express the mathematical concepts or relationships to be learned or handled in some form, such as written symbols, graphs (tables), scenarios, manipulative models, texts (including oral communications), etc., so as to eventually solve the problem.

Mathematical transformation means changing the format of information in the process of mathematical problem solving; that is, transforming the problems to be solved in a mathematical way while maintaining their invariant properties, thus converting complex, unknown, and unfamiliar problems to simple, known, and familiar ones. Therefore, mathematical transformation ability means to be able to use mathematical transformation strategies that can change the format of information in order to simplify or successfully solve a problem.

3.3.4 Mathematical Reasoning and Argumentation

Reasoning is the basic way of thinking in mathematics as well as a common way of thinking in people's learning and daily life. Mathematical reasoning refers to formulating certain judgments about mathematical objects under the function of the mathematical concept system by combining certain mathematical conditions, knowledge, and methods. As one kind of reasoning, it has its own characteristics: first, the object of mathematical reasoning is neither common sense nor a social phenomenon, but mathematical symbols representing quantitative relations and spatial forms; second, in a certain process of thinking, mathematical reasoning is more coherent than general reasoning; third, mathematical reasoning is mainly based on the mathematical system where questions are located. The high abstractness of mathematics and the strictness of logic result in the relative difficulty of mathematical reasoning.

Argumentation is inseparable from reasoning. In the process of argumentation, the fact that the judgment as to whether the statement is true or false can be made based on known judgments is thanks to the establishment of logical connection between known judgments and the statement to be judged, while the latter is derived from the former via reasoning. So, one or a series of reasoning must be applied in the process of argumentation, which means this process is the application of reasoning and reasoning is the tool of argumentation. Based on the above analysis, the concrete definition of the 'mathematical ability of reasoning and argumentation' is: to be able to draw inferences via logical thinking (observation, experimentation, induction, analogy, deduction) about mathematical objects (mathematical concepts, relationships, properties, rules, propositions), and to illustrate the reasonableness of the given inferences by seeking further evidence, or providing proofs or counterexamples.

3.3.5 Mathematical Modelling

Although always associated with mathematical application, mathematical modelling is actually different from mathematical application, as modelling focuses on the establishment of a reversible connection between the real world and the mathematical world, and the process of abstracting mathematical problems and solving practical problems. While doing mathematical application, people will be informed which mathematics should be used to solve problems.

Since mathematical modelling is not a linear process, it is necessary to constantly return from the mathematical world to the real world to test the results and improve the model. Blum (Blum, Galbraith, Henn, & Niss, 2007) proposed that mathematical modelling is a nonlinear, cyclical process consisting of seven steps:

(1) establishing a 'situation model' to understand the real-world problem, (2) developing the situation model into a 'real model', (3) mathematising the real model to construct a 'mathematical model', (4) carrying out mathematical procedures to obtain a mathematical solution, (5) interpreting the mathematical solution in terms of the real-world problem, (6) validating the real-world solution, and (7) presenting the real-world solution.

Therefore, the ability of mathematical modelling means: to be able to comprehend and construct a real situation model when confronted with a comprehensive situation, to translate the model into a mathematical problem, to establish a mathematical model, to use mathematical methods to solve the mathematical problem, and then to interpret and test the mathematical solution according to the specific situation as well as to test the reasonableness of the model.

3.3.6 Mathematical Communication

With the development of science and technology, mathematics has penetrated different aspects of society. Students who are future citizens need to have a certain level of mathematical communication ability. Mathematical communication is a way for students to learn mathematics and also one of the ways to apply mathematics. Students learn mathematical language in communication and use specific symbols, vocabulary, and syntax in mathematical language to communicate, understand the world, and gradually gain the accumulation of common sense.

At present, many countries have explicitly put forward the requirement of developing students' mathematical communication ability in their mathematics curriculum standards. For example, in the 'key concept' section of the British national curriculum, 'effective mathematical communication ability' occupies one of the three major abilities (Department for Education of UK, 2007), which requires students to understand and explain mathematics presented in various forms and to communicate confidently in the most appropriate way. The PISA also regards mathematical communication ability as a requisite part of mathematical ability assessment and describes it as 'the mathematical reading and writing ability accompanying the process of communication' (OECD, 2012, p. 18). The Mathematics Curriculum Standards for Compulsory Education (MOE, 2012) in China also explicitly require students to communicate with others about their own algorithms and processes, and to express their own ideas. Mathematics curriculum standards in each country provide us with explanations of mathematical communication ability.

This study defines the ability of mathematical communication as: to be able to recognise, understand, and comprehend mathematical thoughts and mathematical facts at different levels by reading, listening, and so on; to explain the solution, process, and results of their problems by writing, explaining, and so on; and to analyse and evaluate the mathematical ideas and facts presented by others.

In summary, a core mathematical competency model consisting of six competencies and their related meanings are obtained in our study, which is summarised in Table 3.1.

Table 3.1 Meaning of core mathematical competencies

Components of core competence	The meaning of core competence
Mathematical problem posing	To be able to propose new mathematical problems based on certain contexts or problems, or come up with new sub-problems during or after the process of problem solving, and use mathematical language to present these proposed, creative, and independent new mathematical problems
Solving problems mathematically	To be able to solve problems occurring in mathematics or other situations by using various kinds of appropriate mathematical knowledge, methods, and strategies, and to verify and reflect on the process of mathematical problem solving
Mathematical representation and transformation	To be able to express the mathematical concepts or relationships to be learned or handled in some form, such as written symbols, graphs (tables), scenarios, operational models, words (including oral words), etc., so as to eventually solve the problem To be able to use mathematical transformation strategies that can change the format of information in order to simplify or successfully solve a problem
Mathematical reasoning and argumentation	To be able to draw inferences via logical thinking (observation, experimentation, induction, analogy, deduction) about mathematical objects (mathematical concepts, relationships, properties, rules, propositions), and to illustrate the reasonableness of the given inferences by seeking further evidence, providing proofs or counterexamples
Mathematical modelling	To be able to comprehend and construct a real situation model when confronted with a comprehensive situation, to translate the model into a mathematical problem, to establish a mathematical model, to use mathematical methods to solve the mathematical problem, and then to interpret and test the mathematical solution according to the specific situation as well as to test the reasonableness of the model
Mathematical communication	To be able to recognise, understand, and comprehend mathematical thoughts and mathematical facts at different levels by reading, listening, etc.; to explain the solution, process, and results of their problems by writing, explaining, etc.; to analyse and evaluate the mathematical ideas and facts of others

3.4 An Assessment Framework of Core Mathematical Competencies

3.4.1 Levels of Core Mathematical Competencies

The construction of the core mathematics competencies model in this study will provide a theoretical framework for the practice of academic quality measurement in China. In order to enhance the practical guiding significance of the core mathematics competencies model, the study combed the representative content of mathematics both internationally and in China, and refined the six core mathematics competencies into observable student performance behaviours. Due to different cognitive levels, students will have different behaviours related to core mathematics competencies, and these behavioural differences will reflect the differences of their ability level. The present study attempts to research the stratification of ability levels as well. By referring to a series of international evaluation projects on the classification of ability or cognitive levels, this study divides the mathematical core competencies into three levels: reproduction, connection, and reflection.

Level one, reproduction, refers to the ability to memorise basic mathematical concepts, theorems, and methods, and to apply these contents by imitation.

Level two, connection, refers to the ability to process familiar content by using acquired knowledge, skills, and techniques, meanwhile drawing connections between different mathematical content strands.

Level three, reflection, refers to being adept in processing complex content and obtaining solutions, meanwhile proving, reasoning, interpreting, or evaluating solutions.

In view of the different fields of mathematical content, the core mathematics competencies have specific performance at three levels. Therefore, we include three dimensions in the core mathematics competencies framework: mathematical content, core competencies, and ability levels, as shown in Fig. 3.2 (Xu, Zhu, Bao, & Kong, 2015).

For instance, A represents the level of connection related to mathematical representation and transformation ability in the field of figures and geometry; B represents the level of reflection on mathematical communication ability in the field of statistics and probability.

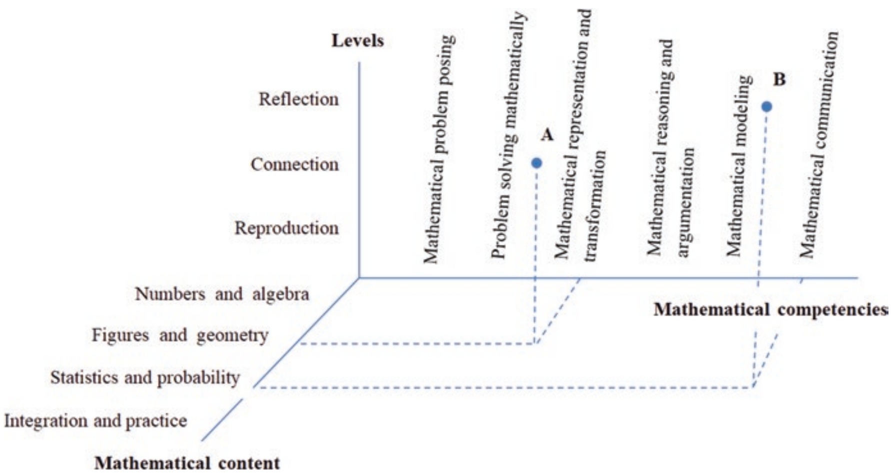

Fig. 3.2 Assessment framework of core mathematical competencies

3.4.2 *Behavioural Performance of Core Competencies at Different Levels*

With reference to comparative international results and in consideration of talent development goals in China and the country's mathematics education characteristics, this study puts forward the meaning and level stratification of core mathematics competencies during the compulsory education period. In order to examine and develop specific testing tasks to measure students' core mathematics competencies and an analysis tool for assessing such tests, it is necessary to describe the behaviours of core mathematics competencies at different levels. We have worked out practical behavioural indicators for different levels of competencies, which are presented in Table 3.2.

Table 3.2 Behaviours of mathematical core competencies at different levels

	Levels		
Competencies	Level one: reproduction	Level two: connection	Level three: reflection
Mathematical problem posing	To be able to recognise the structure of a given mathematical problem To be able to imitate or modify a given problem and pose a similar mathematical problem To be able to make appropriate supplements to a mathematical problem based upon the missing elements, turning it to a complete mathematical problem	To be able to find or pose more important mathematical problems (for accomplishing tasks) in real-world situations or tasks To be able to expand and select information according to one's own mathematical knowledge and experience, establish mathematical links, and put forward different mathematical problems	To be able to classify various mathematical problems raised by oneself and explain their bases and processes To be able to evaluate mathematical problems raised by peers To be able to put forward more complex and extended mathematical problems
Solving problems mathematically	To be able to recognise and select familiar mathematical information when faced with simple problem contexts, and solve simple mathematical problems according to known mathematical methods and strategies To be able to express the simple process of mathematical problem solving	To be able to connect with knowledge and expressions (tables, words, symbols, etc.) of different mathematical domains To be able to express thinking procedures, solutions, and results briefly and logically To be able to explain the meaning of one's own mathematical results	To be able to solve complex mathematical problems by integrated use of mathematical knowledge, methods, and strategies, and explain the consistency of mathematical models, results, and reality To be able to reflect on one's own problems, solutions, and strategies To be able to compare, evaluate, and correct others' understanding To be able to select the optimal strategy according to the specific situation

(continued)

Table 3.2 (continued)

	Levels		
Competencies	Level one: reproduction	Level two: connection	Level three: reflection
Mathematical representation and transformation	To be able to directly process and utilise familiar representations in a standardised context, such as translating familiar word expressions into symbols, figures, or charts	To be able to clearly interpret and convert two or more representation formats into relatively familiar situations To be able to design a certain representation format for the problem situation	To be able to understand and apply nonstandard forms of representation To be able to design specific representations for key steps in a complex problem context To be able to compare and weigh different forms of representation
Mathematical reasoning and argumentation	To be able to put forward certain reasonable conjectures To be able to express the reasoning process of the conjecture To be able to verify the correctness of propositions in simple situations To be able to express in relatively appropriate and correct mathematical language	To be able to make higher-level conjectures in relatively complicated problem contexts by relating to connected knowledge To be able to clearly express the process of thinking To be able to conduct argumentation of complex propositions by relating to other people's reasoning and previous experience with a concise and complete process	To be able to make more conjectures, reflect on and test the conclusions, and then systematise the mathematical objects To be able to express reasonably and logically To be able to expand thinking and choose the proper reasoning method according to the specific situation in order to make strict argumentation To be able to express clearly and precisely
Mathematical modelling	To be able to recognise the standard model in a simple and familiar situation, directly translate the real situation into a mathematical model, and try to solve the mathematical problem without testing the reasonableness of the model	To be able to put forward a corresponding realistic model in a relatively familiar yet complex irregular problem situation by comparing with familiar models, then translate it into a mathematical model and attempt to solve the mathematical problem and test the reasonableness of the model even though the process is not complete	To be able to recognise a reasonable realistic model in a complex and unfamiliar situation, then create a mathematical model to solve the mathematical problem, and attempt to test and evaluate the model

(continued)

Table 3.2 (continued)

	Levels		
Competencies	Level one: reproduction	Level two: connection	Level three: reflection
Mathematical communication	To be able to express simple mathematical facts To be able to recognise and select information embedded in short mathematical texts	To be able to recognise and select information embedded in mathematical texts and understand its meaning To be able to transform others' mathematical thinking from one carrier to another To be able to express the thinking process, solutions, and results briefly and logically To be able to explain others' explanation of mathematical texts on the basis of judgment	To be able to understand the meaning of complex mathematical texts, compare and judge others' mathematical thinking To be able to design a plan that can completely present the complex process of solution and demonstration To be able to flexibly transform the carrier of mathematical thinking and select the optimal expression carrier according to the specific circumstance To be able to express the inspection and reflection on the learning process

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Chapter 4
The Development of Problem-Posing in Chinese Mathematics Curriculum

Muhui Li and Binyan Xu

Abstract This chapter is intended to examine the mathematics curriculum documents from 1902 to 2018 that focused on the requirement of mathematical problem-posing. The research question addressed in this study is as follows: What are the characteristics of the conceptual development that mathematical problem-posing has gone through in China's mathematics curriculum since 1902? A content analysis was conducted to answer this research question. The results showed that the development of mathematical problem-posing in the mathematics curriculum documents of Chinese schools can be divided into three phases: From 1902 to 1977, the curriculum documents paid little attention to problem–posing; from 1978 to 2000, the development of problem-posing abilities was emphasised to some extent in the mathematics curriculum (syllabi) and was linked to the cultivation of students' independent thinking and self-learning abilities. Since 2001, mathematical problem-posing in the curriculum standards has penetrated into the curriculum objectives, contents, suggestions, assessment, etc. It has been explicitly pointed out that creating problems is the basis of innovation.

Keywords Mathematical problem-posing · Mathematics curriculum standards · Syllabus · Conceptual development · Content analysis · Independent thinking · Self-learning ability · Create problems · Innovation

4.1 Introduction

Over the years, China has greatly benefited from good traditional education. The training of students' mathematical thinking and the development of students' abilities have always been the focus of mathematics education. With the increase of grade levels, on the one hand, students' problem-solving ability improves progressively; on the other hand, their problem-posing ability emerges as well. However,

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their ability to pose good questions is relatively weak (Cai, 1998; Nie, Wang, & Lv, 2003; Wang, 2009). In the era of information explosion, the expansion of mathematical knowledge and skills should not imply the reduction of problems. On the contrary, it becomes even more critical for students to identify good problems in the knowledge reserves. When examining the real-world circumstances, it is found that teachers are unqualified to fulfil the role of the guider. It seems that the teaching method of 'preparing everything for the students' results in a comprehensive and refined summary of the mathematics learning process. However, over time, it inhibits students' abilities to innovate and pose problems (Ning & Wang, 2012; Xu, 2013, 2015). This forces us to rethink the strategy of how to 'teach students to ask'. In fact, as early on as more than 2000 years ago, our sages wrote a summary in the Record on the Subject of Education (Dai, 1989), which was the skilful problem-poser is like a carpenter cutting hard wood, starting with the smoother texture and, then, moving to the knotted. Over a long time, students understand with practice. The unskilled problem-poser takes the opposite course. The master who skilfully waits to be questioned may be compared to a bell when it is struck. When struck with a small hammer, it produces a soft sound. When struck with a big hammer, it produces a loud sound. But if it is struck leisurely and properly, it gives out all the sound of which it is capable. This text likens posing and answering questions to cutting down trees and striking a bell. It reveals the inner laws of teaching the arts of posing and answering questions between teachers and students. It points out that teachers can only reveal the answers when students really do not understand them, so that the students' problem consciousness and problem-posing abilities can be developed (Yue & Feng, 2009). Therefore, discovering and posing problems is helpful for thinking, and it is also one of the footholds of mathematics teaching. If students focus solely on how to memorise knowledge, they will lose their opportunity to expand their ability to explore and think. It makes sense to consider the manner in which problem-posing can be integrated as an effective element of mathematics instruction. In China, as the intended curriculum, the mathematics curriculum standards play a central role in implementing said curriculum. It is important to investigate how curriculum standards have incorporated mathematical problem-posing in the past 100 years and to provide a window into the development of problem-posing in the Chinese mathematics curriculum.

4.2 Connotation of Mathematical Problem-Posing

More than 30 years of research on mathematics teaching and learning has prompted mathematics educators to pay an increasing amount of attention to the value of problem-posing in the mathematics curriculum as well as to integrate problem-posing into the daily mathematics teaching, which has in turn led to discussions on mathematical problem-posing ability and associated issues.

Some scholars also used 'problem sensing', 'problem formulation', 'problem finding', and 'creative problem discovering' (Yuan & Sriraman, 2011). Shulman

(1965) presented a model that recognised four components of human inquiry: problem sensitivity, problem formulation, search behaviour, and resolution. Allender (1969) said, 'For Shulman, problem formulation is reflected in the number of different information sources used by a subject, but the measure is confounded with search behavior scores' (p. 544). Dillon (1982) concluded, 'The theme of these comments is that finding (discovering, formulating, posing) a problem represents a distinct and creative act, equal to or more valuable than finding a solution' (p. 98). The above verbs' etymology reflects the importance of the process and innovation of problem-posing. The word 'BianTi' used in the early Chinese education documents (Curriculum and Teaching Materials Research Institute (CTMRI), 2001) and the words for 'proposing problems' or 'organising problems' put forward by scholars (Leung, 1993) are generally consistent with the definition of problem-posing in English.

Since researchers explore the connotation of problem-posing from different perspectives, the definitions or concepts defined by the researchers are not the same. In previous research, several scholars started from the perspective of the problem-solving process and regarded the problem-posing as a part of problem-solving. Duncker (1945) proposed that a problem-solving process included a restatement of the original problem. Since then, numerous scholars have been attracted towards studying the refinement and retelling in the process of solving complex problems (Silver, Mamona-Downs, Leung, & Kenney, 1996). The first step in most strategies of problem-solving is, simply, a given problem. That step may accordingly be viewed as the last step of problem-finding, the process which eventuates in a problem to be solved (Dillon, 1982). Moreover, Polya (1957) explained the concept of 'problem' from two aspects: One is that it refers to a way of solving the problem, while the other states that it is a new problem conceived after one problem is solved. 'Re-recognition' of problem-solving activities refers to reviewing the knowledge and methods involved in the solution process. Similarly, other researchers believed that for good problem-solvers, the process of solving problems often produces new problems (Nie et al., 2003; Xia, 2005). They proposed a 'problem chain' model with problems as the nodes, i.e. posing problems – solving problems – raising higher-level problems – solving higher-level problems – raising highest-level problems.

There are also researchers who consider problem-posing as an independent mathematical activity to illustrate concepts. For example, Stoyanova and Ellerton (1996) consider problem-posing as students providing their own understanding of specific contexts based on their own mathematical experience and constructing meaningful and well-structured mathematical problems. In this definition, the situation can be divided into three categories: free context, semi-structured context, and structured context (Si, 2014; Stoyanova & Ellerton, 1996). Leung (1993) also defined problem-posing as thinking about a mathematical problem from one's own perspective. In the process of proposing the problem, the problem-posers use their own mathematical knowledge and life experience to establish and organise relationships between characters, events, figures, and graphics to propose a mathematical problem.

As research progressed, Silver (1994) defined problem-posing in two manners: (1) creating new problems from a situation or experience and (2) forming a mathematical problem by the formulation and reformulation of the original problem in the process of problem-solving, which may occur before, during, or after the problem is solved (Silver, 1994). In addition, posing problems mathematically is included as one of the six core mathematics competencies proposed by Xu (2013), which refers to (1) producing new mathematical problems based on certain contexts or problems or producing new sub-questions in or after the process of problem-solving and (2) expressing these generated, creative, and independent mathematical problems in mathematical language (Xu, 2013).

Based on the researchers' definitions of problem-posing mentioned above, in this study, the term 'mathematical problem-posing' is defined as follows: posing new mathematical problems based on certain contexts or problems or creating new subproblems in or after the process of problem-solving and, then, expressing these newly generated, creative, and independent mathematical problems in mathematic language.

There has not been a substantial amount of research examining whether the curricula themselves incorporate problem-posing and, if so, then how they do it (Cai, Hwang, Jiang, & Silber, 2015). This chapter is intended to examine the mathematics curriculum documents from 1902 to 2018 that focused on the requirement of mathematical problem-posing. The research questions addressed in this study are as follows: (1) What are the characteristics of the conceptual development that mathematical problem-posing has gone through in China's mathematics curriculum since 1902? (2) How does the connotation of mathematical problem-posing change in the middle school mathematics curriculum?

This study will provide researchers, curriculum developers, and textbook writers with rich information about how to incorporate problem-posing into school mathematics.

4.3 Research Design

4.3.1 Research Subjects

To answer the research questions, we examined the mathematics syllabus or curriculum standards at the elementary schools, junior high schools, and high schools in China in the past 100 years. The documents from 1902 to 2000 were selected from the *Collection of Primary and Secondary School Curriculum Standards and Syllabus of the Twentieth China-Mathematics volume* (hereinafter referred to as Collection-Mathematics), which is compiled by the Curriculum and Teaching Materials Research Institute (hereinafter referred to as CTMRI) of the People's Education Press. The Collection-Mathematics contains 67 separate curriculum documents. The Curriculum Standards after the year 2000 include four standards issued

Table 4.1 Research subjects

Publishing year	Curriculum documents
1902–2000	Sixty-seven separate documents from *Collection of primary and secondary school curriculum standards and syllabus of the twentieth China-Mathematics volume, such as:* *Rules on Implementation of Secondary School Decree issued in 1912; Curriculum Standards for Secondary Schools issued in 1913; Interim Arithmetic Curriculum Standards for Junior Secondary Schools of 1929; Mathematics Syllabus for Full-Time Ordinary Senior High Schools (trial) issued in 1996*
2001	*Mathematics curriculum standards for full-time compulsory education (Experimental version)*
2003	*Mathematics curriculum standards for ordinary high schools (Experimental version)*
2012	*Mathematics curriculum standards for compulsory education (2011 version)*
2018	*Mathematics curriculum standards for ordinary high schools (2017 version)*

by the Ministry of Education of the People's Republic of China. Table 4.1 presents information about the research subjects that we analysed.

4.3.2 Research Methods

A content analysis was conducted to answer the research questions. Generally, the process of a content analysis includes the summarising and reporting of written materials, i.e. extracting the core content of written materials and information. Strictly speaking, it is a rigorous and systematic process of analysing, examining, and verifying the content of written materials (Mayring, 2000). The content analysis can not only be used to describe the relative frequency and relative importance of certain topics but also be used to assess the errors, biases, and religious tendencies in textual material (Anderson & Arsenault, 1998). When conducting a content analysis of written documents, it is necessary to simplify the problems into manageable and understandable material. In this study, we will follow the principles of content analysis and examine the content of various documents. The general structure of curriculum documents comprises four parts: (a) curriculum objectives, which include general objective and stage objective, (b) curriculum content (mathematics content), (c) suggestions for curriculum implementation, which include suggestions for teaching and textbook-writing, and (d) learning and teaching assessment.

4.3.3 Data Processing and Analysis

We (both authors) first checked every sentence that exhibited semantic integrity in the curriculum documents that we will analyse and, then, identified sentences that were associated with the definition of problem-posing under study.

For example, we checked the sentence 'In addition, children should be trained gradually to draw up application word problems that are similar to problems solved already in classrooms' (CTMRI, 2001, p. 74), which appeared in a section of the teaching implementation of the *Arithmetic Syllabus for Primary School (Revised Draft)* issued in 1956. We recognised that this sentence implied that students were required to construct problems similar to the ones they solved in classrooms. According to the definition of problem-posing, this sentence expressed a type of requirement of mathematical problem-posing. Another example was taken from the *Mathematics Syllabus for Ten-Year Full-Time High Schools (trial version)* issued in 1978, from the section of suggestions for teaching implementation, the syllabus emphasised, 'It is necessary to inspire students to continuously recognise, raise, and solve problems' (CTMRI, 2001, p. 455). We determined that this sentence indicated a requirement of problem-posing.

When there was a disagreement between both authors regarding the identification of sentences, a third expert would be invited to participate in our work and analyse sentences with us until a consensus was reached. In total, we obtained 113 sentences that indicate mathematical problem-posing.

4.4 Results on Conceptual Development

4.4.1 The Summary of the Conceptual Development of Problem-Posing

4.4.1.1 General Information on Conceptual Development of Problem-Posing over Time

By examining the mathematics curriculum documents that focused on expressions of mathematical problem-posing, a total of 113 valid sentences (expressions) were identified. The problem-posing expressions were distributed in syllabi or curriculum standards over time as follows (Fig. 4.1).

Figure 4.1 shows that before 1978, the problem-posing expression appeared only once in the mathematics curriculum documents, which was in the *Arithmetic Syllabus for Primary School (Revised Draft)* published in 1956. From 1978 to 1999, problem-posing expression has more appearances in the syllabi. For secondary education, the problem-posing was mentioned once or twice in every document. However, the expression was used without further clarifications. For example, the *Mathematics Syllabus for Ten-year Full-time High Schools (trial version)* issued in

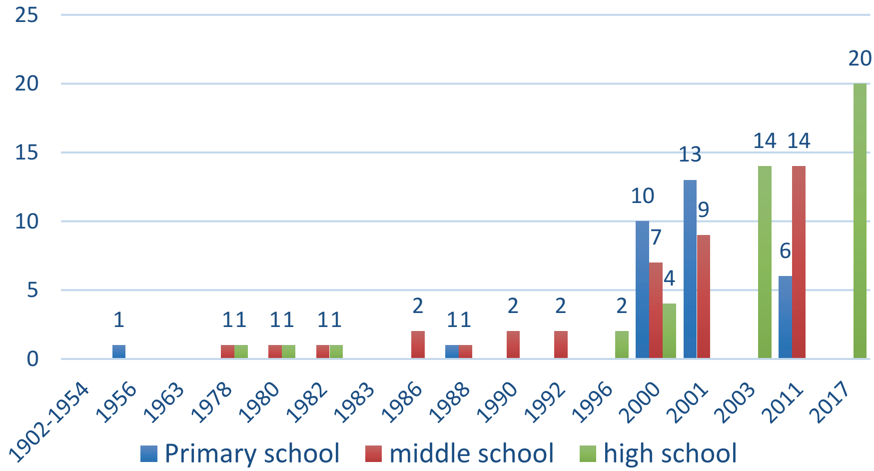

Fig. 4.1 Expressions of problem-posing distributed over time and by schools

1980 emphasised the 'need to inspire students to find out problems, propose problems and solve problems' only once (CTMRI, 2001, p. 473). The syllabi of the subsequent year were similar to each other in terms of the layout and writing of the texts, and there was no major change.

In 2000, the *Mathematics Syllabus for Nine-Year Full-Time Compulsory Education in secondary school (Revised Trial)* and *Mathematics Syllabus for Full-time Ordinary Senior High Schools (Revised Trial Edition)* were issued but not actually implemented. In these documents, the appearance of expressions regarding problem-posing increased dramatically, in that 21 sentences were identified, with special attention given to students' ability of determining problems in real life and expressing them. In the same year, however, many investigative works related to the mathematics curriculum reform were completed. The Ministry of Education of the People's Republic of China issued the *Mathematics Curriculum Standards for Full-Time Compulsory Education (experimental version)* in 2001 (MOE, 2001), and this marked a turning point in the development of mathematics curriculum.

Since 2001, four curriculum standards have been issued by the MOE. There have been more than 67 appearances of sentences related to problem-posing in these four curriculum standards. When we considered the standards for senior secondary school, we found 14 problem-posing expressions in the standards issued in 2003. In 2018, the *Mathematics Curriculum Standards for Ordinary High Schools (2017 version)* (MOE, 2018) issued by MOE contained 20 sentences that referred to problem-posing. It seemed that the standards had increased the emphasis on problem-posing.

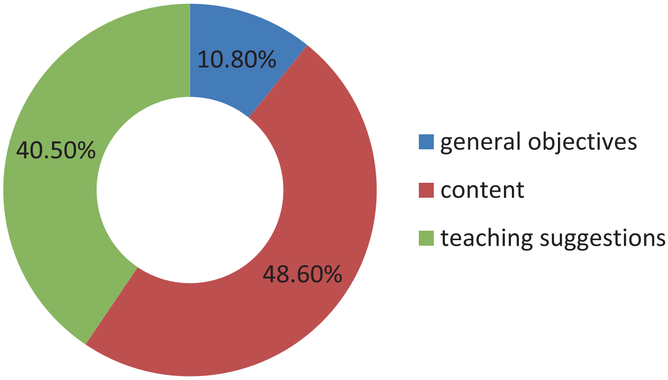

Fig. 4.2 Distribution of problem-posing in standards from 1978 to 2000

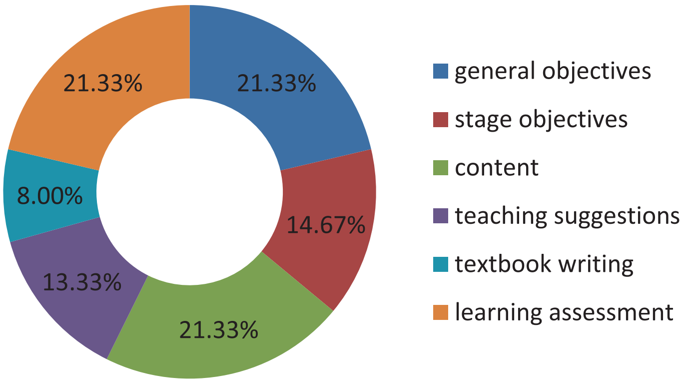

Fig. 4.3 Distribution of problem-posing in standards from 2001 to 2018

4.4.1.2 Distribution of Problem-Posing

We observed that the requirement of problem-posing was expressed in different sections of the curriculum documents. We already mentioned that the standards comprised four parts: objectives, content, suggestions, and assessment. The distribution of problem-posing is illustrated in Figs. 4.2 and 4.3.

In the 1902–1977 syllabi, problem-posing only appeared once, which was in the teaching suggestions of *Arithmetic Syllabus for Primary School (Revised Draft)* in 1956. The distribution of problem-posing in the syllabi or curriculum standards in the latter two periods of time is presented in Figs. 4.2 and 4.3. It was found that, in the syllabi from 1978 to 2000, the three sections of 'objectives, content, and teaching suggestions' involve the cultivation of students' problem-posing ability. Most of the expressions were presented in the 'curriculum content' and 'teaching suggestions', accounting for 48.6% and 40.5% of all statements of problem-posing from 1978 to 2000, respectively. For example, in the syllabus issued in 1986, it was stated in the 'algebraic content' that 'students should be cultivated to be able to transform simple real problems into mathematics problems by solving triangle word problems' (CTMRI, 2001, p. 535); it was also stated in the 'geometric content' that 'students need to know how to explore and find new problems using analogy methods by learning about similar figures' (CTMRI, 2001, p. 539). In the section of teaching suggestions, problem-posing was also heeded to. For example, in the syllabus issued in 2000, the section of teaching implementation suggested that 'teaching should encourage students to think independently, to pursue new knowledge, and to find, pose, and analyse problems' (CTMRI, 2001, p. 650).

The standards of 2001–2018 incorporate the problem-posing from the content into the objectives. As can be seen from Fig. 4.3, during this period of time, problem-posing was included in the different sections of the curriculum standard, such as 'objectives, content, teaching implementation, and assessment'. The proportion accounted for 36%, 21%, 22%, and 21% of all statements of problem-posing from 2001 to 2018, respectively. The curriculum objectives were clearly stated as 'improving the ability to mathematically pose, analyse and solve problems (including simple practical problems), the ability to express and communicate in mathematics, and develop the ability to acquire mathematical knowledge independently'. In the curriculum content setting, the task of cultivating students' problem-posing ability was mostly placed in comprehensive content, such as mathematical inquiry and mathematical modelling by the curriculum standards. In the suggestions section, the curriculum standards did not only stress the teaching methods but also offered certain suggestions for textbook writing, which required that textbooks should pay attention to the setting of situations and present the process of occurrence and development of mathematics knowledge using real examples to help

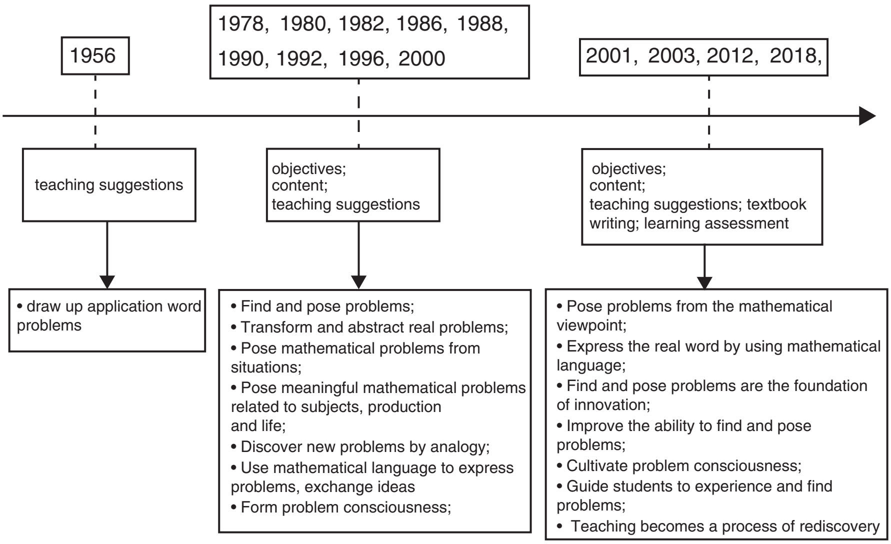

Fig. 4.4 Development of specific expression of problem-posing

students identify and raise problems. More importantly, the curriculum standards also provided assessment suggestions for students' problem-posing ability. In particular, teachers were required to actively pay attention to whether students are able to find and ask questions from life experience while dealing with comprehensive content.

The study explored the development of meaning and requirements related to problem-posing from the syllabi or curriculum standards during different periods. Some results are summarised in Fig. 4.4.

4.4.2 The Connotation Change of Problem-Posing in the Middle School Curriculum

In the next chapter, we focus on the assessment of middle school students' problem-posing ability. In order to understand the assessment results of problem-posing here, we have analysed the development of problem-posing in middle school in more depth. The development of the connotation expressions of problem-posing in the curriculum standards needs to be categorised.

- Beginning to attach importance to posing questions while solving problems

Before 1978, the curriculum documents for middle schools ignored problem-posing in mathematics education. In 1978, China's *Mathematics Syllabus for Full-Time Ten-Year High Schools (trial version)* suggested that teachers should continuously inspire students to find, pose, and solve problems. Further, teachers should also cultivate students' ability to think independently and learn by themselves, which is considered to be cultivating students' mathematical problem-solving ability. In 1980, the syllabus explained the teaching instructions regarding problem-posing and problem-solving again. It can be seen that since the 1970s, China has formed a chain on training students' learning, such as 'finding problems – posing questions – solving problems'.

- Paying attention to problem-posing while students learn specific knowledge

Growing up: In 1982, the *Full-Time Six-Year Key Middle School Mathematics Syllabus (*Draft *for Comment)* pointed out that 'Based on the basic requirements of strengthening students' basic mathematics knowledge and basic skills, it was also necessary to train students how to turn practical problems into mathematical problems and solve them' (p. 486). Since then, the 'mathematicalisation' of practical problems has become an important interpretation indicator for the cultivation of students' mathematical problem-posing ability in the syllabus. Based on this, the subsequent syllabi added some regulations regarding the cultivation of students' problem-posing ability regarding the study of certain specific mathematics content. For example, the *Mathematics Syllabus for Full-Time Secondary Schools* in 1986 and its revised version in 1990 both required teachers to further develop students' ability to convert simple practical problems into mathematical problems to help students understand how to use the analogy method to explore, discover new problems, and gradually develop the habit of researching problems. At the same time, the formulation of the mathematics syllabus for junior high schools was also in full swing.

- Regarding problem-posing as bridge between theory and practice

From 1988 to 2000, the three syllabi followed the idea of 'persisting with linking the theory with practice', and emphasized that the teaching should stem from students' life, relate to social situation and other disciplines to carry out scientific abstraction and logical reasoning, so that students could be trained to abstract practical problems into mathematical problems. After that, students' ability to analyse and solve problems should be cultivated before their mathematical awareness would be shaped.

In 2000, the syllabus linked the practical mathematics problems with innovative consciousness and practical ability for the first time, pointing out that 'In teaching, students' curiosity to learn mathematics should be inspired. Through independent thinking, they could constantly pursue new knowledge, discover, propose, analyse and solve problems creatively and make mathematics learning a process of rediscovery and re-creation.' (CTMRI, 2001, p. 650)

In general, during this period, the syllabus paid increasing attention to problem-posing. Cultivating the mathematical problem-posing ability of students was established in the teaching objectives of the syllabus issued in 2000. The requirement for problem-posing became niche, targeting specific mathematics content.

- Cultivating problem-posing ability became explicit in curriculum objectives

After 11 years, the *Mathematics Curriculum Standards for Compulsory Education (2011 version)* (MOE, 2012) re-emphasised that the basis of innovation lies in students' finding and asking about problems on their own. Students in the third stage of schooling (Grades 7–9) were required to 'preliminarily learn to find problems and ask questions from the perspective of mathematics in specific situations' (MOE, 2012, p. 9) in terms of problem-solving. As per the curriculum content, it is also a requirement to attempt to find and pose problems while designing and implementing solutions to specific problems based on actual situations. Further, the curriculum standards also provide suggestions for teaching, assessment, and the textbook writing process. For example, when assessing students' ability of posing and analysing problems, one should adopt a flexible method for recording, retaining, and analysing the students' performance in different aspects. It is a pity that despite the several objectives and suggestions, the curriculum standards do not provide good examples of how to guide students to find and pose mathematical problems. In addition, the guidance for practical operations needs to be improved.

4.5 Summary and Discussion

The connotation development of mathematical problem-posing in Chinese school mathematics curriculum documents can be divided into three stages: shedding of light on the teaching component, objective-oriented initiatory guidance, and going from intended to implemented requirements.

1902–1977: A shedding of light on the teaching component
Among all the mathematics syllabi or standards issued during this period, only the 1956 *Elementary School Arithmetic Syllabus (Revised Draft)* stated in the teaching instructions that students should learn to raise problems imitatively. Although the instruction limited children's problems to the simple level of imitative compiling, it had begun to heed to the multi-transformation of the subject from teachers to students. Unfortunately, in the following 30 years, this small but enlightening instruction was not valued and developed.

1978–2000: Objective-oriented initiatory guidance
Influenced by the education philosophy of pragmatism, the mathematics syllabi in this period were objective-oriented and implicitly guided the development of problem-posing ability. Before 1990, the related content about problem-posing started from 'a few points/problems that should be paid attention to in teaching', and advocated teachers to inspire students to 'find and ask problems' and 'transform practical problems into mathematical problems' (CTMRI, 2001, p. 555). Although it stood out of the narrow connotation of the compiling question, the expression was too general and, thus, lacked practical guidance. However, it is worth mentioning that mathematical problem-posing at this time was linked to the cultivation of students' independent thinking and self-learning abilities, and the expressions reflected the significance of problem-posing. The syllabi issued after 1990 clarified the teaching purposes, such as guiding students to 'abstract practical problems into mathematical problems' and 'using mathematical language to express problems and to communicate' (CTMRI, 2001, p. 649). Since then, problem-posing have risen to a new heights. At the same time, the syllabus also stated requirements for the source and quality of the problems, providing teachers with guidance regarding the implementation of the teaching, which was in combination with the specific content and knowledge that is closely related to daily life.

2001–Present: Going from intended to implemented requirements
Since 2001, the mathematical problem-posing in the standards has penetrated into the curriculum objectives, contents, suggestions, assessment, etc. This indicates its ability to serve as the benchmark, as it required to improve students' ability of finding and posing problems in multiple aspects, such as mathematics and life. In addition, it is explicitly highlighted in the M*athematics Curriculum Standards* that creating problems is the basis of innovation, and it is necessary to develop students' problem awareness beyond the operational level. The standards also include the requirement that teachers should guide students to experience problems-finding and lead them to a process of re-finding by creating appropriate situations.

In the next chapter, we will investigate whether the inclusion of problem-posing in curriculum documents reflects a systematic approach to the development of problem-posing abilities in students.

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Chapter 5
Chinese Eighth Graders' Competencies in Mathematical Problem-Posing

Binyan Xu and Muhui Li

Abstract This chapter focuses on the research question 'How do eighth-grade Chinese students perform in an assessment of mathematical problem-posing ability?' To answer this question, a set of indicators and frameworks for evaluating students' mathematical problem-posing abilities as well as instruments for testing mathematical problem-posing ability were developed. In this study, 1210 eighth graders were selected as research subjects using two-staged cluster sampling method. The results indicated that the overall performance of students' mathematical problem-posing abilities was not significantly good. Most students were able to imitate given problem structures and generate problems. Few students had the capacity of dealing with problem-posing tasks in free situations and integrating other knowledge or experience into creating mathematical problems. The study also examined the differences in problem-posing abilities in terms of region and gender and stated certain implications.

Keywords Mathematical problem-posing · Ability · Framework for evaluating abilities · Instrument for testing · Staged cluster sampling · Overall performance · Imitate given structure · Integrate knowledge into creating problems · Region difference · Gender difference

5.1 Introduction

The posing of problems plays an important role in the development of science and technology as well as in social development. The formulation of a problem is more essential than its solution. To pose new problems, to raise new possibilities, and to perceive old questions from a new angle all require a creative imagination and indicate a real advance in science (Einstein & Infeld, 1938). Due to the advancement of mathematics education and the need for talent cultivation in innovative

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societies, researchers began to explore certain issues associated with mathematical problem-posing ability and its teaching in school mathematics in the early 1980s (Xia, 2005). It was acknowledged that problem-posing not only develops students' mathematical reasoning abilities in general but also enhances their mathematical creativity and interest in mathematics learning. Further, it improves students' mathematical communication abilities and cultivates their cooperative learning abilities (Barlow & Cates, 2006; English, 1997; Lavya & Shriki, 2010; Ponte & Henriques, 2013; Van Harpen & Sriraman, 2013). For decades, numerous countries have stated explicit requirements for problem-posing in their mathematics curriculum standards. Problem-posing is a significant component of the mathematics curriculum. For example, *Principles and Standards for School Mathematics* published by National Council of Teachers of Mathematics (NCTM) of the United States have recommended that students should be able to formulate interesting problems based on a wide variety of situations both within and outside of mathematics (National Council of Teachers of Mathematics [NCTM], 2000). China has also advocated the ability of students to discover and pose problems in the *Mathematics Curriculum Standards for* Compulsory *Education (2011 version)*. The curriculum objectives of the standards require that students learn to identify problems, pose problems mathematically and apply mathematics knowledge to solve practical problems (MOE, 2012). The changes in curriculum have the potential to alter classroom instruction and student learning (Cai & Howson, 2013). We hoped that students would be able to meet the requirements of the intended curriculum while observing classroom teaching. In order to examine the effectiveness of the implementation of the curriculum, this chapter presents an empirical study that is aimed to assess eighth-grade Chinese students' performance in mathematical problem-posing.

5.2 Literature Review

The assessment of problem-posing ability has been a perennial challenge for the mathematics education researchers. Different studies provide various perspectives and thoughtful results.

In early studies, researchers analysed the characteristics of the problems generated by students and examined students' problem-posing performances. Silver (1994) pointed out that the difficulty and novelty of the mathematical problems posed by students were both considered when determining the level of students' problem-posing performance. Silver and Cai (1996) investigated the mathematical problems generated by 509 middle school students and examined the parameters of solvability, linguistic and mathematical complexity, and relationships within the sets of the posed problems. The results illustrated that the students constructed a large number of solvable mathematical problems, many of which were syntactically and semantically complex. In the study conducted by English (1997), a framework was developed for studying fifth-grade children's abilities with regard to problem-posing and development of diverse mathematical thinking. The mathematical problem-posing ability in this study was related to the recognition and utilisation of problem structures and the perceptions as well as preferences associated with different problem types. Studies from this period have a special focus on the relationship between problem-posing and problem-solving abilities. The second aim of the study carried out by English (1997) was to investigate the extent to which children's sense of numbers and novel problem-solving skills govern their problem-posing abilities in routine and unique situations. Cai (1998) explored the mathematical problem-posing and problem-solving abilities of American and Chinese sixth-grade students. The study analysed the students' responses to mathematical problem-posing and problem-solving tasks and found that Chinese students outperformed American students in computational tasks, but there were differences between the two when performing relatively novel tasks.

In the twenty-first century, researchers have gained new insights into the evaluation of problem-posing, and these evaluations have drawn on the multidimensional analysis around indicators such as quantity of the problems being raised and the creativity and complexity of problems. While certain analyses include multiple levels within each dimension (e.g. quantity, speed, and quality) to gauge students' problem-posing abilities (Zheng, Wang, & Lv, 2007), some contain nested dimensions, i.e. each main indicator (e.g. quantity, category, and novelty) contains subdimensions. In such cases, the ability scores were derived by synthesising weighted assignments (Silver & Cai, 2005; Xia, Wang, & Lv, 2008).

Some studies applied mathematical problem-posing as an intermediary tool in measuring the effectiveness of mathematics curriculum implementation instead of evaluating mathematical problem-posing directly. Cai et al. (2013) used problem-posing as a measure of the effect of the middle school curriculum on students' learning. In the study, a qualitative rubric was developed to assess the different characteristics of students' responses to the tasks posed. The results bolster the feasibility and validity of problem-posing as a measure of the curriculum's effect on student learning. Problem-posing is also used as a formative assessment tool. Kwek (2015) examined students' thinking processes, understandings, and competencies using problem-posing tasks. The study analysed the problems posed by high-ability secondary school students. Then, the students' performances were analysed and evaluated with respect to the complexity of the problems. Recently, researchers have studied creativity in the context of problem-posing (Pelczer, Singer, & Voica, 2013; Voica & Singer, 2014). Problem-posing was considered a parameter of mathematical creativity. Singer, Voica, and Pelczer (2016) and colleagues provided students who were prospective teachers with geometry-problem-posing tasks. Students were asked to propose different mathematical problems that were related to either geometry or conceptual dispersion. After analysing the students' responses (i.e. the posed problems), the results could be used to gauge cognitive flexibility, which is a basic indicator of creativity.

In summary, the research on the evaluation of problem-posing ability has grown increasingly comprehensive, and the characteristics of problem-posing have become abundant. The assessment of problem-posing abilities is valuable with regard to the understanding and further promotion of students' mathematical competencies. In this chapter, students' problem-posing abilities were assessed in order to understand how their performance is related to the curriculum standards. The research question is "How do eighth-grade Chinese students perform in an assessment of mathematical problem-posing ability?" To answer this question, a set of indicators and frameworks for evaluating students' mathematical problem-posing abilities as well as instruments for testing mathematical problem-posing ability were developed. The different performances of students and the differences in their mathematical problem-posing abilities were compared.

5.3 Methodology

5.3.1 Research Participants

The research participants comprise eighth-grade students recruited from Mainland China. In order to obtain results that could portray students' mathematical abilities in this region, two-staged cluster sampling method was adopted for selecting participants. First, the study identified eight representative provinces according to the geographical location of regions (including East China, Central China, North China, South China, Northwest China, and Southwest China). For each province, the provincial capital city was selected as the research sample. Normally, there is a teaching research instructor for mathematics in each city. These individuals are responsible for the mathematics teacher training and scientific events related to mathematics teaching in schools. Second, we invited them to support our selection of participants. In the eight cities, each instructor selected three schools based on senior high school entrance examination performance. All schools exhibited above-average performance. Third, in each selected school, one or two classes were selected, and the students from the selected classes were asked whether they wanted to be research participants. Most of them agreed to participate in the study. The number of the participants is shown in Table 5.1. With reference to the Chinese Statistics Yearbook 2017, cities were divided into three categories based on their corresponding economic development levels: developed, moderately developed, and less developed.

Table 5.1 Research participants

Developed region			Moderately developed region		Less developed region			Total number of research participants
BJ(4)	SH(3)	GZ(3)	CD(3)	DL(3)	XA(3)	ZZ(6)	XJ(3)	
174	121	80	153	114	145	247	176	1210

Note: Each city is indicated by two letters, and the numbers in the parentheses indicate the number of schools participating in the investigation from these cities

5.3.2 Framework and Instruments

In order to investigate the aforementioned research question, a mathematical problem-posing test was administered for the research participants. For this process, six tasks were developed based on the framework mentioned in Chap. 3 and other literature. In the framework, mathematical problem-posing ability was defined as the ability to propose new mathematical problems based on certain contexts or problems or to formulate new sub-problems during or after the process of problem-solving and use mathematical language to present these proposed, creative and independent new mathematical problems. The framework described problem-posing abilities at three different levels, which are presented in Table 5.2.

Mathematical problem-posing abilities will be measured by means of a mathematical problem-posing test. While developing the tasks for this test, the present study considered the problem-posing task classification proposed by Stoyanova and Ellerton (1996). They classified a problem-posing situation as either free, semi-structured, or structured. The present study applied this classification model to the development of assessment tasks related to the three levels of problem-posing abilities. Table 5.3 presents the developmental criteria of tasks.

According to this classification, a problem-posing situation is referred to as structured when the problem-posing activities are based on a specific problem. Students were asked to generate mathematics problems that align with a given problem situation or that have a similar structure as the given structure. In the present study, two tasks with structured situation were developed.

Task 1 (Structured problem-posing situation) Please propose a mathematics word problem according to the given binary linear equations $\{\begin{matrix} y-x=1 \\ 2x+y=16 \end{matrix}$ using graphic representation or word repres**entation.**

Table 5.2 Framework of mathematical problem-posing abilities at three levels

Competency	Levels		
	Level I: Reproduction	Level II: Connection	Level III: Reflection
Mathematical problem-posing	To be able to recognise the structure of a given mathematical problem To be able to imitate or modify a given problem and pose a similar mathematical problem To be able to make appropriate supplements to a mathematical problem based on the missing elements, thus turning it to a complete mathematical problem	To be able to find or pose more important mathematical problems (for accomplishing tasks) in real-world situations or tasks To be able to expand and select information according to one's own mathematical knowledge and experience, establish mathematical links, and put forward different mathematical problems	To be able to classify various mathematical problems raised by oneself and explain their bases and processes To be able to evaluate mathematical problems raised by peers To be able to put forward more complex and extended mathematical problems

Table 5.3 The development criteria of tasks for assessing mathematical problem-posing abilities

Content	Level		
	Level I	Level II	Level III
Mathematical problem-posing	In a well-structured context, the proponent is able to identify the mathematical structure of the situation and propose a complete mathematical problem that has a consistent or similar structure	In a semi-structured situation or a real-life situation, it is possible to screen and organise information from the context and pose mathematical problems, conjectures or propositions related to the original situation	In the free setting, it is possible to supplement the necessary conditions or conclusions with the actual needs of the situation, complete the mathematical structure of the situation, and pose different mathematical problems that are complex and difficult
Characteristics of task situations	Well-structured situations: mathematical symbolic operations, mathematical formulas, graphics and text, or well-structured mathematical problems	Semi-structured or realistic situations: The amount of information is sufficient, but the mathematical structure is not clear, and it is necessary for the proponent to screen and organise it by himself/herself	Free situations: The amount of information is not sufficient, and the proponent needs to grasp from the whole, expand, and supplement the reasonable and non-overlapping information

This task situation provides a structured mathematical symbolic representation that is familiar for students. Based on the binary linear equation, students were required to generate another form of mathematics problem, called a word problem. This was used to predict whether the students found it easy to begin the task based on their learning experiences. Level I abilities are required to formulate new problems.

Task 2 (Structured problem-posing situation) Imitate the following problem to propose another practical mathematical problem (no need to answer.)

The figure below shows a pair of symmetrical stepladders. The distance between the upper and lower step is 40 cm. The length of the upper step is 5 cm shorter than that of the lower step. The length of the lowest step is 50 cm. If the ladder is 480 cm long, what is the length of the step on the top?

This task provides a well-structured problem in which the situation is based on students' real life. The problem contains some numerical relationships that need to be identified. Students were asked to imitate the problem structure and propose a mathematical problem with a similar structure. This was carried out to predict whether Level I abilities are required for posing new problems.

The present study considered a semi-structured situation. A problem-posing situation is referred to as semi-structured when students are given an open situation and are invited to explore the structure of that situation and complete it by applying the knowledge, skills, concepts, and relationships they have learned from their previous mathematical experiences. In the present study, two tasks with semi-structured situations were developed.

Task 3 (Semi-structured problem-posing situation) A story about selling crabs.

Wang is selling crabs in a market. The sale price is 100 RMB per 500 g. Two customers are talking to Wang. One customer says, 'The crabs are of good quality. But I just want to buy crab bodies.' The other customer says, 'I just want to buy crab pincers and legs.' They say to Wang, 'We will buy all the crabs. One needs the crabs' bodies, while the other needs pincers and legs. Now, your sale price is 100 RMB per 500 g. We suggest that the sale price of crab bodies is 70 RMB per 500 g, and the sale price of crab pincers and legs is 30 RMB per 500 g. Further, 70 plus 30 is 100. We haven't bargained with you. Could you please divide the crabs into bodies and pincers and legs and, then, weigh them separately. What do you think?' Wang considered it and thought that it seemed to be right. He agreed with the suggestion and weighed the different parts of the crabs separately. The crab pincers and legs weighed 500 g, while the crab bodies weighed 1500 g. Then, one customer paid 30 RMB, and the other paid 210 RMB. Wang received 240 EMB in total.

Please read the story and pose problems mathematically. Write down your mathematical problems.

This task was designed for assessing mathematical problem-posing abilities at Level II. The situation came from real life and provided some information about the mathematical variables and relations instead of presenting a structured problem. The students were required to screen and organise the information related to mathematical aspects based on the situation.

Task 4 (Semi-structured problem-posing situation) The following number array is called a 'Pythagorean triple'.

$$3^2 + 4^2 = 5^2$$
$$6^2 + 8^2 = 10^2$$
$$9^2 + 12^2 = 15^2$$
$$\cdots$$

Please generate a mathematical proposition around this Pythagorean triple.

This task was also designed for assessing mathematical problem-posing abilities at Level II. Certain mathematical laws govern the mathematical situations, but the situation did not provide structured proposition conditions and conclusions. The students were asked to search for the mathematical laws and generate the mathematical propositions that reflect these laws.

The present study also considered a free situation. A problem-posing situation is referred to as free when students are asked to generate a problem from either a given, contrived, or naturalistic situation. There were two such tasks designed in the present study.

Task 5 (Free problem-posing situation) As shown in Figure a, fold the vertex B of a square to the midpoint E of the edge AD to obtain one crease FG. As shown in Figure b, fold the vertex C of a square to the midpoint E of the edge AD to obtain one crease LM. Please put forward at least two complicated mathematical problems according to the given conditions.

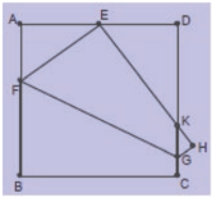

Figure a

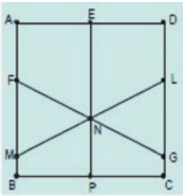

Figure b

This task was designed for assessing mathematical problem-posing abilities at Level II. The situation contains certain mathematical conditions and information but no complete structure. In order to formulate mathematical problems, students were required to integrate mathematical information within the situation.

Task 6 (Free problem-posing situation) Given that the two triangles are both equilateral triangles (figure c), please formulate certain conditions and pose at least two complicated mathematical problems

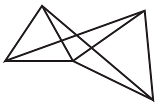
Figure c

This task was designed for assessing mathematical problem-posing abilities at Level II. In this situation, the mathematical information is incomplete, and students needed to expand or supplement some conditions. This situation is open with regard to problem-posing, and students have opportunities to exhibit their thinking.

5.3.3 Coding Approach

All responses were collected. In order to analyse participants' responses, the responses for the six tasks were coded as per the corresponding coding criteria. The coding criteria were constructed based on Tables 5.2 and 5.3. First, the problems posed by participants were judged based on whether they met the coding criteria. When a response met the coding criteria, it was coded as 1, and if it did not meet the criteria, it was coded as 0. Thus, the first code referred to the performance of problem-posing. Second, the problems posed by participants were analysed to identify what forms or features they contained. While completing the analysis of the performance for every response, a second code was used to represent the features of the response. Let's take Task 2 as an example. Table 5.4 presents the coding criteria for Task 2.

The following are typical examples of responses:

Response coded as 11: *As shown in the figure, it is a symmetrical stepladder. It is known that the distance between the upper and lower step is 40 cm. The length of the upper step is 3 cm shorter than that of the lower step, and the length of the top step is 30 cm long. If the total length of the ladder is 420 cm, what is the length of the lowest step?*

This response, posed by one student, retained the same situation and imitated the original problem structure. However, it changed the condition and asked a different question. Therefore, the response is coded as 11.

Response coded as 12: *As shown in the figure, it is a small symmetrical water channel. It is known that the distance between the upper and lower scales is 40 cm.*

Table 5.4 Coding criteria and explanation for Task 2

Explanation of Task 2 (coding criteria) Task 2 requires students to master the structure of the original problems and the relation between their variables. The posed mathematical problems contain complete conditions and conclusions; or the posed problems exhibit certain similarity and relevance to the original problems in terms of mathematical structure	
First code	**Second code and explanation**
1: meet the coding criteria	11: A response retains the background situation of the original problem and imitates the original problem structure but changes the condition or question 12: A response has a similar background situation as the original problem and imitates the original problem structure
0: didn't meet the coding criteria	01: No problems are posed 02: No changes in the original problem or only quantity was changed within original problem 03: Situation or structure of a response has nothing to do with the original problem

The width at the lower scale is 5 cm narrower than that at the upper scale, and the width at the uppermost scale is 50 cm. If one hypotenuse of the water channel is 480 cm long, what is the bottom width of the water channel?

This response changed the problem situation but had a similar problem structure, so it was coded as 12.

Response coded with 02: *The figure on the right shows a pair of symmetrical stepladders. The distance between the upper and lower step is 30 cm. The length of the upper step is 4 cm shorter than that of the lower step. The length of the lowest step is 40 cm. If the ladder is 450 cm long, what is the length of the step on the top?*

This response, posed by one of the students, duplicated the original problem and only changed certain quantities within the original. Hence, it was coded as 02.

Response coded with 03: *It is known that the mid-section of a right-angled isosceles triangle is 2 cm smaller than the bottom edge, and it is known that the length of the hypotenuse is 6 cm. What is the length of the side?*

The response mentioned some mathematics elements but did not imitate any situations or structures of the original problem. Thus, it was coded as 03.

5.4 Results

By analysing the problems generated by the eighth-grade students, the present study obtained results while focusing on the following aspects: (1) the overall performance of students' mathematical problem-posing ability, including the ability level classification and the specific performance in tasks of different ability levels; (2) differences in the performance of students' mathematical problem-posing ability based on regional and gender differences.

5.4.1 The Overall Performance of Students' Mathematical Problem-Posing Ability

5.4.1.1 Students' Performance of Problem Posing Abilities for Six Tasks

In total, 1210 participants attempted the mathematical problem-posing test, which included six tasks. After coding the six tasks based on the corresponding criteria, all the data were managed and analysed using Microsoft Excel. As mentioned above, when responses met coding criteria, they were coded as 1 and were referred to as 'accurate responses' in the present study. Whereas, if responses did not meet the coding criteria, they were coded as 0 and were called 'inaccurate responses'. First, the accuracy of the responses was calculated. These results are presented in Fig. 5.1.

This figure indicates that 8.1% of all participants generated accurate problems for all six tasks, 8.8% of all participants posed accurate problems for five tasks, and 18.10% for four tasks. If we consider that students could generate accurate

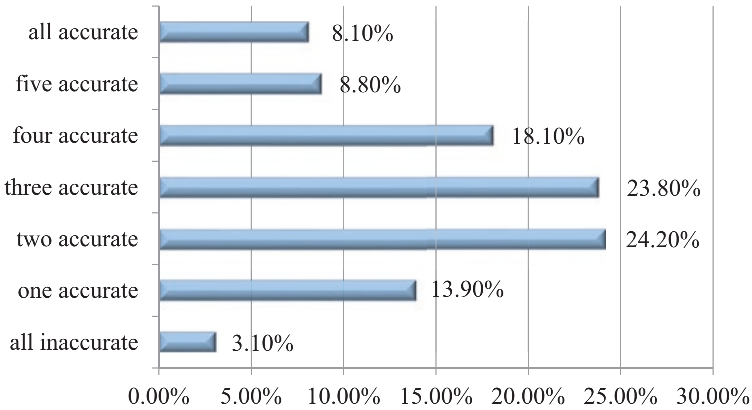

Fig. 5.1 Percentage of students with accurate responses to tasks

problems for more than three tasks, it can be said that these students displayed good performance in terms of problem-posing abilities. It was found that 35.8% of students performed well. If we consider that students could generate accurate problems for less than three tasks, it can be said that these students exhibited poor problem-posing abilities. Thus, 40.3% of students were found to have performed poorly. Finally, 23.8% of students displayed moderate performance of problem-posing abilities.

5.4.1.2 Students' Performance in Tasks at Different Ability Levels

The results also revealed the specific performance of students for different task situations and different levels of abilities. Task 1 (T1) and Task 2 (T2) gauged Level I abilities, Task 3 (T3) and Task 4 (T4) gauged Level II abilities, and Task 5 (T5) and Task 6 (T6) gauged Level III abilities. Figure 5.2 illustrates the percentage of students who provided accurate responses for different tasks.

As can be seen from the above figure, the students' performance varied depending on the task level. The percentage of students who posed accurate problems decreased as the task became more complicated. The percentage of students who provided accurate responses for the first and second tasks are 73.2% and 60.4%, respectively, while the percentage of students who provided accurate responses for the third and fourth tasks declined by about 20% each compared with that of the second task. There is little difference between the accuracy of the fifth take and the fourth task. Further, the percentage of students who generated accurate responses for the sixth task dropped to about 21.4%. The above data revealed that the students' problem-posing abilities are essentially consistent with the setting of the task situation and structure. For structured problem-posing situations, students performed much better than in free situations. Students displayed a strong ability to imitate problem

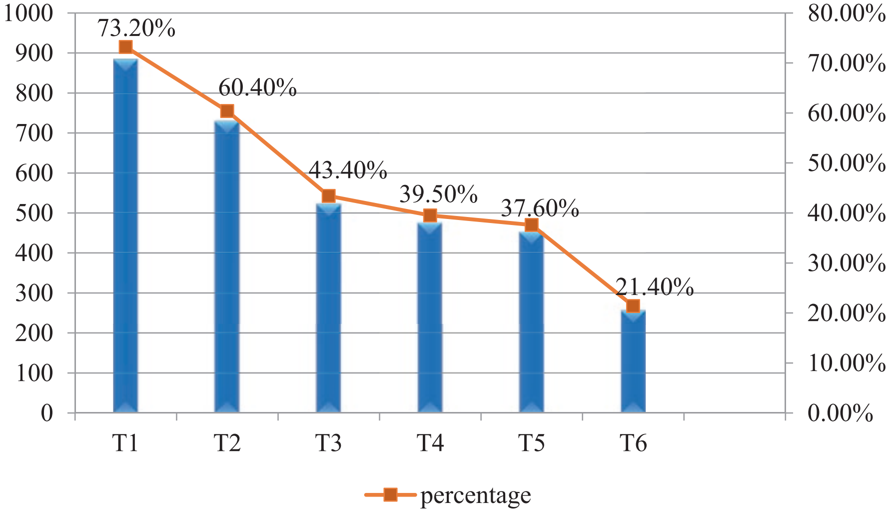

Fig. 5.2 Percentage of students who posed accurate problems for tasks

structures while posing mathematical problems. However, as task situations became more open and free, the students' problem-posing abilities became weaker.

5.4.1.3 Characteristics of the Problems Posed by Students

The present study analysed the different forms and features of the problems posed by students in cases where they posed accurate problems. As previously mentioned, the characteristics of problem were also represented by a code (second code). The results for the characteristics are illustrated in Table 5.5.

Table 5.5 illustrates that most students followed the original task structure and made slight changes in the situation or mathematical condition when generating accurate problems. For Tasks 1, 2, and 3, 94%, 76%, and 95% of the accurate responses, respectively, had a similar problem structure to that of the original tasks. Thus, it can be assumed that students primarily imitated the given problem structure when they were asked to pose mathematical problems. They lacked the ability to integrate comprehensive knowledge and thinking into the process of posing problems. However, when analysing the tasks for Level III (T5 and T6), it was found that nearly half of the accurate responses were novel and exploratory, although only few of the students could formulate accurate problems while dealing with a free problem-posing situation.

Table 5.5 Characteristics of the *accurate problems* posed by students for the six tasks

Level	Task	Percentage	Second code Characteristic of problem	Example of responses
I	T1	94%	1: Word problems based on daily life situations	*Xiao Ming went to buy fruit. If a catty of peaches costs 1 yuan more than a catty of apples, and Xiao Ming bought two catties of apples and a catty of peaches for a total of 16 yuan, then how much is a catty of peaches and apples?*
		6%	2: Problems posed while using expressions (equations) as a whole	*A given rectangle has side lengths xy and x/y, and they fit in the equations of Task 1. Calculate the area of the rectangle*
	T2	76%	1: The response retains the background situation of the original problem and imitates the original problem structure but alters the condition or question	*Mentioned in* Sect. 5.3.3
		24%	2: The response has a similar background situation as the original problem and imitates the original problem structure	
II	T3	95%	1: Structured mathematical problem based on the story's context	*The total weight of crabs should be 1500 + 500 = 2000 g. If they are sold whole at 100 yuan per 500 g, the total should be 400 yuan. Why did the two customers only pay 240 yuan for buying them separately?*
		5%	2: Enlightening or reflective mathematical problems related to the story context	*Is the money paid by the two people right? If so, please explain the reason; if not, please state how much each person should have paid? Also, explain why the payment made was incorrect*
	T4	57%	1: Mathematics propositions posed by students that summarised or improved the given propositions	*If* $a^2 + b^2 = c^2$, *then* $(xa)^2 + (xb)^2 = (xc)^2$
		43%	2: Mathematics propositions posed by students that combined given propositions and other knowledge	*For a Pythagorean triple, if each of its elements is expanded or reduced by the same multiple, the three numbers obtained would still form a Pythagorean triple*
III	T5	56%	1: Well-structured problems related to proof or judgment with a certain degree of difficulty and based on the conditions	*Prove that* $\triangle FMN \cong \triangle NLG$; *connect FL and MG, and prove that the quadrilateral MGLF is a rectangle*

(continued)

Table 5.5 (continued)

Level	Task	Percentage	Second code Characteristic of problem	Example of responses
		44%	2: Mathematical problems with a certain difficulty in terms of solvability (such as finding angles, line lengths, etc.)	*If AD = m, find the value of MB; find the value of NP*
	T6	52%	1: Compilation of some conditions and their use to pose relevant well-structured mathematical problems	*Given that ∠DAC = ∠BEC, AC = 4, CE = 6, find the ratio of BE to AD*
		48%	2: Posing exploratory mathematical problems based on added conditions	*If the area of ⊿DCE is 2 and the area of ⊿ABC is 1, can the areas of ⊿BCF and ⊿CDG be obtained? If not, explain the reason, and, if so, calculate it*

5.4.2 Differences in the Performance of Students' Mathematical Problem-Posing Ability Based on Region and Gender

The impacts of regional and gender differences on students' mathematical performance has been a topic of interest in the field of mathematics for a long time. International comparative studies have found that students' mathematical problem-proposing abilities vary with cultural backgrounds or mathematics curriculum (Cai et al., 2013). In this study, the effects of regional and gender differences on students' mathematical problem-proposing ability were also examined.

5.4.2.1 Regional Differences

By analysing the accurate problems posed by students from the three regions, which are developed, moderately developed, and less developed, the overall performance for the different regions was found. These results are shown in Fig. 5.3.

The data has already illustrated that only less than 10% of the students were able to pose accurate problems for all six tasks for the three levels. The overall performance of the problem-posing abilities was relatively weak. In order to identify the effects of regional differences on problem-posing ability, we analysed the performance on three different levels, respectively. Figure 5.3 shows that the students from the developed regions performed better than those from the other two regions when dealing with problem-posing tasks. For tasks for Level I, Level II, and Level III, 55.2%, 25.1%, and 11.5% of the students from developed regions, respectively, generated accurate mathematical problems. For tasks for Level I, the students from less developed regions performed slightly better than those from moderately developed regions. However, for tasks for Level II and Level III, 22.1% and 9.0% of the

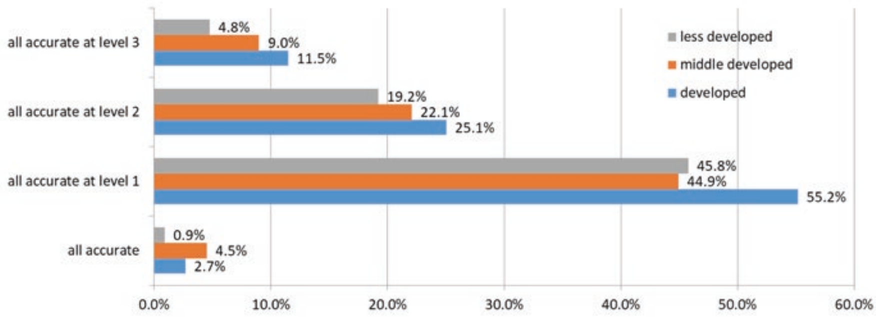

Fig. 5.3 Percentage of students in different regions who posed accurate problems for the six tasks

students from moderately developed regions, respectively, posed accurate problems, while the lowest percentage of students who could generate accurate problems were from less developed regions. The economical situation seemed to have an impact on the development of students' problem-posing abilities.

5.4.2.2 Gender Differences

In this study, 51% of the total 1210 participants were boys, 47% were girls, and 2% did not indicate their gender. In the same manner as previously described, the accurate problems generated by boys and girls were examined. Figure 5.4 illustrates the percentage of boys and girls who posed accurate problems for the tasks for three levels.

The above figure reveals that the girls performed better than the boys. For the tasks for Levels I, II, and III, 51.8%, 22.8%, and 9.0% of the girls, respectively, generated accurate problems. However, the effects of gender difference on problem-posing abilities reduced as the posing task situations become increasingly open and free. When faced with a free context of posing tasks, boys and girls were challenged to a similar degree and displayed weak capabilities of posing mathematical problems.

5.5 Summary and Discussion

5.5.1 A General Description of Students' Mathematical Problem-Posing Ability

The present study developed three kinds of test tasks with structured problem-posing situations, semi-structured problem-posing situations, and free problem-posing situations. As previously described, the tasks with structured situations aimed at examining mathematical problem-posing abilities at Level I, those with

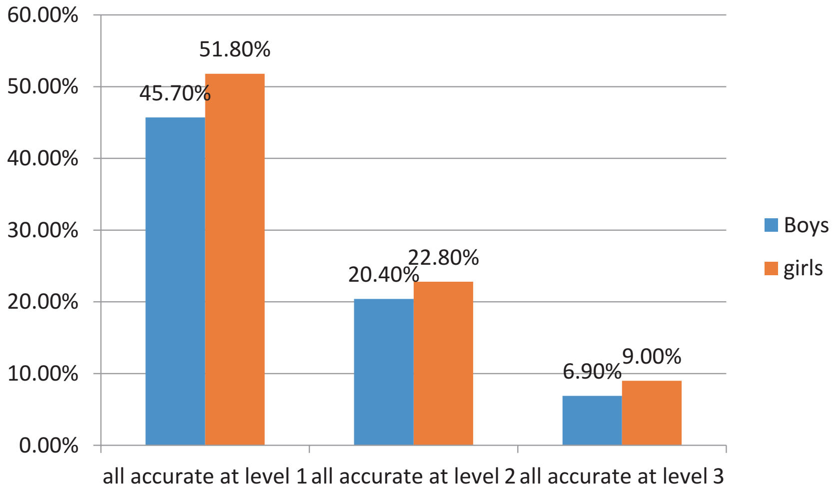

Fig. 5.4 Percentage of boys and girls who posed accurate problems for the tasks for three levels

semi-structured situations were associated with problem-posing abilities of Level II, and those with free situations were aimed at investigating abilities for Level III. The results of the study revealed that the students' problem-posing abilities for tasks of Level I were much better than those for tasks for Level II and Level III. For tasks of Level III, only a few of the students were able to formulate accurate mathematical problems.

For tasks for Level I, students were asked to generate problems that had the same or similar structures as those provided in the task situations. It was not difficult for students to imitate the mathematical situation structure and pose problems, so their problem-posing performance was much better at this level. For tasks for Level II, students were asked to integrate their knowledge or experiences with posing problems mathematically. They were required to recognise and sum up the mathematical structure based on ill-structured situations and generate mathematical problems or propositions. The results indicated that only less than half of the students were able to determine and formulate accurate mathematical problems. They lacked the ability to generate reasonable problems when dealing with ill-structured situations. The tasks for Level III provided the students with open situations, and some situations contained redundant mathematical information. Students were required to select information and pose difficult problems accordingly. However, certain situations included insufficient information, and students needed to add some mathematical conditions or conclusions before posing the problems. The results indicated that most students were unable to complete these problem-posing tasks.

Although students' performance of problem-posing was generally consistent with the task levels, some findings indicate that certain students performed 'abnormally'. These students did not generate accurate problems with structured or ill-structured situations but were able to create interesting mathematical problems when given free situations. It was observed that such students could engage with creating and combining conditions and posing mathematical problems. The results indicated that these students could be good problem-posers, although they could not imitate situation structures. The cultivation of students' problem-posing abilities may not always begin with an easy situation.

5.5.2 Analysis of the Characteristics of Students' Mathematical Problem-Posing Ability

The present study also analysed the characteristics of the problems posed by the students and marked them using a second code. An in-depth explanation of these second codes has been previously provided, and the characteristics of the problems have been summarised. Primarily, the second code indicates the context the students used while posing problems. The situations of the mathematical problems posed by students were similar to their current life experiences. These situations fall under four categories: First is family life experiences, such as daily life with parents and siblings. There are quite a lot of mathematical problems that use daily life as material and background. Second is school life experience. As students are familiar with the academic life and interpersonal processes at school, there are a considerable number of mathematical problems that use school life as their background. The third category is social life experiences. These mainly consist of certain knowledge and experiences about social production and operation, which is also important background information used by students to pose mathematical problems. Finally, the fourth category is existing mathematics knowledge. Mathematics knowledge itself also contains a lot of material that can be used to pose questions. Several students associate it with other mathematical concepts and principles that they have learned, such as those in algebra and geometry, and pose mathematical application problems.

Further, the second code indicates how the mathematical problem should be organised. Students generated the mathematical problems in different ways even though they used the same type of information. In this study, two such ways can be identified. Some students directly selected the relevant mathematical information in the task situation and integrated it while creating problems. For example, in Task 3, there are multiple relationships between the two quantities in mathematics, and students directly used these relationships to express a mathematical problem. Whereas, for some tasks, they directly imitated the mathematical structures and generated mathematical problems. Other students were able to identify the mathematical relationships and structures that were embedded in the tasks and used them to pose problems in a relatively implicit and indirect manner. These students were also able to make certain changes to the original mathematical relationships and indirectly use the mathematical structure to create mathematical problems.

Last, the second code indicates the characteristics of the problems formulated by the students. It was found that the mathematical problems raised by the same student in a certain task situation have the following characteristics: (1) Similarity. Mathematical problems posed based on the same task situation share the homogeneous structure, organisation of the problem, and even conclusions of different problems. These similarities were also found in the mathematical problems posed for different task situations. (2) Chaining. After posing certain types of mathematical problems, the students tended to pose the same type of mathematical problems continuously. There was a certain degree of linkage between the problems that they posed for different task situations. (3) Solvability. Most students tended to only raise mathematical problems that they could solve, even though they were encouraged to raise difficult mathematical problems for the task situation.

5.5.3 Analysis of the Effects of Regional and Gender Differences on Students' Mathematical Problem-Posing Ability

In terms of regions of origin, it was found that the students from developed regions performed better than those from moderately developed regions, but the performance gap between the two was not notable. There is no significant difference between the performance of the students from underdeveloped regions and that of students from moderately developed regions. This means that the development of problem-posing abilities does not always depend on the economic situation of students. The result showed that students from less developed regions performed better than those from moderately developed regions while dealing with the tasks for Level I. The study revealed that most of students did not have enough capacity to generate problems, especially for the tasks with open situations, for which students had no idea about how to pose accurate mathematical problems. This implicated that mathematics teaching needs to play a role in improving students' problem-posing abilities.

Further, the differences in the mathematical problem-posing ability between boys and girls were also observed, and it was found that the girls performed better than the boys. The study did not analyse the effects of gender difference on problem-posing in depth, which inspired us to consider what the specific characteristics of the problems posed by girls and boys are and what the relationship between mathematics learning and mathematical problem posing is with regard to gender differences.

The study implicated a certain gap between the intended mathematics curriculum and the implemented curriculum. The development of mathematical problem-posing abilities required by the current mathematics curriculum is reflected in mathematics textbooks (Zeng et al., 2006). Certain textbooks organise structured content and provide suggestions for how problem-posing can be taught to students. It is anticipated that the curriculum objective of developing mathematical problem-posing ability will be implemented in the mathematics classroom, and further evidence can be collected to explore students' mathematical problem-posing abilities.

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Chapter 6
The Development of Problem Solving in Chinese Mathematics Curricula

Xiang Gao

Abstract This chapter opens with a review of the concept of mathematical problem-solving competency and defines it based on three aspects: different mathematics content, contexts and cognitive demands. By combing the content about mathematical problem-solving competency from syllabi and curriculum standards in China since 1902, this chapter explores the conceptual development of mathematical problem-solving competency in China. It concludes that the connotation of mathematical problem-solving competency in Chinese mathematics texts has changed greatly over five time phases: 1902–1922, 1923–1951, 1952–1977, 1978–2000 and 2001–present. Meanwhile, a longitudinal analysis of the changes in the requirements of mathematical problem-solving competency is also discussed, focusing on three developments: (1) from 'independence' to 'integration', from 'unbalanced' to 'gradual balance'; (2) from 'general' to 'concrete', from 'adapting to national conditions' to 'close to life'; and (3) from 'low level' to 'high level'.

Keywords Mathematical problem solving · Mathematics content · Contexts · Cognitive demands · Syllabus · Curriculum standards · Conceptual development · Content analysis · Analytical framework · PISA · TIMSS · China

6.1 Introduction

In September 2016, the Core Literacy Research Group in China released a report on developing Chinese students' core literacy, and 'problem-solving competency' was put forward as one of the fundamental literacy concerns (Core Literacy Research Group, 2016). In the field of mathematics, China's Mathematics Curriculum Standards for Compulsory Education (2011 version) explicitly included 'problem solving' as one of the overall goals of the mathematics curriculum (Ministry of Education of China, 2012). In 2018, the Ministry of Education of China released the

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Mathematics curriculum standard for Ordinary High Schools (2017 version), which integrated the development of mathematical problem-solving competency into the cultivation of students' mathematical abstraction, logic reasoning, mathematical modelling, intuitive imagination, mathematical operation and data analysing literacy, and bringing students' mathematical problem-solving competency to a new level (Ministry of Education of China, 2018).

As the programmatic text guiding teaching and learning, mathematics curriculum documents play an important part in mathematics curriculum areas in China. It is necessary to analyse the changes in the requirements for mathematical problem-solving competency in Chinese mathematics curriculum documents to improve understanding of the topic in Chinese mathematical education and to discover the reasons for the changes in students' mathematical problem-solving competency.

6.2 Literature Review of Research on Mathematical Problem Solving

6.2.1 The Concept of Mathematical Problem Solving

In the 1980s, 'An Agenda for Action: Recommendations for School Mathematics of the 1980s' was issued by the National Council of Teachers of Mathematics (NCTM) in the United States; the document stated that 'problem solving must be the focus of school mathematics in the 1980s' (NCTM, 1980, p. i), which quickly received a positive response from educators in many countries around the world. For example, the 1982 Cockcroft Report in the UK pointed out that mathematics teaching should provide students with the opportunity for 'problem solving, including the application of mathematics to everyday situations' (Cockcroft, 1982). Furthermore, some countries have begun to pay more attention to 'mathematical problem solving' in mathematics curriculum standards (Clarke, Goos, & Morony, 2007; Fan & Zhu, 2007).

At present, there is no unified definition of 'mathematical problem solving', and researchers have explored the topic based on different approaches. The first focuses on the definition of 'mathematical problem'. Before the 1980s, problem solving usually referred to solving the routine one- or two-step word problem (Schoenfeld, 2007); after the 1980s, the more generally accepted understanding of 'mathematical problem' in problem solving was that 'mathematical problems should be solved in a certain situation, but the problem solvers have no ready-made solutions to get the answers to the problems' (Reiss & Törner, 2007). This kind of question is usually not closed, but is instead a non-routine mathematical problem.

The second focus has been to define mathematical problem solving as a kind of competency. The 2000 report 'Principles and Standards for School Mathematics' by the NCTM required K–12 students to 'build new mathematical knowledge through problem solving', 'solve problems that arise in mathematics and in other contexts', 'apply and adapt a variety of appropriate strategies to solve problems', and 'monitor and reflect on the process of mathematical problem solving' (NCTM, 2000). In the mathematics curriculum standard issued by Germany in 2003, the competency of 'solving problems mathematically' was defined as: having appropriate mathematical strategies to discover and reflect on the ideas or methods of solving problems (Xu, 2007). In the eight standards for mathematical practice from the Common Core State Standards for Mathematics (CCSSM), it is noted that students should be able to 'make sense of problems and persevere in solving them' (Common Core State Standards Initiative, 2010). In the process of solving mathematical problems, students should analyse the givens, constraints, relationships and goals, plan the solution pathway, do self-monitoring and evaluation, use various forms of representation to seek relations and patterns and reflect on the results (Common Core State Standards Initiative, 2010).

The curriculum standard of China (2011 version) defines mathematical problem-solving competency from four aspects: the application of mathematical knowledge, the methods of solving mathematical problems, the communication with others and the consciousness of reflection. It requires students to 'initially learn to find and pose problems from a mathematical point of view, comprehensively use mathematical knowledge to solve simple practical problems, enhance application awareness, and improve practical ability'. It also expects students to 'get some basic methods of analysing and solving problems, experience the diversity of methods of solving problems, and develop innovative consciousness'; 'learn to cooperate and communicate with others'; and 'initially form a sense of evaluation and reflection' (Ministry of Education of China, 2012). Danish scholar Niss (2003) has proposed eight mathematical abilities. Among them, 'the ability to mathematically solve problems' is defined as the ability to solve different kinds of mathematical problems (pure or applied, open-ended or closed), whether posed by others or by oneself, and, if appropriate, in different ways (Niss, 2003).

PISA 2012 considers 'devising strategies for solving problems' as a manifestation of students' mathematical literacy and explains it along three dimensions: (1) formulating, employing and interpreting, and requiring students to select or devise a plan or strategy to mathematically reframe contextualized problems; (2) activating effective and sustained control mechanisms across a multi-step procedure leading to a mathematical solution, conclusion or generalization; and (3) devising and implementing a strategy to interpret, evaluate and validate a mathematical solution to a contextualized problem (OECD, 2013). At the same time, PISA 2015 points out that choosing appropriate mathematical strategies and representation forms for different contexts puts forward additional requirements for problem solvers. Therefore, PISA 2015 constructs personal, occupational, societal and scientific contexts to test students' problem-solving competency (OECD, 2016). Finally, TIMSS reflects three levels of mathematical problem-solving competency from cognitive demands: knowing, applying and reasoning (Mullis & Martin, 2017).

To sum up, mathematical problem solving focuses on three aspects: first, how students solve problems in different mathematics content and build new mathematics knowledge; second, how to solve problems in various contexts, such as giving a mathematical solution to a contextualized problem; and third, how to solve problems related to different cognitive demands, including problems that may require students to choose appropriate mathematical strategies and representation forms, interpreting, evaluating and validating a mathematical solution.

6.2.2 Problem Solving in Mathematics Curriculum

At present, the trend of globalization and internationalization of mathematics curriculum is becoming increasingly prominent. The development of mathematics curriculum in many countries focuses on the cultivation of mathematical problem-solving competency. Some countries also put mathematical problem-solving competency at the core of mathematics curriculum. For example, Singapore's mathematics curriculum has formed a pentagonal framework cantered by mathematical problem solving and with five unique components: concepts, skills, processes, attitudes and metacognition (Fan & Zhu, 2007).

From the perspective of international mathematics curriculum, the development of problem solving presents the following characteristics, for which problem solving is considered the overall goal, or learning goals, of mathematics curriculum. Curriculum reform in the compulsory education stage of 2001 in China presented the characteristic of 'shifting from overemphasizing knowledge transmission to placing more emphasis on students' active participation and to developing such mathematical abilities as collecting and processing new information, gaining new knowledge independently, analysing and solving problems, and communicating and cooperating with others' (Cai & Howson, 2012). Researchers have analysed the recent mathematics curriculum standards of China and found that the overall goal covers helping students acquire important knowledge and the basic problem-solving skills in mathematics that are important for their lifelong learning (Ni, Li, Cai, & Hau, 2015).

The Curriculum and Evaluation Standards for School Mathematics issued by NCTM regards 'becoming a mathematical problem solver' as one of the five basic objectives of students' mathematics learning (NCTM, 1989). NCTM has also made significant adjustments in mathematics teaching to achieve this goal, requiring conjecturing, inventing and problem solving, and moving away from an emphasis on mechanistic answer-finding (Cai & Howson, 2012). Subsequently, NCTM 2000 standards and recent CCSSM guidelines put mathematical problem-solving competency in the important position of teaching and practice objectives in curriculum standards. The Cockcroft Report (1982) emphasized the importance of mathematical problem solving; thus, subsequent mathematics curriculum standards also gradually stressed their importance. In the latest national curriculum framework document of 2014 in the United Kingdom, in the area of 'numeracy and mathematics', it was proposed that teachers should teach students to apply their mathematics to both routine and non-routine problems, including breaking down more complex problems into a series of simpler steps (UK Department for Education, 2014).

Under the influence of Realistic Mathematics Education, problem solving in the mathematics curriculum of the Netherlands requires students to 'develop informal, highly context-specific solution strategies' to support the construction of mathematical concepts, and 'strengthen the concepts that have been developed and integrate them into mathematical problem solving activities to develop and deepen the use of strategies' (Gravemeijer & Doorman, 1999).

To sum up, the mainstream definition of mathematical problem solving in mathematics curriculum starts from the perspective of competency and mainly focuses on how students transfer information in specific contexts to mathematical problems; apply appropriate mathematical knowledge, methods and strategies to find solutions to mathematical problems; and check and reflect on the process of problem solving. During the process of problem solving, students must use appropriate and reasonable forms of representation and express the whole thinking process effectively and smoothly. These characteristics of mathematical problem-solving competency can also be categorized as how students solve different problems with different mathematics content, contexts and cognitive demands.

6.3 Research Questions and Methodology

6.3.1 Research Questions

By reviewing the literature, we have shown that mathematical problem-solving competency focuses on three aspects: how students solve different problems with different mathematics content, contexts and cognitive demands. Specifically, mathematical problem-solving competency mainly focuses on how students transfer problems in specific contexts to mathematical problems; how they apply appropriate mathematical knowledge, methods and strategies to reach solutions; and how they check and reflect on the process of problem solving. During this process, they should use appropriate and reasonable forms of representation and express the whole thinking process effectively and smoothly. It is clear that these characteristics are also present in China's latest curriculum standards. However, what remains unanswered is: What was the connotation of mathematical problem-solving competency in China's programmatic texts (syllabus and curriculum standards) of mathematics curriculum 100 years ago? What has been the conceptual development of mathematical problem-solving competency during the last 100 years in China?

By combing the content about mathematical problem-solving competency from syllabus and curriculum standards in China since 1902, this chapter explores the conceptual development of mathematical problem-solving competency. At the same time, it identifies clues in the historical context of the development of the concept of mathematical problem-solving competency. As a result, the reasons why our understanding of Chinese mathematical problem-solving competency has developed this way, and changes in students' competency requirements can be better understood.

Taking junior middle school as an example, the specific research questions in this chapter are as follows:

- What are the characteristics of the evolution of the meaning of mathematical problem-solving competency across Chinese history?
- What changes have occurred in mathematical problem-solving competency requirements in Chinese mathematics curriculum?

6.3.2 Research Design

6.3.2.1 Subjects

The research subject in this chapter is the syllabus and curriculum standards of mathematics curriculum in junior middle schools from 1902 to the present. The texts of mathematics curriculum from 1902 to 2000 were selected from the *Collection of Primary and Secondary School Curriculum Standards and Syllabus of the Twentieth Century China. Mathematics Volume*, compiled by the Curriculum and Teaching Materials Research Institute (CTMRI, 2001). The texts after 2000 were selected from the ‘Mathematics Curriculum Standards for Full-time Compulsory Education (Experimental version)’ (Ministry of Education of China, 2001), and ‘Mathematics Curriculum Standards for Compulsory Education (2011 version)’ (Ministry of Education of China, 2012).

6.3.2.2 Methodology: Content Analysis

The primary method used in this chapter is content analysis. Mayring (2015) simplifies the process into three steps: (1) *Reducing procedures*: reduce the material such that the essential content remains; (2) *Explicating procedures*: provide additional material on individual doubtful text components (terms, sentences, etc.); and (3) *Structuring procedures*: filter out particular aspects of the material and assess the material according to certain criteria, usually using the categories of ‘inductive’ and ‘deductive’ to encode the content and count the frequency of keywords, finishing by reanalysing the material (Mayring, 2015). Based on these guidelines, we first deleted and filtered the content in the mathematics curriculum texts according to the definition and characteristics of mathematical problem-solving competency; second, we determined the text analysis framework to code the texts; finally, we analysed the results of the coding process.

This chapter defines the analytical framework of mathematical problem-solving competency as three categories: (1) the content domains of mathematical problems, (2) the situational background of mathematical problems and (3) the cognitive demands of mathematical problem solving. Among them, the first includes indistinguishable (comprehensive requirements), arithmetic, algebra, geometry and probability statistics; the second adds 'no context' on the basis of personal, occupational, societal and scientific contexts proposed by PISA in 2015 (OECD, 2016); and the third originated from the cognitive domains of TIMSS 2019, including knowing, applying and reasoning (Mullis & Martin, 2017). The specific analytical framework is shown in Table 6.1.

The encoding unit is based on a sentence. For example, 'Use the locus method to solve geometric construction problems'. The problem belongs to the content domain of 'geometry', so it is coded as A3; the situational background of the mathematical problem belongs to 'no situation', so it is coded as B0; and in the cognitive demand domain, it belongs to 'implement strategies and operations to solve problems involving familiar mathematical concepts and procedures', so it is coded as C23. Thus, the code of this sentence is A3B0C23. If the expression of a sentence involves multiple mathematical content domains, situational backgrounds and cognitive domains, it is given multiple codes. The encoding process was divided into two stages: in the first stage, three coders independently encoded 20 randomly sampled sentences according to the analytical framework, discussing and negotiating the divergent parts; the consistency of the initial encoding was 85%. In the second stage, two coders separately encoded 398 units of all curriculum texts, and the consistency was 95.7%. After negotiating the inconsistent parts, final agreement was reached.

6.4 Research Results on Conceptual Development

6.4.1 The Connotation Evolution of Mathematical Problem-Solving Competency

According to the content analysis framework of mathematical problem-solving competency, this research involved content screening, coding, keyword frequency counting and analysis of mathematics curriculum texts from 1902 to the present. The results indicate that, over time, the meaning of problem-solving competency in mathematics curriculum texts has changed greatly, and these developments can be divided into five time phases: (1) 1902–1922: mathematical problem-solving competency as a 'plan for making a living'; (2) 1923–1951: the ability to solve application problems with 'operation' as the kernel; (3) 1952–1977: emphasis on 'linking with reality', based on solving application problems; (4) 1978–2000: comprehensive use of knowledge and skills to solve problems based on the 'three basic abilities'; and (5) 2001–present: components that cover 'key abilities'. The specific connotation of mathematical problem-solving competency in each historical stage is described in the following sections.

Table 6.1 Content analytical framework and corresponding coding for mathematical problem-solving competency

Dimension		Coding	Description
Content domains		A0	Indistinguishable (comprehensive requirements)
		A1	Arithmetic
		A2	Algebra
		A3	Geometry
		A4	Probability statistics
Situational background		B0	No contexts (problems are directly expressed by mathematical forms)
		B1	Personal context (problems focus on activities of one's self, one's family or one's peer group)
		B2	Occupational context (problems are centred on the world of work)
		B3	Societal context (problems focus on one's community perspective, such as voting systems, public transport, government, public policies, demographics, advertising, national statistics and economics)
		B4	Scientific context (problems relate to the application of mathematics to the natural world and issues and topics related to science and technology)
Cognitive domains	Knowing (C1)	C11	*Recall* Recall definitions, terminology, number properties, units of measurement, geometric properties and notation
		C12	*Recognize* Recognize numbers, expressions, quantities, and shapes. Recognize entities that are mathematically equivalent
		C13	*Classify/order* Classify numbers, expressions, quantities and shapes by common properties
		C14	*Compute* Carry out algorithmic procedures, or a combination of these with whole numbers, fractions, decimals and integers. Carry out straightforward algebraic procedures
		C15	*Retrieve* Retrieve information from graphs, tables, texts or other sources
		C16	*Measure* Use measuring instruments and choose appropriate units of measurement
	Applying (C2)	C21	*Determine* Determine efficient/appropriate operations, strategies and tools for solving problems for which there are commonly used methods of solution

(continued)

Table 6.1 (continued)

Dimension		Coding	Description
		C22	*Represent/model* Display data in tables or graphs; create equations, inequalities, geometric figures or diagrams that model problem situations; and generate equivalent representations for a given mathematical entity or relationship
		C23	*Implement* Implement strategies and operations to solve problems involving familiar mathematical concepts and procedures
	Reasoning (C3)	C31	*Analyse* Determine, describe or use relationships among numbers, expressions, quantities and shapes
		C32	*Integrate/synthesize* Link different elements of knowledge, related representations and procedures to solve problems
		C33	*Evaluate* Evaluate alternative problem-solving strategies and solutions
		C34	*Draw conclusions* Make valid inferences on the basis of information and evidence
		C35	*Generalize* Make statements that represent relationships in more general and more widely applicable terms
		C36	*Justify* Provide mathematical arguments to support a strategy or solution

6.4.1.1 1902–1922: Mathematical Problem-Solving Competency as 'A Plan for Making a Living'

In 1902, the first statutory school system in the history of modern Chinese education, 'School Rules Made by Emperor Order in 1902', was promulgated but not implemented. In 1904, the guidelines imitated the Japanese school system and laid the foundation for the establishment of a modern school system in China, which continued until the founding of the Republic of China (Dai, 2015).

Although there is no direct expression of mathematical problem-solving competency in the texts of the mathematics curriculum at this stage, the standard can be inferred from some expressions that the competency requirement at that time aimed to enable students to 'master the basic skills of earning a living'. For example, in 1904, the 'General Constitution of the School' of the 'Approved School Articles of 1904' emphasized that 'those who are not official after graduation are engaged in various industries'. It dictated that education by middle schools should focus on teaching 'bookkeeping' so that students could learn 'the usage of bookkeeping' and 'the format of various calculating tables'. In specific areas of learning, students were expected to grasp the principle of operation, be familiar with quick calculation and apply measurement and quadrature (CTMRI, 2001, p. 206). That is, for the vast majority of students who did not work as public officials in the future, they only needed to ascertain the knowledge of arithmetic, geometry and other basic living skills.

6.4.1.2 1923–1951: The Ability to Solve Application Problems with 'Operation' as the Kernel

At this stage, the expression of mathematical problem-solving competency clearly reflects the requirement to be able to solve application problems in various fields based on students' 'operation' ability. Figure 6.1 shows the percentage of cognitive demands in 103 coding units from 1923 to 1951.

It is clear that, in the coding of this stage: the percentages of the three cognitive requirements of 'Compute' in Level 1 and 'Determine' and 'Implement' in Level 2 are much higher than other cognitive demands. The mathematical problem-solving competency at this stage is based on 'operation' as the kernel, and has students solve familiar mathematics application problems using familiar problem-solving methods.

In different fields of mathematics content, the curriculum texts at this stage are more consistent in the requirements of solving the mathematics application problems, which is shown in the following: in arithmetic, students are required to solve application problems related to fractions, decimals, proportions and percentages; in algebra, students are required to create and solve application problems related to linear equations with one unknown, quadratic equations with one unknown, and binary quadratic equations; in geometry, students are required to solve basic construction problems related to triangles, quadrilaterals and simple measurement problems; finally, in statistical probability, students are required to solve application

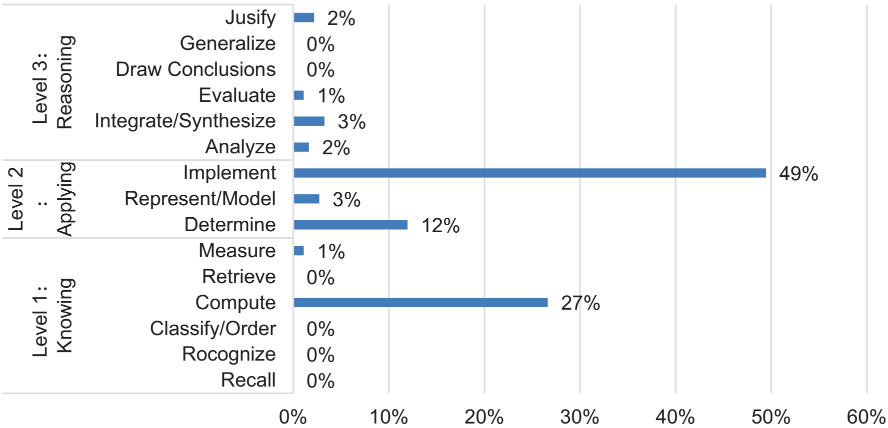

Fig. 6.1 The percentage of cognitive demands, 1923–1951

problems related to the average and price index, and to understand statistical charts, etc.

6.4.1.3 1952–1977: Emphasis on 'Linking With Reality', Based on Solving Application Problems

In 1952, under the guidance of 'taking Russia as a teacher' and 'the one-sided' policy, the Ministry of Education compiled the first mathematics syllabus of New China, 'Mathematics teaching syllabus for middle schools (Draft)', based on the 10-year high school mathematics teaching syllabus of the Soviet Union. However, since 1958, China has begun to reflect on the limitations of copying the educational experience of the Soviet Union, rethink the phenomenon of 'achieve less but cost more' in the field of mathematics curriculum, and enter the stage of exploring the independent development path of Chinese mathematics curriculum. At the same time, Chinese researchers began to reflect on the problems brought about by blindly copying the Soviet Union's syllabus and ignoring the specific realities of China, such as the narrow scope of knowledge and low degree of teaching content, which cannot meet the needs of students for future production and labour (Zhang & Dai, 2017).

When encoding the 86 units of mathematics curriculum texts at this stage, we found that the percentages of the three cognitive demands of 'Compute', 'Determine' and 'Implement' remained high compared with 1923–1951 (32%, 4% and 43%, respectively). This indicates that the requirements in curriculum texts from 1952 to 1977 for solving mathematics application problems have not decreased, but the percentage of 'Integrate/Synthesize' in Level 3 has increased from 3% to 7%; likewise, 'Evaluate' increased from 1% to 2%. This shows that the curriculum texts during 1952–1977 began to emphasize the comprehensive application of mathematical knowledge and skills. Specifically, the preface of the curriculum texts at this stage clearly pointed out that students should apply the mathematical knowledge and skills comprehensively, based on mastering basic knowledge and skills, so as to solve practical problems.

In addition, the mathematical problem-solving competency at this stage also requires students to evaluate the results of problem solving according to the actual situation. For example, the 1956 syllabus clearly states that 'when solving application and compute problems, students must learn how to write calculation procedures reasonably and acquire the skills of checking answers' (CTMRI, 2001, p. 400). At the same time, in the specific requirements of the mathematics learning field, the curriculum texts at this stage point out that students should be 'linking with reality'. For example, in algebra, students are required to 'apply algebraic knowledge to solve simple problems related to physics, chemistry, astronomy, technology, and agriculture'; in geometry, they are expected to 'apply the knowledge they have learned to solve practical problems: to measure the surface area and volume of various buildings, and do simple measurement of military aspects'. During the Cultural Revolution from 1966 to 1976, China's education system suffered a heavy blow, which led to the disappearance of the national unified mathematics curriculum and the stagnation of the compilation and revision of the syllabus.

To summarise, mathematical problem-solving competency of 1952–1977 emphasized that students should be linked with practical problems in reality, based on solving application problems.

6.4.1.4 1978–2000: Based on the 'Three Basic Abilities', Comprehensive Use of Knowledge and Skills to Solve Problems

With the end of the Cultural Revolution, China entered a new era of re-exploring the development path of Chinese mathematics curriculum. Since September 1977, the mathematics compiling group composed of primary and secondary school mathematics experts, such as Buqing Su, began to draft mathematics syllabi for primary and secondary school teaching. In February 1978, the 'Mathematics Syllabus for Full-time Ten-year High Schools (trial version)' was promulgated and proposed that the purpose of middle school mathematics teaching was to 'enable students to have correct and rapid operation ability, certain logical thinking ability and certain spatial imagination ability (three basic abilities), so as to gradually cultivate students' ability to analyse and solve problems step by step'. For the first time, the 'three basic abilities' are regarded as the basis of students' mathematical problem-solving skills. The syllabus for the middle school stage adopted similar expressions 'to train students' ability to analyse and solve practical problems, based on the training of students' ability of operation, logical thinking and spatial imagination'.

In 1992, the 'Mathematics Syllabus for Nine-year Full-time Compulsory Education in Secondary School (Trial)' offered a definition of 'can solve practical problems' for the first time: 'can solve mathematical problems which are of practical significance and related to relevant disciplines, and solve practical problems about production and daily life' (CTMRI, 2001, p. 605). It emphasizes training students to abstract practical problems into mathematical problems, gradually developing students' ability to analyse and solve problems, and helping them form a sense of using mathematics while also highlighting thinking ability as the core of cultivating ability.

By analysing 107 coding units at this stage, it was found that, compared with the percentages for 1952–1977, the percentage of Represent/Model in Level 2 of cognitive demands increased from 1% to 10%, and the percentage of Integrate/Synthesize in Level 3 increased from 7% to 21%. These data, to a certain extent, can reflect that the connotation of mathematical problem-solving competency for 1978–2000 focused on students' possession of 'three basic abilities', and on this basis, they could make comprehensive use of relevant knowledge and skills.

6.4.1.5 2001–Present: Components that Cover 'Key Abilities'

In June 1999, the CPC Central Committee and the State Council promulgated the 'Decision of the CPC Central Committee and the State Council on Deepening Educational Reform and Promoting Quality Education in an All-Around Way', aiming at cultivating talents from many aspects of comprehensive quality, innovative spirit and practical ability. At the same time, many countries have begun to attach importance to multi-component mathematical core competence or mathematical core literacy (Si & Zhu, 2013). In this context, 'Mathematics Curriculum Standards for Full-time Compulsory Education (Experimental version)', promulgated in 2001 (Ministry of Education of China, 2001), clearly regarded problem solving as the overall goal of the curriculum, requiring students to 'preliminarily learn to use mathematical thinking to observe and analyse the real society, to solve problems in daily life and other disciplines, and to enhance the awareness of applying mathematics'. At the same time, four specific requirements were put forward:

> To preliminarily learn to pose and understand problems from a mathematical point of view, and to comprehensively use the knowledge and skills learned to solve problems, develop the awareness of application; to form some basic strategies to solve problems, experience the diversity of problem-solving strategies, develop practical ability and innovation spirit; to learn to cooperate with others, and be able to communicate with others about the process and the result of thinking; to initially form the consciousness of evaluation and reflection. (Ministry of Education of China, 2001).

These reflect that mathematical problem-solving competency covers the key ability components, such as posing mathematical problems and mathematical communication, and the goal of problem solving is similarly stated in the 'Mathematics Curriculum Standards for Compulsory Education (2011 version)'.

At the same time, the 2001 curriculum standard designed the content field of 'practice and comprehensive application', and the 2011 curriculum standard designed 'integration and practice', both aiming at strengthening students' comprehensive use of the knowledge of arithmetic, algebra, geometry and probability statistics. They also sought to mobilize the key abilities of posing mathematics problems, reasoning and argumentation, representation and transformation, mathematical communication, and the ability of solving problems comprehensively.

6.4.2 The Change in Requirements of Mathematical Problem-Solving Competency

After encoding the texts of the mathematics curriculum according to the content analysis framework of mathematical problem-solving competency, a longitudinal analysis of the changes in the requirements was conducted. Since there is no expression of mathematical problem-solving competency in the texts of mathematics curriculum from 1902 to 1922, this section only encodes the texts of the four historical stages from 1923 to the present (1923–1951; 1952–1977; 1978–2000;

2001–present). The analysis indicates that, in the curriculum text, the requirements for students' mathematical problem-solving competency have changed in the following three ways: (1) from 'independence' to 'integration', from "unbalanced' to 'gradual balance', including changes in the requirements of mathematical content domains; (2) from 'general' to 'concrete', from 'adapting to national conditions' to 'close to life', including changes in the requirements of the context of mathematical problems; and (3) from 'low level' to 'high level', or changes in cognitive demands. The specific changes are described below.

6.4.2.1 From 'Independence' to 'Integration', from 'Unbalanced' to 'Gradual Balance': Changes in the Requirements of Mathematical Content Domains

By encoding the mathematical content domains using 398 codes and counting the percentage of each category during the four historical stages (as shown in Fig. 6.2), the results indicate that mathematical problem-solving competency in the field of mathematical content domain has changed from 'independence' to 'gradual integration', and the internal requirements of each content field have changed from 'unbalanced' to 'gradually balanced'.

First, in terms of mathematics curriculum content, when encoding the category 'the indistinguishable (comprehensive requirements)' for the comprehensive use of mathematical knowledge, skills and methods for solving mathematical problems in the fields of arithmetic, algebra, geometry and probability statistics, the findings suggest that the percentage has gradually increased from 1% in 1923–1951 to 9% in 1952–1977, to 20% in 1978–2000, and to 58% from 2001 to the present, indicating that requirements for students' mathematical problem-solving competency have gradually moved from the focus of independent requirements within each content field to cross-content topics, and finally to integration.

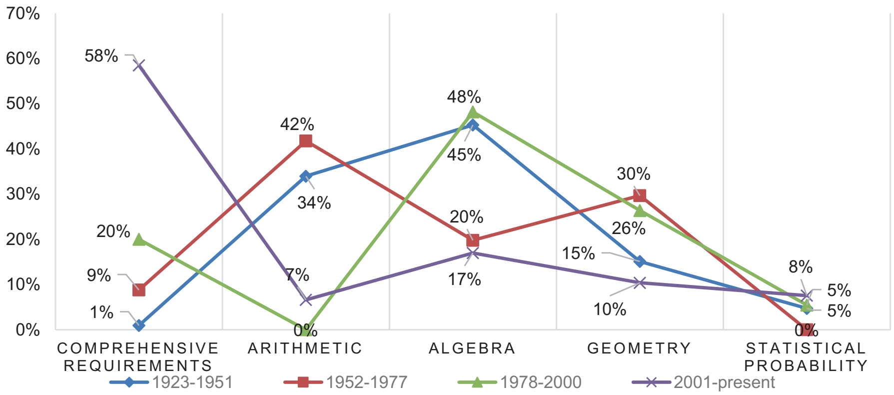

Fig. 6.2 Changes in requirements for mathematical content domains in four historical stages

Second, the proportion of the requirements in each content domain has gradually become more balanced. The demand for solving problems in arithmetic has declined from about 40% before 1978 to less than 10% today. Although the algebra and geometry fields have shown some fluctuations, the proportions since 2001 have dropped to the lowest points among the four historical levels, while the requirements for probability statistics have increased slightly. Until 2001, the proportions of the requirements of arithmetic, algebra, geometry and probability statistics had been balanced, each accounting for about 10%. At the same time, even within the various content domains, the requirements for mathematical problem-solving competency have gradually increased. In the field of arithmetic, previous mathematics curriculum texts emphasised computing problems about fractions, decimals, proportions, percentages and applications. However, the curriculum standards since 2001 proposed, ‘When solving practical problems, students can use calculators for approximate calculation and approximate the results as required by the problem’ (Ministry of Education of China, 2001, 2012). In probability statistics, the previous curriculum texts required students to understand statistical charts and some issues related to the average, price index, etc., but the standards from the most recent period suggest that students ‘have the statistical concept to think over problems related to data information from a statistical perspective’ (Ministry of Education of China, 2001); they should ‘experience the process of collecting and processing data, using data to analyse problems, and accessing information in solving practical problems’ (Ministry of Education of China, 2012).

These changes confirm that, in China, the focus of mathematical problem-solving competency has transformed from students’ operation ability to solving application problems, and then to the connection with practical problems on the basis of solving application problems—from ‘three basic abilities’ to the cultivation of students’ ‘key ability’.

6.4.2.2 From ‘General’ to ‘Concrete’, from ‘Adapting to National Conditions’ to ‘Close to Life’: Changes in the Context Requirements of Mathematical Problems

By analysing the percentages of students’ mathematical problem-solving competency required in different problem contexts (i.e. non-context, or directly presented as mathematical form; personal contexts; occupational contexts; societal contexts; and scientific contexts) in China’s four historical stages, the results show that the problem situation without context accounts for approximately 90% of the cases (87% in 1923–1951; 89% in 1952–1977; 83% in 1978–2000; and 89% since 2001). The percentage of the other four contexts is very small.

Despite this, we also note that the relevant expressions on mathematical problem contexts in curriculum texts have changed from ‘general’ to ‘concrete’, and from ‘adapting to national conditions’ to ‘close to life’. In the curriculum texts before 2001, students were often required to ‘solve mathematical problems with practical meaning or related to disciplines’, ‘solve practical problems in production and life’, or solve problems related to ‘price index’ and ‘field measurement’. Although these expressions exemplify the requirements for students to solve problems involving ‘personal context’, ‘societal context’ and ‘scientific context’, more specific expressions are not given; the expressions in the context of specific mathematical problems are more concrete in the curriculum standards after 2001. For example, in 2001, mathematical problem-solving competency in students’ ‘personal contexts’ is expressed as ‘observing and understanding similarity of objects by typical examples, and solving some practical problems by similarity of graphics (such as measuring the height of the flagpole with similarity)’. Likewise, in the ‘scientific context’, competency is expressed as ‘creating contexts by mining resources that can be utilized from other disciplines (such as natural phenomena, social phenomena, and human heritage), and solving problems in other disciplines’.

At the same time, in the early days of the founding of the People’s Republic of China (after 1949), curriculum texts emphasized that mathematical problems should reflect China’s national conditions. For example, the 1952 syllabus indicates that problems should reflect the new democratic and socialist constructions when receiving training in application problems (CTMRI, 2001, p. 359). The 1956 syllabus states that teachers should ‘adopt a wide range of technical and agricultural materials when selecting and compiling application problems, and that they should combine the content of the application with the situation and achievements of socialist construction’ (CTMRI, 2001, p. 400). The guidelines since 2001 have been more closely related to the lives of students. For instance, in the 2011 curriculum standards, teachers are advised to choose meaningful topics from newspapers, magazines, TV broadcasts and online media, which should be closely related to current events as well as to students’ lives; this is to aid in exploring materials suitable for students’ learning, and to improve students’ ability to apply mathematical concepts in solving problems (Ministry of Education of China, 2012).

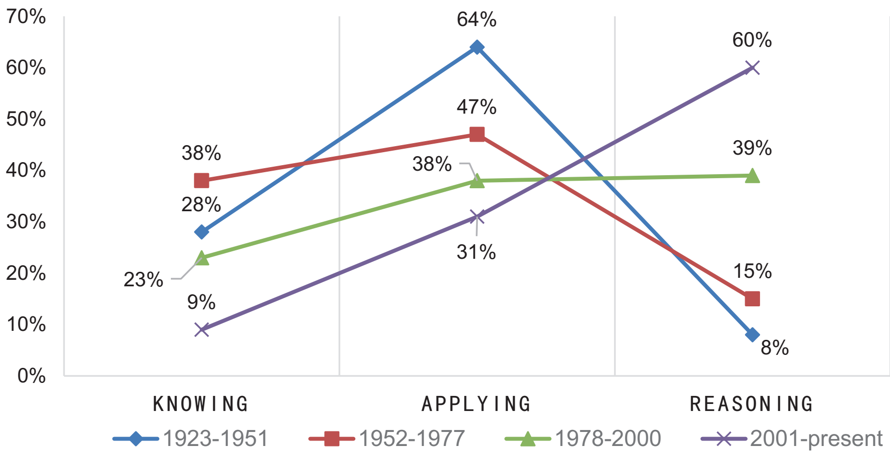

Fig. 6.3 Changes of requirements for cognitive demands in four historical stages

6.4.2.3 From 'Low Level' to 'High Level': Changes in Cognitive Demands

By encoding the cognitive demands and counting the percentages of the codes of knowing, applying and reasoning (as shown in Fig. 6.3), we find that the cognitive demands for mathematical problem-solving competency have gradually improved over the four historical stages since 1923.

Since 1923, the proportion requiring 'reasoning' has gradually increased, from 8% in 1923–1951 to 15% in 1952–1977, to 39% in 1978–2000, and to 60% in 2001, while the proportion of 'knowing' has declined, reaching its lowest point (9%) in 2001 to the present.

Looking at the various sub-components of cognition demands, the proportion of 'compute' under Level 1 'knowing' dropped from 27% in 1923–1951 to 32% in 1952–1977, to 21% in 1978–2000, and to 5% after 2001. The corresponding values for 'analyse' under Level 3 'reasoning' remained at 2% during the first two periods, increased to 16% in 1978–2000, and declined slightly to 13% during the last phase. At the same time, 'integrate/synthesize' under Level 3 'reasoning' also increased substantially, from 3% to 7%, 21%, and 34%, respectively. Clearly, China's expectations for students' cognitive ability to solve mathematical problems have gradually increased from the low-level cognitive demands of applying computing skills in solving application problems to the high-level cognitive demands of comprehensively adopting mathematical knowledge and mathematical skills, and selecting reasonable representation forms and methods in solving more complex problems in mathematical and other contexts.

After 1980, Chinese scholars analysed and reflected on problem-solving theories abroad and suggested higher requirements for students' cognition in teaching based on teaching practices in China. These were mainly reflected in two aspects. First, they emphasized the practical application of mathematical knowledge while paying attention to traditional problem-solving techniques. Second, as indicated by the International Assessment of Educational Progress (IAEP), organized by the US Educational Testing Service in 1991, Chinese students' performance in the application problems was only at the middle level. Yan, Zhang, and Su (1993) argued that, in recent years, the college entrance mathematics exam 'is all about pure mathematics skills without applying mathematics knowledge'; 'Chinese students lack the ability to apply mathematics, and the ability to use mathematical knowledge creatively is declining'. Thus, the following curriculum standard emphasized the understanding and comprehensive application of mathematical knowledge in the process of solving mathematical problems. It also expanded the types of mathematical problems, and focused more on open-ended and mathematical modelling problems. In 1992, at the invitation of the National Institute for Educational Policy Research (NIER), Dianzhou Zhang and colleagues attended the Sino-Japanese Mathematical Education Joint Research Association meetings in Tokyo, where the teaching of open-ended problems was discussed. They gradually promoted open-ended problems in China, and the teaching requirements were written into the national curriculum standards. Cases on open-ended problems appeared in the 2001 curriculum standards (Dai, 2012, p. 2), and expectations for cognitive demands were raised in mathematical problem-solving competency.

6.5 Conclusion and Discussion

6.5.1 Relevant Conclusions on the Evolution of Mathematical Problem-Solving Competency in China

Through the analysis of China's mathematics curriculum since 1902, this study finds that a profound change has occurred in the connotation of mathematical problem-solving competency at each historical stage. This is shown in the following changes during each historical period:

1. 1902–1922: the ability to solve mathematical problems as 'a plan for making a living'.
2. 1923–1951: the ability to solve problems with 'operation' as the kernel.
3. 1952–1977: emphasis on the 'link with reality' on the basis of solving application problems.
4. 1978–2000: comprehensive use of knowledge and skills to solve problems based on the 'three basic abilities'.
5. 2001 to the present: components covering most parts of the 'key abilities'.

The requirements for students' mathematical problem-solving competency also have the following three characteristics: (1) the content areas involved in mathematical problems change from 'independence' to 'integration', from 'unbalance' to 'gradual balance'; (2) the mathematical problem context changes from 'general' to 'concrete', from 'adapting to national conditions' to 'close to life'; and (3) the cognitive demands of mathematical problem-solving competency gradually develop from 'low level' to 'high level'.

Based on China's national conditions, we can identify three characteristics of the evolution of China's mathematical problem-solving competency: (1) being rooted in Chinese historical tradition (before the 1950s); (2) adapting to China's national conditions (from the 1950s to the beginning of the twenty-first century); and (3) integrating into the world (from the twenty-first century to the present).

Two basic characteristics are present in the content of traditional Chinese mathematics teaching, namely, 'practicality-based' and 'algorithm-centred' (Cao & Leung, 2018, p. 28). Capturing these two characteristics, the 'Nine Chapters of Arithmetic' covers 246 questions and their corresponding solutions, with problems embedded in the real-life context of the time and exerting a profound impact on the formulation of future mathematics education in China (Cai & Nie, 2007). As an important part of mathematics education, mathematical problem-solving competency is also deeply branded with an emphasis on practicality and basic algorithms. Before the 1950s, mathematical problem-solving competency in the curriculum emphasized that students should master 'bookkeeping' to solve the basic problems related to making a living, or that students should become 'familiar with arithmetic representation and be able to apply them to daily life'. It focused on the basic algorithm (CTMRI, 2001), which gradually evolved into the ability to solve the relevant application problems, with 'operation' at its core. Before the 1950s, the cultivation of students' mathematical problem-solving competency was consistent with the basic purpose of traditional mathematics education—'serving for reality'. That is, at that time, students mastered basic mathematics to meet the basic needs of society and life with no ambition for meeting higher requirements for advanced mathematics; this is completely different from the ancient Greek approach to mathematics, which was characterized by the pursuit of a deductive system (Cao & Leung, 2018, p. 30).

From 1949 to 1957, after learning from the model of mathematics education in the Soviet Union, China focused on teaching students practical knowledge—basic mathematical ideas and skills—thus cultivating students' ability to solve practical problems. However, disadvantages gradually came to light, such as failing to meet the needs of students' future production labour and achieving only a weak understanding of what they had learned (Zhang & Dai, 2017). After recognizing this problem, in view of China's national conditions and the real situation of mathematics education, Chinese mathematics education researchers suggested that the mathematics knowledge learned by students should be related to 'actual life' and to 'actual industrial and agricultural production'. They further emphasised the requirements of cultivating students' 'three basic abilities'—computing, logical thinking and spatial imagination. These requirements aimed to improve students' mathematical problem-solving competency, and required students to 'pose, analyse and solve problems of practical significance or related to disciplines, production, and daily life; be able to express problems and communicate in mathematical languages, and form mathematics-application awareness' (CTMRI, 2001). These changes in connotation reflect the fact that China's mathematics education—combined with the national situation and adhering to the tradition of cultivating students' basic knowledge and skills—now pays more attention to students' mathematical abstraction, mathematical communication and expression, and awareness of mathematics application, as is the case with other countries in the 'Confucian cultural circle', such as Singapore (Fan & Zhu, 2007).

NCTM first proposed the core competence of mathematics in 1989 and promulgated 'Principles and Standards for School Mathematics' in 2000, in which mathematical abilities, including problem solving, were put forward from the perspective of the relationship between mathematical understanding and mathematical ability (NCTM, 2000). Later, countries like Germany, Singapore, and Japan proposed mathematics curriculum standards that were incorporated with mathematical problem-solving competency and oriented by mathematical core competence (Xu, 2013). Since 2000, the status of mathematical problem-solving competency in Chinese mathematics curriculum standards has been continuously improved. It was even juxtaposed with the traditional 'three basic abilities', emphasizing students' mathematical problem-solving competency by a comprehensive application of problem-proposing ability, mathematical abstract ability, mathematical representation and transformation ability, and mathematical communication ability. It has also been gradually connecting with international guidance on mathematics education, with discipline core competence as its orientation.

In 2001, 'solving problems' was established as the overall goal in the curriculum standards of China's compulsory education. Then, the 2011 compulsory education curriculum standards continued to emphasize that, in the process of solving mathematical problems, students should experience a series of thinking processes that embody several mathematical core competencies. In 2018, six core competencies were proposed in the curriculum guidelines, including mathematical abstraction, logical reasoning, mathematical modelling, intuitive imagination, mathematical operations, and data analysis. The whole process of solving mathematical problems is carried into the cultivation of these six core competencies, and the importance of mathematical problem-solving competency has been raised to an unprecedented height (Ministry of Education of China, 2018). At the same time, this change is in line with the basic trend of international mathematics education to require teachers to pay attention to students' higher-order mathematics thinking and the application of information technology while cultivating their problem-solving competency.

6.5.2 Discussion: Mathematical Curriculum Texts Need an Operative Description of Some of the Requirements

By combing the curriculum since 1902, we find that there is no specific and operable expression of the requirements of mathematical problem-solving competency in the mathematics curriculum in China, and most have been highly summarized. For example, 'when solving practical problems, students should be trained to abstract practical problems into mathematical problems, to gradually develop the ability to analyse and solve problems, and to form an awareness of using mathematics'. However, how can students abstract practical problems into mathematical problems? How do they gradually develop the ability to analyse and solve problems, or form an awareness of using mathematics? There are no detailed operational instructions for these questions in the curriculum text, which causes some uncertainty in teachers' instruction of mathematical problem solving.

At the same time, the mathematics curriculum does not provide a clear evaluation index of the ability to solve mathematical problems; thus, it is difficult to quantify and evaluate students' levels or progress in different learning phases and grades. Even though there is an attempt to divide the six core competencies in the curriculum promulgated in 2018 in China, no mathematical problem-solving competency is included, so evaluation of the ability to solve mathematical problems is still not clear.

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Chapter 7
Chinese Eighth Graders' Competencies in Mathematical Problem Solving

Xiang Gao and Yingkang Wu

Abstract This chapter begins with a review of the assessment of mathematical problem-solving competency and the construction of the assessment framework through three levels: memory and reproduction, connection and variation, and reflection and expansion. The analysis included 1185 Chinese Grade-8 students selected based on the geographic location of the city in which they live (including east, central, north, south, northwest, southwest and northeast China) and the corresponding level of economic development (including developed, medium-developed, less-developed, etc.). The student participants completed the mathematical problem-solving competency test, and the results indicate that nearly 80% of the eighth-grade Chinese students have reached the medium level. This chapter also summarizes some characteristics and problems of students' mathematical problem-solving competency in China.

Keywords Mathematical problem solving · Assessment framework · PISA · TIMSS · Large-scale student assessments · Open-ended problems · Grade 8 · Mathematical materials · Problem solving strategies

7.1 Introduction

China has a long history and rich experience in cultivating students' mathematical problem-solving competency, but the current situation is still not optimistic. Lingyuan Gu, a famous mathematics educator, conducted two-level tests in 1990 and 2007 on mathematics teaching objectives of Grade-8 students. The findings indicated that the level of 'analysis-inquiry understanding', which embodies students' ability to analyse and solve problems, has not significantly improved, and has

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even declined to some extent (Qingpu Experimental Research Institute, 2007). Considering the international assessment project, PISA, the proportion of Chinese students who can achieve the highest level of mathematics problem-solving ability—'mathematical proficiency' level 6—dropped from 30.8% in 2012 to 9.0% in 2015 (OECD, 2014, 2017). Scholars must focus urgently on students' mathematical problem-solving competency.

What is the current situation of Chinese eighth graders' mathematical problem-solving competency? Through the analysis of empirical data, this chapter explains to what extent the present situation of Chinese eighth graders' mathematical problem-solving competency reflects the requirements of the mathematics curriculum and what shortcomings exist. It then reflects on the advantages that need to be maintained and the sections that still need to be improved in the development of mathematics curriculum in China.

7.2 Assessment of Mathematical Problem-Solving Competency

Chinese students are outstanding in the large-scale international evaluation of mathematics, showing strong mathematical problem-solving competency. Whether they come from mainland China or from Hong Kong, Macao or Taiwan, their performance in TIMSS and PISA is eye-catching. Many researchers study Chinese students' mathematical problem-solving competency by means of assessment (see, for example, Cai, 2000, 2004; Stevenson et al., 1990), and research has shown that Chinese students perform better than their international competitors in problem solving in almost all grades and in different fields of mathematics (Cai & Nie, 2007). However, Chinese students are not as strong in complex, open-ended tasks that test creativity, problem-posing competency and unconventional problem-solving competency (Cai & Hwang, 2002; Chen et al., 2002). They may be better at solving computational tasks and simple problem-solving tasks (Cai & Cifarelli, 2004), and their mathematical problem-solving competency has the following six characteristics (Cai & Cifarelli, 2004). Chinese students:

1. Perform unevenly on various tasks—better on tasks assessing computation skills and basic knowledge than on tasks assessing open-ended, complex problem solving
2. Are more likely to use generalized strategies and symbolic representations
3. Usually provide more conventional solutions
4. Can generate more solutions if they are asked for them
5. Frequently commit errors involving unjustified symbol manipulations
6. Are less willing to take risks in problem solving.

Some large-scale student assessments in China also explore students' mathematical problem-solving competency. When Binyan Xu, Yan, Bao, and Kong (2015) evaluated the core competence of the eighth-grade students in China, they examined the identification of information, strategy application, mathematical communication, reflection and evaluation in the process of problem solving. They found that the students' mathematical problem-solving competency is below average, and that boys' problem-solving competency is significantly better than that of girls (Xu et al., 2015). In the investigation of mathematical ability of students from Grades 8–12, Yiming Cao, Xiaoting, and Kan (2016) found that students can express concepts with appropriate representations in the process of solving mathematical problems, identify and solve specific problems in simple problem contexts, and interpret the meaning of the results.

The assessment of students' mathematical problem-solving competency mainly focuses on different grades' problem-solving behaviour in different content areas of mathematics. It also evaluates their mathematical problem-solving competency by analysing problem-solving strategies, representation forms and mathematical communication. From the perspective of students doing mathematical activities, the evaluation of students' mathematical problem-solving competency focuses on three processes that students experience: organizing empirical materials mathematically, organizing mathematical materials logically and applying mathematical theories (Kruteskii, 1983). From the perspective of cross-national comparisons, these assessments around the world also explore the strategies used by students in solving mathematical problems, the forms of representation and students' practice of mathematical thinking (Doorman et al., 2007; Fan & Zhu, 2007; Hino, 2007).

7.3 Research Question and Methodology

7.3.1 Research Question

By constructing the assessment framework of students' competency in mathematical problem solving, and using this framework to analyse Chinese eighth graders' performance on the mathematical problem-solving competency test, the research question in this chapter is:

- What is the current performance of Grade-8 students in China in terms of their mathematical problem-solving competency?

7.3.2 Research Design

7.3.2.1 Subjects

The study participants are Grade-8 students in China. Considering that the unbalanced level of economic development in different regions may cause differences in the evaluation results, this study adopted the method of stratified cluster sampling when selecting the samples. First, based on the geographical location of cities (including east, central, north, south, northwest, southwest and northeast China) and their corresponding level of economic development (including developed, medium-developed, less-developed, etc.), eight representative cities were identified. Second, in each city, one high-performing school, one general-performing school and one low-performing school were selected. Third, in each sample school, 2–3 classes of students were randomly selected to participate in the test. The relevant information on the participants is shown in Table 7.1.

7.3.2.2 Methodology: Assessment Research Method

This research adopts the assessment research method by constructing the assessment framework of mathematical problem-solving competency, designing test items, implementing assessment and encoding and analysing students' responses. The assessment framework is the 'mathematical problem-solving competency' part of the mathematical core competence evaluation model constructed in Chap. 3 of this book. The performance of mathematical problem-solving competency at different levels is shown in Table 7.2.

According to the three levels of mathematical problem-solving competency, an overview of the designed test items in this research is shown in Table 7.3.

The analyses of test items are mainly based on the 0–1 dichotomy, which means that the correct answer is scored 1 point, and 0 points otherwise. When analysing the performance of the subjects in the mathematical problem-solving competency assessment, this study refers to the double scoring system of TIMSS, and gives diagnostic codes for the responses of the subjects to determine specific representation methods, problem-solving strategies and routine errors. The relevant scoring rules and student response codes for Test Item 1 are given in Fig. 7.1.

Table 7.1 The subjects of this research

Developed area			Medium-developed area		Less-developed area			Total
A(4)	D(3)	E(3)	B(3)	C(3)	F(3)	G(3)	H(6)	
159	86	109	147	118	177	131	258	1185

Note: The city codes are in alphabetical order, and the numbers in parentheses indicate the number of schools participating in the test in their cities

Table 7.2 The performance of mathematical problem-solving competency

Level 1: Memory and reproduction	Level 2: Connection and variation	Level 3: Reflection and expansion
When facing simple problem situations, to be able to identify and select familiar mathematical information and solve simple mathematical problems according to the mathematical methods and strategies learned before To be able to express simple mathematical problem-solving processes (To be able to select strategies that can be easily thought of, and to solve well-structured mathematical problems.)	To be able to relate knowledge and expressions in different mathematical fields (such as charts, words, symbols, etc.) To be able to briefly and logically express the process of thinking, solutions and results To be able to explain the meaning of one's own mathematical results on the basis of judgment (Through the analysis of the problem, to be able to identify proper strategies and solve well-structured mathematical problems; to use various unambiguous methods to solve well-structured mathematical problems; and to test the reasonableness of the problem-solving plans.)	To be able to comprehensively apply mathematical knowledge, methods and strategies to solve complex mathematical problems and explain the consistency of mathematical models, model results and reality To be able to reflect on solutions and strategies To be able to compare, evaluate and correct other people's understanding To be able to choose the best solution strategy according to the specific situation (Through the analysis and investigation of the problems, to be able to identify proper strategies and solve ill-structured problems, and to test the reasonableness of problem-solving plans.)

Table 7.3 Overview of test items of mathematical problem-solving competency

Item number	Content	Level of target competency	Situation type	Item type
1	Numbers and algebra: Pythagorean theorem	Level 1	Life situation	Problem solving
2(1)	Numbers and algebra: equation	Level 1	Life situation	Blank filling and problem solving
2(2)	Numbers and algebra: equation	Level 2	Life situation	Problem solving
3	Graphics and geometry: circumference and height	Level 2	Life situation	Open-ended question
4	Graphics and geometry: parallel	Level 3	Life situation	Problem solving

If the student's answer is correct, the code is 1 for the correctness dimension, and 0 otherwise. On the student's problem-solving strategy dimension, 1, 2, 3 and so on are used, respectively, to label the different strategies, and a brief description of the student's specific problem-solving behaviour is given, as shown in Table 7.4.

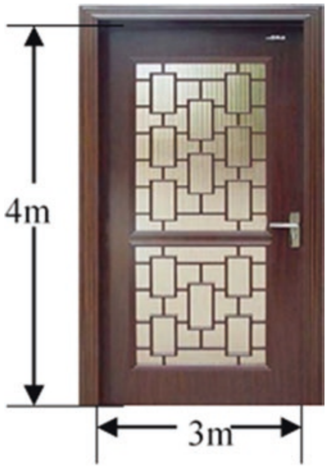


Fig. 7.1 Test item 1 The size of a door frame is as shown in the figure. Can a thin board with a length of 6 m and a width of 4.4 m pass through the door frame? Why?

7.4 Research Results for the Empirical Investigation

7.4.1 The Overall Level of Students' Mathematical Problem-Solving Competency

Mathematical problem-solving competency is the ability to analyse and extract problems in the materials that have been given, transform them into mathematical language, and then logically organize and reason, calculate and so on to find the solutions to the problems. The overall ability of participants is shown in Table 7.5. The number of students in target competency level 2 accounts for 76.0% of all participating students, while the number of students in target competency levels 1 and 3 accounts for 13.6% and 10.4%, respectively. Students' mathematical problem-solving competency reaches a medium level; that is, the majority of students are in level 2. In other words, most students can understand the meaning of the identified and selected mathematical information; relate knowledge and expressions in different mathematical fields (such as charts, words, symbols, etc.); briefly and logically express the process of thinking, solutions and results; and explain the meaning of their mathematical results to the situation on the basis of judgment.

A small percentage of students (10.4%) reach level 3, 'reflection and expansion'. This indicates that these students can use appropriate methods and strategies to solve problems based on understanding the relationship between general information and information in the problem, and can evaluate and explain the results of their own problem solving, showing the rationality, integrity, indirectness and harmony of the entire mathematical problem-solving process. A small number of students (13.6%) are at the level of 'memory and reproduction'; these students can only transfer specific situations into mathematical form, identify and select mathematical information from the description of mathematical problems, organize existing

Table 7.4 Scoring rules and coding examples

Score and code			
Correctness	Categories	Problem-solving strategies	Students' answers
1	1	Compare the diagonal of the door frame with the width of the thin board	The length of the diagonal of the door frame is $\sqrt{3^2+4^2}=5(m)$. Because the width of the thin board is 4.4 m, and 4.4 m < 5 m, the board can pass through the door frame.
0	0	Blank	
	1	No key problem-solving steps	Only a final answer, such as 'It can pass through the door frame' or 'It cannot pass through the door frame'.
	2	Compare the length of the thin board with the diagonal of the door frame	The length of the diagonal of the door frame is $\sqrt{3^2+4^2}=5(m)$. Because the length of the thin board is 6 m, and 6 m > 5 m, the board cannot pass through the door frame.
	3	Compare the diagonal of the thin board with the diagonal of the door frame	The length of the diagonal of the thin board is $\sqrt{6^2+4.4^2}\approx 7.4(m)$ and that of the door frame is $\sqrt{3^2+4^2}=5(m)$
	4	Compare the length and the width of the thin board with those of the door frame	Impossible, because both the length and the width of the board are longer than the length of the door frame.
	5	Compare the width of the thin board with the length of the door frame	Impossible, because the width of the board is 4.4 m, and 4.4 m >4 m.
	6	Compare the area of the thin board with the area of the door frame	Because $4 \times 3 = 12$ cm^2, $6 \times 4.4 = 26.4$ cm^2, position the board horizontally and it can pass through the door frame.
	7	Make the thin board pass by cutting or deforming it	For example, cut the board into small squares with a side length of 0.2 m, or bend the board into a certain arc so that the thin board can pass through the door frame.

Table 7.5 Percentage of participants' overall competency

Level 1	Level 2	Level 3
13.6%	76.0%	10.4%

Table 7.6 Average correct rate of student responses in the three target competency level tasks

Level 1	Level 2	Level 3
64.4%	54.5%	40.8%

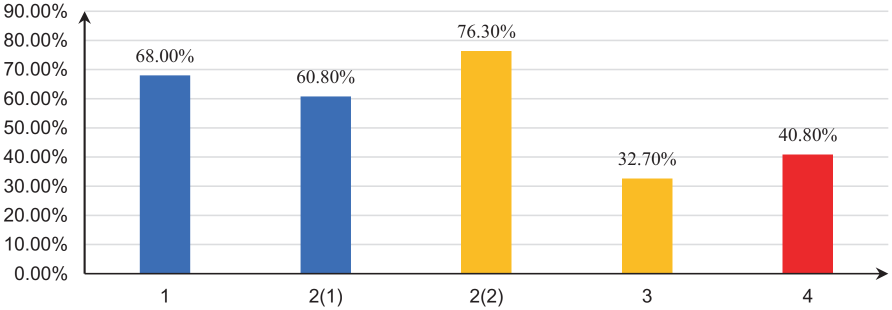


Fig. 7.2 Students' specific correct rate of each item

mathematical information, and express some simple mathematical facts in a clear way.

Next, there is a need to examine, in depth, the differences and existing problems of students at these three levels of competency to further analyse students' level of mathematical problem-solving competency and provide targeted advice and suggestions for mathematics education instructors and researchers.

7.4.1.1 Analysis of the Overall Correct Rate of Test Questions

As shown in Table 7.6, the average correct rates of students in target ability level 1 and level 2 is 64.4% and 54.5%, respectively, which are higher than the average correct rate of those in target ability level 3 (40.8%). In other words, as the target ability level increases, the rate of correct answers decreases.

The specific correct rate of each item is shown in Fig. 7.2.

In the 2(2) item of target level 2, students' correct rate (76.30%) is higher than for target levels 1 (68.00%) and 2(1) (60.80%). It is possible that the problem situation of the 2(2) item is more closely related to students' actual lives and is more likely to appear in daily homework and practice. Moreover, the setting of the question is relatively open, so students can try a variety of methods to get answers. Problem 2(1) requires the use of more stringent algebraic operations; thus the correct rate of student answers declines to some extent.

In addition, the correct rate of the fourth item in target ability level 3 (40.8%) is slightly higher than the correct rate for the third item (32.7%), which is lower than the target ability level. The explanation is similar to item 2(2). Students often encounter similar questions in their daily practice, so they are familiar with the methods to solve such problems, and the problem context is also similar to students' lives.

7.4.1.2 Overall Analysis of Students' Problem-Solving Characteristics

The code of each test item consists of two digits. The first digit indicates whether students answer correctly or not, and the second represents the diagnostic code of students' problem-solving characteristics (including problem-solving strategy, mathematical representation, error type, etc.). By analysing the diagnostic codes of each item, the characteristics of students' mathematical problem-solving competency can be obtained. The following four characteristics are presented in the whole process of students' mathematical problem solving.

- *Lack of mathematical organization ability of empirical materials.*

Level 1 of mathematical problem-solving competency, 'memory and reproduction', requires students to organize and reproduce real-life materials, solve simple mathematical problems with learned mathematical methods and strategies, and express simple mathematical problem-solving processes. The source of the test items selected is not only the empirical materials, but also test items that offer obvious mathematical information with relatively low difficulty. The results show that the situation is not optimistic.

For example, the 2(1) item involves shopping point redemption: 'Xiaohua has a total of 8,200 points, and 7,000, 2,000 and 500 points are required for the exchange of electric teapots, mugs and toothpaste, respectively. Xiaohua exchanges two kinds of gifts in the end. Which two gifts does he exchange and why?' The following answers emerged:

Student A: 'It may be a mug and toothpaste, because it will use as much points as possible.'

Student B: 'It may be four mugs, because four mugs are 8,000 points, so you can try to use up the points.'

Student C: 'Option 1, seven things are exchanged—three mugs and four toothpastes: $3 \times 2000 + 4 \times 500 = 8{,}000$; Option 2, three things are exchanged—one electric teapot, two toothpaste: $2 \times 500 + 7000 = 8000$.'

In an open-ended question like this, students are more likely to estimate and guess than to solve problems mathematically. All three students considered using as much of the 8200 points as possible, but they only offered their own guesses without giving all possible exchange options in a rigorous mathematical argument. The students simply organized their empirical materials based on their own life experiences and certain consumption concepts, whereas a more rigorous mathematical organization is lacking.

- *The logical organization ability of mathematical materials needs to be improved.*

Students often encounter three types of problems when they perform mathematical problem-solving activities. One is the problem presented by the actual situation, one is the combination of situational and mathematical descriptions and another is pure mathematical problems. Students will go through the same process when solving these three types of problems—logical organization of mathematical materials, which is also a very important requirement in the second level, 'connection and variation', of this test target level. After students organize or accumulate empirical materials from a mathematical perspective, they also need to abstract the original concepts and axiom systems and deductively establish theories based on these concepts and systems (Xu, 2013). Digging into the relationship between problem information and disorganized mathematical information or materials and then clarifying the solution to the problem is the ultimate goal of organizing mathematical materials. The results here suggest that some students are unable to clarify the relationship between different quantities and the amount of change in the face of extensive mathematical information; thus, their thinking is not clear enough in solving mathematical problems.

For example, in the first test item (see Fig. 7.1), 'Can a thin board with a length of 6 m and a width of 4.4 m pass through a door frame with a height of 4 m and a width of 3 m?', some students gave the following answers:

Student A: 'The diagonal length of the door frame is $\sqrt{3^2+4^2}=5$ m. Since the length of the thin board is 6 m, which is longer than the diagonal length of the door frame, the board cannot pass through the door frame.'

Student B: 'The diagonal length of the thin board is $\sqrt{6^2+4.4^2}\approx 7.4$ m, and the diagonal length of the door frame is $\sqrt{3^2+4^2}=5$ m. As 7.4 m > 5 m, the board cannot pass through the door frame.'

Student C: 'No, because the length and the width of the thin board are longer than the door frame.'

Student D: 'No, because the width of the board is 4.4 m, and 4.4 m > 4 m.'

Student E: 'Because $4 \times 3 = 12$ cm^2, and $6 \times 4.4 = 26.4$ cm^2, you can pass the board horizontally.'

All of the above students use the mathematical information given in the item, but there is a problem in their logical organization. They do not figure out which two key quantities must be compared, resulting in the final mathematical problem being answered incorrectly.

The same problem also appears in the third test item: 'Wrap a neon hose around a tree, four rounds on each tree. The cross-section perimeter of the tree is about 0.6 m, and the winding height is 2.5 m. Please calculate the length of neon hose needed for each tree, and write down the specific calculation process.' It appears that students only consider the number of windings regardless of the height of the tree, or simply process mathematical information without logic, as they cannot figure out the related properties of the planar graphics after the three-dimensional graphics are expanded.

- *The application level of mathematical theory is not high.*

Mathematics is a discipline that requires a high degree of abstraction. This plays a decisive role when there are practical problems to be solved. In solving some seemingly non-mathematical problems, we need to first translate them into mathematical language and turn the problem into mathematical problems, thus solving the problem in the mathematical world. At this stage, students need to use mathematical theory to think, analyse and solve complex and ever-changing problems in real life. The third level of mathematical problem-solving competency—'reflection and expansion'—puts high demands on students solving complex mathematical problems under the guidance of mathematical knowledge, methods and strategies, and requires students to reflect, compare, evaluate and correct others' understanding, and to choose the best solution according to the specific situation.

In this study, students' application level of mathematical theory is not high, as is shown in the following aspects.

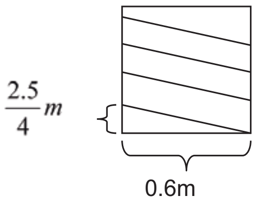
First, students tend to use arithmetic methods when they haven't reached a deep understanding of mathematical problems. In this study, students mainly adopted algebraic methods to solve these four real-life problems. This is especially true of the fourth item, in which only a small number of students used arithmetic methods. This is because students have transformed their focus from arithmetic thinking to algebraic thinking since the fifth grade; thus, the eighth-grade students are already very familiar with using algebraic methods to solve problems. At the same time, we find that, when students cannot solve the problem because they do not understand the meaning of the test item, they rarely choose the algebraic method, but use the arithmetic method to simply add, subtract, multiply or divide the numbers appearing in the test item—see, for example, the third test item. Because students cannot sort out the relationship between the various quantities in the item, they operate without understanding the meaning of the operation at hand.

Second, the selection effectiveness of students' mathematical problem-solving strategies is insufficient. The third test item is more abstract, requiring students to calculate the length of the neon hose around the cylinder. This question requires students to translate the three-dimensional graphics problem into a more intuitive problem. Most of the students use the problem-solving strategy of finding the length of one circle plus the height of the circle, and finally multiplying by 4, as shown in Fig. 7.3.

Solution: It can be obtained from the side view of the trunk.

The length of the neon hose is

$$4\sqrt{\left(\frac{2.5}{4}\right)^2 + 0.6^2} = \frac{\sqrt{1201}}{10} \approx 3.47\ (\mathrm{m})$$



Answer: A neon hose of approximately 3.47 m is required for the tree.

Fig. 7.3 Test item 3

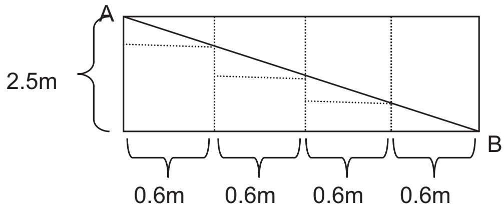
Test item 3:  Wrap a neon hose around a tree, four rounds on each tree. The cross-section perimeter of the tree is about 0.6 m, and the winding height is 2.5 m. Please calculate the length of neon hose needed for each tree, and write down the specific calculation process.	Re-splicing the side view $AB = \sqrt{(4 \times 0.6)^2 + 2.5^2} = \frac{\sqrt{1201}}{10} \approx 3.47\ (m)$ Solution: From the meaning of the title, AB is the request $AB = \sqrt{(4 \times 0.6)^2 + 2.5^2} = \frac{\sqrt{1201}}{10} \approx 3.47\ (m)$ Answer: A neon hose of approximately 3.47 m is required for the tree. 

Fig. 7.4 Problem-solving strategy for test item 3

Although this method is rated as correct, the result is not accurate. It does not consider that each circle around the cylinder is not a straight line, but a diagonal line, so it can only be regarded as an estimate. This shows that the selection effectiveness of students' mathematical problem-solving strategies is not enough.

Third, students have a negative transfer of mathematical knowledge in the process of solving mathematical problems. The first item of this test involves the application of the Pythagorean Theorem, and the third item needs to be transformed and then applied to the Pythagorean Theorem. Many students use the problem-solving strategy shown in Fig. 7.4.

This problem-solving strategy requires a high level of cognition. It involves two transformations to make the mathematical problem intuitive to use the Pythagorean Theorem more accurately. However, the study finds that, due to the negative transfer of the knowledge of Pythagorean Theorem, students make mistakes in the process of solving mathematical problems. After the students simply convert the cylindrical model into a flat figure, they do not find the correct right-angled triangle, so the Pythagorean Theorem is used incorrectly.

- *The awareness of rethinking and reusing after solving mathematical problems is weak.*

An important indicator of mathematical problem-solving competency is that students have the awareness of rethinking and reusing. From the point of view of students' mistakes in solving problems, the problem that most students have in common is the lack of reflection on the results of their work. Reflection allows students to find out whether the problem-solving strategies and calculations they used are correct. If, in the reflection stage, students understand that they cannot explain the meaning of the equations or expressions listed, they will find errors in their problem-solving process. For example, in the first item of the test, many students consider the thickness, length and width of the door, which are not factors that determine whether the thin board can pass through the door frame. If students can reflect on the meaning of their listed expressions and item requirements after solving the problem—and by analysing and comparing the methods used, reveal the ideas and methods contained in them, as well as their respective characteristics and scope of application (Zhang, 2008)—such errors can be easily avoided.

In addition, the accumulation of mathematical problem-solving methods and the re-use or transfer in other situations are the most direct purposes of students' problem solving. The purpose of problem-solving instruction is not only for students to solve problems, but also to cultivate their way of thinking in doing so. This way of thinking enables students to use the existing problem-solving strategies and ways of thinking to process and transform problems in different situations and problem presentations, from unknown to known. The four test items used in the study involved the knowledge that the students had already learned, but from the perspective of problem solving, the students' use of existing knowledge is not ideal. For example, the fourth item in the test is shown in Fig. 7.5.

The students' problem-solving errors are shown in the following three aspects: (1) students cannot apply the mathematical knowledge of 'parallel projection'; (2) students can only use the relevant knowledge of similarity to calculate the complete length of the tree shadow of the entire tree, and cannot reuse the existing mathematical knowledge; and (3) students incorrectly use mathematical knowledge and problem-solving strategies they assume are 'right', but that are actually far from the requirements of the test item.

Test item 4:
Someone wants to use the shadow of a tree to measure its height. He measures the shadow length of a 1.2-meter-long bamboo pole as 0.9 m at a certain time of the day. However, when he tries to calculate the height of the tree, the tree's shadow is not all on the ground, as the tree is close to a building and part of its shadow projects on the wall. He measures the tree's shadow's length on the ground as 2.7 m, while the one on the wall is 1.2 m. How tall is the tree?

Fig. 7.5 Test item 4

7.5 Conclusion

7.5.1 Overall Description of Students' Mathematical Problem-Solving Competency

The results of the research show that nearly 80% of the eighth-grade Chinese students participating in the study have reached level 2; that is, students can relate knowledge and expressions in different mathematical fields (such as charts, texts, symbols, etc.); they can express the thinking process, the solution and the result in a brief and logical way; and they can explain the meaning of their own mathematical results on the basis of judgment.

From the perspective of the rate of correct responses, as the level of target ability rises, the correct rate of students' answers gradually decreases. However, in some test items, the pattern is reversed; this is likely because the contexts of the test items and students' daily lives are similar, due to students encountering these issues in daily work and practice, and the open-ended nature of the problems is high. Students perform noticeably well in problems emphasizing basic mathematical knowledge and skills, and they perform well in problems that need to be calculated in arithmetic methods; nevertheless, they perform poorly in open-ended and unconventional problems. These results are similar to those of relevant international mathematics education research (Cai, 2000; Cai & Cifarelli, 2004; Cai & Hwang, 2002; Chen et al., 2002).

7.5.2 Analysis of the Characteristics of Students' Mathematical Problem-Solving Competency

Chinese students perform well in international large-scale evaluation projects, such as PISA and TIMSS, which have received attention from mathematics educators around the world, but their performance in problem solving is slightly worse. In PISA 2012 and PISA 2015, Chinese students outperformed their international counterparts in terms of scores in mathematics achievement, but they lagged behind students from other East Asian countries in problem solving, with scores at least 12 points lower (Cao & Leung, 2018). Some researchers (Leung, 2005) pointed out that, in the mathematics classroom in Hong Kong, the problems that teachers ask students to solve are complicated. Students are expected to use problem-solving methods to answer questions that are not related to life—questions that are mainly about procedural applications. This reflects that there is a certain degree of deficiency in the classroom teaching of mathematics problems in China when it comes to solving complex, open-ended and unconventional problems.

This study also reflects on some of the problems of the eighth-grade students in China in mathematical problem-solving competency. First, very few students participating in the assessment lacked the mathematical organization ability of the empirical materials when answering target level 1 test items, as they merely relied on life experience to guess and organize related materials in mathematics problems; thus, a more rigorous mathematical organization was lost. Some scholars point out that 'partial empirical understanding often restricts people's general understanding of meaning' (Hu, 2011). Barely relying on partial life experience will restrict students' more general and comprehensive understanding of mathematics problems.

Second, some students' logical organization ability of mathematical materials needs to be improved, as they only have a shallow understanding of the constants, variables and relationships between them, and some even make mistakes. Lester (1994) points out that good mathematical problem solvers' knowledge is well connected and composed of rich schemata, and they can notice the structural features of mathematical problems rather than only their surface features. Students participating in the assessment had problems with the logical organization of mathematical materials, as there was a deficiency in students' mathematical knowledge, a loose relationship among mathematical knowledge, and a shallow understanding of the essential structural features of mathematical problems.

Third, students' application level of mathematics theory is not high, which is manifested in the following three aspects: (a) students tend to use arithmetic methods when they cannot fully understand mathematical problems; (b) students are not effective in choosing mathematical problem-solving strategies; and (c) students have a negative transfer of mathematical knowledge in the process of solving mathematical problems. These problems reflect Grade-8 Chinese students' lack higher-order thinking ability. In the transition from arithmetic thinking to algebraic thinking, students' ability of identifying and selecting mathematical problem-solving strategies needs to be guided and cultivated, while their basic mathematical knowledge and skills need to be further consolidated.

Fourth, students' awareness of rethinking and reusing after solving mathematical problems needs to be strengthened. Most of the students involved in the assessment failed to reflect on the results of the problem. If students can reflect on the meaning of their expressions and the requirements of mathematical problems after solving them by analysing and comparing the methods used, they can find errors and correct them before the problem-solving process is completed. In addition, students also lack the ability to transfer and reuse mathematical problem-solving methods in other situations, which is manifested in mistakenly using the mathematics knowledge and problem-solving strategies that they assume are correct and the inability to flexibly use different effective problem-solving strategies and ways of thinking in different problem situations. As is pointed out by some researchers, 'promoting learners to develop a deeper understanding in mathematical contexts' is a feature of students' mathematical thinking (Kieren & Pirie, 1991). Students can promote the development of mathematical higher-order thinking by rethinking after solving problems and reusing problem-solving methods and strategies in different situations.

7.6 Discussion

The findings presented in this chapter indicate that there is still a gap between the current status of students' mathematics problem-solving competency and the curriculum text requirements in China. With the development of mathematical problem-solving competency in the curriculum, the requirements of students are also gradually improving, from the initial simple acquisition of mathematics knowledge for basic life skills; to the ability to comprehensively apply mathematical knowledge and skills based on operation ability, logical thinking ability and spatial imagination; to the current multifaceted ability containing many key competencies. In 2012, the Ministry of Education of China released the '*Mathematics curriculum standards for compulsory education (2011 version)*'; students are required to 'preliminarily learn to find problems and pose questions from the perspective of mathematics, comprehensively apply mathematical knowledge to solve simple practical problems, enhance application awareness, improve practical ability'; they also are expected to 'acquire some basic ways to analyse and solve problems, experience the diversity of problem-solving methods, develop innovative consciousness', 'learn to communicate with others', and 'preliminarily form the consciousness of evaluation and reflection' (Ministry of Education of China, 2012). The results of this assessment show that the vast majority (nearly 80%) of the Grade-8 students in China are at level 2; that is, they can 'relate knowledge and expressions in different fields of mathematics'; 'simply and logically express thinking processes, solutions, and results'; and 'explain the meaning of their own mathematical results to the situation on the basis of judgment'. However, problems also exist, such as a poor ability to abstract practical problems into mathematical problems, a low level and poor feasibility of strategies in the comprehensive use of problem-solving strategies, and a weak ability to compare, evaluate and reflect on problem-solving results.

In the future, in the relevant expressions of mathematical problem-solving ability in the mathematics curriculum in China, we need to further clarify the connotation of mathematical problem-solving competency, determine the development requirements of mathematical problem-solving competency of students in different learning phases and grades, and come up with an operational evaluation indicator and framework that identifies mathematical problem-solving competency.

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Chapter 8
The Development of Representation in Chinese Mathematics Curriculum

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Abstract Representation is an important concept in cognitive psychology and is the way in which knowledge is presented and expressed in the learner's mind. Mathematical representations help students understand concepts, relate the mathematics used in their own problem-solving processes, and grasp connections between mathematical concepts. The purpose of this study is to investigate how the three functions of mathematical representations have changed in the mathematics syllabi of Chinese primary and secondary schools since 1902. Content analysis was the primary approach used in the study. Results show that the functions of mathematical representations change with the syllabi to suit the needs of the times. The focus and interrelatedness of the three representational functions also vary across school years. The representational communication function relies on the model application function at the elementary school level and on the operation transformation function at the middle school level, but the model application function does not receive much attention at the elementary and middle school levels. Therefore, future curricula should pay more attention to the balance of the three functions across school years.

Keywords Functions of Mathematical Representation · Chinese Syllabus · Content Analysis · Representation Categories · Expressive Communication · Operational Transformation · Model Applications

8.1 Introduction

Abstraction is a core feature of mathematics. A single mathematical concept often has multiple possible representations. As a part of mathematical processes, it is important for learners to deepen their understanding of mathematical concepts through the interpretation and transformation of external representations (mathematical symbols, tables, images, models, etc.) of the same structure (isomorphisms), developing and establishing connections between the abstract mathematical world

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and their real, experienced lives (Even, 1998). *An Action Plan for Mathematics Education in Primary and Secondary Schools in the twenty-first Century* (Gu et al., 1997) further expands the requirements for basic mathematical skills for schools in China. These skills are no longer confined to what are commonly referred to as computational skills, logical reasoning skills, and spatial imagination skills. They also include the abilities to make mathematical abstractions, symbolic transformations, and mathematical applications. Specifically, the ability to make mathematical symbolic transformations includes quantitative computations, logical deductions, empirical inductions, and even spatial associations, all of which are the basic methods used in mathematics.

By synthesizing related research on mathematical representation and transformation abilities and based on *Collection of primary and secondary school curriculum standards and syllabus of the twentieth century China (Mathematics volume)* (Curriculum and Teaching Materials Research Institute (CTMRI), 2001), this study explores the historical evolution of mathematical representations along with mathematics curriculum reforms, identifying patterns and implications for future practice.

8.2 Literature Review

Thanks to the efforts of mathematics education researchers and practitioners, research on the concept of mathematical representations has been successfully completed, resulting in theoretical and practical breakthroughs. Markman and Dietrich (2000) argues that cognitive science employs so many different kinds of representations that it would be nearly impossible to provide a complete overview of all of them. Therefore, this paper does not attempt to provide a complete overview of mathematical representations, but rather a systematic account of what mathematical representations mean in relation to mathematical representational abilities.

Representation is a central concept in the development of cognitive psychology. Since the "cognitive revolution" in the 1950s, researchers in cognitive science have attempted to describe the nature of information processing in the brain in terms of the processes by which representations occur. According to Ralmer (1978), representation is a process of cognitive activity: how people construct, combine, and represent in the brain what they have learned. Representation is also the result of cognitive activity: the form in which knowledge or information is stored in the brain. Thus, representations are the combination of process and outcome. Hiebert and Carpenter (1992) divided representations into internal and external representations from a cognitive psychology perspective. Internal representations refer to mental representations that exist in the learner's mind that cannot be directly observed or mental structures that the learner possesses, such as knowledge networks and objects that the individual constructs in the mind; external representations refer to observable presentations of objects in the form of words, symbols, graphics, concrete operations, or actual situations. Eysenck and Keane (2005) describe representation as "a form (physical or mental) of a symbol or a set of symbols that can repeatedly refer to an object; the re-presentation of things, objects, ideas, knowledge in a new form, which essentially establishes a mapping between the domain of 'representation' and some feature or element of the domain of 'represented'."

Around the same time as the beginning of the cognitive revolution, many researchers began to slowly shift their attention to the meaning of mathematical representations. Greeno and Hall (1997) analyzed the representational behavior in mathematical problem-solving activities from the perspective that cognitive individuals or learning groups draw graphs, take notes, construct tables or equations in the process of solving problems. These representations help learners understand mathematical concepts and theorems, draw valid conclusions, and optimize ongoing logical thinking processes. Goldin and Shteingold (2001), through a review of research on representations in the field of mathematics education, argue that extra mathematical representations are external forms that reflect the objects of mathematical learning. These include mathematical symbol systems as well as specific structured learning contexts. Internal representations of mathematics are conceptual processes that make connections between different external forms of mathematical objects, and it is this conceptual process that is the true meaning of representation in mathematics. Cai and his colleagues (2005) summarized the meaning of mathematical representations in their study of problem solving and pedagogy. They found that mathematical representations exist in the process of expressing mathematical objects or relationships, and that the use of appropriate forms of mathematical representations helps students understand the concepts, relationships, and mathematical knowledge used in problem solving. To gain a deep understanding of a new mathematical object, a student must establish a mapping between the structure in which the object appears and another more understandable structure, and mathematical representations are such mapping processes. A mathematical representation is neither the object of the representation (the mathematical structure being represented) nor the purpose of the representation (the more comprehensible mathematical structure); it exists within this mapping activity as a "component" that contains the transformation of objects into other objects.

Although the above definitions of mathematical representations are different, each includes a mapping process from what is represented to what is used to represent it. Based on the essential elements common to these definitions, we characterize representation as the process of expressing mathematical concepts or relations in some way, and as the process by which an individual establishes a mapping between the structure of the problem to be learned or dealt with and another more understandable structure of the problem.

8.3 Research Questions and Methodology

8.3.1 Research Questions

This chapter is primarily a study of the following questions: What are the changing trends in mathematical representations in primary and secondary school mathematics curriculum standards in China in the 20th and 21st centuries, and what are the main causes of such changes?

The main research method used for this study is "historical research." The goal of a historical research is to systematically search and organize data to better understand historical phenomena and their possible causal relationships (Gower, Gower, Borg, & Hou, 2016). The process of a general historical research includes three steps: screening historical sources, examining historical evidence, and interpreting historical data. Because China is a centralized state, the mathematics syllabus promulgated by the state and the test syllabus that it adapts to enjoy are viewed as having a high degree of authority. As the historical sources for this paper are *Collection of primary and secondary school curriculum standards and syllabus of the twentieth China (Mathematics volume)* and the curriculum standards implemented after the year 2000, all of which are unified standards issued by the state, the sources were deemed to have a considerably high level of relevance and reliability. Therefore, only the third step in the general historical research process was performed, the interpretation of historical data. A combination of qualitative and quantitative approaches was used during this step. The quantitative aspect included the interpretation of data based on a coding framework, while the qualitative aspect included the explanation of quantitative research results. This method is called sequential-explanatory research design (Manion, Cohen, & Morrison, 2011). Through this research, we hope to identify patterns in the development of mathematical representation in the school curricula in China during different eras and in different subject areas and to promote a better understanding of the new curriculum standards and subsequent revisions.

8.3.2 Research Design

8.3.2.1 Research Objects

The main objects of this research include middle school mathematics curriculum standards and syllabi used in China from 1902 to the present. The mathematics curriculum documents from 1902 to 2000 were selected from *Collection of primary and secondary school curriculum standards and syllabus of the twentieth China (Mathematics volume)*, which was edited by the Curriculum and Teaching Material Research Institute of People's Education Press, while the curriculum documents for after 2000 were selected from the *Mathematics curriculum standards for full-time* *compulsory education (Experimental version)* (MOE, 2001) promulgated in 2001, as well as the *Mathematics curriculum standards for compulsory education (2011 version)* (MOE, 2012).

8.3.2.2 Framework of Representation

Principles and Standards for School Mathematics (NCTM, 2000) of the United States characterizes competency in mathematical representation into three core abilities: "to create and use various mathematical representations to organize, record, and communicate mathematical concepts"; "to select, apply, and interchange various mathematical representation methods to solve problems"; and "to apply representations to simulate and explain phenomena in physics, society, and mathematics." Niss (2003) identifies mathematical representation as one of eight mathematical competencies and enumerates three required abilities for representing mathematical entities (objects and situations): to understand and use (decode, interpret, distinguish) mathematical objects, different kinds of phenomena-based and situations-based representations; to understand and use the relationships between different representations of the same entity, including understanding their relative advantages and limitations; and to select and switch representation as appropriate. The mathematical evaluation framework of PISA 2018 describes various aspects of representation as part of the fundamental mathematical capabilities underlying three basic mathematical processes, specifically, mathematically expressing the situation; using mathematical concepts, facts, procedures and reasoning; and explaining, applying and evaluating mathematical results (OECD, 2019).

Mathematical representations have rich and complex connotations (Brachman & Levesque, 2004) and are generally defined from two perspectives: the cognitive perspective of mathematical knowledge and the functional perspective of mathematical representation. The cognitive perspective mostly focuses on three aspects: internal representation, external representation, and representation transformation (Zhang, Jiang, & Xie, 2016). Internal representation refers to psychological structure of an individual's knowledge learned, which is generally not easily observed (Bao & Zhou, 2009). For example, psychologists believe that information is stored, represented, and reproduced in a person's working memory and long-term memory through knowledge representation (Yu, 2004). External representations are physical and observable behaviors or objects. The mathematical education encyclopedia describes mathematical representations as visible and tangible products, such as charts, numbers, graphs, concrete objects or aids, physical models, mathematical expressions, descriptions on computer screens or computer coding, etc., which represents mathematical ideas and mathematical relationships (Lerman, 2014). The notion of representation transformation was first introduced by Lesh, Post, and Behr (1987). Niss (2003) later emphasized its role, noting that mathematical representation ability refers to the understanding, interpretation, and identification of multiple representations of mathematical objects, phenomena, and situations; the ability to understand and use the relationships of representations and the grasp of advantages and limitations of those different representations; as well as the ability to select and transform various representations. Xu (2013) and other researchers (Zhang et al., 2016) define mathematical representation and transformation abilities separately. They suggest that the goal of mathematical representation is to express mathematical concepts or relationships so that they can be learned or processed in some form, such as written symbols, graphics (tables), situations, operational models, and words (including spoken words), etc. The ultimate goal is to solve the problem in hand. Mathematical transformation ability refers to maintaining some invariant nature of a mathematical problem while changing the information form in the process of solving mathematical problems, mathematically transforming the problem to be solved.

From a functional point of view, the most inclusive definition for mathematical representation comes from NCTM in the United States, which added the standard of representation for the first time in the *Principles and Standards for School Mathematics* published in 2000. Bao and others (Bao & Zhou, 2009) summarized two functions for representation: communication tools and materials for thinking. As a communication tool, a specific form of representation is used to describe experiences in an activity; as a material for thinking, a representation can be used to denote the mathematical concept of materialization or the inherent activity type and perform thinking operations on the meanings manifested by the representations.

"Pragmatism" is not only a great tradition of Chinese mathematics, but also an important value choice for the study of mathematics teaching in China (Cao & Leung, 2018). Due to the requirements of the subject, this chapter needs a compilation of all school stages and content sections (here only for arithmetic and algebra) in the *Collection of primary and secondary school curriculum standards and syllabus of the twentieth China (Mathematics volume)* (primary school, junior high school are included in the first and 11th editions). From the perspective of representation functions, we analyzed the contents for teaching and explored the development of arithmetic, algebraic, and geometric representations in primary and secondary schools in the twentieth century, with the hope that the results could shed light on the current status and future development of mathematics representations. The functions of the two representations summarized by Bao in the above literature include expressive communication and operational transformation. The functions of the model applications are also described in NCTM (2000) publications. Combining these perspectives, this paper differentiates among three aspects of representations based on their functions: expressive communication, operational transformation, and modeling applications. Expressive communication refers to the use of symbols (written symbols, oral expression, etc.) and graphical charts for communication. Operational transformation refers to calculations (measures) and logical thinking such as reasoning or model operations on physical devices (e.g., an abacus, computer, or calculator). Model application refers to the use of mathematical knowledge to solve problems in real life situations.

Table 8.1 is a framework for the functions of mathematical representations which encompasses three levels of indicators. The first level is the functions of representations, including expressive communication, operational transformation, and model

Table 8.1 A framework for the functions of mathematical representations

Functions of representations	Connotations	Samples of representations
Expressive communication	Symbolic expressions	Written symbols (scores, decimals, notation), spoken expressions (reading, numbering)
	Graphical charts	Graphic (function image, floor plan) chart (multiplication ninety-nine table, prime number table, anti-log table)
Operational transformation	Thinking operations	Computation/metrics (four operations, solving equations, time calculation, currency calculation), logical reasoning (cosine theorem, mathematical induction, Newton's formulas)
	Model operations	Physical devices (abacus and logarithmic slide rule, computers, calculators, computer simulations)
Model applications	Situation application	Situational problems (taxation problems, interest problems), practical applications (internship assignments, research topics)

applications. The second level shows subcategories of the functions of representations: expressive communication includes symbolic expression and graphical diagram, operational transformation refers to thinking operation and model operation, and model application is situation application. The third level is a more detailed description of keywords based on *Collection of primary and secondary school curriculum standards and syllabus of the twentieth China (Mathematics volume)* as well as indicators at the second level.

8.3.2.3 Content Analysis Method

This research adopted the content analysis method, and its coding process can be divided into three stages: (1) preliminary formation of coding table, (2) validity test, and (3) use of coding table and reliability analysis. In the first stage, researchers established the three-level indicators of representation and their corresponding connotations by reviewing literature and formed a coding table using the analysis framework. Then two members of the research team used the same set of content from 3 years of geometry syllabi to analyze the content of teaching, find the corresponding keywords in the text, review the connotation to make further adjustments to the keyword sample, and establish a preliminary yet consistent coding table and keyword selection principle (e.g., select the most appropriate classification according to the connotation and examples of the analysis framework. If a keyword has multiple meanings, assign at most two sample classifications). In the second stage, a third member of the team who had not participated in the first stage checked the validity of the coding table previously established by the two members by randomly selecting and analyzing a content that had already been coded. If the coding accuracy for the same content is less than 80%, indicators at the third level in the analysis framework were further revised and coded by the first two members, and the table was checked by the third member until the coding was validated. Once the table was deemed valid, data analysis entered the second stage. In the third stage, one researcher used the coding table to search, classify, and count keywords throughout the geometry content. Another researcher randomly selected more than three course standards, performed word frequency statistics on the arithmetic, algebra, plane geometry, and solid geometry coding tables, and conducted reliability analyses based on these tables.

The left half of Fig. 8.1 is a flow chart for designing the coding framework for the functions of representations. First, three members of the research team each individually read and coded the text. After several rounds of discussion, the first-level indicators were unified, the members re-encoded based on the determined first-level indicators, and the codes were ultimately determined after further discussions. The right half of Fig. 8.1 is a flow chart of the coding process. First, the fourth indicator in the coding framework was determined, and part of the content of the file was coded, then re-coded after multiple discussions. Finally, some of the year codes were randomly selected and a reliability analysis was performed. The fourth indicator was a keyword extracted from *Collection of primary and secondary school curriculum standards and syllabus of the twentieth China (Mathematics volume)* based on the third indicator and varied according to the grade and knowledge content (*Arithmetic and Algebra, Planar Geometry and Solid Geometry*). The data derived from coding was the number of times that the fourth indicator keyword was mentioned in the text. During the coding process, in order to fully demonstrate the mathematical representations in each year, the following principles were followed: if there were multiple schemes in a certain year, first scheme was followed; if the syllabus for a certain year included content for both five-year systems and six-year systems, the content for six-year system was used. If there was a division between liberal arts and sciences in a certain year, sciences were used. If there was a division between advanced standard and lower level standards, the syllabus for the advanced standard was used. The minimum unit of coding was an independent phrase, which typically ended with a comma; if a word in the coding standard reference table had two meanings, the code was counted twice, but not more than twice.

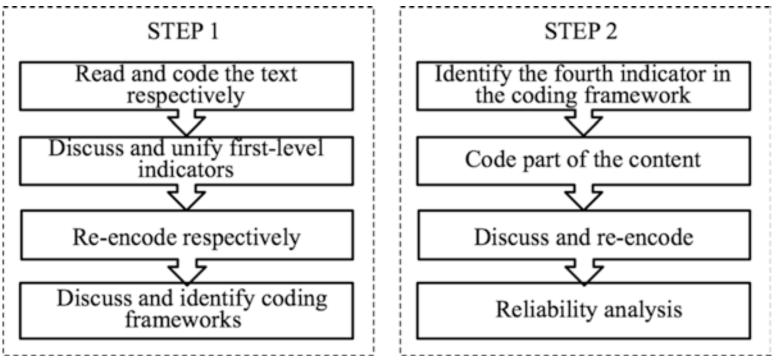

Fig. 8.1 Flow charts for designing the coding framework for representation function

Table 8.2 Paired sampled t-test statistics table for representation function coding

School level	Correlation coefficient	Sig.	Pairwise difference					*t* value	Sig. (two sides)
			Mean	Standard deviation	Standard error of mean	95% confidence interval for difference			
						Lower limit	Upper limit		
Primary school	0.966	0.000	−0.35354	3.42057	0.34378	−1.03576	0.32868	−1.028	0.306
Junior high school	0.995	0.000	−0.84615	2.89772	0.46401	−1.78548	0.09318	−1.824	0.076

The final step of the coding process was to perform a differential test on primary and middle school standards to determine whether the coding was valid. First, 6 years were randomly selected from different school sections to be re-coded, and SPSS was used to perform paired sample t-test between the obtained data and the data corresponding to the original codes. The correlation coefficients for primary and middle are shown in Table 8.2:

Table 8.2 shows that the correlation coefficient between the randomly selected codes and the original codes in the two school levels is relatively high, and the significance level is less than 0.05, which means the two sets of coded data are significantly correlated, and the significance levels of the t-test values are both greater than 0.05, which means no difference was considered to exist between the two coding processes. Therefore, the coded data is trustworthy.

8.4 Research Results on Conceptual Development

8.4.1 Primary and Junior High School (Arithmetic and Algebra)

As shown in Figs. 8.2 and 8.3, the line charts for representation in primary school and junior high school arithmetic demonstrate very similar trends. On the whole, the measure of expressive communication functions at the primary school level changed along with the measure of model application functions and tended to change along with the measure of operational transformation at junior high school level. This indicates that the training of students' communication skills relied more on practical applications in primary schools and relied more on thinking operations and model operations in junior high schools. The time periods can be roughly divided into three stages. (1) Before 1923, education in China witnessed the introduction of moral education and Dewey pragmatism, the functions of representation remained almost unchanged. (2) Between 1923 and 1978, foreign educational concepts were introduced, the educational system in Soviet Union was studied and adapted, and "the two basics" (basic knowledge and basic skills) became the center of attention. This period saw two relatively strong fluctuations, especially in the operation transformation function. (3) After 1978, basic education reform with Chinese characteristics was steadily implemented, and the representation function gradually stabilized.

8.4.1.1 The First Stage (Before 1923)

Modern history in China before 1923 can be divided into two periods: the Qing Dynasty (before the Revolution of 1911) and the Republic of China established in 1912. During the first period, only the *School Education System in the Year of Renyin* (1902) and the *School Education System in the Year of Kuimao* (1903) were

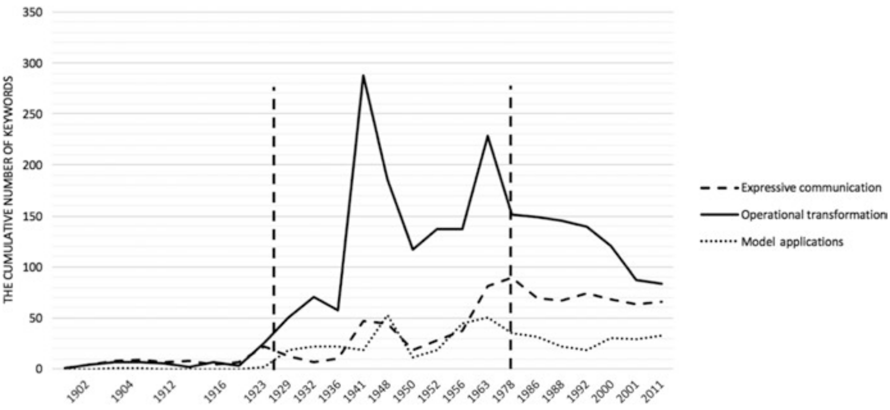

Fig. 8.2 Line chart showing functions of representation in primary school arithmetic

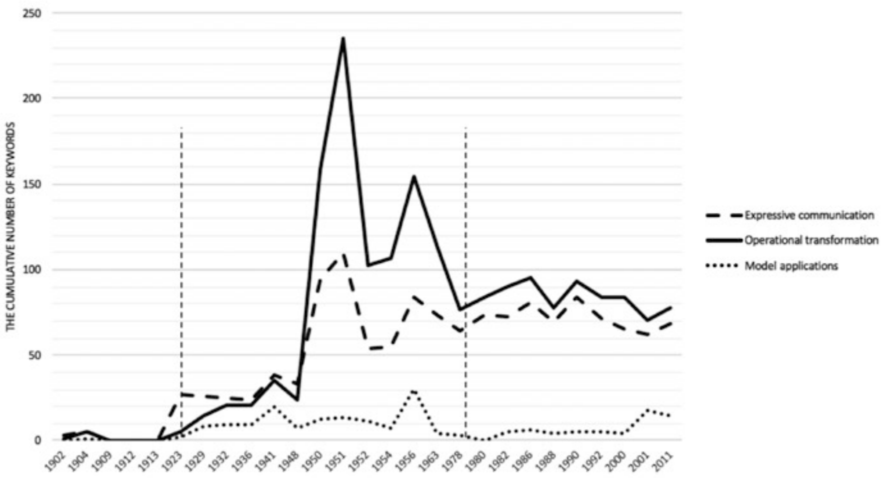

Fig. 8.3 Line chart showing functions of representation in junior high school arithmetic

enacted, and representation was not significantly present in the mathematics curriculum. At the beginning of the second period, in the years immediately following 1912, the Ministry of Education was established with the aim of "paying attention to moral education, supplemented by practical education, military national education, and aesthetic education to complete its morality" (Wei, 1987). During the First World War, in 1914–1918, imperialist forces temporarily relaxed their economic aggression against China. The "May 4th" Movement broke out in 1919, leading to an unprecedented educational reform movement in the country, contributing to the development of education nationwide (Wei, 1987). During this period, the educational philosophy of the great American educator J. Dewey (1859–1952) were introduced to China through the efforts of Yuanpei Cai and Yanpei Huang, and many

Chinese scholars such as Xingzhi Tao, Xiaotong Zheng, and Shi Hu actively promoted Dewey's theories. During his visit in 1919, Dewey gave lectures all around China, and his pragmatic education theory was widely known in China (Wang, 2009; Zhang, 2018). For example, Dewey's pragmatism was the guiding ideology of *School Education System in the Year of Renxu* announced by the Ministry of Education of the Republic of China in 1922. It was a modern school system based on the American school infrastructure, also known as the "six-three-three" school system. Before 1923, although there were no changes in the three functions of mathematical representations, these other developments paved the way for changes in the latter periods.

8.4.1.2 The Second Stage (1923–1978)

The events of 1952 were critical for both primary and junior high schools in China, with significant changes occurring both before and after. Hence, we divide this stage into two periods. In the first period (1923–1952), various teaching theories from abroad and new development from international mathematics education reforms in the early twentieth century were first introduced. In the second period (1952–1978), the education system in the Soviet Union was comprehensively and systematically studied, beginning in 1952. It was proposed that China should "learn from the advanced experiences of the Soviet Union, first moving it over and then re-Sinicizing it" (Cao & Leung, 2018), and the importance of basic knowledge and basic skills was emphasized in the national syllabus in 1963.

At the beginning of the first period (1923–1952), the 1923 syllabus was formulated on the basis of the *School System Reform Order* (Curriculum and Teaching Materials Research Institute, 2001), in which expressive communication, operational transformation, and model application functions of mathematical representation began to be valued. The War of Resistance Against Japan broke out in 1936, a year after the publication of the revised curriculum standards, and educators in China had to adjust the content to meet wartime needs. The document *Design of the Main Items of the Various Subjects in the War of Resistance Against Japan* indicated that problems unrelated to students' lives should be omitted; this change was mainly based on the practicality of national defense, using digital and military calculations on national defense as the context for problems. For example, the "engineering problem" in the mathematics application section was changed to "digging trenches" and "ship power" (Wei, 1987). Therefore, all three functions were seriously considered at the time, but the operational transformation function was more significant. It is possible that, because the level of knowledge required at that time was low, the operational transformation function increased significantly in 1941 for primary and secondary schools, while for junior high schools, it only increased slightly. After the People's Republic of China was founded in 1949, the country began to learn from all aspects of the Soviet Union's education system. The national curriculum syllabus was written based on the Soviet Union's ten-year school education system (Lv, Wu, & Chen, 2009). China adopted a teaching system that was "knowledge-centered," "classroom-centered," and "teacher-centered." Chinese students became overburdened, in part because physics, chemistry, and mathematics textbooks included too many topics, and the arrangement was not consistent with the students' perceptions (Wei & Zhang, 1996). To resolve the issue, the Ministry of Education organized a symposium and developed the following principles: (1) the purpose of streamlining contents is to pursue effective teaching, rather than to lower expectations on student learning; (2) mathematics textbooks should be related to the real world as much as possible and combined first with the learning of physics and chemistry, and then with the scientific knowledge required for economic development (Wei & Zhang, 1996). The three functions represented in the "1952 Outline" of elementary and junior high school education reduced requirements and, for the first time, mentioned improving students' basic knowledge and skills through training and practice, emphasizing the systematic and logical characteristics of mathematics.

In the second period (1952–1978), China underwent the "great leap forward" movement, after which the Central Committee of the Communist Party issued the "Guidelines for Educational Endeavors" and launched a national educational reform movement that focused on the academic system and content for teaching and learning. Due to excessive reforms of the course content, the burden on students and the quality of education declined (Liu, Xu, & Zhao, 2006). At that time, a "Part Work (Agricultural) and Part Study" movement was proposed, as political education and productive labor education were highly regarded. For example, the curriculum plan included basic knowledge of industry and agriculture, improving operational transformations. In 1962, the Ministry of Education issued the *Notice on the Textbooks for Primary and Secondary Schools of 1962–1963*, and all "Middle School Arithmetic" was reallocated to become part of primary school content (Wei & Zhang, 1996). When the People's Education Press published a set of primary and secondary school mathematics textbooks in 1963, the guiding ideology was "to strive to avoid one-sided emphasis on getting in contact with the reality so much so that basic knowledge is weakened to pay attention to the enrichment of basic knowledge and the strengthening of basic training" (Zhang, 2006). As the result, all arithmetic courses were completed in primary school, and more attention was given to basic knowledge and skills. In the "1963 Outline," the functional transformation, expressive communication, and model application functions of representation were strengthened, while they were reduced at the junior high school level. During the period of the "Cultural Revolution" (1966–1976), there was no unified mathematics syllabus. Provinces and municipalities created their own outlines and compiled their own teaching materials (Lv et al., 2009) that did not emphasize the two basics, resulting in a significant decline in the quality of teaching in the country (Zhang, 2006).

8.4.1.3 The Third Stage (After 1978)

After 1976, the country entered a period of comprehensive rectification, summing up the lessons learned from the reform of mathematics teaching materials at home and from abroad. For example, in the 1970s, after the failure of the "New Math" movement, the United States advocated "Back to Basics" (Zhang, 2018), and the "1978 Outline" in China was updated based on the "1963 Outline," emphasizing the strengthening of the two basics. In 1985, when the Ministry of Education examined junior high school education, it determined that students' burden of study had increased, so in the "1988 Outline," the function of each type of representation was downplayed. In the twenty-first century, the "2001 Standard" emphasized "independence, cooperation, exploration, and innovation," and the theme of "practice and synthesis" was added to the content. It focused on creating learning situations in mathematics and de-emphasized the two basics, reducing the importance of basic knowledge and basic skills in the curriculum (Zhang, 2018). In the meantime, the model application function received more attention, whereas the operation conversion function was de-emphasized. An investigation of the implementation status of the "2001 standard" found that the new teaching method made students' achievement polarization in the lower grades of primary schools. Therefore, the "2011 standard" returned to the "two basics" and expanded on them by proposing an emphasis on the "four basics": mathematical learning in the compulsory education stage enables students to acquire the basic knowledge, basic skills, basic ideas, and basic experiences of mathematical activities which are necessary for students' adaptations to social life and further development. This meant the operational transformation function was picked up again, and emphasis on the model application function decreased. However, in general, the three functions gradually became balanced.

8.4.2 Junior High School (Planar Geometry)

As shown in Fig. 8.4, levels of expressive communication and operational transformation fluctuated greatly, and the trends of changes in these two areas are generally consistent, with both gradually increasing, while the level of model application fluctuated only slightly. Quantitatively speaking, keywords related to operational transformation were found most frequently, followed by keywords related to expressive communication. Keywords associated with model applications appeared least frequently.

8.4.2.1 The First Stage (1923–1951)

In the first stage, the instructional content described in the curriculum standards put emphasis on expressive communication. It was not further enriched possibly because these specific instructional contents had only recently been added to the

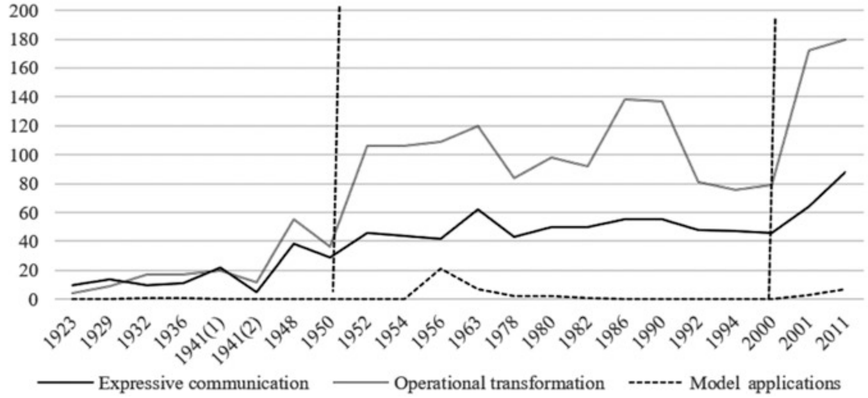

Fig. 8.4 Line chart showing geometry teaching content in junior high school mathematics curriculum standards

curriculum standards. Beginning in 1932, experimental geometry was included in the curriculum standards, and intuition in geometry was emphasized; "developing students' skills in calculating and constructing, cultivating the habit of calculating skillfully and accurately, constructing neatly and precisely" was included among the course objectives (Curriculum and Teaching Materials Research Institute, 2001). These objectives reflect the greater importance attached to learning at the level of operational transformation. After China's victory in the *War of Resistance Against Japan*, the Ministry of Education revised the curriculum standards in 1948 to adapt mathematics education to the needs of rebuilding the country, and the instructional content was made more specific. After the founding of the People's Republic of China in 1949, the Ministry of Education found that topics in mathematics textbooks were not arranged logically and that students were burdened by frequent symposia. In 1950, a draft briefing was issued to streamline unnecessary and repetitive content in textbooks, resulting in a reduction of knowledge included.

Although the instructional content section in the curriculum standards did not involve knowledge and skills at the level of model application, these skills were included in other sections. Due to the War of Resistance, specific applications in national defense and military were proposed as part of the implementation methods, including measurement, construction, and so on. The instructional objectives also emphasize the connection between mathematics and daily life.

8.4.2.2 The Second Stage (1952–2000)

Generally speaking, the instructional content in the curriculum standards gradually matured through the process of adapting to national conditions in China. The "1952 Outline" was different from previous syllabi of China and was based on the mathematics syllabi of the Soviet Union over the previous 10 years. It adopted the principle of "moving over first and then sinicizing" and only modified and supplemented content that did not conform to China's situation. As a result, the quantity of the instructional content was not consistent with the previous standards. The outline also focused on the systematic study of the nature of geometric figures so that students could answer computational and construction questions. As the result, content at the level of operational transformation was greatly enhanced. While the Chinese curriculum was based on the Soviet Unions', requirements for geometry teaching in the three syllabi mentioned the need to "use the knowledge learned to solve practical problems." However, only the "1956 Outline" added content related to production techniques, corresponding to the level of model application.

In 1960, the *Report on the Revision of the Mathematics Syllabus for Primary and Secondary Schools and the Compilation of General Mathematics Textbooks for Primary and Secondary Schools*, submitted by the Ministry of Education, pointed out that "according to the experiences of teachers in China and her allies, it is possible to finish plane geometry in junior high school stage." In 1963, the instructional content of geometry in high school curricula in China was transferred to the junior high school curriculum, greatly increasing the number of topics in the syllabus. At the same time, the teaching standards required that basic knowledge of plane geometry must be taught, and learners must be able "to construct proofs in plane geometry, answer computational questions, complete drawing problems, and make simple measurements to adapt to the needs of participating in productive labor and further studying in high school mathematics, physics, chemistry, etc.," which corresponded to the three levels of representation.

Since 1986, in order to improve the quality of compulsory education, China has pursued reducing the academic burden on students in the transitional period and during the implementation of the compulsory education curriculum. For example, the 1986 syllabus adjusted the instructional content of the 1978 syllabus following the principle of "appropriately reducing difficulty, reducing students' burden, and making teaching requirements as clear and as specific as possible" (Ministry of Education, 1989). The 1990 revision of the "1986 Outline" recommended "removing excessive content and relaxing high requirements" (Curriculum and Teaching Materials Research Institute, 2001).

8.4.2.3 The Third Stage (2001–Present)

The third stage is the period of curriculum reform in the new century. In comparison to the previous stage, this period of reform is characterized by the presentation of instructional content and requirements that further refine the instructional content, and expectations for both the expressive communication and operational transformation levels have grown significantly. In particular, the instructional content is expressed more in the form of operational requirements. New applications were added to the instructional content, and the level of model applications has increased slightly.

In the "2001 Curriculum Standards," Euclidean geometry was replaced by "Space and Graphics." Content and requirement related to deductive geometry were reduced, whereas those for experimental geometry were greatly increased, and a significant amount of content related to transformational geometry was added, corresponding more to the operational transformation level. Although the curriculum standards are more closely related to real life, and require students to explore, feel, and recognize the graphics from the examples (Kong, Liu, & Sun, 2001), the requirements for solving practical application problems are still minimal. In other words, the instructional content is less focused on model application.

Compared to the "2001 Curriculum Standards," the 2011 edition changed the theme "Space and Graphics" to "Graphics and Geometry." The number of topics for learning also increased, with the number of topics related to expressive communication seeing the largest increase. In relation to student development, the 2011 Standards removed some content that was repetitive, unrelated to students' real-life experiences, or difficult to learn (such as trapezoids) and added related content supplements that further refined concepts, and highlighted topics at the corresponding communication level (Shi, Ma, & Liu, 2012). The curriculum standards also added content related to the graphic proof in order to provide students with more opportunities to learn geometry in depth, strengthening content related to operational transformation.

8.4.3 Junior High School (Solid Geometry)

As shown in Fig. 8.5, since 1932, the junior high school mathematics curriculum has included a limited number of topics related to solid geometry. Except for in 1948, expressive communication topics were included in the mathematics curriculum in each period. In contrast, curricular content corresponding to the operational

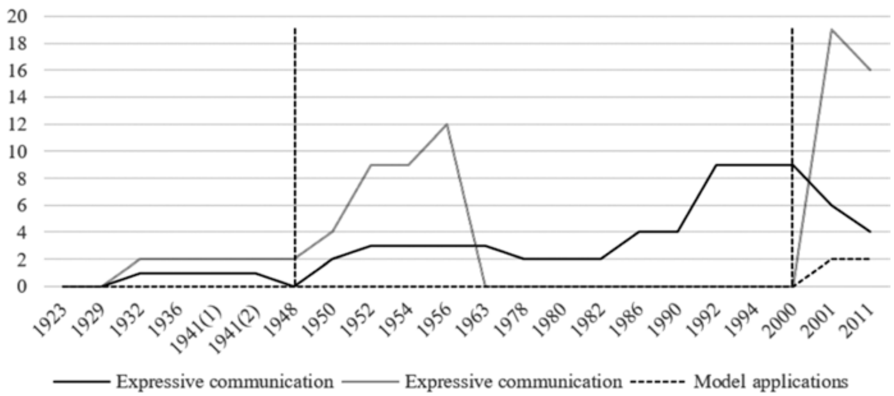

Fig. 8.5 Line chart showing solid geometry instructional content in the junior high school mathematics curriculum standard

transformation level was entirely omitted from standards issued between1963 and 2000; content at the level of model application appears only in the curriculum standards published in the twenty-first century.

8.4.3.1 The First Stage (1923–1949)

Before the founding of People's Republic of China in 1949, solid geometry content included in the junior high school curriculum was in its infancy, and was represented only by experimental geometry, including themes such as "spatial geometry" and "measurement of solid area and volume," which corresponded to one of the graphics at the communication level and two of the calculations at the operational transformation level. After the victory of the *War Against Japan* in 1945, only "simple three-dimensional area and volume calculations" remained in the revised 1948 curriculum standards.

8.4.3.2 The Second Stage (1950–2000)

In the early days of the newly founded People's Republic of China, content around expressive communication and operational transformation broke their previous balance, with keywords related to both appearing significantly more frequently. In the 1950–1956 syllabi, experimental geometric was included in the arithmetic discipline, and solid geometry mainly focused on the formulas for calculating volume and area, and other basic geometry. Later when China began modeling its education system after the Soviet Union's, calculation of the surface area of several simple objects was added to the curriculum. Additionally, the "1956 Outline" further required the production of a cube model, a cuboid model as well as a corresponding expanded view, which corresponded to the geometry teaching method of "using graphics, models, and common physics to help students intuitively recognize various geometric figures" in the arithmetic course of the curriculum. From 1963 to 2000, the junior high school mathematics curriculum had only included content at the level of expressive communication. In 1963, mathematics courses in junior high schools were almost all cancelled, with only plane geometry taught, and the correspondence of "volume" and "area" with solid geometry appeared only in the geometric introduction of the outline. In 1978, "a preliminary understanding of the views of simple objects" was newly added to the teaching requirements, whose corresponding teaching content was "preliminary knowledge of the view" and "view of simple objects," focusing on cultivating students' spatial imagination abilities, and the geometric introduction was deleted. The "1986 Outline" complemented the "two views" and "three views," further expanding the knowledge content. In 1992, China officially launched the *Full-time Junior High School Mathematics Syllabus of Nine-year Compulsory Education (Trial)*. The development of students' spatial concepts became more prominent and was distributed across three parts, namely, geometric introduction, spatial straight lines and positional relationships, and views on a plane.

8.4.3.3 The Third Stage (2001–Present)

In the “2001 Outline,” teaching content related to solid geometry focused on view and projection. Since the teaching content was mixed with the specific requirements, content related to “views” was described from the perspective of operation, so the prevalence of operational transformation was drastically increased, and the expressive communication was correspondingly reduced. This may also have been related to prominence of the concept of space at the time that the standard was changed from traditional geometry to “Space and Graphics.” In the “2011 Outline,” the previous theme “Views and Projections” was changed to “Graphic Projections.” The requirements of the three views were not changed, and content related to shadows, viewpoints, and blind spots were deleted, so the coverage of expressive communication declined again. Content related to model application was introduced for the first time, which both reflected the real-life applications of the views and expanded views of basic geometry.

8.5 Discussion

In this chapter, we summarize our analysis of mathematical content areas in elementary and junior high school, including arithmetic, algebra, plane geometry, and solid geometry, in the *Collection of primary and secondary school curriculum standards and syllabus of the twentieth China (Mathematics volume),* from the perspective of the functions of the mathematical representations. In addition, we reviewed some related literature and historical materials to provide context for the results revealed in the analysis. Results show that the functions of mathematical representations changed as curriculum standards changed. Emphasis on and correlations among the three representation functions (expressive communication, operational transformation, and model applications) exhibited different features during different stages of mathematics curriculum development.

In terms of arithmetic and algebra content, elementary school students develop their expressive communication ability mainly through solving contextual questions and practical applications. Junior high school students’ expressive communication ability is cultivated through operational transformations and model applications that are more abstract. The operation conversion function is emphasized in both elementary school and junior high school. However, the model application function of mathematical representations is not considered seriously. The various functions of mathematical representations played different roles during different periods. For example, between 1923 and 1978, there were two dramatic fluctuations. The first instance of volatility occurred when China adopted Soviet teaching theory after

1949 and basic knowledge received attentions. These changes resulted in the line of operation conversion and expression rising quickly. They were later rapidly reduced after the Ministry of Education requested schools and teachers ease students' academic burdens. The second wave arrived when the requirements for student learning were raised as a reflection of the country's growing political ambitions, but expectations on students' representation ability were reduced again as the quality of teaching decreased.

In terms of plane geometry, junior high school mathematics curriculum always focused on two mathematical representation abilities: expression communication and operational transformation. In particular, due to the importance of mapping and calculation, operational transformation function was increasingly valued in plane geometry. Requirements related to expressing communication functions were relatively stable. However, the mathematics curriculum standards rarely focused on the model application function of mathematical representations before the twenty-first century, and this function was not directly mentioned until the twenty-first century.

Solid geometry is an important subject for developing students' reasoning abilities and spatial imaginations but is the subject of constant debates in China because of its complexity and difficulty. Content related to solid geometry is often added to or removed from the curriculum standards. The operational transformation function was not mentioned for a long time, until it was added and emphasized in the twenty-first century, when the content related to "view" was added. Logical reasoning and expressive abilities are significant in three-dimensional geometry, so the expression communication function has always maintained a dominant position. The balance between the model application function and the operation transformation function stabilized after the beginning of the twenty-first century.

Outside of algebra or geometry, requirements placed on students in the curriculum standards are closely related to the context of the times. The knowledge content and systems changed in order to adapt to the societal changes and personal development. For example, before 1923, China gradually shifted from moral education to pragmatic education with the introduction of Dewey's theory. In the 1950s, The Chinese education system entered a stage of comprehensive study of the Soviet Union. In the 1980s, education in China began to fully recover and develop. The current revision process for course standards is based on China's national status and continuously draws on experiences in practice. The curriculum standards emphasize basic knowledge and basic skills. The mathematics curriculum standards issued in the twenty-first century gradually increased attention to three representation functions.

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Chapter 9
Chinese Eighth Graders' Competencies in Mathematical Representation

Jinyu Zhang and Na Li

Abstract Mathematical representation as a core competency in mathematics has been the focus of research in mathematics education, and it is used by many international educational organizations and educators as an indicator of students' mathematical ability. The intent of this study was to use literature analysis as the basis for an evaluation framework for the mathematical representation competencies of students at certain stages of compulsory education. A set of test questions was developed to test the mathematical representation competencies of 1197 eighth-grade students in eight regions of China, and the results showed the following: (1) the overall mathematical representational competencies of Chinese students were at the transition stage from the second level (connection) to the third level (reflection); (2) intra-system representations were better than inter-system representations, and these two were highly correlated; (3) the average level of mathematical representational competencies was lower for boys than for girls and varied considerably; and (4) there was greater variability in the performance of different regions on some items than on others.

Keywords Mathematical Representational Competencies · Assessment Framework · Intra-system Representation · Inter-system Representation · Reflection Level · Connection Level · Reproduction Level

9.1 Introduction

In mathematics classrooms, students are often asked to prove that two parts that were previously considered to be completely separate are actually two parallel yet different examples of a more abstract expression. This requires that students are able to understand mathematical representations. Some researchers emphasize the importance of the ability to understand mathematical representations, as it plays a significant role in understanding concepts and problem solving (Gagatsis &

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Shiakalli, 2004). In the evaluation framework for mathematical literacy from the Program for International Student Assessment (PISA), the ability to represent and transform mathematical expressions has always been considered an important indicator of mathematical competence (OECD, 2013). Other studies have shown that mathematical representation and transformation abilities play important roles in the processes of mathematics learning and problem solving and are among the factors that ensure students' smooth entry into higher level learning such as algebra and geometry (Goldin, 1998a).

Through the synthesis of related research on mathematical representation and transformation abilities, in this chapter, we aim to use the synthesis of related research to construct an evaluation framework and an indicator system suitable for assessing students' competencies in mathematical representation and transformation in the compulsory education system in China. It is the researchers' hope that, through the preparation of the tasks, students' mathematical representation and transformation ability levels can be categorized using different evaluation dimensions and indicators, providing ideas for a more in-depth study of the developmental characteristics of mathematical representation and transformation abilities of primary and secondary school students in China.

9.2 Literature Review

In 1973, Bruner (1973) introduced the concept of "representational systems" in order to study the ways in which diversity is represented and the interactions between them. He argued that there are three systems of representation at work during the growth of human intelligence: "gestural representation," "pictorial representation," and "symbolic representation," which is the general term used to refer to the representation of the human mind. The interplay of these three systems of representation is central to cognitive and intellectual growth. In his view, these three types of representation are essentially the three systems of information processing by which humans make sense of the world.

Most researchers have built on this foundation to further create and refine representational systems, and most of the goals of mathematics instruction and assessment have focused on students learning to understand these representations and use them as tools to solve mathematical problems. Lesh and Landau (1983) classify representations as written symbolic representations, graphical representations, situational representations, operational representations, and linguistic representations. Together they form a system of representations. There is not necessarily a sequential order in which competency in each must be developed; rather, it is the transitions and interactions between them that require attention and that are important for students' concept formation and understanding. Hitt (2002) found that translating between and within representational systems is not an easy task for learners. It is necessary for educators to teach learners how to translate between and within representational systems in order to help learners establish the relationships between representational forms in their minds and to make use of the functions of representations. In recent years, with the in-depth study of the mathematical representations in mathematical teaching and problem solving, the concept of "representation and transformation" has gradually developed. Its definition is derived from a variety of sources: mathematics, mathematics teaching psychology, problem solving psychology, classroom teaching, and research on teaching integrated with technology.

The importance of mathematical representation in overall mathematical processes has led a number of scholars to focus on the nature of representational competence as one of the key indicators of students' ability in mathematics, and to explore ways of improving students' competence.

National Council of Teachers of Mathematics' NCTM Standards for the Mathematics Curriculum (2000) identify representational skills as an important stage in the problem solving process and elaborate on what is expected of students in grades 9–12: (1) the ability to create and use representations to organize, record, and communicate mathematical concepts in problem solving and (2) the ability to select, apply, and transform mathematical representations to solve problems and use representations to construct models and explain natural, social, and mathematical phenomena. In the *Atlas of Science Literacy* (American Association for the Advancement of Science, 2001), after summarizing the characteristics of disciplines like "statistical reasoning," "computer," and "design system," the United States' Project 2061 presents designs and proposals for teaching objectives of learning and using symbolic systems and image systems in $K-12$ curricula. Danish mathematician Niss (2003) believes that competence in mathematical representation includes the following abilities: to understand, interpret, and identify various representations of mathematical objects, phenomena, and situations; to understand the relationship between different representations of the same mathematical object, and grasp the advantages and limitations of different representations; and to select and transform representations.

Based on Niss' framework, the definition of representation has gradually developed from that presented in Programme for International Student Assessment (PISA) 2000 to that in PISA 2012. PISA (OECD, 2013) stated that the development of mathematical competency is inseparable from the individual's representation of mathematical objects and situations and various transformation among representations. Representation involves all aspects of the modeling and problem-solving processes. For example, in the face of mathematical situations and objects, an individual grasps the mathematical essence of the question by selecting, representing, and transforming various representations to solve problems. Mathematical representation includes images, diagrams, graphs, and the specifics of the problem.

In 2005, the German Council of Ministers of Culture (KMK) has issued new educational standards for tenth-grade graduates in German, mathematics, and English and requires German states to measure the competencies of their pupils against the corresponding educational standards. Core competencies in the standards for mathematics education include the "Application of mathematical representations." The KMK states that to have the ability to make mathematical representations is "to be able not only to formulate one's own representations of mathematical objects, but also to apply the given mathematical representations comprehensively," and the standards include different forms of mathematical representations, such as pictorial, symbolic, and verbal representations. (Xu, 2007).

Cai and Lester (2005) designed open-ended mathematical questions to examine students' representational behavior in problem solving in order to obtain the types of representations and levels of competence used in problem solving by sixth graders in China and the United States. Using this as a reference to determine students' cognitive level, the study noted that "mathematical representational competence exists in the use and transformation of mathematical concepts or relationships."

Although there are different approaches to assessing representational mathematical skills, the current assessment processes focus on students' external representations of mathematics; i.e., they use external representations to assess students' representational mathematical skills. At the same time, the definition of representational competence varies, but the requirement to be able to identify, interpret, select, apply, and transform different forms of representation to solve problems in a given situation is always included in assessments of representational competence. Through analysis of related research literature, we found that the consensus of representation and transformation abilities is concentrated on three aspects: external representation, internal representation, and representation transformation.

9.2.1 External and Internal Representation

The development of cognitive science has contributed to the study of the internal cognitive rules of individuals. Researchers (Arcavi, 2003) started with visual representation and found that diversity in and concreteness of representations were conducive to promoting students' understanding of mathematical concepts and improving students' problem solving and reasoning abilities (Pape & Tchoshanov, 2001). As computers enter people's lives and study, computer-assisted instruction has also become a popular research area (Ainsworth, 1992). From the perspective of cognitive sciences, computers can present abstract mathematical concepts and principles in intuitive and dynamic ways, injecting new elements into the diversity of representation (Ainsworth, Bibby, & Wood, 1997).

However, the diversification and concreteness of representation does not always promote individuals' understanding of mathematical concepts. Some may not be able to recognize the same mathematical structure across different backgrounds and representations, and exhibit "non-conservation of operations" behavior (Greer, 1998). Research shows that the diversification and concreteness of external representations have different effects on individual learners. Therefore, it is necessary to explore the reasons for the formation of individual differences from an internal cognition perspective.

Exploring the differences in individuals' representation abilities is another topic of increasing interest in representation research, with studies focusing on how individuals manipulate different representations in their minds. Perkins and Unger

(1994) argued that the act of characterization is the process of representing the entire symbolic system with representative mathematical notations, mathematical definitions, mathematical languages, diagrams, etc. From a cognitive psychology perspective, representation can reduce cognitive load, help individuals quickly sort out problem spaces, and aid in explaining, forecasting, and correcting steps in the problem-solving process.

Goldin (1998b) further classified representation into internal representations and external representations. External representations include traditional mathematical symbol systems (such as decimal systems, formal algebraic symbols, real-numbered axes, Cartesian coordinate systems) and structured learning environments (such as mathematical learning situations with specific operational materials, computer-based micro learning environments). Internal representations include individuals' constructions of symbolic meanings, the meaning-giving of mathematical symbols, and students' natural language, visual imagination and spatial representation, problem-solving strategies, heuristic method, and emotions about mathematics.

At the same time, related research has discussed the role of analogy, imagery, and metaphor in the construction of individuals' external representations, concluding that because the various representations of mathematical concepts are not as accurate as mathematical definitions, they are highly likely to hinder individuals' accurate understanding of mathematical concepts (Bagni, 2006). An in-depth discussion of the construction of internal representations would bridge the gap between behaviorism and cognitivism (Augusto, 2014).

9.2.2 Representation Transformation

Lesh and Landau (1983) discussed five types of knowledge representation: written symbol representation, graphical representation, situation representation, operational representation, and linguistic representation. Together, these representations constitute the representation system. Different representations are transformed into and from each other, and they promote students' understanding of mathematical concepts (Mcintosh, 1984).

Representation transformations occur mostly in the problem-solving process. In other words, it is easier to observe individuals' representation transformation behaviors in problem-solving activities. The mathematics curriculum standards from NCTM (2000) emphasize the importance of mathematical representation ability in the mathematical problem solving process, stating that in the problem solving process, students can organize, record, and communicate mathematical concepts by creating and using representation, can solve problems by choosing, applying, and transforming mathematical representations, and can construct models and explain phenomena of nature, society, and mathematics by using representation. Research on representation based on problem solving has developed rapidly in recent years. In the process of problem solving, the characteristic behavior is not static, but a dynamic process in which the individual's cognitive level of mathematical concepts influences his or her success in mathematically representing problem backgrounds and real situations (Gérard, 1998). In the transformation between different representations, the individual's recognition of "correspondence"in different representations is very important. Identifying the same structures and relationships in different situations without being affected by surface information is an important stage in the development of mathematical cognition (Greer, 1998).

Based on the above analysis of the understanding of representation in the existing literature, we developed a definition of mathematical representation and transformation abilities; mathematical representation ability is the ability to express a mathematical concept or relationship to be learned or dealt with in some way, such as written symbols, figures (tables), situations, operational models, words (including spoken words), etc., so as to ultimately solve problems. Mathematical transformation refers to the process of, in the course of solving mathematical problems, maintaining certain invariant properties of mathematical problems while changing the forms of information and mathematically transforming the problems to be solved, so as to reach the purpose of changing from complicated to simple, from unknown to known, from unfamiliar to familiar (Xu, 2013).

9.3 Research Questions and Methodology

9.3.1 Research Questions

Research described in this chapter mainly focuses on the following question: How well do eighth grade students in China perform on evaluations of mathematical representation ability?

The development of test items was based on a framework for evaluating the mathematical representation abilities of Chinese students. This framework is organized into three dimensions, namely the contextual dimension, the content dimension, and the competency dimension (Fig. 9.1). The contextual dimension is divided into four categories: personal life situation, educational situation, social situation, and natural science situation. The content dimension includes specific mathematical topics that are consistent with the content requirements of Mathematics Curriculum Standards for Compulsory Mathematics in China, including (1) Numbers and Algebra, (2) Graphs and Geometry, (3) Statistics and Probability, and (4) Synthesis and Practice. The competency dimension divides the test questions into task types and representations to determine which of the three levels of competency best describes students' representational abilities.

Task types are divided into standard solution questions and open-ended questions (open conclusion or open strategy). The forms in which representation ability manifests are divided into inter-system representation and intra-system representation. (See Sect. 9.3.2.2 for explanations and examples of these two types of presentation.)

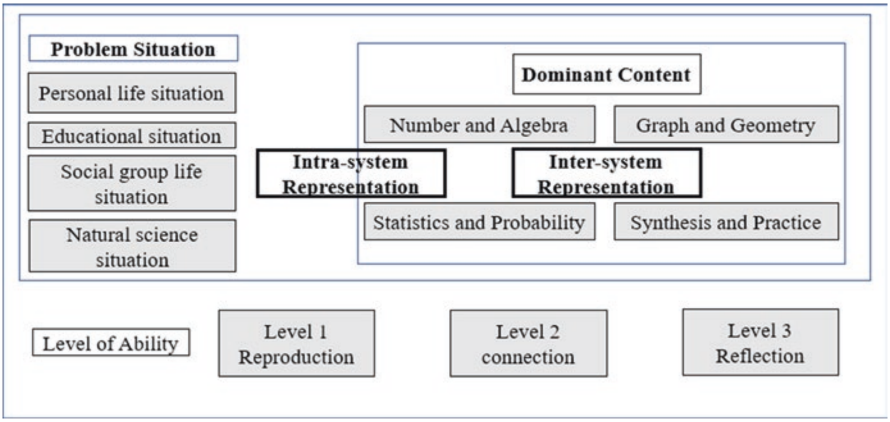

Fig. 9.1 Evaluation framework for students' mathematical representation ability. *Note.* Inter-system representation refers to mapping processes across different representation systems. Intra-system representation refers to transformation processes within the same representation system

9.3.2 Research Design

9.3.2.1 Research Subjects

Considering that imbalances in the economic development across various regions of China may influence the results of the evaluation, the study adopted the method of staged cluster sampling for sampling. First, eight representative cities were identified based on the geographic location of the city (including East China, Central China, North China, South China, Northwest China, Southwest China, and Northeast China) and corresponding levels of economic development (including developed, medium, underdeveloped, etc.); second, three schools were selected in each city; third, 2 to 3 classes of students at each participating school were randomly selected for inclusion in the test. We recruited 1197 eighth grade students in China as participants of this study.

The relevant information of the participants is shown in Table 9.1.

9.3.2.2 Evaluation Research Method

The intent of this study was to use literature analysis as the basis to construct an evaluation framework for the mathematical representation abilities of students at compulsory education stage. This framework is shown in Fig. 9.1.

The study was designed to use test tasks to evaluate students' mathematical representation abilities. Mathematical representation ability and its level dimension also determine different types of tasks required. The test tasks evaluated students' mathematical representation and transformation ability against five indicators:

Table 9.1 Information of the participants in mathematical problem-solving ability test

Region	A	B	C	D	E	F	G	H	Total
Number of schools	4	3	3	3	3	3	3	6	28
Number of students	167	146	119	92	158	109	153	253	1197

dominant content, problem situation, task type, manifestation of ability, and ability level.

In terms of the dominant content dimension, as the study was aimed at abilities of students in various stages of compulsory education. The four major strands (Numbers and Algebra, Graphs and Geometry, Statistics and Probability, and Synthesis and Practice) specified in the mathematics curriculum standards for compulsory education were used as the main content components to guide the design of evaluation tasks. The processes of gaining knowledge and ability are intertwined. Knowledge acquisition and ability development are especially important for teaching mathematics with abstract symbols as the carriers and the training of high-level thinking ability as the objective.

The study divides problem into four categories based on the situation involved: (1) personal life situations–the immediate, personal living environment and behaviors related to, such as behaviors related to specific personal operations like origami; (2) educational situations–the individual's school education environment, such as subject-related knowledge; (3) societal situations–issues related to the markets or certain elements of society, such as issues involving stock market fluctuations or supermarket promotions; (4) natural science situations–scientific issues related to nature, such as the use of sound to measure distance (OECD, 2013). The problem situations involved in the evaluation tasks included in this chapter are mainly personal life situations and educational situations.

In the problem-solving process, behavioral patterns and ability levels of mathematical representation vary with the situation and content of the problem. Therefore, the study design uses standard tasks and open tasks to comprehensively assess students' representation and transformation abilities. The two types of tasks require different levels of ability for representation and transformation. Open tasks are based on realistic problems, so individuals must analyze the essence of the situation, and map the mathematical content within the situation, taking problem situations in a mathematical way and applying mathematical knowledge and methods to problem solving. Open tasks are aimed at ability level 2 and level 3 (Fig. 9.1). Standard tasks require general answers, covering four major strands of mathematical content, and include two types corresponding to the different forms of individual representations. One type of test tasks is aimed at intra-system representation; that is, students need to solve problems through mapping activities between different mathematical symbol systems, geometric representation systems, linguistic systems, and operational representation systems. The other type of test tasks is focused on evaluating transforming problems by applying constant deformation and elementary geometric transformation to mathematical problems in the same representation system so as to solve the problem. Standard tasks point to inter-system and intra-system transformations (Fig. 9.1), with a focus on evaluating representation and transformation abilities levels 1 and level 2.

Inter-system representation Inter-system representation is a mapping process, that is, the mathematical transformations of actual life situations and the multi-transformations among the four mathematical representation systems with the same mathematical structure, including the written mathematical representation system, the geometrical (table) representation system, the linguistic (verbal) representation system, and the operational representation system. Inter-system representations reflect the individual's ability and level of competency in representations involving complex problems and real-life situations. Take the following task as an example:

[Example] A circle with a radius of 1 unit is in the center of a square with sides 3 units long. Xiao Ming throws a bean into the square. If the bean will definitely fall into the square, is the center of the bean (area not considered) more likely to fall inside the circle or outside the circle? Explain why this is true and the thought process that brought you to this conclusion.

This task is an open-ended task with a real-life experiences as the problem situation and probability, statistics, and geometry as the main testing content. As tasks are presented in text form, individuals must transform the problem through models, images, etc., treating the bean as a single point, reflecting positional relationships of the square and circle, and assess their relative size relationship in order to solve the problem. Students must then calculate the probability that the beans fall inside the circle by applying their knowledge of area. The specific behaviors required of students are described in Table 9.2. In the process of solving the problem, the individual must perform transformations across the four aforementioned representation systems.

Intra-system representation Intra-system representation is a mathematical transformation process. Within the same representation system, through mathematic transformation, understanding of situations can be changed from complex to sim-

Table 9.2 Description of specific behaviors of inter-system representation abilities

Inter-system representation	Description of specific behaviors
	Solve problems that arise in other systems by directly or indirectly transforming them into written mathematical representations such as numbers, algebraic symbols, operators, etc.
	Solve problems that arise in other systems by directly or indirectly transforming them into geometric representations such as line segment distance, graphs, etc.
	Describe the structural nature of original problems or solve problems that arise in other systems by directly or indirectly transforming them into operational representations such as specific gesture expression, physical enumeration, etc.
	Express, explain, or solve problems that arise in other systems by directly or indirectly transforming them into linguistic (verbal) representations such as spoken language, words, etc.

ple, from unknown to known, and from strange to familiar. Intra-system representations commonly found in mathematics at the basic education level include trigonometric transformation, constant deformation, elementary geometric transformation, segmentation transformation, and parameter transformation. Based on subject content analysis and teacher interviews, we adopted four areas of intra-system representation to serve as the task design indicators for the evaluation framework: variable substitution, elementary geometric transformation, identity transformation, and mapping transformation. Variable substitution and constant deformation are related to the Numbers and Algebra section, elementary geometric transformation is related to the Graphs and Geometry sections, and mapping transformation is covered in all four content sections.

Among the above four mathematical transformation methods, mapping transformation is especially important for transforming unfamiliar problem situations and simplifying corresponding problems. In relation to of sets and correspondence, mapping is the establishment of a special correspondence between two sets. The following task addresses mapping transformation:

[Example] Given an equation about x: $2x^2 - (3m + n)x + mn = 0$*, where m and n are real numbers and* $m > n > 0$*, prove that for the two roots of the equation, one root must be greater than n, and the other must be less than n.*

This task is a standard task, with content related to the Numbers and Algebra section. Students need to prove that the two roots of the quadratic equation belong to certain ranges of values. If the two roots are directly expressed using the root formula and proved by inequalities, the process is complicated. Therefore, by using the mapping transformation method, according to the relationship between the root and the coefficient, the problem of finding the ranges of two values is transformed into finding the ranges of the coefficients, and the original problem is simplified. The specific behaviors necessary to the process of solving such problems are described in Table 9.3.

In order to assess the different levels of representation and transformation ability that individuals exhibit in the problem-solving process, the context and content on

Table 9.3 Description of specific behaviors in intra-system representation

Intra-system representation	Description of specific behaviors
	In the numbers and algebra system, replace the original variable with another, for instance, using the replacement parameter to solve problems, in order to simplify the mathematical problem.
	In the Graphs and Geometry system, the original conditions are more concentrated or directly used for problem solving by rotating, moving, balancing, or adding auxiliary lines on the basis of the original geometry.
	In the Numbers and Algebra system, elements of the original algebraic equation are converted or flexibly interpreted to equivalent, simplified expressions, such as transforming fractions to decimals.
	In each of the systems, the original proposition is constancy transposed or the original problem is transformed into another one that makes it easier to solve the problem.

which the task design is based are very important. Due to the importance of functional concepts for algebraic learning, many researchers have evaluated the level of individuals' representation abilities based on function. Based on observations of how 14 university freshmen solved a series of similar algebraic text representation problems, Cifarelli (1998) summarized three levels of conceptual structure development in the problem-solving process, including recognition, reinterpretation, and structural abstraction.

Based on relevant literature analysis, the framework we developed for this study differentiates representation ability into three levels corresponding to students' cognitive abilities:

Usage of standard mathematical representations (level 1: reproduction). This level involves the most basic elements of the mathematical learning: mathematical processes, mathematical knowledge, and mathematical skills. The corresponding tasks contain basic mathematical representations (formulas, charts, etc.) that are commonly used in the daily learning and practice of the individual. The problem situation contains hints to help guide students to recall and reproduce commonly used representations.

[Example] Given that a, b and c represent the three sides of a triangle where a < c and b < c, please devise a mathematical expression containing a, b and c to show "this triangle is a right triangle."

The design of the task focuses primarily on leading students to recall relevant knowledge, in this case, the Pythagorean theorem. The content is related to the sections Graphs and Geometry. The knowledge point examined is the three-sided relationship of right triangle, and the symbolic representation belongs to representational representation.

Application of diversified mathematical representations (level 2: connection) This level reflects a higher degree of ability in mathematical representation: individuals can solve problems in situations that are unconventional but contain familiar information. The corresponding test tasks might integrate content knowledge that students are familiar with into an unconventional problem situation, and the individual student needs to identify and transform unfamiliar elements of the problem into familiar representations.

[Example] A piece of paper has the shape of a regular triangle. How can it be cut and assembled into a parallelogram? Please describe at least two different parallelograms and the processes used to create them.

The content of the task is related to the Graphs and Geometry section. Knowledge points examined include regular triangles, parallelograms, etc. The problem is based on personal life situations, and students need to transform operational representation into graphic representation, produce solutions with existing mathematical knowledge, and then verify them.

Transfer and construction of mathematical representation (level 3: reflection) This level of cognition represents a high level of mathematical ability, at which the student can extract and transform representations that are conducive to problem solving in relatively complex problem situations. The corresponding evaluation tasks are highly abstract, and individuals need to construct and creatively transfer different representation methods to solve problems through analysis, coding, decoding, and other processes.

[Example] A number plus 168 produces the square of a positive integer, and the original number plus 100 produces the square of another positive integer. what is the original number?

The basis of the task is relatively abstract. There is no clear information to guide students to identify and apply familiar representation methods, so it is necessary to construct appropriate representation methods to transform the problems based on analysis. The problem is gradually simplified by means of intra-system mathematical representation, and the unknown is transformed into the known as the problem is solved.

Specific behaviors associated with the three problem-solving competency levels are described in Table 9.4:

This study compiled and optimized the test tasks based on the mathematical representation ability levels and used the tasks to evaluate the mathematical representation ability of eighth grade students in China.

The students' collected answers were coded based on four elements designated in the framework: the task number; the form of ability expression, with inter-system representation recorded as R and intra-system representation recorded as T; proposed ability level, coded from 1 to 3; and the score of students' answer, with a correct answer recorded as 1 and an incorrect or missing answer recorded as 0. For example: 4_R_2_1 indicates that the data was derived from the fourth task, which evaluated inter-system representation abilities, was designed as level-2 task, and was answered correctly.

The third element in the coding system, level coding, was assigned through the researchers' subjective evaluations. In order to judge the rationality of the proposed

Table 9.4 Descriptions of specific behaviors associated with levels of mathematical representation and transformation ability

Level 3 (reflection)	Can understand and apply non-standard forms of representation (i.e., extensive decoding and transformation needed to create a familiar representation); can produce specific representations for critical steps in relatively complex problem situations; can compare and evaluate different forms of representation.
Level 2 (connection)	Can clearly interpret and transform two or more different representations in an unconventional problem setting given some familiar information, such as adjusting a certain representation, or autonomously choosing to use a more familiar representation.
Level 1 (reproduction)	Can directly process and use relatively familiar representations in relatively familiar and standard situations; can transform mathematical representations when given hints, such as changing familiar expressions into numbers, algebraic equations, graphs, and charts, or completing familiar mathematical transformation processes specified by the question design.

Table 9.5 IRT measurement results

Original task code	Objective level of tasks	Task difficulty (B)	Task discrimination (A)
1_R_1	−1.846_L1	Easy	0.988-Moderate
3_R_2	−1.398_L1	Easy*	0.585-Low
4_R_2	−0.991_L1	Easy*	0.844-Moderate
5_R_3	1.476_L3	Hard	0.726-Moderate
6.1_R_2	0.185_L2	Medium	0.776-Moderate
7.2_R_2	0.278_L2	Medium	0.772-Moderate
2.1_T_1	−0.057_L2	Medium*	1.145-Moderate
2.2_T_2	0.692_L2	Medium	1.567-High
6.2_T_3	1.515_L3	Hard	1.569-High
7.1_T_2	0.224_L2	Medium	0.806-Moderate

Note: [a]Difficulty division: B: <−2.25 very easy; −2.25 ~ −0.76 easy; −0.75 ~ 0.75 medium; 0.75 ~ 2.25 hard; >2.25 very hard
[b]Discrimination division: A: 0 none; 0.01 ~ 0.34 very low; 0.35 ~ 0.64 low; 0.65 ~ 1.34 moderate; 1.34 ~ 1.69 high; >1.70 very high; +infinity perfect

test level, test results were analyzed using IRT (item response theory). When used to analyze achievement tests, traditional CTT (Classical Test Theory) can only classify the corresponding level of the test item at the proposed level of the test task. However, the IRT method can objectively determine the corresponding level of the test task based on the performance of the test subjects on the test task.

After summarizing the test results with SPSS (Statistical Product and Service Solutions) and analyzing the difference between the objective level and the proposed level of each task using the IRT measurement method, the research team came up with the results shown in Table 9.5:

The objective levels of most tasks were shown to be consistent with the proposed levels, and the degree of discrimination was mostly moderate and high. However, there were three tasks for which the objective levels differed slightly from the proposed levels: task 3, 4, and 2.1. The differences between their objective difficulty coefficients and proposed difficulty coefficients were 0.648, 0.241, and 0.693, respectively, all of which are less than one-half of each difficulty interval. Therefore, we concluded that the mathematical representation ability evaluation tool is relatively reasonable in the horizontal dimension.

9.4 Empirical Investigations

9.4.1 An Overview of the Representation Levels of Eighth Grade Students in China

Figure 9.2 roughly depicts the levels of mathematics representation abilities of Chinese eighth grade students. The graph indicates that 13.1% of students have not reached level 1 and that 15.4% have, at most, answered all the questions of level 1

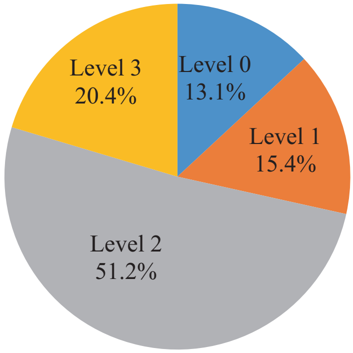

Fig. 9.2 Overview of representation ability levels of eighth grade students in China

but failed to complete level 2 or level 3 tasks. A majority of students, 51.2%, were assessed at level 2, indicating they had failed to correctly answer all the level 3 questions but had correctly responses to most of the questions at level 1 and level 2; 20.4% of students had not only answered the level 1 and level 2 tasks correctly, but had also successfully completed all level 3 tasks.

Test results show that most students' mathematical representation abilities were at level 2, which is the "connection" level, defined in this study as the ability to "clearly interpret and transform two or more different representations in an unconventional problem setting, given some familiar information." For example, test task 4: *The lengths of two line segments are 6 cm and 12 cm. How long is the third line segment when the three line segments form a right triangle?* This is a level 2 task. The trilateral relationship of right triangles is familiar information for students, but the task does not clearly indicate whether the third line segment is a leg or the hypotenuse, as a conventional problem would. Students needed to consider the two possible situations separately according to geometric representation and associate them with algebraic representation to solve the problem. In this study, 78.7% of students solved the problem correctly, indicating that most students could successfully identify the unconventional information and understood the geometric representation in the trilateral relationship and the "isomorphic" nature of algebraic representation. This suggests most students can link unconventional situations with familiar information and easily solve problems by transforming between graphical systems and symbolic systems.

The proportion of students who reached level 3, the level of reflection, was higher than the proportion of students at levels 0 and 1. This chapter defines the "reflection" level as the ability to "understand and apply non-standard forms of representation, produce specific representations for critical steps in relatively complex problem situations, and compare and evaluate different forms of representation." For example, test task 6: *a number plus 168 produces the square of a positive integer, and the original number plus 100 produces the square of another positive integer. What is the number?* There are two elements to this problem. The first is at level 2: transform the verbal representation into a symbol representation expressing the unknown relationship. The second task is at level 3: determine the value of the unknown integer. There are three unknowns in the problem, a condition students have rarely encountered, and the process of using two equations to find three unknowns is difficult, as students must be familiar with various intra-system representations in order to identity and map the necessary transformations. In total, 50.5% of students correctly expressed $a^2 = x + 168$, $b^2 = x + 100$, or other equivalent forms, while only 15.7% of students correctly solved for x. Overall, the average mathematical representation ability of the tested students was in the transition from level 2 to level 3, in other words, in the process of upgrading from the "connection" to the "reflection" level.

9.4.2 Distribution of Ability Levels for Inter-system and Intra-system Representation in Chinese Eighth Grade Students

As indicated in Fig. 9.3, the correct response rate for inter-system representation test tasks was higher than for intra-system representation test tasks at both level 2 and level 3; level 1 cannot be compared as no intra-system representation test tasks were included. Take task 6 as an example. Its first element involves inter-system transformation between the language system and the symbolic system, and the rate of correct answers was high. Its second element involves intra-system transformation, which requires higher level techniques, so the correct response rate was reduced.

As seen in Fig. 9.3, as the task ability level increased and the correct response rate for inter-system problems decreased, the correct response rate for intra-system problems also decreased. Therefore, we suspected that inter-system representation

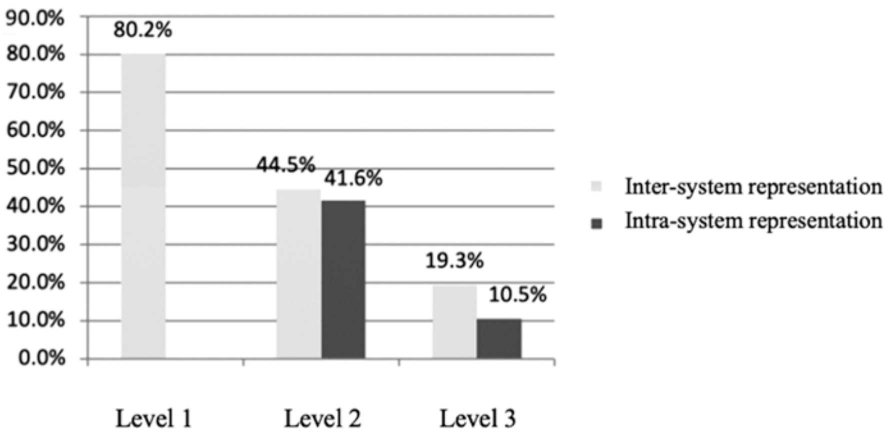

Fig. 9.3 Correct responses rates for test questions by level and type of expression

ability might be related to intra-system representation ability. In order to verify this correlation, we set the ability value calculated by IRT as a continuous order numerical variable, and the value interval was defined as (−1, 1). Greater ability values indicate stronger student abilities. The following analysis (Table 9.6) was based on a Spearman (rho) level correlation test:

Table 9.6 shows that the correlation coefficient between the students' inter-system representation ability value and the intra-system representation ability value is 0.523, $P = 0.000 < 0.01$, indicating inter-system representation ability is highly correlated with intra-system representation ability, which means if a student performed well on the inter-system representation test, they were likely to perform well on the intra-system representation test.

9.4.2.1 Gender Differences (Table 9.7)

Students were classified according to gender. Statistical analysis showed that the average level of male students' overall representation abilities ($M = -0.038$) was lower than that of female students' abilities ($M = 0.021$), and the average level of male students' inter-system representation abilities ($M = -0.058$) was also lower than that of females students' abilities ($M = 0.037$), while the average level of male students' intra-system representation abilities ($M = 0.006$) was higher than that of females students' abilities ($M = -0.020$). In addition, for both inter-system representation and intra-system representation, variability in performance level was higher among female students than among male students.

Table 9.6 Inter-system and intra-system representation ability value correlation

			Ability_R	Ability_T
Spearman's rho	Ability_R	Correlation coefficient	1.000	0.523**
		Sig. (2-tailed)		0.000
		N	1191	1190
	Ability_T	Correlation coefficient	0.523**	1.000
		Sig. (2-tailed)	0.000	
		N	1190	1190

Note: **Correlation is significant at the 0.01 level (2-tailed)

Table 9.7 Average ability values of male and female students and their standard deviation

		Male ($N = 543$)	Female ($N = 560$)
Ability _R	M	−0.058	0.037
	Std.	0.818	0.766
Ability _T	M	0.006	−0.020
	Std.	0.880	0.839
Ability_RT	M	−0.038	0.021
	Std.	0.932	0.838

Table 9.8 Statistical table for regional test differences

	1_R_1	3_R_1	4_R_1	6.1_R_2	7.2_R_2	5_R_3
Chi square	67.074	55.306	59.824	104.277	31.939	92.868
df	7	7	7	7	7	7
Asymptotic significance	0.000	0.000	0.000	0.000	0.000	0.000
	2.1_T_2	2.2_T_2		6.2_T_3	7.1_T_2	
Chi square	227.955	192.905		77.877	44.243	
df	7	7		7	7	
Asymptotic significance	0.000	0.000		0.000	0.000	

Note: Kruskal Wallis Test

9.4.2.2 Regional Differences

As indicated in the Table 9.8, students' mathematical representation ability varied significantly by region for each item (see Table 9.8, $p < 0.01$). This is in line with China's vast territory and different characteristics of local education environments, school settings, teacher resources, and management. Students in Shanghai had slightly stronger representational skills and the highest percentages of correct answers on almost all levels of the test. They also led in proficiency values among the eight regions. Students in Guangzhou had slightly weaker mathematical representation skills and had the lowest percentage of correct answers on all levels of the test among the eight regions. It also ranked at the bottom of the IRT mathematical representation ability value rankings.

9.5 Discussion

Results show that the majority of students were at the second highest level of mathematical representation ability and were able to perform on problems with unconventional elements containing some familiar information and interpret and transform two or more different representations in a given context. At the same time, the proportion of students who achieved level 3 was higher than the proportion of students who achieved levels 1 and 0, so more students were able to interpret and transform two or more different representations in non-standard settings or even more complex problem situations, to understand and translate different forms of representations, or to design some form of representation for use in solving problems. Thus, overall, the students tested were slightly above level 2, in the transition stage from level 2 to level 3, i.e. from the "connection" to the "reflection" level.

This chapter characterizes representation and transformation abilities as encompassing three main aspects: external representation, internal representation, and transformation between representations. On this basis, the different transformations were further divided into inter- and intra-system representation transformations (Fig. 9.1). Differences and correlations in students' ability to perform the two forms of representations were assessed. Students' answers were better on items that appeared in the form of inter-system representations. In contrast, students' performance was more variable on items presented in the form of intra-system representations, and the two forms of representational ability were highly related. There were some differences in inter- and intra-system representation abilities across regions. Students in Shanghai, Beijing, and Guangzhou were more consistent in their inter- and intra-system representation abilities, while students in Hanzhong and the Dalian performed more variably on the inter- and intra-system representations test items. Differences in students' mathematical representational skills by gender were assessed. Female students had slightly higher overall mathematical representational ability than male students and showed better stability than male students. Male students had lower representational ability in inter-system forms than girls, but higher representational ability in high level intra-system forms than girls.

At the K-12 educational level, inter-system representation has been the focus of the research community in recent years. The whole process of mathematical modeling can be distilled (Fig. 9.4) into the following stages: a given real-world situation is simplified or structured to produce a real-world model. The real-world model is then mathematized to produce a mathematical model. The mathematical model is analyzed, mathematical results are obtained, and the results are returned to the real situation for examination, i.e. validation. If the results are not satisfactory, the process must be repeated (Kaiser, Schwarz, & Tiedemann, 2010, p. 435). From a representational point of view, the process can be divided into inter-system representation (from the real world to the mathematical world and from the mathematical world to the representational world) and intra-system representation (analyzing the mathematical model to get mathematical results). The results show (Fig. 9.3) that as the difficulty level of the task increases, inter-system representation ability decreases as intra-system representation ability decreases, so we speculate that there may be a correlation between competency in inter-system representations and intra-system representations. To verify this correlation, we set the IRT-calculated proficiency values as continuous sequential numerical variables with the interval defined as $(-1,1)$, where larger proficiency values indicate stronger student proficiency. The test

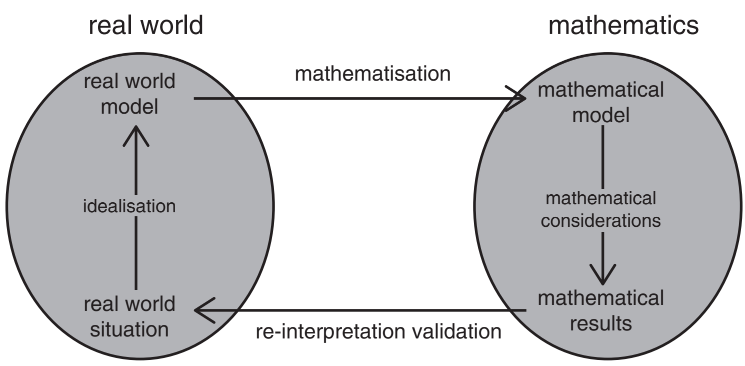

Fig. 9.4 Description of modelling process (from Kaiser, Schwarz, & Tiedermann, 2010, p. 435)

correlation coefficient table (Table 9.6) shows that the correlation coefficient between inter-system and intra-system performance values is 0.523 ($P = 0.000 < 0.01$), which means that students who perform better on the intra-system performance test are likely to perform better on the inter-system performance test.

The above results suggest that stronger mathematical modeling performance is dependent on better mathematical knowledge and skills (and vice versa). Traditional mathematics problem tends to emphasize the transformation between mathematical representations within the system, and the information students receive during the learning process is mostly mathematical concepts and definitions. Therefore, mathematics teaching should focus on both modeling problem situations and traditional mathematical problem situations, and students who do not perform well in modeling problems should pay attention to their proficiency in mathematical knowledge and skills.

From the results of the study, there were significant differences in the mathematical representational competence for students from different regions (Table 9.8, $p < 0.01$). Differences in educational environment, teachers, and management levels may be responsible for the variability, which needs to be investigated in further research.

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Chapter 10
The Development of Reasoning in Chinese Mathematics Curriculum

Xin Zheng and Jing Cheng

Abstract Reasoning-and-proving competency is indispensable for not only doing mathematics but also for solving real-world problems. In order to gain insights into the changing trend of reasoning-and-proving competency in the Chinese mathematics curriculum, the mathematics curricular programmatic documents for junior secondary education in mainland China since 1923 were selected and reviewed. From the content analysis and the review of historical documents, we found that reasoning has always been one of the most important goals of the mathematics curriculum. Moreover, we divided the development of mathematical reasoning-and-proving competency in the Chinese mathematical curriculum into four stages and revealed the pendulum phenomenon between plausible reasoning and deductive reasoning over the past 100 years. Reflecting on the development of reasoning in the Chinese mathematical curriculum, we may need to seek a balance between plausible reasoning and deductive reasoning in mathematical curriculum.

Keywords Mathematical reasoning · Reasoning-and-proving competency · Proving · Plausible reasoning · Deductive reasoning · Curriculum reform · Mathematics curriculum standards · Content analysis · Word frequency · Pendulum phenomenon

10.1 Introduction

Mathematical reasoning helps us develop lines of thinking or argument about and with the objects of mathematics (Brodie, 2010). Proving is often considered as the final stage of developing new knowledge in mathematics learning (Stylianides, 2008). Firstly, mathematical reasoning is essential to understanding mathematical concepts, the use of mathematical ideas, the flexibility of procedures and the

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reconstruction of some already understood but forgotten mathematical knowledge (Ball & Bass, 2003). Secondly, mathematical proving contributes to verifying and explaining why a statement is correct, exchanging mathematical knowledge, discovering or creating new mathematics (Knuth, 2002) and systematising the statement into an axiomatic system (e.g., Bell, 1976; de Villiers, 1999; Hanna, 1983, 1990; Schoenfeld, 1994).

Therefore, mathematics curricula in different countries all regard reasoning and proving as important ability goals (Si & Zhu, 2013). Meanwhile, international large-scale research into tests of students' mathematical abilities, such as TIMSS and PISA, also consider reasoning as an important dimension of its evaluation. Given that most literature on mathematics education defines this ability as 'reasoning and proof' (Hanna, 2014, p. 404–408), the terminology 'mathematical reasoning and proving' is employed to describe the capability discussed in this chapter.

In China, the school mathematics curriculum has long focused on the development of students' reasoning and proving abilities. In recent years, the standards of the mathematics curriculum particularly emphasise the advancement of students' abilities of plausible reasoning and deductive reasoning. The purpose of this chapter is to examine the development of the mathematical ability of reasoning and proving in the programmatic documents of junior high school mathematics in China.

10.2 Literature Review

10.2.1 Definition of Mathematical Reasoning

At present, researchers have different opinions on the definition of mathematical reasoning and proving ability. They carry out research from different perspectives. Broadly speaking, we can regard reasoning as the ability to make inferences or deductions. However, reasoning in a general sense is significantly different from mathematical reasoning in terms of process and result (Fischbein, 1999). In mathematics, the essence of mathematical reasoning lies in the generation, proving and application of mathematical generalization (Russell, 1999). We need clear proof to ensure the accuracy of the conclusion, and this series of proof which leads to the conclusion can help us construct mathematical proving. Explanatory and convincing proof provides us with the knowledge of mathematical truth (Weber, 2002). Therefore, proving is not separate from mathematics. It is a basic component of mathematical operation, communication and recording. Proving can be embedded in all levels of our curriculum (Schoenfeld, 1994).

Researchers led by Piaget, from the perspective of students' cognitive development, believe that human reasoning, based on propositions, applies logical rules or mental logic according to the form of argument rather than the content of it (Bao & Zhou, 2009). Therefore, mathematical proving is a logical derivation of a given proposition, starting from an axiom and deducing through an explicit chain of reasoning that follows accepted rules of reasoning (Hanna, 2014).

Lithner derives the definition of mathematical reasoning through empirical research on how students participate in various mathematical activities. He believes that mathematical reasoning is a thinking process during which the reasoner generates ideas or conclusions based upon mathematical experience when solving problems. This thinking process only needs to be reasonable for the reasoner, not necessarily to be precisely logical (Lithner, 2000, 2008). Olsson further points out that students' reasoning is the student's thinking route, that is, the thinking process in which the students attempt to handle the task regardless if it ends in success or failure. Students' reasoning is generated in a social and cultural environment and is limited and guided by the students' own abilities (Olsson, 2018). Johansson also proposes the definition of mathematical reasoning on the basis of Lithner's research. He holds the idea that mathematical reasoning is an extension of the rigorous mathematical proving used to prove a condition. It is regarded as the product of an independent sequence of reasoning (Johansson, 2016).

From the perspective of mathematical ability, Ball and Bass (2003) think that reasoning is the basic skill of mathematics. Xu (2013) states that mathematical reasoning is a comprehensive ability to make inferences by means of logically thinking (observation, experiment, induction, analogy and deduction) over mathematical objects (mathematical concepts, relations, properties, rules, propositions, etc.), which is furthered through seeking evidence, giving proof or a counterexample to illustrate the rationality of the given inferences.

In any case, though there is no unified conclusion about the specific definition of mathematical reasoning and proving, the researchers' views are roughly similar. Mathematical reasoning must be applied in the field of mathematics, and it is the inference or connection between mathematical objects. The process of inferring or connecting can be either rigorous proving or reasonable conjecture. Based on that, this study draws on the definition of mathematical reasoning and proving ability from Xu's research.

10.2.2 Classification of Mathematical Reasoning

Similar to the definition of mathematical reasoning and proving, different perspectives and different classification criteria bring different types of mathematical reasoning.

Based on the differences in students' cognitive processes and many years of teaching experience, Sternberg proposes three forms of mathematical reasoning: analytical reasoning, creative reasoning and practical reasoning (Sternberg, 1999). Analytical reasoning tends to be deductive logic analysis; creative reasoning is the activity of guessing and discovering; and practical reasoning means inferring and working out solutions to the problems in specific and real-life question contexts. He believes that analytical reasoning is the basic element of mathematical reasoning, for it has obvious promotion and restriction to creative reasoning and practical reasoning to a certain extent (Sternberg, 1999). Similarly, the *Mathematics Curriculum Standards for Compulsory Education* divides reasoning into plausible reasoning and deductive reasoning (MOE, 2012). When it comes to plausible reasoning, it is necessary to mention Polya (1954), who divided reasoning as plausible reasoning and deductive reasoning. The achievement of the mathematician's creative work is deductive reasoning, or proving. But the proof is discovered through plausible reasoning and guessing. On this basis, Cheng, Sun, and Bao (2016) divided plausible reasoning into observation and experiment, intuition and association and induction and analogy, while dividing deductive reasoning into syllogism reasoning, relational reasoning, mathematical induction, etc., according to different operational methods of reasoning.

The mathematical basis of reasoning can be either superficial or intrinsic, and one object has different recognised mathematical properties under different conditions (Johansson, 2016), which gives rise to the surface property and intrinsic property of reasoning. Therefore, based on Schoenfeld's research on mathematical problem-solving, Lithner (2000, 2008) divides mathematical reasoning into imitative reasoning and creative reasoning, while imitative reasoning can be further divided into memorised reasoning and algorithmic reasoning.

There are also other classifications from different scholars. For example, according to the different inductive reasoning processes, reasoning is divided into three categories: similarity, dissimilarity and integration (Christou & Papageorgiou, 2007). In accordance with the language of mathematical logic, deductive reasoning is divided into two categories: immediate inference and proof by contradiction; and there are nine basic types of reasoning (Bell, 1990). According to the purpose of mathematical proofs, they can be distinguished as proofs that convince, proofs that explain, proofs that justify the use of a definition or axiomatic structure and proofs that illustrate technique (Weber, 2002).

10.2.3 Reasoning in the Mathematics Curriculum

The mathematics curriculum standards promulgated in various regions have invariably taken reasoning ability as an important aspect of cultivating students' mathematical abilities. In the United States, Virginia takes mathematical reasoning as one of the five criteria for mathematics learning. Singapore's Ministry of Education regards reasoning as an important process in mathematical problem-solving; International Baccalaureate Assessment Objectives and Australian F-10 Mathematics Curriculum Key Ideas used reasoning as a key word (Hattie et al., 2016).

The United States' Common Core State Standards Initiative for the mathematics curriculum does not directly expound the definition of reasoning and proving but puts forward the requirements for reasoning and proving ability at the same time. The first requirement is reasoning by abstraction and quantification (Common Core State Standards Initiative [CCSSI], 2010). Reasoning by abstraction emphasises students' ability to de-contextualize and contextualize, while reasoning by quantification stresses the cultivation of habitually symbolising questions in a coherent manner. The second requirement is to construct viable proving and judge the reasoning of others (CCSSI, 2010). It is highlighted that students understand and use the given assumptions, definitions and the known conclusions to construct their own proving and actively respond to the questions of others when communicating.

The mathematics curriculum standards of California in the United States emphasise the significance of logical reasoning skills: mathematics teaching from kindergarten to seventh grade should make students recognise the importance of logical reasoning in mathematics; from the eighth grade, students should understand that logical reasoning is the backbone of all mathematics (California State Board of Education, 2005). In other words, any assertion in mathematics can be confirmed by logical deduction based on known facts. Students must learn to prove each judgment they make.

The mathematics curriculum standards of China's compulsory education elaborate on the definition of the ability of reasoning and proving from both the aspects of plausible reasoning and deductive reasoning. Plausible reasoning starts from the existing facts and, based on experience and intuition, some results are inferred by means of induction and analogy. Deductive reasoning, based on existing facts and established rules, is to prove and calculate according to the rules of logical reasoning (MOE, 2012).

Although the competence of reasoning plays an important role in the mathematics intended curriculum in almost every country, there is little research focusing on the changes of the demands of reasoning in the mathematics curriculum standards during different historical periods.

10.3 Research Question

The research question of this chapter is the following:

Since the twentieth century, how has the goal of 'mathematical ability of reasoning and proving' changed in the programmatic document of the junior high school mathematics curriculum in China?

The sub-questions are the following:

(a) In the objectives of the mathematics curriculum, how has the interpretation of the mathematical abilities of reasoning and proving changed?
(b) In the objectives of the mathematics curriculum, what change has taken place in terms of the proportions of mathematical plausible reasoning and of deductive reasoning?

10.4 Methods

10.4.1 Objects of Content Analysis

The research object selected for the research question was 23 programmatic documents of the junior high school mathematics curriculum in China since the twentieth century, including 2 curriculum guidelines, 12 syllabuses and 9 curriculum standards. This is because, during different periods, the names of the programmatic documents of the mathematics curriculum are different, but their core function is invariably to point out the expectation courses for junior high school mathematics in the corresponding period. For the convenience of expression, the following section will refer to all of the documents as the programmatic documents of the curriculum. Other labelling will be expressly stated if necessary.

The programmatic documents of the curriculum before 2000 are from *Collection of Primary and Secondary School Curriculum Standards and Syllabus of the Twentieth Century China (Mathematics Volume),* compiled by Curriculum and Teaching Materials Research Institute of People's Education Press. The curriculum standards after 2000 are separate editions published respectively in 2002 and 2012.

10.4.2 Procedures of Content Analysis

The analysis of all the research objects went through the following process: Firstly, based on the literature, the content related to the mathematical abilities of reasoning and proving in the programmatic documents was extracted and interpreted; then, the text analysis framework of mathematical reasoning and proving ability was constructed and the text was encoded; finally, the results of the encoding were analysed.

As far as each of the programmatic documents of the mathematics curriculum was concerned, among the overall goals of the curriculum, the descriptions related to the abilities of reasoning and proving were mainly extracted, while in the objectives of the curriculum content, attention was paid to the frequency of plausible reasoning and deductive reasoning in different domains of content (arithmetic, algebra, geometry, probability and statistics), for such frequency statistics can reflect, to a certain extent, the degree of emphasis on the two types of reasoning in the mathematics curriculum of junior high schools during a certain period.

Table 10.1 shows the index system of coding and corresponding examples. The first-level index includes the mathematical abilities of reasoning and proving, and the second-level index includes plausible reasoning and deductive reasoning, on which the research focuses. Since these two types of reasoning are decomposed into specific behaviours in the content objectives of the mathematics curriculum, the three-level indexes were formulated corresponding to the viewpoints of literature and the characteristics of the text. Under the three-level indexes, plausible reasoning includes observation, experimentation, induction and analogy, while deductive

Table 10.1 Coding framework for the mathematical ability of reasoning and proving

First-level index	Second-level index	Third-level index	Examples
Mathematical reasoning and proving	Plausible reasoning	Observation	'Find the functional relation between the corresponding values of these words'
		Experimentation	'Participate in experiments and develop the ability of plausible reasoning'
		Induction	'Inform students of discovering formulas by induction'
		Analogy	'Able to reason by analogy'
	Deductive reasoning	Analysis	'Able to find the proof of geometry theorems by analysis methods'
		Deductive reasoning	'Able to reason by deduction'
		Relational reasoning	'Analyse the relations between different quantities'
		Counterexamples	'Know that a proposition can be proved false by counterexamples'
		Proof by contradiction	'Realize the definition of proof by contradiction by means of instances'

reasoning consists of analysis, deductive reasoning, relational reasoning, counterexamples and proof by contradiction. The smallest unit of coding is a specific teaching objective.

In order to verify the reliability of the coding, a double consistency test was conducted on the coding framework obtained from the last revision. After the researchers explained the analysis framework to them, the two coders separately encoded the six curriculum standards which were randomly drawn. Through testing, the consistency between the two was 90.9%. Regarding the divergences, the two coders reached an agreement after negotiation.

10.5 Results

10.5.1 Changes in the Statements of Mathematical Reasoning and Proving

In November 1922, the Beiyang Government stipulated a new educational system. The system did not change until the founding of the People's Republic of China in 1949. In 1922, the *Outline of New Educational Curriculum Standards* was issued, which included arithmetic courses in primary and secondary schools. Among them, the *Outline of Junior High School Arithmetic Curriculum* added 'develop students' reasoning ability by mathematical methods' (Hu, 1923, p.7) into the course objectives. This is the first time since modern times that the ability of reasoning and proving has appeared in the syllabus of China. The *Outline of Primary School Arithmetic Curriculum* drafted by Yu (1922) put forward that the teaching of methods and principles should be constructed with induction instead of the promotion of deduction. Afterwards, the programmatic documents of the primary, junior high and senior high school curriculum were revised simultaneously. A total of five versions have come into being. The programmatic documents of the first four editions of primary school (respectively 1929, 1932, 1936, and 1941) and the first three editions of junior high school (1929, 1932 and 1936) absorbed the essence of the 1922 edition. According to those documents, for the students of primary and junior high schools, the teaching of new methods and principles should be practised step-by-step through induction and avoiding deduction. In 1941, the programmatic documents of the junior high school curriculum were revised, and the cultivation of students' ability of induction (Ministry of Education of the Republic of China, 1941) was written into the curriculum goals. In 1948, it was modified into 'cultivate students' ability to infer the unknown based on the known' (Curriculum and Teaching Materials Research Institute [CTMRI], 2001, p. 275).

After the founding of the People's Republic of China, the *Syllabus for Middle School Mathematics* drafted in 1952 was revised based on the then Soviet outline at that time, which considered 'developing students' ability of logical thinking and judging' (CTMRI, 2001, p. 356) as an objective of preventing formalistic teaching. Meanwhile, the draft also pointed out that 'developing students' logical thinking and imagination of space' (CTMRI, 2001, p. 361) is the goal of geometric teaching in the guidance instructions for geometry. Since then, the programmatic documents of the two versions in 1954 and 1956 have been amended on the basis of the 1952 draft, using the expression of 'logical thinking ability'. Although the full-time ten-year schooling was adjusted as a transitional period from 1978 to 1986, there was no change in the expression of reasoning and proving ability. It still put 'making students foster certain logical thinking ability' (CTMRI, 2001, p. 453) as one of the purposes of mathematics teaching in middle schools and so did the later four editions for the programmatic documents of secondary school curriculum.

China has implemented compulsory education since 1986, including primary and secondary schooling in the scope of compulsory education, and it separately laid down mathematics curriculum standards. In 1988, China's first mathematics curriculum standard of compulsory education came into being. On the basis of cultivating students' preliminary ability of logical thinking in primary school, it interpreted 'further developing students' logical thinking ability' (CTMRI, 2001, p. 553) as one of the teaching objectives of junior high school mathematics. Afterwards, the 1992 edition of the junior high school curriculum standard was the same as the 1988 edition. After the reform of the compulsory education system, the programmatic documents of China's mathematics curriculum stated the requirements for different levels of logical thinking ability for students at different stages in contrast with the past. However, these documents were interpreted merely by ambiguous vocabulary such as 'preliminary' and 'further', without any clearer explanation.

In 1996, the *Mathematics Syllabus for Full-time General Senior High Schools* (for trial only) divided the reasoning and proving abilities into two aspects. First, the basic skills that students need to develop include simple reasoning (CTMRI, 2001). Second, students also need to further develop thinking ability including reasoning by induction, deduction and analogy (CTMRI, 2001). The curriculum standard for junior high school, revised 4 years later, retained the expression of 'thinking ability' and defined it more clearly: 'able to observe, experiment, compare, conjecture, analyze, synthesize, abstract and generalize; to reason by induction, deduction and analogy; to logically and accurately articulate their own thoughts and opinions; to distinguish between mathematical relations based on mathematical concepts, principles, ideas and methods' (CTMRI, 2001, p. 648).

At the beginning of the twenty-first century, the programmatic documents of curriculum at the compulsory education stage in China no longer distinguished between primary school and junior high school and the two were uniformly revised. The Mathematics Curriculum Standard of Compulsory Education in 2011 proposed ten mathematical abilities, including reasoning ability. This version of the curriculum standards pointed out that reasoning generally includes plausible reasoning and deductive reasoning. Plausible reasoning is to infer certain results through induction, analogy, etc., by means of experience and intuition based on existing facts. Deductive reasoning is the calculation and proving of rules based on logical reasoning, which starts from existing facts (including definitions, axioms, theorems, etc.) and determined principles (including the definition, rules, orders, etc., of operations) (MOE, 2012).

10.5.2 Change of Word Frequency in Terms of Different Index Levels

From the angle of the first-level index of reasoning and proving, there have been four major fluctuations in word frequency. As Fig. 10.1 reveals, in the complete 23 programmatic documents between 1923 and 2011, word frequency of mathematical reasoning and proving in junior high school reached periodic peaks in 1941, 1963, 1988 and 2001, respectively 24, 45, 118 and 194 times. After the peaks, sharp falls occurred.

Figure 10.2 further compared the word frequency of plausible reasoning and deductive reasoning. It is not difficult to find that the proportions of the two have also undergone many changes. In the 1929 tentative standard, the word frequency of plausible reasoning was only 25%, compared to deductive reasoning. Since then, in the six editions of programmatic documents of curriculum from 1932 to 1950, the proportion of plausible reasoning has been larger than that of deductive reasoning and has been increasing. In 1950, it reached the historical maximum, accounting for 80%. After that, it dropped to 47.37% in the 1951 standard draft. Since then, the word frequency ratio of plausible reasoning was generally on the decline until 1982.

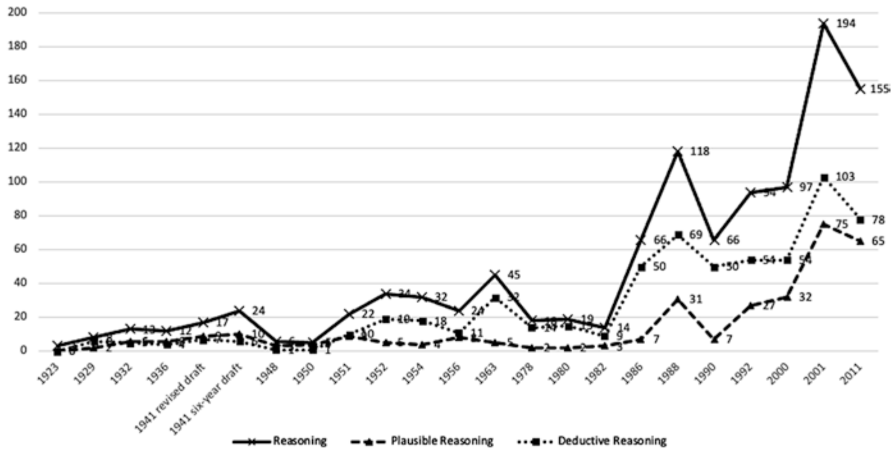

Fig. 10.1 Word frequency of mathematical reasoning and proving in the programmatic documents of junior high school curriculum

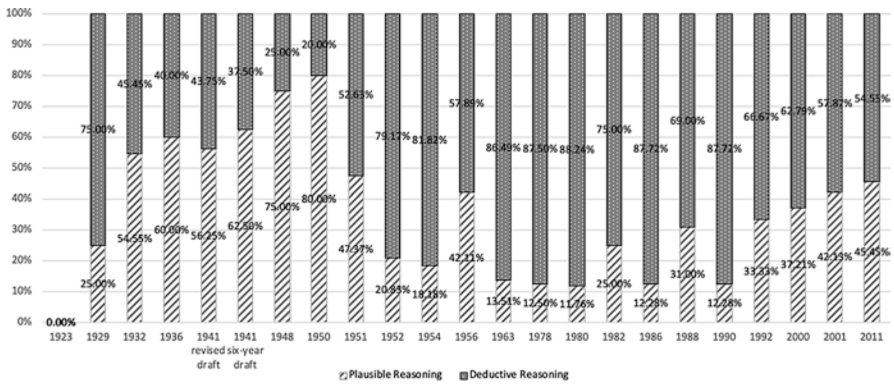

Fig. 10.2 Comparison between word frequency of plausible reasoning and deductive reasoning in the programmatic documents of curriculum over the years

In 1980, it reached the lowest level in history, at only 11.76%. After 1990, the proportion of plausible reasoning shows a steady increase, reaching 45.45% in 2011.

We can find some similar changing trends between plausible reasoning and deductive reasoning in mathematics curricula by observing the third-level indexes. As Figs. 10.3 and 10.4 show, the frequency of all the third-level indexes in the programmatic documents did not exceed 5 between 1929 and 1986. Particularly between 1963 and 1986, all indexes touched the lowest point. In the 1988 edition, attention to observation, induction, analogy and relational reasoning was enhanced. In 1990, there existed a fall after a rise. Since then, all the indexes have been continuously strengthened in the programmatic documents of curriculum, especially

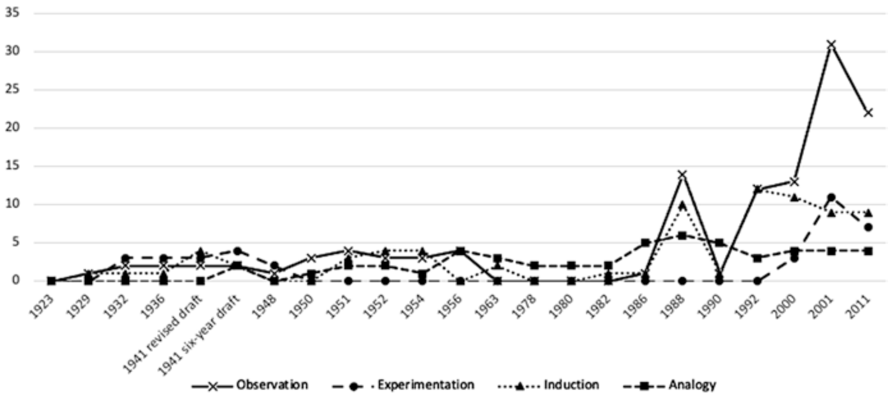

Fig. 10.3 Word frequency of the second-level index of plausible reasoning in the programmatic documents of curriculum over the years

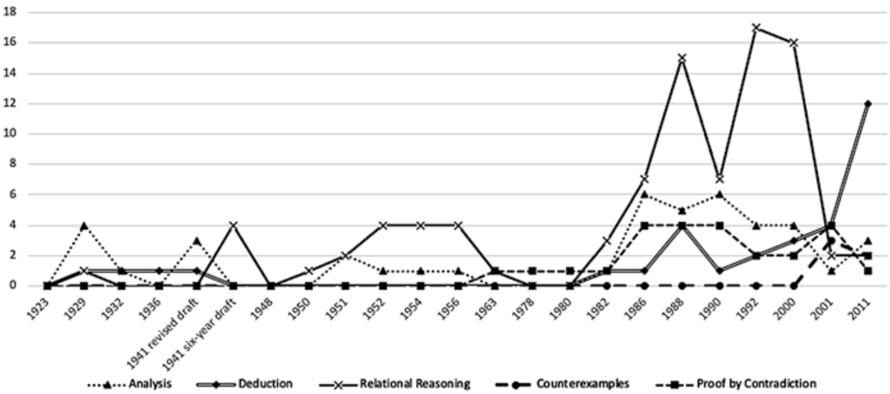

Fig. 10.4 Word frequency of the second-level index of deductive reasoning in the programmatic documents of curriculum over the years

observation. It is worth mentioning that the teaching requirements for experiment teaching were seen in the programmatic documents of curriculum before the founding of the People's Republic of China. However, during the 50 years from 1950 to 2000, the documents mentioned nothing about the requirements of experiment. It was reintroduced in the 2000 syllabus.

10.6 Discussion

China's mathematics curriculum has exhibited different characteristics in different historical periods. From 1923 to 2012, it has generally gone through four stages. The first stage is the period of the Republic of China before the founding of the People's Republic of China; the second stage is from the founding of the People's Republic of China to its early period before the reform and opening up; the third stage is from the reform and opening up to the promulgation of China's Compulsory Education Law; the fourth stage is after the implementation of the compulsory education system in China. In these four different historical periods, the objectives of reasoning and proving ability in the China's mathematics curriculum also present different features. The corresponding historical documents confirm the results of the word frequency analysis in this research to a certain extent.

10.6.1 Stage 1: Paying Attention to Induction

Before 1941, mathematics was called the science of calculation. The science of calculation, based on the axioms and definitions that do not contradict daily experience, deduces theorems, formulas and rules with strict reasoning methods, in order to study the application and theories of numbers and shapes (Wang, 1932). 'Reasoning' refers to 'logic', as we say today. Most of the scholars at that time did not use transliteration. They thought that transliteration was appropriate in English, but it was not appropriate in German and French (Peng, 1931). Therefore, they translated the word 'logic' to '*lunli*', which means 'reasoning' in Chinese. At that time, logic was set up as a course in general high schools. Logic, to put it simply, is the science of thinking. If further explained in detail, it is the science to study the patterns and rules of thinking and to decide what norms should be observed (Wang, 1928). The teaching content of logic included the analysis of human thoughts, the gist of scientific methods, induction, deduction and so on. The methods of reasoning contained various ways introduced in logic courses such as observation, experimentation, analysis, hypothesis, deduction, etc. The science of calculation gradually developed, relying on rigorous reliance reasoning methods; that is to make use of the major premise and the minor premise to derive the conclusion or to employ the axiom and definition as the premise and derive formulas and theorems. The proved formulas and theorems are then used as the basis of reasoning according to the rules. Students' logical ability is the ability to use the methods above, which has a lot in common with the ability of mathematical reasoning and proving as this chapter defines. So, the ability of logical reasoning can be regarded as the origin of the ability of mathematical reasoning and proving. Influenced by Dewey's educational thought of pragmatism, at this stage, China's mathematics curriculum paid more attention to induction and experimental operation. In the programmatic documents of curriculum, the proportion of plausible reasoning was greater than that of deductive reasoning.

10.6.2 Stage 2: Emphasizing Deductive Reasoning

In 1949, the People's Republic of China was founded. The development of China's mathematics curriculum faced two routes: one was to inherit and develop the original system of the mathematics curriculum; the other was to find another way to establish a new curriculum system. After the Communist Party of China determined the 'one-sided' policy, announcing 'taking Russia as the teacher and relying on the help of the Soviet Union to carry out various aspects of construction' (Fang, Li, Bi, et al., 2002, p. 71), the junior high school mathematics curriculum experienced a short transitional period. Then, the original mathematics curriculum was abandoned and turned to the comprehensive study of the Soviet mode of mathematics curriculum. Therefore, influenced by Soviet mode of mathematics curriculum, mathematics teaching pursued formalised interpretation for a long time (Ke & Liu, 2017). So, in the programmatic documents of this stage, the importance of plausible reasoning was in decline.

From 1958 to 1961, China's mathematics educational world reflected on the consequences of studying the Soviet Union (Lv, 2013) and conducted a short-term exploration of the development of the Chinese mathematics curriculum. In March, 1963, after the Central Committee of the Communist Party of China summarised the lessons from the previous period, three major abilities, including logical reasoning ability, were clearly put forward in the *Full-time Middle School Mathematics Syllabus (Draft)* promulgated in May. In the promulgated syllabus, the word frequency of reasoning and proving ability reached the maximum at this stage.

It is worth mentioning that the 1963 version of the outline reaches a peak. The plane geometry textbooks adapted to this version formed a complete system, clearly bringing up the idea of cultivating students' logical reasoning ability at different stages: stage 1: develop judging ability; stage 2: cultivate the ability to simply reason and demonstrate; stage 3: foster the ability to analyse more complex proof questions, thereby improving the logical reasoning ability; stage 4: continue to enhance the logical reasoning ability through the learning of various proof methods (Zhang, 2014). In this system, deductive reasoning was particularly prominent.

10.6.3 Stage 3: Developing Logical Thinking

After the Cultural Revolution, the Ministry of Education organised a revision of mathematics syllabus. As a result, *The Mathematics Syllabus of Full-time Ten-year Middle Schools (tentative draft)*, promulgated in 1978 and based on the 1963 outline, changed the logical reasoning ability to logical thinking ability. The possible reason for this modification was that logical reasoning ability in Chinese context was usually thought as deductive reasoning, while logical thinking ability including not only deductive reasoning, but also plausible reasoning (Lv & Ye, 2012). In other words, the focus on plausible reasoning began to increase gradually at this stage.

Influenced by the mathematics education of the Soviet Union, China's mathematics education respected the scientificity of mathematics and pursued the formalised definition of mathematics. The formalised logic of mathematics, emphasised since the 1980s, required students to recite the formalised definitions and rules of mathematics. The background of exam-oriented education led students to learn knowledge points mechanically and ignore the cultivation of abilities (Studio of New Young Mathematics Teachers, 2015). In this context, the cultivation of reasoning and proving ability encountered obstacles at all levels, which laid the groundwork for China to try to explore the Chinese mathematics curriculum system in the 1990s.

In 1986, the National People's Congress passed the Compulsory Education Law of the People's Republic of China, which stipulated that China implemented a nine-year compulsory education and that primary and junior high schooling were compulsory. *The Full-time Junior High School Mathematics Syllabus of Nine-year Compulsory Education (preliminary draft)* issued in 1988 emphasised the cultivation of students' proper personalities. In terms of value orientation, its focus shifted from social standards to student standards. This explains the culmination of the 1988 outline that paid attention to the ability to reason and prove.

10.6.4 Stage 4: Attaching Equal Importance to Plausible Reasoning and Deductive Reasoning

After the enactment of the compulsory education law, China entered the period of trying to establish a Chinese mathematics curriculum system (Lv, 2014). At this stage, many advanced ideas of mathematics education in the West had gradually been recognised by people. Among them, the concept of 'non-formalisation of mathematics' emerged, which led to the controversy of 'formalisation and non-formalisation' in research of mathematics education in the 1990s (Ke & Liu, 2017). *The Full-time Junior High School Mathematics Syllabus of Nine-Year Compulsory Education (tentative edition)* promulgated in 1992 embodied the spirit of diluting the form and focusing on the essence (Chen & Song, 1993). For the first time, the 1992 outline specifically explained the requirements of logical thinking ability, emphasising plausible reasoning to a certain extent and pointing out that bias should be prevented in teaching. It avoided being only satisfied with a certain method (including observation, experimentation and conjecture) in the process of teaching and learning and failed to further investigate the corresponding explanation, let alone contemplating other solutions and whether further deduction could be carried out (Zheng, 1994).

The Mathematics Curriculum Standard of Full-time Compulsory Education (experimental draft), promulgated in 2001, changed the logical thinking ability into reasoning ability and proposed it as a mathematical idea. At the same time, it triggered a debate in China's mathematics educational community on real-life application and reasoning and proving (Ke & Liu, 2017). The debate led to two sharply opposite views: one party believed that the 2001 version of the curriculum standard was lower in level than the previous programmatic documents, which was manifested by replacing reasoning and proving with life experience, making mathematics lose its soul (Cai, Zhou, & Jiang, 2005); the other side believed that formalised deductive proof was diluted compared to the old curriculum, but plausible reasoning used for scientific discovery was emphasised, which resulted in the enhancement of the level (He, 2006). Figure 10.1 shows that the word frequency of the content concerning plausible reasoning and deductive reasoning both reached a peak in the 2001 curriculum standard. As can be seen from Fig. 10.2, since 1992, the proportion of plausible reasoning has steadily increased, and both plausible reasoning and deductive reasoning have tended to balance.

10.6.5 Enlightenment from the Evolution of Objectives

Throughout the programmatic documents of curriculum over 100 years, the goal of reasoning and proving has always been one of the most important goals of the mathematics curriculum. However, whether it is the overall developmental trend of reasoning and proving or the relationship between plausible reasoning and deductive reasoning, the phenomenon of a 'pendulum' in curriculum development is presented. Curriculum designers are always seeking a balance between plausible reasoning and deductive reasoning. Polya (1954) pointed out that the study of mathematics must include both proof and guess at the same time. According to this, in terms of cultivating students' ability of mathematical reasoning and proving, it is necessary to pay attention to both plausible reasoning and deductive reasoning, both of which cannot be emphasised or neglected.

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Chapter 11
Chinese Eighth Graders' Competencies in Mathematical Reasoning

Jing Cheng, Xin Zheng, and Yan Zhu

Abstract Mathematics curricula around the world have attached great importance to the development of mathematical reasoning ability, but many studies have reported that students have difficulties in mathematical reasoning. The purpose of this chapter is to show Chinese students' performance of mathematical reasoning in junior secondary schools. A total of 1464 eighth graders from five regions in mainland China were selected as participants in the study. The findings reveal that the distribution of students' plausible reasoning competency is concentrated, while the distribution of their deductive reasoning competency is relatively dispersed. The two competencies show a significant positive correlation. Moreover, students demonstrated significant regional and gender differences in the two types of reasoning. Students' rigorousness in reasoning related to arithmetic, algebra and geometry is also depicted via case analysis.

Keywords Mathematical reasoning · Proving · Mathematical competency · Plausible reasoning · Deductive reasoning · Arithmetic reasoning · Algebraic reasoning · Geometric reasoning · Large-scale study · IRT · MANOVA · Gender differences · Regional differences

11.1 Introduction

Mathematical reasoning is a comprehensive ability to make inferences by means of logical thinking (such as observation, experiment, induction, analogy and deduction) over mathematical objects (such as mathematical concepts, relations, properties, rules and propositions). Reasoning is furthered through seeking evidence, giving

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proof or offering a counterexample to illustrate the rationality of the given inferences (Cheng, Sun, & Bao, 2016; Xu, 2013).

In the mathematics curriculum standard for compulsory education in mainland China, reasoning is classified into plausible reasoning and deductive reasoning. Plausible reasoning starts from the existing facts and, based on experience and intuition, some results are inferred by means of induction and analogy. Deductive reasoning, based on existing facts and established rules, is used to prove and calculate according to the rules of logical reasoning (MOE, 2012, p. 6). Although mathematics curricula around the world attach importance to reasoning and proving, a majority of studies reveal that students have great difficulty in logical reasoning.

The purpose of this chapter is to present the status quo of the mathematical reasoning competence of Grade 8 students from different regions of mainland China based on a paper-and-pencil test, as well as students' performance of rigour during the process of proving.

11.2 Related Literature

Moshman (1997, pp. 947–978) found that children aged 7–8 years can logically reason from specific facts; at the age of 11–12, they can make deductions and apply rules systematically. However, after conducting in-depth interviews with 17 high school students, Chazan (1993) concluded that students think of empirical evidence as proof. Healy and Hoyles (2000) reached similar conclusions after investigating students who did well in algebraic proving around the age of 15. Although most students were aware of the limitations of the empirical argument, it dominated the students' structure of proof. Moreover, in the evaluation of students' proof structures and given arguments, research has found that it is difficult for students to judge the certainty and necessity of a conclusion (Morris, 2002).

In addition, other researchers have examined mathematical reasoning. Senk's study (1989) is known for its large sample, and it required 1520 middle school students to prove four geometric theorems, two of which only needed to go beyond one deduction of the hypothesis. It turned out that only 30% of the students were able to prove at least three theorems, and 29% could not even construct one proof.

International comparative research has also examined students' mathematical reasoning abilities, including Trends in International Mathematics and Science Study (TIMSS) and the Programme for International Student Assessment (PISA).

TIMSS, initiated by the International Education Association (IEA), proposes the concept of TIMSS Numeracy, which is divided into numeracy content domains and numeracy cognitive domains. The former contain three cognitive skills: recognition, application and reasoning. These are progressive, in which the field of reasoning goes beyond conventional problem solving and incorporates unfamiliar situations, complex backgrounds and multi-step problems. Specifically, reasoning includes analysis, synthesis, evaluation, conclusion, induction, proof, etc. (Mullis & Martin,

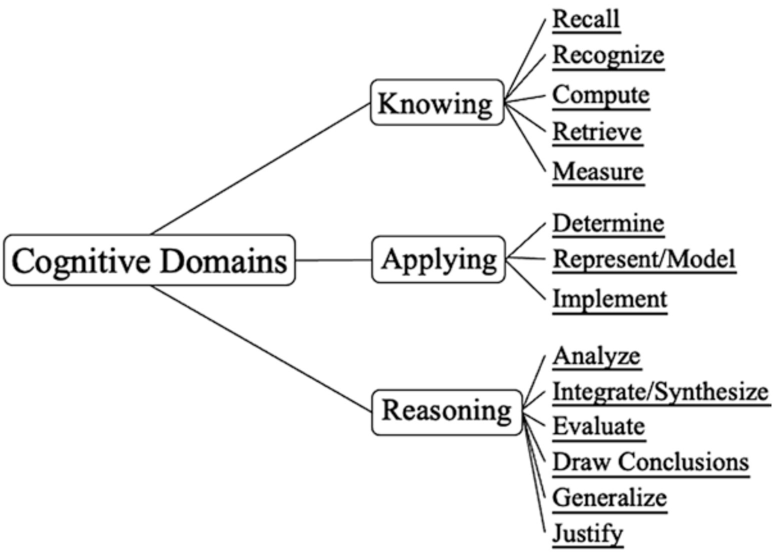


Fig. 11.1 TIMSS 2015 Cognitive Domains framework

2013), as shown in Fig. 11.1. According to the 2015 TIMSS assessment report, students at the eighth grade in 13 of the 39 countries involved in the survey were relatively strong in mathematical reasoning, while those from 16 countries were relatively weak; compared with 2011, 18 countries showed an upward trend, while 4 countries moved downwards (Mullis, Martin, Foy, & Hooper, 2016).

Meanwhile, PISA was developed and launched by the OECD in 2000. It proposes the concept of mathematical literacy, which refers to an individual's capacity for making mathematical deductions and applying mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena (OECD, 2017). PISA divides mathematical literacy into basic mathematical abilities and processes, including mathematical reasoning and proving. Reasoning and proving refer to the process of logical thinking in which question elements are explored and connections are established between them to make inferences, examine a given argument, and provide arguments for propositions or offer solutions to the problems (OECD, 2017).

The results of research on international mathematical achievement reflect a phenomenon labelled the "Chinese learner paradox" (Huang, 2008). Mathematical teaching in most East Asian countries is described as exam-driven teaching based on rules and factual memories. In contrast, math classes in Western countries aim to achieve meaningful and personalized learning. However, different studies on international mathematical achievement show that East Asian students are often better than Western students at dealing with routine questions, complicated mathematical problems and proving tasks. Therefore, researchers conducted a series of surveys of students in Taiwan and Germany, as typical representatives of East and West Europe, and supported the research results of TIMSS and PISA in terms of their reasoning and proving projects (Heinze, Cheng, & Yang, 2004).

Although in China there is also some research on students' competence in mathematical reasoning (Cheng et al., 2016), what has not been systematically analysed is the contrast between the students' attained mathematical reasoning and proving ability and the ability requirements of intended curricula.

11.3 Research Question

The research question of this chapter is: What is the present performance of China's Grade 8 students in the test of mathematical reasoning and proving?

The mathematical ability of reasoning and proving refers to the comprehensive capacity to think logically (observation, experiment, induction, analogy and deduction) over mathematical objects (mathematical concepts, relations, properties, rules, propositions, etc.), thereby making inferences, followed by seeking evidence, giving proof or giving counterexamples to justify the given inferences (Xu, 2013).

This ability can be divided into plausible reasoning and deductive reasoning. Plausible reasoning is used for guessing and discovering, which means the reasoning process involves speculating about certain results based on existing facts and correct conclusions (definitions, axioms, theorems, etc.), experimental and practical results, and personal experience and intuition. Deductive reasoning is applied to rigorous proving. It is based on existing facts and correct conclusions (definitions, axioms, theorems, etc.) to arrive at new conclusions in accordance with strict rules of logic (MOE, 2012, p. 6).

11.4 Methods

11.4.1 Participants

Grade 8 students from different regions of China were chosen as research participants. Details about the participants are shown in Table 11.1. First, according to the economic and educational development levels of different geographical locations, five regions were selected nationwide: East China, Central China, South China, Southwest China and Northwest China. One core city was selected from each region. Then, in each city, three or more schools were randomly chosen. Finally, two or more classes were selected at random for the purpose of conducting the tests of mathematical reasoning and proving ability.

Table 11.1 Regional backgrounds of the tested students

Region	Central China	East China	South China	Southwest China	Northwest China	Total
City	A	B	C	D	E	5
Number of schools	3	4	3	3	3	16
Number of students	365	295	216	325	263	1464

The formal test time was at the beginning of the student's entry into the eighth grade. At that time, the students had a foundation in algebra and were studying proofs of plane geometry. The final number of effective test papers collected was 1464.

To eliminate the interference of any external unrelated factors, the test was carried out in class. The time available was 45 minutes. Teachers and researchers monitored the testing, but they did not explain the content of the exam.

11.4.2 Instrument

In a large-scale proficiency test, single-question exams will reduce the validity of the test. Therefore, to ensure the reliability and validity of the test and the statistical analysis of large-scale test results, multi-question exams and framework-oriented test questions were used to evaluate the students' mathematical levels of reasoning and proving.

Based on the overall design of the major project, Research on the Core Competence Model and Evaluation Framework of Mathematics at the Stage of Compulsory Education, launched by the Key Research Institute of Humanities and Social Sciences of the Ministry of Education of the PRC, this study built a framework for assessing the ability of mathematical reasoning and proving along three dimensions: mathematical content, mathematical reasoning types and the ability level of reasoning and proving.

Given the limitations of the present grade of the tested students, as for the mathematical content dimension, this evaluation only took arithmetical, algebraic and geometric content into account, regardless of statistical and probabilistic content. The dimension of mathematical reasoning types was divided into two categories: plausible reasoning and deductive reasoning. Concerning the dimension of the ability level of reasoning and proving, it was divided into three levels, from low to high: reproduction, connection and reflection. Specific descriptions are shown in Table 11.2.

The test paper contained six questions altogether, each of which provided students with an opportunity to reason plausibly, and asked the students to demonstrate while giving the answers. Table 11.3 shows the content areas to which different test questions belong, as well as the ability level, pre-set based on experts' proof and pre-tests.

Table 11.2 Evaluation indexes of the ability of mathematical reasoning and proving

	Plausible reasoning	Deductive reasoning
Level 1 (reproduction)	By the methods of observation, operation, comparison and contrast, induction, analogy, etc., some reasonable guesses can be made; the reasoning process of guessing can be expressed.	Able to demonstrate the correctness of a proposition in a simple situation and give expressions in a relatively standardized symbolic language.
Level 2 (connection)	Able to connect relevant knowledge, garner useful information, obtain higher-level conjectures and clearly articulate the thinking process in relatively complex problem situations.	Able to connect others' reasoning and existing experience to demonstrate complex propositions, with the reasoning process being concise and complete.
Level 3 (reflection)	Able to get more conjectures, reflect on and examine the conclusions, and further systematize mathematical knowledge; the reasoning is sufficient and logical.	Able to flexibly change the train of thought and select appropriate methods of reasoning and proving according to specific problem situations for strict demonstration; the statements are clear and rigorous.

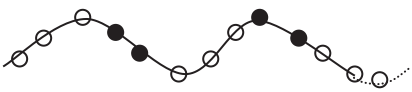
Table 11.3 Distribution of test questions

Question number	Mathematical content	Target ability level	Intended ability of reasoning and proving
1	Arithmetic	Level 1	Plausible reasoning/ deductive reasoning
2	Algebra	Level 1	Plausible reasoning/ deductive reasoning
3	Algebra	Level 2	Plausible reasoning/ deductive reasoning
4	Geometry	Level 2	Plausible reasoning/ deductive reasoning
5	Geometry	Level 3	Plausible reasoning/ deductive reasoning
6	Arithmetic	Level 3	Plausible reasoning/ deductive reasoning

The complete test is as follows:

Question 1:

As shown in the following figure, there are a series of beads arranged in white and black. What colour should the 36th bead be if arranged continuously in this way? Please write down the reason.



Question 2:
True or False: Is the product of two consecutive natural numbers odd or even? Please give the process of proof.
Question 3:
Observe the following equations:

$$① \sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}} \quad ② \sqrt{3\frac{3}{8}} = 3\sqrt{\frac{3}{8}} \quad ③ \sqrt{4\frac{4}{15}} = 4\sqrt{\frac{4}{15}}$$

Can you draw any conclusion? Please prove it.

Question 4:
Please write down the inverse proposition of the proposition, 'The median on the hypotenuse of the right triangle is equal to half of the hypotenuse', and judge whether it is true or false. Please give the proving process.
Question 5:
As shown in the following figure, please choose two appropriate relations as conditions in the following four relations to deduce that the quadrilateral, ABCD, is a parallelogram and prove it (list all the possible conditions).

Relations: ① $AD \parallel BC$, ② $AB = CD$, ③ $\angle A = \angle C$, ④ $\angle B + \angle C = 180°$.
Given: In the quadrilateral ABCD, ___, ___;
Prove: The quadrilateral ABCD is a parallelogram.

Question 6:
There are five scoring areas on the dart target, which are 2, 3, 5, 11 and 13. Xiaoming just got 150 points. How many darts did he cast, at minimum?

11.4.3 Data Analysis

The study used a multiple coding system to encode the students' answers. The students' performance code for each question consists of two parts. The first part of the code contains two digits, where the first digit represents whether the answer is correct, and the second digit represents whether the proof is right. The second part is a multi-digit diagnostic code, which is used to distinguish between different performances of reasoning and proving. The results of a code consistency test for each question all exceed 85%. As regards inconsistent coding, a consensus was reached after discussion.

For the 0–1 scoring of the first part of the coding, this research exploited the Item Response Theory model, the BILOGMG3.0 software and the two-parameter logistic model to calculate separately each question's difficulty coefficient and discrimination coefficient, and the estimated value for the ability of the subjects. Then the study analysed the distribution of the subjects' ability and used MANOVA to analyse regional and gender differences.

With respect to the diagnostic codes that reflect the answer types of the tested students, primarily qualitative analysis was conducted.

11.5 Results

11.5.1 Distribution of Students' Abilities of Plausible Reasoning and Deductive Reasoning

Figures 11.2 and 11.3 show the overall distribution of students' reasoning ability, with 0 in the horizontal axis representing the average level. In Fig. 11.2, the ability values were calculated based on the correctness of students' reasoning results, which to some extent represent their ability of plausible reasoning. In Fig. 11.3, the ability values were calculated based on the correctness of the student's reasoning processes to reflect their ability of deductive reasoning. It is worth noting that the majority of students whose performance of plausible reasoning is below average have ability values at −0.25 (similar to the average), and the students whose ability values are much lower than the average are few. In contrast, the distribution of students' deductive reasoning ability is more discrete, with the ability estimates not lower than +1. The number of students whose ability estimates are not higher than −1 exceeds 15% of the total sample. The results of further correlation analysis reveal that there is a significant positive correlation between students' plausible and deductive reasoning abilities (Pearson correlation = 0.67, $p < 0.01$), with no significant difference between the mean values.

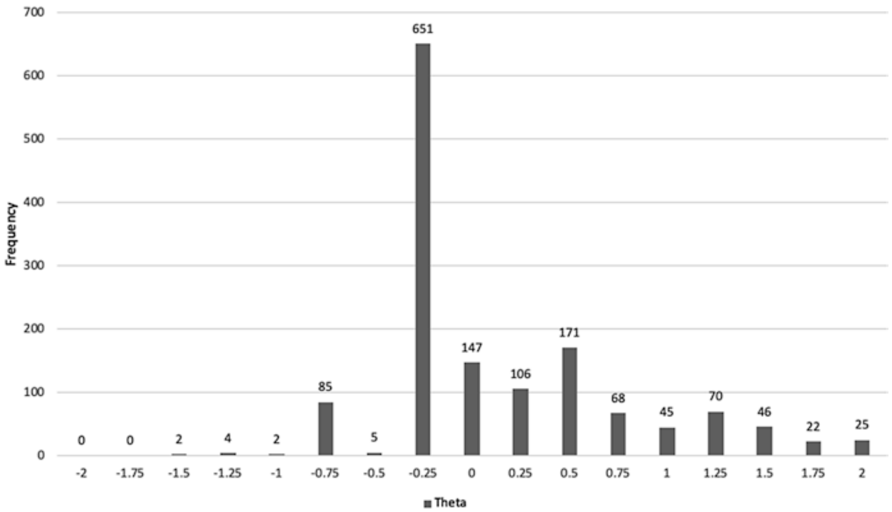


Fig. 11.2 Distribution of estimates of plausible reasoning ability

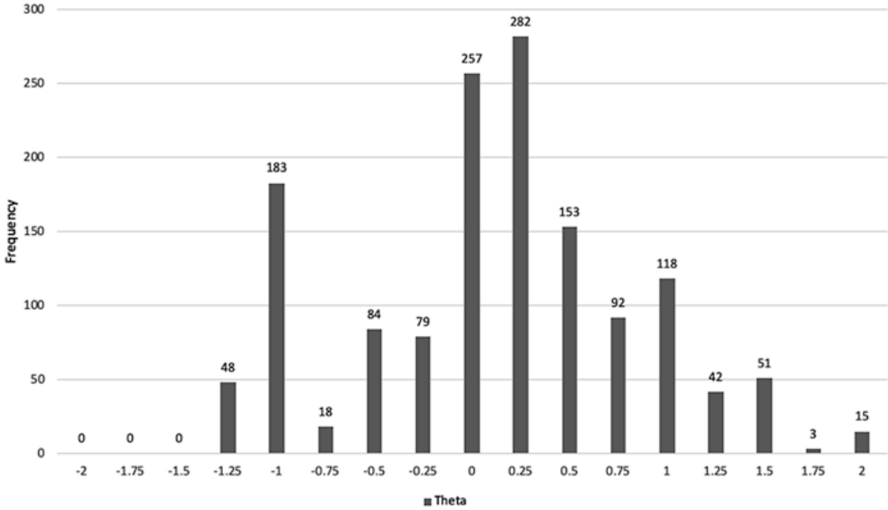


Fig. 11.3 Distribution of estimates of deductive reasoning ability

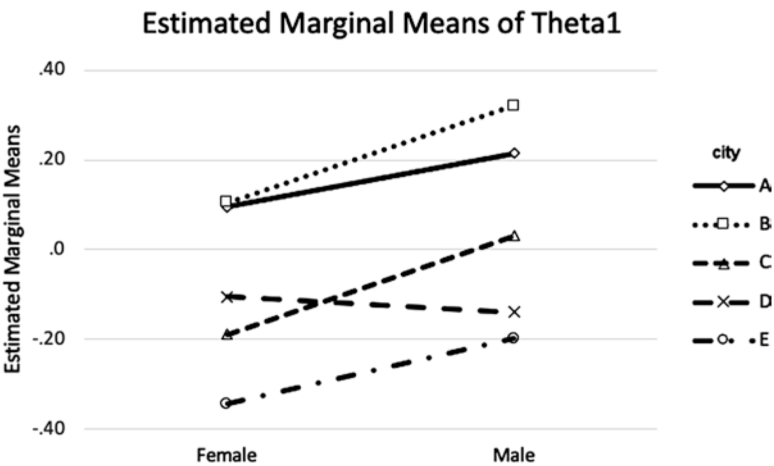


Fig. 11.4 Regional and gender differences in plausible reasoning ability

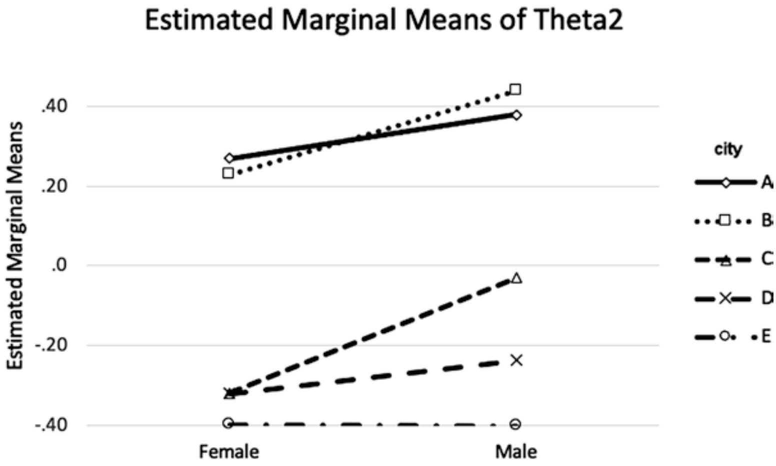


Fig. 11.5 Regional and gender differences in deductive reasoning ability

Table 11.4 Multivariate tests with regional and gender effects in plausible and deductive reasoning

Effect		Value	*F*	Hypothesis *df*	Error *df*	*p*
City	Wilks' lambda	0.845	31.738	8	2894	0.000
Gender	Wilks' lambda	0.990	7.054	2	1446	0.001
City*gender	Wilks' lambda	0.992	1.477	8	2892	0.160

11.5.2 Regional and Gender Differences in Students' Abilities of Plausible and Deductive Reasoning

It can be observed from Figs. 11.4 and 11.5 that students in city A and city B have better reasoning and proving abilities while students in city E perform worse. Further statistical analysis displays that there are significant regional differences in students' plausible and deductive reasoning abilities (see Tables 11.4 and 11.5). In the meantime, Figs. 11.4 and 11.5 show the differences between male and female students in the abilities of mathematical reasoning and proving. Whether it relates to plausible reasoning or deductive reasoning, the average ability value of boys is higher than that of girls, and the difference is statistically significant (see Tables 11.4 and 11.5).

Table 11.5 Regional and gender differences in plausible and deductive reasoning

Source	Dependent variable	df_1	df_2	F	p
City	Plausible reasoning	4	1456	22.489	0.000
	Deductive reasoning	4	1456	65.007	0.000
Gender	Plausible reasoning	1	1456	11.628	0.001
	Deductive reasoning	1	1456	11.566	0.001

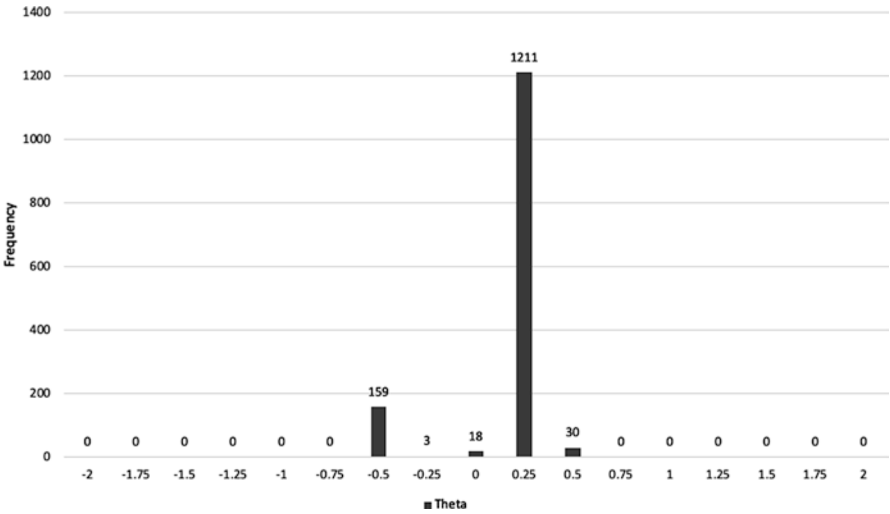


Fig. 11.6 Distribution of arithmetic reasoning ability

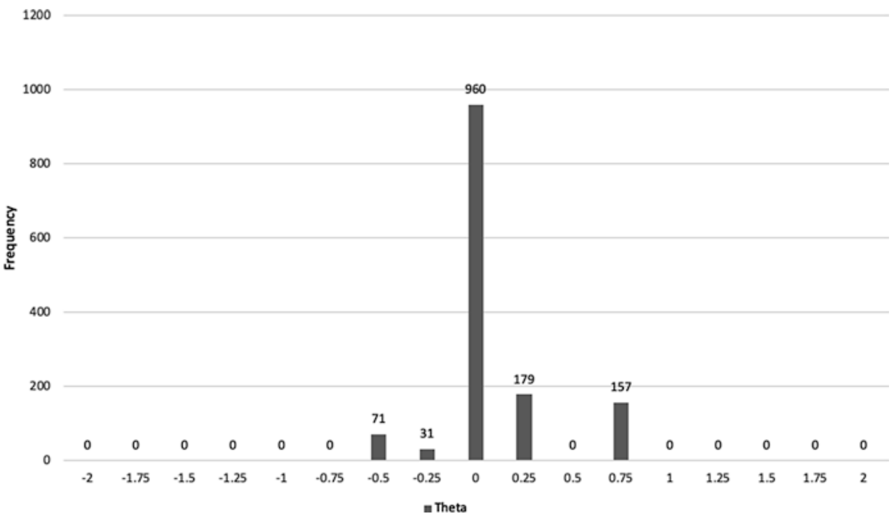


Fig. 11.7 Distribution of algebraic reasoning ability

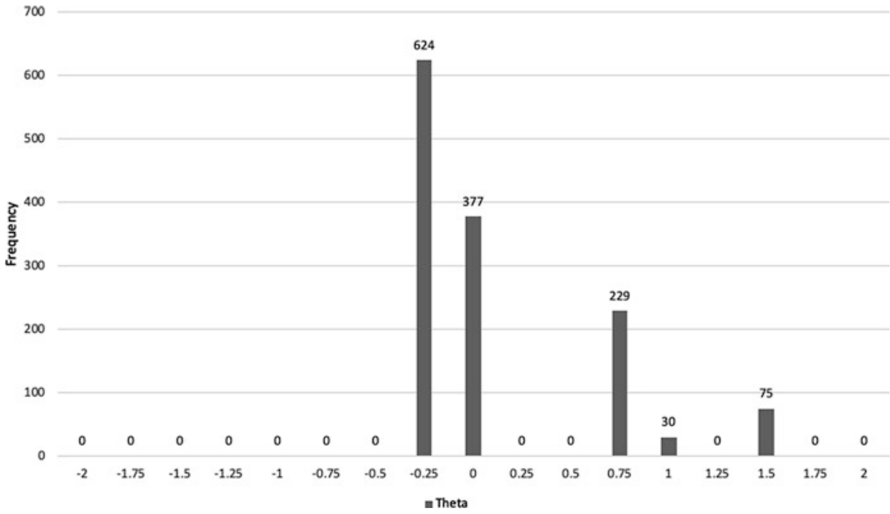


Fig. 11.8 Distribution of geometric reasoning ability

Table 11.6 Correlation analysis of different reasoning abilities ($N = 1464$)

	Plausible reasoning	Deductive reasoning	Arithmetic reasoning	Algebraic reasoning	Geometric reasoning
Plausible reasoning	1	0.673***	0.412***	0.732***	0.546***
Deductive reasoning		1	0.445***	0.503***	0.928***
Arithmetic reasoning			1	0.313***	0.234***
Algebraic reasoning				1	0.351***
Geometric reasoning					1

Note. ***$p < 0.001$

11.5.3 Distribution of Students' Abilities of Arithmetic, Algebraic and Geometric Reasoning

After investigating the reasoning performance of students based on different subject content, ability distribution diagrams can be drawn as Figs. 11.6, 11.7 and 11.8. According to these figures, more students performed slightly above average in arithmetic reasoning, and most students are at the average level for algebraic reasoning. The performance of geometric reasoning is the most dispersed. More than one-fifth of the students' ability estimates are not less than 0.75, some of which can reach 1.5. The specific performance of these students can be seen in the analysis of the rigour of students' reasoning later in this chapter.

The results of the correlation analysis indicate that students' abilities of arithmetic, algebraic and geometric reasoning are significantly and positively correlated (see Table 11.6), and there are no significant differences among the means of the three. It is noted that students' geometric and deductive reasoning abilities are not only significantly and positively correlated, but the correlation coefficient is as high as 0.93, which may reflect the important role played by plane geometry courses in junior high school in the cultivation of deductive reasoning ability.

11.5.4 Regional and Gender Differences in Students' Abilities of Arithmetic, Algebraic and Geometric Reasoning

Further concentration on students' regional and gender differences in reasoning abilities in different learning contents leads to the following findings (see Figs. 11.9, 11.10 and 11.11).

First, there was a statistically significant regional difference in students' arithmetic, algebraic and geometric reasoning abilities (see Tables 11.7 and 11.8). In city A

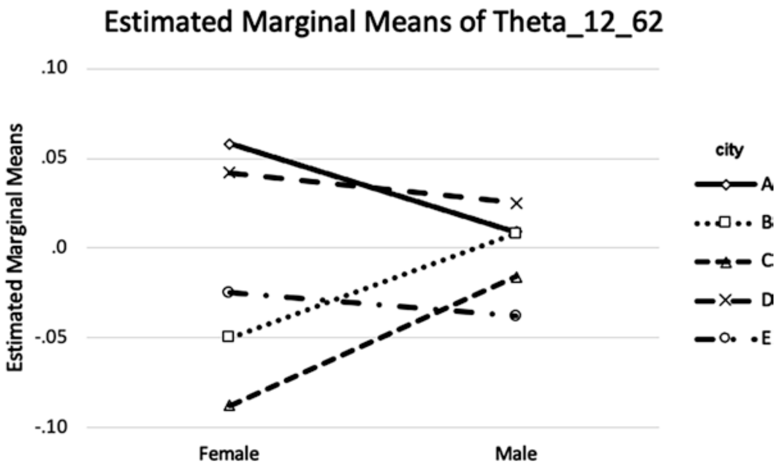


Fig. 11.9 Regional and gender differences in arithmetic reasoning ability

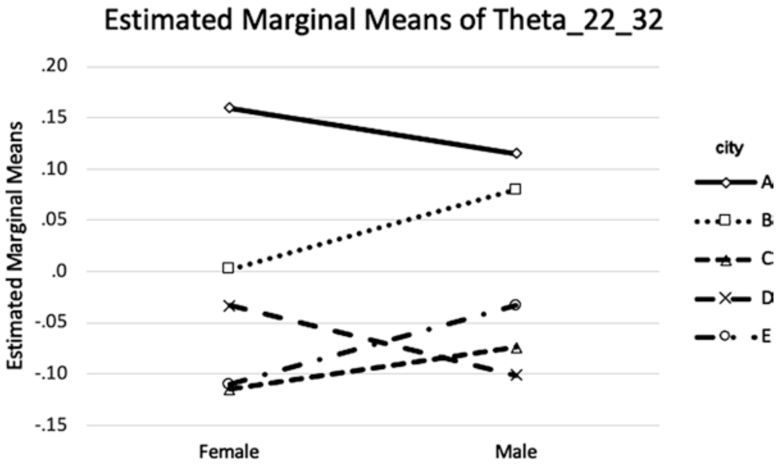


Fig. 11.10 Regional and gender differences in algebraic reasoning ability

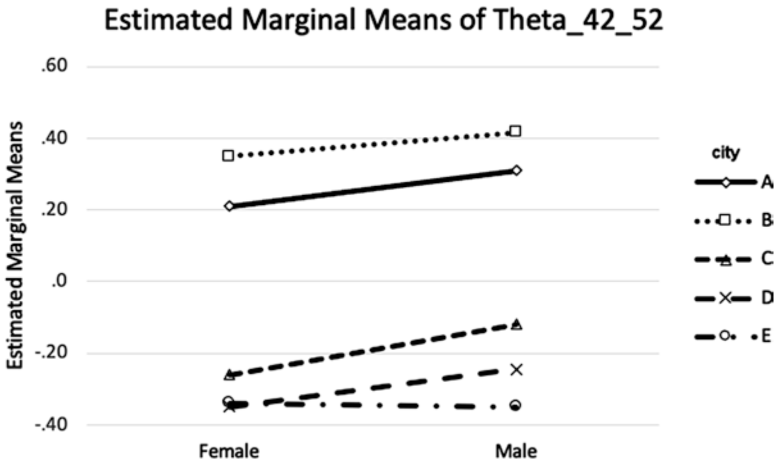


Fig. 11.11 Regional and gender differences in geometric reasoning ability

Table 11.7 Multivariate tests with regional and gender effect in arithmetic, algebraic and geometric reasoning abilities

Effect		Value	*F*	Hypothesis *df*	Error *df*	*p*
City	Wilks' lambda	0.821	24.656	12	3823	0.000
Gender	Wilks' lambda	0.997	1.248	3	1445	0.291
City*gender	Wilks' lambda	0.987	1.569	12	3823	0.093

Table 11.8 Regional and gender difference in arithmetic, algebraic and geometric reasoning abilities

Source	Dependent variable	df_1	df_2	*F*	*p*
City	Arithmetic reasoning	4	1456	5.916	0.000
	Algebraic reasoning	4	1456	12.945	0.000
	Geometric reasoning	4	1456	65.023	0.000
Gender	Arithmetic reasoning	1	1456	0.474	0.491
	Algebraic reasoning	1	1456	0.422	0.516
	Geometric reasoning	1	1456	3.700	0.055

and city B, as cities with students of relatively strong reasoning ability, students' performances in different content areas differ: students in city A perform better than those in city B in numbers and algebra, while students' performances in geometric reasoning in the two regions are exactly the opposite. This may be occurring because the mathematics curriculum in city B somehow differs from that in other regions of mainland China.

Besides, comparing Figs. 11.9, 11.10 and 11.11, it can be seen that the gender difference between males and females in reasoning ability lies primarily in geometric reasoning, while gender differences in arithmetic reasoning and algebraic reasoning are not significantly different (see Tables 11.7 and 11.8).

11.5.5 Students' Performance in the Rigour of Reasoning and Proving

11.5.5.1 Students' Performance in the Rigour of Algebraic Reasoning and Proving

The test results for Questions 2 and 3 reflect to some extent the rigorous performance of students when solving algebraic problems. Most students (over 70%) were aware of the need to demonstrate mathematical propositions in a general sense. However, there are still some students who employed the special cases as proof.

Take Question 2 as an example. Here, 28.1% of the students justified their results solely by giving examples rather than by providing general proof. For instance:

Answer: It is even.

$$\text{E.g. } 1\times 2=3,\ 3\times 4=12,\ 5\times 6=30$$

They are all even.

So the product of two consecutive natural numbers is even.

However, in Question 3, the proportion of giving special examples drops to 14.0%. For instance:

$$\text{Answer}: \sqrt{n+\frac{n}{n^2-1}}=n\sqrt{\frac{n}{n^2-1}},$$

$$\text{If } n=5,\ \sqrt{5\frac{5}{24}}=\sqrt{\frac{125}{24}}=5\sqrt{\frac{5}{24}},$$

$$\text{If } n=6,\ \sqrt{6\frac{6}{35}}=6\sqrt{\frac{6}{35}}.$$

The reason for the differences above may be related to the presentation of the proposition. Compared with Question 2, Question 3 has more obvious symbolic features, and students are more inclined to demonstrate propositions in the general sense of symbolic operations. The proposition to be proved in Question 2, 'The product of two consecutive natural numbers is even', appears in the form of character representation. Therefore, students also exhibit multiple choices in their proofs.

Many students (about 20.5%) used character representation in this question to demonstrate that: two consecutive natural numbers must be an odd number and an even number; because the product of an odd number and an even number is even, the product of two consecutive natural numbers is even. Nearly half of the students employed different representations in their proving. Interestingly, 6.3% of them consciously used symbolic operation to prove the proposition. Among them, a few students were able to use the idea of classified discussion and symbolization to make their proofs rigourous, as in the following example.

Answer:

1. If the first one in the two consecutive natural numbers is even, presume that the two numbers are $2n$ and $2n + 1$, then $2n(2n + 1) = 2[n(2n + 1)]$ is even;
2. If the first one in the two consecutive natural numbers is odd, presume that the two numbers are $2n + 1$ and $2n + 2$, then $(2n + 1)(2n + 2) = 2[(n + 1)(2n + 1)]$ is even.

Based on (1) and (2), the product of two consecutive natural numbers is even.

In addition, from students' performance in Question 3, the proper method of symbolic representation is another aspect that needs to be considered for the strictness of algebraic reasoning. Several students used $\sqrt{n\frac{n}{n^2-1}}$ to refer to $\sqrt{n+\frac{n}{n^2-1}}$, without realizing the difference between the number representation $\sqrt{2\frac{2}{3}}$ and the letter representation $\sqrt{n+\frac{n}{n^2-1}}$.

Meanwhile, inductive reasoning from special to general needs to consider the value range of n that can make the proposition true. Some students were aware of this point and noted that $\sqrt{n+\frac{n}{n^2-1}}=n\sqrt{\frac{n}{n^2-1}}$, in which n is a natural number no less than 2, showing a rather strong consciousness of rigour.

11.5.5.2 Students' Performance in the Rigour of Geometric Reasoning and Proving

The test results for Questions 4 and 5 reflect, to some extent, the rigourous performance of students in geometric reasoning. Considering that students needed intuitive figures in the proving of plane geometry, researchers combined the figures drawn by students and their proof to judge whether their demonstrations were conducted in a general sense. It was found that less than 10% of students mistook the proving of special circumstances for a general one.

Question 5 required students to complete the proposition and prove it. More than 60% of the students were able to find at least one reasonable condition and complete the proof. Furthermore, 13.9% were able to combine the four given conditions in pairs (six combinations in total) and identify all the conditions that make the proposition true (four in total). Some students not only demonstrated the four true propositions, but also used the counterexamples to illustrate that the other two propositions are not true (see Fig. 11.12).

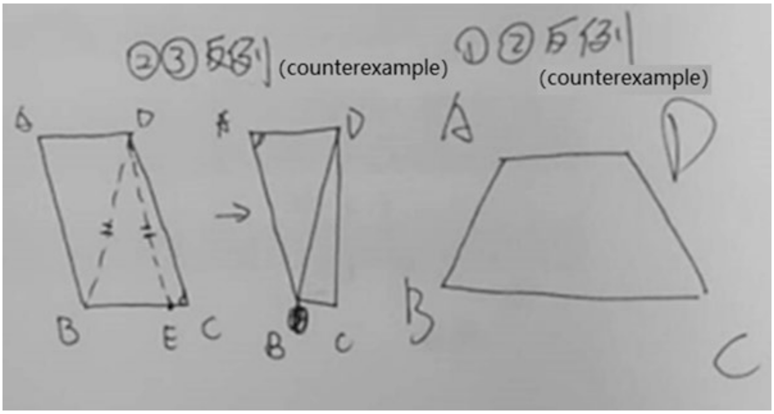


Fig. 11.12 Counterexamples shown in question 5

In contrast, students encountered greater obstacles in Question 4. Here, 13.1% of the students mistakenly believed that the inverse proposition of the proposition in Question 4 is, ‘Half of the hypotenuse of the right triangle is equal to the median of the hypotenuse’, and nearly one in four students did not give any answer. Of the students who gave an answer, one-third were able to complete the demonstration. Although these students could find the inverse proposition of the original proposition and complete the proving, nearly half of them expressed the inverse proposition in an unstrict way.

For instance:

If the median on the hypotenuse of a triangle is equal to half the hypotenuse, the triangle is a right triangle.

In the example above, the student misused the term ‘hypotenuse’, unaware that this term only applies to right triangle instead of all triangles. In terms of this question, students who were able to rigorously state the inverse proposition account for 13.9% of the students tested.

11.5.5.3 Students’ Performance in the Rigour of Arithmetic Reasoning and Proving

Questions 1 and 6 examined students’ performance in arithmetic reasoning and proving. Here, 93.4% of the students chose to take five beads as a group in the first question. Three students chose to treat 13 beads as a group. There may be different answers if no students asked questions. In the process of demonstration, more than 70% of the students only listed the formula and directly gave the answer about the colour of the bead. Other students tried to explain the meaning of the formula in words, and 7.8% of the students were able to solve the problem with clear and complete word expressions.

For instance:

$$\text{Answer}: 36 \div 5 = 7 \cdots 1,$$

In every 5 beads, the first 3 are white and the last 2 are black. The 36th one is the one more bead after 7 times’ repetition according to this rule. And in every 5 beads, the first one is white. So the 36th bead is white.

In the proving of Question 6, students needed to use specific formulas to explain that casting 12 darts is a possible situation and it is necessary to demonstrate that 12 darts is the minimum number. This means that, when casting fewer than 12 darts, it is impossible to reach 150 points. It is worth mentioning that 64.9% of the students consciously attempted to argue that fewer than 12 darts is impossible, but only 6.8% were able to complete the proof rigourously (see the example below).

For instance:

Cast at least 12 darts: $13 \times 9 + 11 \times 3 = 150$.

If casting 11 darts only, at most get $11 \times 13 = 143$ points, which is less than 150.

So at least cast 12 darts.

11.6 Discussion

11.6.1 Balanced Development of Plausible and Deductive Reasoning Abilities

This chapter finds that most eighth-grade students can conduct simple plausible reasoning and demonstrate propositions in the general sense. The results of this study are consistent with the goals of the current mathematics curriculum standards in China. However, within the country, there are significant regional differences in students' abilities of reasoning and proving. Under the unified national expectation curriculum, do the regional differences in students' reasoning and proving abilities stem from the teachers or from the family environment? Further research is needed on this question. In addition, in terms of different mathematical content, students only exhibit gender differences in geometric proofs. This may be related to the greater degree of discrimination among the geometric questions of this test and the content characteristics of the geometric problems.

It is necessary to draw attention to the fact that there is a more obvious individual difference in students' ability of deductive reasoning than in plausible reasoning, especially when the proposition that needs to be proved does not contain symbolic information itself. For instance, in Question 2, about one-quarter of the students used special cases instead of a generalized proof. This phenomenon is similar to Morris's findings (2002, 2007), where students were empirically proven to use experience as a general proof. Students seem to lack an effective understanding of the proof strategy, or lack the ability to discern what is an invalid demonstration. Although the ability to construct proofs in the field of mathematics is important (Weber, 2001), it is of equal significance for students to judge whether the proof is correct or not (Selden & Selden, 2003).

Interestingly, when faced with symbolic propositions (such as Question 3), the proportion of students with similar performance drops significantly. Similarly, work by Knuth et al. (2009, pp. 153–170) found that, when the quantity of proof carried by different problems is different, it will in turn affect the proof produced by students. Therefore, in the teaching and evaluation of mathematical demonstrative ability, what are the roles of different questions in promoting or reflecting the degree of students' understanding and mastery? That is a question worth further research.

Additionally, the ability of students to discover new propositions with the help of plausible and deductive reasoning needs to be improved. When there are multiple conjectures in plausible reasoning, further trials are needed to overturn the false conjectures so that the correct propositions can be proved by demonstration; here, many students still have obvious difficulties. For example, when answering Question 3, one in five students got the false proposition through inductive conjecture, but failed to further overthrow it with a counterexample. Only one in ten students succeeded in guessing the true proposition, and almost all of these students were able to demonstrate the correctness of their guess. This also shows that the high-level plausible reasoning ability is inseparable from the ability of deductive reasoning. School instruction needs to provide students with the opportunity to feel that 'examples' can be used to overturn propositions, which can be employed to reinforce confidence in a conjecture, but cannot serve as an argument for generalized outcomes.

11.6.2 Students' Performance in the Rigour of Mathematical Reasoning and Proving

Although this chapter finds that some students did not achieve the expected goals of the mathematics curriculum in terms of the abilities of reasoning and proving, some top students showed a rigorous pursuit in the process of deductive reasoning, which makes educators more confident in providing high-class opportunities to learn mathematical reasoning and proving for the junior high school students, especially the gifted ones.

Previous studies have pointed out that the reason why students cannot distinguish between deductive and inductive reasoning in high school and even university may be the lack of corresponding learning opportunities in junior high school. It is feasible to develop students' plausible and deductive reasoning abilities in junior high school as long as appropriate teaching methods are used (Stylianides, 2009).

The study also finds that there are some eighth-grade students who can explain the meaning of the formulas in clear wording, demonstrate the propositions in a rigourous symbolic language, carry out an unbiased classified discussion in the proving process, pay attention to the value range of alphabetic symbols from the special to the general reasoning process, use accurate terminology in expressing the propositions, overturn false conjectures with counterexamples and employ proof by contradiction in the proving process, among other abilities. If the teacher consciously develops students' abilities mentioned above, it is reasonable to expect an increase in the proportion of these types of students.

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Chapter 12
Mathematical Modelling in China: How It Is Described and Required in Mathematical Curricula and What Is the Status of Students' Performance on Modelling Tasks

Xiaoli Lu and Jian Huang

Abstract This chapter starts with a review of the mathematical modelling competency in mathematics curricula all over the world and puts its emphasis on the development and promotion of mathematical modelling competency in China's mathematical curricula through analysing the curricular standards or syllabi from 1902 to 2018. Moreover, an investigation of students' modelling competency was conducted to 1359 eighth-grade students from five cities located in the south-western, south-eastern, eastern, north-western, and midland regions of China. With the students' paperwork on three modelling tasks, the chapter reports to which stage of modelling cycle the students achieved to represent their modelling competency. The results show an increasing attention to the promotion of mathematical modelling in the development of mathematics curriculum in China. The students were still used to a problem-solving process rather than modelling. There were significant differences at the individual level, but not the school or district level when considering their modelling competency.

Keywords Mathematical competency · Mathematical modelling competency · Mathematics curriculum · Two basics · Modelling cycles · The teaching and learning of mathematical modelling · Modelling cycles · Students' performance · Solving real-world problems · Mainland China · East and West · Text analysis

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12.1 Introduction

Mathematical modelling has gained considerable influence in mathematics education worldwide. It is believed that mathematical modelling has the potential to facilitate critical citizenship with the key skills to face the complex and rapidly changing world, due to its aims at promoting young people's competencies to solve real-world problems via mathematics (Kaiser, 2017). Because of its increasing importance, mathematical modelling has become a hot topic among mathematics curricula, especially in Western countries, such as Australia, Finland, France, Germany, the United States (US), and the United Kingdom (UK). Recently, mathematical modelling has also been increasingly emphasised in China's centralised mathematics curricula, especially since 2003, when it published *Mathematics curriculum standards for senior secondary schools (experimental version)* (MOE, 2003).

In the mathematics curricular standards for Shanghai's primary and secondary schools (Shanghai Municipal Education Commission, 2004), it has emphasised the importance of the mathematical thinking of mathematical modelling in the curricular objectives during the phase of grade 6 to 9, and specifically claimed that grade 10 to 12 students should be 'conducting mathematical modelling, solving [the problems from the real-world] and interpreting [the real-world situations]' (p.38, translated by the authors). Furthermore, viewing modelling as an essential component of mathematical competence was included in the 2010 version of China's mathematical curricular standards for students in grades 1–9 (Zhang, 2011). More recently, mathematics modelling was recognised as one of the six core mathematical competencies for students to develop in the *Mathematics curriculum standards for senior secondary schools (2017 version)* (MOE, 2018). Due to these developments, addressing the question of how mathematical modelling can be integrated into the teaching and learning of mathematics with the aim of promoting students' modelling competence has become the focus of many studies in mathematics education.

Although its importance has been well recognised, a consensus regarding the most effective approaches to integrating modelling into mathematics education has not been reached, especially as the approaches relate to cultural differences. Cultural influences have been considered an essential factor of East Asian students' consistently outstanding performance on large-scale international assessments (e.g. *Programme for International Student Assessment* [PISA]) when compared to their Western counterparts. Shanghai's students ranked highest on the first PISA they participated in, which drew the international community's attention to China's approaches to mathematics education. China has consistently emphasised the curricular idea of 'two basics' (i.e. basic knowledge and skills in mathematics) for a long time, which may have contributed to Chinese students' mathematics achievement. However, it is unclear whether this strength also impacts Chinese students' mathematical modelling knowledge as it has only recently been focused on in the mathematics curriculum. Very little information regarding the historical development and state-of-the-art of mathematical modelling in the Chinese mathematics curriculum is available.

This chapter investigates how mathematical modelling has been described or required in China's mathematics curriculum over the past 100 years, as well as Chinese students' performance in mathematical modelling activities. By doing so, this study aims to promote effective practices and empirical studies of the teaching and learning of mathematical modelling in China. It also seeks to provide insights into modelling practices and theories in the international community by focusing on the case of China.

12.2 Literature Review

To situate our study within the international community, we review the literature on modelling cycles, which is considered a key feature of modelling activities and is related to recent perspectives of the teaching and learning of mathematical modelling (Kaiser, 2017), mathematical modelling in different countries' curricula, and students' mathematical modelling competencies.

12.2.1 Modelling Cycles

According to different goals of learning and teaching mathematics at various educational levels, a classification of theoretical perspectives of mathematical modelling in schools was developed by Kaiser and Sriraman (2006): realistic or applied modelling, epistemological or theoretical modelling, educational modelling, contextual modelling or model-eliciting perspective, socio-critical and sociocultural modelling, and cognitive modelling as a meta-perspective. These various perspectives show different characterisations of the modelling process as idealised in different modelling cycles. Therefore, theoretical and empirical studies of these cycles have become a central part of the teaching and learning of mathematical modelling (Borromeo Ferri, 2006). Five prevalent types of modelling cycles addressed in the literature are listed below and briefly introduced.

The modelling cycle of applied mathematics. Mathematical modelling is considered a cycle between the mathematics world and the rest of the world (Pollak, 1979). The cycle begins in 'the rest of the world', moves on to classical applied mathematics, followed by 'applicable mathematics' and, finally, back to 'the rest of the world' (p. 256). The cycle has a long-term, significant impact on the teaching and learning of mathematics.

The didactical or pedagogical modelling cycle. In this type of modelling cycle, mathematical modelling is emphasised, along with its process- and content-related goals. Pedagogical modelling not only encourages the learning process of modelling but also deals with modelling examples used to introduce and practise mathematical methods, through which modelling is incorporated into the teaching and learning of mathematics (Greefrath & Vorhölter, 2016). Blum (1996) and

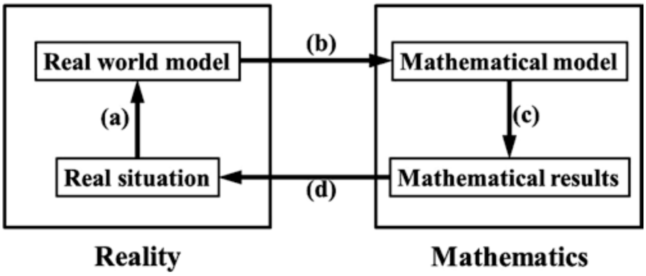

Fig. 12.1 A modelling cycle proposed by Blum (1996) and Kaiser (1995)

Kaiser (1995) described a modelling cycle from this perspective (Fig. 12.1) in German publications. Their proposed cycle includes four important steps in modelling: idealising, mathematising, investigating the model, and interpreting the results (cited in Borromeo Ferri, 2018).

The psychological modelling cycle. Historically, based on psychological research, the psychological modelling cycle emphasises a 'situation model' that connects real-world problems with a mathematical model. The situation model is integrated into the modelling cycle as an additional step between the real situation and the model (Blum & Leiß, 2007).

The mathematical modelling cycle from a cognitive perspective. With the inclusion of situation modelling in the modelling cycle, individuals' cognitive processes during modelling activities became the focus of modelling research. The cognitive perspective of modelling cycles emphasises the situation model and the mental representation of a situation for diagnostic purposes (e.g. Borromeo Ferri, 2007).

*The mathematical modelling cycle integrated with specific aspects (*e.g. *metacognitive strategies)*. Stillman (2011) proposed the application of metacognitive knowledge and strategies in modelling tasks at the secondary school level.

These five types of modelling cycles show the common features of mathematical modelling: a real-world situation is incorporated into mathematical modelling, and interpreting the mathematical models to gain real-world results also needs to be validated (as characterised in Fig. 12.2). Differences between modelling cycles reflect the theoretical perspectives underlying them and the characteristics of the teaching and learning of mathematical modelling they support.

12.2.2 Mathematical Modelling in Curricula

Modelling competencies play an essential role in many countries' national curricula, such as Australia, Germany, the US, and China, which supports the relevance of mathematical modelling at the international level (Kaiser, 2017).

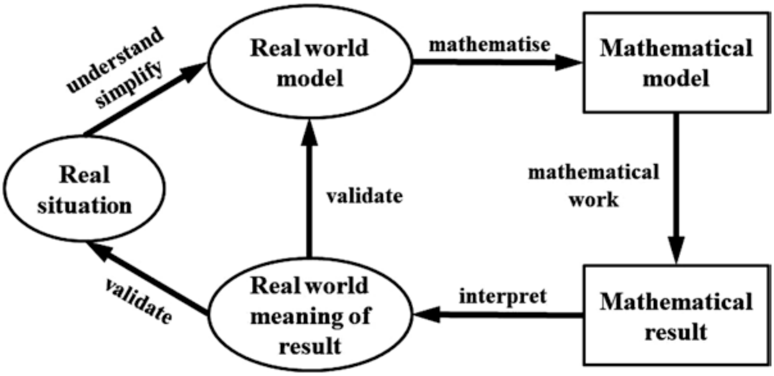

Fig. 12.2 Mathematical modelling cycle (Kaiser & Stender, 2013)

12.2.2.1 Australian Curriculum

Real-world problem-solving expertise is an espoused goal for all students according to the Australian curriculum. Specifically, the country's mathematics curriculum focuses on the development of increasingly sophisticated and refined mathematical understanding, fluency, reasoning, and problem-solving skills (ACARA, 2019). Mathematical modelling has been advocated for in Australian mathematics curriculum since 1990 when *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990) was published. In the curriculum, mathematical modelling is often described as a cycle processed from a real situation problem to its mathematical representation and solution through the application of mathematics (p. 61).

Despite its long-term promotion in the national curriculum, mathematical modelling uptake has ranged from minimal or none to mandatory inclusion in the teaching, learning, and assessment practices of mathematics. The regions of Australia have experienced varying levels of uptake due to cultural differences. For example, the teaching and learning of mathematical modelling have attracted less attention in New South Wales than in Queensland because it emphasises rigorous mathematics learning in schools (Geiger, 2005).

12.2.2.2 German Curriculum

Mathematical modelling has been advocated for in the teaching and learning of mathematics in Germany since the 1980s. The country's mandatory mathematical standards for mathematical modelling were introduced in 2003 and have developed into one of the six general mathematics competencies in school mathematics in Germany. Many empirical studies have been performed and international cooperative initiatives were undertaken in Germany regarding the teaching and learning of mathematical modelling. These have paid particular attention to students' modelling competencies, which will be introduced in more detail in the next section. Due to the important role teachers play in delivering effective teaching and learning activities, empirical studies have increasingly focused on teacher competencies in teaching modelling and constructing the learning environment (Greefrath & Vorhölter, 2016).

12.2.2.3 The US' Common Core Curriculum

Modelling with mathematics is one of the eight standards for mathematical practice in the US' *Common Core State Standards for Mathematics*. It is also one of the six categories of mathematical content and has strong connections to the other categories: *number and quantity*, *algebra*, *functions*, *geometry*, and *statistics and probability*. Lu, Cheng, Xu, and Wang (2019) reviewed these standards and identified connections between modelling and other mathematical contents. They found that modelling is connected to all of the content in *statistics and probability*, 1/9 of the content in *number and quantity*, 8/27 of the content in *algebra*, 3/7 of the *function* content, and 6/43 of the *geometry* content. Modelling appears to be an essential component of the curriculum, which requires independent development and is promoted via the teaching and learning of other mathematical content.

There has been a long history of ‘modelling eliciting activities’ in mathematics education in North America (Kaiser, 2017). These activities are problem-solving activities constructed using instructional design principles so that students can make sense of meaningful situations and invent, extend, and refine their own mathematical constructs during the activities (Lesh & Doerr, 2003). In this sense, American students may have more opportunities to practice mathematical modelling and applications than Chinese students. From Lu’s (2011) analysis of the mathematical examples and exercises in mathematics textbooks used in high schools in China and the US, she found more mathematical problems with real-world situations in American textbooks than in the Chinese ones.

12.2.2.4 China’s Curriculum

In its long-term historical development, China’s curriculum has focused on sound mathematical knowledge in the teaching and learning of mathematics. Like New South Wales in Australia, mathematical modelling has not been emphasised until recently. Due to the increasing needs of prompting responsible and qualified citizenship in such a rapidly changing world, mathematical modelling, as well as five other key competencies, has become the focus of mathematics education in China, with the newly promulgated national curricular standards of high school mathematics (MOE, 2018) as a landmark. However, compared to Western countries, such as Germany and the US, there is little information available on the state-of-the-art of teaching mathematical modelling in China. Hence, theoretical and empirical studies are needed in this context.

12.2.2.5 Summary

In the past, in some contexts, such as China and New South Wales in Australia, which emphasised ensuring mathematics teaching maintained a certain level of rigorousness and that students had a sound mathematical foundation, little attention was paid to modelling in the teaching and learning of mathematics. Due to globalisation and rapid societal and technological changes, mathematical modelling has become a central topic worldwide. Western countries, such as Germany and the US, have promoted mathematical modelling in their curricula for a much longer time. Attention should be paid to making use of the advantages of Western experiences in promoting mathematical modelling and the combination of the Eastern strengths in mathematical foundations to develop students' mathematical modelling competencies. Therefore, mathematical modelling in Eastern countries' curricula needs to be explored.

12.2.3 Students' Mathematical Modelling Competencies

Although there has been a widespread consensus on the necessity of promoting the teaching and learning of mathematical modelling, researchers still have not reached agreement on the measurement of mathematical modelling competencies. In recent decades, many efforts have been made to define or describe the construct of modelling skills, abilities, and competencies, and several aspects of modelling competencies have been recognised: the global modelling competency referring to abilities for one to successfully perform the entire modelling process, sub-competencies referring to competencies necessary to individual phases of modelling cycle, and additional competencies such as metacognitive strategies, and non-cognitive aspects.

A holistic approach has, typically, been employed to measure the global modelling competency involved in performing a modelling process. The measures used are students' written work and audio/video recordings of students' working processes. Data analyses have focused on categorising the levels of modelling competency. Stillman (2019) presented five levels of modelling competency according to the nature of the real-world situation and the complexity of modelling:

- Level 0: No constructive solution approach and no reasonable solution.
- Level 1: Implementing a representational change between a context and mathematical representation. Using familiar and directly recognisable standard models for describing a given situation with an appropriate decision.
- Level 2: Describing the given situation by mathematical standard models and mathematical relationships. Recognising and setting general conditions for the use of mathematical standard models.
- Level 3: Applying standard models to novel situations, finding a suitable fit between mathematical modelling and real situations.

- Level 4: Complex modelling of a given situation; reflecting on solution variants or model choice and assessing the accuracy or adequacy of underlying solution methods.

Ludwig and Xu (2010) also described students' modelling competency from a holistic perspective. They assumed that how far students go in the modelling cycle when performing mathematical tasks will determine to what extent the students overcame cognitive obstacles. They, therefore, categorised six types of student performance according to the modelling process (p. 79):

- The student has not understood the situation and is not able to sketch or write anything concrete about the problem.
- The student understands the given real situation but is not able to structure and simplify the situation or cannot find connections to any mathematical ideas.
- After investigating the given real situation, the student finds a real model through structuring and simplifying but does not know how to transfer this into a mathematical problem (i.e. the student creates a kind of world problem about a real situation).
- The student is able to not only find a real model but also translate it into a proper mathematical problem, but cannot work with it clearly in the mathematical world.
- The student is able to develop a mathematical problem from the real situation, work with this mathematical problem in the mathematical world, and achieve mathematical results.
- The student is able to experience the mathematical modelling process and validate the solution of a mathematical problem in relation to the given situation.

This kind of holistic approach describes, in general, students' performance of modelling tasks but may miss some important aspects of the complex modelling process. Moreover, completing a modelling task is usually time-consuming and, sometimes, requires teamwork. Hence, the holistic approach is not quite suitable for large-scale assessments.

Atomistic approaches, such as multiple-choice items, are employed to assess the sub-competencies of mathematical modelling, e.g. simplifying real-world problems. These approaches are more quantitative and use various psychometrical models, such as one-factor-model and multi-dimension models (Stillman, 2019).

Many studies have focused on the atomistic approaches. Two of the four well-known strands of international discussions on modelling competencies (Kaiser & Brand, 2015) focus on sub-competencies. A British and Australian assessment research group worked on developing assessment instruments for modelling competencies (e.g. Haines & Izard, 1995); they identified various sub-competencies and developed multiple-choice items to evaluate them. This allowed a pre- and post-design in many studies. German researchers have also focused on the study of modelling sub-competencies. Kaiser (2007) proposed five sub-competencies developed during the various phases of the modelling cycle (p. 111):

- Competency to understand real-world problems and construct a reality model
- Competency to create a mathematical model out of a real-world model

- Competency to solve mathematical problems within a mathematical model
- Competency to interpret mathematical results in a real-world model or a real situation
- Competency to challenge solutions and, if necessary, perform an additional modelling process

In the development of modelling competencies, metacognition is recognised as an important influencing factor, and it 'describes thinking about one's own thinking and controlling one's own thought processes' (Maaß, 2006, p. 118). Metacognition is also emphasised by an Australian modelling group that employs reflective metacognitive approaches during modelling activities. More recently, the integration of metacognition into the teaching and learning of mathematical modelling has also been promoted in German classes (Vorhölter, 2018).

Atomistic approaches allow researchers to investigate specific aspects of mathematical modelling in-depth, but these approaches are not readily applicable to demonstrating a student's global modelling competency. In future studies of mathematical modelling, combining holistic and atomistic approaches will probably become the focus to obtain comprehensive measurements.

Various methods have been used in many countries to measure students' mathematical modelling competencies. However, few studies have evaluated Chinese students' modelling competencies, and little is known about which methods are suitable for Chinese students.

12.3 Research Questions

To capture a comprehensive picture of the state-of-the-art of the teaching and learning of mathematical modelling in China, we analysed the content of mathematical modelling in the curricular syllabi/standards of mathematics from 1902 to 2018 to determine the trajectory of its development in the curriculum. We also conducted a general exploration of current Chinese students' modelling competencies by assigning modelling tasks to grade 8 students. In this study, the following research questions were addressed:

1. How is mathematical modelling described or required in the curricular syllabi/standards of mathematics in China from 1902 to 2018?

   How many textual references to mathematical modelling are in the different curricular syllabi/standards?

   Do the texts reflect a kind of evolution of mathematical modelling in China's mathematics curriculum? If yes, what is it?
2. What is the status of current students' mathematics competency in China?

   Which stage of the modelling process can the students achieve while performing each modelling task?

   Are there gender, regional, school, or individual differences in the students' modelling competency?

12.4 Research Methods

To answer the research questions, we carefully examined the past 100 years of China's mathematics curricular documents. An investigation was also conducted to explore Chinese students' modelling competency by administering a test to 1359 grade 8 students (ages 13–14) in 2017.

12.4.1 Text Analysis: Mathematical Modelling in Mathematics Curricula

The curricular documents analysed in the study are mathematics curricular syllabi and standards published in China from 1902 to 2018. During 1902–2000, 24 primary mathematics syllabi and 43 secondary mathematics syllabi were published in the *Collection of primary and secondary curriculum standards and syllabus of the twentieth century in China (Mathematics volume)* (Curriculum and Teaching Materials Research Institute, 2001). After 2000, four national mathematics curricula were published: two for grade 1–9 students (in 2001 and 2011) and two for grade 10–12 students (in 2003 and 2017). A total of 71 mathematics curricular syllabi and standards were analysed; for the number of pages analysed, see Table 12.1.

We searched for the terms 'model' (*mo-xing*) and 'modelling' (*jian-mo*) in the 71 syllabi/standards to identify content in the text related to mathematical modelling. Some were excluded because they referred to geometric or physical objects instead of mathematical modelling. For instance, text referring to a 'geometric object model (cube or cuboid)' in the 1952 syllabus was not analysed.

The textual analysis was both data- and theory-driven. In the initial reading and coding of the texts, we identified text related to two aspects of modelling cycles (i.e.

Table 12.1 The number of pages of curricular documents analysed

Year	Publication	Number of Pages
1902–2000	Mathematics Volume of the Collection of Primary and Secondary Curriculum Standards and Syllabus of the twentieth century in China	685
2001	Mathematics Curriculum Standards for Full-Time Compulsory Education (experimental version)	102
2003	Mathematics Curriculum Standards for Senior Secondary Schools (experimental version)	122
2012	Mathematics Curriculum Standards For Compulsory Education (2011 version)	132
2018	Mathematics Curriculum Standards for Senior Secondary Schools (2017 version)	180

mathematics and modelling and reality and modelling), as well as a third category of affective aspects. Qualitative text analysis was used to produce descriptive codes (Kuckartz, 2014), as shown in Table 12.2. Using these codes, the texts were coded and categorised and, thus, were quantified to answer the first sub-question of the first research question. Then, by examining the texts repeatedly, we identified what the concepts, meanings, or processes/cycles of mathematical modelling were described to determine whether there was an evolution of mathematical modelling in the curricula (i.e. the second sub-question).

Using the analytic framework, we gained a comprehensive understanding of how mathematical modelling is described or required in the curricular documents in China from 1902 to 2018 according to the curriculum phases described in Chap. 1 of this book (see Table 12.6).

12.4.2 An Investigation of Students' Modelling Competency

To investigate the teaching and learning of mathematical modelling, which has been advocated for in China's recent national mathematics curricular standards, a study of grade 8 students' performance on modelling tasks was conducted in 2017. A total of 1359 students participated. These students were selected via stratified sampling (Cohen, Manion, & Morrison, 2000) from 15 schools in five cities (i.e. Cities A–E) located in the south-western, south-eastern, eastern, north-western, and midland regions of China, respectively; for the distribution of this sample, see Table 12.3.

The students were asked to complete a paper-based test consisting of three modelling tasks involving different contexts, mathematical content, and difficulty levels. Additional details regarding these tasks are presented in Table 12.4. Based on Blum and Kaiser's (1997, cited in Maaß, 2006) five modelling sub-competencies and a qualitative text analysis of some of the students' written work (about 300 of the 1359 students who participated), a coding scheme for the modelling stage students achieved in performing each task was constructed (see Table 12.5). Examples in Table 12.5 are from the students' written work on the first modelling task, 'Lanzhou noodles'.

With the students' written work coded, we can present which stages of competency the students have demonstrated in performing each modelling task. To summarise the students' achievement in performing the modelling tasks, we generated modelling competency scores scaled with the graded response model, one of the three polytomous item response theory models (Nerling & Ostini, 2010). Next, we determined whether the students' demographic characteristics (e.g. gender, school, or region) were related to their demonstrated level of modelling competency.

Table 12.2 Analytic framework for the description or requirements of mathematical modelling in the curricular content

Category	Code	Description	Example (translated texts)
C1 Mathematics and modelling	C11: Mathematical models	Using mathematical models or mathematical knowledge to construct mathematical models	Using these functions to develop models; selecting proper functional models; constructing probability models; using mathematical methods to construct models; establishing mathematical relations
	C12: Solving problems mathematically	Using mathematical knowledge to obtain a solution	Solving the model; calculating and obtaining the solution
	C13: Promoting mathematical learning	Enhancing the understanding of mathematical knowledge, acquiring skills, etc.	Developing mathematical knowledge; learning mathematical concepts and rules; strengthening the student's understanding of related knowledge; acquiring necessary knowledge and skills (through modelling)
C2 Reality and modelling	C21: Understanding real situations	Understanding the relationship between the real situation and mathematics	Posing mathematical problems; identifying mathematical relationships; finding proper objects to study from a mathematical perspective
	C22: Mathematising	Proposing mathematical problems based on a real situation; mathematising the quantity and relationships in the real situation	Expressing the problem using mathematical language; translating problems into mathematical problems; mathematising
	C23: Verifying the model	Verifying and improving the model	Improving the model; justifying the rationale of the model; verifying the solutions (in a real situation); reflecting on the modelling process
	C24: Applying it to the real world	Applying the result of modelling or models to the real world	Interpreting and applying; explaining economic phenomena
	C25: Solving real-world problems	Applying modelling to solve real-world problems	Dealing with realistic problems; solving real-world problems; solving simple real-world problems

(continued)

Table 12.2 (continued)

Category	Code	Description	Example (translated texts)
C3 Affective aspects	C31: Increasing interest	Stating the importance of modelling in increasing students' interest	Inspiring students' interest in learning mathematics
	C32: Improving attitude	Stating the importance of modelling in improving students' attitudes towards mathematics	Acquiring a relatively comprehensive understanding of mathematics; feeling familiar with mathematics; gaining emotional feeling improved when experiencing modelling; appreciating the value of applying mathematical theory

Table 12.3 Distribution of student sample by city

Cities		A	B	C	D	E	
Schools	S101	93	0	0	0	0	
	S102	93	0	0	0	0	
	S201	0	60	0	0	0	
	S202	0	86	0	0	0	
	S203	0	88	0	0	0	
	S301	0	0	64	0	0	
	S302	0	0	71	0	0	
	S303	0	0	70	0	0	
	S304	0	0	104	0	0	
	S401	0	0	0	84	0	
	S402	0	0	0	97	0	
	S403	0	0	0	86	0	
	S501	0	0	0	0	40	
	S502	0	0	0	0	242	
	S503	0	0	0	0	81	
Total		186	234	309	267	363	1359

12.5 Results

12.5.1 Mathematical Modelling in the Mathematics Curricula

We identified 310 codes related to mathematical modelling in the texts, and the distribution of the codes in the analysed curricular documents are presented in Table 12.6.

Based on the results presented in Table 12.6, it is clear that mathematical modelling is a fairly new concept in China's mathematical curriculum. The terms 'model' or 'modelling' first appeared in the 1996 mathematical curriculum for high school

Table 12.4 Description of the three modelling tasks

	Simple description	Mathematical content	Difficulty level
Task 1	The Lanzhou noodle problem: Lanzhou noodles are a well-known dish in China and originated in the north-western region of China. To make noodles, a chef kneads dough into a long strip, stretches it, folds it, and repeats the stretching and folding process 7–8 times until the noodles become thin and long. Please estimate how long the noodles would be if they are folded and stretched 4 times?	Numbers and algebra	Level 1 The easiest level. It is easy to use a mathematical model learned in class to solve the problem
Task 2	Big shoes problem: To estimate the size of big shoes	Space and graphs; Numbers and algebra	Level 2 It is not difficult to find a similar model learned in class, but it needs to be modified
Task 3	Gas station problem: Gas prices at a nearby station are more expensive than prices at a station located far away from you. Decide whether it is worth driving a further distance to buy gas based on the conditions provided	Numbers and algebra	Level 3 The most difficult level. Students may not be familiar with the problem since there are no similar ready-made models available to them

mathematics and involved four codes (i.e. C11, C22, C24, and C25), which reflect a simple modelling process. Then, the terms appeared in both the junior and high middle mathematics curricula, which contained 9 codes that reflect a simple modelling process. The remaining 288 codes were found in four curricular standards: the 2001 and 2011 versions for grade 1–9 students and the 2003 and 2017 versions for grade 10–12 students.

In the 1996 syllabus, modelling was referred to as '. . . [letting] students acquire the basic knowledge, enhance mathematical awareness, and initially use mathematical modelling to solve some problems in the real world' (p. 644). It appears that the concept of modelling in Chinese mathematics curricula was stemmed from the concept of real-world problem-solving, which has been advocated for in mathematical curricula for a long time, as stated in Chap. 5.

Since the beginning of the twenty-first century, mathematical modelling has been strongly advocated for in the curricular documents. It is also evident in the distribution of codes in the curricular documents in Table 12.6. The 2001 version of the standards for grade 1–9 students made a rather complete statement on the process of modelling: 'abstracting mathematical problems from a real problem situation using various types of mathematical language to express the problem, establish mathematical relationships, obtain proper solutions, and understand and acquire the meaning of corresponding mathematical knowledge and skills' (MOE, 2001, p. 61), which is a four-stage process (Fig. 12.3). The standards also require allowing

Table 12.5 Coding scheme used to analyse the modelling stage the students demonstrated in performing the assigned tasks

Code	Description	Examples (translated texts from task 1)
Stage 0	The student cannot identify any quantities or relationships in the real situation, does not attempt the task, or uses unrelated/nonsensical numbers in the response	*Blank* '14'
Stage 1	The student attempts to structuralise the real situation and presents ideas but is unable to develop a mathematical model (e.g. only lists some variables or identifies some relationships between the variables)	'About 1.7 meters'. 'As an adult is about 1.7 m tall, which is similar to the length he opens his arm'.
Stage 2	The student proposes some reasonable hypotheses and develops a mathematical model, but the model is not properly developed	'$2 \times 7 \times 2 = 28$ (m)'
Stage 3	The student develops a realistic model and converts it into a mathematical model but is unable to reach an accurate mathematical solution or solves the model incorrectly	'When a person opens his arm, the length is more than 1 meter. As shown in the picture, the person did not completely open his arm, so we hypothesise it was 1 meter. Folding and stretching 7 times: 1: $1 + 1 = 2$ (m), 2: $2 + 2 = 4$(m), 3: $4 + 4 = 8$ (m) 4: $8 + 8 = 16$ (m), 5: $16 + 16 = 32$ (m), 6: $32 + 32 = 74$ (m), 7: $74 + 74 = 148$ (m) So, it is about 148 m'
Stage 4	The student proposes a proper mathematical model and obtains the correct solution but does not interpret the solution using the real situation	'The length of an adult's arms when he opens the arms is about his height, about 1.8 m. So, $1.8 \times 27 = 1.8 \times 128 = 230.4$ Answer: the total length is about 230.4 m'
Stage 5	The student develops a realistic model, converts it into a mathematical model, and solves it correctly. The student also interprets and verifies the model in the real situation and assesses the rationale of the model	*[No examples provided]*

Table 12.6 Number of codes identified in the curricular documents

Phases[a]	Before 1949	1949–1975	1976–1985	1986–1999	2000–2018	Total
No. of paragraphs	0	0	0	4	306	310

[a]According to the phases of mathematics curricula development in China, which are described in Chap. 1

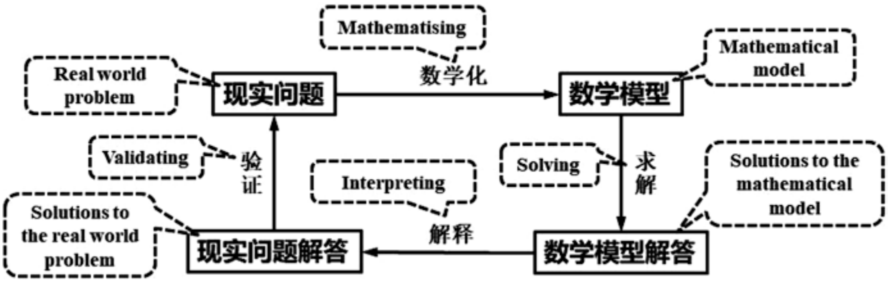

Fig. 12.3 A four-stage modelling cycle in China's mathematics curriculum

students to experience, by themselves, the procedure of abstracting a real problem to mathematical models, as well as interpretation and application procedures (MOE, 2001).

The modelling process was then visualised in the 2003 standard for grade 10–12 students, and it was added to two stages of *posing problems* and *verifying* (Fig. 12.4), which is much more similar with the modelling cycle proposed by Blum (1996) and Kaiser (1995; see Fig. 12.1). Moreover, the standard placed more emphasis on the importance of 'cultivating [students'] mathematical modelling ability' (MOE, 2003, p. 89).

The 2011 version of China's standards for grade 1–9 students followed the same modelling process described in the 2003 version. Furthermore, 'modelling thinking' became one of the ten keywords; the other nine are number sense, symbol sense, space conception, geometric visualisation, data analysis sense, the skill of mathematical operations, reasoning ability, application, and creativity.

In the newest released standards for grade 10–12 students, the 2017 version, mathematical modelling is considered one of the six core mathematical competencies. The modelling process includes seven stages: 'discovering problems in realistic situations from a mathematical perspective, posing problems, analysing problems, constructing models, determining parameters, calculating and solving, verifying results, improving models, and finally, solving realistic problems' (MOE, 2018, p.35).

To summarise, mathematical modelling in China's mathematics curricula stems from problem-solving and has been significantly and increasingly emphasised since 2001. The modelling process is described from a four-stage cycle (Fig. 12.3) to a seven-stage cycle (Fig. 12.4). To gain a comprehensive understanding of the concept of modelling in China's mathematics curricula, we performed an in-depth comparison of the codes of content in text identified in the four versions of standards published in the twenty-first century.

As shown in Table 12.7, there are, generally, far more modelling codes in the mathematics curricula for grade 10–12 students than for grade 1–9 students. Thus, modelling is emphasised much more heavily in the high school curricula.

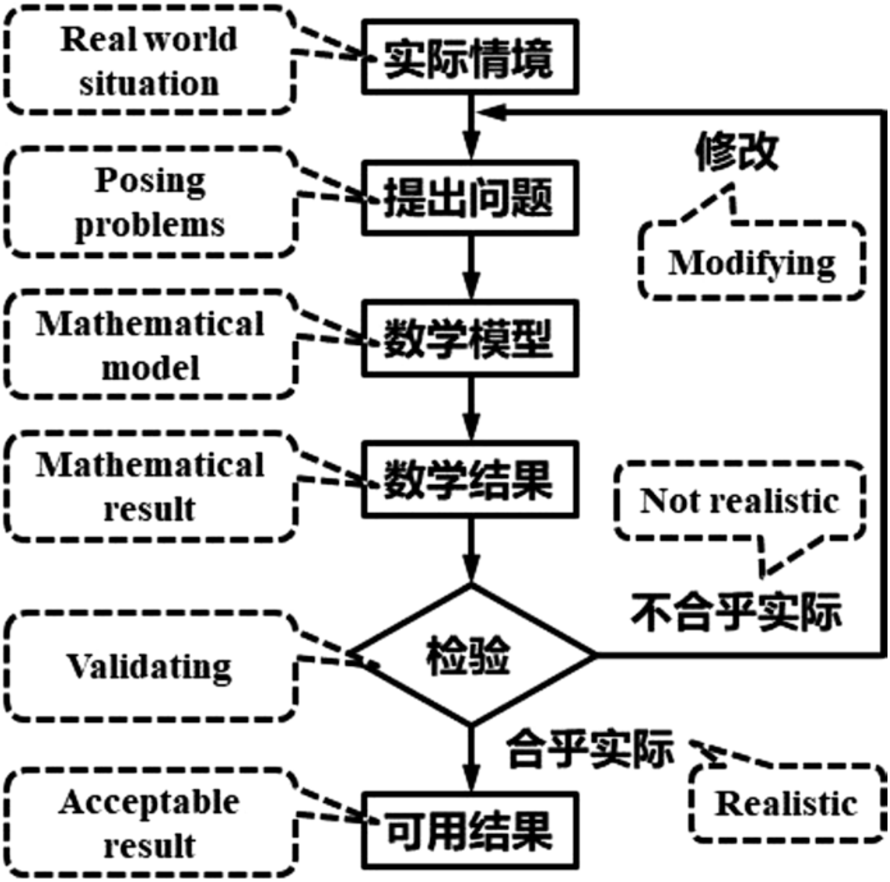

Fig. 12.4 The modelling cycle presented in the 2003 version of China's mathematical standards

Specifically, the cognitive level for modelling required by the grade 1–9 curricula, using the 2011 version as an example, is described by terms such as 'experiencing' and 'realising' (MOE, 2012, p. 14). Comparatively, the high school ones have higher requirements. For instance, the 2003 version requires that students can select 'effective ways and strategies to collect information, connect with relative knowledge, propose how to solve the problem, establish proper mathematical models, and then try to solve the problems' (MOE, 2003, p. 89). The 2017 version specifically points out that the teaching objectives of mathematical modelling are 'through learning high school mathematics, students can use mathematical language to express the real world consciously, discover and propose problems, make sense of the connections between mathematics and the reality, and learn how to use mathematical models to solve real problems' (MOE, 2018, pp. 5–6).

Table 12.7 The distribution of codes in the four versions of standards by year of publication

Categories	Codes	2001	2003	2011	2017	Total	
Mathematics and modelling	C11	3	38	4	38	87	121
	C12	2	7	2	7	18	
	C13	1	6	1	5	16	
Reality and modelling	C21	3	11	2	34	50	156
	C22	3	9	4	17	36	
	C23	1	1	1	4	7	
	C24	3	3	0	9	16	
	C25	1	16	3	25	47	
Affective aspects	C31	1	21	1	7	30	33
	C32	0	2	0	1	3	
Total		18	114	18	147	310	

Note: The 2001 and 2011 versions are for grade 1–9 students, and the 2003 and 2017 versions are for grade 10–12 students

By examining the codes in the high school curricula, it is apparent that the codes on affective aspects are the minority. Specifically, the number of affective codes in the two curricula decreased from 23 to 8, probably because '*affects and attitudes'* is one of the three basic curricular ideas of the 2003 version, and thus, almost every curricular requirement is connected to it. The 2017 version promotes six core mathematical competencies that do not specifically emphasise '*affects'*, but the construct of competencies *per se* contains affeactive aspects. It is reflected in the statements of the curicular standards that there are fewer descriptions of the affective aspects; instead, the focus is on modelling competencies, especially '*reality and modelling*'. The number of *reality and modelling* (C2) codes increased significantly in the 2017 version when compared to the 2003 version, from 40 to 89, which is consistent with the curricular idea that emphasises connections between mathematics and reality.

In addition to the differences discussed above, there are also similarities in the descriptions related to modelling in these curricular documents. As shown in Table 12.7, the 2001 and the 2011 versions of the curricular standards for grade 1–9 students are quite similar, both in the total number of codes and the distribution of codes in the three categories, and there were no significant changes in the numbers of the first category, '*mathematics and modelling',* in the two versions of high school mathematics curricula.

To summarise, mathematical modelling in China's mathematics curricula stemmed from problem-solving and was not emphasised until the twenty-first century. In the curricular standards published after 2000, modelling is much more strongly emphasised in high school mathematics than in the lower grades. In high school mathematics, the importance of modelling seems to be increasingly recognised. Therefore, more research on the teaching and learning of mathematical modelling to promote effective practices in China are needed.

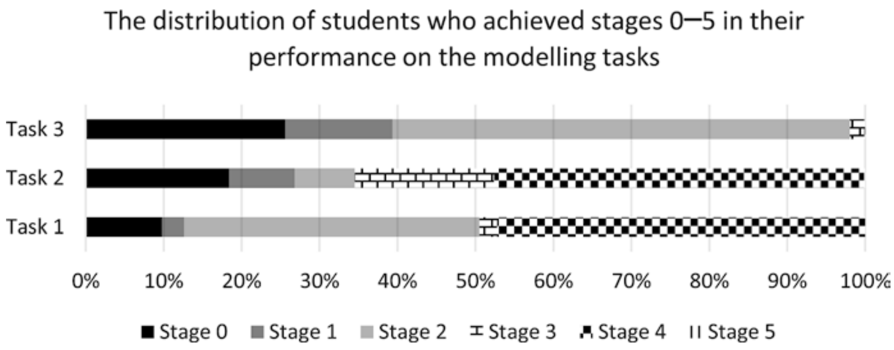

Fig. 12.5 Stages of the modelling process the students achieved while performing the three tasks

12.5.2 Grade Eight Students' Modelling Competency

12.5.2.1 An Overview

Figure 12.5 presents the distribution of stages achieved by the 1359 students who completed the three assigned modelling tasks. In this figure, it is clear that on tasks 1 and 2, almost half of the students achieved the fourth stage of the modelling process (i.e. they proposed proper mathematical models and obtained the correct solutions but did not interpret the solutions using the real-world problem or assess/verify their models). On task 3, most of the students were unable to develop a proper model, much less obtain the correct answers or modify the model used. Generally, it seems that the students treated these tasks as mathematical word problems rather than real-world problems to solve and, thus, did not assess or critique the models they proposed. This made them stop at obtaining a mathematical solution on the easier tasks (i.e. tasks 1 and 2) or propose incorrect models on the difficult task (i.e. task 3).

The students' level of performance on each task has a statistically significant correlation with the other two tasks (see Table 12.8).

Using the graded response model, we assigned the students modelling competency scores on all three of the modelling tasks. Since two students' performance data were missing, modelling competency scores were assigned for 1357 students. As shown in Table 12.9, the students' mean score was 0.0009, and the highest score was 1.75.

12.5.2.2 Gender Differences and Individual, School, and City Differences

The girls had lower modelling competency scores ($M = -0.059$, $SD = 0.781$) than the boys ($M = 0.056$, $SD = 0.833$), with $t\,(1350) = -2.617$, $p < 0.01$. But the magnitude of the difference is quite small with Cohen's $d = 0.14$.

Table 12.8 Correlations among the students' level of performance on each of the three tasks ($N = 1357$)

	Task 1	Task 2	Task 3
Task 1	1	0.425***	0.414***
Task 2		1	0.383***
Task 3			1

Note. All correlations are based on weighted data; *** $p < 0.001$

Table 12.9 Descriptive statistics of the students' modelling competency scores

	No. of students	Range	Minimum	Maximum	Mean	Std. Deviation
Students' assessed modelling competency	1357	3.59	−1.84	1.75	0.0009	0.81065

Table 12.10 Variances of Graded Response at different level

	Task 1	Task 2	Task 3
Individual level	1.40	1.94	0.57
School level	0.34	0.54	0.20
City level	0.00	0.00	0.00

The students' individual performance on each modelling task is significantly different; and the differences in schools are significant but not so significant at the individual level. More than 70% of the variance in attainment can be attributed to differences between individual students, and around 20% of the variance can be attributed to differences between schools. There are insignificant differences between cities (see Table 12.10).

12.6 Summary, Discussion, and Conclusion

12.6.1 Mathematical Modelling in China's Mathematics Curriculum

In the findings presented in Sect. 12.5.1, we showed that in China's mathematics curriculum, mathematical modelling was not specifically advocated for until 2000. The country's curriculum does have a long history of emphasising mathematical problem-solving and the application of mathematics, which may promote the development of mathematical modelling in the teaching and learning of mathematics. However, the focus on problem-solving, as well as China's long history of placing a heavy emphasis on foundational mathematical knowledge, may also constrain the promotion of mathematical modelling competency, especially in dealing with real-world situations which usually contain a rich source of information.

In the textual analysis findings, it is fairly obvious that the concept of mathematical modelling developed from a four-stage modelling cycle (i.e. beginning with a real-world problem, followed by creating and finding a solution to a mathematical model, and then applying the solution to the real-world problem) to more precise modelling that places greater emphasis on translating a real situation into a mathematical model, as well as validating and evaluating the modelling process. This developmental trajectory is also consistent with the conceptual development of the mathematical modelling cycle in international research focused on the teaching and learning of mathematical modelling. The four-stage modelling cycle mentioned in the 2001 version of the standards for grade 1–9 students (Fig. 12.3) is almost the same as the modelling cycle proposed by Blum (1996) and Kaiser (1995; Fig. 12.1). The newly developed modelling cycle (Fig. 12.4) is closer to the dominate modelling cycles in the current literature (e.g. the model presented in Fig. 12.2). However, unlike most of the cycles, it does not emphasise the situation model and simply mentions ‘posing problems’. It appears that minimal attention has been paid to translating real-world problems into the mathematical world, which is probably the biggest challenge in the promotion of mathematical modelling in China’s mathematics curriculum, considering the characteristics of its development, as discussed above.

Specifically, we analysed, in great detail, three categories of texts related to mathematical modelling in curricular documents: ‘mathematics and modelling’, ‘reality and modelling’, and ‘affective aspects’. It was found that the connection between reality and modelling is much more frequently mentioned in the most recent promulgated curricular standards for high school mathematics, published in 2017. Mathematical foundations are more heavily emphasised in the curricular standards for high school mathematics than those for middle school mathematics, and affective aspects seem to be mentioned less frequently in the curricular standards for both high school and middle school mathematics education. These findings provide implications for understanding mathematical modelling in the intended mathematical curriculum in China: (1) mathematical modelling seems to be more heavily emphasised in the curriculum of high school mathematics; (2) the characteristics of connections between reality and mathematical modelling have been recognised, and mathematical foundations have been emphasised in the promotion of mathematical modelling; and (3) affective aspects, such as students’ interest in mathematics, seem to be less important than other aspects, such as mathematical foundations. These characteristics are consistent with the common understanding of mathematics education in China, which is widely known for heavily emphasising the learning of mathematical content and students’ performance instead of stimulating students’ interest in learning (e.g. Leung, 2001).

12.6.2 From Curricular Standards to Student Performance

Mathematical modelling has not been strongly advocated for in China's mathematics curriculum standards for very long. Mathematical modelling tasks have not been included in mathematics textbooks until the 2017 version curricular standards released, and it is considered as a challenge for textbook developers to compile modelling tasks for teachers to conduct related teaching (Zhang, Zhang, & Jin, 2020); although mathematical textbooks in China have emphasised solving real-world problems with mathematical means for a long time, which provided bases for the promotion of modelling (Du, 1998). Assessing Chinese students' mathematical modelling performance is an important step toward promoting this competency in the future, for example, providing insights for textbook developers and for teachers to develop or use modelling tasks to conduct related teaching practices. By assigning 1359 students with three modelling tasks involving different real-world situations and mathematical content, we found that they appear to be accustomed to solving regular word problems rather than real-world problems. Thus, they usually stopped after obtaining a mathematical solution and did not recognise the necessity of validating their mathematical solutions and critiquing the mathematical means they used according to the complex requirements of real-world problems.

Regarding gender differences, the boys seemed to have performed better than the girls, which is not consistent with the results of traditional examinations. Currently, girls commonly perform better than boys on routine examinations, perhaps because girls tend to be much more diligent. However, from our analyses, it seems that the boys seemed to go further in the modelling process, possibly because they tend to be more brave and open-minded in solving real-world problems. Concerning the individual, school, or city differences, it was found that students' performance on mathematical modelling is more individualised, and there were no significant city differences in our results, which may be because the promotion of mathematical modelling has not been well implemented in the teaching and learning of mathematics in China, and the students relied much more heavily on their daily experiences while performing the tasks. From another perspective, it may also imply that the promotion of mathematical modelling could contribute to students' individual development and, furthermore, achieve the educational goal of cultivating professional citizenship in all walks of life.

12.6.3 Conclusion

To more effectively promote mathematical modelling in the teaching and learning of mathematics (i.e. the implemented curriculum) in China, this study investigated the historical development of mathematical modelling by focusing on the use of related terms and requirements in curricular documents, as well as the state-of-the-art through in-depth analyses of content related to mathematical modelling in the current curricular standards (i.e. the intended curriculum). We also assessed and analysed students' performance on three mathematical modelling tasks (i.e. attained curriculum). The findings are consistent with the common knowledge of the characteristics of mathematics education in China. Thus, the mathematical modelling in China's intended curriculum places greater emphasis on mathematical foundations and pays minimal attention to students' non-cognitive aspects and additional competencies, such as communicative and metacognitive aspects. The students seemed to have little knowledge about mathematical modelling and, therefore, lacked the awareness and skills required to critique and validate their work on the tasks, which are very important in mathematical modelling competency. These findings could provide insights for the research and practical design of teaching and learning mathematical modelling in China in the future.

As previously mentioned, few studies have focused on the teaching and learning of mathematical modelling in China, probably because it has not been emphasised in the country's centralised curriculum until recent years. Our investigation may stimulate further research focused on the theoretical and practical exploration of mathematical modelling in similar contexts as China in which there is a long history of the curriculum heavily emphasising mathematical foundations. However, there are some limitations in our study. For example, we did not conduct an in-depth exploration of students' modelling competencies, including sub-competencies and additional competencies, such as metacognition. However, we believe that more significant theoretical and practical contributions could be made in further analyses of the study's results and in future research on the topic of mathematical modelling competencies, especially in the context of East Asian countries.

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Chapter 13
The Development of Communication in Chinese Mathematics Curricula

Yuelan Chen, Xiaoyan He, and Binyan Xu

Abstract This chapter examines the development related to mathematical communication abilities in math syllabus and curriculum standards at the junior high level in China since 1902. This chapter analyses curriculum documents in China from 1902 to 2011 using keyword frequency analysis and text analysis. The study found that mathematical communication abilities in curriculum standards over the past hundred years are defined in four ways: teacher-student communication, student-self communication, student-student communication, and student-text communication. The analysis of the changes to the curriculum requirements provides a better understanding of mathematical communication abilities in China and offers insights on the key factors that affect the development of students' mathematical communication abilities.

Keywords Mathematical communication abilities · Math syllabus · Curriculum standards · Types of communication · Teacher-student communication · Student-self communication · Student-student communication · Student-text communication · Keyword frequency analysis · Text analysis

13.1 Introduction

With the growing usage of mathematics in modern society, *mathematical communication ability* has become an important part of math competency. Niss (2015) explained the following:

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> Considering the fact that anyone who is learning or practising mathematics has to be engaged, in some way or another, in receptive or constructive communication about matters mathematical, either by attempting to grasp others' written, oral, figurative or gestural mathematical communication or by actively expressing oneself to others through various means, a mathematical communication competency is important to include. (p. 40)

Mathematical communication is the process in which students learn and use mathematical language to communicate and understand the world, such as using specific mathematical symbols and terminologies. With mathematical communication abilities, students are expected to build common sense regarding mathematics (Shi, 1998; Niss, 2003; Xu, 2013). As future citizens, students need to achieve certain levels of mathematical communication abilities. However, such abilities are not innate. The higher a student's grade level is, the more complicated and instructive his or her mathematical communication abilities are. It is imperative for educators to establish a set of explicit, detailed and measurable mathematical communication abilities to evaluate students' current communication ability levels and to promote their mathematical communication abilities.

In China, syllabus and curriculum standards play an important role in guiding curriculum writing, teaching and learning. The latest mathematics standards contain modified requirements of mathematical communication abilities for students. The analysis of the changes in the requirements provides a better understanding of mathematical communication abilities in China and offers us insights on the key factors that affect the development of students' mathematical communication abilities.

13.2 Literature Review

13.2.1 Definition of Mathematical Communication

Communication is a process of receiving and communicating through language, symbols, diagrams and artistic forms, which requires listening, speaking, reading and writing as the main means. In many curriculum standards, mathematical communication abilities entail the processes of receiving and expressing. For example, the German mathematics standards state that mathematical communication abilities include the understanding of mathematical text or expression as well as the written or verbal communication of mathematical thinking and solutions. Reading and understanding mathematical texts is a process of *receiving*, while interpreting and presenting mathematical ideas in written or oral form belongs to the *expressing* process (Kultusministerkonferenz, 2004). The German standards require that students be able to receive, understand and evaluate mathematical facts as well as present one's own mathematical ideas and assess and correct others' ideas (Xu, 2007). The United Kingdom's national curriculum guide requires students to understand and interpret mathematics in multiple representations and to communicate mathematics with confidence in the most appropriate way (U.K. Department of Education, 2007). Students should be able to choose the most effective way to communicate in different contexts. Students are also required to provide explanations and assess the correctness of expressing. Such processes involve an understanding of mathematical information and help develop students' mathematical thinking.

Some standards define *mathematical communication abilities* with a focus on either the process of *receiving* or the process of *expressing*. For example, the *Curriculum and Evaluation Standards for School Mathematics* in the United States has a focus on the process of expressing. It requires students to "reflect upon and clarify their thinking about mathematical ideas and relationships, and formulate mathematical definitions and express generalisations discovered through investigation, and to express mathematical ideas orally and in writing" (National Council of Teachers of Mathematics, 1989, p. 140). Singapore's secondary school syllabus also focuses on the expressing process. It states that a critical skill in education is the ability to use mathematical language to express the process of mathematical thinking and argumentation accurately, concisely and logically (Singapore Ministry of Education, 2011). Recently, Singapore has paid more attention to mathematical communication and has mentioned that "communication of mathematics is necessary for the understanding and dissemination of knowledge within the community of practitioners as well as general public" (Singapore Ministry of Education, 2019, p. 6). The *Mathematics Curriculum Standards for Compulsory Education* in China (Ministry of Education of the People's Republic of China, 2012) lists four requirements as mathematical communication abilities:

1. Students will be able to communicate about their own algorithms and processes to solve the problem and to express their own ideas.
2. Under the guidance of teachers, students will be able to choose the appropriate strategy to solve the problem through communicating with others.
3. Students will be able to explain and communicate the statistical results and make simple assessments and predictions based on the results.
4. Students will be able to rethink the whole process of mathematical participation, to write a report or short paper about the research process and results, and to communicate so as to further obtain mathematical practice experience.

The sequence of the four requirements implies the assumption that a good receiving process serves as the basis for the improvement of the expressing skill.

13.2.2 Classification of Mathematical Communication

Students with strong mathematical communication abilities can explain a large amount of quantitative data encountered in daily life and make reasonable evaluations of the data. They can also fully reflect on their own problems and understand arguments from others. Niss (2003) defined *mathematical communication* as involving two processes. The first process is to understand the mathematical meaning of the texts presented in various representations, including written, visual or verbal. The second process is to present one's own mathematical ideas in multiple representations at different levels of precision.

The Common Core State Standards for Mathematical Practice (National Governors Association and Council of Chief State School Officers, 2010) include eight standards that apply to students from kindergarten to 12th grade. Students should be able to perform the following important tasks: make sense of problems, reason abstractly, construct arguments and critique the reasoning of others, construct mathematical models, use appropriate tools, attend to precision, make use of structure and look for and express regularity in repeated reasoning. Communication is key to many of these tasks. To construct mathematical models, students must construct representations of mathematical thinking—a crucial element of communication. To construct *arguments, critique* the reasoning of others, *attend to precision* or *express* regularity in repeated reasoning, students must be able to clearly communicate their mathematical thinking. Mathematical communication skills include mathematical dialogue, writing and reading.

Mathematical dialogue is the conversation of mathematics between two or more persons. It is a two-way process involving listening and speaking. For example, a teacher-student dialogue and dialogue among students in the classroom are mathematical dialogues. Regarding the purpose of student dialogue in mathematics classrooms, Pimm (1987) categorised mathematical dialogue as *mathematical dialogue with others* and *mathematical dialogue of self-reflection*. Students use mathematical dialogue with others to convey their own mathematical ideas. Through self-reflective mathematical dialogue, students can effectively organise their own thinking and clarify mathematical meanings and ideas, thus gaining further understanding of mathematics. For example, when solving a mathematical problem, students read the mathematical questions repeatedly to clarify or correct the problem-solving model. The repeated reading method indicates that self-reflective dialogue can promote student reflection on mathematical thinking. Self-reflective dialogue is implicit and serves as the basis of conversations with others.

Mathematical writing is an important complement to verbal communication. When students write in mathematics, they are actively involved in the process of absorbing mathematical knowledge, developing mathematical understanding, and improving math-learning attitudes. Common mathematical writing in classes includes diary writing and explanatory writing. One type of diary writing asks students to reflect on the entire learning process by debriefing the math they have learned. Clarke et al. (1993) conducted a study on diary writing for 4 years with a focus on mathematical debriefing. They asked seventh-grade students to write a math diary with three prompts at the end of each math class. The three prompts were as follows: What did you do in class? What did you learn? What were the examples and questions? The purpose of explanatory writing is to describe and explain the process of solving a mathematical problem or the validation of a mathematical solution to a given question. Shield and Galbraith (1998) studied two explanatory writing tasks: (1) writing a letter to a classmate who'd missed the class to explain what was learned in the class and (2) writing to help a student who had difficulties with the math in class.

Mathematical reading involves reading and understanding texts containing words, forms, figures, illustrations, timetables, etc. (Organisation for Economic Co-operation and Development, 2009). In mathematical reading, students need to process and transition among multiple representations, including symbols, diagrams, graphics and forms. It is a nonlinear process and is the main difference between mathematical reading and other reading (Bosse & Faulconer, 2008).

13.3 Research Question

This chapter examines the requirements related to mathematical communication abilities in math syllabus and curriculum standards at the junior high level since 1902. Two research questions are explored:

1. What are the definitions of mathematical communication abilities in math syllabus and curriculum standards used throughout the past 100 years in China?
2. What are the changes in requirements for mathematical communication abilities in math syllabus and curriculum standards?

To answer the two research questions, we reviewed literature and analysed curriculum documents in China from 1902 to 2011. Findings illustrate the changes in defining mathematical communication abilities and the requirements for student mathematical communication abilities in China.

13.4 Research Methods

13.4.1 Objects of Content Analysis

The data for this study are math syllabus and curriculum standards at the junior high level in China from 1902 to 2011. In particular, the documents from 1902 to 2000 were selected from the *Collection of primary and secondary curriculum standards and syllabus of the twentieth century in China (Mathematics volume)*, published by People's Education Press and edited by Curriculum and Teaching Materials Research Institute. The curriculum documents after 2000 were selected from *Mathematics Curriculum Standards for Full-Time Compulsory Education (Experimental version)* (Ministry of Education of the People's Republic of China, 2001) and *Mathematics Curriculum Standards for Full-Time Compulsory Education (2011 version)* (Ministry of Education of the People's Republic of China, 2012).

13.4.2 Procedures of Content Analysis

13.4.2.1 Content Analysis

The content analysis method was used to analyse documents. Mayring (2015) simplified content analysis into three steps: deletion, interpretation and structuring. Texts were assessed with predetermined criteria and were coded in both inductive and deductive classifications. Frequency of the keywords was counted. In this study, we first filtered documents with the keyword expression. An analysis framework was then developed to code the filtered documents.

13.4.2.2 Analysis Framework of Mathematical Communication

Mathematical communication abilities are a set of abilities revolving around receiving mathematical information through reading and understanding of mathematical texts and expressing mathematical ideas in written or verbal form (including mathematical thinking processes, problem-solving strategies and mathematical answers).

There are three types of mathematical communication: teacher-student communication, student-student communication and student-text communication (Nührenbörger & Steinbring, 2009). Teacher-student communication is a conversation led by the teacher, usually with a rapid introduction, and passively received by the students. In such a conversation, the teacher dominates the delivery of mathematical concepts and mathematical thinking. Student-student communication entails conversations involving various levels of mathematical understandings and practices. Participating students are open to communicate their mathematical ideas, no matter the correctness or completion of the mathematical idea. Student-text communication is the communication with mathematical texts, such as solving mathematical problems, reading textbooks and learning mathematical concepts (Nührenbörger & Steinbring, 2009). In addition, students' self-communication and reflection is becoming more and more important, and should be an important part of mathematical communication ability. In the present study, four types of communication were investigated (see Fig. 13.1).

Based on the information processing theory, mathematical communication is a process of receiving, processing and expressing (Zeng & Lian, 2017). Figure 13.2 shows the various activities involved in the three phrases of the mathematical

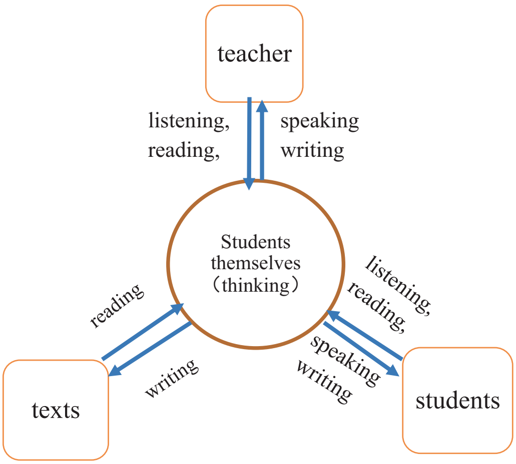

Fig. 13.1 The four types of mathematical communication

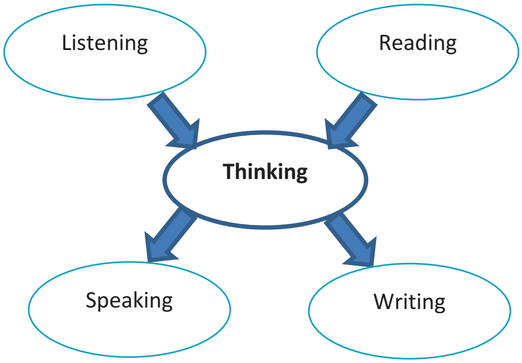

Fig. 13.2 The process of mathematical communication

communication process. Receiving exists in all three types of mathematical communication: teacher-student, student-student and student-text. Processing is mainly implicit self-reflective thinking and communication within an individual student. Expressing is the process of a student presenting mathematical ideas in verbal or written form after receiving and processing mathematical information.

13.4.2.3 Coding Framework for Mathematical Communication

As discussed in Chap. 3, the cognitive requirements in the process of mathematical communication include three levels: reproduction, connection and reflection. Reproduction is when students express or present simple mathematical content and recognise information embedded in short mathematical texts. Connection is the transfer of others' mathematical thinking from one carrier to another and students' explanations of their thinking processes, solutions and results briefly and logically. Reflection is the process of understanding the meaning of complex mathematical texts, comparing and judging others' mathematical thinking, and expressing one's own inspection and reflection on the learning process. The coding system of mathematical communication was developed using the following analysis framework (Table 13.1).

Every single sentence from the curriculum documents was a coding unit. For example, the sentence "using the trajectory method to solve the drawing problem" was one coding unit. The content area mentioned in this sentence is geometry, coded as A3. The context of mathematical communication was coded as B2, since it is an educational context. The communication form was coded as C3, which is student-based communication. The cognitive requirement is a conversion (D21). Thus, the code for this sentence is A3B2C3D21. In a case where a sentence involved multiple mathematical contexts or cognitive requirements, all suitable codes were applied to the sentence. Two researchers who had background knowledge and experience in curriculum content analysis independently coded the same 20 sentences randomly selected from the curriculum documents. Comparison revealed that 90.7% of the coding results were consistent. The researchers discussed and reconciled the

Table 13.1 The coding framework for mathematical communication

Dimension		Code	Description
(A) Content domains		A0	Comprehensive requirements
		A1	Arithmetic
		A2	Algebra
		A3	Geometry
		A4	Probability and statistics
(B) Communication context		B1	Personal context
		B2	Educational context
		B3	Social context
(C) Communication types		C1	Student-teacher: The teacher asks students to answer questions and discuss the process of mathematics, mathematical thinking and mathematical methods with other students. It is mainly about the process by which students receive and understand information
		C2	Student-self: Students answer questions and give results by accepting information and carefully thinking and expressing mathematical conjectures or feelings about the speech of mathematical topics
		C3	Student-text: Communication occurs between students and texts when students do mathematical problems, review textbooks and learn mathematical concepts
		C4	Student-student: Students express their opinions to the communication objects (teachers, peers or texts) and use relevant mathematical knowledge and concepts to prove their ideas, convince and understand the objects of communication, listen to the mathematical ideas and strategies of communication objects, understand their methods of thinking, analyse the mathematical views expressed by others and judge others' abilities to express, listen and absorb others' ideas. It includes processes of acceptance, processing and expression
(D) Cognitive domains	(D1) Recognise & imitate	D11	Recognise: Be able to identify and select information from short mathematical texts
		D12	Imitate: Be able to clearly express simple mathematical facts, such as understanding of simple mathematical content
	(D2) Connect & transform	D21	Transform: Recognise and select information from mathematical texts and understand its significance and be able to convert the mathematical ideas of others from one carrier (chart, text, symbol, object or action, etc.) to another, so as to facilitate further understanding
		D22	Connect: Be able to express the thinking process, the solution and the result in a brief and logical way and be able to explain the explanation (correct or wrong) of the mathematical text made by others

(continued)

Table 13.1 (continued)

Dimension		Code	Description
	(D3) Reflect & extend	D31	Reflect: Comprehend the meaning of complex mathematical texts and compare and judge other people's mathematical ideas
		D32	Extend: Be able to fully present the process of a complex solution and argumentation; be able to compare, evaluate and correct the understandings of others; be able to flexibly transform the carrier of mathematical ideas and select the optimal expression carrier according to the specific situation; and be able to express the examination and reflection of the learning process so that the problem-solving process is rational, complete, concise and harmonious

remaining 9.3% of the coding results and reached an agreement in the end. Then all relevant sentences ($N = 306$) were coded by both researchers.

13.5 Results

We analysed curriculum and syllabus standards from 1902 to 2011 using keyword frequency analysis and text analysis. Findings were categorised into five time periods: 1902–1922, 1923–1951, 1952–1977, 1978–2000 and 2001–2011. The division of the time periods was based on the year when one curriculum reform started (see Chap. 1).

13.5.1 The Emergence of Mathematical Communication Abilities: 1902 to 1922

From 1902 to 1922, China reformed school curricula, mirroring academic systems in Japan, Germany, and America (Curriculum and Teaching Materials Research Institute, 2001). The phrase *mathematical communication ability* was not used in the syllabi or standards during this period of time (Fig. 13.3). However, some of the statements in these texts imply that the required mathematical communication abilities at that time were abilities regarding teacher-student communication and student-self communication. For example, *Middle School Rules Approved by Emperor*, published in 1904, pointed out that teachers should "teach the bookkeeping . . . and then teach plane geometry and three-dimensional geometry, and also teach algebra" (Curriculum and Teaching Materials Research Institute, 2001, p. 206) so that students could "know the application of knowledge of bookkeeping" and "the format of the calculation table" (Curriculum and Teaching Materials Research Institute, 2001, p. 206). Although the term *communication* was not used, the statement "teachers should teach" indirectly indicated that students needed to receive mathematical

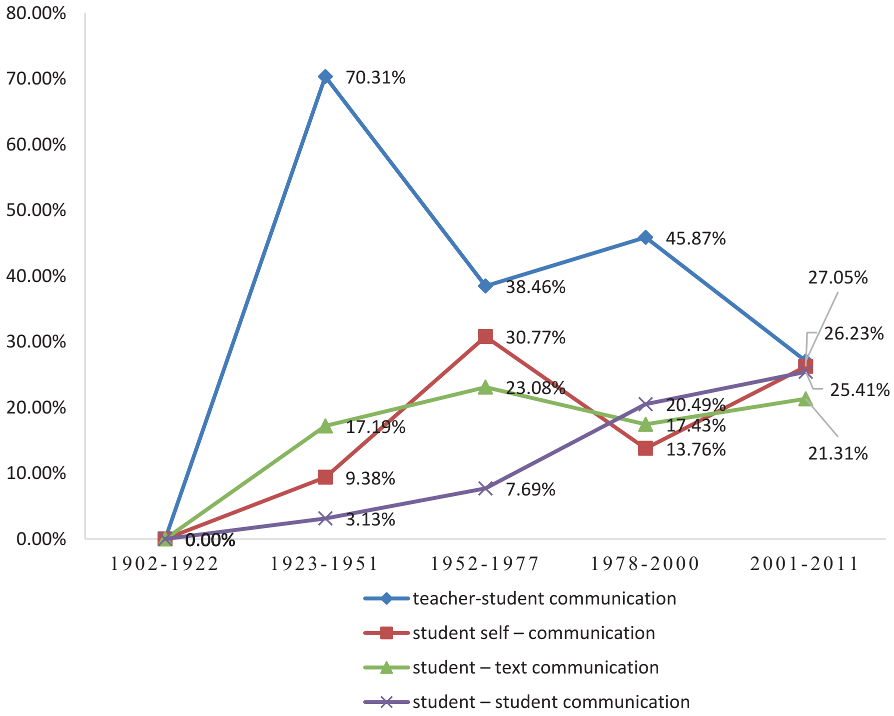

Fig. 13.3 Changes of mathematical communication ability requirements in curriculum standards

information and study it. After that, students needed to talk with themselves to grasp mathematical knowledge and understanding.

13.5.2 The Rise of Mathematical Communication Abilities: 1923 to 1951

In 1923, the Ministry of Education of China released revised curriculum standards for primary, middle and high schools. The new curriculum standards listed requirements for mathematical communication abilities, such as requiring a teacher to guide, question, and teach mathematics to students. After that point, mathematical communication abilities started to become explicitly required in curriculum standards.

From 1923 to 1951, the keywords related to mathematical communication abilities in curriculum standards included *oral answering*, *asking*, *discussion*, *critical questioning* and *explanation*. The different types of communication – including teacher-student communication, student-self communication, student-student communication and student-text communication – appeared in the documents. Among these, teacher-student communication had the largest percentage (70.31%) of relevant sentences coded using the coding framework for mathematical communication. Student-text communication accounted for 17.15%. The percentages of student-self communication and student-student communication were less than 10% (9.38% and 3.13%, respectively).

The percentages show that during this period of time, the curriculum standards emphasised the importance of teacher-student communication in mathematics teaching. Students were expected to receive mathematical information from the guidance of teachers. The standards required students to process mathematical information and express mathematical ideas according to the way trained by teachers. There was little emphasis on student-self communication and student-student communication.

13.5.3 Student-Oriented Mathematical Communication Requirement: 1952 to 1977

From 1952 to 1977, the keywords that reflected mathematics communication in the curriculum standards were *posing mathematics questions* and *Q&A lectures*. The requirement of *expressing one's ideas in mathematical language* was listed in the standards for the first time.

The percentages of relevant sentences which focused on student-self communication, student-student communication and student-text communication increased. As Fig. 13.3 shows, student-self communication increased from 9.38% during 1923 to 1952 to 30.77% during 1952 to 1977. Student-text communication increased gradually. Student-text communication consists of students' interactions with textbooks, mathematical problems and other written mathematical texts. At this stage, mathematical communication requirements were oriented around students' behaviours; they emphasised that students should deal with written mathematical information and express their ideas to others.

13.5.4 The Emphasis of Student-Student Communication: 1978 to 2000

Since 1978, curriculum standards increased the emphasis on communication among students, stating that students should be able to express their views in mathematical language to others and discuss with each other. The proportion of student-student communication in curriculum standards increased from 7.69% to 20.49% (Fig. 13.3). For example, in 1988, the mathematical syllabus listed "expressing one's thoughts and opinions concisely" as one of the purposes of schooling (Curriculum and Teaching Materials Research Institute, 2001, p. 553). In 1992, the syllabus put forward that "students have the ability to expound their thoughts and conceptions using mathematical language correctly" (Curriculum and Teaching Materials Research

Institute, 2001, p. 605). In 2000, teacher-student interactions and student-student interactions were prioritised in the curriculum standards.

The shift from teacher-student communication to student-oriented communication in curriculum standards shows the increasing recognition of student-centred learning in mathematics. Students are expected to express their mathematical ideas to teachers or classmates. They should use relevant mathematical knowledge and abilities to prove their ideas and convince others. At the same time, students are required to listen to others to understand their mathematical ideas, strategies and ways of thinking.

Some examples of the keywords related to mathematical communication abilities during this period included *explanation using examples*, *heuristic teaching*, and *explaining mathematical ideas*. The frequency of keywords focused on mathematical communication abilities in curriculum standards offered us some insights on the emphasis of student-student communication from 1978 to 2000 (Fig. 13.4). All three phases of the mathematical communication process can be found in the curriculum standards. Teacher-student communication is found in teachers' understanding, guiding, and conducting heuristic teaching as students receive mathematical information. Student-self communication is found in students reflecting and thinking on the information they received. Student-student communication is found in students questioning, expressing, and communicating their mathematical ideas to others.

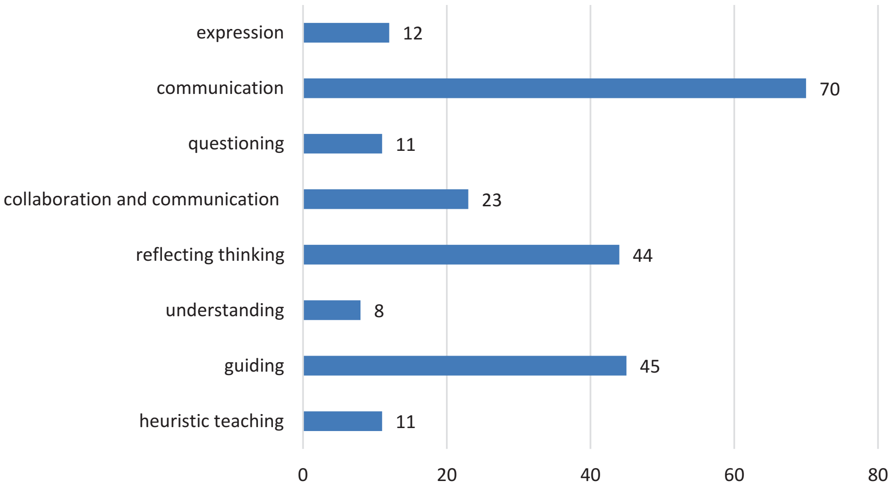

Fig. 13.4 Frequency of requirements for mathematical communication abilities in curriculum standards

13.5.5 Collaboration-Oriented Mathematical Communication: 2001 to 2011

From 2001 to 2011, attention to the phases of mathematical communication changed. The requirement for *expressing* mathematical ideas increased from 34.48% to 43.48%, while attention given to *receiving* mathematical information decreased from 53.45% to 31.06% (Fig. 13.5). Some examples of the keywords on mathematical communication in curriculum standards included *inspirational teaching*, *communication and interaction*, *communicating with mathematical languages*, *collaboration* and *questioning*.

The focus on collaboration and communication was one significant feature during this period of time. The terms *collaboration* and *communication* appeared 23 times in the curriculum standards. The *Mathematics Curriculum Standards for Compulsory Education (2011 version)*, published in 2012 (Ministry of Education of the People's Republic of China, 2012), highlighted that the goal of mathematical communication is to learn to communicate with others.

13.5.6 Other Changes in Requirements for Mathematical Communication Abilities

Since there was no clear expression on mathematical communication abilities in curriculum standards from 1902 to 1922, the changes in requirements on mathematical communication abilities presented here are from 1923 to 2011. We looked at the

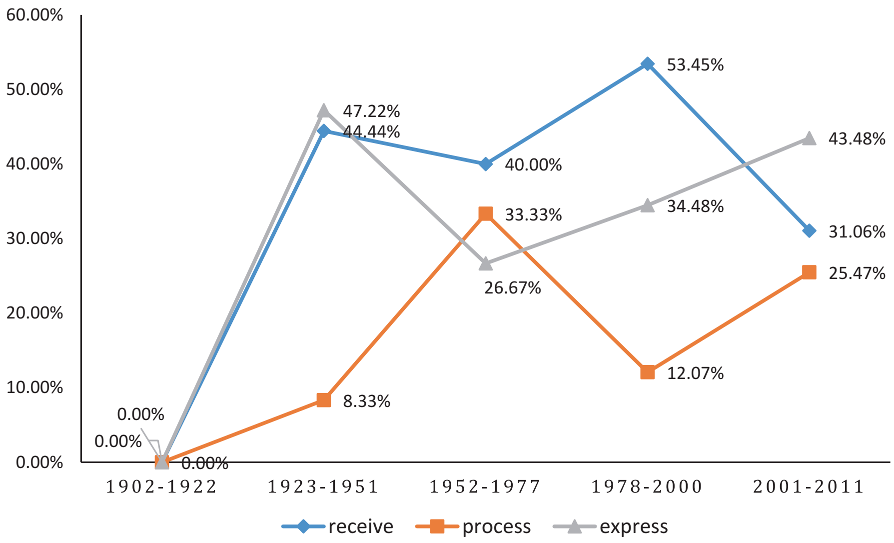

Fig. 13.5 Changes in the phases of mathematical communication in curriculum standards

changes from three perspectives: mathematical content areas, cognitive requirements and communication contexts.

13.5.6.1 Changes in Terms of Mathematical Content Areas

Throughout nearly 100 years, the requirements for mathematical communication abilities in different mathematical content areas have changed dramatically. Some mathematical content received almost 15 times more attention in 2011 as compared to 1923, while the emphasis on some math content dropped 20% (Fig. 13.6).

As shown in Fig. 13.6, the *comprehensive requirements* content area received the most attention from 1923 to 2011. This illustrates that mathematical communication skills are a set of comprehensive abilities, such as mathematical reasoning and mathematical representation, which cannot be achieved overnight (Cai & Xu, 2016). The content areas that increased the most in attention were probability and statistics. The percentage of requirements for mathematical communication abilities in probability and statistics increased from 2% (in the period from 1923 to 1951) to 31% (in the period from 2001 to 2011). Such a huge increase reflected the changes of requirements for the teaching and learning of probability and statistics in curriculum standards. With the rapid development of economy in China, people likely realised the importance of attaining certain probability knowledge, such as the difference between uncertainty thinking and mathematical certainty thinking,

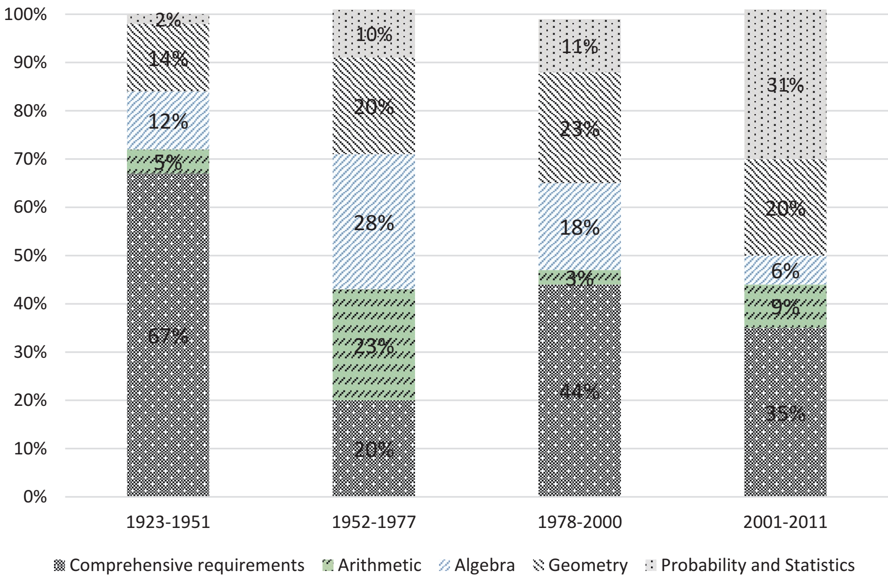

Fig. 13.6 Changes in requirements for mathematical communication abilities based on content areas

statistical thinking and inductive inference in probability statistics (Ministry of Education of the People's Republic of China, 2018). The standards also stated that teachers should let students experience simple data collection and organising processes. Student would then understand some data collection methods such as surveys and assessments and could present the results in various representations, such as texts, pictures and tables. Students would engage in activities such as collecting, describing and analysing data; evaluating and communicating; understanding the necessity of sampling; and experiencing the use of samples to make estimations or predictions. Students would accumulate relevant mathematical-activity experience in collaborating and communicating with others.

In mathematical content areas such as algebra, arithmetic and geometry, there were few changes in the requirements. When examined in detail, most of the changes were to requirements for basic abilities such as reading tables, performing calculations, and validating solutions. China issued a series of notices and notifications to adjust the teaching requirements on various content areas between 1952 and 1977. Although more than 70% of the teaching requirements focused on algebra, arithmetic and geometry, there were few requirements on mathematical communication abilities. For instance, when solving fraction equations, students were required to test whether there was an extraneous root. No discussion was needed (Curriculum and Teaching Materials Research Institute, 2001, p. 360).

13.5.6.2 Changes in Cognitive Demands

A total of 306 coding units with a focus on cognitive demands were analysed. As mentioned in the methods section, we categorised three levels of cognitive demands: recognise and imitate (level 1), connect and transform (level 2), and reflect and extend (level 3). In general, there was an increased requirement for the high-level cognitive demands throughout the past 80 years (Fig. 13.7).

The percentage that referred to the highest-level cognitive demands for mathematical communication increased almost 20% over the past 80 years. In the period from 1923 to 1951, only 27% of the 306 units in curriculum documents related to the level 3 cognitive demands, but in 2001, the percentage of level 3 reached 43%, which was the highest among all three levels. The percentage of level 1 cognitive demands decreased from 35% to 18% over the past 80 years, except for an unexpected rise to 59% in the period of 1952 to 1977. Similar changes happened to level 2. From 1923 to 2011, the percentage of *connect and transform* related to mathematical communication remained fairly stable at about 40%, except for a dramatic drop to 12% during the period of 1952 to 1977. Further studies could be conducted to explore the potential reasons for the substantial changes during that period.

After coding and classifying the 306 units in curriculum documents, we compiled the results in Fig. 13.8.

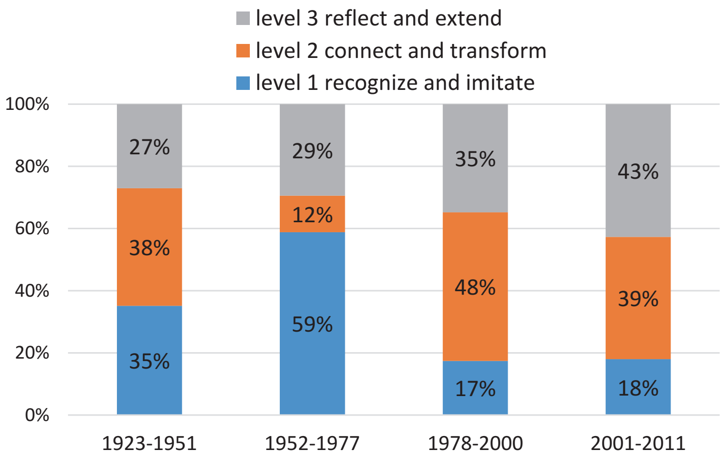

Fig. 13.7 Changes in requirements for mathematical communication abilities in terms of cognitive demands

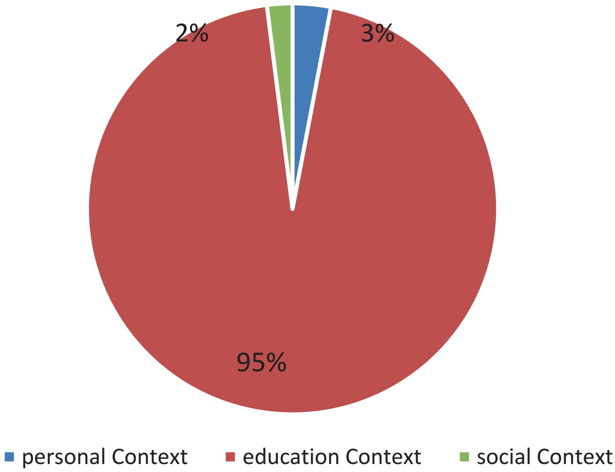

Fig. 13.8 Distribution of communication contexts

The majority of the mathematical communication context in curriculum standards is the educational context (95%). The personal and social contexts mainly appeared after 2001. Within the educational context, almost half of contexts require teacher-student communication (48.80%). 21.31%, 11%, and 18.9% of educational contexts initiate student-text communication, student-student communication, and student-self communication respectively. Since teacher-student communication plays a dominant role in mathematical communication, it is vital to promote teacher-student communication to support students' mathematical learning.

13.6 Conclusion

In general, we found that mathematical communication abilities in curriculum standards over the past hundred years were defined in four types: teacher-student communication, student-self communication, student-student communication, and student-text communication. The development of the definitions and requirements of mathematical communication abilities in China went through five phases: the emergence of mathematical communication abilities from 1902 to 1922; the rise of mathematical communication abilities from 1923 to 1951; student-oriented mathematical communication abilities from 1952 to 1977; the emphasis of student-student communication from 1978 to 2000; and collaboration-oriented mathematical communication from 2001 to 2011.

Mathematical communication is defined as a process of receiving, processing and expressing mathematical information and ideas. Among all four types of mathematical communication abilities, teacher-student communication plays a dominant role in curriculum documents in China. Starting from the curriculum reform in 1952, there was a shift from teacher-student communication to student-oriented communication in the curriculum standards requirements. More emphasis was placed on student-student, student-self, and student-text communication.

In thinking of the research question regarding changes to the requirements for mathematical communication abilities, we found considerable changes to the requirements in terms of mathematical content areas, cognitive demands, and communication contexts. Over the past hundred years, there has been a substantial increase in the requirements for mathematical communication abilities in probability and statistics and high-level cognitive demands (e.g., level 3, reflect and extend). The percentages of mathematical communication abilities requirements for the comprehensive requirements content area and educational context remain at half or above.

With the development of mathematics curricula, the standards have put emphasis on the requirements of mathematical communication abilities comprehensively. The four objectives of the current mathematics curriculum for compulsory education all have a focus on mathematical communication abilities. For example, in relation to objectives of problem-solving, students “should experience problem-solving collaboratively with others and explain their own thinking ways . . . and communicate with others and can understand others’ thinking ways and conclusions” (Ministry of Education of People’s Republic of China, 2012, p. 14). In addition, the objectives of emotion, attitudes and values include requirements for mathematical communication abilities – namely, students should “dare to express their own ideas. . . develop habit for collaborative communication” (p. 15).

The current curriculum standards place high demands on mathematical communication. After implementing the mathematics curriculum, the development of students’ mathematics communication ability has reached the curriculum goal to a certain extent. The next chapter will investigate and analyse the students’ mathematical communication ability.

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Chapter 14
Chinese Eighth Graders' Competencies in Mathematical Communication

Yuelan Chen, Binyan Xu, and Xiaoyan He

Abstract This chapter examines students' mathematical communication abilities and explores their performances in mathematics communication. In the study, a stratified sampling method was adopted to select 1192 eighth-grade students as a sample. Four tasks, which included seven items of mathematical communication, were developed and provided to students to solve them. After data analysis, the study found that eighth-grade students in China performed well in the mathematical communication tasks at level 1 and level 2. That is, students can understand the meaning of complicated mathematical texts, can express complicated mathematical understandings and can explain other people's (correct or wrong) mathematical thoughts, but there is a lack in reflective thinking and obstacles in evaluating and correcting others' opinions. The study showed that there is a gender difference in solving strategies of mathematical communication tasks. The boys tended to use concise and clear mathematical language, while the girls were good at using verbal expressions and mathematical symbols to express their own opinions.

Keywords Mathematical communication · Ability · Performance · Stratified sampling method · Mathematical text · Mathematical understanding · Mathematical thought · Reflective thinking · Gender difference · Solving strategy · Mathematical language · Verbal expression

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14.1 Introduction

Chapter 13 traced the history and the development of mathematical communication abilities in mathematical curriculum standards in China. Since mathematical communication ability is listed as a core requirement in curriculum standards in China and other countries, it is reasonable and necessary to examine the status quo of students' mathematical communication abilities. In this chapter, we present a study of a group of eighth-grade students' mathematical communication abilities in China. The purpose of the study was to investigate the level of mathematical communication abilities among this group of middle school students in China and explain the implementation of the intended curriculum in regard to mathematical communication abilities.

14.2 Literature Review

Mathematical communication abilities are included in mathematics curricula in different countries. On one hand, mathematics curriculum standards clarified connotations of mathematical communication abilities, on the other hand, have paid attention to the assessments of such abilities.

Mathematics curriculum standards for middle schools in Germany propose one mathematical competencies assessment model which includes mathematical core ideas and three requirement dimensions (Kultusministerkonferenz, 2004). They propose that in the dimension of reproduction, students should express simple mathematical facts orally and in written form; in the dimension of relationship construction, students should understand and formulate mathematical consideration, solving processes and solutions; and in the dimension of generalisation and reflection, students should represent complex mathematical facts orally and in written form. *Curriculum and Evaluation Standards for School Mathematics* states that students' mathematical communication abilities should be assessed in three aspects: (1) expression of mathematical ideas orally or through mathematical writing, proof, and visual description; (2) understanding and commentary on mathematical ideas presented in words, verbal communication or visual presentations; and (3) use of mathematical language to present mathematical ideas, describe relationships and build models (National Council of Teachers of Mathematics [NCTM], 1989). All three aspects in the standards emphasise the way students are expected to communicate mathematical ideas. Teachers should pay attention not only to how students communicate their own mathematical ideas, but also to how students understand others' mathematical ideas (NCTM, 1989).

International comparative programs have also provided assessments of mathematical communication. For example, PISA identified three levels of mathematical communication abilities focused on cognitive development. The first level is reproduction, in which students present their understanding of basic mathematical content in oral or written form. The second level is connection, referring to students interpreting the connection between mathematical facts and understanding others' oral or written narratives of the mathematical facts. The third level is reflection, meaning students read mathematical materials and then present their mathematical ideas about these mathematical materials in oral or written form (Organisation for Economic Co-operation and Development, 2009). Students' mathematical communication abilities could be assessed with these three levels.

Researchers have also focused on assessments of mathematical communication abilities. Cai et al. (1996) pointed out that from the perspective of assessment instruments, open-ended questions could provide students with the opportunity to demonstrate their thinking processes, interpretations and argumentation. Open-ended questions could also be used to investigate the mathematical communication abilities of students. Cai et al.'s study identified five levels of student performance. Cai et al. defined the quality of students' mathematical communication in terms such as *accuracy* and *clarity*. Representation was also examined to study the way students communicated mathematics (e.g., mapping, mathematical representation). Santos and Semana (2015) focused on concrete components of mathematical communication abilities. Their study concerned expository writing in mathematics and investigated how four eighth-grade students performed three expository writing tasks. The four students worked in a group and were assisted by feedback and the use of supporting assessment documents. The study found that the assessment strategies contributed to development in the students' expository writing, particularly regarding interpretation and justifications.

Technology has also played an important role by investigating and assessing mathematical communication abilities. Herheim (2015) investigated students' mathematical communication while students worked with geometry at a computer and especially focused on identifying and characterising good communication practices. The study was concerned with how to communicate mathematics through joint reflections between students and teachers while they shared a stand-alone computer with Geoboard software.

Studies on mathematical communication abilities implemented by Chinese scholars were reviewed. There were few studies focused on assessing communication abilities. Su (2003) examined the status of mathematical communication among middle school students. He mainly developed one questionnaire and implemented it in one school. Results showed that students had not attached importance to mathematical communication in the classroom. Outside the classroom, students' communication aimed at asking questions instead of talking and discussing interactively. Hu and Zhao (2007) explored assessments of mathematical communication abilities. They concluded that there were five elements of mathematical communication – which were *application of mathematical language*, *application of mathematical basic knowledge*, *visualisation of mathematics*, *problem-solving aspect* and *attitude of communication of mathematics* – then suggested that every element could be assessed from three different cognitive levels. They proposed one assessment framework but did not examine it.

Recently, some case studies on developing mathematical communication have been published. Zhang et al. (2019) were concerned with students' communication while posing mathematical problems. They developed teaching cases for mathematical communication in problem-posing. Deng and Xia (2019) explored a teaching model for mathematical expression. They explained that the mathematics curricula in China paid more and more attention to communication in mathematics. In order to make the intended curriculum realistic, on the one hand, corresponding teaching cases should be developed and analysed; on the other hand, the status quo of students' mathematical communication abilities need to be investigated. It is important for teachers to try to understand students first.

This chapter examines students' mathematical communication abilities and explores their performance in mathematics communication. The research questions include the following: Which level of mathematical communication abilities have eighth-grade students reached? Are there gender differences in mathematical communication?

14.3 Methods

14.3.1 Participants

In this study, a stratified sampling method was adopted to select eighth-grade students as participants. First, based on the geographical location and economic development status (including developed, medium-developed, and less-developed), eight regions were identified nationwide. One city (capital city of province) was selected from each region. Then in each city, three or more schools were randomly chosen. Finally, at each selected school, one or two classes were selected for the purpose of participating in the test of mathematical communication. All students from the selected classes were asked whether they would like to participate in the test. A few students went to attend activities because they did not want to participate in the math test. Details about participants are shown in Table 14.1.

Table 14.1 Information of participants

	Developed area			Medium-developed area		Less-developed area			Total
Cities	A	D	E	B	C	F	G	H	8
Number of participating schools	4	3	3	3	3	3	3	6	26
Total number of participating students	171	80	107	137	119	180	154	244	1192

14.3.2 Instrument

14.3.2.1 Framework for Developing Test Items

We first reviewed literature about investigating and assessing mathematical communication abilities. Then we analysed the characteristics of Chinese mathematics curricula and determined three mathematical communication ability levels: namely, recognition and imitation (level 1), connection and transformation (level 2), and reflection and extension (level 3). At each level, there were particular associated ability performances. All these are shown in Table 14.2.

According to the descriptions of the ability performances, we developed four test tasks which corresponded to each level respectively. When students finished one task at a certain level correctly, it revealed that students' communication abilities reached the given level.

Table 14.2 Mathematical communication ability levels and corresponding performance

Ability level	Connotation	Ability performance
Level I: recognition and imitation	Be able to understand the meaning of simple texts and express simple mathematical facts and the process of simple mathematical communication	(a) Be able to recognise and select information from short mathematical texts (b) Be able to clearly express simple mathematical facts, such as understanding of simple mathematical content
Level II: connection and transformation	Be able to understand the meaning of complex texts, express complex mathematical understanding and explain the mathematical ideas of others	(a) Be able to recognise and select information from mathematical texts and understand its significance and be able to transform the mathematical ideas of others from one carrier (chart, text, symbol, object, action, etc.) to another, which is convenient for further understanding (b) Be able to express the thinking process, the solution and the result in a brief and logical way and be able to explain the description of the text of the mathematical class (correct or incorrect) on the basis of judgment
Level III: reflection and extension	Be able to understand the meaning of complex texts, express one's own solution with an appropriate carrier, and evaluate others' and their own mathematical ideas	(a) Be able to grasp the meaning of complex texts and compare and judge others' mathematical thoughts (b) Be able to design a program that completely presents a complex solution and be able to compare, evaluate and correct the understanding of others; flexibly transform the carrier of mathematical thought; choose the optimal expression carrier according to the specific situation; and express the inspection and reflection of the learning process so that the problem-solving process is reasonable, complete, concise and harmonious

Situations Used in Items

Three kinds of situations were considered while developing test items – namely, individual situations, educational situations and social situations. Individual situations referred to the daily behaviour of students–for example, watching TV or playing computer games. Educational situations related to school life – for example, subject knowledge or examinations. Social situations referred to community or economic development – for example, financial problems or census of population.

Type of Test Items

Fill-in questions and open-ended questions were developed for this study. Open-ended questions could provide students with opportunities for developing their higher-order thinking. The questions were related to real complex problems. While solving such problems, students should identify problems using mathematical language. Open-ended questions were intended to investigate and assess mathematical communication abilities at higher levels (Shepard, 2000). The fill-in questions were developed for assessing whether students could understand the text and explain their ideas briefly. So fill-in questions were used for assessing communication abilities at the lower level.

Type of Communication

This study adopted three types of mathematical communication: mathematical dialogue, writing and reading. Different types of communication could provide opportunities for students to perform at different levels of communication abilities. For instance, the task 2 and 3 used in the study stimulated different mathematical dialogues.

[Task 2] Number transformation

In a mathematics transformation game, we call the integers 0, 1, 2 . . . 100 "old numbers". The game's transformation rule is: the old number is squared first and then divided by 100; the resulting number is called the "new number".

1. *How do you transform the old number 80 into a new number according to the above rules? Please describe the transformation process.*
2. *After the above rule transformation, we found that many new numbers have become smaller. Xiao Min asserted: "According to the above transformation rules, all new numbers are not equal to its old numbers." But Xiaoping disagrees with Xiao Min's statement. She believes: "There must be situations where the new numbers equal the old numbers." You support whose opinion? Please give further reasons.*

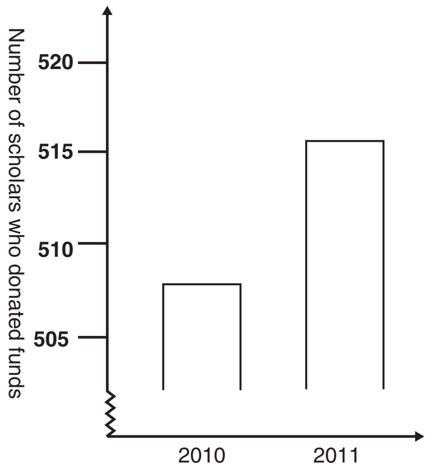


Fig. 14.1 The number of scholars who donated funds

This task invited students to have dialogue with each other mathematically. For the first question, students could directly use the rules to get a result. It was a simple process that students could explain. This item reflected level 1 of mathematical communication abilities. For the second question, students needed to propose their conclusions and argue them. This required students to make a judgement based on dialogue instead of recognising results directly.

[Task 3] Donation for students

The number of scholars who donated funds in 2010 and 2011 is shown in Fig. 14.1. Xiaonan said: "This picture reflects the dramatic increase in the number of donations to schools from 2010 to 2011." Do you think this picture does reflect Xiaonan's so-called dramatic increase? Please give a specific explanation.

This task described a public event from a social situation. The opinion of Xiaonan could initiate students to think and explain their viewpoint. The task simulated a mathematical dialogue between Xiaonan and the students. Students needed to evaluate and correct Xiaonan's mathematical understanding on the basis of comprehending the diagrams and Xiaonan's views. From the perspective of calculating the growth rate or the amount of growth, students could point out that the diagram showed a "slight increase" situation. So they could deny Xiaonan's view. Or they could explain the irrationality of the drawing. This task was aimed at assessing students' mathematical communication abilities at the level of reflection and extension.

The task 4 in this study presented a social situation which would be discussed through mathematical writing.

[Task 4] Two views that are contradicting

The education expenditure and public finance expenditure of the government of a certain place for two consecutive years are shown in Table 14.3 (Note: Public finance expenditure includes education expenditure; at the same time, the inflation rate during the 2 years is not considered).

Table 14.3 Education expenditure and public finance expenditure

	2010	*2011*
Education expenditure	*75 million yuan*	*80 million yuan*
Public finance expenditure	*500 million yuan*	*600 million yuan*

Some people say that from the above table, it can be seen that from 2010 to 2011, education expenditure increased, but some people say it decreased. Please use the above data to explain the perspective from which these two seemingly contradictory views were put forward. Please give further explanation.

The two mathematical viewpoints in the task seem to be contradictory, but they are actually caused by different perspectives: absolute expenditure and relative expenditure. This task requires students to flexibly combine data according to their needs and explore the source of contradictory views, which is a requirement for reflection and expansion. Students needed to write text using mathematical language to explain different viewpoints. For example, in order to explain the "decrease" viewpoint, it could be stated that "education expenditures account for a decrease in the proportion of public finance expenditures". It could also be stated that "the growth rate of education expenditures is smaller than that of public finance expenditures", and it could also be stated that "for every 100 million yuan of public finance expenditure, the proportion of education expenditure has decreased". This required a high level of mathematical communication abilities.

The task 1 in this study required students to read and understand mathematics information in the situation.

[Task 1] Meeting a cat during an outing
Xiaolin drove out for an outing. A cat rushed to his car on the way. Xiaolin braked hard, and the cat slipped away. Xiaolin was frightened and decided to drive back and take a shorter way. Figure 14.2 is a partial image of the speed change during this period.

1. *According to Fig. 14.2 (speed-time image), the fastest speed driven by Xiaolin was ______________; to avoid the cat, Xiaolin hit the brake at the time of ______ (fill in the time).*
2. *Use the following description to complete the speed-time image:*
   When Xiaolin hit the brake, the car's speed was reduced to 12 km/h. Then, he started to step on the accelerator, and the speed reached 36 km/h at 9:09. He slowed down gradually and arrived home at 9:12.
3. *From 9:00 to 9:12, during which period of time do you think Xiaolin's driving speed increased the fastest? ______________*

The task required students to read the text and graph and identify mathematical information to solve problems. The first question required students to directly extract information from the speed-time graph, which is a requirement for recognition and imitation. In order to complete the second and third questions, students needed to establish the connection between the text and graph, convert the text information into graph information, and convert the graph information into text information. This is a requirement for connection and transformation.

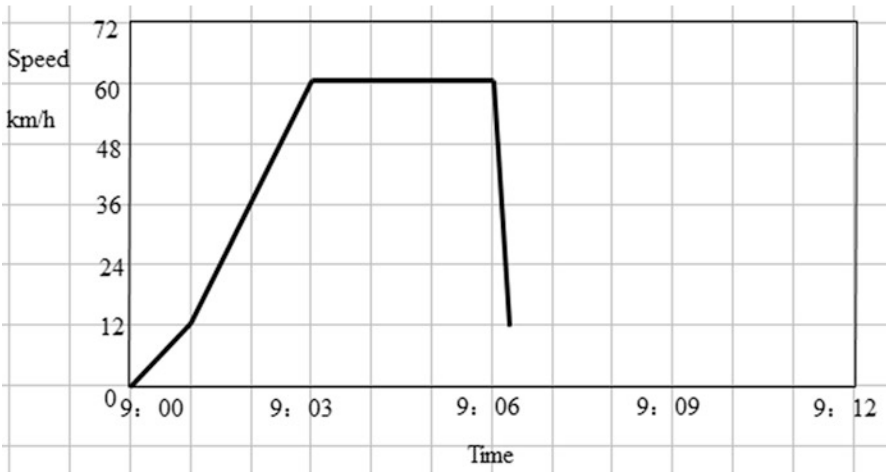


Fig. 14.2 Speed change during Xiaolin's outing

14.3.2.2 Information of Test Items

The test comprised four tasks that referred to seven mathematical items, which covered different mathematical content – namely, numbers, algebra, probability and statistics. Table 14.4 lists the information of the test items.

14.3.3 Data Analysis

A total of 1192 students from eight cities attended the test on mathematical communication abilities. Each response made by participating students was coded using a double coding system. The students' responses to each item corresponded to a two-digit number. The first number represented the correctness of solving tasks: 1 for a correct result, 0 for an incorrect result. The second number represented a diagnostic code to identify the form of representation (word, number, or graphic representation) and the problem-solving strategy (solving equations or conjecture).

Task 1 consisted of three question items. The first question included two blanks which were to be filled in. The first blank asked students to read the graph and find out the quickest speed from Fig. 14.2; students needed to fill in the value and unit. So the coding framework distinguished the correctness of the value or unit. The second blank asked students to identify the moment when the brake was used. Table 14.5 shows the coding framework for the first question of task 1.

The second question of task 1 required students to make graphs based on the situation. There were two correct solving strategies. Some students determined the speed at turning point 9:09 and the final speed at 9:12, the two speeds were connected by the line segment (#1). Some students determined the speeds at 9:09

Table 14.4 Information of test items

Task/ item	Mathematical content	Mathematical communication ability level	Type of situation	Type of test item	Type of communication
1(1)	Numbers and algebra	Level 1	Individual situation	Fill-in question	Mathematical reading
1(2)	Numbers and algebra	Level 2	Individual situation	Open-ended question	Mathematical reading
1(3)	Numbers and algebra	Level 2	Individual situation	Fill-in question	Mathematical reading
2(1)	Numbers and algebra	Level 1	Educational situation	Open-ended question	Mathematical dialogue
2(2)	Numbers and algebra	Level 2	Educational situation	Open-ended question	Mathematical dialogue
3	Probability and statistics	Level 3	Social situation	Open-ended question	Mathematical dialogue
4	Number and algebra	Level 3	Social situation	Open-ended question	Mathematical writing

Table 14.5 The coding framework for the first question of task 1

Task (item) 1(1)	The first code		The second code	
The first blank	Code		Code	
	1	Correct result	11	Both the values and units are correct, such as 60 km/h
			12	The value is correct, while the unit is missing
			13	The value is correct. The unit is in both Chinese and English, such as 60 km//小时
	0	Incorrect result	01	The value is correct, but the unit is incorrect, such as 60 km/t.
			02	The value is incorrect./。
			03	Irrelevant content
			04	Blank
The second blank	1	Correct result	11	Correct result, 9:07
	0	Incorrect result	01	Wrong time
			02	Wrong time interval, such as 9:03–9:06
			03	Blank

and 9:12, and the two speeds were connected by a curve (#2). Table 14.6 shows the coding framework for the second question of task 1.

To solve the third question of task 1, students needed to consider the ratio of speed increment and the time interval. There were two correct solutions. Some students calculated the slope of each line segment and then found that the second segment had the largest slope, so the result was 9:01–9:03 (#1). Some students treated the first two segments as a whole and compared the acceleration with the acceleration after braking; then they found that the former acceleration was larger, so the result was 9:00–9:03 (#2). Table 14.7 shows the coding framework for the third question of task 1.

In the study, seven items from the four tasks corresponded to the coding framework. All responses made by participating students were coded based on such frameworks. All data driven by coding approaches were analysed with reference to performance of mathematical communication abilities.

14.4 Findings

14.4.1 The Status Quo of Participating Students' Mathematical Communication Abilities

Students' performances on each task or item were analysed. The data showed that the average rates for students correctly solving the items at level 1 and level 2 were 86.3% and 83.3%, respectively. The average rates for correctly solving the items at level 3 was 37.6%. The students performed very well while solving the second task

Table 14.6 The coding framework for the second question of task 1

Task (item) 1(2)	The first code		The second code	
	Code		Code	
	1	Correct result	11	Solving strategy #1
			12	Solving strategy #2
	0	Incorrect result	01	Half drawing correctly (correct speed at 9:09)
			02	Only showing the trend on "add first and then subtract"
			03	Irrelevant drawing
			04	Blank

Table 14.7 The coding framework for the third question of task 1

Task (item) 1(3)	The first code		The second code	
	Code		Code	
	1	Correct result	11	Solving strategy #1
			12	Solving strategy #2
	0	Incorrect result	01	Irrelevant period of acceleration, such as 9:08–9:09
			02	No period of acceleration
			03	No time period
			04	Blank

(number transformation), which referred to level 2 of mathematical communication abilities. The average rates of correctly solving the first and second questions of the second task were 96.0% and 89.8%. The problem setting of the second task was a pure mathematical situation in the educational context, which was similar to daily mathematical exercises experienced by students. It can be speculated that the students performed well because they were more familiar with the situation. However, the accuracy rate of the second fill-in for the first question of task 1, which referred to level 1 of mathematical communication abilities, was only 67.8%, which was far from the other two items at the same target level (95.1%, 96.0%). The overall performance was shown in Fig. 14.3.

Further, we found that the majority of participating students (71.2%) solved mathematical communication tasks at level 1 and level 2 correctly. In other words, most students can understand the meaning of relatively complex mathematical texts, can express relatively complex mathematical understanding, and can explain other people's (correct or incorrect) mathematical thoughts. A small number of students (15.4%) solved all tasks correctly. Such students can understand the meaning of complex mathematical texts, can express their own solutions with appropriate carriers, and can evaluate other people's and their own mathematical ideas, especially via the inspection and reflection of the learning process, which strengthens the rationality, integrity, simplicity, and harmony of the problem-solving process. There were also a small number of students (13.4%) who could only understand the meaning of simple texts, organise existing mathematical information, and express simple mathematical facts in a relatively clear way.

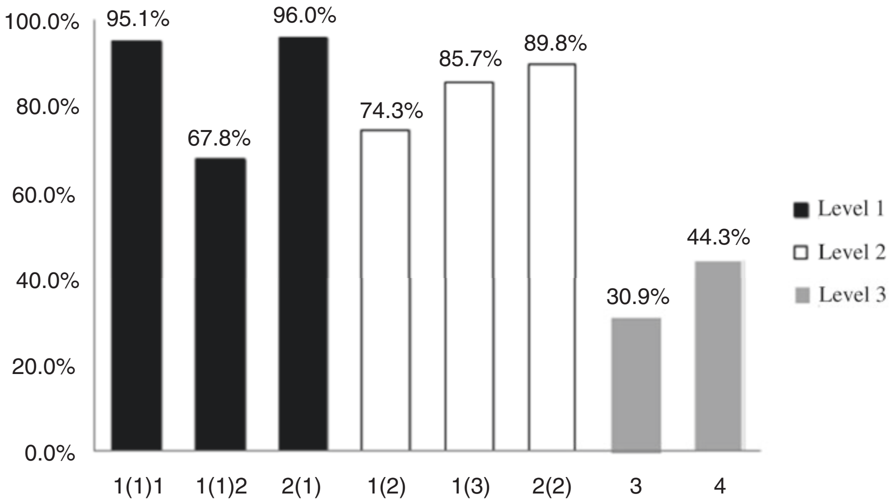


Fig. 14.3 The rates for students correctly solving tasks at three different levels

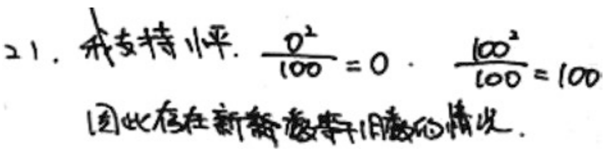
14.4.2 Characteristics of Solving Mathematical Communication Tasks

As mentioned above, the students' performances on each item corresponded to a two-digit code. The first digit indicated the correctness, and the second code represented students' problem-solving strategies, forms of mathematical communication, etc. Through the analysis of the second code of each item, we can investigate the characteristics of students' mathematical communication.

The data indicated that if the questions of the tasks contained numbers, students were more inclined to choose the operations of these numbers to express mathematical information. For example, the first question of task 2 (number transformation) asked students to describe the process of transforming the number 80 into a new number (see the original question mentioned under Sect. 14.3.2.2). 85.9% of students used numerical formulas to perform operations to express the transformation process. The second question of task 2 described a dialogue situation and required students to make a judgement. The question was as follows:

After the above rule transformation, we found that many new numbers have become smaller. Xiao Min asserted: "According to the above transformation rules, all new numbers are not equal to its old numbers." But Xiao Ping disagrees with Xiao Min's statement. She believes: "There must be situations where the new numbers equal the old numbers." You support whose opinion? Please give further reasons

Earlier in the task, it was mentioned that the integer contained 0, 1, 2 . . . 100. 66.6% of the students selected one or two concrete numbers and calculated results, then made a judgement about "*there must be situations where the new numbers equal the old numbers*". The following is one example of a student's solution.

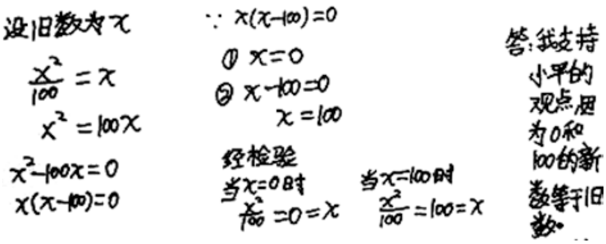


I supported Xiao Ping's statement.

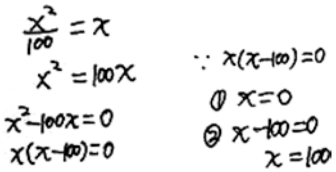
$$\frac{0^2}{100} = 0, \ \frac{100^2}{100} = 100$$

So there are situations where the new number is equal to the old number.

We observed that 15.5% of students set unknowns, formulated equations and solved them, then made a judgement. One example of such a solution is as follows.



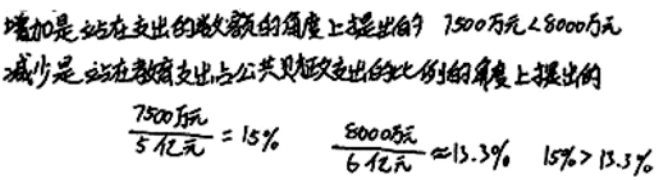
Let the old number be x



Tested, while $x = 0$, $\frac{x^2}{100} = 0 = x$; while $x = 100$, $\frac{x^2}{100} = 100 = x$

Answer: I supported Xiao Ping's opinion, because the new numbers of 0 and 100 are equal to the old numbers.

By solving other mathematical communication tasks, the students identified similar characteristics. Task 4 was about two views that were contradicting. This task gave two sets of determined data from 2010 and 2011. Students tended to choose different combinations of the data to solve problems. The data analysis found that 62.3% of students focused on number calculations to analyse the clues behind the two views. Among them, 51.1% of the students correctly used numbers to illustrate the two seemingly contradictory views. One example is shown in the following:



The increase is proposed from the perspective of the amount of expenditure, namely 75 million yuan <80 million yuan.
The reduction is proposed from the perspective of the ratio of education expenditure to public financial expenditure, namely

$$\frac{75\ \textit{million yuan}}{500\ \text{million yuan}} = 15\%,\ \frac{80\,\textit{million yuan}}{600\,\textit{million yuan}} = 13.3\%,\ 15\% > 13.3\%$$

However, when students explained their opinion using pure word expression, only 33.2% of them could explain their statements correctly. For example, they explained, "The increase in education expenditure is based on the amount of expenditure; the reduction in education expenditure is based on the ratio of education expenditure to public financial expenditure." Most students could not clarify their opinions using word expression; for example, they provided unclear explanations such as the following: "The first opinion considered education expenditures, and the second considered education expenditures and public financial expenditures together."

The analysis showed that students seemed to be unfamiliar with social situations in a mathematical context. When they dealt with task 3 and task 4, students did not realise how to translate social situation into relevant mathematical problems. They could not find mathematical information behind situations; they used explicit social information or life experiences that they could read to explain statements or make judgements. Facing task 3 and task 4, students could not perform mathematical communication abilities to some extent.

14.4.3 Gender Differences in Mathematical Communication Problem-Solving Strategies

14.4.3.1 Gender Difference in Performance of Mathematical Communication

The gender difference in mathematical learning is a hot topic in mathematics education research. One view is that girls' language expression ability is better than boys', but that they are inferior to boys in terms of spatial ability. Some studies showed that compared with boys' tendencies to use abstract problem-solving strategies, girls prefer specific strategies (Liu & Sha, 2012); that is, boys are more inclined to choose simple and concise mathematical expressions than girls. However, with the development and implementation of large-scale international evaluation projects, such conclusions have been refuted by researchers, who pointed out that gender differences are gradually decreasing and are far from being as big as imagined (Spelke, 2005). Actually, the debates driven by different researchers raised attention for gender research on mathematics teaching and learning.

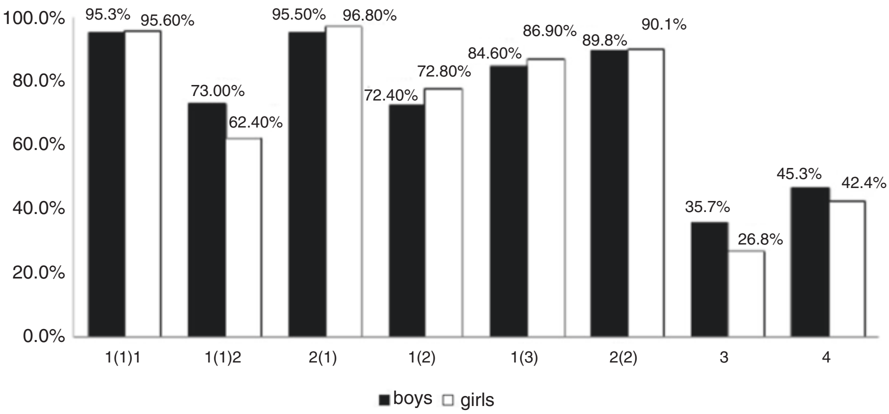


Fig. 14.4 Correctness rate of girls and boys in different tasks

In this study, we paid more attention to characteristics of problem-solving strategies of mathematical communication tasks from the gender perspective. Of the 1192 students, 51 did not indicate gender, so 1141 students were analysed. Among them, 564 students are girls.

The data analysis showed that in the tasks at level 2 (task 1[2], 1[3], 2[2]) the correctness rate of girls was higher than that of boys. For performance of the tasks at level 3 (tasks 3, 4), the correctness rate of boys was higher than that of girls. The overall correctness rate of girls and boys is shown in Fig. 14.4.

From the analysis of the correctness rate of each item, we found that the performance gap between boys and girls in the second fill-in of the first question of task 1 is the largest, with a difference of 10.6%. From the perspective of situation, girls performed better than boys in educational situation tasks. For example, in task 2, girls' correctness rates for the two questions were 96.8% and 90.1%, respectively, slightly higher than the boys' 95.5% and 89.8%. But in social situation tasks, girls' performances were inferior to boys. For example, in tasks 3 and 4, the gap between boys and girls was larger than that in the educational situation tasks, with a difference of 8.9% and 3.9%, respectively.

14.4.3.2 Gender Differences in Mathematical Communication Problem-Solving Strategies

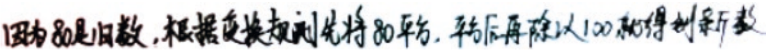
We found that compared with boys' tendencies to use abstract problem-solving strategies, girls preferred specific strategies. Boys are more inclined to choose simple and capable mathematical expressions than girls. This feature was mainly reflected in task 2.

Take the first question of task 2 as an example. More than half the students set a column formula and found a solution, such as the following:

$$\frac{80^2}{100} = 64$$

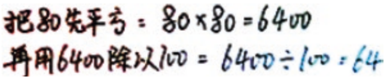
This is a simple and clear strategy, and 65.4% of boys and 54.7% of girls used this strategy.

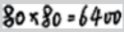
Alternatively, 18.9% of the girls chose to use a word-expression strategy, which was slightly higher than the ratio of boys (14.8%). Following is an example:

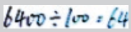

因为80是旧数，根据变换规则先将80平方，然后再除以100就得到新数

Because 80 is an old number, according to the transformation rules, first square 80, and then divide by 100 to get the new number.

The third strategy is to combine the methods of setting column formula and using word expression. Girls were more inclined to choose words expressions than boys and even supplement words in the column formula, such as the following:


把80先平方：80×80=6400
再用6400除以100 = 6400÷100=64

First square 80, 
80×80=6400

then 6400 is divided by 100, . 
6400÷100=64

Take the second question of task 2 as an example. Of the boys, 67.5% just guessed the counterexample directly without formulating the equation, but 43.1% of girls formulated equation and express their solution. For the strategy of formulating and solving an equation, 49% of girls supposed an unknown number and set an equation, but only 27.5% of boys used such a strategy. Since task 2 did not stipulate that students must choose a certain strategy or exhaustive counterexamples – that is, as long as they could express the correct counterexamples – it can be judged that students met the question's requirements for mathematical communication. Boys tended to choose a relatively simple strategy, which only required guessing the counterexample and then verifying it, but this strategy may lead to the inability to exhaust all counterexamples; the *solving equation* strategy that girls tended to choose required relatively lengthy writing.

In solving task 4, students did not perform very well and made different kinds of mistakes. Among the boys, 43.3% described the two perspectives in a reasonable way, but the descriptions were too general and the correspondence was not clear – 37.5% of the girls made such mistakes, slightly lower than that of the boys. Mathematical communication pursues simple and accurate mathematical expression, but too much pursuit of simplicity will bring the risk of reducing accuracy. Therefore, students need to learn how to keep the balance in communication.

14.5 Conclusion and Discussion

The results of this study found that eighth-grade students in China performed well in the mathematical communication tasks at level 1 and level 2. That is, students can understand the meaning of complicated mathematical texts, can express complicated mathematical understandings, and can explain other people's (correct or incorrect) mathematical thoughts, but there is a lack of reflective thinking, and there are obstacles to evaluating and correcting others' opinions.

Mathematics provides powerful, concise and accurate tools to exchange information. Students should be given the opportunity to read, write and talk about mathematics in a variety of ways (Cockcroft, 1982). However, traditional mathematics teaching styles tend to neglect the descriptiveness and transitivity of mathematics. Students have few chances to reflect on mathematics as a way of communication (Xu, 2006). Reflective thinking is high-level thinking. It is not just memorisation, repetition, or simple application. Reflective thinking is a complex thinking process that involves specific goals, sustained psychological efforts and cognitive activities, including divergence and reflection (Liu, 2002). Other studies have shown that the memory and comprehension skills of Chinese students are developed; the analytical and application skills are average, and the reflective and creative skills need to be improved (Liu, 2008). The above analysis can partially explain why the eighth-grade students did not perform well in the mathematical communication tasks at level 3.

On the other hand, mathematics education in China has the tradition of variation teaching. Teachers emphasise the connection and sequence of different mathematical concepts. With the new math curriculum reform, there is an increased amount of high-cognitive-level mathematical tasks in curriculum materials. Students are expected to describe processes and provide explanations. This study found that the participating students' mathematical communication abilities reflected that students are good at connection and transformation of mathematical information. This aligns with the expectations in the math curriculum reform.

The text used for communication includes continuous text and discontinuous text. The latter specifically refers to text that contains not only words but also graphics and other formats (Wang & Tian, 2007). The text used for task 1 and task 3 belonged to a kind of discontinuous text. Task 1 and task 3 provided word information and graphic information. When students dealt with these tasks, they needed to read and understand mathematical information within words or graphs and connect different forms of texts. Communication with discontinuous text is more complicated than with continuous text. In this study, students performed weaker in task 3 and task 4 because different forms of text in the tasks may interfere with students' mathematical communication. Students are accustomed to grabbing mathematical information (data, formulas, etc.) in texts to express mathematical opinions. However, when there is no direct mathematical expression in the text, students may not be able to accurately express their mathematical opinions.

This study showed that there are no significant gender differences in mathematical communication abilities. But there is a gender difference in problem-solving strategies of mathematical communication tasks. The boys tended to express their statements using concise and clear mathematical language, while the girls were good at using word expressions or combining words and mathematical symbols to express their own opinions. Girls performed better while solving tasks in the educational situation than boys, but boys were better at mining mathematical information from social situations. Different situational problems will affect the performances of girls' or boys' mathematical communication abilities.

Mathematical communication abilities involve both written and oral communication. Considering the convenience of the written test in the large-scale study, this study did not collect any data on oral communication. Therefore, the study presented here offers part of the picture of students' mathematical communication abilities. Future research could study students' mathematical communication abilities in oral form.

With the development of mathematics curricula, more and more attention has been paid to mathematical communication abilities in China. How to cultivate students' mathematical communication abilities in mathematics education – and how to let students observe the world from the mathematical perspective, analyse the world with mathematical thinking, and express the world with mathematical language – are the research and practice issues. Some researchers have explored structures of classroom teaching focused on instilling mathematical communication abilities (Xu, 2012; Zha, 2001; Qiao & Gao, 2005). Some common teaching aspects were emphasised – namely, collaboration, interaction, coexisting, looking for technical support, and communicative dialogue. We look forward to a win-win situation for the evaluative research of mathematical communication and communication teaching practices.

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Chapter 15
Chinese Eighth Graders' Self-Related Beliefs During Mathematical Modelling

Yan Zhu

Abstract It is suggested that belief in one's ability to complete a mathematics task is an important predictor of one's subsequent performance in the task. Paradoxically, students in high-achieving East Asian educational systems generally exhibit a low level of self-related beliefs in mathematics. However, interestingly, PISA reported that such a phenomenon does not appear in Shanghai. Given these inconsistent findings, an investigation of a total of 1359 Chinese eighth graders was conducted to examine their self-judgements about their modelling performance before and after they work with three modelling tasks of varying difficulty. The results showed that students' self-efficacy is consistently higher than their self-evaluation for the modelling task, and both beliefs have a positive correlation with students' actual performance. A hierarchical analysis reveals that the variances in students' beliefs are mainly based on differences between individual students and that gender and actual modelling performance have important impacts.

Keywords Self-related beliefs · Self-efficacy · Self-evaluation · PISA · Mathematics performance · East Asia · Chinese students · Mathematical modelling · Eighth grade · Gender gap · One-child status · ANOVA · Effect size · Hierarchical analysis · Collective culture

15.1 Self-Related Beliefs

How students think and feel about themselves is an important predictor of how they act and make decisions when they are challenged by tasks and situations (Bandura, 1997). When it comes to learning and teaching mathematics, Henry Ford suggested that there is a two-way relationship between belief in one's ability to complete a mathematics task and subsequent performance in the task (Champion, 2010). From the social cognitive perspective of learning, Albert Bandura addressed this potential

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relationship as *perceived self-efficacy* or self-evaluation of one's ability to achieve certain performance under specific constraints (Champion, 2010). Self-efficacy was developed by Bandura as part of a larger theory, *social learning theory* (Ashford & LeCroy, 2010), which in turn evolved into *social cognitive theory* (Levin et al., 2001). Social cognitive theory was developed in response to the lack of attention paid to the role of cognition in motivation and the role of situation in behaviourism and psychoanalysis (Redmond, 2010). The theory is composed of four processes of goal realisation: self-observation, self-evaluation, self-reaction and self-efficacy. These four components are interrelated, and all have effects on motivation and goal attainment (Redmond, 2010).

Self-observation refers to learners' systematic monitoring of their own performance (Walter, 2012). It can be used to assess one's progress toward goal attainment as well as motivate behavioural changes. *Self-evaluation* refers to the act of 'comparing self-monitoring information with a standard or goal' (Zimmerman, 2000, p. 21). It is affected by set standards and the importance of goals; thus, the goals must be specific and important. *Self-reaction* refers to one's cognitive, affective and tangible responses to performance evaluations, which may involve self-corrections and affective and motivational self-inducements (Zimmerman & Schunk, 2003). According to Bandura (1991), a person becomes more motivated when a positive self-reaction is anticipated as a result of achieving a given goal. Therefore, the behaviour or performance producing a positive self-reaction is more likely to create a future incentive to repeat the action (Thompson, 2007). *Self-efficacy* refers to an individual's belief in his or her capacity to execute the behaviours that are necessary to attain specific performance goals (Bandura, 1997). In other words, it reflects individuals' capacity to take measures to achieve targeted goals. Axtell and Parker (2003) remarked that self-efficacy increases one's effort and persistence for challenging tasks, thus increasing the likelihood that the tasks will be accomplished.

According to Chen (2003), self-efficacy is distinct from other self-related beliefs due to its specificity and close relation to overt performance. Zimmerman (1995) remarked that the predictability of self-efficacy depends on its specificity and relation to actual tasks. Some researchers suggest that self-efficacy actually influences one's academic performance in addition to their prior knowledge and skills (e.g. Pajares, 2008; Zimmerman, 2002). Furthermore, Pajares (1996) remarked that self-efficacy is a better predictor of one's performance than self-concept. Similarly, Bandura (1997) commented that a reasonable agreement between one's self-efficacy and action is desirable.

While self-efficacy concerns pre-performance judgement, self-evaluation concerns post-performance judgement. It is not unusual for students to make inaccurate self-evaluations. In particular, low-achieving students are often less accurate and more overconfident (Bol & Hacker, 2001; Kruger & Dunning, 1999). Overconfident students are more prone to select difficult problems to solve and are more likely to fail, which can undermine their subsequent self-efficacy and desire to continue learning (Bandura, 1986; Schunk & Parjares, 2004). Schunk (1996) argued that when one self-evaluates his/her capabilities or progress toward learning a particular task, he/she develops a higher level of competence, which in turn strengthens his/her perceived self-efficacy.

The main purpose of the present study is to investigate Chinese eighth graders' self-judgements about their modelling performance before and after they work on modelling tasks with different levels of difficulty. Three research questions are addressed in this study:

1. What are Chinese eighth graders' levels of self-efficacy and self-evaluation during a mathematical modelling task?
2. How do Chinese eighth graders' self-efficacy and self-evaluation relate to their actual modelling performance?
3. How do Chinese eighth graders' self-efficacy and self-evaluation, as well as the relations of these to their modelling performance, vary across cities, schools and students with different demographic characteristics?

15.2 Findings About Students' Mathematics Self-Related Beliefs from PISA 2003 and PISA 2012[1]

PISA 2003 and PISA 2012 are the two mathematics-focused studies that have been conducted so far in the PISA cycles. Both studies look at students' beliefs about mathematics learning in terms of students' self-concept (i.e. confidence in mathematics ability, which is constructed based on students' responses regarding their perceived competence in mathematics) and self-efficacy (i.e. belief in the capacity to tackle difficult mathematics tasks, which is constructed based on students' responses regarding their perceived ability to solve a range of pure and applied mathematical problems).

According to PISA 2003, across OECD educational systems, on average, 67% of students claimed that they do not understand the most difficult work in their mathematics class. The percentages range from 84% or more in Japan and Korea to 57% or less in Canada, Mexico, Sweden and the United States. Similarly, roughly half of the students across OECD systems do not think that they learn mathematics quickly. More than 62% of students report this self-concept in Japan and Korea, while only 40% do so in Denmark and Sweden. There is a comparatively large gender difference in students' self-concept. For instance, while one-third of boys do not think they are good at mathematics, the average for girls is 47%. In Japan, Korea, Hong Kong SAR and Macao SAR, 50–70% of girls agree with this statement. Regarding students' self-concept, PISA 2003 found that students in Canada, Denmark, Germany, Mexico, New Zealand, the United States and Tunisia have great confidence in their mathematics abilities, while students in Japan, Korea and Hong Kong have the lowest self-concept. In all the systems, boys tend to show a statistically higher level of mathematics self-concept than girls.

[1] This section is mainly based on *Learning for Tomorrow's World: First Results from PISA 2003* (OECD, 2004) and *Ready to Learn: Students' Engagement, Drive and Self-belief* (OECD, 2013).

Students' mathematics self-efficacy goes beyond how good they think they are in mathematics; instead, it relates to the kind of confidence students need to successfully resolve specific mathematics tasks. PISA 2003 reports that, on average, students in Greece, Japan, Korea, Mexico, Brazil, Indonesia, Thailand and Tunisia express the least mathematics self-efficacy, while students in Canada, Hungary, the Slovak Republic, Switzerland and the United States express much higher self-efficacy. PISA 2003 also shows that students' mathematics self-efficacy is more closely related to their performance on a mathematics assessment than their mathematics self-concept. In fact, within OECD countries, an average increase of one index point in mathematics self-efficacy corresponds to an increase of 47 points in mathematics performance, which is nearly equivalent to one school year.

Compared to PISA 2003, students' mathematics self-concept is slightly improved in PISA 2012. In particular, the students in PISA 2012 are four percentage points more likely to believe they can understand the most difficult work and three percentage points more likely to think that mathematics is one of their best subjects. Eighteen educational systems observed significant improvements in mathematics self-concept, including Iceland, Spain, Hong Kong SAR, Indonesia, Portugal, Norway, the Netherlands and the United States. The magnitude of gender differences in mathematics self-concept remained stable between PISA 2003 and PISA 2012. In eight educational systems, including Uruguay, Mexico and Hong Kong SAR, the gender gap widened in favour of boys.

Between 2003 and 2012, students' mathematics self-efficacy increased slightly across OECD systems, particularly in Portugal, Germany, Thailand, Turkey and Spain. However, decreases are observed in New Zealand, Hungary, the Slovak Republic and Uruguay. PISA 2012 reports that girls are more likely to have lower levels of self-efficacy than boys, although both boys and girls showed some improvement in their mathematics self-efficacy between 2003 and 2012. The gender gap in mathematics self-efficacy in 2012 is, on average, in favour of boys by over 0.3 points. This gap widened in favour of boys in France, Hong Kong SAR, Iceland and Australia. However, the gender gap in this self-related belief narrowed in Macao SAR, the Slovak Republic, Greece and Finland.

Reflecting on students' self-related beliefs and mathematics performance, Lee (2009) highlights a paradox among students in East Asia systems: while Japan and Korea are top-performing educational systems, their students exhibited some of the lowest scores for mathematics self-efficacy (see also Han et al., 2015). Chiu and Klassen (2010) attributed this phenomenon to the collectivist culture of East Asia, in which family members and the community assume some of the responsibility for students' success. To a certain extent, students are less concerned about their self-efficacy and self-concept and more about their mastery of skills and academic achievement. Some researchers pointed out that students who overestimate their self-efficacy tend to show less effort and poor performance, while students who underestimate their self-efficacy are more likely to show more effort and better performance (Chen, 2003; Chen & Zimmerman, 2007). Chiu and Klassen (2010) suggested that high-achieving East Asian students may have underestimated their self-efficacy, and low-achieving students abroad may have overestimated their self-efficacy, leading to the result that East Asian students exhibit low self-efficacy.

The trend of lower self-efficacy in East Asia does not appear in Mainland China. In fact, PISA 2012 reports that the mathematics self-efficacy of Shanghai students is above the OECD average, and their achievement is the highest among all the participating systems (OECD, 2014). Wu (2016) argued that the high self-efficacy found with Shanghai students may be attributed to the specific educational system of Shanghai and other demographic factors. Ma (1999) related this difference to different understandings of the measurement items. In particular, Ma commented that Shanghai students may have treated the self-efficacy items more as mathematics problems than psychological traits. Another possible explanation suggested by Wu (2016) is that Shanghai is the most developed region in China which may benefit students to get higher self-efficacy. More in-depth investigations in Shanghai and other cities in China are needed.

15.3 Research Methods

15.3.1 Participants

A total of 1359 eighth graders from five Chinese cities were invited to work on three mathematical modelling tasks with different levels of difficulty. Each task was accompanied by one self-efficacy item and one self-evaluation item. The students were selected using a stratified sampling method. They were enrolled in 15 schools in five cities, which were located in the east (309), south (234), middle (363), southwest (186) and northwest (267) of China. For the majority of the schools, two classes of students participated in the study, and the average class size was 48 students. The gender distribution is similar across the five cities, and the proportion of boys ranges from 45.3% (northwest) to 57.8% (south). Interestingly, a high percentage of students in cities from the east (81.8%) and northwest (78.9%) come from a one-child family, while the percentages of such students from the other three cities are low (middle: 59.6%, southwest: 58.6%, south: 45.7%).

15.3.2 Measures of Self-Efficacy and Self-Evaluation During Mathematical Modelling

Mathematical modelling performance The research team designed three mathematical modelling tasks. All the tasks are rooted in the context of real-life situations and cover different mathematical content (i.e. *numbers and algebra* and *space and graphs*). Moreover, the three tasks have different levels of difficulty: easy (i.e. a learnt model), moderate (i.e. a modified learnt model) and difficult (i.e. an unfamiliar model). More details can be found in Chap. 8.

Mathematical self-efficacy Students' mathematics self-efficacy was measured based on students' ratings of their confidence in solving each modelling task individually. The self-efficacy item was 'I believe I can solve this task', and it was rated on a four-point Likert scale ranging from 1 (totally agree) to 4 (totally disagree).

Mathematical self-evaluation Students' self-evaluative judgement was measured after they attempt to solve each mathematical modelling task. The self-evaluation item was 'I believe I solved this task', and it was rated on a four-point Likert scale ranging from 1 (totally agree) to 4 (totally disagree).

Gender and one-child status All the students were asked to provide their demographic information, such as gender and one-child family status. In the data analysis, boys were coded as 1 and girls were coded as 0. Additionally, students from one-child families were coded as 1 and those from multi-children families were coded as 0.

15.3.3 Data Processing and Analysis

For each modelling task, students' performance was first evaluated based on a six-point rating scale, which is mainly aligned with Blum and Kaiser's (1984) modelling process-oriented cycle. A descriptive data analysis was conducted to summarise students' modelling performance and related self-efficacy and self-evaluation. A one-way repeated-measures ANOVA was then used to examine whether students' performance and two self-related beliefs about their performance differ across the three tasks. When overall significant differences were detected, planned contrasts were carried out. Third, a two-way repeated ANOVA was conducted to compare students' self-efficacy and self-evaluation across different modelling tasks. After this, a series of one-way repeated measures ANCOVA were performed to replicate the earlier comparison with a control representing students' actual performance on the respective modelling tasks.

A two-way mixed ANOVA was used to examine students' modelling performance, self-efficacy and self-evaluation across the three tasks in relation to students' demographic characteristics (i.e. gender and one-child status). When students with different demographic characteristics had significantly different self-judgements, ANCOVA was performed to explore the influence of the demographic characteristics on students' self-judgements when their modelling performance is taken into account.

After that, a three-way mixed ANOVA was performed using students' self-judgements at different time points related to different modelling tasks as the within-subjects factors and their demographic characteristics as the between-subjects factors. This was followed by a three-way mixed ANCOVA using students' modelling performance as the covariance. Separate analyses were conducted for gender and one-child status. When significant differences were detected, effect sizes were calculated.[2]

Given the hierarchical nature of the data (i.e. students nested within schools and schools nested within cities), a random effects ANOVA model, also known as the fully unconditional model, was used to examine the heterogeneity of students' self-efficacy and self-evaluation across individuals, schools and cities. Such an analysis is generally used to partition subjects' variances in measures into three components: among students at level 1, among schools at level 2 and among cities at level 3 (Raudenbush & Bryk, 2002). Further, students' gender (girls = 1 vs. boys = 0), one-child status (one-child = 1 vs. multiple-children = 0) and actual modelling performance were used as predictors in the analysis with an intercepts- and slopes-as-outcomes model.

15.4 Results

15.4.1 Students' Overall Self-Efficacy and Self-Evaluation

Table 15.1 presents the means and standard deviations of students' modelling performance as well as their corresponding self-efficacy and self-evaluation for each modelling task. The results of the one-way repeated measures ANOVA show that students have significantly different performance across the three modelling tasks (F [1.883, 2553.477] = 777.336, $p < 0.001$, $\eta_p^2 = 0.364$). Planned contrasts reveal that students' performance on Task 1 is considerably better than that on Task 2 ($p = 0.060$) and significantly better than that on Task 3 ($p < 0.001$, $\eta_p^2 = 0.544$). It is understandable that students showed higher self-efficacy and self-evaluation for Task 1 than the other, more difficult tasks. Students have higher self-efficacy before Task 3 than before Task 2, but they show a similar level of self-evaluation for the two tasks. Students performed significantly better on Task 2 than Task 3 ($p < 0.001$, $\eta_p^2 = 0.435$).

Table 15.1 Means and standard deviations of students' modelling performance, self-efficacy and self-evaluation by task

	Performance	Self-efficacy	Self-evaluation
Task 1 (easy)	3.74 (1.33)	3.12 (0.76)	2.88 (0.91)
Task 2 (moderate)	3.66 (1.57)	2.91 (0.85)	2.71 (0.94)
Task 3 (difficult)	2.37 (0.89)	3.03 (0.79)	2.70 (0.95)

Note. As the Likert scale for self-efficacy and self-evaluation is designed in descending order, students' responses were reverse-scored before all the analyses in this study for easier interpretation

[2]The rules of thumb for magnitudes of effect sizes can be seen at http://imaging.mrc-cbu.cam.ac.uk/statswiki/FAQ/effectSize

The results of the one-way repeated measures ANOVA show that which task students read has a significant effect on their self-efficacy (F [1.972, 2596.920] = 54.240, $p < 0.001$, $\eta_p^2 = 0.072$). Planned contrasts reveal that students' self-efficacy is significantly higher for Task 1 than the other two tasks at the 0.001 level. The difference in students' self-efficacy for Task 2 and Task 3 also reached significance ($p < 0.001$). A smaller difference is observed for students' self-evaluation across the three tasks, although it still reaches significance (F [1.976, 2550.506] = 39.408, $p < 0.001$, $\eta_p^2 = 0.030$). In particular, the significant differences between Task 1 and the other two tasks exist at the 0.001 level.

The two-way repeated measures ANOVA compared students' self-efficacy and self-evaluation across the three modelling tasks. The results show that students display significantly different self-judgements before and after they actually perform the tasks (F [1, 1260] = 204.113, $p < 0.001$, $\eta_p^2 = 0.139$). Such differences demonstrate significantly different, though trivial in terms of effect size, patterns across the tasks (F [2, 2520] = 9.487, $p < 0.001$, $\eta_p^2 = 0.007$). An inconsistent pattern appears for Task 3; students have the largest drop from self-efficacy to self-evaluation for this task ($\Delta = 0.33$).

Correlation analysis confirms an important relation between students' self-related beliefs and modelling performance. The correlation between students' self-evaluation and their actual performance is higher than that between self-efficacy and actual performance. It will be interesting to compare students' self-judgements when controlling for their modelling performance. Thus, a one-way repeated measures ANCOVA was conducted using students' performance on the modelling tasks as a covariance. The results show that the difference between self-efficacy and self-evaluation for Task 1 and Task 3 becomes smaller in terms of their effect sizes (η_p^2; Task 1: 0.088 to 0.044; Task 3: 0.120 to 0.049), but it nearly maintains for Task 2 (0.055 to 0.068). This verifies the important influence of students' cognitive ability to perform modelling tasks on their self-judgements.

15.4.2 Demographic-Related Differences in Students' Self-Efficacy and Self-Evaluation

Not only do students' actual abilities have an important influence on their self-efficacy and self-evaluation but also their demographic characteristics may have an effect on their beliefs. This study looks into two such characteristics: gender and one-child status. The results of the two-way mixed ANOVA reveal that boys consistently performed significantly better than girls on all three modelling tasks, although the magnitude of the significant difference is trivial ($p = 0.013$, $\eta_p^2 = 0.005$). A significant difference is also revealed between students with different one-child status ($p = 0.001$, $\eta_p^2 = 0.008$) in favour of students from one-child families.

On the two self-related beliefs, boys consistently reported significantly higher levels than girls (self-efficacy: $p < 0.001$, $\eta_p^2 = 0.017$; self-evaluation: $p < 0.001$,

$\eta_p^2 = 0.028$). Further analyses with a control for students' actual performance on the modelling tasks again reveal that boys had a significantly higher level of self-efficacy than girls, although the magnitude of this difference is small ($\eta_p^2 \approx 0.01$). The gender differences in self-evaluation after controlling for modelling performance are larger than the differences in self-efficacy, with the largest difference observed for Task 3 ($\eta_p^2 = 0.022$), followed by Task 1 ($\eta_p^2 = 0.018$) and Task 2 ($\eta_p^2 = 0.014$). Although the self-efficacy and self-evaluation of students from one-child families are consistently higher than those from multiple-children families, both differences are insignificant (self-efficacy: $p = 0.121$; self-evaluation: $p = 0.133$).

The three-way mixed ANOVA, which used students' self-judgements at different time points for different modelling tasks as the within-subjects factors and students' gender as the between-subjects factor, reveals significant main effects for tasks (F [1.987, 2495.192] = 62.857, $p < 0.001$, $\eta_p^2 = 0.048$), self-judgements at different time points (F [1, 1256] = 207.369, $p < 0.001$, $\eta_p^2 = 0.142$) and gender (F [1, 1256] = 35.163, $p < 0.001$, $\eta_p^2 = 0.027$). The interactions between the three main effects are either insignificant or trivially significant ($\eta_p^2 \approx 0.01$). In fact, the differences between self-efficacy and self-evaluation for all three tasks are larger among girls than boys. Moreover, the magnitudes of the differences are nearly the same across the three tasks, and all the differences are significant at the 0.001 level (Task 1: $\eta_p^2 = 0.022$; Task 2: $\eta_p^2 = 0.019$; Task 3: $\eta_p^2 = 0.023$). The gender-related differences become slightly smaller after controlling for students' performance on the three modelling tasks (Task 1: $\eta_p^2 = 0.019$; Task 2: $\eta_p^2 = 0.014$; Task 3: $\eta_p^2 = 0.021$).

Similar analyses were conducted to examine whether students with different one-child status have different self-judgements before and after they perform the modelling tasks. While the main effects of task and self-judgement at different time points remained significant, the main effect of one-child status is insignificant ($p = 0.094$). Moreover, all the interactions are insignificant, which suggests that students' self-efficacy, self-evaluation and the relationship among these factors are similar for students from both types of families. The findings are consistent even after controlling for students' actual modelling performance.

15.4.3 Variances of Students' Self-Efficacy and Self-Evaluation

Given the hierarchical nature of the data (i.e. students nested within schools and schools nested within cities), a hierarchical linear model was employed to examine the differences among students in terms of self-efficacy and self-evaluation at the student, school and city levels. Further, students' gender (girls = 1 vs. boys = 0), one-child status (one-child = 1 vs. multiple-children = 0) and actual modelling performance were used as predictors in the analysis.

Table 15.2 Sources of variance in students' self-efficacy and self-evaluation by modelling task

% of variances		Student level	School level	City level
Task 1	Self-efficacy	95.1***	2.2**	2.7**
	Self-evaluation	92.3***	7.7***	0.0
Task 2	Self-efficacy	94.5***	3.9***	1.6*
	Self-evaluation	90.9***	9.1***	0.0
Task 3	Self-efficacy	95.9***	2.1***	2.0*
	Self-evaluation	95.4***	4.6***	0.0

Note. *$p < 0.05$, **$p < 0.01$, ***$p < 0.001$

Table 15.2 displays the variances in self-efficacy and self-evaluation related to the modelling tasks at the student, school and city levels. More than 90% of the differences occur at the student level, and the proportions of variance in self-efficacy are consistently larger than those for self-evaluation. The school-level variances in the two types of self-judgement are much smaller than those at the student level. However, it is interesting to observe that the proportions of school-level variance in self-evaluation for all the three tasks are larger than those for self-efficacy. Such differences become smaller for the more challenging tasks. Some significant variances are found at the city level, but they are only related to self-efficacy. In other words, students from different cities do not have great differences in their performance judgements after they actually work on the modelling tasks.

An intercepts- and slopes-as-outcomes model was used to explore how students' individual characteristics (i.e. gender, one-child status and modelling performance) at the student level and school mean performance on modelling tasks at the school level contribute to the variances in students' self-efficacy and self-evaluation. Table 15.3 presents the detailed results.

Consistent with the results reported earlier, students' one-child status does not show a significant influence on their levels of self-efficacy and self-evaluation, while being a boy and having better modelling performance have significantly positive impacts. Moreover, school mean modelling performance significantly positively contributes to students' high levels of self-efficacy and self-evaluation. The magnitudes of the influences of school mean performances are generally greater than those of students' individual performance. That indicates that students' self-related beliefs related to mathematics modelling increase more within a school with higher average modelling performance and that this increase is greater than the increase caused by students' own high performance.

15.5 Summary and Conclusions

According to Bandura (1977), the way students think and feel about themselves shapes their behaviour, especially when they face challenging scenarios. In the case of mathematics learning, students' self-related beliefs could determine how well they motivate themselves and persevere in the face of challenging tasks, influence

Table 15.3 Multilevel model with student-level and school-level factors affecting students' self-efficacy and self-evaluation during mathematical modelling

	Task 1		Task 2		Task 3	
	Self-efficacy	Self-evaluation	Self-efficacy	Self-evaluation	Self-efficacy	Self-evaluation
Fixed						
Intercept	3.183***	2.946***	2.930***	2.825***	3.074***	2.807***
Student level						
Girl	−0.166***	−0.212***	−0.131**	−0.200***	−0.154***	−0.260***
One-child status	−0.029	0.00570	0.0285	−0.0380	0.00210	−0.0139
Performance	0.146***	0.195***	0.126***	0.214***	0.197***	0.313***
School level						
Performance_S	0.125***	0.360***	0.168***	0.345***	0.236**	0.364***
Random (Variance, SD)						
Student level	0.505 (0.711)	0.691 (0.831)	0.645 (0.803)	0.710 (0.842)	0.566 (0.752)	0.789 (0.888)
School level	0.00374 (0.0612)	0.0133 (0.115)	0.0116 (0.108)	0.0106 (0.103)	0.00280 (0.0529)	0.00852 (0.0923)
City level	0.0150 (0.122)	0.00007 (0.00812)	0.00621 (0.0788)	0.00134 (0.0367)	0.00785 (0.0886)	0.00013 (0.0115)

Note. $*p < 0.05$, $**p < 0.01$, $***p < 0.001$

their emotional lives and affect their coursework choices and further educational and career paths (Bandura, 1997; Wigfield & Eccles, 2000; see also OCED, 2013). In the two mathematics-focused PISA studies (i.e. PISA 2003 and PISA 2012), students' mathematics self-related beliefs are measured in terms of self-efficacy and self-concept. It is found that both types of self-related beliefs are generally positively correlated with mathematics performance and that the association with performance is stronger for self-efficacy than for self-concept. However, some East Asian systems (e.g. Japan and Korea) exhibit a paradox; that is, their students performed well in mathematics but had a low level of belief in their own mathematics abilities, both in general (self-concept) and specifically (self-efficacy). Researchers have attempted to interpret this observation as linked to the collectivist culture of East Asia. While Shanghai students in PISA 2012 also report a low level of mathematics self-concept, their self-efficacy is nearly one standard deviation higher than the OECD average. Ma (1999) associates Shanghai students' high self-efficacy with their different understandings of the measurement items, suggesting that Shanghai students may have treated the self-efficacy items more as mathematics problems than as psychological traits and that the problems displayed in the PISA measures are relatively easier for Shanghai 15-year-olds.

Given these inconclusive findings and interpretations, this study is designed to examine students' self-related beliefs about their mathematical modelling, a more challenging type of mathematical task. As PISA only measures students' self-efficacy, it is unclear whether students can actually resolve related tasks and how they will evaluate their actual performance afterwards. Using a stratified sampling method, a total of 1359 eighth graders were selected from five Chinese cities in a variety of locations. All the students were given three modelling tasks of different difficulty levels. They were asked to pre-judge their future performance, solve the tasks and then post-judge about their performance.

It was found that students' level of self-efficacy is consistently higher than their self-evaluation for the modelling tasks. The lowest level is observed on self-evaluation for the most challenging task (i.e., Task 3). For both self-related beliefs, students generally show higher levels for easier modelling tasks. Consistent with the findings of the PISA studies, this study reveals that students' self-related beliefs have a positive correlation with their actual performance and that this association is stronger with self-evaluation. This is verified by the shrinking gap between self-efficacy and self-evaluation when students' actual modelling performance is taken into account.

Two demographic characteristics are investigated in this study to determine their impact on students' self-judgements of their mathematical modelling performance. It is revealed that boys not only perform significantly better than girls but also have a significantly higher level of self-related beliefs. Interestingly, the gender gaps are larger for self-evaluation than self-efficacy, regardless of whether one controls for students' modelling performance. Meanwhile, the differences between the two types of self-related beliefs on all three tasks are larger for girls than boys. One-child status, the other demographic characteristic, does not show a significant influence on students' self-judgements, though students from one-child families appear to have a slightly higher level of both self-related beliefs.

Further hierarchical analysis revealed that the between-city differences in the two self-related beliefs account for no more than 3% of the total variance, while between-school variances in self-evaluations are much larger than those for self-efficacy. Meanwhile, the variances in differences still mainly come from individual students. A full model analysis again shows that students' gender and actual modelling performance have an important impact on their self-efficacy and self-evaluation. This impact appears greater for self-evaluation than for self-efficacy. Moreover, the school's average modelling performance generally has a larger influence on students' self-related beliefs than individual students' performance.

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Chapter 16
Math Anxiety in the Context of Solving Mathematical Modeling Tasks in China

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Abstract In this study, cluster analysis and hierarchical regression were used to analyze the relationship between math anxiety (MA) and mathematical modeling ability in a sample of 1359 eighth-grade students from five places in China. Our results showed students tended to worry more about having difficulty in mathematics class (58%) than reported by the Shanghai PISA 2012 (53%); our finding is closer to the PISA 2012 global average (59%). Our analysis also revealed students were less nervous when solving mathematics problems (20%), felt less helpless when faced with a mathematical problem (14%), and were less worried about getting poor math grades (32%) than the PISA 2012 Shanghai average (27%, 28%, and 71%, respectively) and global average (31%, 30%, and 61%, respectively). Female students reported higher MA than male students; cluster analysis showed substantial gaps in mathematical modeling between high and low MA students. After controlling for family SES, gender, and task difficulty, MA explained an additional 3.5% of the variance of mathematical modeling ability, with task difficulty accounting for 19% of the variance. The relationship between MA and mathematical modeling depended on the difficulty of the tasks. The applications of the results are further discussed in the study.

Keywords Math anxiety · Mathematical modeling · Math performance · Hierarchical regression · Cluster analysis · PISA · Shanghai · China · Gender difference · Difficulty of mathematical modeling

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16.1 Introduction

Math anxiety (MA) is commonly defined as "a feeling of tension, apprehension, or fear that interferes with math performance" (Ashcraft, 2002). It undermines an individual's math performance by influencing their learning motivation, confidence, and cognitive mechanisms (Eysenck, Derakshan, Santos, & Calvo, 2007; Pekrun, Goetz, Frenzel, Barchfeld, & Perry, 2011). It has also been shown to have detrimental effects on students' mental health, learning behaviors, and self-regulation (Diener, 2000; Pekrun, Goetz, Titz, & Perry, 2002; Zeidner, 1998). MA can also lead students to avoid math-related contexts, for example, they intentionally do not select math-related courses or math-related careers (Hembree, 1990).

MA is common in both eastern and western countries, and it affects a large proportion of people. Ashcraft and Kirk (2001) found that about one-fifth of the US population suffers from some degree of MA when confronting a task related to the manipulation of numerical information. In China, 87.05% of 224 secondary students in rural areas have high or moderate MA (Han, Zhang, & Zhang, 2016). Among 315 Chinese pre-service math teachers, 5% and 50% suffered from high and moderate MA, respectively (Wang, 2009). The evidence described above suggests that understanding and resolving MA in China calls for the attention of researchers.

Since joining the Programme for International Student Assessment (PISA), which is used in the evaluation of education systems worldwide, Shanghai has twice achieved the best math scores compared to all PISA countries and regions, in 2009 and 2012. Due to these results, China's math education, especially in Shanghai, has attracted attention from all over the world. In particular, Shanghai students achieved the top math scores, and their MA ranked in the middle, at 28th among 68 countries (ranked from the lowest rate of MA to the highest). Finland, previously the reigning champion for the highest math achievement scores, ranked 12th in 2009. For MA, the Finnish students ranked fifth (ranked from lowest to highest) among all of the participating countries and regions. Shanghai students' MA seemed higher than expected, which suggested further examination of MA was required in China.

16.1.1 Gender Difference in MA

Gender differences have been reported in most of the literature regarding MA; however, the results reported by various studies are mixed. Many studies showed that female students reported higher MA than male students (e.g., Ashcraft & Faust, 1994; Hembree, 1990; Wigfield & Meece, 1988). Ashcraft and Faust (1994) found that female college students scored significantly higher for MA than male students. After analyzing data from two major international surveys (TIMSS 2003 and PISA 2003), Else-Quest, Hyde, and Linn (2010) showed that, among 13- to 15-year-old students, the male students reported a more positive attitude toward math than the females despite having similar levels of mathematics achievement. After reviewing 70 studies which included 126 samples, Hyde, Fennema, Ryan, Frost, and Hopp

(1990) also concluded that female students hold more negative attitudes toward math than male students. However, a few studies have shown that there is no gender difference in MA. For example, Meece, Wigfield, and Eccles (1990) examined gender differences in a sample of 250 7th-, 8th-, and 9th-grade students and found no gender difference in MA. Birgin, Baloğlu, Çatlıoğlu, and Gürbüz (2010) also showed no gender difference for 6th- to 8th-grade students, and a longitudinal study using LSAY data (Ma & Xu, 2004) also failed to indicate gender differences in MA for 7th- to 12th-graders. However, there have been a few studies wherein male students reported higher MA than female students (Abed & Alkhateeb, 2001; Reavis, 1989; Sandman, 1979). These examples illustrate that reports of gender differences in MA are inconsistent in the literature.

16.1.2 Relations Between MA and Math Performance

MA had been consistently demonstrated to be related to poor math performance (Hembree, 1990; Ma, 1999; Namkung, Peng, & Lin, 2019; Ramirez, Gunderson, Levine, & Beilock, 2013; Vukovic, Kieffer, Bailey, & Harari, 2013; Wigfield & Meece, 1988). Cognitive interference theory, as related to math performance, suggests that MA undermines three aspects of students' math performance: preprocessing, processing, and retrieval of information (Carey, Hill, Devine, & Szücs, 2016; Deutsch & Tobias, 1980; Namkung et al., 2019; Tobias, 1986). Students with MA avoid participating in math-related tasks; consequently, they have fewer math learning opportunities. Ashcraft (2002) demonstrated that while working on math tasks, students with MA are interrupted by intrusive thoughts and worries. These students have to spare part of their working memory capacity to conquer those worries and intrusive thoughts. Working memory, which is responsible for storing and processing information when solving math problems, is a limited-capacity system. Therefore, when overcoming intrusive thoughts created by MA, a student has less available working memory for solving the math problems, leading to longer response times and higher error rates. Thereby, students with MA often perform poorly in math.

The relationship between MA and math performance might depend on the difficulty or complexity of a math task. For example, in contrast to a simple arithmetic problem, Ching (2017) found complex math problems elicited higher MA among 2nd- and 3rd-grade students with high working memory capacity. Beilock and Willingham (2014) found a strong link between MA and mathematical problem solving (complex math problems) while Harari, Vukovic, and Bailey (2013) demonstrate a weak link between MA and digital calculation (simple math problems). Namkung et al. (2019) analyzed 131 studies with 478 effect sizes using meta-analysis. They compared the relationship between MA and foundational math skills (e.g., number sense and computation) to the relationship between MA and advanced mathematics domains (e.g., algebra, measurement, geometry). They found a stronger link ($r = 0.35$) between MA and advanced math domains than between MA and foundational domains ($r = -0.20$). Zhang, Zhao, and Kong (2019) also compared computation to problem solving, finding a stronger link between MA and problem solving ($r = -0.33$) than between MA and computation ($r = -0.21$). As complex math problems require more cognitive processing and information retrieval than simple math problems, they require more working memory. Thus, MA might influence difficult math tasks more than simple tasks. However, scant research is available comparing the relationship between MA and levels of math performance in the same domain.

Furthermore, the relationship between MA and math performance might differ depending on the type of math performance or, as Harari et al. (2013) suggest, on the measurement of math performance. Vukovic et al. (2013) found that MA uniquely explained the variance of children's calculation skills and math applications but did not explain variances in children's geometric reasoning. In a meta-analysis of 84 studies from 2000 to 2019, Zhang et al. (2019) found that the relationship between MA and math performance varies across different types of math performance. Zhang's findings were consistent with those of Namkung et al. (2019) which also indicate that differences in MA occur depending on the type of math task. Therefore, MA might have a stronger impact on some types of math performance than others.

However, few, if any, studies have examined the relationship between MA and mathematical modeling. According to the meta-analysis mentioned above, 47% of the 84 studies assessed general skills of math performance, 35% assessed computation, and 8% assessed problem solving (Zhang et al., 2019). None of the included studies examined MA in the context of solving mathematical modeling tasks. Mathematical modeling can consider as a creative mathematical activity. It employs abstract, conceptualize, and math language to simplify a real situation or natural phenomenon into a proposed mathematical model, and uses mathematical methods to solve the proposed model. The results are then applied to interpret the examined phenomenon (Kaiser, 2014). More simply, it is a process for converting a real-world problem into a math problem and then using the math results to interpret the real-world condition. Thus, it requires students to use math knowledge to solve a real-world problem. Mathematical modeling is widely used in science, technology, engineering, and mathematics education (STEM). Mathematical modeling is recommended for use in math curricula to provide students a real-world situation that connects personal experience and math knowledge; this connection to real life improves student's interest in math and increases their math-learning motivation (Mogens, 2012). To maximize student instruction in the use of mathematical modeling, further research is needed to clarify the relationship between MA and mathematical modeling.

16.1.3 The present study

The present study aimed to examine MA in regard to four aspects of mathematical performance: the MA level of students in China and how it differs from the PISA 2012 results; gender differences in MA; the relationship between MA and math performance; and whether the relationship between MA and mathematical modeling performance is dependent upon the difficulty of mathematical modeling tasks.

16.2 Methods

16.2.1 Sample

We collected data from five places in China that represent the country's southwest, southeast, east, northeast, and midland regions. They also represent differences in economic status in China. The total participant sample included 1359 eighth-grade students in 33 classes from 14 schools. The students' mean age was 14 years. Participants included 46.7% females, 53.3% males, and 0.4% who did not indicate a gender; 65.7% of the students were the only child in the family, 34.0% were not from single-child families, and 0.3% of students did not provide this information. The rate of missing data ranged from 0.1% to 4.9%.

16.2.2 Procedures and Instrument

Three modeling tasks and questionnaires were administered by trained math teachers. First, they assigned modeling tasks to the students. Next, MA questionnaire was assigned after students finished solving the modeling tasks.

Students' MA was measured using items on a four-point scale. The items included, "I often worry that it will be difficult for me in mathematics classes," "I get very nervous doing mathematics problems," "I feel helpless when doing a mathematics problem," and "I worry that I will get poor grades in mathematics." These items were the same as those included in the PISA (OECD, 2015). However, we did not include the PISA item "feeling worrisome about doing homework," as it was not relevant. The Cronbach's alpha, based on standardized items, was 0.850.

Family demographic indicators included family economic status (SES) and gender; family SES was measured by parents' highest education level. The correlation between mothers' and fathers' highest level of education level was 0.70 ($p < 0.001$). Gender information was also collected.

Mathematical modeling ability was measured by three modeling tasks (Q_1, Q_2, and Q_3, for details, please refer to Chap. 12, Table 12.4). Q_1 was about the "Lanzhou noodle problem." The question read, "Lanzhou noodles are a famous and well-known kind of noodle that originated in the northwest of China. To make the noodles, the chef needs to knead the paste into a long strip, stretch it, fold it and stretch, and fold and stretch it again and again, seven or eight times in total. Then the noodles become thin and long. Please estimate how long the noodles would be if the paste were folded and stretched four times." Q_1 was expected to be the easiest question in that students could easily find a mathematical model learned in class to solve the problem. Q_2 was the "big shoes" problem for which students were asked to estimate the size of big shoes. Q_2 was designed to be more difficult than Q_1, as students were expected to indicate, “It is not hard to find a similar problem strategy learned in class, but I need to modify it to answer this question.” Q_3 was the “gas station problem,” which read, “Gas at a nearby station is more expensive than it is at a station further away. How do you decide whether it is worth it to drive far away to buy gas, according to conditions provided?” It was the most difficult one among the three tasks because students may not be familiar with the problem and there is no ready-made mode or strategy.

A coding scheme was developed to evaluate students’ mathematical modeling tasks (for details, please refer to Chap. 12, Table 12.5), and six stages were defined. For example, 0 was used when no part of an answer was correct or when the student left it blank; 1 stands for “the student tried to structuralize the real situation but was not able to find an appropriate mathematical strategy”; 2 stands for “the student proposed a reasonable hypothesis and figured out a mathematical strategy, but did not use the proper method.” A code score of 3 stands for “the student could find the realistic strategy and transfer it to this mathematical problem, but they were not able to reach an accurate mathematical solution or it was incorrectly solved”; 4 stands for “the student proposed the proper mathematical strategy and got right solution, but their interpretation of the solution was not appropriate to a real situation”; and 5 stands for “the student found the realistic model, transferred it to the mathematical problem and solved it, interpreted and verified the model according to a real situation, and assessed the rationale of the model.”

16.2.3 Data Analysis

We first described the MA level using frequencies, means, and standard deviations. Gender differences and task difficulty were examined using t-test, ANOVA, and MANOVA. Effect sizes were also calculated. Second, two-step cluster analysis, which was primarily designed for analyzing large datasets, was used to cluster the students based on the four items of mathematics anxiety, identifying different groups of students by MA. ANOVA paired with effect size was used to compare the differences in math achievement in each category with levels of MA. Third, we used hierarchical multiple regression to analyze the relations between difficulty and MA, their interaction effects, and mathematical modeling abilities. All analyses were performed in SPSS 23.0.

16.3 Results

Mean scores and standard deviations of the three questions are presented in Table 16.1. Their mean scores were compared using ANOVA. Results showed significant differences in mean scores across the three questions, $F(4070, 2) = 480.00$, $p < 0.001$. Post hoc tests showed that the mean score for Q_1 had no significant

Table 16.1 Mean, standard deviations and correlations of the variables

		Means	SD	Min	Max	N	Q_1	Q_2	Q_3	MA	Gender
1	Q_1	3.74	1.334	1	5	1358	1.00				
2	Q_2	3.66	1.568	1	6	1358	0.42***	1.00			
3	Q_3	2.37	0.888	1	5	1357	0.41***	0.38***	1.00		
2	MA	2.09	0.713	1	4	1350	−0.25***	−0.21***	−0.27***	1.00	
3	Gender	1.53	0.499	1	2	1354	0.04	0.07*	0.04	−0.15***	1.00
4	SES	4.32	1.284	1	6	1341	0.12***	0.13***	0.28***	−0.14***	−0.02

Note. ***, $p < 0.001$; **, $p < 0.01$; *, $p < 0.05$

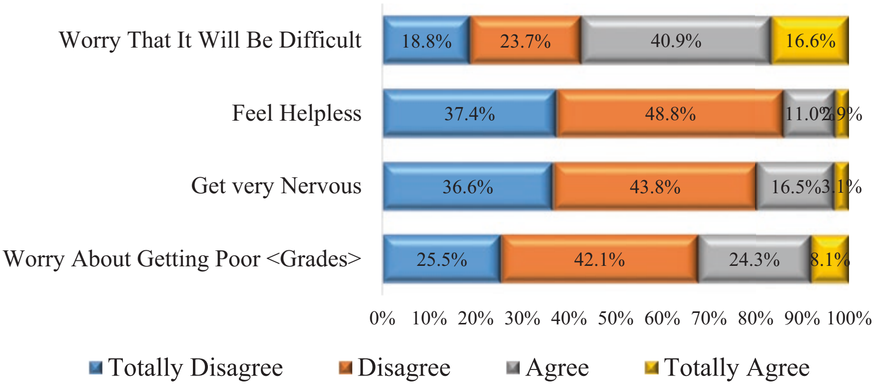

Fig. 16.1 The percentage of degree of anxiety in each item

difference with that of Q_2 (mean difference (MD) = 0.80, $p = 0.239$). Q_3 had significant lower mean score than Q_1 ($MD_{Q1-Q3} = 1.37$, $p < 0.001$) and Q_2 ($MD_{Q2-Q3} = 1.29$, $p < 0.001$). Mean scores' differences suggested that Q_1 and Q_2 might present the same difficulty level for students; Q_3 was the one they found the most difficult.

16.3.1 MA in the Context of Solving Modeling Tasks

Figure 16.1 presents students' MA in the context of solving modeling tasks. The majority of the students (57.5%) indicated that they often worried that mathematics class would be difficult for them, and one-third (32.4%) worried that they would get poor grades in mathematics. About one-fifth of the students (19.6%) get very nervous while some (13.9%) feel helpless when solving mathematics problems. Results showed that difficulty in mathematics class is what students worry about most.

Means and standard deviations of mathematical modeling questions (Q_1–Q_3), MA, gender, and family SES; their correlation coefficients are presented in Table 16.1. MA was negatively correlated with mathematical modeling abilities (r ranged from −0.21 to −0.27, $p < 0.001$). Male students had slightly higher performance in mathematical modeling abilities than female students ($r = 0.06$, $p < 0.05$). Female students reported higher MA than male students on average ($r = 0.15$, $p < 0.001$). SES had positive correlation with mathematical modeling abilities ($r = 0.21$, $p < 0.001$) and it had negatively correlation with MA ($r = -0.14$, $p < 0.001$).

The correlation matrix is presented on the right-hand side of the table.

16.3.2 Gender Difference in MA

We examined gender differences relating to MA. Female students showed higher MA in general than male students, F(1317, 4) = 9.32, $p < 0.001$; partial $\eta^2 = 0.028$ (Table 16.1). Female students showed substantially higher MA in each indicator

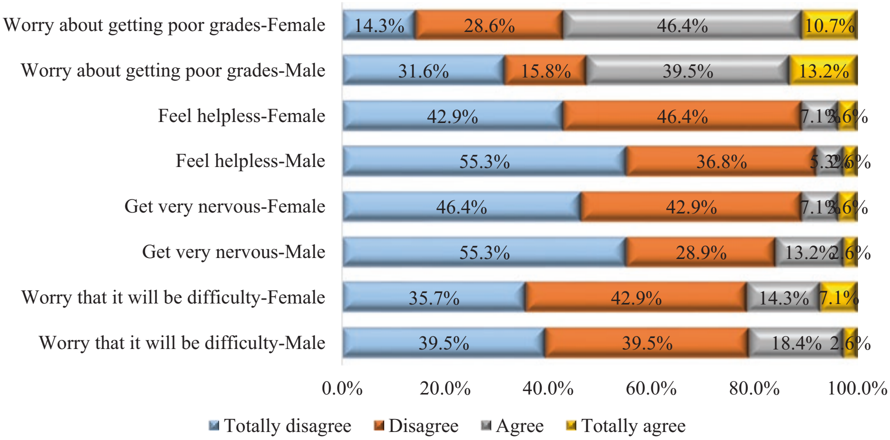

Fig. 16.2 The percentage of male's and female's MA in each indicator

than male students, the effect sizes were ranging from 0.21 to 0.32. Among the four indicators, female showed much higher anxiety than male students in "worrying that it will be difficult for me in mathematics classes" (57% vs 53%), the effect size was 0.32. The degree of MA for each indicator was presented in Fig. 16.2.

We also examined gender differences in mathematical modeling abilities using MANOVA. Results showed that there was no significant gender difference in mathematical modeling abilities, F(1348, 3) = 2.37, $p = 0.069$; partial $\eta^2 = 0.005$. Effect sizes of the three questions (Q_1 to Q_3) were 0.07 to 0.14.

16.3.3 Mathematical Modeling Abilities Gap Between High and Low MA

Cluster analysis resulted in two clusters: high MA (36%) and low MA (64%). The four indicators showed substantial differences between the high and low MA groups. Effect sizes ranged from 1.31 to 2.39. The profiles of the four indicators between low and high MA are presented in Figs. 16.3 and 16.4.

MANOVA was used to examine the differences in three mathematical modeling questions between the low and high MA groups, F(3,1320) = 33.19, $p < 0.001$; partial $\eta^2 = 0.07$. The effect sizes between high and low MA for Q_1, Q_2, and Q_3 were 0.46, 0.34, and 0.48, respectively. Results seemed to reveal that the gap between high and low MA was related to the difficulty of the mathematical modeling tasks. That is, the more difficult the mathematical modeling question, the larger the gap between the high and low MA groups. Further examination was conducted with multiple regression.

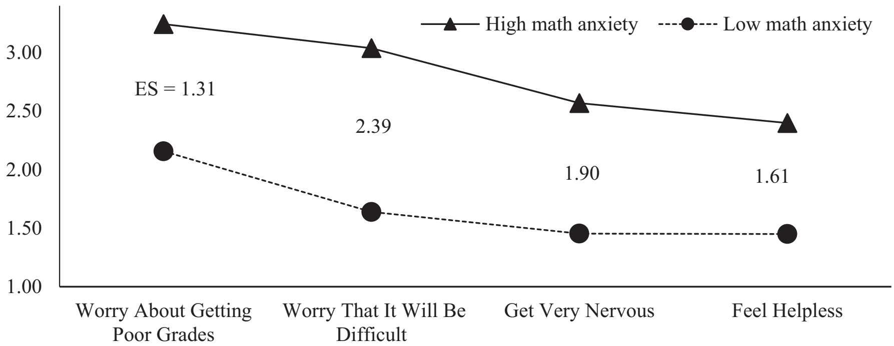

Fig. 16.3 Means of five indicators of MA in high and low MA.
Note: Effect size for each indicator is presented between the two lines

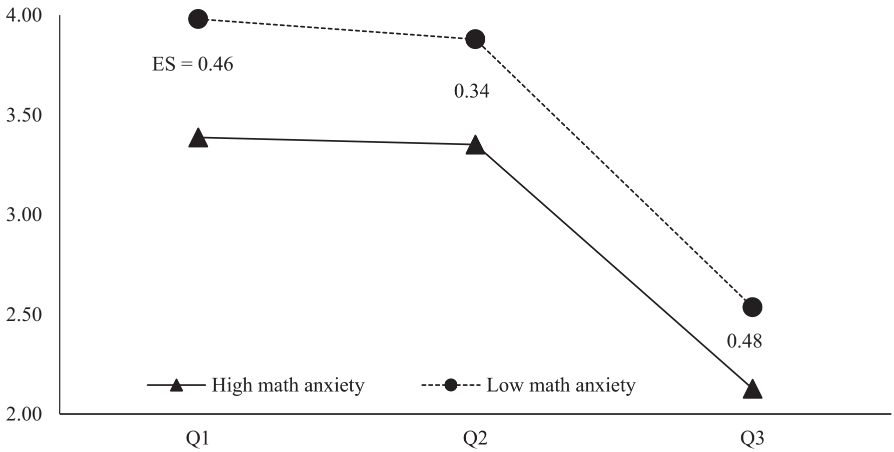

Fig. 16.4 *Differences in three mathemati*cal modeling tasks between high and low MA.
Note: Effect size for each indicator is presented between the two lines

16.3.4 Linking MA and Difficulty to Mathematical Modeling Abilities

We applied four-stage hierarchical multiple regression with mathematical modeling abilities as the dependent variable. Gender and parental highest education level were entered at stage one of regression, labeled as $M_{0,}$ MA (M_1), to control for demographic characteristics. The difficulty of mathematical modeling (M_2) and interaction effects (M_3) were entered successively at stage two, stage three, and stage four, labeled as M_1, $M_{2,}$ and M_3, respectively.

Hierarchical multiple regression revealed that at stage one, parents' highest education level and gender contributed significantly to the regression model, F(2, 3983) = 44.06, $p < 0.001$, and accounted for 2.2% of the variance in mathematical modeling abilities. MA explained an additional 3.5% of the variance in mathematical modeling abilities; the change in R^2 was significant, $F(1, 3982) = 146.76$, $p < 0.001$. The difficulty of mathematical modeling questions explained an additional 19.2% of the variance in mathematical modeling abilities, and the change in R^2 was significant, $F(1, 3981) = 1016.79$, $p < 0.001$. The interaction between the difficulty of mathematical modeling questions and MA explains the extra 0.1% of the variance in mathematical modeling abilities; the change in R^2 was significant, $F(1, 3980) = 4.37$, $p = 0.037$.

Table 16.2 presents the unstandardized (B) and standardized (β) regression coefficients for stage one through stage four. Students with a parent having a higher education level obtained higher mathematical modeling abilities scores ($\beta = 0.14$, $p < 0.001$). Male students had slightly higher mathematical modeling abilities score than female students ($\beta = 0.05$, $p < 0.01$). MA was negatively related to students' mathematical modeling abilities score ($\beta = -0.19$, $p < 0.001$) after controlling for parents' highest education levels and gender. The more difficult the mathematical modeling question, the lower the mathematical modeling scores ($\beta = -0.44$, $p < 0.001$) after controlling for MA, parents' highest education levels, and gender. A significantly positive but trivial interaction effect was found between the difficulty of mathematical modeling and MA ($\beta = 0.04$, $p = 0.037$).

Table 16.2 Mathematical modeling abilities regressed by gender, SES, MA, and question difficulty

	Mathematical modeling abilities							
	M_0		M_1		M_2		M_3	
	B (SE)	β	B (SE)	β	B (SE)	β	B (SE)	β
Constant	2.37***	/	2.62***	/	3.07***	/	3.07***	/
	(0.106)		(0.106)		(0.096)		(0.096)	
SES	0.16***	0.14	0.12***	0.11	0.13***	0.11	0.13***	0.11
	(0.018)		(0.017)		(0.016)		(0.016)	
Male	0.15**	0.05	0.06	0.02	0.06	0.02	0.06	0.02
	(0.045)		(0.045)		(0.040)		(0.040)	
MA			−0.38***	−0.19	−0.38***	−0.19	−0.29***	−0.15
			(0.032)		(0.028)		(0.027)	
Difficulty					−1.34***	−0.44	−1.34***	−0.44
					(0.042)		(0.042)	
Difficulty *Anxiety							0.12*	0.04
							(0.035)	
R^2	2.20%		5.60%		24.80%		24.9%	
ΔR^2			3.50%		19.20%		0.10%	

Note. *$p < 0.05$; *** $p < 0.01$; *** $p < 0.001$

16.4 Discussions

This study examined the relationship between MA and mathematical modeling abilities in eighth-grade students in China. Results showed that female students had substantially higher MA than male students, but no gender difference was found in mathematical modeling ability. Substantial gaps in mathematical modeling abilities were found between high- and low-MA students. After controlling for family SES and gender, MA significantly explained the variance of mathematical modeling abilities. The gap between high and low MA seems to relate to the difficulty of the mathematical modeling questions. Hierarchical multiple regression further demonstrated that the relationship between MA and mathematical modeling was dependent on the difficulty of the modeling tasks.

In the context of solving modeling tasks, students' worrying about the difficulty in mathematics class was consistent with that shown for students in Shanghai and the global average score in PISA 2012. Results showed that 58% of students worried about the difficulty of their mathematics class. This result was a bit higher than seen in the Shanghai PISA 2012 results (53%), but it is close to the results of the average percentage in PISA 2012 (59%) (OECD, 2015).

However, the scores for other items of MA were generally lower than the Shanghai average and the global average level in PISA 2012 (OECD, 2015). Results showed that 20% of students were nervous when solving mathematics problems. This result was lower than that for Shanghai students (27%) and even lower than the world average (31%). Results showed that 14% of students felt helpless when faced with a mathematics problem. This result was also lower than for Shanghai students (28%) and the global average (30%). Notably, our results only showed 32% of students worried about getting poor grades in math, which is much lower than that of Shanghai students (71%) and the global average (61%). Our results generally indicate a lower level of MA for the students in this study than those in Shanghai and the global average indicated by PISA 2012. This might be because we used a customized test specially designed for measuring mathematical modeling abilities and because it is not a high-stakes test. We stated clearly before the test that we would not release any individual student's information about the test or students' rank according to this test, and, therefore, students felt less anxiety in this context. PISA also did not rank specific students, but it is well-known by students and teachers. Additionally, PISA tested general math abilities. When compared, the differences are consistent with the findings that MA was higher in school-like math tasks than verbally mediated tasks (Ashkenazi & Danan, 2017).

In our study, female students reported substantially higher MA than male students. This result might not be surprising, as this result is consistent with most of the literature (e.g., Birgin et al., 2010; Devine, Fawcett, Szucs, & Dowker, 2012). The sex-role socialization hypothesis argues that math is traditionally viewed as a male domain subject (e.g., Duckworth & Seligman, 2006). Females may be socialized to think of themselves as mathematically incompetent; therefore, females may avoid mathematics and experience more anxiety when participating in math-related activities (Bander & Betz, 1981). Another explanation is that females may be more willing to express their emotions whereas males may not, because expressing negative emotions is viewed as immature or less acceptable for males (Flessati & Jamieson, 1991; Hunsley & Flessati, 1988). Fear suppression is more desirable for males than females in most cultures (Diener & Lucas, 2004; Peterson, 2006). Future studies should further explore gender differences in MA expression in China or Eastern culture.

MA was negatively related to students' mathematical modeling. Students with high and low MA showed substantial gaps in mathematical modeling abilities in all three modeling tasks, which is consistent with hierarchical multiple regression results. This result is consistent with the literature (Hembree, 1990; Ma, 1999;Namkung et al., 2019; Zhang et al., 2019) and supports the cognitive interference theory. The cognitive interference theory points out that MA taps an individual's working memory, therefore, undermines the individual's math performance (Ashcraft, 2002). We also provide a different perspective by our use of cluster analysis to support this theory. However, there are two competing theories: deficit theory (Carey et al., 2016; Tobias, 1986) and bidirectional theory (Ashcraft & Krause, 2007; Carey et al., 2016). The deficit theory explains that the negative experience triggers students' MA. The bidirectional theory combines deficit theory and cognitive interference theory. Because of the limitations of our data, we cannot examine the bidirectional theory. Further longitudinal studies are needed to examine these three theories together in the context of solving modeling tasks.

The gap between high and low MA seems to be related to the difficulty of modeling tasks. A significant interaction effect between the difficulty of modeling tasks and MA further shows that the relationship between mathematical modeling and MA depends on the difficulty of the modeling tasks. Our results are consistent with previous studies in that the relationship between MA and math performance was dependent on the complexity of math problems. For example, complex math problems elicited higher MA than simple arithmetic problems (Ching, 2017). Also, MA-problem-solving skill links are stronger than MA-digital calculation links (Beilock & Willingham, 2014; Harari et al., 2013). Also, there are stronger links between MA and advanced mathematics domains than between MA and the fundamental domain (Namkung et al., 2019; Zhang et al., 2019). Unlike the above studies, our study compared the different complexities of tasks in the same domain of mathematical modeling. Our results revealed that the relationship between MA and mathematical modeling depends on the difficulty of the modeling tasks.

In summary, this is the first study about the MA in the context of solving mathematical modeling tasks. Our results are consistent with the studies of MA and math performance. Therefore, the suggestions from intervention studies for relieving MA can also be applied to mathematical modeling areas. This finding adds to the growing empirical evidence of the relationship between math anxiety and math performance. We provided evidence that the relationship between MA and math performance depends on the difficulty of the math problems in the mathematical modeling domain.

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Chapter 17
Mathematical Competencies of Chinese Students: An International Perspective

Kaye Stacey

Abstract This chapter identifies aspects of the study of mathematical competencies of Chinese students that are likely to be of special interest to international readers. Perhaps the most striking feature is the scale of the work. Within an overarching framework for conceptualizing mathematical competencies, there are detailed reviews of the treatment of each of these cognitive and non-cognitive competencies in the Chinese curriculum since 1902, and a comprehensive set of snapshots of the current performance of Grade 8 students on each. The assessment tools provide powerful base-line data for monitoring students' mathematical competencies into the future. The detail in the studies will assist international researchers to more deeply understand some of the paradoxes in PISA results, such as Chinese students reporting low classroom exposure to "applied problems" while also demonstrating outstanding performance on items emphasizing PISA's formulate process. The book gives insight into the strong Chinese tradition of mathematics education, changing over time in response to dramatic social and economic forces but retaining unique features and also becoming increasingly well integrated with international thought.

Keywords Affective competencies · Applied problems · Assessment · Two basics · Four basics · Employ process · Formulate process · History · Mathematical communication · Mathematical competencies · Mathematical modeling · Mathematical problem posing · Mathematical reasoning and argumentation · Mathematical representation and transformation · PISA · Reasoning · Solving problems mathematically · Mathematical literacy

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17.1 Introduction

For an international audience, this book provides a valuable summary of the evolution of thinking about mathematical competencies in China, and how these ideas have developed over time and are presented in curriculum documents. In addition, it provides an overall framework for conceptualizing mathematical competencies and reports a comprehensive set of snapshots of the performance of Grade 8 students. There is a unity within the book, as the early chapters set the scene, and the later chapters present similarly structured studies of each of the mathematical competencies.

The international interest in the book is very likely to be stimulated by the outstanding results that Chinese students have achieved in international assessments. The authors draw significantly on the conceptual framework of PISA and to some extent TIMSS, and the empirical approaches that these studies use. In a report of the influence of PISA on thought and action in mathematics in ten countries, Stacey et al. (2015) noted that, as expected, the mean scores of students and their distributions have been an important stimulus for change or confirmation of direction in many countries. However, that article clearly demonstrated that the mathematics framework for PISA 2012 and earlier (OECD, 2013) has been equally or even more influential on thought and action across the world. This influence has been especially through PISA's emphasis on mathematical literacy as its goal, the embedded modeling processes of formulate, employ and interpret and the theoretical foundation of the underlying mathematical competencies (the fundamental mathematical capabilities of PISA 2012). The identification of PISA processes and capabilities aims to move the understanding of what students should learn in mathematics beyond knowing specific content and being able to solve closely defined classes of problems towards being able to use mathematics in their lives outside and beyond school. Although the PISA assessment has focused on using mathematics, the PISA capabilities also apply to mathematics as a discipline in its own right. This book demonstrates that China has also been influenced in these ways.

International readers will be impressed with many features of the book. There are detailed reviews of specific mathematical competencies that draw on research from around the world. While embedding the arguments within Chinese scholarship, the book demonstrates that these Chinese authors are well connected to international scholarship. The frameworks developed for the specific competencies deserve consideration for work elsewhere; in developing practical assessment tools to monitor progress in teaching for mathematical competencies. Readers will also be impressed by the scale and co-ordination of this set of studies, undertaken to support the intention of new aspects of the curriculum and standards. The studies have direct implications for curriculum implementation across China. They are designed to show the extent to which Chinese students achieve the intended Chinese curriculum. As Chap. 2 stresses in its careful review of the findings from China's participation in PISA, in China, the PISA success is understood not just as a cause for celebration, but also as a call to action to improve on identified weaknesses.

The fundamental ideas that underlie Chinese mathematics education are discussed in Chap. 1 by Jing Chen and colleagues. It discusses the evolution of the "four basics" principle, an expansion of the "two basics" (knowledge and skills) of earlier years. This transition corresponds to the underlying concern of educators around the world to expand the narrow view of school mathematics that is too frequently held by teachers, students, and the wider community. This concern has motivated progressive thought about mathematics education for at least 50 years, naturally with variation over time and place.

The expansion from two basics to four basics demonstrates the place that mathematical thinking processes take alongside the development of knowledge and skill, and it also emphasizes how mathematical activity is the glue that holds the other three basics together. Knowledge, skills, and thinking processes are used together in mathematical activity. Included in this message is the observation that students learn mathematics by engaging in mathematical activity at an appropriate level of challenge, not just by being told facts and practicing copied skills. The aim of the four basics is to create students with broad mathematical competence. This sets the scene for the following chapters which present a detailed account of how the thinking processes can be conceptualized, assessed and achievement monitored.

The central contribution of the book is to lay out a comprehensive framework for mathematical competence and its assessment. The framework is introduced in Chap. 3 by Binyan Xu and colleagues. Subsequent chapters address each of the components in turn, tracing their rising importance within Chinese curriculum documents over a century and reporting empirical results using a consistent set of levels. The motivation for this work is related to the policy decision to use evaluation and monitoring as a way to improve the quality and equity of education. Many other countries have also adopted policies of monitoring educational achievement in recent decades. It is clear that if mathematical competence beyond just knowledge and skills is to be valued and taught in all schools, then it must feature in the high-stakes assessments that are used. However, assessing more than knowledge and skills is a complex task, and hence the task is often put aside in national programs. Providing a well-researched, practical way for such assessment of broad mathematical competence can therefore be of national, possibly international, importance.

17.2 Evolution of Chinese Mathematics Education

Every chapter includes a discussion of the changes that have occurred in Chinese mathematics education since 1902. Together, these well-documented accounts show how the mathematics curriculum and goals of China have been influenced by the social, economic, and political environment within China and also by the international environment. For an international audience, the dramatic economic, social, and political changes that China has undergone in the last 100 years serve to highlight how school mathematics changes in response to social needs and the capacity of a society to provide education. Because of the strong unified national curriculum, the changes can be tracked very clearly.

The chapters document the shift from the ancient classics-focused imperial examination system to a more modern system under the influence of Western and Japanese ideas, the adoption of mathematics from the Soviet Union after the Second World War until 1958, and then a change (until 1963) where mathematics was focused narrowly on the needs of industrial and agricultural work. After the turmoil of the Cultural Revolution, from 1976 the curriculum gradually began to emphasize students' ability to analyze and solve problems as the goal of learning knowledge and skills. In this way, developing "basic ideas" was added to the two basics. This trend continued and accelerated to today, as the economy supported a rise in the school leaving age and increasingly ambitious goals for schooling. Concurrently, international trends, such as the shift from syllabus to curriculum standards, have been more influential. The "four basics" became the overall goal for compulsory education in 2011.

Chapter 1 identifies the unique characteristics of Chinese mathematics, and the key to its success, as a focus on the basics (attributed to China's farming tradition and Confucian culture) and the deliberate choice of ideas from other countries. The extensive international literature reviews in Chaps. 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, and 16 demonstrate the care taken in selecting perspectives and approaches from other traditions. There is no one dominant set of research studies, but a selection of the most appropriate from around the world. Because these literature reviews are very clearly focused on specific competencies, they can be valuable to beginning scholars from many countries. These later chapters make the ideas behind the four basics very concrete, and illustrate the mathematical depth intended.

Chapter 1 also provides an explicit contrast between Chinese and Western mathematics education from a Chinese point of view. It discusses a tendency for Western countries to ignore the foundation for mathematics and points out that building strong "basics" in Chinese mathematics is not simply reducing difficulty and focusing on mathematics in daily life. The chapter also identifies a tendency of Western child-centric education to give excessive priority to the students' interests and happiness, avoiding the fact that a strong foundation requires "basic mathematics knowledge and skill [that] is quite boring content and does not stimulate children's interest" (page 11 of Chap. 1). It is pointed out that attempts to adopt Chinese ideas in Western mathematics education are often based on a misunderstanding of the foundation as repetition only, ignoring the need to combine "laying the foundation with seeking development" (page 12 of Chap. 1). This book may contribute to dispelling this misconception by showing readers from other countries a bigger picture of how Chinese teachers attend to both foundation and development. Perhaps future books by this team will provide a deeper understanding for audiences outside China by explaining with examples how teaching sequences can fully develop the four basics and mathematical competencies.

17.3 Describing and Organizing Core Mathematical Competencies

Chapter 3 is the key chapter of the book. Drawing on Chinese and international literature, it describes the model that features throughout the book, identifying core mathematical competencies that contribute to three fundamental mathematical activities (organizing empirical materials mathematically, organizing mathematical materials logically, and applying mathematical theory). Six core competencies have been selected for the model: mathematical problem posing, mathematical representation and transformation, mathematical reasoning and argumentation, solving problems mathematically, mathematical communication, and mathematical modeling. A unifying aspect of the model is that each competency is described in terms of three levels, always labeled reproduction, connection, and reflection. By choosing these labels, the levels are implicitly linked to other descriptions of increasing cognitive depth, right back to Bloom's taxonomy as well as current TIMSS and pre-2012 PISA names for cognitive levels. Describing competencies in terms of levels is essential for assessment, but the general labels are insufficient and so specific descriptions are given for each competency (and further developed in subsequent chapters). This set of six bears a strong resemblance to other sets of mathematical competencies (e.g., Niss, 2015; Turner, Blum, & Niss, 2015), although each set has its own unique characteristics to meet its own purpose. Turner et al. (2015), for example, created a scheme to predict the difficulty of PISA items and hence describe what students who perform at different levels of the PISA assessment can do. They refined a larger set of competencies (derived from Niss) to the empirically demonstrated six best predictors of item difficulty and described four levels for each competency. The specific descriptors were selected to describe the competency and how students progress through it, but also to best match the assessment purpose (the PISA pen-and-paper written test, with relatively short items set in real-world contexts). Other descriptions of exactly the same competency but written for a different purpose may sensibly include other things.

Chapters 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, and 16 examine each of the competencies in turn, following a standard pattern. A content analysis of the major curriculum documents from 1902 to the present-day maps changes in the treatment of the competencies over time. In each case, the general pattern of the data indicates there has been an increase in attention to the competency, reflecting an increasingly deeper set of goals for schooling. It is likely that most countries would exhibit a similar general trend, although the timelines for change and development would be very varied. I wondered if the content analysis, mainly based on the incidence of particular words in the documents, gave a really reliable picture. Words come in and out of fashion (influencing frequency) and can change their meaning and connotation over time. A deeper analysis might have gone beyond the documents to show how the ideas were operationalized in the mathematics that students were asked to do at each stage.

More generally, across the book, I wanted to see more examples of mathematical tasks to get a better idea of how I should interpret the words written on the page. As an international reader, I found it very instructive to carefully examine the relatively few mathematical tasks that were given. They reveal large differences in the intentions of the Chinese curriculum compared to the Australian curriculum (the one with which I am most familiar). Tasks with sample student responses were even more illuminating. For example, Chap. 11 presented six questions used to assess reasoning; two at each of the three levels and two involving each of arithmetic, algebraic and geometric reasoning. Were these questions posed to Australian students, I expect that only the two arithmetic reasoning questions would be answered correctly by more than a tiny percentage of students. Interestingly, one of these questions has been placed at level 1 and the other at level 3. The other items (at levels 1, 2, and 3) required an understanding of two aspects of reasoning of which Australian Year 8 students have very little experience: proving with algebra and identifying the logical relationships between statements. This interesting case made me reconsider what I mean by mathematical reasoning; a valuable experience. Since looking at these problems, I am less sure than before that I have a clear conception of a general mathematical reasoning competence independent of the content being taught. Mathematics has its own techniques to learn (and teach) for both plausible and deductive reasoning.

Throughout the book, the authors share their challenge of trying to understand the precise definitions of key terms that other mathematics educators have used. What a difference from mathematics itself, where the definition is central and names are used consistently with due regard to formal definition! The words describing the various aspects of "working mathematically" are a hot spot of confusion. Perhaps mathematics education literature in English is particularly difficult because, in addition to the challenges of clear definition, the English literature includes the very significant contributions from people from many cultural and linguistic backgrounds, drawing on additional connotations of each word from their own language and didactical traditions.

The use of the word "reasoning" in PISA is an informative example. The PISA 2012 framework used the word "reasoning" in two senses. One is a carefully defined meaning, naming the "fundamental mathematical capability" (competency) of "reasoning and argument". The definition, later refined by Turner et al. (2015, p. 114), is "Drawing inferences by using logically rooted thought processes that explore and connect problem elements to form, scrutinize or justify arguments and conclusions." The second use is conversational. "Reasoning" is used as a synonym for general thinking about anything mathematical. An example is "Aspects of quantitative reasoning – such as number sense, multiple representations of numbers, elegance in computation, mental calculation, estimation and assessment of reasonableness of results – are the essence of mathematical literacy relative to quantity." (OECD, 2013, p. 35). In the (second draft) framework for PISA 2021/2022 reasoning is again used in this overarching conversational sense: "Having an appreciation of abstraction and symbolic representation supports reasoning in the real-world applications of mathematics…." (OECD, 2018, p. 17). However, in a new initiative, mathematical literacy is to be assessed in four domains (rather than three), now including “mathematical reasoning” weighted at 25% (p. 33) which is a much broader concept than the “reasoning and argument’ competency of PISA 2012. “Mathematical literacy therefore comprises two related aspects: mathematical reasoning and problem solving” (OECD, 2018, p. 9) and “Mathematical reasoning (both deductive and inductive) involves evaluating situations, selecting strategies, drawing logical conclusions, developing and describing solutions, and recognizing how those solutions can be applied” (OECD, 2018, p. 14). The example items on the website give the impression that mathematical reasoning will be assessed through intra-mathematical items without a real-world context. One of the example questions on the website (https://pisa2021-maths.oecd.org/) asks students to calculate $(-5)^{43} + (-1)^{43} + (5)^{43}$.

The example of the word “reasoning” in PISA has been discussed to show how even within one project, words relating to the process aspects of mathematics are used with substantially different meanings. The consequences of this are (i) that reading the international literature for meaning is very difficult; (ii) the progress of mathematics education as a disciplined study is held back by this; (iii) individual groups need to make or choose their own clear definitions, and explain them with examples so that others can understand their work. The work put into defining the competencies in this book can be used as a strong basis for future research on competencies in China.

17.4 Empirical Studies of Cognitive Mathematical Competencies

The central section of the book (Chaps. 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, and 16) reports a series of empirical studies of students’ performance related to the cognitive and non-cognitive competencies. Each of the studies devised a set of relevant items, which were administered to some Grade 8 students. Samples varied, but were generally large (e.g., over a thousand students in intact classes in about 3 schools in each of 5 regions selected to represent schools of different standards in developed, medium-developed, and less-developed areas of China). The results are analyzed in various ways (e.g., using item response theory) so that the ability distribution of students can be reported numerically and against the defined levels. In some cases, “double-digit coding” reveals students’ common errors and strategies. Further analysis of the results in some chapters reveals differences between geographical areas and gender differences.

This comprehensive and unified approach to the assessment of mathematical competencies is impressive. While it is clear that these representative samples cannot provide statistically solid data for the huge population of China, the studies provide a valuable set of items for future monitoring of students’ mathematical competencies, directions for future research and curriculum development, and information about how and where to target future support for teachers.

There is something of interest to international audiences in every chapter, especially for researchers interested in the theorization and assessment of mathematical competencies. Perhaps the main international interest will be in better understanding some of the puzzling results of international assessments. One relates to the affective dimensions of mathematics. The PISA 2012 survey reported on five aspects of self-related cognition related to mathematics: self-efficacy, self-concept, anxiety, intrinsic motivation, and instrumental motivation. Implicit behind these choices is the common-sense proposition that achievement is promoted if students are low on mathematics anxiety and high on the other four factors. Given the "top of the class" achievement scores of students in China, it was therefore surprising to see that Shanghai PISA 2012 students were very close to the OECD average on three of the five self-related cognition measures (self-concept, anxiety, instrumental motivation). The other two measures (self-efficacy and intrinsic motivation) were high, as would be predicted. Chapters 2, 15, and 16 provide a theoretical analysis and further data for a detailed exploration of these unexpected results with a wider Chinese sample. International readers will see a well-document example of how measures of self-related cognition of anxiety and self-concept are strongly related to the local, classroom, and family situation in which students find themselves.

Another paradox in the PISA 2012 data relates to national scores on the formulate-employ-interpret processes of solving real problems with mathematics. This is described by Stacey in Cai, Mok, Reddy, and Stacey (2017). Stereotypes of East Asian education assume an emphasis on routine procedures, so it would have been predicted that they would perform best on the intra-mathematical "employ" process. However, those countries tended to achieve their own best score on the formulate process (translating from the real world to the mathematical world), which was internationally the hardest process. East Asian students achieved their own lowest score on the interpret process, which was the easiest process in many Western countries and for the OECD average. This result was even more surprising in conjunction with the accompanying survey of students' confidence in solving a selection of problems and how frequently they had encountered similar problems in class. The selected problems included some "formal" items lacking any real-world context (e.g., find the volume of a prism) and "applied" items set in a real-world context like almost every PISA 2012 item. Students in Asian countries on average reported low classroom exposure to the applied problems. Despite this low exposure, they excelled in the formulate process, translating a real-world problem into mathematical terms. The studies of mathematical modeling, problem solving, and reasoning in this book begin to provide the depth of data needed to understand puzzling phenomena like this, and hence to reveal what type of school experiences produce students who can use the mathematics they have learned in all spheres of their lives.

17.5 Conclusion

As the sections above have illustrated, this book has much to interest an international reader. First, it provides international readers with insiders' views of the famous Chinese PISA performances. One of the clear messages is that there is a definite intention to improve, despite this success. The book also provides international readers with insight into Chinese mathematics education and the history of curriculum over a momentous century. It shows how the mathematics curriculum is tied to the goals and the educational and economic conditions of society. It also shows a capacity to maintain a strong national tradition, while adopting trends from countries around the world. For researchers, the book documents a comprehensive and unified attempt to establish theoretically sound tools and some base-line data to monitor Chinese students' cognitive and non-cognitive mathematical competencies. Beyond the monitoring progress, these tools and the data that result can inform future curriculum development and identify the professional learning needs of teachers to make even more mathematically competent students.

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Chapter 18
Mathematics Core Competencies of Chinese Students – What Are They?

Frederick K. S. Leung

Abstract China has been implementing mathematics curriculum reform driven by "subject competencies," and the Chinese term used for competencies is *he-xin-su-yang*, often translated as "core competency." In this commentary chapter, concepts related to *he-xin-su-yang* are discussed. The concept of "core" (*he-xin*) competencies addresses not only the issue of "what mathematics competencies are," but also the question of "among the mathematics competencies, which are core competencies and which are just peripheral competencies," a question which has not really been addressed in China. For the concept of competencies or *su-yang*, the term implies something of high quality being developed deep inside human being, different from how competencies are used in the literature which often convey a sense of being adequate or sufficient for a certain purpose. The strengths of the book and how well Chinese students performed in cognitive and non-cognitive mathematics competencies are then discussed. Students' attitudes toward mathematics and mathematics teaching and learning, one important non-cognitive competency which is not covered in this book, are then discussed with reference to the TIMSS data. A cultural explanation of the negative attitudes of students of Chinese origin is then presented, and the impact of the Chinese language on mathematics learning is also mentioned.

Keywords Competencies · Mathematics competencies · Cognitive competencies · Non-cognitive competencies · Core competencies · Mathematics literacy · Culture · Confucian heritage culture · Student attitudes · Chinese language · TIMSS · *he-xin-su-yang*

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18.1 Introduction

The spectacular performance of Shanghai students in mathematics (and science and reading) in Programme for International Student Assessment (PISA) 2009 caught the attention of educators, policy makers, and the general public around the world (see Table 2.6) (Tan, 2017; Yang & Fan, 2019). Many people are asking: what is happening in Shanghai that makes it apparently so successful in education? In particular, how is mathematics taught and learned in Shanghai that produces such remarkable results of their students in PISA? Shanghai is of course not the whole of China, it is a very special city in China (some refer to Shanghai as the Paris of the Orient, the New York of China, or the Great Athens of China, see, for example, Cumming, 1899). The PISA results thus prompt us to ask what (mathematics) education in China is like beyond Shanghai and PISA. This book intends to "open a window for relevant parts from the world to understand Chinese students' mathematics competencies" (Preface), and so provides at least a partial answer to the important question of what education in China is like.

Are Chinese students on the whole highly competent in mathematics, like what the PISA results show for the Shanghai students? Various books and papers have been written on the mathematics achievements of Chinese students (Leung, 2001, 2005; Stevenson et al., 1990; Stevenson, Chen, & Lee, 1993) and students of the Confucian Heritage Culture (CHC) more generally (Leung, Graf, & Lopez-Real, 2006; Stankov, 2010; Watkins & Biggs, 2001; Wong, 2004). But this book is about mathematics *competencies* of Chinese students, not their mathematics achievement in general. To understand mathematics education in China, it is important and illuminating to go beyond simply learning about mathematics achievements as measured by international studies such as PISA (or even worse, merely focusing on the ranking of the country in such studies) and look at different aspects of students' mathematics competencies, in order to gain an insight into the strengths of the Chinese students. That is, if Chinese students are really strong in mathematics achievement, how do they flair in different aspects of their mathematics competencies? And why are they strong in particular aspects of competencies?

"The study of mathematics competencies (is) an internationally hot topic in mathematics education research" (Preface), and the term "competencies" has become a catchword in recent literature (Enderson & Ritz, 2016; Lee, 2016; Leung et al., 2006; Niss, Bruder, Planas, Turner, & Villa-Ochoa, 2016; Pettersen & Braeken, 2019; Wintermute, Betts, Ferris, Fincham, & Anderson, 2012). Many countries and studies are advocating the development of competencies and their assessment, and China has been implementing mathematics curriculum reform driven by "subject competency" (see Chap. 1). As with most catchwords, the term "competencies" is so pervasively used that sometimes we do not have a clear sense of what it means. In the case of China, there is a further complication in the concept of "competencies" because of the Chinese language used to describe this concept, as will be seen below.

Chapter 1 of this book described the history of how this advocacy of competencies in China has evolved from the advocation of "the two basics" and "the three abilities", to "the four basics" and then finally "the six core competencies" in recent years. This historical perspective lays the foundation for readers to understand Chaps. 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, and 14 on the various aspects of competencies of Chinese students.

18.2 The Concept of Competencies

Various conceptions of the idea of competencies in the international literature were discussed in Chap. 1 in the context of justifying the advocacy of mathematics competencies in China against the "overall environment of the development of international mathematics education" (Chap. 1). And Chap. 3, in establishing "a framework of mathematical competencies in China" for the purpose of this book, also reviewed the various ideas of competencies and its definitions, connotations, and usage adopted in different international studies of mathematics achievement or mathematics curricula around the world.

Different terms akin to the idea of "competencies" were used in these international studies and mathematics curricula. PISA, for example, used the term (mathematics) "literacy" as the core concept underlying its assessment of student achievement. Chapter 2 presented the mathematics achievement of Shanghai students in PISA, and as background information, the idea of mathematics literacy was discussed. There it was pointed out that "PISA 2000 defines mathematical literacy as 'an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgments and to engage in mathematics, in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen'" (OECD, 1999, p.41, quoted in Chapter 2). The assessment framework in the first PISA consisted of two major aspects, namely, *mathematical competencies* and *mathematical big ideas* (OECD, 1999). "*Mathematical competencies* includes eight general mathematics skills in a non-hierarchical order (e.g., *problem posing and solving skills*, *symbolic, formal and technical skills*, and *modeling skills*), which are further organized into three larger classes of competency: … *reproduction, definitions and computations* (class 1), *connections and integration for problem solving* (class 2), and *mathematical thinking, generalization and insight* (class 3)" (Chap. 2). So how is the idea of competencies related to other concepts such as literacy? Is competencies a kind of literacy, as seemed to be implied in the assessment framework of PISA?

18.2.1 Core Competencies or **he xin su yang***?*

The Chinese term denoting the concept or attribute akin to competencies being advocated in China is *he xin su yang*, which is often translated as "core competencies". The first official appearance of the term was in 2013, in a project on "A study on the overall framework of students' core literacy in basic education and higher education in China"[1] commissioned by the Ministry of Education to a research team headed by Professor Lin Chongde of Beijing Normal University. Subsequently in 2014, a document entitled "Opinions on Comprehensively Deepening Curriculum Reform and implementing the Fundamental Task of Developing Student's Moral Values"[2] issued by the Ministry of Education explicitly introduced the concept of *he xin su yang* and promoted its use in the country.

In introducing or promoting a new measure or initiative, the term adopted to describe the measure is important. Usually, either a new term is coined to denote the new measure, or an existing term that best captures the meaning of the measure is picked as the official term for the measure. An important issue is how this newly coined term, roughly translated as Core Competencies in English, is related to the various terminology related to the idea of core competencies in the international literature.

As mentioned above, the Chinese term *he xin su yang* is translated as "core competencies". The translation of *he xin* as core poses not controversy. The contentious issue here is not the translation but which competencies are considered as core in mathematics. I will return to this point below. On the other hand, the translation of *su yang* as competencies *is* controversial.

Chapter 1 explicitly discussed the translation of the term *su yang*, and argued that "'养' (or *yang*) represents 'accomplishment', i.e., a certain level of thought, theory, knowledge and art". The authors further pointed out that the Chinese official curriculum document defines Mathematical Competency as "a comprehensive reflection of the basic characteristics of mathematics in terms of thinking quality, key abilities, emotional attitudes and values" (Ministry of Education of the People's Republic of China, 2018). This translation of the character *yang* as an accomplishment is however not uncontroversial. The origin of the term *su yang* is from a Chinese classic, "*Li Xun Zhuan*, the Book of *Han*".[3] There, *su yang* refers to a quality acquired or developed through sustained training and practice over time. It often refers to moral character, and it is more an internal or internalized attribute than an external "accomplishment". The term *su yang* implies something of high quality being developed deep inside human being, while accomplishment is often

[1]"我国基础教育和高等教育阶段学生核心素养总体框架研究", headed by Professor Lin Chongde (林崇德)

[2]《关于全面深化课程改革落实立德树人根本任务的意见》

[3]《汉书・李寻传》:"马不伏历, 不可以趋道; 士不素养, 不可以重国", roughly translated as "They need to feed the horses every day, otherwise the horses cannot run, and if they do not train the people's *su yang*, they cannot hold the country strong.".

understood as a person's attributes observable externally. Accomplishment may be considered as an external manifestation of competencies or *su yang*, but it should not be equated with *su yang*.

18.2.2 "Adequate" Competencies for a Certain Purpose or High-level Qualities?

Moreover, the term competencies often convey a sense of being adequate or sufficient for a certain purpose (e.g., someone who has computer programming competencies, or a competent computer programmer, is someone who has sufficient knowledge and skills to do the job of a computer programmer well). This is similar to the idea of Literacy as used in PISA, where, as pointed out above, it is defined as "an individual's capacity … (to) meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen" (OCED, 1999, p.41). So literacy, like the word competencies, conveys a sense of being *adequate* for a certain purpose (in the case of PISA, to meet the needs of an individual to live as a citizen). But *su yang* is an attribute at a higher level. The idea embodied in the Chinese character *yang* is something that is beyond being merely adequate, something that is desirable but not essential or necessary. The authors of Chap. 1 tried to capture this aspect of the character *yang* by using the words "a certain level of (thought, theory, knowledge and art)", but I personally think that *yang* conveys the meaning of being at a high level, not just at "a certain level" (which could be a rather low level!).

As far as mathematics *su yang* is concerned, it would of course be good if an individual acquires the *su yang* of mathematics *in addition to* being literate or having competencies, but we should not expect all individuals to have acquired *su yang* (e.g., to be able to appreciate the beauty of mathematics). To draw an analogy with language, I think everyone will agree that an individual will need certain language competencies (as demonstrated by accomplishment?) in order to live meaningfully in this modern world, competencies including reading, listening (and understanding), speaking, and writing – this is exactly what is meant by being literate. Of course, it would even be better if the individual knows some literature too, but this is something which is desirable and not essential or necessary. I don't think many people will argue seriously for everyone learning Shakespeare! For the more general idea of *su yang* in ancient China, the arts[4] of music, chess, calligraphy, and painting were considered as typical *su yang*. Clearly, these are good examples of something which are desirable but not necessarily essential (Han, Li, & Fu, 2005; Liu & Zhu, 2012; Su, 2010).

[4] In ancient China, music, chess, calligraphy and painting were known as the Four Arts or four *su yang*, originated from *Lanting Ji*, in the book *Fashu Yaolu* by Zhang Yanyuan of the Tang Dynasty (唐張彥遠《法書要錄》載唐玄宗朝何延之〈蘭亭記〉) (Cheung, 2011).

18.2.3 Competencies and Related Concepts

So when talking about competencies, are we referring to the minimum capacity or accomplishments of a child that enable him or her to survive or live effectively and meaningfully in the modern world, or are we looking for qualities beyond being adequate or sufficient? More importantly, how does this concept of *he xin su yang* relate to the variously relevant concepts in the international literature, terms such as literacy, accomplishment, attainment, ability, capability, competence, competency, proficiency, skill, knowledge, and standards. NCTM, for example, used the term Standards or Process Standards, rather than competencies (National Councils of Teachers of Mathematics [NCTM], 2000); and in Singapore, the term "skills" is used (American Institutes for Research [AIR], 2005; Kaur, 2010). Denmark seems to be one of the few countries where the term competency or competencies is explicitly used (Niss & Højgaard, 2019). But how are these terms or concepts related to the Chinese term *he xin su yang*?

Clarification of the meaning of the term *he xin su yang* in the Chinese context is crucial. Jan de Lang argued that different conceptions of "mathematics literacy (ML) will lead to different curricula in different cultures. ML will need to be culturally attuned and defined by the needs of the particular country. This should be kept in mind as we attempt to further determine what mathematics is needed for ML" (De Lange, 2001). Although Jan de Lang used the term Mathematics Literacy instead of Mathematics Competencies, the idea he expressed is clear. Conceptions of mathematics competencies in different cultures, and indeed the very terms used in different languages to denote the concept, will lead to different mathematics curricula and different ways mathematics is conceptualized, taught, and assessed. It is in the context of the different conceptions of the term *he xin su yang* in China versus the conceptions of mathematics competencies in different cultures as found in the international literature that we could and should understand the mathematics "accomplishment" of Chinese students. I hope a "by-product" of this book is to initiate discussion of and research into this conceptual aspect of mathematics competencies across different cultures, enriching the understanding of the people around the world on what are the essential things to learn in mathematics, and indeed what does it mean by learning mathematics in different cultural contexts.

18.2.4 What Are "Core" Competencies?

The problem becomes even more complicated when we consider the adjective "core" that qualifies "competencies", because we need to address not only the issue of "what mathematics competencies are", we also need to deal with the very important question of "among the mathematics competencies, which are core competencies and which are just peripheral competencies?" As far as I know, this important question has not really been addressed in China, as all the relevant literature merely dealt with the issue of what are core competencies in mathematics but never addressed the issues of what non-core mathematics competencies are. If there are no non-core competencies, then using the adjective "core" to qualify competencies is pointless.

This discussion on the conceptual understanding of *he xin su yang*, or core competencies, is essential to this book which is on the (cognitive and non-cognitive) competencies of Chinese students in mathematics, and as can be seen from the sections above the discussion is of paramount importance to mathematics education in China more generally as well. Since the current curriculum reform in China is driven by the idea of developing "competencies" or *he xin su yang* in our students, an elucidation of the conceptual understanding of mathematics competencies clarifies our understanding of the nature of school mathematics (as opposed to other school subjects such as Physics, for example); and clarifying what *core* competencies direct us to understand *what* features among the various characteristics of mathematics are essential for our students to learn, and thus *how* such mathematics features should be taught or inculcated. Such clarification of concept obviously is extremely important for research into this important area in mathematics education too.

18.3 Strengths of the Book

Chapter 3 is pivotal to the book, as it proposes a theoretical framework of mathematical competencies for the rest of the chapters in the book. Based on an extensive review of the international literature on the idea of competencies and related concepts, the chapter discussed the different conceptualizations and components of mathematics competencies adopted in international studies and national curricula around the world, and came up with a comprehensive model of mathematics competencies. As the author claimed, "to provide a meaningful and operable reference for the evaluation of mathematics education in China, this study intended to establish a mathematical competence model which take into account not only the essential characteristics of the mathematics subject, but also the new requirements for the mathematics education brought by social development" (Chap. 3). In addition to providing the conceptual underpinnings for the book, the framework presented in Chap. 3 is making a substantial contribution to the literature, especially the literature on mathematics education in China. The chapter however did not tackle the issue of the equivalence between the Chinese concept of *he xin su yang* and the core competencies as discussed above, nor was the intricate relationship between the concept of competencies and the variously related concepts in the international literature discussed at length. The author of the chapter also did not touch on the issue of what "core", as opposed to peripheral, competencies are.

This book provides rich and comprehensive information on both the curriculum development and the achievements of Chinese students in the area of mathematics competencies, and offers a lens for understanding the superior achievement of

Chinese students in international studies of mathematics achievements more generally. Seven aspects of "competencies" are studied in this book: problem posing, problem solving, representation, reasoning, mathematical modeling, and communication, and self-efficacy. Although *core* competencies or *he xin su yang* is the prevailing concept of curriculum reform in China today, the adjective "core" is not used to describe the competencies in this book, and so the authors do not explicitly claim that these competencies are "core" ones. Suffice to say is that through the inclusion of these seven competencies in this book, they are considered important competencies, and the authors took pain to argue for their importance in the literature review or theory part of the chapters. In this regard, this book may constitute a contribution to the promotion of students' core mathematics competencies development in China, and trigger a discussion or debate on whether the competencies included in this book are the core ones, or indeed whether there is such a concept of "core" competencies at all, or that the adjective "core" should be dropped in future discussions of competencies in the country. For example, do these seven aspects rightly belong to the realm of "competencies"? Have the seven aspects covered the most important or the "core" competencies in mathematics? Are there any important or core competencies missing from the coverage? These are important and interesting issues that are worthy of further exploration.

Each of the main chapters included a section analyzing the development of the curriculum from the early twentieth century to date. This is highly informative, as well as important for understanding the second section of the chapters in reporting the performance of Chinese students in the respective competencies. As pointed out in the beginning of this chapter, in contrast to most international literature on the mathematics achievements of Chinese students or students of CHC, this book focuses on mathematics competencies, providing readers with rich information based on rigorous analysis of curriculum documents from the beginning of the twentieth century till now and results of empirical studies on different aspects of competencies conducted in the country. As far as I know, this is the most comprehensive treatment of Chinese students' mathematics competencies in the international literature.

18.4 How Well Do Chinese Students Perform in Mathematics Competencies?

As pointed out above, the second section of each of the main chapters reported the performance of Chinese students in the respective competencies. Students' performances were assessed according to a set of scoring rubrics described in each chapter. Many chapters provided a comparison of achievements of students in different regions of China (e.g., sampling students according to how developed the region they are from, or according to geographic location (eastern China, central China, southern China, southwest China, etc.) – although the classifications of regions are not the same in the chapters), and between boys and girls. These give an idea of the distribution of the performance within the country and hence a better understanding of Chinese students' mathematics competencies. But since the reported studies were national studies and not designed as comparative studies with students from other parts of the world or against any international standards, it is not possible to tell how competent Chinese students are in these competencies compared to their counterparts elsewhere. Given the absence of an international comparison group, the within country comparisons are perhaps the best the authors could do in providing a more nuanced picture of the achievement of Chinese students in these competencies. Actually, given the structure of the chapters which all started with an analysis of the historical development of the curriculum and ending with the specifications of expectations in the latest curriculum in that particular aspect of mathematics competencies, a very instructive comparison would be between student achievement and the curriculum intention (or using the terminology of the IEA studies, alignment study between the intended and the attained curricula). That is, the results of the second parts of the main chapters should inform readers on the achievement of Chinese students in these areas of mathematics competencies as measured against what the national curriculum intends the students to achieve. This was done more explicitly in some chapters than others, and I look forward to more studies, analysis, and reports on such alignment studies on mathematics competencies.

18.4.1 Non-cognitive Competencies

Although the sub-title of this book is "Cognitive and non-cognitive competencies of Chinese students in mathematics", there is a clear lop-sidedness in the book in the sense that there are many more chapters on the cognitive aspects than the non-cognitive aspects of mathematics competencies. This is probably due to the different amount of research done in these two different domains in China, but could this also reflect the bias of the Chinese mathematics education community toward the cognitive aspects of mathematics competencies at the possible expense of relative negligence of the non-cognitive aspects of mathematics competencies? If this is indeed the case, then the chapters of this book may constitute a rectifying pointer to increased attention to the non-cognitive aspects of mathematics competencies.

18.4.2 Attitudes of Chinese Students

One important non-cognitive competency is the attitudes of students toward mathematics and mathematics teaching and learning, a competency which is not covered in this book. One may of course argue that attitudes are not a competency. Although I do not share this view (I do think that attitudes are a kind of competency), it is not my intention to defend my view here. Suffice to say is that attitudes are definitely an important part of *su yang*, the internal attributes associated with a certain area (mathematics *su yang* in this case). If we think of PISA's goal of Literacy as being to "meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen" (OCED, 1999, p.41), attitudes are clearly an extremely important aspect of literacy or competency or *su yang*. Students' attitudes are also an important component of the attained curriculum, since in all school systems, students' positive attitudes are invariably one of the goals of education. Indeed, as argued elsewhere, "in this era when life-long learning is so much stressed, some people think that a positive attitude is even more important than attaining high test scores. A positive attitude will motivate students to continue to learn even after they have left school" (Leung, 2014, p.603).

Attitudes of Chinese students toward mathematics and mathematics learning were only touched upon slightly in Chap. 2 when the PISA results of Shanghai students were discussed. There, in discussing the Shanghai students' self-related cognition in mathematics, it was pointed out that consistent with their stellar performance in mathematics, Shanghai students had very high mathematical self-efficacy (see Fig. 2.3). In big contrast, their mathematical self-concept was rather low, even lower than the OECD average. These findings are consistent with the results of the Trends in International Mathematics and Science Study (TIMSS), where students of Chinese origin or under the influence of CHC performed extremely well in the mathematics test, and yet had rather negative attitudes toward mathematics and mathematics learning (including liking of mathematics, valuing of mathematics and confidence in mathematics) (Callingham, 2013; Leung, 2006, 2014). In particular, their confidence in mathematics was exceptionally low, in sharp contrast to their high mathematics achievement. (Note however that the anxiety of Shanghai students in PISA 2012 toward mathematics was not particularly high, see Chap. 2.)

The author of Chap. 2 went on to discuss Shanghai students' disposition toward mathematics in PISA 2012. The salient findings here are that Shanghai students had a very high level of "mathematics work ethic, … (which) suggests that Shanghai students have relatively high ability to dedicate time, hard work and persistence to attain mathematics competency". However, Shanghai students were much "more likely to attribute their failure in mathematics to themselves rather than external factors (e.g., bad luck, bad guess or the teacher)". This is consistent with the observations of Leung (2006), who commented that Chinese or CHC students are more likely to attribute success and failure in mathematics to external rather than internal factors.

The findings of Chap. 2 on the PISA 2012 results are largely consistent with findings in previous studies, except for the finding that Shanghai students had much "higher level of intrinsic motivation to learn mathematics … than instrumental motivation, … a pattern … also observed in all the other top-performing East Asian systems". This seems to contradict results of the analysis by Zhu and Leung on the TIMSS 2003 results, where they found that for students in Hong Kong (and Korea), their product-oriented motivation (a kind of instrumental motivation) was exerting a greater influence on their mathematics achievement than their pleasure-oriented motivation (a kind of intrinsic motivation), compared to their western counterparts. Zhu and Leung attributed this finding to the fact that while "educators in the West advocate more of the role of internally oriented motivation in students' learning but deemphasize that of externally oriented motivation … educators in East Asia … highly promoted extrinsic motivation" (Zhu & Leung, 2011, p.1206). Note however that Zhu and Leung were analyzing the TIMSS 2003 results of students from Hong Kong, Chinese Taipei, Japan, Korea, and Singapore; whereas the PISA 2012 results were obtained 9 years later, and were for Shanghai students only. As argued earlier in this chapter, Shanghai is a very exceptional city in China in many regards, and it should not be held as representative of the rest of China. Given these consistent and inconsistent findings, it is all the more important for the non-cognitive aspects of competencies of Chinese students from different parts of the country to be investigated, to see how well or how badly the Shanghai students represent their fellow countrymen, and hence arrive at a comprehensive picture of this non-cognitive aspect of competencies – the attitudes of Chinese students toward mathematics and mathematics learning.

Figures 18.1, 18.2, 18.3, 18.4, and 18.5 show the attitudes of Chinese students or students of Chinese origin (Hong Kong and Chinese Taipei) in the latest cycle of TIMSS, as compared to a number of western countries. Together with the PISA results presented in Chap. 2, it is clear from even a cursory look of the data that in contrast to Chinese students' high achievement in mathematics, their attitudes toward mathematics and mathematics learning are rather negative. In the literature,

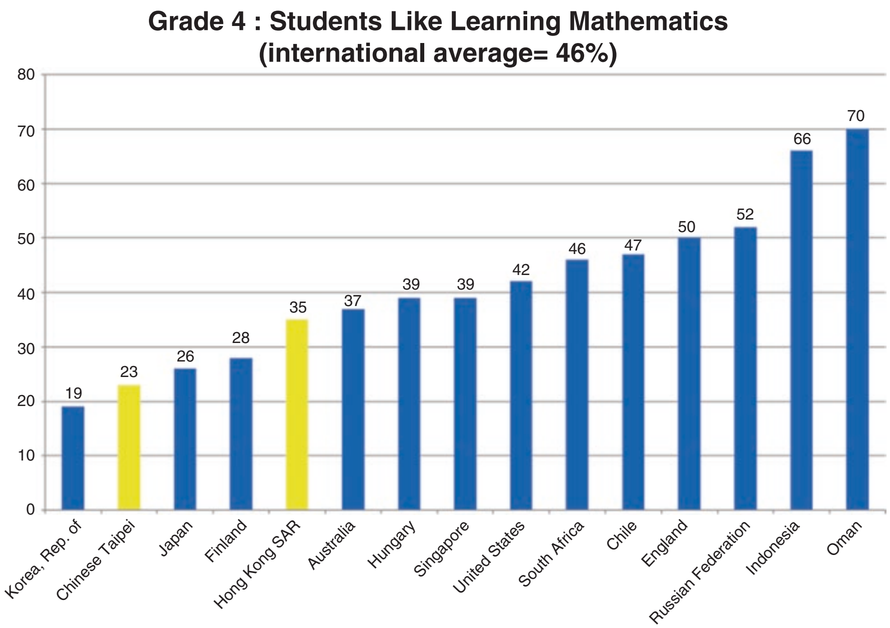

Fig. 18.1 Grade 4 students like learning mathematics (TIMSS 2015)

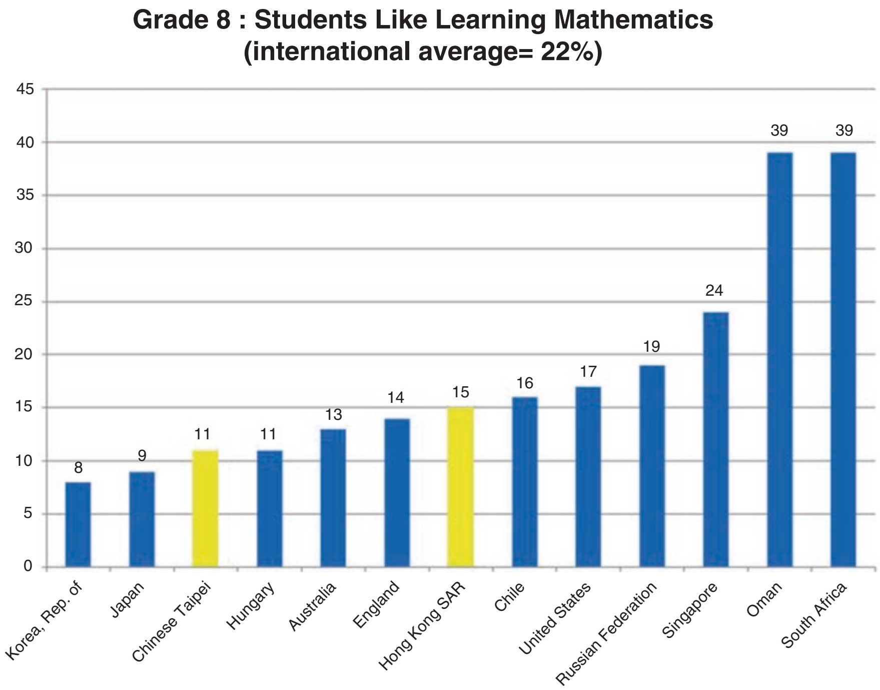

Fig. 18.2 Grade 8 students like learning mathematics (TIMSS 2015)

student achievement is usually positively related to student attitudes (Papanastasiou, 2000). While this is true at the individual student level, even for students of Chinese origin, the relation is not found at the national level. Why is it the case?

18.5 Culture

18.5.1 CHC Cultural Values

Elsewhere I have argued that both the high achievements of CHC students and their negative attitudes could be explained by the common cultural values that they share (Leung, 2006). In fact one observation I have for this book is that although it is a book about (the mathematics competencies of) Chinese students, no reference at all has been made to the cultural values held by students (and their teachers) when discussing either the historical development of the curriculum in China or the achievements of the Chinese students. Chapter 1 merely provided the historical context for the understanding of Chinese students' (cognitive and non-cognitive) competencies in mathematics without any reference to culture. Chapter 10 on self-related beliefs did touch on the collectivist culture in East Asia slightly when discussing the

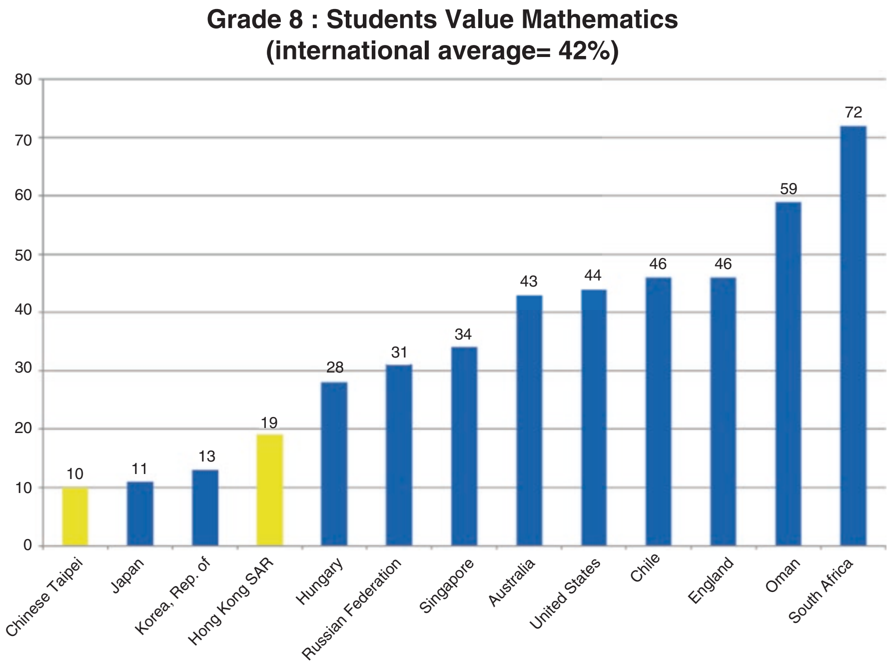

Fig. 18.3 Grade 8 students valuing mathematics (TIMSS 2015)

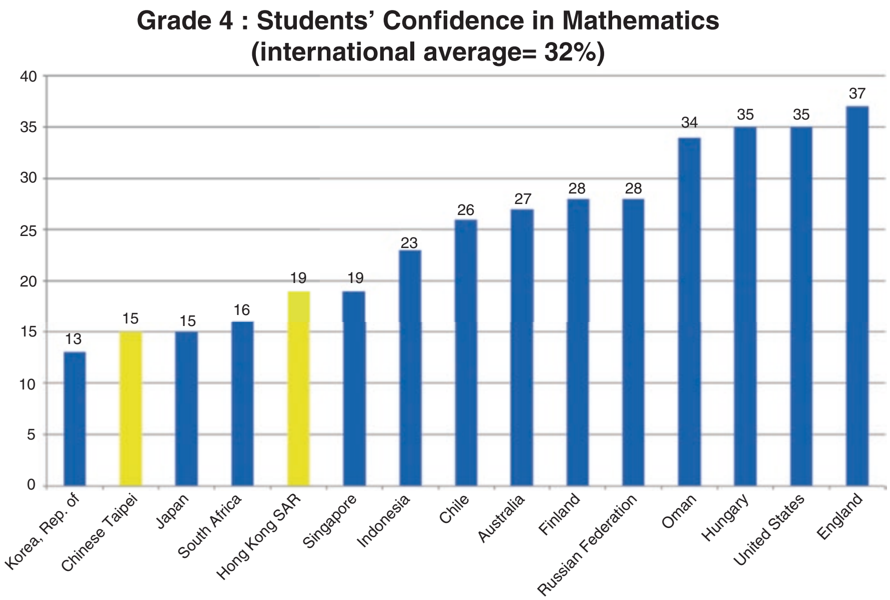

Fig. 18.4 Grade 4 students' confidence in mathematics (TIMSS 2015)

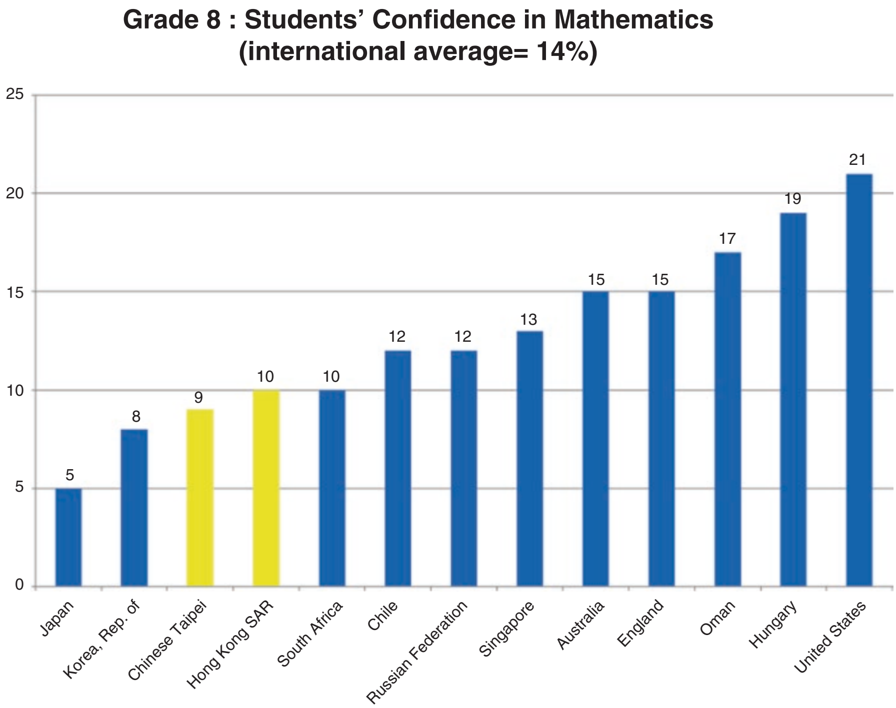

Fig. 18.5 Grade 8 students' confidence in mathematics (TIMSS 2015)

"paradox" of the high achievement and negative attitudes of East Asian students, a paradox which is sometimes referred to as "the Chinese Learners' Paradox" or "East Asian Learners' Paradox". But the Chinese cultural values have not be explored in explaining the negative self-related beliefs. Nor has the cultural perspective been utilized in discussing the findings on the cognitive and non-cognitive competencies of Chinese students in the other chapters of the book.

From the literature, we may summarize the relevant CHC cultural values for explaining achievement as follows (Lai & Leung, 2012; Leung, 2001, 2006; Leung & Park, 2009; Leung, Park, Shimizu, & Xu, 2012):

1. Strong emphasis on the importance of education and high expectation to achieve
2. Examination culture
3. Belief in effort
4. The role of practice and memorization in learning
5. Pragmatic philosophy
6. Reflection
7. The Chinese language

The first six items in this list apply to all subject disciplines of learning, not just to mathematics. However, there is a widespread belief in the universality of mathematics, that is, of all academic subject disciplines, mathematics is the most culture free (although there are scholars such as Bishop (1988) who strongly opposed this view), so the cultural beliefs or elements listed above should apply to mathematics learning the least. Elsewhere the author has argued against this assertion (Leung, 2006) and the arguments will not be repeated here. If cultural values, as the author argued, do affect students' mathematics competencies, then they will surely affect students' achievement in other "less universal" subject disciplines as well.

18.5.2 Language

The last item in the list above, language, may be the most relevant as far as the impact of culture on mathematics learning is concerned. Language is of course an important element of culture, and all experience, including that of mathematics learning, is mediated by language (Gadamer, 1979). As Laborde remarked, "the semantics of an expression are constructed by the student by means of his or her mental representations and of linguistic features of the expression. The role that natural language plays in these processes appears to be very strong" (Laborde, 1990, p.61).

Ng and Rao (2010) argued that the simplicity of Chinese mathematical language and Chinese number system contributes to more efficient learning of mathematics, especially in learning "number structure, counting, and arithmetic operations" (Ng & Rao, 2010, p.190), which are important basic "competencies" for further development and learning in mathematics. There is also emerging evidence from neuroscience research that language is influencing the learning of mathematics. For example, Ge et al. (2015) found that there are commonality and specificity in how language is processed in the brain by native speakers of Chinese and English languages, and Tang et al. (2006) used fMRI to demonstrate a differential cortical representation of numbers between native Chinese and English speakers. Could these differences contribute to the difference between Chinese students and their western counterparts in their mathematics competencies, and in their mathematics achievement more generally?

Of course, the discussion above is not meant to be a substantial treatment on the issue of the influence of culture on Chinese students' (cognitive and non-cognitive) mathematics competencies, but it is the contention of the author that without an appreciation of the Chinese cultural values, and how they differ from values in the West, understanding of the competencies of Chinese students cannot be complete. This cultural lens in interpreting both the development of the Chinese curriculum and the achievement of Chinese students in various kind of mathematics competencies (and in mathematics more generally) is something that needs to be explored further based on the findings of this book so as to arrive at a deeper level of understanding of the mathematics competencies of Chinese students.

18.6 Conclusion and Further Research

This informative book provides a comprehensive introduction and analysis on the mathematics competencies of Chinese students, especially the cognitive competencies. It fills an important research gap in the literature, and should prove to secure a significant place in the literature on mathematics competencies. Based on the findings of the book, further research is suggested, including a more thorough inquiry into the non-cognitive aspect of competencies, something implied by the title of the book but yet not fully actualized.

Another further line of inquiry is to explore into the deep-rooted reasons in explanation of the competencies of Chinese students, something that is touched upon in some chapters in the book, but not fully examined. In particular, since this is a book on the achievements of students in China, naturally we should look into the historical and socio-cultural setting of the Chinese students and the curriculum they follow in order to gain a deep understanding of factors that impact their competencies. While the book provided an informative and systematic introduction to the historical context of Chinese students' mathematics competencies, the socio-cultural background or factors are not given due attention. Hopefully, the content of this book will prompt further research into this interesting and important aspect in seeking an explanation of the mathematics competencies of Chinese students.

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Chapter 19
From "Qingpu Experience" to Investigating Chinese Students' Mathematical Competencies

Lingyuan Gu

Abstract The chapter illustrates the influential mathematics teaching and learning reform in China called the "Qingpu Experiment" which focused on guiding teachers to "learn to teach" and improve the quality of mathematics teaching since the 1970s. Entering the twenty-first century, Qingpu experiment's main contribution is to develop an innovative paradigm for teachers' professional development – action education. In addition, "Qingpu Experiment" created a typical Chinese mathematics teaching model: teaching with variation. The experiment reveals the cultural foundation of "two basics" and "four basics" in Chinese mathematics education. International scholars have shown interests in China's basic education and basic mathematics education through the PISA results since 2009. The chapter addresses that more scientific data should be needed to report the more complete mathematical competencies of Chinese students, and the project in this book helps international scholars to understand Chinese students' mathematical performance.

Keywords Mathematics teaching and learning · Qingpu Experiment · Learn to teach · Quality of mathematics teaching · Professional development · Action education · Teaching with variation · Two basics · Four basics · Scientific data · Mathematical competencies · Chinese students

19.1 Starting from "Qingpu Experiment"

Since 2009, Shanghai students have participated in the PISA tests and ranked the first in mathematics. As a result, Chinese students' mathematics learning and their performance have again received world attention (OECD, 2010). In fact, Chinese students' outstanding performance in mathematics learning is inseparable from

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society, schools, teachers, education administrators, parents, and researchers, who care about mathematics learning in all aspects.

In the 1970s, I worked as a mathematics teaching-research-staff in Qingpu County, Shanghai. At that time, we assessed the basic level of mathematics knowledge for all the regional secondary school graduates. The results showed that for more than 4300 graduates, their average score is only about 11.1%, while the percentage of graduates who scored zero was as high as 23.5% (Gu, 1997). That is the bleakest period in the history of Chinese mathematics education, while the case of Qingpu County can be viewed as a microcosm of the whole China. In order to change such a lagging education situation, we must turn to educational scientific research and carry out reformatory experiments. My research team and I then started an experimental study called the Qingpu Experiment.

After 3 years of investigation, 1 year of screening experience, 3 years of experimentation, and 8 years of promotion, our research team has finally improved the quality of regional mathematics teaching. In 1996, the experience of "Qingpu Experiment" got on the international platform. In particular, at the eighth International Congress on Mathematics Education (ICME-8) which was held in Spain, we reported this reformatory experience of Chinese mathematics education to the scholars from all over the world. In the next 20 years (1977–1997), the mathematics education experiment focused on guiding teachers to "learn to teach" and improve the quality of mathematics teaching. The "Qingpu Experiment" also provides a model for Chinese educational experimental research.

Entering the twenty-first century, under the overall background of Chinese mathematics curriculum reform, Qingpu experiment continues its exploration in practice and launched an experiment entitled "New Century Action of Qingpu Experiment". Its main contribution is to develop an innovative paradigm for teachers' professional development – action education, which intends to help a group of ordinary teachers to reach the professional level of "learning to learn". This achievement has been promoted through the project on the development of Chinese school-based teaching research system and has become an effective way of teacher action learning (Yang & He, 2007). "Action Education" uses lessons as one carrier to stimulate teachers' learning, design, and reflection at different stages. Many cases have shown that the action education practice model can really promote teachers' professional development, and teachers will actively adjust their teaching behaviors from the perspective of students (Yang & He, 2007).

"Qingpu Experiment" created a typical Chinese mathematics teaching model: teaching with variation, which contains a series of effective strategies: (1) using conceptual variation to help students understand mathematical concepts from multiple perspectives, (2) using variation foreshadowing to set up suitable steps in the zone of students' proximal development, (3) using procedural variation to provide a way to resolve mathematical problems, and (4) using the variation expansion of typical examples to construct a hierarchical experience system (Wu & Bao, 2017). The discussion of the Shanghai experience triggered by Shanghai PISA again acknowledged the significance of teaching with variation.

In addition, I also gave a brief description of the experimental research of mathematics education which has been persisted for more than 40 years, aiming at explaining the development of mathematics education in China, accompanied by scholars’ variety of research as well as practitioners’ efforts (Gu, 2014). The research results presented in this book provide another new perspective for the recognition and understanding of Chinese mathematics education.

19.2 Chinese Students’ Mathematics Competencies Need Research

The research team of this book, in the continuous research of more than 10 years, systematically analyzes the traditional characteristics of Chinese mathematics education by using scientific and reasonable research tools, and reveals the cultural foundation of “two basics” and “four basics” in Chinese mathematics education. The “two basics” refer to basic knowledge and basic skills. The “four basics” include basic knowledge, basic skills, basic mathematical thoughts, and methods, as well as basic activity experience. The first chapter of the book points out that the “four basics” in mathematics education in China mainland plays an indispensable role in the development of mathematical core competencies, and its importance may be equivalent to that of cells in humans’ organs. I also strongly agree with the status of the mathematical basic activity experiences discussed in this chapter. The mathematical activity experience that students acquire is based on their basic knowledge, basic skills, and basic thoughts and methods throughout the entire learning process.

Today’s Chinese mathematics curriculum aims to develop the “four basics” while emphasizing the implementation of goals through the development of core competencies. Therefore, how do the current Chinese students perform on the core competencies? PISA tells us the results which we are proud of. However, we believe that Chinese students’ competencies are far more than that. Based on our experience, Chinese students should have special performance on mathematical reasoning and argumentation as well as mathematical problem solving. The research team of this book, based on Chinese unique characteristics, the international development trend, as well as the instructional perspective of mathematical activities, proposes six mathematical competency components: problem posing from a mathematical perspective, mathematical representation and transformation, mathematical reasoning and argumentation, solving problems mathematically, mathematical communication, and mathematical modeling. Although the expressions of these six competencies are not the same as those documented in “Ordinary High School Mathematics Curriculum Standards (2017 version)” issued in 2018, we shall be able to make a good match between the two by a comparison analysis.

Based on the mathematical competencies framework generated by the research, this book conducted a qualitative analysis of the evolution of each competence component from a perspective of historical development, and presents the readers with the development of each competence in the Chinese curricula, which could help to understand the specific connotation of current mathematical competences. For example, the qualitative study of the mathematical reasoning and argumentation in the curriculum documents makes us realize that in the development of Chinese mathematics curriculum, due to the influence of Eastern and Western educational thoughts, inductive methods and logical thinking play an important role in the development of mathematical reasoning and argumentation.

While analyzing the contents of curriculum documents, the research team developed assessment tasks to evaluate the six competencies components based on the constructed mathematical competencies framework. Through stratified sampling, more than 7000 students across the country were tested. It can be seen from the test results that there is a long way to go to achieve the goals of the intended curriculum. For example, based on the experience, we believe that Chinese students' mathematics problem solving ability should be at a relatively high level, but the test results show that Chinese students' ability to solve mathematical problems is as follows: they can connect knowledge and representation (such as charts, texts, symbols, etc.) in different mathematical domains; they can express their thinking processes, the solutions and the results in a brief and logical way; they can explain the meaning of mathematical results corresponding to the situation based on their judgments. However, students are not effective enough to have strategic choices when solving complex problems and they also lack higher-order thinking abilities. These findings are similar to those obtained from the "New Century Action of Qingpu Experiment".

Since the twenty-first century, with the reform of the Chinese mathematics curriculum, students have also shown new features in their ability. The data in this book show that Chinese students have great development in mathematics communication, can understand the meaning of more complex mathematical texts, can express more complex mathematical understandings, can explain others' (correct or wrong) mathematical thoughts and methods, and can make appropriate evaluations.

19.3 The Modern Significance of the Study of Chinese Students' Mathematical Competencies

Based on the characteristics of Chinese mathematics teaching and learning, this book draws on the ideas and methods of international mathematical competencies research and develops an analytical and evaluation framework of mathematical competencies that reflects the characteristics of Chinese mathematics education. The research processes and methods have made some contributions to the research field of students' mathematical competencies. In recent years, most research on mathematical competencies focuses on the connotation and educational value of competencies, while there is no much empirical research on students' performance of mathematical competencies.

In classroom teaching, Chinese teachers pay attention to the development of all students and teach in accordance with students’ abilities. They also expect to have a set of tools to understand the level of students’ mathematical competencies, so that they can have a better organization of their teaching based on students’ cognitive characteristics. The research tools developed by the research team of this book will provide advanced resources for teachers’ school-based training. Although the results here are more of academic-oriented, I believe that the frontline teachers have the ability to understand these research results because of the system of teaching research developed in China as well as the existing “action education” mode.

International scholars have shown interests in China’s basic education and basic mathematics education through the PISA results since 2009. We should have more scientific data to report the more complete mathematical competencies of Chinese students. This research helps international scholars to understand Chinese students’ mathematical performance. I am also honored to receive an invitation from the IPC of ICME-14 to give a plenary lecture about a 40-year reform experiment on mathematics teaching at the 14th International Congress of Mathematics Education held in Shanghai in July 2020, which covers the early Qingpu experiment (1977–1992) and the subsequent measurement of students’ mathematical cognitive ability (1990–2018). The measurements of student’s cognitive abilities have accumulated more than 800,000 standardized data and related micro-experimental materials. The technique of big data factor analysis is used to draw conclusions followed by a discussion of the experience and challenges of Chinese mathematics education. Let us look forward to the gathering in 2021, in Shanghai, at ICME-14.

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Chapter 20
Summary and Conclusion

Binyan Xu

Abstract The chapter points out that the main goal for China to attend PISA test is not to evaluate the quality of education but to better understand present educational situations and questions that deserve the attention. If PISA opens a window for people to access math education in China, the study in this book invites all people to experience another authentic situation about the math educational progress in China and some major achievements of the effort in improving students' core competencies in math. On the one hand, grade 8 students' math competencies were investigated; on the other hand, math-modeling competence was used as an example to first try to investigate students' performance in non-cognitive aspects, including self-related belief or math anxiety and so on. The further study will be focused on non-cognitive factors that raise few people's attention. The conclusion of the investigation of the two aspects reflects the relationships between students' math competence performance and intended curriculum demands.

Keywords Quality of education · Core competencies in math · Math-modeling competence · Students' performance · Non-cognitive aspect · Self-related belief · Math anxiety · Intended curriculum demands · PISA · Grade 8 students

20.1 PISA 2018 and Math Curriculum Reform in China

20.1.1 Chinese Students' Outstanding Math Performance in PISA

Before the draft of the book was finished, we are astonished to learn that PISA 2018 results published by OECD show that the performance of reading, math, science (three core areas of students' competences) of students from two major cities (Beijing and Shanghai) and two provinces (Jiangsu and Zhejiang) ranked the first within all countries and regions involved. The results of PISA show that Chinese

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students who won excellent scores in math also outperformed many other students from other participant regions and nations. Referring to figures of tests of foundational competence, the average math score of students from the four major cities and provinces was 591, ranking the first among all the countries and regions involved.

OECD (2019) reported:"Around one in six 15-year-old students in Beijing, Shanghai, Jiangsu and Zhejiang (China) (16.5%), and about one in seven students in Singapore (13.8%), scored at Level 6 in mathematics, the highest level of proficiency that PISA describes. These students are capable of advanced mathematical thinking and reasoning. On average across OECD countries, only 2.4% of students scored at this level." (p.15)

Chinese students' math performance was outstanding because of comprehensive reasons, including their school teaching and management, their learning input, and family support (Xin et al., 2020).

Despite the outstanding PISA 2018 performance of Chinese students, we also found some distinct problems especially regarding the figures of students' school belonging and the extent of their satisfactions that ranked No. 51 and No. 61 among all the nations involved. Furthermore, the data also reported that those students with lower scores in tests had the least sense of belonging. Albeit, such a result, to some extent, reflected that more effort should be made in supporting students' emotional wellbeing.

The main goal for China to attend PISA test is not to evaluate the quality of education, but to better understand present educational situations and questions that deserve our attention. For example, by taking reference to PISA test experience, we reflected on how to further diagnose and improve the quality of education. We diagnosed students' comprehensive competencies, including their creative thinking and collaborative problem-solving skills that could be applied to interdisciplinary subjects and fields. We also tracked multiple family, school, and personal factors that affect students' performances. Last but not the least, we diagnosed and will further our long-term track on key factors influencing students' study burdens and low sense of happiness.

20.1.2 Key Points of 2018 Math Curriculum Reform

If PISA opens a window for people to access math education in China, this study invites all people caring about math education in China to enter another door. After entering the door, we welcome all of you to experience another authentic situation about the math educational progress in our nation and some major achievements of our effort in improving students' key competencies in math.

Through the literature review, the first chapter introduces the latest characteristics of math curriculum reform and its cultural foundation. Accompanied with the publishing of The "Mathematics Curriculum Standard for Senior Secondary Schools (2017 Edition)" (hereinafter High School Curriculum), innovations and inherited characteristics coexist in Chinese math curriculum reform (Shi, 2018). “Innovation” refers to six math core competences in math high school curriculum that include mathematical abstraction, logical reasoning, mathematical modeling, intuitive imagination, mathematical operation, and data analysis, all of which have become the core goals for the present curriculum.

Another innovation emphasizes test evaluation based on core competences, which raises the evaluation system including “the principle of satisfaction” and “the principle of bonus point”. The biggest characteristic of the new math curriculum reform is “inherence”, which concerns about the relation between math core competences and traditional math education. Chapter 1 systematically analyses the characteristic that relates to the “Four Basic Teaching” cultural characteristics (hereinafter Four Basic) involving the highlight of basic knowledge, basic skills, basic thoughts, and basic activity experience. “Four Basic” accentuates the process of accumulating foundational math knowledge, the practice of math basic skills, and the formation of basic math ideas. Through ubiquitous activities, students gain experiences interwoven with math foundational knowledge, basic skills, and basic ideas, permeating through the whole math learning process.

The cultivation of core math competences is the goal of “Four Basic” teaching. In typical Chinese math classrooms, students struggle in practices for solid foundation and abundant math variations and exploration in which students’ math learning is no longer a fixed, but as a solid and flexible process. Accordingly, students possess huge flexible knowledge and skills to solve new math tasks. Hence, Chinese students (e.g., students in Shanghai) performed outstandingly in math tests (such as PISA).

Indeed, the success of Chinese students’ math learning is also attributed to their hard work and endeavor as non-cognitive factors in PISA Mathematics. Chapter 2 also summarized multiple factors including self-related cognitions in mathematics and dispositions toward mathematics based on the investigation of PISA 2012. The data reported that Shanghai students’ mathematical self-efficacy (M = 0.94, SD = 1.10) is nearly one-half standard deviation higher than that in the second highest system (Singapore: M = 0.47, SD = 1.02). This result suggests that students in Shanghai are more confident when they are facing a mathematics task. The second conclusion of this chapter is that Shanghai works for the second highest level on mathematics work ethic among the top-performing East Asian systems (M = 0.32, SD = 0.02) and it scores about one-third standard deviation higher than the OECD average. It suggests that Shanghai students have relatively high ability in devoting time, hard work, and persistence to attaining mathematics competences. As the whole, Shanghai students’ self-belief about their mathematics learning and dispositions to mathematics has significantly positive impacts on their performance in the mathematics assessment in a direct way.

In addition to these data and Chinese students’ rankings from PISA, we also learn from empirical studies on students’ math performance and competence, which reported students’ math competences from different dimensions. Some relevant chapters in this book make a deep literature review on these empirical studies.

20.2 Chinese Students' Competences and Performance in Math Curriculum Development in China

20.2.1 Reference to Math Competences by High School Curriculum

Studies of this book concur with the ideologies and cores in math curriculum reform in China, focusing on students' math competences. The data of this book and conclusions critically illuminate math curriculum reform in China.

Research of this book raises math competence framework, including math problem posing and solving, mathematical representation and transformation, math reasoning and argumentation, math modeling, and math communication. On one hand, this book took reference to international comparative studies and experience. On the other hand, this book also reflects characteristics of Chinese math teaching practices, i.e., how some math activities are emphasized in Chinese class learning. These activities highly accentuate students' access to basic knowledge, basic skills, and mathematics ways of thinking and simultaneously facilitate students' accumulation of experience in exploration, creation, innovation, and communication in math activities. Chapter 3 gives a detailed narration of the math competence framework.

By taking reference to the connation of core math competences of High School Curriculum, we found connections between two math competence frameworks. First, both frameworks emphasize logic reasoning and math modeling. Shi (2018) argues that the essence of math thinking is logical reasoning, through which math conclusion could be made. In other words, logical reasoning is a process in which students start from premises or fact and follow certain rules to achieve or verify questions. In addition, the essence of math language is math modeling, which applies math competence to real-life situations, thereby constructing the bridge between math and the real world. Hence, in high school, logical reasoning and math modeling are two important components of math competences as well as two frameworks.

Furthermore, High School Curriculum math competences include math abstraction, which demands students to draw connections between different math concepts through studying the relationships within different quantities and graphs, raise general laws and structure via specific contexts and then use math language to express the characterization (MOE, 2018, p.4). Therefore, the cultivation of math abstraction could not go without the grasp of such competences as math problem solving, math representation, and math communication.

High School Curriculum also highlights intuitve imagination competence, which refers to problem-inquiry and solving thinking modes employing the forms of spaces to understand the relationship of object positions, changes in pattern, and laws of motions. It also uses the descriptions of graph to analyze math problems and form the relationships between patterns and figures. It helps to construct the intuitive models of math problems to explore how to solve problems (MOE, 2018, p.6). This competence not only accentuates the use of intuitive models to solve math problems, but also stresses the conversions between forms of math representations. Furthermore, High School Curriculum raises the competence of data analysis which refers to collecting data from research subjects, use math methods to organize, analyze data, make deduction, and form the competence of understanding the knowledge of research subjects (MOE, 2018, p.7). The main embodiments of data analysis are: collecting and organizing data, understanding and analyzing data, gaining and explaining conclusion, summarizing, and forming knowledge. This competence is related to how to raise math problems, math representation, and math communication. Based on the interrelationships between math core competences raised in this book and High School Curriculum, we hereby explain the implications of our studies.

20.2.2 Relationship Between Chinese Students' Math Competence and Expected Curriculum Demands

This book reviews guidelines of math curriculum documents (math curriculum standard or math curriculum outline) in the past one hundred years, summarizing different historical evolutions of math competence connotations in the expected curriculum. On one hand, we tested and investigated grade 8 students' math competence (cognitive) level. On the other hand, we used math-modeling competence as an example to first try to investigate students' performance in non-cognitive aspects, including self-cognition or math anxiety, and so on. Our further study will be focused on non-cognitive factors that raise few people's attention. The conclusion of the investigation of the two aspects reflects the relationships between students' math competence performance and expected curriculum demands. Now we will present the conclusion of our study.

Regarding the competence in posing math problems (see Chaps. 4 and 5), we found that there were not any expected curricula demanding raising questions before twenty-first century. We seldom found the demands for raising question in math abstraction and math expression and communication. Since the Math Curriculum enacted in 2001, more emphasis has been gradually put on students' raising questions from math and life contexts. According to the present math curriculum standard, discovering problems is the foundation of innovation, and therefore, students' initiative in posing problems should be encouraged. Referring to the test data, grade 8 students were weak at posing math problems. For their familiar scenarios, students were more capable of raising questions based on the math concepts and theories that they had grasped but the patterns of their questions were still monotonous. The data illustrate that more effort has been made in cultivating students' awareness in posing problems. However, in the meantime, we also argue that raising less complex questions does not promote students' abilities in innovation.

Regarding math problem-solving competence (see Chaps. 6 and 7), the evolution of hundreds of years curriculum expectations illustrates there has always been definite demand for math problem-solving competence in Chinese math curriculum. From the beginning of twentieth century, the expected curriculum highly regarded students' abilities in solving practical problems or math application questions. Until the end of twentieth century, math curricula began to highlight students' comprehensive knowledge application skills and math problem-solving competence, which reflected the emphasis of "Two Basic" in Chinese math curriculum. The results revealed in the large-scale international comparative studies (IAEP1, IAEP2) showed us that Chinese students were overall among the top performers by international standards (Fan & Zhu, 2004). Since the twenty-first century, the expected curriculum has accentuated the cultivation of high cognitive math problem solving competence, whose cultivation demands are integrated with the formation of math thinking strategies and innovative competence. According to test data, nearly 80% of research participants were capable of connecting and employing math knowledge in different fields and various forms of expression (e.g., diagrams, characters, and symbols) to solve problems and logically expressing their thinking process, ways of solving problems, and results. Nevertheless, the data also show that when facing different problems in different settings, students were easily to mistakenly use the strategies to solve problems that they believed correct. Participant students did not understand strategies in solving problems thoroughly, and therefore the application abilities that math curriculum demands should also be further improved.

Regarding math representation and transformation competences (see Chaps. 8 and 9), the evolution of Chinese expected curricula shows the close integration between the demands of math representation and math transformation, requiring students to employ their math representation and transformation competences to solve relevant math problems. Of all 21st math curriculum requirements after twenty-first century, students not only have been required to face different settings and be capable of using specific forms of math representation (figure, graph, diagram, and symbols, etc.) to express, but also have been demanded to transform unknown/unfamiliar math expressions to known/familiar ones so as to effectively solve math problems. According to the test data, when facing complicated but common problem settings, most students were capable of recognizing familiar math representation form and turning it to another familiar one. Nevertheless, only tiny portions of students are able to construct novel representation form or flexibly change ways of representation and efficiently solve problems.

Regarding the competence of math reasoning and argumentation (see Chaps. 10 and 11), hundreds of years of math curriculum guideline have specific requirements. Math curricula of the beginning of twentieth century emphasized ways of induction, the cultivation of reasoning strategies, and the practicality of argumentation. In the later twentieth century, math curricula accentuated the development of students' logical thinking skills, and the cultivation of logical reasoning and argumentation competence in the process of deduction. After twenty-first century, math curricula highlight innovation and exploration, demanding the cultivation of math reasoning and argumentation competences that combine rational inference and deductive inference. Test data reported that even though students were capable of bravely employing logical inference, concluding supposition, or raising hypotheses, and then further testing hypotheses or opposing hypotheses, they were still hard to get the conclusion. The data demonstrate the match between the cultivation of students' reasoning and argumentation competence and the demand of expected curriculum.

With regard to math communication competence (see Chaps. 13 and 14), it did not appear in the hundreds of years of Chinese math curriculum goal and content. Albeit, since the twenty-first century, the math curricula put more emphasis on students' math expression and communication, aiming at students' more opportunities to express, reflect, and revise math concepts and their capability in using math to communicate and fluently express their reflection on the learning process. The test data show that grade 8 students were competent at math communication, able to recognize and choose information from math texts and interpret the meanings. In the meantime, they were also able to transfer other people's math ideas from one modality to another. The data also indicated that students' math communication competence should be enhanced according to the demands of expected curricula.

Concerning math modeling competence (see Chap. 12), the expected math curriculum of the mid twentieth century demanded that students were able to use math models to solve practical problems given the goal of cultivating students' math knowledge application competence. At that time, the requirements for math modeling focused on the application of known math models. Since the enactment of the math curriculum requirements of the twenty-first century, a new concept on math modeling was raised, requiring students to discover problems and construct models according to authentic settings. Students were demanded to build, examine, and revise models to solve problems in real-life situations. However, the math curriculum requirement did not explain the connections between math modeling and other math content.

In 2018, the issued high school math curriculum requirement firstly made math modeling as the compulsory math curriculum content. All the participants did not experience systematic math modeling training. The data reported that most of the students experienced barriers in the initial stages of math modeling, which means that they were not capable of modeling in authentic scenarios, based on which they felt challenged to raise relevant questions, make assumption, and seek parameters. Therefore, they were unable to construct and examine models and enter the following-up stages of modeling. These data offered reference for us to take appropriate strategies to implement math-modeling teaching.

This study investigated students' non-cognitive dimension of math competence, enabling us to gain more complete understanding of students' math competences.

Chapter 15 looks into students' self-related beliefs about their mathematical modeling, particularly regarding self-efficacy and self-evaluation. The results showed that students' self-efficacy level is consistently higher than their self-evaluation level on the respective modeling task with the largest decrement on the most challenging modeling task. In general, students had a higher level of the two self-related beliefs with easier tasks. Boys did better than girls on modeling tasks and they also held a significantly higher level of self-related beliefs. Compared to between-city differences, students' self-related beliefs varied greater between schools, while the majority of the differences occurred at the individual student level. Moreover, school average modeling performance illustrates a generally larger influence on students' self-related beliefs than students' individual performance.

Chapter 16 looks into students' math anxiety in mathematical modeling ability in China. Results showed that more than half of students worried about the difficulty of the math problems. Girls had slightly higher math anxiety than male students; however, no gender difference was found in mathematical modeling ability. Low and high math anxiety students showed substantial difference in mathematical modeling ability. Moreover, after controlling family socioeconomic status and gender difference, math anxiety also significantly explains the variance of mathematical modeling ability.

20.3 Future Research Project for Chinese Math Curriculum Reform

Our ongoing and nearly completed project has brought us new ideas, which contribute to solutions for present Chinese math curriculum reform. We will proceed with our research in the following aspects:

We will conduct more in-depth research on the historical development of Chinese math curriculum, which may give insights to the transformations in the characteristics of Chinese math curriculum in different historical periods. What factors led to the changes?

Regarding these math competences, how about the performance of primary school and high school children? Whether amendments should be made in these test assignments that were used to investigate the performance of primary school and high school children? If yes, how to make amendments?

The research above discovered students' weakness in posing problems and modeling, all of which are heatedly discussed issues in math education since twenty-first century. These problems have been realized and corresponding requirements have been put forward in the expected math curricula. We also need to ponder about how to turn test accomplishments into practical teaching experience, as a way to improve students' competences in posing problems and math modeling.

Although students performed well in math problem solving, mathematical representation, and math reasoning, how much effort have they made? How to evaluate their effort? What are the influences of non-cognitive factors?

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