Acyclic Coloring Parameterized by Directed Clique-Width

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Abstract. An acyclic r-coloring of a directed graph $G = (V, E)$ is a partition of the vertex set V into r acyclic sets. The dichromatic number of a directed graph G is the smallest r such that G allows an acyclic r-coloring. For symmetric digraphs the dichromatic number equals the well-known chromatic number of the underlying undirected graph. This allows us to carry over the W[1]-hardness and lower bounds for running times of the chromatic number problem parameterized by clique-width to the dichromatic number problem parameterized by directed clique-width. We introduce the first polynomial-time algorithm for the acyclic coloring problem on digraphs of constant directed clique-width. From a parameterized point of view our algorithm shows that the Dichromatic Number problem is in XP when parameterized by directed clique-width and extends the only known structural parameterization by directed modular width for this problem. Furthermore, we apply defineability within monadic second order logic in order to show that Dichromatic Number problem is in FPT when parameterized by the directed clique-width and r. For directed co-graphs, which is a class of digraphs of directed clique-width 2, we even show a linear time solution for computing the dichromatic number.

Keywords: Acyclic coloring · Directed clique-width · Directed co-graphs · Polynomial time algorithms

1 Introduction

In this paper, we consider an approach for coloring the vertices of digraphs. An *acyclic r-coloring* of a digraph $G = (V, E)$ is a partition of the vertex set V into r sets such that all sets induce an acyclic subdigraph in G. The *dichromatic number* of G is the smallest integer r such that G has an acyclic r-coloring. Acyclic colorings of digraphs received a lot of attention in [4,28,29] and also in recent works [26,27,32]. The dichromatic number is one of two basic concepts for the class of perfect digraphs [1] and can be regarded as a natural counterpart of the well known chromatic number for undirected graphs.

In the Dichromatic Number problem (DCN) there is given a digraph G and an integer r and the question is whether G has an acyclic r-coloring. If r is constant

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and not part of the input, the corresponding problem is denoted by DCN_r. Even DCN_2 is NP-complete [12], which motivates to consider the Dichromatic Number problem on special graph classes. Up to now, only few classes of digraphs are known, for which the dichromatic number can be found in polynomial time. The set of DAGs is obviously equal to the set of digraphs of dichromatic number 1. Further, every odd-cycle free digraph [29] and every non-even digraph [27] has dichromatic number at most 2.

The Dichromatic Number problem remains hard even for inputs of bounded directed feedback vertex set size [27]. This result implies that there are no XP-algorithms[1] for the Dichromatic Number problem parameterized by directed width parameters such as directed path-width, directed tree-width, DAG-width or Kelly-width. The first positive result concerning structural parameterizations of the Dichromatic Number problem is the existence of an FPT-algorithm[2] for the Dichromatic Number problem parameterized by directed modular width [31].

In this paper, we introduce the first polynomial-time algorithm for the Dichromatic Number problem on digraphs of constant directed clique-width. Therefore, we consider a directed clique-width expression X of the input digraph G of directed clique-width k. For each node t of the corresponding rooted expression-tree T we use label-based reachability information about the subgraph G_t of the subtree rooted at t. For every partition of the vertex set of G_t into acyclic sets $V_1, \ldots, V_s$ we compute the multi set $\langle \text{reach}(V_1), \ldots, \text{reach}(V_s) \rangle$, where $\text{reach}(V_i)$, $1 \leq i \leq s$, is the set of all label pairs (a, b) such that the subgraph of G_t induced by V_i contains a vertex labeled by b, which is reachable by a vertex labeled by a. By using bottom-up dynamic programming along expression-tree T, we obtain an algorithm for the Dichromatic Number problem of running time $n^{2^{\mathcal{O}(k^2)}}$ where n denotes the number of vertices of the input digraph. Since any algorithm with running time in $n^{2^{o(k)}}$ would disprove the Exponential Time Hypothesis (ETH), the exponential dependence on k in the degree of the polynomial cannot be avoided, unless ETH fails.

From a parameterized point of view, our algorithm shows that the Dichromatic Number problem is in XP when parameterized by directed clique-width. Further, we show that the Dichromatic Number problem is W[1]-hard on symmetric digraphs when parameterized by directed clique-width. Inferring from this, there is no FPT-algorithm for the Dichromatic Number problem parameterized by directed clique-width under reasonable assumptions. The best parameterized complexity, which can be achieved, is given by an XP-algorithm. Furthermore, we apply defineability within monadic second order logic (MSO) in order to show that Dichromatic Number problem is in FPT when parameterized by the directed clique-width and r, which implies that for every integer r it holds that DCN_r is in FPT when parameterized by directed clique-width.

[1] XP is the class of all parameterized problems which can be solved by algorithms that are polynomial if the parameter is considered as a constant [9].

[2] FPT is the class of all parameterized problems which can be solved by algorithms that are exponential only in the size of a fixed parameter while being polynomial in the size of the input size [9].

Since the directed clique-width of a digraph is at most its directed modular width [32], we reprove the existence of an XP-algorithm for DCN and an FPT-algorithm for DCN_r parameterized by directed modular width [31]. On the other hand, there exist several classes of digraphs of bounded directed clique-width and unbounded directed modular width, which implies that directed clique-width is the more powerful parameter and thus, the results of [31] does not imply any parameterized algorithm for directed clique-width.

In Table 1 we summarize the known results for DCN and DCN_r parameterized by width parameters.

Table 1. Complexity of DCN and DCN_r parameterized by width parameters. We assume that P $\neq$ NP. The "///" entries indicate that by taking r out of the instance the considered parameter makes no sense.

Parameter	DCN		DCN_r	
Directed modular width	FPT	[31]	FPT	[31]
Directed clique-width	W[1]-hard XP	Corollary 1 Corollary 3	FPT	Corollary 5
Directed clique-width + r	FPT	Theorem 4	///	
Directed tree-width	$\not\in$ XP	[27]	$\not\in$ XP	[27]
Directed path-width	$\not\in$ XP	[27]	$\not\in$ XP	[27]
DAG-width	$\not\in$ XP	[27]	$\not\in$ XP	[27]
Kelly-width	$\not\in$ XP	[27]	$\not\in$ XP	[27]
Clique-width of $un(G)$	$\not\in$ FPT	by Corollary 1	open	

For directed co-graphs, which is a class of digraphs of directed clique-width 2 [23], we even show a linear time solution for computing the dichromatic number and an optimal acyclic coloring.

2 Preliminaries

We use the notations of Bang-Jensen and Gutin [2] for graphs and digraphs.

2.1 Directed Graphs

A *directed graph* or *digraph* is a pair $G = (V, E)$, where V is a finite set of *vertices* and $E \subseteq \{(u, v) \mid u, v \in V,\ u \neq v\}$ is a finite set of ordered pairs of distinct vertices called *arcs* or *directed edges*. For a vertex $v \in V$, the sets $N^+(v) = \{u \in V \mid (v, u) \in E\}$ and $N^-(v) = \{u \in V \mid (u, v) \in E\}$ are called the *set of all successors* and the *set of all predecessors* of v. The *outdegree* of v, outdegree(v) for short, is the number of successors of v and the *indegree* of v, indegree(v) for short, is the number of predecessors of v.

A digraph $G' = (V', E')$ is a *subdigraph* of digraph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. If every arc of E with both end vertices in V' is in E', we say that G' is an *induced subdigraph* of digraph G and we write $G' = G[V']$.

For some given digraph $G = (V, E)$ we define its *underlying undirected graph* by ignoring the directions of the arcs, i.e. $un(G) = (V, \{\{u, v\} \mid (u, v) \in E, u, v \in V\})$. There are several ways to define a digraph $G = (V, E)$ from an undirected graph $G' = (V, E')$. If we replace every edge $\{u, v\} \in E'$ by

- both arcs (u, v) and (v, u), we refer to G as a *complete biorientation* of G'. Since in this case G is well defined by G' we also denote it by $\overleftrightarrow{G'}$. Every digraph G which can be obtained by a complete biorientation of some undirected graph G' is called a *complete bioriented graph* or *symmetric digraph*.
- one of the arcs (u, v) and (v, u), we refer to G as an *orientation* of G'. Every digraph G which can be obtained by an orientation of some undirected graph G' is called an *oriented graph*.

For a digraph $G = (V, E)$ an arc $(u, v) \in E$ is *symmetric* if $(v, u) \in E$. Thus, each bidirectional arc is symmetric. Further, an arc is *asymmetric* if it is not symmetric. We define the symmetric part of G as $\text{sym}(G)$, which is the spanning subdigraph of G that contains exactly the symmetric arcs of G. Analogously, we define the asymmetric part of G as $\text{asym}(G)$, which is the spanning subdigraph with only asymmetric arcs.

By $\overrightarrow{P_n} = (\{v_1, \ldots, v_n\}, \{(v_1, v_2), \ldots, (v_{n-1}, v_n)\}), n \geq 2$, we denote the directed path on n vertices, by $\overrightarrow{C_n} = (\{v_1, \ldots, v_n\}, \{(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}), n \geq 2$, we denote the directed cycle on n vertices.

A *directed acyclic graph (DAG)* is a digraph without any $\overrightarrow{C_n}$, for $n \geq 2$, as subdigraph. A vertex v is *reachable* from a vertex u in G if G contains a $\overrightarrow{P_n}$ as a subdigraph having start vertex u and end vertex v. A digraph is *odd cycle free* if it does not contain a $\overrightarrow{C_n}$, for odd $n \geq 3$, as subdigraph. A digraph G is planar if $un(G)$ is planar.

A digraph is *even* if for every 0-1-weighting of the edges it contains a directed cycle of even total weight.

2.2 Acyclic Coloring of Directed Graphs

We consider the approach for coloring digraphs given in [29]. A set V' of vertices of a digraph G is called *acyclic* if $G[V']$ is acyclic.

Definition 1 (Acyclic graph coloring [29]). *An* acyclic r-coloring *of a digraph $G = (V, E)$ is a mapping $c : V \to \{1, \ldots, r\}$, such that the color classes $c^{-1}(i)$ for $1 \leq i \leq r$ are acyclic. The* dichromatic number *of G, denoted by $\vec{\chi}(G)$, is the smallest r, such that G has an acyclic r-coloring.*

There are several works on acyclic graph coloring [4,28,29] including several recent works [26,27,32]. The following observations support that the dichromatic number can be regarded as a natural counterpart of the well known chromatic number $\chi(G)$ for undirected graphs G.

Observation 1. *For every symmetric directed graph G it holds that $\vec{\chi}(G) = \chi(un(G))$.*

Observation 2. *For every directed graph G it holds that $\vec{\chi}(G) \leq \chi(un(G))$.*

Observation 3. *Let G be a digraph and H be a subdigraph of G, then $\vec{\chi}(H) \leq \vec{\chi}(G)$.*

Name: Dichromatic Number (DCN)
Instance: A digraph $G = (V, E)$ and a positive integer $r \leq |V|$.
Question: Is there an acyclic r-coloring for G?

If r is a constant and not part of the input, the corresponding problem is denoted by r-Dichromatic Number (DCN_r). Even DCN_2 is NP-complete [12].

3 Acyclic Coloring of Directed Co-graphs

As recently mentioned in [31], only few classes of digraphs for which the dichromatic number can be found in polynomial time are known. The set of DAGs is obviously equal to the set of digraphs of dichromatic number 1. Every odd-cycle free digraph [29] and every non-even digraph [27] has dichromatic number at most 2. Thus, for DAGs, odd-cycle free digraphs, and non-even digraphs the dichromatic number can be computed in linear time. Furthermore, for every perfect digraph the dichromatic number can be found in polynomial time [1].

We next show how to find an optimal acyclic coloring for directed co-graphs, which are defined below, in linear time.

Definition 2 (Directed co-graphs [8]). *The class of* directed co-graphs *is recursively defined as follows.*

1. *Every digraph with a single vertex $(\{v\}, \emptyset)$, denoted by v, is a* directed co-graph.
2. *If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are vertex-disjoint directed co-graphs, then*
   (a) the disjoint union $G_1 \oplus G_2$, which is defined as the digraph with vertex set $V_1 \cup V_2$ and arc set $E_1 \cup E_2$,
   (b) the series composition $G_1 \otimes G_2$, which is defined by their disjoint union plus all possible directed edges between V_1 and V_2, and
   (c) the order composition $G_1 \oslash G_2$, which is defined by their disjoint union plus all possible directed edges from V_1 to V_2, are directed co-graphs.

Every expression X using the four operations of Definition 2 is called a *di-co-expression*. For every directed co-graph we can define a tree structure denoted as *di-co-tree*. This is an ordered rooted tree whose leaves represent the vertices of the digraph and whose inner nodes correspond to the operations applied on the subexpressions defined by the subtrees. For every directed co-graph one can construct a di-co-tree in linear time [8]. Directed co-graphs are interesting from an algorithmic point of view since several hard graph problems can be solved in polynomial time by dynamic programming along the tree structure of the input graph, see [3,18,19].

Lemma 1 (★[3]). *Let G_1 and G_2 be two vertex-disjoint directed graphs. Then, the following equations hold:*

1. $\vec{\chi}(G_1 \oplus G_2) = \max(\vec{\chi}(G_1), \vec{\chi}(G_2))$
2. $\vec{\chi}(G_1 \oslash G_2) = \max(\vec{\chi}(G_1), \vec{\chi}(G_2))$
3. $\vec{\chi}(G_1 \otimes G_2) = \vec{\chi}(G_1) + \vec{\chi}(G_2)$

Lemma 1 can be used to obtain the following result.

Theorem 1. *Let G be a directed co-graph. Then, an optimal acyclic coloring for G and $\vec{\chi}(G)$ can be computed in linear time.*

The *clique number* $\omega_d(G)$ of a digraph G is the number of vertices in a largest complete bioriented subdigraph of G and the *clique number* $\omega(G)$ of a (-n undirected) graph G is the number of vertices in a largest complete subgraph of G. Since the results of Lemma 1 also hold for ω_d instead of $\vec{\chi}$ we obtain the following result.

Proposition 1. *Let G be a directed co-graph. Then, it holds that*

$$\vec{\chi}(G) = \chi(un(sym(G))) = \omega(un(sym(G))) = \omega_d(G)$$

and all values can be computed in linear time.

4 Parameterized Algorithms for Directed Clique-Width

For undirected graphs the clique-width [7] is one of the most important parameters. Clique-width measures how difficult it is to decompose the graph into a special tree-structure. From an algorithmic point of view, only tree-width [30] is a more studied graph parameter. Clique-width is more general than tree-width since graphs of bounded tree-width have also bounded clique-width [5]. The tree-width can only be bounded by the clique-width under certain conditions [22]. Many NP-hard graph problems admit polynomial-time solutions when restricted to graphs of bounded tree-width or graphs of bounded clique-width.

For directed graphs there are several attempts to generalize tree-width such as directed tree-width, DAG-width, or Kelly-width, which are representative for what people are working on, see the surveys [16,17]. Unfortunately, none of these attempts allows polynomial-time algorithms for a large class of problems on digraphs of bounded width [16, Table 2]. This also holds for DCN_r and DCN since even for bounded size of a directed feedback vertex set, deciding whether a

[3] The proofs of the results marked with a ★ are omitted due to space restrictions, see [20].

directed graph has dichromatic number at most 2 is NP-complete [27]. This result rules out XP-algorithms for DCN and DCN_r by directed width parameters such as directed path-width, directed tree-width, DAG-width or Kelly-width, since all of these are upper bounded by the feedback vertex set number.

Next, we discuss parameters which allow XP-algorithms or even FPT-algorithms for DCN and DCN_r. The first positive result concerning structural parameterizations of DCN was recently given in [31] using the directed modular width (dmw).

Theorem 2 ([31]). *The Dichromatic Number problem is in FPT when parameterized by directed modular width.*

By [16], directed clique-width performs much better than directed path-width, directed tree-width, DAG-width, and Kelly-width from the parameterized complexity point of view. Hence, we consider the parameterized complexity of DCN parameterized by directed clique-width.

Definition 3 (Directed clique-width [7]). *The* directed clique-width *of a digraph G, d-cw(G) for short, is the minimum number of labels needed to define G using the following four operations:*

1. *Creation of a new vertex v with label a (denoted by $a(v)$).*
2. *Disjoint union of two labeled digraphs G and H (denoted by $G \oplus H$).*
3. *Inserting an arc from every vertex with label a to every vertex with label b ($a \neq b$, denoted by $\alpha_{a,b}$).*
4. *Change label a into label b (denoted by $\rho_{a \to b}$).*

An expression X built with the operations defined above using k labels is called a directed clique-width k-expression*. Let digraph(X) be the digraph defined by k-expression X.*

In [23] the set of directed co-graphs is characterized by excluding two digraphs as a proper subset of the set of all graphs of directed clique-width 2, while for the undirected versions both classes are equal.

By the given definition every graph of directed clique-width at most k can be represented by a tree structure, denoted as *k-expression-tree*. The leaves of the k-expression-tree represent the vertices of the digraph and the inner nodes of the k-expression-tree correspond to the operations applied to the subexpressions defined by the subtrees. Using the k-expression-tree many hard problems have been shown to be solvable in polynomial time when restricted to graphs of bounded directed clique-width [16,23].

Directed clique-width is not comparable to the directed variants of tree-width mentioned above, which can be observed by the set of all complete biorientations of cliques and the set of all acyclic orientations of grids. The relation of directed clique-width and directed modular width [32] is as follows.

Lemma 2 ([32]). *For every digraph G it holds that d-cw$(G) \leq$ dmw(G).*

On the other hand, there exist several classes of digraphs of bounded directed clique-width and unbounded directed modular width, e.g. even the set of all directed paths $\{\overrightarrow{P_n} \mid n \geq 1\}$, the set of all directed cycles $\{\overrightarrow{C_n} \mid n \geq 1\}$, and the set of all minimal series-parallel digraphs [33]. Thus, the result of [31] does not imply any XP-algorithm or FPT-algorithm for directed clique-width.

Corollary 1. *The Dichromatic Number problem is* W[1]*-hard on symmetric digraphs and thus, on all digraphs when parameterized by directed clique-width.*

Proof. The Chromatic Number problem is W[1]-hard when parameterized by clique-width [13]. An instance consisting of a graph $G = (V, E)$ and a positive integer r for the Chromatic Number problem can be transformed into an instance for the Dichromatic Number problem on digraph $\overleftrightarrow{G}$ and integer r. Then, G has an r-coloring if and only if $\overleftrightarrow{G}$ has an acyclic r-coloring by Observation 1. Since for every undirected graph G its clique-width equals the directed clique-width of $\overleftrightarrow{G}$ [23], we obtain a parameterized reduction. □

Thus, under reasonable assumptions there is no FPT-algorithm for the Dichromatic Number problem parameterized by directed clique-width and an XP-algorithm is the best that can be achieved. Next, we introduce such an XP-algorithm.

Let $G = (V, E)$ be a digraph which is given by some directed clique-width k-expression X. For some vertex set $V' \subseteq V$, we define reach(V') as the set of all pairs (a, b) such that there is a vertex $u \in V'$ labeled by a and there is a vertex $v \in V'$ labeled by b and v is reachable from u in $G[V']$.

Within a construction of a digraph by directed clique-width operations only the edge insertion operation can change the reachability between the present vertices. Next, we show which acyclic sets remain acyclic when performing an edge insertion operation and how the reachability information of these sets have to be updated due to the edge insertion operation.

Lemma 3 (★). *Let $G = (V, E)$ be a vertex labeled digraph defined by some directed clique-width k-expression X, $a \neq b$, $a, b \in \{1, \ldots, k\}$, and $V' \subseteq V$ be an acyclic set in G. Then, vertex set V' remains acyclic in digraph$(\alpha_{a,b}(X))$ if and only if $(b, a) \notin$ reach(V').*

Lemma 4 (★). *Let $G = (V, E)$ be a vertex labeled digraph defined by some directed clique-width k-expression X, $a \neq b$, $a, b \in \{1, \ldots, k\}$, $V' \subseteq V$ be an acyclic set in G, and $(b, a) \notin$ reach(V'). Then, reach(V') for digraph$(\alpha_{a,b}(X))$ can be obtained from reach(V') for digraph(X) as follows:*

- *For every pair $(x, a) \in$ reach(V') and every pair $(b, y) \in$ reach(V'), we extend reach(V') by (x, y).*

For a disjoint partition of V into acyclic sets $V_1, \ldots, V_s$, let $\mathcal{M}$ be the multi set[4] $\langle$reach$(V_1), \ldots,$ reach$(V_s)\rangle$. Let $F(X)$ be the set of all mutually different

[4] We use the notion of a *multi set*, i.e., a set that may have several equal elements. For a multi set with elements $x_1, \ldots, x_n$ we write $\mathcal{M} = \langle x_1, \ldots, x_n\rangle$. The number

multi sets $\mathcal{M}$ for all disjoint partitions of vertex set V into acyclic sets. Every multi set in $F(X)$ consists of nonempty subsets of $\{1,\dots,k\}\times\{1,\dots,k\}$. Each subset can occur 0 times and not more than $|V|$ times. Thus, $F(X)$ has at most

$$(|V|+1)^{2^{k^2}-1} \in |V|^{2^{\mathcal{O}(k^2)}}$$

mutually different multi sets and is polynomially bounded in the size of X.

In order to give a dynamic programming solution along the recursive structure of a directed clique-width k-expression, we show how to compute $F(a(v))$, $F(X \oplus Y)$ from $F(X)$ and $F(Y)$, as well as $F(\alpha_{a,b}(X))$ and $F(\rho_{a\to b}(X))$ from $F(X)$.

Lemma 5 (★). *Let $a,b \in \{1,\dots,k\}$, $a \neq b$.*

1. *$F(a(v)) = \{\langle\{(a,a)\}\rangle\}$.*
2. *Starting with set $D = \{\langle\rangle\} \times F(X) \times F(Y)$ extend D by all triples that can be obtained from some triple $(\mathcal{M},\mathcal{M}',\mathcal{M}'') \in D$ by removing a set L' from $\mathcal{M}'$ or a set L'' from $\mathcal{M}''$ and inserting it into $\mathcal{M}$, or by removing both sets and inserting $L' \cup L''$ into $\mathcal{M}$. Finally, we choose $F(X \oplus Y) = \{\mathcal{M} \mid (\mathcal{M},\langle\rangle,\langle\rangle) \in D\}$.*
3. *$F(\alpha_{a,b}(X))$ can be obtained from $F(X)$ as follows. First, we remove from $F(X)$ all multi sets $\langle L_1,\dots,L_s\rangle$ such that $(b,a) \in L_t$ for some $1 \le t \le s$. Afterwards, we modify every remaining multi set $\langle L_1,\dots,L_s\rangle$ in $F(X)$ as follows:*
   - *For every L_i which contains a pair (x,a) and a pair (b,y), we extend L_i by (x,y).*
4. *$F(\rho_{a\to b}(X)) = \{\langle\rho_{a\to b}(L_1),\dots,\rho_{a\to b}(L_s)\rangle \mid \langle L_1,\dots,L_s\rangle \in F(X)\}$, where we use $\rho_{a\to b}(L_i) = \{(\rho_{a\to b}(c),\rho_{a\to b}(d)) \mid (c,d) \in L_i\}$ and $\rho_{a\to b}(c) = b$, if $c = a$, and $\rho_{a\to b}(c) = c$, if $c \neq a$.*

Since every possible coloring of G is realized in the set $F(X)$, where X is a directed clique-width k-expression for G, it is easy to find a minimum coloring for G.

Corollary 2. *Let $G = (V,E)$ be a digraph given by a directed clique-width k-expression X. There is a partition of V into r acyclic sets if and only if there is some $\mathcal{M} \in F(X)$ consisting of r sets of label pairs.*

Theorem 3. *The Dichromatic Number problem on digraphs on n vertices given by a directed clique-width k-expression can be solved in $n^{2^{\mathcal{O}(k^2)}}$ time.*

Proof. Let $G = (V,E)$ be a digraph of directed clique-width at most k and T be a k-expression-tree for G with root w. For some vertex u of T we denote by T_u the subtree rooted at u and X_u the k-expression defined by T_u. In order to solve

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how often an element x occurs in $\mathcal{M}$ is denoted by $\psi(\mathcal{M},x)$. Two multi sets $\mathcal{M}_1$ and $\mathcal{M}_2$ are *equal* if for each element $x \in \mathcal{M}_1 \cup \mathcal{M}_2$, $\psi(\mathcal{M}_1,x) = \psi(\mathcal{M}_2,x)$, otherwise they are called *different*. The empty multi set is denoted by $\langle\rangle$.

the Dichromatic Number problem for G, we traverse k-expression-tree T in a bottom-up order. For every vertex u of T we compute $F(X_u)$ following the rules given in Lemma 5. By Corollary 2 we can solve our problem by $F(X_w) = F(X)$.

Our rules given Lemma 5 show the following running times. For every $v \in V$ and $a \in \{1, \ldots, k\}$ set $F(a(v))$ can be computed in $\mathcal{O}(1)$. The set $F(X \oplus Y)$ can be computed in time $(n+1)^{3(2^{k^2}-1)} \in n^{2^{\mathcal{O}(k^2)}}$ from $F(X)$ and $F(Y)$. The sets $F(\alpha_{a,b}(X))$ and $F(\rho_{a\to b}(X))$ can be computed in time $(n+1)^{2^{k^2}-1} \in n^{2^{\mathcal{O}(k^2)}}$ from $F(X)$.

In order to bound the number and order of operations within directed clique-width expressions, we can use the normal form for clique-width expressions defined in [11]. The proof of Theorem 4.2 in [11] shows that also for directed clique-width expression X, we can assume that for every subexpression, after a disjoint union operation first there is a sequence of edge insertion operations followed by a sequence of relabeling operations, i.e. between two disjoint union operations there is no relabeling before an edge insertion. Since there are n leaves in T, we have $n-1$ disjoint union operations, at most $(n-1)\cdot(k-1)$ relabeling operations, and at most $(n-1)\cdot k(k-1)$ edge insertion operations. This leads to an overall running time of $n^{2^{\mathcal{O}(k^2)}}$. □

The running time shown in Theorem 3 leads to the following result.

Corollary 3. *The Dichromatic Number problem is in XP when parameterized by directed clique-width.*

Up to now there are only very few digraph classes for which we can compute a directed clique-width expression in polynomial time. This holds for directed co-graphs, digraphs of bounded directed modular width, and orientations of trees. For such classes we can apply the result of Theorem 3. In order to find directed clique-width expressions for general digraphs one can use results on the related parameter bi-rank-width [24]. By [2, Lemma 9.9.12] we can use approximate directed clique-width expressions obtained from rank-decomposition with the drawback of a single-exponential blow-up on the parameter.

Next, we give a lower bound for the running time of parameterized algorithms for Dichromatic Number problem parameterized by the directed clique-width.

Corollary 4. *The Dichromatic Number problem on digraphs on n vertices parameterized by the directed clique-width k cannot be solved in time $n^{2^{o(k)}}$, unless ETH fails.*

Proof. In order to show the statement we apply the following lower bound for the Chromatic Number problem parameterized by clique-width given in [14]. Any algorithm for the Chromatic Number problem parameterized by clique-width with running in $n^{2^{o(k)}}$ would disprove the Exponential Time Hypothesis. By Observation 1 and since for every undirected graph G its clique-width equals the directed clique-width of $\overleftrightarrow{G}$ [23], any algorithm for the Dichromatic Number problem parameterized by directed clique-width can be used to solve the Chromatic Number problem parameterized by clique-width. □

In order to show fixed parameter tractability for DCN_r w.r.t. the parameter directed clique-width one can use its defineability within monadic second order logic (MSO). We restrict to MSO_1-logic, which allows propositional logic, variables for vertices and vertex sets of digraphs, the predicate $\text{arc}(u, v)$ for arcs of digraphs, and quantifications over vertices and vertex sets [6]. In [16, Theorem 4.2] it has been shown that for every integer k and MSO_1 formula ψ, every ψ-LinEMSO_1 optimization problem (see [16]) is fixed-parameter tractable on digraphs of clique-width k w.r.t. the parameters k and length of the formula $|\psi|$. Next, we will apply this result to DCN.

Theorem 4. *The Dichromatic Number problem is in FPT when parameterized by directed clique-width and r.*

Proof. Let $G = (V, E)$ be a digraph. We can define DCN_r by an MSO_1 formula

$$\psi = \exists V_1, \ldots, V_r : \left(\text{Partition}(V, V_1, \ldots, V_r) \wedge \bigwedge_{1 \leq i \leq r} \text{Acyclic}(V_i) \right)$$

with

$$\begin{aligned} \text{Partition}(V, V_1, \ldots, V_r) = \forall v \in V : (\textstyle\bigvee_{1 \leq i \leq r} v \in V_i) \wedge \\ \nexists v \in V : (\textstyle\bigvee_{i \neq j,\ 1 \leq i,j \leq r} (v \in V_i \wedge v \in V_j)) \end{aligned}$$

and

$$\text{Acyclic}(V_i) = \forall V' \subseteq V_i, V' \neq \emptyset : \exists v \in V' (\text{outdegree}(v) = 0 \vee \text{outdegree}(v) \geq 2)$$

For the correctness we note the following. For every induced cycle V' in G it holds that for every vertex $v \in V'$ we have $\text{outdegree}(v) = 1$ in G. This does not hold for non-induced cycles. But since for every cycle V'' in G there is a subset $V' \subseteq V''$, such that $G[V']$ is a cycle, we can verify by $\text{Acyclic}(V_i)$ whether $G[V_i]$ is acyclic. Since it holds that $|\psi| \in \mathcal{O}(r)$, the statement follows by the result of [16] stated above. □

Corollary 5. *For every integer r the r-Dichromatic Number problem is in FPT when parameterized by directed clique-width.*

5 Conclusions and Outlook

The presented methods allow us to compute the dichromatic number on directed co-graphs in linear time and on graph classes of bounded directed clique-width in polynomial time.

The shown parameterized solutions of Corollary 3 and Theorem 4 also hold for any parameter which is larger or equal than directed clique-width, such as the parameter directed modular width [32] (which even allows an FPT-algorithm by [31,32]) and directed linear clique-width [21].

Further, the hardness result of Corollary 1 rules out FPT-algorithms for the Dichromatic Number problem parameterized by width parameters which can be bounded by directed clique-width. Among these are the clique-width and rank-width of the underlying undirected graph, which also have been considered in [15] on the Oriented Chromatic Number problem.

From a parameterized point of view width parameters are so-called structural parameters, which are measuring the difficulty of decomposing a graph into a special tree-structure. Beside these, the standard parameter, i.e. the threshold value given in the instance, is well studied. Unfortunately, for the Dichromatic Number problem the standard parameter is the number of necessary colors r and does even not allow an XP-algorithm, since DCN_2 is NP-complete [27]. A positive result can be obtained for parameter "number of vertices" n. Since integer linear programming is fixed-parameter tractable for the parameter "number of variables" [25] the existence of an integer program for DCN using $\mathcal{O}(n^2)$ variables implies an FPT-algorithm for parameter n, see [20].

It remains to verify whether the running time of our XP-algorithm for DCN can be improved to $n^{2^{\mathcal{O}(k)}}$, which is possible for the Chromatic Number problem by [10]. Further, it remains open whether the hardness of Corollary 1 also holds for special digraph classes and for directed linear clique-width [21]. Additionally, the existence of an FPT-algorithm for DCN_r w.r.t. parameter clique-width of the underlying undirected graph is open.

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