



On Parameterized Complexity of Liquid Democracy

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Abstract. In *liquid democracy*, each voter either votes herself or delegates her vote to some other voter. This gives rise to what is called a delegation graph. To decide the voters who eventually votes along with the subset of voters whose votes they give, we need to *resolve* the cycles in the delegation graph. This gives rise to the RESOLVE DELEGATION TO MINMAXWEIGHT problem where we need to find an acyclic sub-graph of the delegation graph such that the number of voters whose votes they give is bounded above by some integer λ . Putting a cap on the number of voters whose votes a voter gives enable the system designer restrict the power of any individual voter. The RESOLVE DELEGATION TO MINMAXWEIGHT problem is already known to be NP-hard. In this paper we study the parameterized complexity of this problem. We show that RESOLVE DELEGATION TO MINMAXWEIGHT is para-NP-hard with respect to parameters λ , number of sink nodes and the maximum degree of the delegation graph. We also show that RESOLVE DELEGATION TO MINMAXWEIGHT is W[1]-hard even with respect to the treewidth of the delegation graph. We complement our negative results by exhibiting FPT algorithms with respect to some other parameters. We finally show that a related problem, which we call RESOLVE FRACTIONAL DELEGATION, is polynomial time solvable.

Keywords: Liquid democracy · RESOLVE DELEGATION TO MINMAXWEIGHT · Parameterized complexity

1 Introduction

In a *direct democracy*, agents vote for a candidate by themselves. In *liquid democracy*, the voters can delegate their votes to other agents who can vote on their behalf. Suppose voter 1 delegates her vote to voter 2 and voters 2 and 3 delegate their votes to voter 4. Then voter 4 has a voting power equivalent to 4 individual votes. That is delegations are transitive. This particular feature can make liquid democracy a disruptive approach to democratic voting system. This happens because such a voting system can lead to what we call a super-voter who has a lot of voting power. So now the candidates instead of trying to appease the general public can do behind the closed door dealings with the super-voters and try

to win the election in an unfair manner. In order to deal with this issue, a central mechanism ensures that no super-voter has a lot of voting power. Formally we do it as follows. We create a delegation graph where the set of vertices is the set of voters and we have a directed edge from vertex i to vertex j if voter i delegates her vote to voter j . We observe that delegation graph may contain cycles. Every voter is also allowed to delegate her vote to more than one other voters and let the system decide her final delegation. We use a central mechanism to find a acyclic sub-graph of the delegation graph such that no super-voter (the vertices having out-degree 0) has a lot of voting power. We call this problem RESOLVE DELEGATION TO MINMAXWEIGHT.

1.1 Related Work

An empirical investigation of the existence and influence of super-voters was done by [9]. They showed that the super-voters can be powerful although they seem to act in a responsible manner according to their results. There have been a few theoretical work in this area by [4, 7] and [8]. A detailed theoretical work especially on the approximation algorithms in this setting was done by [6]. Some other important work in Liquid democracy includes [2] and [3].

1.2 Our Contribution

We study parameterized complexity of the RESOLVE DELEGATION TO MINMAXWEIGHT problem with respect to various natural parameters. In particular, we consider the number of sink vertices (t), maximum allowed weight λ of any sink in the final delegation graph, maximum degree (Δ), tree-width, number of edges deleted in optimal solution (e_{rem}), number of non-sink vertices ($|\mathcal{V} \setminus \mathcal{J}|$). The number of sink vertices corresponds to the number of influential voters which is often a small number in practice. This makes the number of sink vertices an important parameter to study. Similarly, the parameter λ corresponds to the “power” of a voter. Since the input to the problem is a graph, it is natural to study parameters, for example, tree-width (by ignoring the directions of the edges) and the number of edges that one needs to delete in an optimal solution. We summarize our results in Table 1. We finally show that RESOLVE DELEGATION TO MINMAXWEIGHT is polynomial time solvable if we allow fractional delegations [Theorem 6].

2 Preliminaries

A directed graph \mathcal{G} is a tuple $(\mathcal{V}, \mathcal{E})$ where $\mathcal{E} \subseteq \{(x, y) : x, y \in \mathcal{V}, x \neq y\}$. For a graph \mathcal{G} , we denote its set of vertices by $\mathcal{V}[\mathcal{G}]$, its set of edges by $\mathcal{E}[\mathcal{G}]$, the number of vertices by n , and the number of edges by m . Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a sub-graph $\mathcal{H} = (\mathcal{V}', \mathcal{E}')$ is a graph such that (i) $\mathcal{V}' \subseteq \mathcal{V}$, (ii) $\mathcal{E}' \subseteq \mathcal{E}$, and (iii) for every $(x, y) \in \mathcal{E}'$, we have $x, y \in \mathcal{V}'$. A sub-graph \mathcal{H} of a graph \mathcal{G} is called a *spanning sub-graph* if $\mathcal{V}[\mathcal{H}] = \mathcal{V}[\mathcal{G}]$ and *induced sub-graph* if $\mathcal{E}[\mathcal{H}] = \{(x, y) \in \mathcal{E}[\mathcal{G}] : x, y \in \mathcal{V}[\mathcal{H}]\}$. Given an induced path P of a graph, we define *end vertex* as vertex with 0 outdegree in P and *start vertex* as a vertex with 0 indegree in P .

Table 1. Summary of results.

<i>Parameter</i>	<i>Result</i>
\mathfrak{t}	para-NP-hard [Theorem 1]
(λ, Δ)	para-NP-hard [Theorem 2]
(λ, \mathfrak{t})	quadratic vertex kernel [Observation 1]
tree-width	W[1]-Hard [Theorem 3]
e_{rem}	FPT by bounded search tree technique [Theorem 5]
$ \mathcal{V} \setminus \mathcal{T} $	FPT by bounded search tree technique [Theorem 4]
<i>Problem under Assumption</i>	<i>Result</i>
fractional delegation	Reduction to LP [Theorem 6]
DAG, Bipartite Graph	W[1]-Hard w.r.t treewidth [Corollary 2]
DAG, Bipartite Graph	para-NP-hard w.r.t λ, Δ [Corollary 1]

2.1 Problem Definition

We now define our problem formally.

Definition 1 (RESOLVE DELEGATION TO MINMAXWEIGHT). *Given a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ (also known as delegation graph) with the set $\mathcal{T} \subseteq \mathcal{V}$ as its set of sink vertices and an integer λ , decide if there exists a spanning sub-graph $\mathcal{H} \subseteq \mathcal{G}$ such that*

- (i) *The out-degree of every vertex in $\mathcal{V} \setminus \mathcal{T}$ is exactly 1*
- (ii) *For every sink vertex $\mathfrak{t} \in \mathcal{T}$, the number of vertices (including \mathfrak{t}) in \mathcal{V} which has a path to \mathfrak{t} in the sub-graph \mathcal{H} is at most λ*

We denote an arbitrary instance of RESOLVE DELEGATION TO MINMAXWEIGHT by (\mathcal{G}, λ) .

In the spanning sub-graph $\mathcal{H} \subseteq \mathcal{G}$, if there is a path from u to v in \mathcal{H} such that all the vertices on this path except v has out-degree 1, then we say that vertex u *delegates* to vertex v . In any spanning sub-graph $\mathcal{H} \subseteq \mathcal{G}$ with the out-degree of every vertex in $\mathcal{V} \setminus \mathcal{T}$ is exactly 1 (we call sub-graph \mathcal{H} a feasible solution), weight of a tree rooted at the sink vertex u is the number of vertices (including u) that have a directed path to u . We study parameterized complexity of RESOLVE DELEGATION TO MINMAXWEIGHT with respect to \mathfrak{t} , λ , and the maximum degree Δ of the input graph as our parameters. In the optimization version of RESOLVE DELEGATION TO MINMAXWEIGHT, we aim to minimize λ .

3 Results: Algorithmic Hardness

Our first result shows that RESOLVE DELEGATION TO MINMAXWEIGHT is NP-complete even if we have only 3 sink vertices. For that, we exhibit reduction from the TWO VERTEX DISJOINT PATHS problem.

Definition 2 (TWO VERTEX DISJOINT PATHS). *Given a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, two pairs (s_1, t_1) and (s_2, t_2) of vertices which are all different from each other, compute if there exists two vertex disjoint paths \mathcal{P}_1 and \mathcal{P}_2 where \mathcal{P}_i is a path from s_i to t_i for $i \in [2]$. We denote an arbitrary instance of it by $(\mathcal{G}, s_1, t_1, s_2, t_2)$.*

We know that TWO VERTEX DISJOINT PATHS is NP-complete [5]. The idea is to add paths containing large number of nodes in the instance of RESOLVE DELEGATION TO MINMAXWEIGHT which we are creating using the instance of TWO VERTEX DISJOINT PATHS. This key idea will make both the instances equivalent.

Theorem 1. *The RESOLVE DELEGATION TO MINMAXWEIGHT problem is NP-complete even if we have only 3 sink vertices. In particular, RESOLVE DELEGATION TO MINMAXWEIGHT is para-NP-hard with respect to the parameter t .*

Proof. The RESOLVE DELEGATION TO MINMAXWEIGHT problem clearly belongs to NP. To show its NP-hardness, we reduce from TWO VERTEX DISJOINT PATHS. Let $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), s_1, t_1, s_2, t_2)$ be an arbitrary instance of TWO VERTEX DISJOINT PATHS. Let $n = |\mathcal{V}|$. We consider the following instance $(\mathcal{G}' = (\mathcal{V}', \mathcal{E}'), \lambda)$.

$$\begin{aligned} \mathcal{V}' &= \{a_v : v \in \mathcal{V}\} \cup \mathcal{D}_1 \cup \mathcal{D}'_1 \cup \mathcal{D}_2 \cup \mathcal{D}'_2 \cup \mathcal{D}_3 \text{ where} \\ |\mathcal{D}_1| &= |\mathcal{D}'_1| = 10n, |\mathcal{D}'_1| = |\mathcal{D}_2| = 5n, |\mathcal{D}_3| = 15n \\ \mathcal{E}' &= \{(a_u, a_v) : (u, v) \in \mathcal{E}\} \cup \mathcal{F} \end{aligned}$$

We now describe the edges in \mathcal{F} . Each $\mathcal{D}_1, \mathcal{D}'_1, \mathcal{D}_2, \mathcal{D}'_2$ and \mathcal{D}_3 induces a path in \mathcal{G}' and thus the edges in these paths are part of \mathcal{F} . The end vertices of the path induced on \mathcal{D}_1 and \mathcal{D}_2 be respectively d_1 and d_2 . The start vertices of the path induced on \mathcal{D}'_1 and \mathcal{D}'_2 be respectively d'_1 and d'_2 . The end vertices of the path induced on $\mathcal{D}'_1, \mathcal{D}'_2$ and \mathcal{D}_3 be t'_1, t'_2 and t'_3 respectively. The set \mathcal{F} also contains the edges in $\{(d_1, a_{s_1}), (d_2, a_{s_2}), (a_{t_1}, d'_1), (a_{t_2}, d'_2)\}$. \mathcal{F} also contains edge $(a_v, t'_3) \forall v \in \mathcal{V}$. This finishes the description of \mathcal{F} and thus the description of \mathcal{G}' . We observe that \mathcal{G}' has exactly 3 sink vertices, namely t'_1, t'_2 and t'_3 . Finally we define $\lambda = 17n$. We claim that the two instances are equivalent.

In one direction, let us assume that the TWO VERTEX DISJOINT PATHS instance is a YES instance. For all $i \in [2]$, let \mathcal{P}_i be a path from s_i to t_i in \mathcal{G} such that \mathcal{P}_1 and \mathcal{P}_2 are vertex disjoint. We build the solution \mathcal{H} for RESOLVE DELEGATION TO MINMAXWEIGHT by first adding the set of edges $\{(u, v) | \text{outdegree of } u \text{ is } 1\}$. Then we add the paths \mathcal{P}_1 and \mathcal{P}_2 . Then we add the edges $(a_{t_1}, d'_1), (a_{t_2}, d'_2)$. Then for each vertex u in the set $\mathcal{V}' = \{a_v | v \in \mathcal{V}\} \cup \mathcal{V}[\mathcal{P}_1 \cup \mathcal{P}_2]$, add the edge (u, t'_3) to \mathcal{H} .

We observe that the out degree of every vertex is exactly 1 in \mathcal{H} except the sink vertices in \mathcal{G}' (which are t'_1, t'_2 and t'_3). Also since \mathcal{H} contains the path \mathcal{P}_i , every vertex in \mathcal{D}_i has a path to t'_i for $i \in [2]$. Of course, every vertex in \mathcal{D}'_i has a path to t'_i for $i \in [2]$ and every vertex in \mathcal{D}_3 delegates to t'_3 . Hence $\forall i \in [3]$,

the number of vertices which has a path to t_i in \mathcal{H}' is at most $16n$ which is less than λ . Hence the RESOLVE DELEGATION TO MINMAXWEIGHT instance is a YES instance.

In the other direction, let us assume that the RESOLVE DELEGATION TO MINMAXWEIGHT instance is a YES instance. Let $\mathcal{H}' = (\mathcal{V}', \mathcal{E}'') \subseteq \mathcal{G}'$ be a spanning sub-graph of \mathcal{G}' such that (i) the out degree of every vertex which is not a sink is exactly 1, (ii) there are at most λ ($= 17n$) vertices (including the sink nodes) in \mathcal{H}' which has a path to t'_i for $i \in [3]$. Note that a_{s_1} must have a path \mathcal{P}'_1 to a_{t_1} in \mathcal{H}' otherwise at least $20n$ vertices have path to either t'_2 or t'_3 in \mathcal{H}' which is a contradiction (since $\lambda = 17n$). Similarly a_{s_2} must have a path \mathcal{P}'_2 to a_{t_2} in \mathcal{H}' otherwise at least $20n$ vertices have path to either t'_1 or t'_3 in \mathcal{H}' which is a contradiction (since $\lambda = 17n$). Since, for $i \in [2]$, we have a path \mathcal{P}'_i from a_{s_i} to a_{t_i} in \mathcal{H}' and the out-degree of every vertex in \mathcal{H}' except t'_1, t'_2 and t'_3 is 1, the paths \mathcal{P}'_1 and \mathcal{P}'_2 are vertex disjoint. We define path $\mathcal{P}_i = \{(u, v) : (a_u, a_v) \in \mathcal{P}'_i\}$ in \mathcal{G} for $i \in [2]$. Since \mathcal{P}'_1 and \mathcal{P}'_2 are vertex disjoint, it follows that \mathcal{P}_1 and \mathcal{P}_2 are also vertex disjoint. Thus the RESOLVE DELEGATION TO MINMAXWEIGHT instance is a YES instance. \square

We next show that RESOLVE DELEGATION TO MINMAXWEIGHT is NP-complete even if we have $\lambda = 3$ and $\Delta = 3$. For that we exhibit a reduction from (3, B2)-SAT which is known to be NP-complete [1].

Definition 3 ((3, B2)-SAT). *Given a set $\mathcal{X} = \{x_i : i \in [n]\}$ of n variables and a set $\mathcal{C} = \{C_j : j \in [m]\}$ of m 3-CNF clauses on \mathcal{X} such that, for every $i \in [n]$, x_i and \bar{x}_i each appear in exactly 2 clauses, compute if there exists any Boolean assignment to the variables which satisfy all the m clauses simultaneously. We denote an arbitrary instance of (3, B2)-SAT by $(\mathcal{X}, \mathcal{C})$.*

For each literal and clause in (3, B2)-SAT we add a node in the instance of RESOLVE DELEGATION TO MINMAXWEIGHT and we add some special set of edges and nodes so that $\lambda = 3$ and both the out-degree and in-degree of every vertex is at most 3

Theorem 2. *The RESOLVE DELEGATION TO MINMAXWEIGHT problem is NP-complete even if we have $\lambda = 3$ and both the out-degree and in-degree of every vertex is at most 3. In particular, RESOLVE DELEGATION TO MINMAXWEIGHT is para-NP-hard with respect to the parameter (λ, Δ) .*

Proof. The RESOLVE DELEGATION TO MINMAXWEIGHT problem clearly belongs to NP. To show its NP-hardness, we reduce from (3, B2)-SAT. Let $(\mathcal{X} = \{x_i : i \in [n]\}, \mathcal{C} = \{C_j : j \in [m]\})$ be an arbitrary instance of (3, B2)-SAT. We define a function $f : \{x_i, \bar{x}_i : i \in [n]\} \longrightarrow \{a_i, \bar{a}_i : i \in [n]\}$ as $f(x_i) = a_i$ and $f(\bar{x}_i) = \bar{a}_i$ for $i \in [n]$. We consider the following instance $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), \lambda)$.

$$\begin{aligned} \mathcal{V} &= \{a_i, \bar{a}_i, d_{i,1}, d_{i,2} : i \in [n]\} \cup \{y_j : j \in [m]\} \\ \mathcal{E} &= \{(y_j, f(l_1^j)), (y_j, f(l_2^j)), (y_j, f(l_3^j)) : C_j = (l_1^j \vee l_2^j \vee l_3^j), j \in [m]\} \\ &\quad \cup \{(d_{i,2}, d_{i,1}), (d_{i,1}, a_i), (d_{i,1}, \bar{a}_i) : i \in [n]\} \\ \lambda &= 3 \end{aligned}$$

We observe that both the in-degree and out-degree of every vertex in \mathcal{G} is at most 3. Also $\Delta = 3$. We now claim that the two instances are equivalent.

Suppose the (3, B2)-SAT instance is a YES instance. Let $g : \{x_i : i \in [n]\} \rightarrow \{\text{TRUE}, \text{FALSE}\}$ be a satisfying assignment of the (3, B2)-SAT instance. We define another function $h(g, j) = f(l, j) \in \{m\}$, for some literal l which appears in the clause C_j and g sets it to TRUE. We consider the following sub-graph $\mathcal{H} \subseteq \mathcal{G}$

$$\begin{aligned} \mathcal{E}[\mathcal{H}] = & \{(d_{i,2}, d_{i,1}) : i \in [n]\} \\ & \cup \{(d_{i,1}, a_i) : i \in [n], g(x_i) = \text{FALSE}\} \\ & \cup \{(d_{i,1}, \bar{a}_i) : i \in [n], g(x_i) = \text{TRUE}\} \\ & \cup \{(y_j, h(g, j)) : j \in [m]\} \end{aligned}$$

We observe that \mathcal{H} is a spanning sub-graph of \mathcal{G} such that (i) every non-sink vertices in \mathcal{G} has exactly one outgoing edge in \mathcal{H} and (ii) for each sink vertex in \mathcal{G} , there are at most 3 vertices (including the sink itself) which has a path to it. Hence the RESOLVE DELEGATION TO MINMAXWEIGHT instance is a YES instance.

In the other direction, let the RESOLVE DELEGATION TO MINMAXWEIGHT instance is a YES instance. Let $\mathcal{H} \subseteq \mathcal{G}$ be a sub-graph of \mathcal{G} such that (i) every non-sink vertices in \mathcal{G} has exactly one outgoing edge in \mathcal{H} and (ii) for each sink vertex in \mathcal{G} , there are at most 3 vertices (including the sink itself) which has a path to it. We define an assignment $g : \{x_i : i \in [n]\} \rightarrow \{\text{TRUE}, \text{FALSE}\}$ as $g(x_i) = \text{FALSE}$ if $(d_{i,1}, a_i) \in \mathcal{E}[\mathcal{H}]$ and TRUE otherwise. We claim that g is a satisfying assignment for the (3, B2)-SAT instance. Suppose not, then there exists a clause $C_j = (l_1^j \vee l_2^j \vee l_3^j)$ for some $j \in [m]$ whom g does not satisfy. We define functions $f_1, f_2 : \{x_i, \bar{x}_i : i \in [n]\} \rightarrow \{d_{i,1}, d_{i,2} : i \in [n]\}$ as $f_1(x_i) = f_1(\bar{x}_i) = d_{i,1}$ and $f_2(x_i) = f_2(\bar{x}_i) = d_{i,2}$. We observe that the sink vertex $f(l_1^j)$ is reachable from both $f_1(l_1^j)$ and $f_2(l_1^j)$ in \mathcal{H} for every $i \in [3]$. Since $\lambda = 3$, we do not have a path from y_j to any of $f(l_i), i \in [3]$ which is a contradiction since the non-sink vertex y_j must have out-degree 1 in \mathcal{H} . Hence g is a satisfying assignment for the (3, B2)-SAT instance and thus the instance is a YES instance. \square

Corollary 1. *Given that the input graph is both bipartite and directed acyclic graph, the RESOLVE DELEGATION TO MINMAXWEIGHT problem is NP-complete even if we have $\lambda = 3$ and both the out-degree and in-degree of every vertex is at most 3 which concludes that RESOLVE DELEGATION TO MINMAXWEIGHT is para-NP-hard with respect to the parameter (λ, Δ) .*

Proof. The corollary follows as the resulting graph \mathcal{G} from reduction of (3, B2)-SAT instance in Theorem 2 is bipartite as \mathcal{V} can be partitioned into 2 independent sets $\mathcal{V}_1 = \{y_j : j \in [m]\} \cup \{d_{i,1} : i \in [n]\}$ and $\mathcal{V}_2 = \{a_i, \bar{a}_i, d_{i,2} : i \in [n]\}$. Also \mathcal{G} is Directed Acyclic graph as it doesn't have directed cycles. \square

Definition 4. *A (positive integral) edge weighting of a graph G is a mapping w that assigns to each edge of G a positive integer.*

Definition 5. An orientation of G is a mapping $\Lambda : E(G) \rightarrow V(G) \times V(G)$ with $\Lambda((u, v)) \in \{(u, v), (v, u)\}$.

Definition 6. The weighted outdegree of a vertex $v \in V(G)$ w.r.t an edge weighting w and an orientation Λ is defined as $d_{G,w,\Lambda}^+(v) = \sum_{(v,u) \in E(G) \text{ with } \Lambda((v,u))=(v,u)} w((v,u))$.

Definition 7. (MINIMUM MAXIMUM OUTDEGREE). Given a graph G , an edge weighting w of G in unary and a positive integer r , is there an orientation Λ of G such that $d_{G,w,\Lambda}^+(v) \leq r$ for each $v \in V(G)$?

Lemma 1. [10] MINIMUM MAXIMUM OUTDEGREE is $W[1]$ -hard when parameterized by the treewidth of the instance graph.

We now show that RESOLVE DELEGATION TO MINMAXWEIGHT is $W[1]$ -hard when parameterized by the treewidth of the instance graph. We reduce from MINIMUM MAXIMUM OUTDEGREE with instance graph G to RESOLVE DELEGATION TO MINMAXWEIGHT by first creating a replica of the G and then taking an edge (u, v) with weight w and replacing it with a path of w nodes with the end vertex having edges to u and v .

Theorem 3. RESOLVE DELEGATION TO MINMAXWEIGHT is $W[1]$ -hard when parameterized by the treewidth of the instance graph

Proof. To prove $W[1]$ -Hardness we reduce from MINIMUM MAXIMUM OUTDEGREE to RESOLVE DELEGATION TO MINMAXWEIGHT. Let a graph $G(V, E)$ with an edge weighting w in unary and a positive integer r be an arbitrary instance of MINIMUM MAXIMUM OUTDEGREE. MINIMUM MAXIMUM OUTDEGREE is considered to be a YES instance if the weighted outdegree of every vertex is upper bounded by r . Now using the instance of MINIMUM MAXIMUM OUTDEGREE we create an instance $(\mathcal{H}, r + 1)$ of RESOLVE DELEGATION TO MINMAXWEIGHT. Let us construct a graph $\mathcal{H} = (V, \mathcal{E})$ where $V = V_1 \cup V_2$. $V_1 = \{b_u : u \in V\}$. $\forall (u, v) \in E$ add the set of vertices $\{a_{uv_1}, a_{uv_2}, \dots, a_{uv_{w(u,v)}}\}$ to V_2 . $\forall (u, v) \in E$, $(a_{uv_1}, b_u) \in \mathcal{E}$, $(a_{uv_i}, b_v) \in \mathcal{E}$ and $\forall i \in [w(u, v)] \setminus \{1\}$, $(a_{uv_i}, a_{uv_{i-1}}) \in \mathcal{E}$. This completes the construction of \mathcal{H} with V_1 as the sink nodes. It is trivial to observe the fact that $\text{tw}(\mathcal{H}) \leq \text{tw}(G) + 2$. We now prove that the MINIMUM MAXIMUM OUTDEGREE is an YES instance iff the RESOLVE DELEGATION TO MINMAXWEIGHT is an YES instance

Let MINIMUM MAXIMUM OUTDEGREE be a YES instance. Let Λ be the orientation of G which makes MINIMUM MAXIMUM OUTDEGREE an YES instance. We consider the following sub-graph $\mathcal{H}' \subseteq \mathcal{H}$

$$\begin{aligned} \mathcal{E}[\mathcal{H}'] = & \{(a_{uv_i}, a_{uv_{i-1}}) : i \in [w(u, v)] \setminus \{1\}, (u, v) \in E\} \\ & \cup \{(a_{uv_1}, b_u) : (u, v) \in E, \Lambda((u, v)) = (u, v)\} \end{aligned}$$

We observe that \mathcal{H}' is a spanning sub-graph of \mathcal{H} such that (i) every non-sink vertices in \mathcal{H} has exactly one outgoing edge in \mathcal{H}' and (ii) for each sink vertex

in \mathcal{H} , there are at most $r+1$ vertices (including the sink itself) which has a path to it. Hence the RESOLVE DELEGATION TO MINMAXWEIGHT instance is a YES instance.

Let RESOLVE DELEGATION TO MINMAXWEIGHT be a YES instance. Let \mathcal{H}' be the spanning sub-graph of \mathcal{H} which make RESOLVE DELEGATION TO MINMAXWEIGHT a YES instance. Let the edges in \mathcal{H}' be denoted by \mathcal{E}' . We consider the following orientation Λ of G

$$\Lambda((u, v)) = \begin{cases} (u, v) & \text{if } (a_{uv_1}, b_u) \in \mathcal{E}' \\ (v, u) & \text{otherwise} \end{cases}$$

Clearly weighted outdegree of every vertex in G is atmost r . Therefore MINIMUM MAXIMUM OUTDEGREE is an YES instance.

This concludes the proof of this theorem. \square

Corollary 2. RESOLVE DELEGATION TO MINMAXWEIGHT is $W[1]$ -hard when parameterized by the treewidth even when the input graph is both Bipartite and Directed Acyclic Graph.

Proof. In the instance of RESOLVE DELEGATION TO MINMAXWEIGHT created in Theorem 3, graph \mathcal{H} is Bipartite as there is no odd cycle in the underlying undirected graph. Also graph \mathcal{H} is Directed Acyclic Graph (DAG) as there is no directed cycle. \square

4 FPT Algorithms

We now present our FPT algorithms.

Observation 1. There is a kernel for RESOLVE DELEGATION TO MINMAXWEIGHT consisting of at most λt vertices. In particular, there is an FPT algorithm for the RESOLVE DELEGATION TO MINMAXWEIGHT problem parameterized by (λ, t) .

Proof. If the number n of vertices in the input graph is more than λt , then the instance is clearly a NO instance. Hence, we have $n \leq \lambda t$. \square

In this section we define the notion of weights for the nodes in the subgraph \mathcal{H} of the delegation graph \mathcal{G} . We define weight of all nodes u in \mathcal{G} to be 1. To get a notion of weight of a vertex u in a subgraph \mathcal{H} , it can be considered as a number which is one more than the number of nodes who have delegated their vote to u and then have been removed from the graph \mathcal{G} during the construction of \mathcal{H} . If \mathcal{H} is a forest such that every non-sink node has an outdegree 1, then clearly the weight of the tree rooted at a sink node say t is sum of the weights of the nodes in the tree. We now show RESOLVE DELEGATION TO MINMAXWEIGHT is FPT w.r.t number of non-sink nodes by using the technique of bounded search tree by the branching on set of vertices satisfying some key properties.

Theorem 4. *The RESOLVE DELEGATION TO MINMAXWEIGHT problem has a FPT with respect to the parameter k which is the number of non-sink nodes in \mathcal{G} (delegation graph).*

Proof. Let us denote the problem instance by $(\mathcal{G}, \lambda, k)$. Now we present the following reduction and branching rules.

Reduction RD.1. If there is a vertex v in \mathcal{V} with only one outgoing edge to a vertex u (u, v are distinct), delete v from graph and increase weight of u by the weight of v . The incoming edges which were incident on v (except the self loops if any) are now incident on u .

Safeness of Reduction RD.1. is trivial as a node v with single outgoing edge can only delegate the votes it has got (this includes v 's own vote and the votes of other nodes who have delegated to v so far) to the only neighbor u it has got.

Reduction RD.2 Remove self loops if any.

Safeness of Reduction RD.2. follows from the fact that no non-sink node can delegate to itself

Reduction RD.3. If \mathcal{G} contains a non-sink node v with outdegree more than $2(k-1)$ and indegree 0, delete v from \mathcal{G} . The new instance is $(\mathcal{G} - v, \lambda, k-1)$

Safeness of Reduction RD.3. is due to the fact that if we have a vertex v with outdegree greater than $2(k-1)$, it implies that it has an outgoing edge to at least k sink nodes. Let us denote these sink nodes by set S . So, irrespective of the delegations made by other vertices, there will exist one sink node $t' \in S$ such that none of the other $k-1$ non-sink nodes have delegated to t' and hence we can delegate v to t' and still not increase the maximum weight of the sink node.

Branching B.1. Pick a vertex v such that the outdegree is more than $2(k-1)$ and indegree is $k' > 0$. Note that $k' \leq k-1$. Each of k' nodes having an outgoing edge to v can either delegate to v or not delegate it. So we have $2^{k'}$ possibilities and hence we can create $2^{k'}$ subproblems. In each possibility if a node u_1 is delegating to v then we delete all the outgoing edges of u_1 except (u_1, v) and if we have a node u_2 which doesn't delegate to v then we delete the outgoing edge from u_2 to v . In each of the $2^{k'}$ instances of graph created first apply R.D.1, then R.D.2, and then finally R.D.3. Now solve the problem recursively for each of the $2^{k'}$ instances created by considering each of them as a subproblem. If a non-sink node u has delegated to v then u gets deleted due to R.D.1 and if none of the non-sink nodes delegate to v then v gets deleted to R.D.3. So therefore, the new parameter (number of non-sink nodes) for the smaller subproblems gets reduced by at least 1.

Given a directed delegation graph \mathcal{G} , the algorithm works as follows. It first applies Reductions RD.1., RD.2., RD.3. and Branching Rule B.1 exhaustively and in the same order. The parameter (number of non-sink nodes) decreases by at least 1 for each of the subproblems as explained earlier. If we can't apply the branching rule B.1 to a given subproblem it implies that there is no non-sink node such that the outdegree is more than $2(k-1)$ and indegree is greater than 0. Also due to R.D.3 we don't have any non-sink node with outdegree more than

$2(k-1)$ and indegree equal to 0. So we can do a brute force by considering every possible delegations and solve this instance in $O(k^k \cdot n^{O(1)})$ running time. Note that our algorithm will only look at the feasible solutions of RESOLVE DELEGATION TO MINMAXWEIGHT while brute forcing for a subproblem.

Also since every node of bounded search tree splits into at most 2^{k-1} subproblems and height of the tree is $O(k)$, we get $f(k)$ leaves (where $f(k)$ is a function of k only). Clearly the time taken at every node is bounded by $g(k) \cdot n^{O(1)}$ where $g(k)$ is a function of k only. Thus, the total time used by the algorithm is at-most $O(f(k) \cdot g(k) \cdot n^{O(1)})$ which gives us an FPT for RESOLVE DELEGATION TO MINMAXWEIGHT. \square

We now show RESOLVE DELEGATION TO MINMAXWEIGHT is FPT w.r.t number of edges to be deleted from delegation graph by using the technique of bounded search tree by the branching on set of edges satisfying some key properties.

Theorem 5. *The RESOLVE DELEGATION TO MINMAXWEIGHT problem has a FPT with respect to the parameter k which is the number of edges to be deleted from delegation graph.*

Proof. The parameter k is the number of edges to be deleted. Given any instance \mathcal{G} of problem, every feasible solution graph $\mathcal{G}_{\mathcal{J}}$ is a forest with trees with set of roots as set of all sink nodes \mathcal{J} . Clearly then $k = |\mathcal{E}| - |\mathcal{V}| + |\mathcal{J}|$. Let us denote the problem instance by $(\mathcal{G}, \lambda, k)$.

Observation 2. *If $k > 0$ and only the sink nodes have outdegree 0, then there is a non-sink node with outdegree atleast 2.*

Proof. Sum of outdegree of all the non-sink nodes is greater than $|\mathcal{V}| - |\mathcal{J}|$. Hence the observation follows from pigeon hole principle.

Branching B.1. Let $k > 0$. Consider the vertex with maximum outdegree. If l is the outdegree of one such vertex v , delete one of the two groups of edges $\{1, \dots, \lfloor l/2 \rfloor\}$ and $\{\lfloor l/2 \rfloor + 1, \dots, l\}$ outgoing from v . Then solve the problem recursively for two new subproblems with new parameter $k' \leq k - 1$.

Now we describe why the Branching B.1 is safe. Note that the Branching B.1 is triggered only when $k > 0$. It follows from Observation 2 that outdegree of v is at least 2. Consider the degree of v to be l and the corresponding outgoing edges from v to be $\{1, \dots, l\}$. Since v can delegate only to exactly one of its neighbours connected by $\{1, \dots, l\}$, other $l - 1$ edges need to be deleted from delegation graph as they can not be a part of feasible solution. If we partition the set of edges into two disjoint sets $\{1, \dots, \lfloor l/2 \rfloor\}$ and $\{\lfloor l/2 \rfloor + 1, \dots, l\}$, only one out of the two groups can be a part of feasible solution. This allows us to delete the other half set say $\{\lfloor l/2 \rfloor + 1, \dots, l\}$. As we know that $l \geq 2$ which comes from the fact that outdegree of vertex v is at least 2. The problem now reduces to a smaller instance \mathcal{G}' with edges $\mathcal{E}'[\mathcal{G}] = \mathcal{E}[\mathcal{G}] \setminus \{\lfloor l/2 \rfloor + 1, \dots, l\}$ and parameter number of edges to be deleted as $k' \leq k - 1$. Thus way we get a bounded search tree with only constant number of subproblems at each branch such that at each recursive step the height of search tree reduces by at least one.

Given a directed delegation graph \mathcal{G} , the algorithm works as follows. As long as $k > 0$, Branching Rule B.1 is applied exhaustively in the bounded search tree. Note that Branching Rule B.1 brings down the parameter k in every call by at least 1. Whenever the parameter k becomes 0, we have a feasible solution as the non-sink nodes have the outdegree of 1. Now we can easily check in polynomial time whether the feasible solution is a YES instance or a NO instance. At every recursive call we decrease the parameter by at least 1 and thus the height of the tree is at most k . Also since every node of bounded search tree splits into two, we get $O(2^k)$ leaves. Clearly the time taken at every node is bounded by $n^{O(1)}$. Thus if $f(k) = O(2^k)$ be the number of nodes in the bounded search tree, the total time used by the algorithm is at most $O(2^k n^{O(1)})$ which gives us an FPT for RESOLVE DELEGATION TO MINMAXWEIGHT. \square

5 Structural Results

Theorem 6. *There exists a linear programming formulation for the optimization version of RESOLVE DELEGATION TO MINMAXWEIGHT where fractional delegation of votes is allowed. Thus the fractional variant is solvable in polynomial time.*

Proof. We consider the fractional variant of Liquid Democracy Delegation Problem where it is allowed to fractionally delegate votes of a source (delegator) to multiple nodes such that total number of votes being delegated is conserved at the delegator. We formally define conservation while formulating the LP for the problem.

LP formulation follows similar to the LP formulation of flow-problems (e.g. Max-Flow-MinCut etc). We assign $x_{u,v}$ as weight to every edge $(u, v) \in \mathcal{E}[\mathcal{G}]$ which corresponds to the fractional weight of votes delegated from vertex u to v (for all $u, v \in \mathcal{V}[\mathcal{G}]$. For all other $x_{u,v}$ where (u, v) pair doesn't correspond to an edge of delegation graph we assign value 0. It immediately follows that for all sink nodes $t \in \mathcal{T}[\mathcal{G}]$, total weight of fractional votes being delegated to each sink-node t (including that of the sink node t) is $\sum_{v \in \mathcal{V} \setminus \mathcal{T}} x_{v,t} + 1 \ \forall t \in \mathcal{T}$. For all

other non-sink nodes $s \in \mathcal{V} \setminus \mathcal{T}$, node s obeys conservation as follows :

$$\sum_{u \in \mathcal{V} \setminus \mathcal{T}} x_{u,s} + 1 = \sum_{v \in \mathcal{V}} x_{s,v}, \forall s \in \mathcal{V} \setminus \mathcal{T}$$

Our aim is to minimize the maximum weight of votes delegated to any sink node (including that of the sink node). The corresponding LP formulation is:

$$\begin{aligned} & \text{minimize } z \\ & z \geq \sum_{v \in \mathcal{V} \setminus \mathcal{T}} x_{v,t} + 1, \forall t \in \mathcal{T} \\ & \sum_{u \in \mathcal{V} \setminus \mathcal{T}} x_{u,s} + 1 = \sum_{v \in \mathcal{V}} x_{s,v}, \forall s \in \mathcal{V} \setminus \mathcal{T} \text{ [Follows from conservation]} \\ & x_{u,v} \geq 0, \forall (u, v) \in \mathcal{E}[\mathcal{G}] \\ & x_{u,v} = 0, \forall (u, v) \notin \mathcal{E}[\mathcal{G}] \end{aligned}$$

\square

6 Conclusion and Future Direction

We have studied the parameterized complexity of a fundamental problem in liquid democracy, namely `RESOLVE DELEGATION TO MINMAXWEIGHT`. We considered various natural parameters for the problem including the number of sink vertices, maximum allowed weight of any sink in the final delegation graph, maximum degree of any vertex, tree-width, the number of edges that one deletes in an optimal solution, number of non-sink vertices. We also show that a related problem which we call `RESOLVE FRACTIONAL DELEGATION` is polynomial time solvable.

An important future work is to resolve the complexity of `RESOLVE DELEGATION TO MINMAXWEIGHT` if the input graph is already acyclic or tree. We know that there exists a $\Omega(\log n)$ lower bound on the approximation factor of optimizing the maximum allowed weight of any sink [6]. It would be interesting to see if there exists FPT algorithms achieving a approximation factor of $o(\log n)$.

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