Algorithmic Aspects of Total Roman and Total Double Roman Domination in Graphs

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Abstract. For a simple, undirected and connected graph $G = (V, E)$, a total Roman dominating function (TRDF) $f : V \to \{0, 1, 2\}$ has the property that, every vertex u with $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$ and the subgraph induced by the set of vertices labeled one or two has no isolated vertices. A total double Roman dominating function (TDRDF) on G is a function $f : V \to \{0, 1, 2, 3\}$ such that for every vertex $v \in V$ if $f(v) = 0$, then v has at least two neighbors x, y with $f(x) = f(y) = 2$ or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then v must have at least one neighbor w with $f(w) \geq 2$ and the subgraph induced by the set $\{u_i : f(u_i) \geq 1\}$ has no isolated vertices. The weight of a T(D)RDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum total (double) Roman domination problem (MT(D)RDP) is to find a T(D)RDF of minimum weight of the input graph. In this article, we show that MTRDP and MTDRDP are polynomial time solvable for bounded treewidth graphs, chain graphs and threshold graphs. We design a $2(\ln(\Delta - 0.5) + 1.5)$-approximation algorithm (APX-AL) for the MTRDP and $3(\ln(\Delta - 0.5) + 1.5)$-APX-AL for the MTDRDP, where Δ is the maximum degree of G, and show that the same cannot have $(1 - \delta) \ln |V|$ ratio APX-AL for any $\delta > 0$ unless $P = NP$. Finally, we show that MT(D)RDP is APX-hard for graphs with $\Delta = 5$.

Keywords: Total Roman domination · Total double Roman domination · APX-complete

1 Introduction

Let $G(V, E)$ be a simple, undirected and connected graph. For a vertex u of G, the (*open*) *neighborhood* denoted $N_G(u)$ is the set $\{v : (v, u) \in E\}$ and its *degree* is $|N_G(u)|$. The *closed neighborhood* of u is $N_G[u] = \{u\} \cup N_G(u)$. *Maximum degree* of G denoted Δ (or clearly $\Delta(G)$) is $max_{u \in V} |N_G(u)|$. A vertex v is called *isolated vertex* if $|N_G(v)| = 0$. A vertex v of G is called *universal vertex* if $N_G[v] = V(G)$. A graph formed with the vertex set $S \subseteq V$ of graph $H(V, E)$ and the edge set $\{(u, v) \in E : u, v \in S\}$ is called an *induced subgraph* of H denoted $\langle S \rangle$. For undefined terminology and notations we refer to [35].

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A *dominating set* (DS) of a graph G is a set D such that $D \subseteq V$ and $\cup_{w \in D} N_G[w] = V$ and further D is called a *total dominating set* (TDS) of G if every vertex in V is adjacent to at least one vertex in D. The (*total*) *domination number* of G denoted by $(\gamma_t(G))$ $\gamma(G)$ is $min\{|Q| : Q$ is a (T)DS of $G\}$. The problem of finding a (T)DS of smallest cardinality in a graph is called the minimum (total) dominating set (M(T)DS) problem. Literature on the concept of, domination has been surveyed in [16], total domination has been surveyed in [17].

In 2004, Cockayne et al. in [11] introduced the concept of Roman domination (RDOM). A function $f : V \to \{0, 1, 2\}$ is a *Roman Dominating Function* (RDF) on G if every vertex with label zero is adjacent to at least one vertex with label two. We refer to [3,11,13,14,18–21,25,28,31,32] for the literature on RDOM in graphs.

The notion of total Roman domination (TRDOM) was introduced in 2013 by Liu et al. in [22]. A total Roman dominating function (TRDF) is a Roman dominating function with the additional property that the subgraph of G induced by the set $\{k \in V : f(k) \geq 1\}$ is without isolated vertices. The concept of TRDOM has been studied in [1,7,9,27].

Double Roman domination was introduced in 2016 by Beeler et al. in [30]. A Double Roman Dominating Function (DRDF) on G is a function $g : V \to \{0, 1, 2, 3\}$ such that for every vertex $k \in V$ if $g(k) = 0$, then k has at least two neighbors $x, y \in N_G(k)$ with $g(x) = g(y) = 2$ or one neighbor w with $g(w) = 3$, and if $g(k) = 1$, then k must have at least one neighbor w with $g(w) \geq 2$. The double Roman domination has been studied in [2,4,5,8,24].

Total double Roman domination (TDRDOM) was introduced in 2019 by Shao et al. in [33], which is a variant of double Roman domination. A total double Roman dominating function (TDRDF) is a double Roman dominating function with the additional property that the subgraph of G induced by the set $\{k \in V : g(k) \geq 1\}$ is without isolated vertices. The concept TDRDOM has been studied in [15,33].

The weight of a RDF (TRDF, DRDF, TDRDF) g is the value $g(V) = \sum_{v \in V} g(v)$. The *Roman domination number*, *total Roman domination number*, *double Roman domination number*, *total double Roman domination number*, respectively, equals the minimum weight of a RDF, TRDF, DRDF and TDRDF, respectively, denoted by $\gamma_R(G)$, $\gamma_{tR}(G)$, $\gamma_{dR}(G)$ and $\gamma_{tdR}(G)$. The minimum total (double) Roman domination problem (MT(D)RDP) is to find a T(D)RDF of minimum weight in the input graph.

2 Bounded Tree-Width Graphs

A *tree decomposition* of a graph H is a tree T_1 with the vertex set $V(T_1) = \{Z_1, Z_2, \ldots, \}$, where each Z_i is a subset of $V(H)$ with the following requirements.

i) $V(H) = \bigcup_{Z_k \in V(T_1)} Z_k$
ii) $\forall (u, v) \in E(H)$, there exists a vertex $Z_t \in V(T_1)$ such that $u, v \in Z_t$ and

iii) $\forall v \in V(H)$, the induced subgraph $\{Z_t : v \in Z_t \textit{ and } Z_t \in V(T_1)\}$ is a subtree of T_1.
Then the tree decomposition T_1 of H is said to have *width* equals to $max\{|Z_t|-1 : Z_t \in V(T_1)\}$ [29]. The *treewidth* is the smallest width of a tree decomposition of a graph.

Theorem 1. *Given a graph G and a positive integer k, TRDP can be expressed in CMSOL.*

Proof. Let $f : V \rightarrow \{0, 1, 2\}$ be a function on a graph G, where $V_i = \{v|f(v) = i\}$ for $i \in \{0, 1, 2\}$. The CMSOL formula for the RDF problem is expressed as follows.

$Rom\_Dom(V) = \exists V_0, V_1, V_2, \forall p(p \in V_1 \vee p \in V_2 \vee (p \in V_0 \wedge \exists q \in V_2 \wedge adj(p, q))),$

where $adj(p, q)$ is the binary adjacency relation which holds if and only if, p, q are two adjacent vertices of G.

Next, we give a CMSOL formula for the $Total\_Rom(V)$, which says that every vertex $p \in V_1 \cup V_2$ is adjacent to some vertex q in $V_1 \cup V_2$, as follows.

$Total\_Rom(V) = \exists V_0, V_1, V_2, \forall p, \exists q(p \in (V_1 \cup V_2) \wedge q \in (V_1 \cup V_2) \wedge adj(p, q)).$

Let k be a positive integer, then the CMSOL formula for the TRDP is expressed as follows.

$Total\_Rom\_Dom(V) = (f(V) \leq k) \wedge Rom\_Dom(V) \wedge Total\_Rom(V).$

Now, from Theorem 1 and Courcelle's result in [12], the theorem below follows.

Theorem 2. *MTRDP for graphs with treewidth at most a constant is solvable in linear time.*

Theorem 3. *Given a graph G and a positive integer k, TDRDP can be expressed in CMSOL.*

Proof. Let $g : V \rightarrow \{0, 1, 2, 3\}$ be a function on a graph G, where $V_i = \{v|g(v) = i\}$ for $i \in \{0, 1, 2, 3\}$. The CMSOL formula for the DRDF problem is expressed as follows.

$Double\_Rom\_Dom(V) = \exists V_0, V_1, V_2, V_3, \forall p((p \in V_0 \wedge ((\exists q, r \in V_2 \wedge adj(p, q) \wedge adj(p, r)) \vee (\exists s \in V_3 \wedge adj(p, s))) \vee (p \in V_1 \wedge (\exists t \in V_2 \wedge adj(p, t) \vee (\exists u \in V_3 \wedge adj(p, u))))) \vee (p \in V_2) \vee (p \in V_3)),$

where $adj(p, q)$ is the binary adjacency relation which holds if and only if, p, q are two adjacent vertices of G.

Next, we give a CMSOL formula for the $Total\_Double\_Rom(V)$, which says that every vertex $p \in V_1 \cup V_2 \cup V_3$ is adjacent to some vertex q in $V_1 \cup V_2 \cup V_3$, as follows.

$Total\_Double\_Rom(V) = \exists V_0, V_1, V_2, V_3, \forall p, \exists q(p \in (V_1 \cup V_2 \cup V_3) \wedge q \in (V_1 \cup V_2 \cup V_3) \wedge adj(p, q)).$

Let k be a positive integer, then the CMSOL formula for the TDRDP is expressed as follows.

$Total\_Double\_Rom\_Dom(V) = (g(V) \leq k) \wedge Double\_Rom\_Dom(V) \wedge Total\_Double\_Rom(V).$

Now, from Theorem 3 and Courcelle's result in [12], the theorem below follows.

Theorem 4. *MTDRDP for graphs with treewidth at most a constant is solvable in linear time.*

3 Threshold Graphs

Here, we solve MTRDP and MTDRDP for connected threshold graphs in linear time. A graph G is *threshold* iff the following conditions hold, see [23]

i) Vertex set of G is partitioned into two disjoint sets, a clique Q and an independent set R
ii) There exists a permutation $(q_1, q_2, \ldots, q_p)$ of vertices of Q such that $N_G[q_1] \subseteq N_G[q_2] \subseteq ... \subseteq N_G[q_p]$ and
iii) There exists a permutation $(r_1, r_2, \ldots, r_i)$ of vertices of R such that $N_G(r_1) \supseteq N_G(r_2) \supseteq ... \supseteq N_G(r_i)$.

Theorem 5. *Let G be a connected threshold graph. Then,*

$$\gamma_{tR}(G) = \begin{cases} 2, & \text{if } G \cong K_2 \\ 3, & \text{otherwise} \end{cases} \tag{1}$$

and

$$\gamma_{tdR}(G) = \begin{cases} 3, & \text{if } G \cong K_2 \\ 4, & \text{otherwise} \end{cases} \tag{2}$$

Proof. Let G be a connected threshold graph with p clique vertices and i independent vertices as described above. Since, q_p is a universal vertex of G, clearly, this implies that $\gamma_{tR}(G) = 3$ and $\gamma_{tdR}(G) = 4$, except when $G \cong K_2$ where $\gamma_{tR}(G) = 2$ and $\gamma_{tdR}(G) = 3$.

Now, the following result is immediate from Theorem 5 and the fact that the ordering of clique vertices of threshold graph can be found in linear time [23].

Theorem 6. *MTRDP and MTDRDP for connected threshold graphs are linear time solvable.*

If threshold graph G is disconnected i.e., G contains isolated vertices, then TRDF and TDRDF can not be defined on G.

4 Chain Graphs

Here, we solve MTRDP and MTDRDP for connected chain graphs in linear time. An ordering $\alpha = (y_1, y_2, \ldots, y_p, z_1, z_2, \ldots, z_q)$ of vertex set of a bipartite graph $G(Y, Z, E)$ is a *chain ordering* if $N_G(y_1) \subseteq N_G(y_2) \subseteq ... \subseteq N_G(y_p)$ and $N_G(z_1) \supseteq N_G(z_2) \supseteq ... \supseteq N_G(z_q)$. A bipartite graph is a *chain graph* iff it has a chain ordering [36].

Theorem 7. *Let $G(Y, Z, E)$ be a connected chain graph. Then,*

$$\gamma_{tR}(G) = \begin{cases} 2, & \text{if } G \text{ is } K_2 \\ 3, & \text{if } G \text{ is } K_{1,s}, \text{ where } s \geq 2 \\ 4, & \text{otherwise} \end{cases} \tag{3}$$

and

$$\gamma_{tdR}(G) = \begin{cases} 3, & \text{if } G \text{ is } K_2 \\ 4, & \text{if } G \text{ is } K_{1,s}, \text{ where } s \geq 2 \\ 6, & \text{otherwise} \end{cases} \tag{4}$$

Proof. Let $G(Y, Z, E)$ be a connected chain graph with $|Y| = p$ and $|Z| = q$ where $p, q \geq 1$. If $G \cong K_2$ or $G \cong K_{1,s}$, where $s \geq 2$, then $\gamma_{tR}(G)$ and $\gamma_{tdR}(G)$ can be determined directly from Theorem 5. Otherwise, define functions $f : V \to \{0, 1, 2\}$ and $g : V \to \{0, 1, 2, 3\}$ as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in \{y_p, z_1\} \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

$$g(v) = \begin{cases} 3, & \text{if } v \in \{y_p, z_1\} \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Clearly, f (g) is a TRDF (TDRDF) and $\gamma_{tR}(G) \leq 4$ ($\gamma_{tdR}(G) \leq 6$). By contradiction, it can be easily verified that $\gamma_{tR}(G) \geq 4$ ($\gamma_{tdR}(G) \geq 6$). Therefore $\gamma_{tR}(G) = 4$ ($\gamma_{tdR}(G) = 6$).

Now, from Theorem 7 and the fact that chain ordering can be computed in linear time [34], the theorem below follows.

Theorem 8. *MTRDP and MTDRDP for connected chain graphs are solvable in linear time.*

If chain graph G is disconnected i.e., G contains isolated vertices, then TRDF and TDRDF can not be defined on G.

5 Approximation Algorithm and Complexity

Here, results related to obtaining approximate solutions to MTRDP and MTDRDP are presented.

5.1 Approximation Bounds

An existing result obtained on lower bound of approximation ratio of MDS is given below.

Theorem 9 ([10])**.** *For a graph $G = (V, E)$, unless $P = NP$, the MDS problem cannot have a solution with approximation ratio $(1 - \delta) \ln |V|$ for any $\delta > 0$.*

Theorem below provides a lower bound on approximation ratio of MTRDP.

Theorem 10. *For a graph H, unless $P = NP$, the MTRDP cannot have a solution with approximation ratio $(1-\delta)\ln|V|$ for any $\delta > 0$.*

Proof. We propose a reduction which preserves the approximation. Let $H(V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ be an instance of the MDS problem. From H, an instance H' of MTRDP is constructed as follows.

Create n copies of P_3 with b_i as the central vertex and a_i, c_i as terminal vertices. Add the edges $\{(v_i, a_i), (v_i, c_i) : 1 \leq i \leq n\}$. An example construction of H' from H is shown in Fig. 1. Next, we prove a claim.

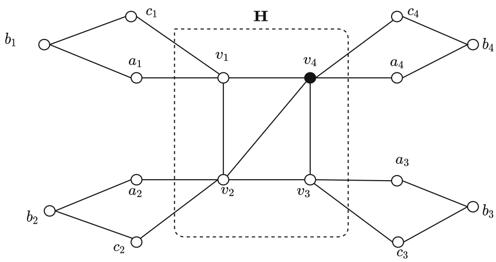

Fig. 1. Construction of H' from H

Claim. $\gamma_{tR}(H') = 3n + \gamma(H)$.

Proof. Let $H(V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ be a graph and $H' = (V', E')$ is a graph constructed from H.

Let M^* be a MDS of H i.e., $|M^*| = \gamma(H)$ and f be a function on H', defined as

$$f(v) = \begin{cases} 2, & \text{if } v \in \{v_i, a_i : v_i \in M^*\} \text{ or } v \in \{b_i : v_i \notin M^*\} \\ 1, & \text{if } v \in \{a_i : v_i \notin M^*\} \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

Clearly, f is a TRDF and $\gamma_{tR}(H') \leq 3n + |M^*|$.

Next, we show that $\gamma_{tR}(H') \geq 3n + |M^*|$. Let g be a TRDF on graph H'. Clearly if $g(v_i) = 0$, then $g(a_i) + g(b_i) + g(c_i) \geq 3$ and if $g(v_i) \geq 1$, then $g(v_i)+g(a_i)+g(b_i)+g(c_i) \geq 4$. Therefore $\gamma_{tR}(H') \geq 3n+|M^*|$. Hence $\gamma_{tR}(H') = 3n + \gamma(H)$.

Suppose that the MTRDP has an approximation algorithm (APX-AL) A which runs in polynomial time with approximation ratio β, where $\beta = (1 - \delta)\ln|V|$ for some fixed $\delta > 0$. Let l be a fixed positive integer. Next, we design an APX-AL, say DOM-SET-APPROX which runs in polynomial time to find a DS of a given graph H.

Algorithm 1. DOM-SET-APPROX(G)

Require: A simple and undirected graph H.
Ensure: A DS M of H.

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if there exists a DS M' of size at most l, then
    M ← M'
else
    Build the graph H'
    Calculate a TRDF f on H' by using algorithm A
    Find a DS M of H from TRDF f (as illustrated in the proof of Claim in
                                   Sect. 5.1)
end if
return M.
```

It can be noted that if M is a DS with $|M| \leq l$, then it is optimal. Otherwise, let M^* be a DS of H with minimum cardinality and g be a TRDF of H' with $g(V') = \gamma_{tR}(H')$. Clearly $g(V) \geq l$. If M is a DS of H obtained by the algorithm DOM-SET-APPROX, then $|M| \leq f(V) \leq \beta(g(V)) \leq \beta(3n + |M^*|) = \beta(1 + \frac{3n}{|M^*|})|M^*|$. Therefore, DOM-SET-APPROX approximates a MDS within a ratio $\beta(1 + \frac{3n}{|M^*|})$. If $\frac{1}{|M^*|} < \delta/2$, then the approximation ratio becomes $\beta(1 + \frac{3n}{|M^*|}) < (1 - \delta)(1 + \frac{3n\delta}{2}) \ln n = (1 - \delta') \ln n$, where $\delta' = \frac{3n\delta^2}{2} - \frac{3n\delta}{2} + \delta$.

By Theorem 9, if there exists an APX-AL for MDS problem with approximation ratio $(1 - \delta) \ln |V|$, then $P = NP$. Similarly, if there exists an APX-AL for MTRDP with approximation ratio $(1 - \delta) \ln |V|$, then $P = NP$. For large values of n, $\ln n \approx \ln(4n)$. Hence, in a graph $H'(V', E')$, where $|V'| = 4|V|$, the MTRDP cannot have an approximation algorithm with a ratio of $(1 - \delta) \ln |V'|$ unless $P = NP$.

Theorem 11. *For a graph H, unless $P = NP$, the MTDRDP cannot have a solution with approximation ratio $(1 - \delta) \ln |V|$ for any $\delta > 0$.*

Proof. The proof is obtained with similar arguments as in Theorem 10, in which replace the assigned value, for the vertices, 2 with 3.

5.2 Approximation Algorithm

Here, an APX-AL for MT(D)RDP is designed based on the approximation result known for MTDS problem below.

Theorem 12 ([37])**.** *The MTDS problem can be approximated with an approximation ratio of* $\ln(\Delta - 0.5) + 1.5$.

Let APP-TD-SET be an APX-AL that produces a TDS D of a graph G such that $|D| \leq (\ln(\Delta - 0.5) + 1.5)\gamma_t(G)$.

Next, we designe APP-TRDF algorithm to determine an approximate solution of MTRDP. In our algorithm, first we determine a TDS D of G using the APX-AL APP-TD-SET. Next, we build a total Roman dominating triple

Algorithm 2. APP-TRDF(G)

Input: A simple, undirected graph G.
Output: A TRDT T_r of G.
1: $D \leftarrow$ APP-TD-SET(G)
2: $T_r \leftarrow (V \setminus D, \emptyset, D)$
3: return T_r.

(TRDT) T_r such that weight 2 is assigned for all vertices in D and weight 0 is assigned for the remaining vertices.

Now, let $T_r = (D', \emptyset, D)$ be the TRDT obtained from the APP-TRDF algorithm. Clearly, every vertex in G is assigned with weight either 2 or 0, T_r gives a TRDF of G and APP-TRDF computes a TRDT T_r of G in polynomial time. Hence, the result follows.

Theorem 13. *The MTRDP in a graph can be approximated with an approximation ratio of* $2(\ln(\Delta - 0.5) + 1.5)$.

Proof. Let D be the TDS from APP-TD-SET algorithm, T_r be the TRDT produced by the APP-TRDF algorithm and W_r be the weight of T_r. Clearly, $W_r = 2|D|$. It is known that $|D| \leq (\ln(\Delta - 0.5) + 1.5)\gamma_t(G)$. Therefore, $W_r \leq 2(\ln(\Delta - 0.5) + 1.5)\gamma_t(G)$. Since $\gamma_t(G) \leq \gamma_{tR}(G)$ [1], it follows that $W_r \leq 2(\ln(\Delta - 0.5) + 1.5)\gamma_{tR}(G)$.

The corollary below follows from Theorem 13.

Corollary 1. *MTRDP* $\in$ *APX for graphs with* $\Delta = O(1)$.

Similar to the Algorithm 2, we propose an APX-AL APP-TDRDF which produces a total double Roman dominating quadruple (TDRDQ).

Algorithm 3. APP-TDRDF(G)

Input: A simple, undirected graph G.
Output: A TDRDQ Q_r of G.
1: $D \leftarrow$ APP-TD-SET(G)
2: $Q_r \leftarrow (V \setminus D, \emptyset, \emptyset, D)$
3: return Q_r.

We also note that the algorithm APP-TDRDF computes a TDRDQ Q_r of a given graph G in polynomial time and the following theorem holds.

Theorem 14. *The MTDRDP in a graph can be approximated with an approximation ratio of* $3(\ln(\Delta - 0.5) + 1.5)$.

Proof. The proof is obtained with similar arguments as in Theorem 13.

The corollary below follows from Theorem 14.

Corollary 2. *MTDRDP* $\in$ *APX for graphs with* $\Delta = O(1)$.

5.3 Approximation Completeness

Here, we prove that the MTRDP and MTDRDP are APX-complete (APXC) for graphs with $\Delta = 5$ using the L-reduction [26]. An optimization problem X is said to be APXC if X belongs to APX and APX-hard classes. By providing an L-reduction from MDS problem with $\Delta = 3$ i.e., DOM-3 which is known to be APXC [6], we show that the MTRDP and MTDRDP belongs to APX-hard for graphs with $\Delta = 5$.

Theorem 15. *MTRDP* $\in$ *APXC for graphs with* $\Delta = 5$.

Proof. From Corollary 1, it is clear that MTRDP is in APX. Given an instance $G = (V, E)$ of DOM-3, where $V = \{v_1, v_2, \ldots, v_n\}$, we construct an instance $G' = (V', E')$ of MTRDP same as in Sect. 5.1. Note that G' is a graph with $\Delta = 5$. First we prove the following claim.

Claim. $\gamma_{tR}(G') = 3n + \gamma(G)$, where $n = |V|$.

Proof. The proof is same as in the Claim in Sect. 5.1.

Let D^* be a MDS of G and $f : V' \to \{0, 1, 2\}$ be a minimum TRDF of G'. It is known that for any graph $G = (V, E)$ with maximum degree Δ, $\gamma(G) \geq \frac{n}{\Delta+1}$, where $n = |V|$. Thus, $|D^*| \geq \frac{n}{4}$. From the above claim it is evident that $f(V') = |D^*| + 3n \leq |D^*| + 12|D^*| = 13|D^*|$.

Now consider a TRDF $g : V' \to \{0, 1, 2\}$ of G'. Clearly, the set $D = \{v_i : g(v_i) \geq 1 \text{ or } g(a_i) \geq 1 \text{ or } g(c_i) \geq 1\}$ is a DS of G. Therefore, $|D| \leq g(V') - 3n$. Hence, $|D| - |D^*| \leq g(V') - 3n - |D^*| \leq g(V') - f(V')$. This implies that there exists an L-reduction with $\alpha = 13$ and $\beta = 1$.

Theorem 16. *MTDRDP* $\in$ *APX-complete for graphs with* $\Delta = 5$.

Proof. The proof is obtained with similar arguments as in Theorem 15, in which replace the assigned value 2 with 3. We get an L-reduction with $\alpha = 18$ and $\beta = 1$.

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