On the Intersections of Non-homotopic Loops

Václav Blažej[1(✉)], Michal Opler[2], Matas Šileikis[3], and Pavel Valtr[4]

[1] Faculty of Information Technology, Czech Technical University in Prague, Prague, Czech Republic
vaclav.blazej@fit.cvut.cz
[2] Computer Science Institute, Charles University, Prague, Czech Republic
opler@iuuk.mff.cuni.cz
[3] The Czech Academy of Sciences, Institute of Computer Science, Prague, Czech Republic
matas.sileikis@gmail.com
[4] Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

Abstract. Let $V = \{v_1, \ldots, v_n\}$ be a set of n points in the plane and let $x \in V$. An *x-loop* is a continuous closed curve not containing any point of V, except of passing exactly once through the point x. We say that two x-loops are *non-homotopic* if they cannot be transformed continuously into each other without passing through a point of V. For $n = 2$, we give an upper bound $2^{O(k)}$ on the maximum size of a family of pairwise non-homotopic x-loops such that every loop has fewer than k self-intersections and any two loops have fewer than k intersections. This result is inspired by a very recent result of Pach, Tardos, and Tóth who proved the upper bounds 2^{16k^4} for the slightly different scenario when $x \notin V$.

Keywords: Graph drawing · Non-homotopic loops · Curve intersections · Plane

1 Introduction

The so-called *crossing lemma*, which was proved independently by Ajtai, Chvátal, Newborn, Szemerédi [1] and by Leighton [2], bounds the number of crossings in any planar drawing of any graph with n vertices and $m \geq 4n$ edges. The crossing lemma has many applications in discrete and computational geometry and other fields. Very recently, Pach, Tardos, and Tóth [3] proved an

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interesting natural modification of the crossing lemma for multigraphs with non-homotopic edges. In the proof of their result, Pach, Tardos, and Tóth [3] applied a bound on the maximum size of certain collections of so-called non-homotopic loops. In this paper, we show that their bound can be significantly improved for a closely related problem.

For an integer $n \geq 1$, let $V_n = \{v_1, \dots, v_n\}$ be a set of n distinct points in the plane $\mathbb{R}^2$. Given $x \in \mathbb{R}^2$, an *x-loop* is a continuous function $f : [0,1] \to \mathbb{R}^2$ such that $f(0) = f(1) = x$ and $f(t) \notin V_n$ for $t \in (0,1)$. We will only consider x-loops that do not pass through x, that is $f(t) = x$ only for $t \in \{0,1\}$. When x is clear from the context we will also call an x-loop simply a *loop*. Two loops f_0, f_1 are *homotopic* (with respect to V_n), denoted $f_0 \sim f_1$, if there is a continuous function $H : [0,1]^2 \to \mathbb{R}^2$ (a *homotopy*) such that

$$H(0,t) = f_0(t) \quad \text{and} \quad H(1,t) = f_1(t) \quad \text{for all } t \in [0,1],$$

$$H(s,0) = H(s,1) = x \quad \text{for all } s \in [0,1].$$

and

$$H(s,t) \notin V_n \quad \text{for all } s,t \in (0,1).$$

A self-intersection of a loop f corresponds to a pair $\{t,u\} \subset (0,1)$ of distinct numbers such that $f(t) = f(u)$, while an intersection of two loops f_1, f_2 corresponds to an *ordered* pair $t, u \in (0,1)$ such that $f_1(t) = f_2(u)$.

Given integers $n, k \geq 1$ and $x \in V_n$, let $g(n,k)$ be the largest number of pairwise non-homotopic loops such that every loop has fewer than k self-intersections and any two loops have fewer than k intersections.

Pach, Tardos and Tóth [3] considered the same quantity, but for x outside of V_n (they also added a restriction that no loop passes through x, which holds trivially in our setting with $x \in V_n$). Although the two settings seem to be very similar, we were able to obtain an upper bound on $g(2,k)$ which is significantly smaller than the upper bound on $f(2,k)$ obtained by Pach, Tardos and Tóth [3]. In the setting of Pach, Tardos and Tóth [3] with $x \notin V_n$, the largest number of pairwise non-homotopic loops so that every loop has fewer than k self-intersections and any two loops have fewer than k intersections is denoted by $f(n,k)$. The two aforementioned quantities are related by the following inequalities.

Proposition 1. *For every $n, k \geq 1$ we have*

$$g(n,k) \leq f(n,k) \leq g(n+1,k). \tag{1}$$

Proposition 1 is proved in Sect. 5.

Pach, Tardos and Tóth [3] showed that for $n \geq 2$

$$f(n,k) \leq 2^{(2k)^{2n}} \tag{2}$$

and

$$f(n,k) \geq \begin{cases} 2^{\sqrt{nk}/3}, & n \leq 2k, \\ (n/k)^{k-1}, & n \geq 2k. \end{cases}$$

Also in [3] it was proved that if $n = 1$, then there are at most $2k + 1$ non-homotopic loops with fewer than k self-intersections (that is, if we do not bound the number of intersections) implying $f(1, k) \leq 2k + 1$.

In our main result we focus on the function g in case $n = 2$. Inequalities (1) and (2) imply that $g(2, k) \leq 2^{16k^4}$. The following theorem improves this bound significantly.

Theorem 1. *Let $n = 2$ and $x \in V_2$. For any k, the size of any collection of non-homotopic x-loops with fewer than k self-intersections is at most $2^{O(k)}$. In particular*

$$g(2, k) \leq 2^{12k}.$$

We believe that the bound in Theorem 1 can be further improved by reducing the exponent to $O(\sqrt{k} \log k)$. We plan to address this in a follow-up paper.

2 Setup and Notation

Depending on the context, we will treat $S := \mathbb{R}^2 \setminus V_n$ either as the plane with n points removed, or as a sphere with $n+1$ points removed (where n of these points is the set $V_n = \{v_1, \ldots, v_n\}$ and the last one, denoted by v_0, corresponds to the "point at infinity"). We refer to the points v_i as *obstacles*.

For convenience, we will always assume the following properties of a finite collection of loops:

1. the set of points of intersections and the set of points of self-intersections are disjoint,
2. every (self-)intersection is simple (that is, no point in S belongs to more than two loops and no loop passes through the same point more than twice),
3. every intersection between two loops is a *crossing*, that is, one loop "passes to the other side" of the other loop (otherwise an intersection is called a *touching*).

Assumptions 1–3 can be attained by infinitesimal perturbations without creating any new intersections or self-intersections.

Given a drawing of the x-loops satisfying the above conditions, we choose a closed curve on the sphere without self-intersections which goes through the points $v_0, \ldots, v_n$ in this order (if $x \notin V_n$, we choose this curve so that it avoids x). We call this loop the *equator*. Removing the equator from the sphere, we obtain two connected sets, which we arbitrarily name the *top half* and the *bottom half*. We refer to the $n+1$ sets into which the equator is split by excluding points v_i as *gaps*. We label the gaps by elements of $A_n := \{0, \ldots, n\}$, assigning label i to the gap between v_i and v_{i+1}, with indices counted modulo $n + 1$.

By a careful choice of the equator, we can assume the following conditions:

4. every x-loop in the collection intersects the equator a finite number of times,
5. each of these intersections (except for, possibly, the intersection at x) is a crossing (as opposed to a touching),
6. no point of self-intersection or intersection (other than x) lies on the equator.

Part of a given loop f between a pair of distinct intersections with the equator (inclusively) is called a *segment* (it is a restriction of f to a closed subinterval of $[0, 1]$). Whenever the two intersections defining a segment are consecutive (along the loop f), the segment is called an *arc*. If some arc intersects itself, we can remove the part of the arc between these self-intersections without changing the homotopy class of the loop; this trivially does not increase the number of (self-) intersections; therefore we can make yet another assumption about the family of x-loops and the equator:

7. there are no self-crossings within any arc (i.e., between consecutive crossings of the equator).

Given a x-loop ℓ, we list the labels of gaps in the order they are crossed by the loop. This way we obtain a word w over alphabet A_n. We say that ℓ *induces* w. The empty word corresponds to a trivial loop.

A segment that intersects the equator k times (including the beginning and the end) consists of $k-1$ arcs, which we order in a natural way so that the segment traverses the arcs in the increasing order. A segment is called a *downsegment* if its first arc is contained in the bottom half, and an *upsegment* otherwise. If a segment does not start nor end at x, by listing the labels of gaps that the segment intersects, we obtain a word w in alphabet A_n, and call such a segment a *w-segment* (or, more specifically, *w-downsegment* or *w-upsegment*, if we want to specify the location of the first arc). For example, a loop with the first arc in the top half that induces the word 01201 has a 01-downsegment and a 01-upsegment, as well as a 012-downsegment but no 012-upsegment.

3 General n

In this section we state and prove several facts that are valid for general n, including all prerequisites for the proof of Theorem 1.

We start with a simple proposition which allows bounding the number of non-homotopic loops in terms of the different words that they induce.

Proposition 2. *Let $x \in V_n$ and suppose that two x-loops ℓ_1 and ℓ_2 start with an arc which belongs to the same half of the sphere. If they induce the same word of length m, they are homotopic.*

Proof. For $i \in \{1, 2\}$ and $k \in \{1, \ldots, m\}$, suppose that the kth arc of ℓ_i ends at point p_i^k (with p_1^k and p_2^k lying in the same gap). Let γ^k be a loop which first goes along the first k arcs of ℓ_2, then goes along a gap from p_2^k to p_1^k and then continues to x along the last $m+1-k$ arcs of ℓ_1. Since the curved quadrilateral consisting of kth arcs of the loops and the two parts of two gaps does not wind around any obstacle, we have $\gamma^{k-1} \sim \gamma^k$ for $k = 2, \ldots, m$. By a similar argument, we have $\ell_1 \sim \gamma^1$ and $\ell_2 \sim \gamma^m$. The proposition follows.

Lemma 1. *Let letters $a, b, c \in A_n$ be distinct. If f_1, f_2 are two, not necessarily distinct, x-loops, then any abc-downsegment in f_1 and any abc-upsegment in f_2 intersect.*

Proof. Choose a cyclic orientation of the equator such that gaps appear in the order a, b, c. Let s_1 be an abc-downsegment of f_1, and s_2 be an abc-upsegment of f_2 and, for contradiction, suppose s_1, s_2 do not intersect.

By removing the point of intersection with s_1 from the gap b, we obtain two disjoint sets, and name them B' and B'' so that — in the same orientation of the equator — gap a is followed by set B' followed by B'' followed by gap c. The bc-arc of s_1 partitions the top half into two connected components. Note that gap a and B'' belong to different components. Since the ab-arc of s_2 belongs to the top half, s_2 must intersect the gap b in B'. Similarly, ab-arc of s_1 partitions the bottom half into two components, so that B' and c belong to different components. Since the bc-arc of s_2 traverses the bottom half, s_2 must intersect the gap b in B''. Since B' and B'' are disjoint, we obtain a contradiction.

Corollary 1. *Given three distinct letters $a, b, c \in A_n$, suppose that loop f_1 has i disjoint abc-downsegments and loop f_2 has j disjoint abc-upsegments. Then the number of intersections between f_1 and f_2 is at least $i \cdot j$. In particular, if $f_1 = f_2$, the number of self-intersections is at least $i \cdot j$.*

In the following lemma we chose $x = v_1$ for simplicity, since in the application we can choose $x = v_1$ without loss of generality. The proof of Lemma 2 shows that if $x = v_i$, then the set $\{2, \dots, n\}$ should be replaced by the set of gaps not incident to x, that is $\{0, \dots, n\} \setminus \{i-1, i\}$.

Lemma 2. *Let $n \geq 1$ and assume $x = v_1$. Any family of non-homotopic x-loops can be redrawn without increasing the number of self-intersections and intersections so that every x-loop induces a word such that (i) no two consecutive letters are equal and (ii) the first and the last letter belongs to $\{2, \dots, n\}$.*

Proof. By an *ear* we mean a segment inducing a word aa for any letter a. Taking into account that the gaps 0 and 1 are incident to $x = v_1$, by *x-ear* we mean a segment, which corresponds to a letter 0 or 1 at the start or end of the word: that is, an x-ear has x as one of its endpoints and a crossing of the gap 0 or the gap 1 as its other endpoint. We will remove ears in the first step (thus deleting the consecutive pairs of equal letters) and x-ears in the second step (thus deleting the wrong letters at the beginning or the end).

For the first step, we choose an ear in some loop (between two points of some gap a) and denote its endpoints by u and v. By *uv-gap* denote the set of points in the gap a strictly between u and v. An ear is minimal if there is no other ear with both endpoints in the uv-gap. We remove ears one by one, always picking a minimal ear.

The chosen ear partitions one of the halves of the sphere into two simply connected sets, one of which, that we denote by P, contains the uv-gap in its boundary.

We remove the chosen ear by continuously transforming it to a path which closely follows the uv-gap inside the other half of the sphere, as shown on Fig. 1. By choosing the new path sufficiently close to the equator we can make sure that if a new (self-)intersection with some loop ℓ appears, then by tracing ℓ from that (self-)intersection in a certain direction we cross the uv-gap, thus entering the set P.

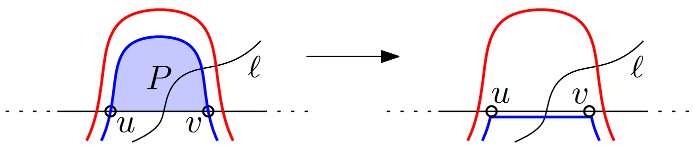

Fig. 1. Removal of a minimal ear

Since $x \notin P$, by tracing ℓ further we must leave P. This cannot happen by crossing the uv-gap again, since that would contradict the fact that we picked a minimal ear. Hence we leave P by crossing the original path of ear. This gives a way to assign, for each newly created intersection with ℓ, a unique intersection with ℓ that was removed, showing that the transformation of the ear does not increase the total number of intersections with ℓ. In particular the number of self-intersections does not increase since we can choose ℓ to be the loop containing the ear in question.

The second step, removing x-ears, is similar to the first one, except that we have to deal with the endpoint x separately. Let v be the point where an x-ear crosses a gap a incident to x (either 0 or 1). Similarly as for ears, by xv-gap we mean the points of gap a strictly between x and v. An x-ear is minimal, if no other x-ear crosses gap a through the xv-gap. We will remove the x-ears one by one, always picking a minimal x-ear.

Since the x-ear is contained in one of the halves of the sphere, it partitions it into two simply connected sets, one of which, that we denote by P, has the xv-gap in its boundary. We remove the x-ear by continuously transforming it into a path that closely follows the xv-gap in the opposite half of the sphere, as shown on Fig. 2. By choosing the new path sufficiently close to the equator, we can make sure that if a new (self-)intersection with some loop ℓ appears, then by tracing ℓ from that (self-)intersection in a certain direction we cross the xv-gap, thus entering set P. Tracing ℓ further we must eventually leave the set P, since $x \notin P$. This cannot happen by crossing the xv-gap again, since that would contradict the fact that we removed all ears in the first step. It also cannot happen by crossing x, since this would contradict that we chose a minimal x-ear. Hence we leave P by crossing the original path of the x-ear, which determines an intersection with the loop ℓ that was removed by transforming the x-ear.

Similarly as in the first step this assigns a unique removed intersection with ℓ to each new intersection with ℓ, showing that removal of a minimal x-ear does not increase the number of (self-)intersections.

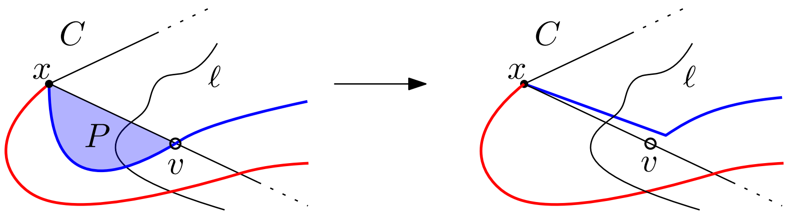

Fig. 2. Removal of a minimal x-ear

We recap what we have proved: by repeatedly removing minimal ears in the first step and removing minimal x-ears in the second step we end up with a drawing which does not have any ears nor x-ears, proving the lemma.

Note that the following lemma holds for $n = 1$ vacuously, since neither of the two conditions can be satisfied.

Lemma 3. *Let $n \geq 1$ and assume $x \in V_n$. Suppose that a, b are adjacent distinct gaps, i.e., $b = a + 1$ or $b = a - 1$ modulo $n + 1$. Let ℓ be an x-loop that induces a word in which no two consecutive letters are equal. Let s be an aba-segment in ℓ. If either (i) x is not a shared endpoint of a and b or (ii) ℓ crosses gaps different from a, b before and after the segment s, then s intersects some other segment of ℓ.*

Proof. Let y be the obstacle incident to gaps a and b. Let u and v be the endpoints of the segment s (thus distinct and both in the gap a) and label them so that u is closer to y than v. Let yu-gap and uv-gap denote part of the a-gap between respective points (or obstacle). The union of s and the uv-gap forms a closed curve without self-intersections (neither the ab arc of segment s nor the ba arc has any self-intersections and the arcs are on the opposite halves of the sphere),which divides the sphere into two parts. Let P be the part of the divided sphere which contains vertex y. As P contains y and its border intersects gap a and gap b, we can partition P by the equator into parts P_1 and P_2, where P_1 denotes the part incident to v, as depicted on Fig. 3.

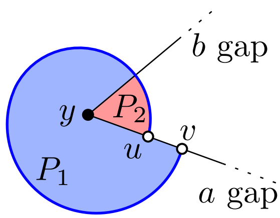

Fig. 3. An aba-segment forces a self-intersection if $x \neq y$ or the loop crosses a gap different from a, b both before and after the segment

Choose an orientation of ℓ so that v precedes u. After passing through u, the loop enters into P_1. Loop ℓ must eventually leave P due to the assumption of the lemma: in the case (i) this is because $x \neq y$ and therefore $x \notin P$ while in the case (ii) it has to reach some gap $c \notin \{a, b\}$ which does not intersect P. Before it leaves the set P, it may cross the equator several times, but only through gaps a and b. Since by assumption ℓ does not cross the same gap twice in a row, the location of the loop before leaving set P is determined by the last crossed gap: it always enters set P_1 after crossing the gap a and enters set P_2 after passing through the gap b.

If the loop leaves P through s, we obtain the desired self-intersection. Otherwise ℓ leaves P through the uv-gap (it cannot leave through point u, since by assumption self-intersections do not occur on the equator). As leaving P through the uv-gap is only possible from set P_1, we obtain that ℓ crosses gap a twice in a row (once to enter P_1 and then to leave through the uv-gap), contradicting the assumption.

4 Case $n = 2$

From now on, we focus on the task at hand and assume that $n = 2$ (meaning that we have exactly 3 obstacles on a sphere, one of which is x), in which case words use letters $0, 1$ and 2.

Proof (of Theorem 1)
Without loss of generality assume that $x = v_1$ and fix an x-loop ℓ. By Lemma 2, we can assume that ℓ induces a word w starting and ending in 2 and with no two consecutive equal letters. Lemma 3 implies that for any two distinct letters a, b in w every aba-segment participates in a self-intersection: for $aba \in \{121, 212, 020, 202\}$ this is because x is not incident to both a and b, while for $aba \in \{010, 101\}$ this is because the word induced by the loop starts and ends in 2.

Since every self-intersection is simple, it occurs in at most two disjoint segments. We claim that the word w induced by ℓ has fewer than $12k$ letters. For contradiction, assume the contrary and partition the first $12k$ letters of w into $2k$ disjoint subwords of length 6. Each of these subwords either contains an aba subword or the word is of the form $abcabc$. Segments in the form $abcabc$ contain an abc-upsegment and an abc-downsegment (for the same word abc) which by Lemma 1 forces a self-intersection. Segments which contain aba subword participate in an intersection. As each intersection may cause at most two participations of disjoint segments, it follows that ℓ has at least k self-intersections, giving a contradiction.

It is easy to see that there are fewer than 2^{12k-1} permitted words with fewer than $12k$ letters. By Proposition 2 at most two non-homotopic x-loops induce the same word. Therefore we conclude that $g(2, k) \leq 2^{12k}$.

5 Proof of Proposition 1

We denote $V_n = \{v_1, \ldots, v_n\}$ the set of points removed from the plane. To see the first inequality in (1), let $x \in V_n$, and fix a family of x-loops that attains the maximum $g(n,k)$. Since the number of (self-)intersections is finite, by continuity of loops, there is a circle centered at x such that (i) each loop intersects it at exactly two distinct points, (ii) inside the circle there are no intersections (other than those at x) and no self-intersections, and (iii) inside the circle there are no points of V_n (other than x). Property (iii) implies that we can homotopically transform the loops inside the circle so that between the circle and x they form straight lines.

Pick a point x' on the circle so that neither x' nor its antipodal point lie on any x-loop. Denoting the points where a loop ℓ crosses the circle by p_l while 'departing' and q_l while 'returning', replace each ℓ by an x'-loop ℓ' in which $p_\ell x$ and $q_\ell x$ are replaced by straight segments $p_\ell x'$ and $q_\ell x'$. Since the pairs p_ℓ and q_ℓ are pairwise disjoint, this does not create additional intersections, see Fig. 4.

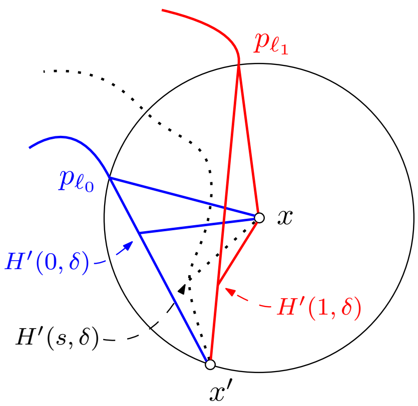

Fig. 4. Constructing homotopy for modified loops, showing only the 'departing' ends of the loops, i.e., for the argument t close to 0.

It remains to check that no two of the resulting x'-loops are homotopic (with respect to V_n). Assuming for contradiction that H' is a homotopy between x'-loops $\ell'_0 = H'(0,\cdot)$ and $\ell'_1 = H'(1,\cdot)$, we will construct a homotopy between original loops ℓ_0 and ℓ_1. Pick $\varepsilon > 0$ such that there are no obstacles in a ball of radius ε around x'. Elementary analysis implies that there is $\delta = \delta(\varepsilon)$ such that

$$\max_{s\in[0,1]} |H'(s,t) - x'| < \varepsilon, \quad t \in [0,\delta] \cup [1-\delta, 1]. \tag{3}$$

Fix $s \in [0,1]$ and define a function $H(s,t)$ as follows. Set $H(s,t) = H'(s,t)$ for $t \in [\delta, 1-\delta]$. On the interval $[0,\delta]$ connects x and $H'(s,\delta)$ by a straight line, that is, set $H(s,t) = x(1-t) + tH'(s,\delta)$, and symmetrically on $[1-\delta, 1]$ connects x

and $H'(s, 1-\delta)$ by a straight line, that is, set $H(s,t) = xt + (1-t)H'(s, 1-\delta)$. By (3) and choice of ε none of these two segments hits any obstacle other than x.

Assuming δ is small enough we can make sure that for $i \in \{0,1\}$, $H'(i,\delta)$ and $H'(i, 1-\delta)$ lie inside the circle on the straight segments of the loop ℓ'_i. It is easy to see that H is a homotopy with respect to V_n, so $H(0,\cdot) \sim H(1,\cdot)$. By replacing the two initial straight segments of $H(0,\cdot)$ (namely x to $H'(0,\delta)$ and $H'(0,\delta)$ to p_{ℓ_0}) by the segment from x to p_{l_0} (and similarly the final two segments at the other end of the loop) we obtain the loop ℓ_0. The three segments form a triangle with no elements of V_n inside it (and similarly for the triangle at the other end), which implies that $H(0,\cdot) \sim \ell_0$. By the same argument $H(1,\cdot) \sim \ell_1$. Recalling that $H(0,\cdot) \sim H(1,\cdot)$, we obtain $\ell_0 \sim \ell_1$, a contradiction.

To see the second inequality in (1), we choose a family of x-loops that attains the maximum $f(n,k)$. Since none of the x-loops passes through x, they are also x-loops with respect to $V_{n+1} := V_n \cup \{x\}$. To show that this family of x-loops gives a lower bound to $g(n+1,k)$, we observe that if two x-loops f_0, f_1 are non-homotopic with respect to V_n, then they are non-homotopic with respect to V_{n+1}. Indeed, assuming for contradiction that that there is a homotopy H between f_0 and f_1 satisfying $H(s,t) \notin V_{n+1}$ for all $s,t \in (0,1)$, it trivially satisfies $H(s,t) \notin V_n$ for all $s,t \in (0,1)$, and thus $f_0 \sim f_1$ with respect to V_{n-1}, giving a contradiction.

This completes the proof of Proposition 1.

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