On the Formal Incompleteness of Reductive Logic

J. Rowan Scott

Abstract A proof of formal incompleteness is spelled-out in relation to 'bottom-up' reductive Logic and three stacked abstract reductive formal system models composing a three-level hierarchy of complexity. Undecidable reductive propositions appear at the 'upper' boundary of each inter-related formal system model of 'sufficient complexity'. The consequence of formal reductive incompleteness demands necessary meta-consideration in the determination of reductive logical consistency. The further implications of formal reductive incompleteness predict that identified undecidable reductive propositions might be decided in adapted reductive formal system models using modified reductive Logic and slightly different axioms and rules that can handle undecidable dynamics. The adapted Logic and reductive formal system models must accept the inevitably of reductive incompleteness. Reductive incompleteness further predicts the exploration of an adjacent possible abstract domain in which 'bottom-up' formal reductive Logic can be preserved as a 'special case'; while multiple adapted forms of reductive Logic as well as multiple inter-related adapted reductive formal system models, develop a deeper understanding of how to abstractly manage undecidable dynamics and reductive incompleteness. The insights, outcome and implications arising from the exploration of the abstract domain, could instigate further scientific work determining whether modified reductive Logic and reductive formal system models sensitive to undecidable dynamics, provides a closer approximation of natural incompleteness driven, novelty generating evolutionary processes. Any abstract or applied, mathematical, computational or information grounded system model, equivalent to a reductive formal system model, will predictably be shown to manifest undecidable dynamics and incompleteness driven novelty generation. The implications of formal reductive incompleteness can be extended to the study of non-linear systems, Chaos Theory, Cellular Automata, Complex Physical Systems and Complex Adaptive Systems. Subsequently, future scientific and mathematical thought may derive incompleteness driven novelty generating formulations of reductive scientific philosophy, epistemology, methods for theoretical modeling and experimental methodology; each of which will unravel the implicit or explicit intention of the Reductive Scientific Paradigm to compose an

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integrated, unified, closed, resolved and complete 'bottom-up' Reductive Scientific Narrative describing our Universe. An encompassing incompleteness driven Meta-Reductive Scientific Paradigm is a possibly, wherein novel approaches to previously unresolvable reductive scientific problems may reveal a unique path toward consilience and integration of the Reductive Sciences and the developing Complexity Sciences.

Keywords Gödel · Formal incompleteness · Formal reductive logic · Formal reductive incompleteness · Reductive science · Complexity science

1 Introduction

Slightly less than 400 years ago, René Descartes developed the first precepts of reductive natural science as well as formulating a *dualistic* philosophy of the human body, brain, mind and consciousness. Descartes precipitated a deep rift; excluding mind and consciousness (*res cogitans*) from reductive natural science and alienating mind and consciousness from brain (*res extensa*), rending open the unresolved Cartesian gap [10].

An unresolved puzzle attributable to Descartes weaves together a Gordian knot involving 'bottom-up' reductive scientific Logic ($\mathcal{L}_{\mathcal{R}}$), the residue of Cartesian dualism, and the historical and modern philosophies of science, mathematics, consciousness and mind. The riddle is most easily visualized in relation to two philosophical propositions, both of which employ, strong, rigorous, well-formulated reductive logic in order to arrive at perplexing statements in the form of *reductive epiphenomenalism of consciousness* [32] and *eliminative materialism* [31]. These two reductive propositions, and the implications associated with their acceptance as well-formed reductive arguments, can be *reframed* as *unresolvable self-referencing paradoxes* [3]. When composed in the form of self-referencing paradoxes, both arguments reveal a structure similar to the Liar paradox, which is stated: "This statement is false". *Reductive epiphenomenalism* can be stated: "*This* logically true reductive epiphenomenal *statement* says the intentional consciousness that composed it *is false*, an epiphenomenal illusion that does not exist"; and, *eliminative materialism* can be stated: "*This* logically true eliminative materialist *statement*, says the intentional consciousness that composed it can be fully reduced to brain, declaring intentional consciousness and mind to be *false*, unnecessary and invalid concepts". These two self-referencing, paradoxical, strong, reductive statements expose potential internal *contradiction* and a clear threat to reductive *logical consistency*. 'Bottom-up' reductive Logic can 'explode' into *triviality,* becoming useless and capable of proving true *anything* [34].

Vigorous attempts, to analytically *question* the logical *truth,* to find error in the reductive logic, to logically resolve the epiphenomenal and the eliminative materialist self-referencing paradoxes, have been unsuccessful [35], leaving scientists and philosophers trying to make convoluted sense of a Universe in which willful, mindful, subjective consciousness and the adaptive agency of human beings are erased from existence [11, 12].

The self-referencing paradoxical formulations of the epiphenomenal and eliminative materialist reductive arguments uncover the possibility that rigorous 'bottom-up' reductive Logic employed in *every* natural science, *could* contain unrecognized *undecidable reductive propositions,* suggesting formal reductive Logic is susceptible to *formal reductive incompleteness.* At a minimum this conclusion can *metaphorically* be related to Gödel's two formal incompleteness theorems [34]. More significantly, formal 'bottom-up' reductive Logic may be susceptible to *formal reductive incompleteness.* Formal proof of undecidable reductive propositions and formal reductive incompleteness could lead to a very different understanding of 'bottom-up' reductive Logic within every reductive natural science and it could also lead to a significant transformation of the entire Reductive Scientific Paradigm.

David Hilbert's 23-problem 1900 Foundations of Mathematics Challenge was clearly intended to advance both mathematics *and* natural science [14]. Kurt Gödel completely solved Hilbert's second problem, which asked for proof of the consistency of the axioms of arithmetic forming the foundations of mathematics. Gödel solved it in a completely unanticipated way by demonstrating the incompleteness of the axioms of arithmetic and the necessity for meta-consideration in order to establish the consistency of the axiom system. Gödel effectively demonstrated that it is not possible to *prove* every *true* mathematical statement.

Kurt Gödel [22] provided a meticulous, detailed logical argument in the 1931 proof of his two incompleteness theorems [9]. In the *first* incompleteness theorem, Gödel proves that any formal mathematical system employing first-order logic (first-order predicate calculus) that contains Peano's axioms of arithmetic [13, 17, 24]; as long as the system is consistent, i.e. allows no inconsistent contradictions, it will inevitably contain undecidable propositions or statements revealing the systems fundamental incompleteness. In his *second* incompleteness theorem, Gödel proves an implication that any such formal system cannot demonstrate its own consistency but must resort to external meta-mathematical analysis and considerations in order to prove consistency [27]. Gödel *generalized* his findings, proving that *formal incompleteness* impacts a wide range of abstract formal systems and computational systems of sufficient complexity [9, 19].

Gödel was deeply committed to the existence of an idealized, abstract Platonic mathematical realm. He was therefore antagonistic and often angered by attempts to apply his two abstract proofs of incompleteness to real world problems in applied mathematics, physics or science [1]. Despite Gödel's expressed reluctance many other authors have explored the implications of the incompleteness theorems in close or distantly related areas of mathematical and scientific interest [7, 9, 20, 24].

Among these explorations, curiously, there are no detailed attempts to apply the two incompleteness theorems to the formalized logic of reduction in natural science [36], despite the obvious importance of knowing whether or not the logic of scientific reduction is susceptible to *incompleteness* and whether or not reduction as a form of formal logic is capable of proving its own *consistency.* The answers to these two questions bear directly on fundamental concerns, involving whether or not reductive natural science is capable of ever creating a closed, resolved, 'decided' and complete scientific model that closely approximates the natural evolutionary logic of Nature and evolution [8, 21, 23, 27].

In experimental reductive science, *Nature always gets the last word*. The natural logic of Nature and evolution may have an unrecognized inherent relationship with *natural undecidable dynamics* and *natural evolutionary incompleteness*. If so, the natural logic of evolution may ultimately be proven to be open, unresolvable, 'undecidable', as well as, forever incomplete. If natural evolutionary incompleteness permeates Nature and the natural logic of the evolutionary process, then *undecidable dynamics* and *natural incompleteness* may drive a natural interplay between, *existent* patterns of order and consistency, *ambiguous* moments of novelty generating incompleteness, and emerging patterns of *meta-order* and *meta-consistency*.

If this is the way Nature works, then, in order to more closely approximate the natural logic of the evolutionary system, 'bottom-up' reductive Logic, the reductive sciences and the Reductive Scientific Paradigm will need to be reformulated.

2 Defining a Context for the Question

The *reductive scientific experimental methodology* ($\mathcal{M}_{\mathcal{R}}$) of modern reductive science proceeds in three steps: (1) *Reduction and Analysis*: A scientist begins by selecting a 'higher-order' phenomena, then taking apart the phenomena into its disjoint elements and individually investigating each of these; (2) *Theoretical Formulation*: By accumulating experimental evidence, using imagination, and trusting luck, a scientist then formulates a model describing how the disjoint components relate to each other and how they interact; (3) *Synthesis and Construction*: Using the composed theory defended by experimental evidence, the scientist again evaluates the theoretical, qualitative and quantitative success of the constructed model of the disjoint parts, their relations and interactions. The scientist again compares the synthesis with the experimental, qualitative and quantitative behavior of the original selected higher-level phenomena of interest, in hopes of demonstrating the scientific understanding of the phenomena is complete [33].

In order to demonstrate the presence of *formal reductive incompleteness* an adequate simplification and idealized model of reductive science, epistemology and the reductive scientific process is necessary (i.e. a 'sufficiently complex' context). The *simplified* model must be *sufficiently complex* that contradiction, self-referencing paradox and potentially threatening logical inconsistency are possible. There must be enough complexity that *unresolvable* and *undecidable concepts* and *propositions*, *undecidable theoretical dynamics* and *undecidable reductive incompleteness* can be spelled-out.

More profoundly, the simplified, abstract, epistemological model must be *sufficiently complex* to model *theoretical undecidable entities* and *processes*, which may reveal experimentally accessible examples of *ontological*, *natural, evolutionary, unresolvable instantiations.* Thus, *theoretical formal reductive incompleteness* may reveal the presence of *natural undecidable reductive systemic dynamics* and *natural reductive evolutionary incompleteness.*

3 *Mathematization* Defines a Reductive Formal System Model

In order to successfully uncover formal reductive incompleteness, a simplified scientific context and mathematized Reductive Formal System Model (RFSM), must be sufficiently complex to formulate the Liar's paradox or a similar *self-referencing paradoxical statement*, thereby establishing what is necessary to instantiate *negation*, formulate *diagonalization* and finally, support a definition of *universality* and *undecidability*. The RFSM used here to model reductive science therefore must conform to the logical rigor and definition of an abstract mathematical Formal System Model defined by Kurt Gödel in his two 1931 formal incompleteness theorems [30], as well as the course-grained examination of incompleteness presented by John Casti in 1995 [5], and the fine-grained explication spelled-out by Prokopenko et al. in 2019 [27]. It must also be consistent with definitions offered for reduction in recent philosophical explorations of reductive Logic in science [36], reductive Logic in biology [4]. Finally, the RFSM must be consistent with the troubling implications of applied reductive Logic when it underpins work on the brain in the neurosciences [2], wherein the implications of strong reductive Logic paradoxically reduce and erase from existence or conceptually invalidate any concept associated with consciousness and mind [31, 32, 35].

Following the path cleared by Kurt Gödel, Casti and Prokopenko et al., let us begin:

An abstract formal mathematical system F can be defined with a minimum of three components:

$$F = \langle A_F, X_F, R_F \rangle \tag{1}$$

where

1. A_F is a set constructing an *alphabet.* In the abstract, A_F is an ordered and finite set of symbols, such that $A*_F$ can be composed as a set of *words* or *strings* of finite linear sequences of symbols, composed in part from the symbols contained in A_F. The *Kleene Star* or *Operator* (*) is an abstract way to form $A*_F$ from A_F.
2. X_F is a finite *subset* of the set $A*_F$, symbolized, $X_F \subseteq A^*{}_F$, creating a specific set of *axioms* (an established statement or proposition from which further defined structures can be composed).
3. R_F is a finite *subset* of the set $A*_F$, symbolized, $R_F \subseteq A^*{}_F$, symbolically defining possible *relations,* called *rules of inference*; from which the alphabet A_F and axioms X_F can be composed into the words or strings in the set $A*_F$.

A similar abstract reductive formal system model (RFSM) can be defined with a minimum of three components. This *general* pattern can be used to create a three-level hierarchy by stacking the RFSM's, allowing us to capture the abstract similarities shared between the levels. The generalization leaves open conceptual space for later filling in *specific* differences and specific events in the sequence of organization, which might be associated with the self-organization of each 'bottom-up' hierarchically organized level.

Therefore, we concentrate first on a *general* description of *any* level (L_1 through L_3), and *any* Theory (T_1 through T_3). The minimal three, *similar* components of *any* level in a *general* Reductive Formal System Model, are:

$$F_R = \langle A_{F_R}, X_{F_R}, R_{F_R} \rangle \tag{2}$$

where

1. A_{F_R} is an abstract epistemological symbol set, symbolically representing ontological, naturally occurring fundamental entities (known or not yet known) that can be finitely (or infinitely) ordered or organized, representative of a scientifically accessible naturally self-organized *fundamental alphabet*. The set $A^*{}_{F_R}$ is the set of *words* or *strings* that can be composed from A_{F_R} using the abstract Kleene Star or Operator (*). Words or strings in $A^*{}_{F_R}$ are symbolic representations of possible (scientifically theorized or methodologically demonstrated) combinations of naturally occurring entities (defined in A_{F_R}). Natural *words* or *strings* (entities) are self-organized in evolution, while parallel reductive scientific conceptual and theoretical *words* or *strings* are composed statements or propositions, intended to create a complementary system model closely approximating the natural system. Scientific *words* or *strings* composed in $A^*{}_{F_R}$ are also *methodologically* intended to closely conform to and model naturally evolved *words* or *strings,* resulting in $A^*{}_{F_R}$ and A_{F_R} being components of an effective Formal System Model, F_{F_R}, capable of mirroring, reducing, predicting and replicating the natural entities found within Level L_1 and Theory T_1. The *words* or *strings*, whether finite or infinite, depend on how natural entities and their possible relations, modeled as an alphabet in A_{F_R}, can be linked, chained together, or concatenated (*) in $A^*{}_{F_R}$.
2. X_{F_R} is a finite (or infinite) *subset* of $A^*{}_{F_R}$, symbolized, $X_{F_R} \subseteq A^*{}_{F_R}$, creating a specific set of *epistemological abstract axioms* modeling what are presumed to be natural *ontological axiomatic entities* (*abstract axioms* and *natural axiomatic entities* create established or accepted reductive statements or reductive propositions from which further defined reductively organized structures can be composed).
3. R_{F_R} is a finite (or infinite) *subset* of the set $A^*{}_{F_R}$, symbolized, $R_{F_R} \subseteq A^*{}_{F_R}$, *symbolically* defining *epistemological defined rules of inference* and *possible relations* within the abstract model of natural entities. *Rules of inference* define *possible relations*; from which the alphabet A_{F_R} and axioms X_{F_R} can be further composed into *words* and *strings*, concatenated (*) in the set $A^*{}_{F_R}$. The natural

evolving system has an underlying logic and inherent evolutionary order to how it self-organizes *entities*, which reveals how Nature instantiates its *possible ontological relations*. Reductive natural science attempts to mimic and model the natural system, through scientific epistemology, experimental methodology and theoretical description; creating a *model* of *possible relations* from a blend of identified, experimentally confirmed natural laws and rules, reductive scientific rules of inductive and deductive reasoning, and abstract rules of inference synthesized from various formulations of abstract formal logic or developed within multiple mathematical system models.

Two broad categories of scientific reduction (Theoretical Reduction and Explanatory Reduction) and three further reductive scientific conceptual frameworks with their associated inter-connections (Ontological reduction, Methodological reduction and Epistemological reduction), can be defined and *embedded* in the overall concept of a *general* Reductive Formal System Model, $F_{F_R} = \langle A_{F_R}, X_{F_R}, R_{F_R} \rangle$.

3.1 Theoretical Reduction

Theoretical reduction refers to a theoretical reduction of *specific* theories: T_3 to T_2 to T_1. It captures the *intention*, to take a fully developed scientific theory describing one semi-isolated level (T_3 for L_3; or T_2 for L_2; or T_1 for L_1), composed in a hierarchical model of an evolved hierarchy of complexity (T_3 and T_2 and T_1); to reduce the selected specific *theory* to a reductive description using only the concepts derived in the next lower level down the hierarchy (T_3 reduces to T_2; T_2 reduces to T_1; with T_1 treated as the most fundamental accessible layer of the evolved hierarchy).

In Theoretical Reduction, as defined here, each layer of the hierarchical description of the evolved hierarchy of complexity, can be defined by its own fully developed Theory and Reductive Formal System Model: Thus, T_3 and T_2 and T_1 can be described by:

1. $T_3 = F_{F_{R_3}} = \langle A_{F_{R_3}}, X_{F_{R_3}}, R_{F_{R_3}} \rangle$ and
2. $T_2 = F_{F_{R_2}} = \langle A_{F_{R_2}}, X_{F_{R_2}}, R_{F_{R_2}} \rangle$ and
3. $T_1 = F_{F_{R_1}} = \langle A_{F_{R_1}}, X_{F_{R_1}}, R_{F_{R_1}} \rangle$ (3)

T_1 describes the composition and properties of 'fundamental' T_1 entities and interactions between T_1 entities, leading to the fabrication of the T_2 entities. T_2 describes the composition and properties of T_2 entities and the interactions between T_2 entities, leading to the fabrication of T_3 entities. T_3 describes the composition and properties of T_3 entities and the interactions of those entities. The expectation and prediction are that T_3 can be fully reduced to T_2; T_2 can be fully reduced to T_1; with T_1 being treated as the most fundamental accessible layer of the evolved hierarchy. Everything can be deduced from T_1.

3.2 Explanatory Reduction

Explanatory Reduction refers to a restricted reduction providing a reductive explanation for any semi-isolated phenomena *x* found *in* level L_3 and/or L_2 and/or L_1, located within the three-level theoretical hierarchy describing the semi-isolated three-level evolved hierarchy of complexity. Explanatory Reduction can also refer to any localized reduction producing an explanation for any semi-isolated phenomena *x* found *in the inter-space* of connection between level L_3 and/or L_2 and/or L_1. Thus, theoretical reduction can itself be reduced to a large number of lesser explanatory reductive explorations.

3.3 Further Subdivision of Theoretical and Explanatory Reduction

Theoretical and Explanatory Reduction can both be further *subdivided*, into *three* inter-related *kinds of reduction.* An implicit or explicit *three-domain* relationship is interwoven into the work of reductive natural science, based on underpinning *reductive premises* associated with *ontological reductive assumptions* (Theory of Reductive Ontological Description), accepted *reductive assumptions about methodological* or *reductive experimental approaches* (Theory of Reductive Observation and Experimental Technique) and *epistemological reductive assumptions* (Theory of Reductive Knowledge). These three *sets* of underpinning *reductive premises* interrelate by *assuming* that an *epistemology* using *abstract reductive scientific logic*, in some important sense, can *theoretically*, *methodologically* and *experimentally* create a *sufficiently close enough approximation* that it will provide a *good-enough model* of *natural ontological systemic evolutionary logic.* 'Good-enough' therefore refers to an, in principle, assumption of reductive closeness of real-world approximation, *and* to an, in practice, acceptance of the usefulness of real-world reductive experimental methods and application.

The *two domains* of reductive scientific interest (Theoretical Reduction and Explanatory Reduction) and the *three* inter-related *kinds of reduction* (Ontological reduction, Methodological reduction and Epistemological reduction), have a relationship with the *Ontology of Nature and the natural evolutionary system* O_n. This can be symbolized in the following way:

1. *Ontological Reduction* (r_o): involves philosophical *ontological* commitment to a meta-physical assumption about the nature of *ontology* and the nature of *being*: There is an underlying ontological assumption that the reality of the universe and the natural evolving system (O_n) conforms to 'bottom-up' reductive principles and 'bottom-up' reductive logic and therefore science must also conform to similar reductive principles and logic in order to succeed at the task of modeling Nature and evolution.

2. *Methodological Reduction* (r_m): involves philosophical and scientific *methodological* commitment, to a reductive system of methods for investigating and formulating results in every field of scientific work: There is an underlying methodological assumption that 'bottom-up' reductive principles and logic, implemented in experimental scientific methods, will effectively and closely approximate the natural logic of Nature and the natural evolving system (O_n).
3. *Epistemological Reduction* (r_e): involves philosophical scientific epistemological commitment, to a reductive theory of knowledge, particularly in regard to the validity and scope of a reductive approach to theory creation and a reductive approach to the selection of experimental methods. The epistemological commitment rests upon decisions about what justifies a scientific belief or opinion: i.e., there is an underlying epistemological assumption that 'bottom-up' reductive principles, and applications of reductive logic, are justified, and can effectively and closely approximate the natural logic of Nature and the natural evolving system (O_n).

The explicit or implicit *reductive premise* and *hope* of reductive natural science, in examining Nature through the lens of Ontological Reduction (r_o), Methodological Reduction (r_m) and Epistemological Reduction (r_e); states that science has in the past succeeded and predicts that science will succeed further in the future, where $r_o \approxeq r_m \approxeq r_e$ and these are found to be synchronous, commensurate and aligned in a pattern of reductive consilience, closely mimicking the natural logic of Nature and the natural evolving system (O_n). Thus:

$$O_n \cong F_{R(r_o \approxeq r_m \approxeq r_e)} \tag{4}$$

The *reductive premise* suggests, that the simplified 3-Theory hierarchically 'stacked' Reductive Formal System Models, as well as the description of the inter-level 'joints' linking one level to another; should be capable of modeling a simplified natural 3-level emergent hierarchy of complexity. The reductive account should therefore provide a *minimal, generalized, fully complete picture* of the 3-level hierarchy of complexity.

Continuing with the exploration, we are particularly interested in finding out whether, within a single Reductive Formal System Model (RFSM) F_R:

$$F_{(r_o \approxeq r_m \approxeq r_e)} = \langle A_{F_{(r_o \approxeq r_m \approxeq r_e)}}, X_{F_{(r_o \approxeq r_m \approxeq r_e)}}, R_{F_{(r_o \approxeq r_m \approxeq r_e)}} \rangle \tag{5}$$

…is it possible, within the structure of a Reductive Formal System Model (RFSM), to construct an *undecidable reductive proposition* declaring *formal reductive incompleteness*? The structure of the following *proof* follows Gödel's incompleteness theorems [30] and recent work by Prokopenko et al. [27].

3.4 Mathematization and Proof of Formal Reductive Incompleteness

A scientific expression W is derivable or provable in F_R, if and only if there is a finite or infinite sequence of expressions $W_1, \ldots, W_n$ in which the statement $W \equiv W_n$ and every W_i is either a member of the alphabet, an axiom or results from the application of an inference rule applied to earlier expressions in the sequence. Where possible or necessary the statement W must also be fully defined within $F_R = F_{(r_o \approxeq r_m \approxeq r_e)}$. Using a standard notation drawn from logic, $F_R \vdash W$, expresses that W can be formulated, derived and proven in the Reductive Formal System F_R. This means there is a *proof* of W in F_R, therefore W is a statement, a proposition, or a theorem of F_R.

Proof and *truth* in reductive natural science are *already* differentiated by the necessity of spelling out W in $F_R = F_{(r_o \approxeq r_m \approxeq r_e)} \cong O_n$. This differentiation of proof and truth in reductive science is defined by the complicated dynamics natural science has with itself and with Nature. Ontological reductive, methodological reduction and epistemological reduction differentiate *proof* and *truth* even before we specifically seek out an analog of Gödel's formal incompleteness and before the necessary separation of *truth* and *proof* specific to formal reductive Logic. In Science, only Nature and ontology 'know' when a scientific statement W is *scientifically true.* Nature 'knows' its *truth* by naturally instantiating a phenomenon in the realized process of natural evolution. Reductive natural science 'knows' its *truth* only through successive theoretical approximations supported by experimental evidence, thus accumulating *proof* from the sum of sequential approximation, reduction, replication and prediction in theoretical and experimental contexts, supported by scientific *epistemology* (r_e), scientific *methodology* (r_m), and a slowly developing understanding of *ontology* (r_o). *Truth* in scientific understanding of *ontology* is never *closed*; it remains *open* and forever susceptible to *falsification,* whenever theoretical and experimental *proof* of a better approximation appears. *Truth* and *proof* in formal reductive Logic has so far been immune to any demonstration that they too are separated by an inevitable gap defined by *formal reductive incompleteness* and its implications. I can only speculate that the absence of any attempt to address this problem directly is in part based on respect for Gödel's strong reluctance to go there and in part because of the complicated and far reaching and difficult to handle consequence and implications of proving reductive incompleteness and the separation of truth and proof in reductive Logic.

Continuing now with the development of the proof in relation to reductive Logic: Just as in the development of a formal or mathematical system, in the Reductive Formal System Model F_R, in order to label W as a *theorem* it would be necessary to define and employ *in advance* some kind of *external* reasoning or measure that is able to characterize W as either an intermediate statement, or as a target expression and theorem. Such a predictive meta-level method must be able to recognize in advance the importance or salience of W as a target theorem. It is not yet clear how this might be achieved in an abstract mathematical Formal System, F, or in the Reductive Formal System Model F_R. Existing formal system and mathematical developments are not able to formally make this particular predictive decision regarding the salience of target theorems [27]. In modern Logic and Formal Systems Theory and in modern mathematical systems theory, there is, however, an awareness of the proof and implications of both the Halting Problem [6] and Gödel's incompleteness theorems [30], both of which provide an explanation for why theorists are not yet able to *predictively* define any external *meta-level criterion* capable of deciding whether W is or is not a target theorem.

Nevertheless, the limitations set by the Halting Problem and Formal Incompleteness have not stopped theorists from deciding from meta-level consideration, whether or not, a '*completed*' or '*final*' well-formed formal or mathematical statement and proof of W, is or is not, a salient target or theorem. It can be decided in relation to a *complete* or *finished* piece of work, it just can't be decided in advance. This realization is particularly significant in the present argument, in relation to Reductive Natural Science and the development of a Reductive Formal System Model F_R.

Past and present scientific developments have not yet formalized a Reductive Formal System Model sufficiently so that the implications of the Halting problem and Gödel's incompleteness theorems can be seen at work in the dynamics of F_R. The implications of proving the Halting problem and Gödel's incompleteness theorems *are* relevant in a Reductive Formal System Model, Reductive Science and the Reductive Scientific Paradigm; includes the realization that, for formal logical reasons, any scientific theory about a 'part' of Nature or the 'whole' reductive scientific narrative describing our Universe, *cannot* ever be closed, resolved, decided or complete.

Whether or not a statement is *intermediate* or a *target* statement *cannot be decided in advance* because there is always the possibility that whatever is now considered *true,* 'closed', 'complete', 'final', or 'finished' in natural science, and thus *believed* to be a 'target' statement; will be found instead to be an *intermediate statement*. Contradictory *proof* and *external meta-level consideration* may reveal a significant separation between *truth* and *proof* and a significant relationship may be found between *completeness* and *incompleteness,* as well between *consistency* and *inconsistency.* This may lead to a subsequent encompassing meta-reductive re-formulation.

Because this argument is about reductive Logic and Natural Science; *Nature must get the last word* regarding whether or not the natural Logic of the evolutionary process also instantiates a significant separation between naturally instantiated *truth* and naturally instantiated self-referencing *proof*. Nature may instantiate a significant relationship between order and disorder, evolutionary *completeness* and *incompleteness,* as well as evolutionary *consistency* and *inconsistency.*

4 Further *Mathematization*

At this point, in forming the set of 'words' and 'strings' of $A^*{}_{F_R}$, the Reductive Formal System Model F_R does not yet conform to any particular syntactical constraints, beyond the restrictive 'bottom-up' assumption, and the restrictions set by the definition of axioms (X_{F_R}) and the scientific rules of inference, defined in R_{F_R}. In order to move further toward a statement matching the complexity of a modern abstract formal system, it is necessary to expand the definitions and components of the Reductive Formal System Model F_R, so as to define and then concentrate only on *well-formed formulas* (abbreviated as wff's). Well-formed formulas, wff's, are fabricated from the reductive alphabet (A_{F_R}), the reductive axioms (X_{F_R}), the reductive rules of inference (R_{F_R}), the 'words' and 'strings' of $A^*{}_{F_R}$ *and* a *formalized* version of a *reductive grammar* (G_{F_R}).

A formalization of a *reductive grammar* (G_{F_R}) consists of the following components:

$$G_{F_R} = \langle A_{F_R}, N_{F_R}, P_{F_R}, S_{F_R}, C_{F_R}, E_{F_R} \rangle \tag{6}$$

where

1. A_{F_R} is an abstract symbol set of 'terminal' symbols, symbolically representing naturally occurring fundamental entities (known or not yet known) that can be finitely (or infinitely) ordered or organized, scientifically representative of a naturally self-organized but scientifically accessible *fundamental alphabet*;
2. N_{F_R} is an abstract symbol set (which *can* include the three *start symbols* such that $(\langle E_{F_R}, C_{F_R}, S_{F_R} \rangle \in N_{F_R})$, symbolically representing a finite (or infinite) set of 'nonterminal' symbols, that can define and form relationships between A_{F_R} symbols but are *disjoint* from $A^*{}_{F_R}$, i.e., the 'words' and 'strings' formed from A_{F_R};
3. P_{F_R} is a finite (or infinite) set of *production rules,* generally of the form $\left(A_{F_R} \cup N_{F_R}\right)^* N_{F_R} \left(A_{F_R} \cup N_{F_R}\right)^* \rightarrow \left(A_{F_R} \cup N_{F_R}\right)^*$. Each production rule maps from one 'word', or 'string' of symbols to another, with the 'head' string containing an arbitrary number of symbols provided that at least one symbol is 'non-terminal';
4. S_{F_R} is a *start symbol such that* $S_{F_R} \in N_{F_R}$. The start symbol S_{F_R} is *non-terminal* and is *disjoint* from $A^*{}_{F_R}$. In modeling Nature and the process of evolution from the presumed point of origin in the dense high-energy state in the Big Bang, S_{F_R} takes the form $S_{F_R} + C_{F_R} = E_{F_R}$, where E_{F_R} *is the dense high energy state from which the entire evolved universe is created through transformations of energy, into lower, evolving energy states and processes defined by* S_{F_R}*,and associated space–time manifestations, represented in* C_{F_R}. In later, evolved, hierarchically organized and semi-isolated systems selected from within the ongoing process of natural evolution (i.e., the three-level hierarchy being explored in the simplified model of the scientific situation), S_{F_R} is defined by the *initial state* of the *necessary* transformations of E_{F_R}*, creating a necessary* 'terminal' set of members of A_{F_R}, composed in $A^*{}_{F_R}$, along with an *initial* 'non-terminal' *process*, defined by the necessary set of production rules in P_{F_R}. The set created by the *initial state and process* defined by S_{F_R} is '*non-terminal'* setting in motion everything that subsequently happens, however, it is 'disjoint' from any specific subsequent arrangement of state and process in the system.

5. C_{F_R} is a *start symbol such that* $C_{F_R} \in N_{F_R}$. This start symbol C_{F_R} is *non-terminal* (or *unknown* as to its limit) in relation to the vastness of space–time but it can be composed and selected to be *terminal* in the choice of semi-isolated system within an experimental context. C_{F_R} is also *disjoint* from $A^*{}_{F_R}$. In modeling Nature and the process of evolution from the presumed point of origin in the original dense high-energy state in the Big Bang, C_{F_R} shares an evolutionary sequence of relationships $E_{F_R} \to S_{F_R} + C_{F_R}$, where E_{F_R} is the dense high energy state from which the entire universe evolved and from which the expanding energy-space–time manifold of the universe, and lower-energy-matter entities are created through transformations of high-energy states and processes into lower-energy states and processes, defined by S_{F_R}, *and* associated space–time manifestations and relationships, represented by C_{F_R}. In later, evolved, hierarchically organized, complex, semi-isolated systems *selected* from within the ongoing process of natural evolution (i.e., the three-level hierarchy being explored), C_{F_R} represents both the background bounded or unbounded space–time encompassing the selected semi-isolated system, *or* the selected finite manifold of space–time (infinite or unknown if the whole universe is the experimental context), contained within the bounded domain or container of the semi-isolated system of interest.
6. E_{F_R} is the initial evolutionary *start symbol such that* $E_{F_R} \in N_{F_R}$, from which C_{F_R} and S_{F_R} evolve through transformation of energy in an abstract, in *principle*, reversible relationship, $E_{F_R} \leftrightarrow S_{F_R} + C_{F_R}$. C_{F_R} and S_{F_R} arise from E_{F_R}, and they are the *start symbols* for all further evolution, which is defined in an abstract, in *practice*, non-reversible evolutionary relationship $E_{F_R} \to S_{F_R} + C_{F_R}$, where energy, space–time and matter transformations, patterns of composition, interactions and relationships, become the further symbolic representation of all subsequent evolution.

The 'complete' 'bottom-up' Reductive Formal System Model F_R can thus be represented:

$$F_R = \langle E_{F_R} \rangle \tag{7}$$

where

$$\langle E_{F_R} \rangle = \langle S_{F_R}, C_{F_R} \rangle \tag{8}$$

such that

$$E_{F_R} \to \langle S_{F_R}, C_{F_R} \rangle \tag{9}$$

and where

$$\langle E_{F_R}, S_{F_R}, C_{F_R} \rangle = \langle A_{F_R}, N_{F_R}, P_{F_R}, X_{F_R}, R_{F_R} \rangle \tag{10}$$

such that this equation

$$\langle E_{F_R} \rangle \rightarrow \langle S_{F_R}, C_{F_R} \rangle \rightarrow \langle A_{F_R}, N_{F_R}, P_{F_R}, X_{F_R}, R_{F_R} \rangle \tag{11}$$

can be used to formally model the *initial start symbols* and the sequence of evolution of the '*whole*' universe.

For practical reasons, associated with formally modeling *local* and *later sequences* of evolution, involving evolved '*part*' systems within the evolved universe (where the original, initial, energy, space–time, matter, 'whole' *start symbols* are now *implicit*; while the *selected* 'part' *start symbols* for a *selected semi-isolated system of interest*, are *explicitly* now defined as consisting of some evolved initial organization of state and process, with an evolved composition and arrangement of energy, space–time, matter, entities, interactions and relationships), the relationship

$$\langle E_{F_R}, C_{F_R}, S_{F_R} \rangle \in N_{F_R} \tag{12}$$

can be applied. This relationship allows a scientist to visualize a local or later sequence of interest within the continuous thread and flow of natural evolution, in which the fundamental relationships associated with the natural system are preserved in a deep pattern of symmetry and self-similarity.

It is now possible to simplify the reductive formal system to conform to more recent handlings of formal systems, in which it is conceivable to include components of the *grammar* G_{F_R} in the definition of the formal system, such that

$$F_R = \langle E_{F_R}, C_{F_R}, S_{F_R}, A_{F_R}, N_{F_R}, P_{F_R}, X_{F_R}, R_{F_R} \rangle \tag{13}$$

can be simplified to the statement

$$F_R = \langle A_{F_R}, N_{F_R}, P_{F_R}, X_{F_R}, R_{F_R} \rangle \tag{14}$$

such that:

1. A_{F_R} is the alphabet, i.e., is a finite or infinite (known or unknown) ordered set of symbols, referring to natural diachronically and synchronically self-organized entities, derived from the *start symbols* $\langle E_{F_R}, C_{F_R}, S_{F_R} \rangle \in N_{F_R}$, located in each subsequent self-organized level or layer in the sequence of change, evolution, emergence, hierarchical organization and complexity;
2. N_{F_R} is a finite or infinite (known or unknown) set of non-terminal symbols, which include the *start symbols* $\langle E_{F_R}, C_{F_R}, S_{F_R} \rangle \in N_{F_R}$, that are disjoint from the set $A^*{}_{F_R}$. The N_{F_R} *start symbols* are energy E_{F_R}, energy-matter S_{F_R} and energy-space–time C_{F_R}, as well as the compositional '*grammatical markers*' for developing states and processes involving the *start symbols*; along with symbols referring to natural and discovered '*grammatical markers*', which define 'grammatically acceptable' patterns of 'alphabet', 'words', 'strings' and 'collections of strings', sequentially forming more complicated self-organized 'grammatically acceptable' patterns of composition, interaction and relationship, effectively fabricating and becoming A_{F_R} and $A^*{}_{F_R}$;

3. P_{F_R} is a finite or infinite (known or unknown) set of *production rules* of the form $\left(A_{F_R} \cup N_{F_R}\right)^* N_{F_R}\left(A_{F_R} \cup N_{F_R}\right)^* \to \left(A_{F_R} \cup N_{F_R}\right)^*$ referring to discovered naturally emergent laws, rules and constraints, defining admissible patterns of production, fabrication, evolutionary self-organization and hierarchical emergence, involving A_{F_R}, $A^*{}_{F_R}$ and N_{F_R}, including the encompassing interaction with evolved states and processes of the *start symbols* $\langle E_{F_R}, C_{F_R}, S_{F_R}\rangle \in N_{F_R}$;
4. X_{F_R} is a specific set of *axioms*, each of which must be a wff, referring to 'self-evident', 'accepted as given', 'established' entities, statements, propositions or levels of organization in F_R, upon which further abstract modeling of diachronic and synchronic sequences of natural evolution, self-organization, emergence and hierarchically organized complexity can be developed;
5. R_{F_R} is a finite or infinite (known or unknown) set of relations within the wff's called *rules of inference*, referring to the fundamental basis of *scientific logic*; the basis supporting 'acceptable evidence', the basis defining 'acceptable inductive and deductive reasoning' and the basis of how one can arrive at an 'acceptable scientific conclusion'. *As well*, the *rules of inference* include the understanding of any fundamental natural basis for the existence of *natural evolutionary logic*; the basis supporting 'an acceptable understanding that there is evidence of Nature instantiating an evolutionary pattern of reasoning', the basis for defining 'an acceptable understanding of any natural instantiated evolutionary pattern of reasoning' and the basis for an understanding of how Nature and evolution can arrive at 'acceptable natural evolutionary conclusions'.

From these general definitions, the specific case of a three-level hierarchy can then be constructed in more formal detail by duplicating the Reductive Formal System Model (RFSM $= F_R$) and creating three stacked Reductive Formal System Models, each one referring specifically to a defined level in the 'bottom-up' hierarchy. Thus,

1. Lower Level 1 in the 'bottom-up' metaphor can be represented by:

$$F_{R(Level\,1)} = \langle A_{F_{R,L1}}, N_{F_{R,L1}}, P_{F_{R,L1}}, X_{F_{R,L1}}, R_{F_{R,L1}}\rangle \tag{15}$$

2. Mid-Level 2 in the 'bottom-up' metaphor can be represented by:

$$F_{R(Level\,2)} = \langle A_{F_{R,L2}}, N_{F_{R,L2}}, P_{F_{R,L2}}, X_{F_{R,L2}}, R_{F_{R,L2}}\rangle \tag{16}$$

3. Upper-level 3 in the 'bottom-up' metaphor can be represented by:

$$F_{R(Level\,3)} = \langle A_{F_{R,L3}}, N_{F_{R,L3}}, P_{F_{R,L3}}, X_{F_{R,L3}}, R_{F_{R,L3}}\rangle \tag{17}$$

The 'joints' between hierarchical levels and the transformations of entities, involving the *composition* (C_{L1-3}) and *interactions* (I_{L1-3}) occurring at the boundaries between hierarchical levels in Levels 1–3, can then be composed:

$$F_{R(L1)} + (C_{L1} + I_{L1}) \to F_{R(L2)} + (C_{L2} + I_{L2}) \to F_{R(L3)} + (C_{L3} + I_{L3}) \tag{18}$$

such that;

The specific lower level, Level 1, *components, properties* and *composition* as well as the specific lower-level, Level 1, *interactions*, are sufficient to explain the appearance of the *components, properties* and *composition* of the mid-level, Level 2, as well as, the *full potential* for that levels specific emergent interactions amongst its components, such that;

$$(C_{L1} + I_{L1}) = \langle A_{F_{R,L1}}, N_{F_{R,L1}}, P_{F_{R,L1}}, X_{F_{R,L1}}, R_{F_{R,L1}} \rangle \rightarrow F_{R(L2)} + (C_{L2} + I_{L2}) \tag{19}$$

The mid-level 2, Level 2, components, properties and composition of components, then engages in specific Level 2 *interactions* that build the *component, properties* and *composition* of the third level, Level 3, as well as the *full potential* for any interactive properties of the third level and its components, such that:

$$(C_{L2} + I_{L2}) = \langle A_{F_{R,L2}}, N_{F_{R,L2}}, P_{F_{R,L2}}, X_{F_{R,L2}}, R_{F_{R,L2}} \rangle \rightarrow F_{R(L3)} + (C_{L3} + I_{L3}) \tag{20}$$

and, Level 3 can then be defined;

$$F_{R(L3)} = (C_{L3} + I_{L3}) = \langle A_{F_{R,L3}}, N_{F_{R,L3}}, P_{F_{R,L3}}, X_{F_{R,L3}}, R_{F_{R,L3}} \rangle \tag{21}$$

The construction of three stacked Reductive Formal System Models describing the three-level evolved hierarchy composed of three levels of specific composition and interaction of entities, with each level of interaction subsequently defining the entities in the next level up the hierarchy of composition, can be summarized in this form:

$$F_{R(L1-3)} = (C_{L1-3} + I_{L1-3}) = \langle A_{F_{R,L1-3}}, N_{F_{R,L1-3}}, P_{F_{R,L1-3}}, X_{F_{R,L1-3}}, R_{F_{R,L1-3}} \rangle \tag{22}$$

The *composition* and *interactions* in the third level, Level 3, might still have the potential to build further complexity but we have decided not to explore this potential at this point. By definition, in the three-level hierarchy of complexity, any emergent properties or phenomenon exhibiting complementarity which may appear in or between each level must be *conservatively* explained employing rigorous reductive logic: i.e., in this simplified example, complementarity at an inter-level interface or any emergent properties must be entirely explained by the composition and interactions described within and between the three levels of the hierarchy. However, as we shall see the devil is in the details of emergence and complementarity and so is the formal reductive incompleteness we shall pursue shortly.

As it is now defined, the entire three-level hierarchy can be effectively reduced, with T_3 reducing to T_2 and T_2 reducing to T_1. The Theory T_3 of Level 3 can be fully reduced to the Theory T_2 of Level 2, and the Theory T_2 of Level 2 can be fully reduced to the Theory T_1 of Level 1. The 'whole' hierarchical system can be reductively reduced to the lowest 'fundamental' level.

The *production rules* in P_{F_R} can be used to produce wff's, with the *rules of inference* also required, in order to derive *theorems*. Theorems in F_R are defined as the statements and propositions describing the components of a complete *scientific theory* modeling selected entities, components, interactions and one or many levels of hierarchical organization. In the *general case*, if we decide to consider all 'words' and 'strings' in $A^*{}_{F_R}$ as wff's, then the grammar G_{F_R} would not constrain the space of possible scientific inference.

In the *special case* of interest here, where the Formal Reductive System must contain *negation* in order to formally model formal reductive incompleteness, the Reductive Formal System Model can be called *consistent* if and only if there is no wff W such that both W and $\neg W$ (*not* W) can both be logically *true* and *proven* in the system.

Further, it is required that there is a *decision procedure* or an *algorithm* (utilizing P_{F_R}), which can be used for *deciding* whether a formula, statement or proposition, *is* or *is not* a well-formed statement (wff). Stated another way, in the modern interpretation of abstract formal systems, there is an expectation that the production rules will be found to be *decidable*: again, an *algorithm* should be available such that, given any arbitrarily chosen 'string' '*x'*, the *decision procedure* can decide whether '*x*' is or is not a wff. As well, *inference rules* must be *decidable*: for every inference rule $R \in R_{F_R}$ there must be an *algorithm,* which can determine whether, given a set of wff $x_1, \ldots .x_n$ and a wff *y*, the *decision procedure* is able to *decide* whether *R* can be applied with input $x_1, \ldots .x_n$ in order to fabricate *y*.

Further, in the modern understanding of abstract formal systems and *recursive abstract formal systems*; *algorithmic*, *decision procedures*, employed in the systems, make the *axioms decidable* and construct a *set of all provable sentences* (i.e., the set of all theorems) that are *recursively enumerable* or *semi-decidable*: In recursively enumerable systems, if you begin with any arbitrarily selected wff, there is an *algorithm* that can correctly determine whether or not the formula is *provable* within the system, but when applied, and the wff is *not provable* in the formal system, the algorithm will produce a *negative answer* or *no answer at all*. Among the statements where the algorithm returns *no answer at all*, there are found to be formulas that are *undecidable* or *cannot be proven for fear of inconsistency,* the hallmark property of a formula associated with *formal incompleteness.*

Gödel's incompleteness theorems specifically refer to formal systems involving first-order logic or first-order predicate calculus, specifically operating with Peano's axioms of arithmetic. This is presented in such a fashion that the results can be generalized to a very wide class formal systems in which, (1) in order to protect the consistency of the system from exploding into inconsistency, there must be wff's that can neither be proven nor disproven, they must be *undecidable*, and thus the system must be *incomplete*; *and* (2) again for fear of inconsistency, any such formal system will not be capable of demonstrating its own consistency from within the system, *consistency must be decided through meta-consideration.*

This completes the construction of a Reductive Formal System Model (RFSM) and the application of the RFSM to the reductive and conservative model of the three-level hierarchy of complexity investigated by the scientist and his research team. We can now dig deeper into an understanding of the inevitability of undecidable dynamics within this simplified but sufficiently complex formal context.

5 The RFSM in the Matrix of Multiple Scientific Languages

The *simplification* (i.e. *restricting* the natural situation being modeled, to linear, local causation and 'bottom-up' reductive examination and accounting, etc.) and *mathematization* (i.e., *translating* the selected natural phenomena being studied, into a simplified, semi-isolated system contained within an experimental context, described by a *demanding, conservative, restrictive*, *strong reductive* framework, in the form of an abstract Reductive Formal System Model that *does not constrain* the space of possible scientific inference and interest), brings into focus an *unsolved but now manageable problem*. Consider 'bottom-up' reductive Logic, reductive formal system models and the experimental methodology employed within Reductive Science, the whole Reductive Scientific Paradigm, the reductive scientific narrative and the philosophy of Reductionism: Are all of these complicated conceptual structures impacted by reductively conceived *formal reductive incompleteness*? Further, consider Nature and the natural evolving system, which *includes* the human brain but *may also include* the *reductively excluded*, evolved, emergent, complex complementarity, associated with, the self-reflective, self-referencing mind and consciousness: Are all these *natural* phenomena prone to, *natural instantiations of evolutionary incompleteness*?

To determine the answer to these questions, the argument will follow Kurt Gödel [13] further down the logical path he set out in the proof of his two incompleteness theorems. In attempting this, it is necessary to *translate* the detailed Logic of Gödel's argument, from the austere World of pure, abstract formal logic and mathematics, into the subjectively experienced and complicated context of scientists and applied mathematicians, consciously living in the convoluted and diverse complexity of the Real-World.

In the complicated modern scientific context, reductive scientific theories are presented in the form of written papers, books or academic presentations, using Power Points and other forms of communicative media, which are all helpful in making the scientist's theory understandable to an interested audience of philosophers, scientists, mathematicians and anyone else interested in making the intellectual journey. A specific Scientific Theory $T_{F_R} = T_1$, when placed in the full complexity of modern natural science, is composed using a number of available 'languages' for scientific communication. These languages include but may not be limited to:

1. *Direct observation of a phenomena and its behavior*;
2. Scientific *narrative* or *story telling*;
3. *Scientific 'bottom-up' Reductive Logic or other selected forms of scientific logic*;

4. *Scientific mathematical languages and models*;
5. *Computational models and demonstrations*;
6. *Experimental evidence*;
7. *Digital information translation.*

In this complicated general scientific context, involving a vast community and convoluted matrix of communicative relationships, we wish to examine one specific example of a scientist composing his ideas about an evolving three-level hierarchy of emergent complexity, using 'bottom-up' reductive Logic formulated within a Reductive Formal System Model (RFSM $= F_R$), composed in the form of a reductive scientific Theory, T_1 ($T_{F_R} = T_1$). We wish to demonstrate that within this simplified context, the presence of an *undecidable reductive proposition,* declaring *formal reductive incompleteness* is inevitable. Our next stop is *arithmetization.*

6 Arithmetization of Reductive Logic

Mathematization created a Reductive Formal System Model (RFSM $= F_R$) that *may* be *sufficiently complex* to be capable of presenting with *undecidable reductive dynamics. Arithmetization* of the mathematized Formal Reductive System Model (RFSM $= F_R$) should now make it possible to *reveal* the presence of *undecidable reductive dynamics.*

To arithmetize an abstract formal system, Kurt Gödel chose *Peano's arithmetic* and the comprehensive mathematical text, *Principia Mathematica,* written by Bertrand Russell and Alfred North Whitehead [24]. Peano's axioms of arithmetic include:

1. Axiom 1: 0 (zero) is a natural number;
2. Axiom 2: Every natural number has a successor informally stated $n + 1$, which is also a natural number;
3. Axiom 3: No natural number has 0 (zero) as a successor;
4. Axiom 4: Different natural numbers have different successors;
5. Axiom 5: If some *property P* holds for 0 (zero), it also holds for every natural number *n*, and then must hold for every natural number $n + 1$ such that *P* holds for all natural numbers.

Gödel's central insight was to *encode* the wff's of a comprehensive formal system (the comprehensive mathematical text, *Principia Mathematica,* written by Bertrand Russell and Alfred North Whitehead) by using '*Peano's arithmetic'* and the *natural numbers*, thus '*arithmetizing*', or '*Gödel numbering*' the wff's of the entire selected formal system.

We can follow Gödel's lead by arithmetizing and encoding the wff's of a comprehensive formal reductive system, purporting to scientifically describe the whole physical universe. In an imaginary book entitled, *The Complete Reductive Narrative of the Universe,* the author (an imaginary version of myself committed to a closed, resolved, decided and complete model of the Universe) presents a comprehensive scientific history and description of natural evolution involving a 'bottom-up' reductive picture of the entire Universe. Within that imaginary book there is a chapter which presents a narrative describing a scientist composing his ideas about an evolving three-level hierarchy of emergent complexity, using formal 'bottom-up' reductive Logic, composed in a Reductive Formal System Model (RFSM = F_R), and spelled-out in a reductive scientific Theory, T_1 ($T_{F_R} = T_1$).

In the present context *another* version of myself is closely aligned with Gödel's line of reasoning and is committed to exploring the concept of an open, unresolved, often undecidable and forever incomplete scientific model of the Universe. Like Gödel, we can also use *Peano's arithmetic* to *encode* every well-formed English statement, formal logical argument, mathematical formulation or theoretical statement, in the comprehensive scientific theory and narrative composed in the imaginary book, including the chapter on the scientist and his theory. Within the specific chapter containing our scientist and his theory, we can *encode* every wff of the *mathematized Reductive Formal System Model* (RFSM = F_R), which narratively and mathematically is used to describe the simplified reductive picture of the three-level evolved 'bottom-up' hierarchy of complexity.

More formally stated: for every well-formed statement $\mathcal{W}$ in the Theory, T_1 (T_{F_R} = T_1) and every well-formed wff, $\mathcal{W}$, composed by the mathematized and recursive Reductive Formal System Model, (RFSM = F_R); Gödel's *encoding* and *numbering* scheme will produce $\mathcal{G}(\mathcal{W})$, i.e., the encoded '*Gödel number*', which can then be additionally *encoded* as *one* of the *natural numbers*. In such a code, the *name* of the '*Gödel number*' $\mathcal{G}(\mathcal{W})$ of any statement or formula $\mathcal{W}$, is denoted as $\ulcorner \mathcal{W} \urcorner$.

To demonstrate this: first assign a natural number, to each and every primitive symbol s in Peano's arithmetic, to each and every primitive symbol s in the English language used in Theory $T_{F_R} = T_1$ and to each and every primitive symbol s used in the mathematized and recursive Reductive Formal System Model, F_R. Call these the '*symbol numbers' of* 's', e.g., to start with Peano's arithmetic, the symbol '0′ is given the natural number 1 and symbol ' = ' is given the natural number 5. Then consider the wff $\mathcal{W}$: '0 = 0'. The *Gödel number* for this formula $\mathcal{W}$ is uniquely produced as the corresponding product of powers of consecutive prime numbers (2, 3, 5, ...), as $\mathcal{G}(\text{'}0 = 0\text{'}) = 2^1 \times 3^5 \times 5^1 = 2 \times 243 \times 5 = 2430$. The *name* of the *Gödel number* $\ulcorner\text{'}0 = 0\text{'}\urcorner$ is the *numeral* 2430. Carry on assigning Gödel numbers to every primitive symbol s and formula in the mathematized and recursive Reductive Formal System Model, F_R and then to every primitive symbol s and English statement in Theory T_{F_R}. Knowing the Gödel number of *any* symbol, formula or statement, specifically allows us to *uniquely decode* back into the original English statements, the mathematized formulas and into the elements of Peano's arithmetic. This is possible because of Gödel's unique coding and decoding scheme using the prime-factorization theorem, which allows us to locate and define the unique sequence of prime factors with their associated exponents. Most significantly, Gödel numbers are *computable* and it is also important to note that it is *decidable* whether or not any given number is a Gödel number [28].

Gödel's coding scheme produces *sufficient complexity* to fulfill the necessary criteria supporting *formal incompleteness* in the context of an abstract Formal

System. *Undecidability* will be produced, and *undecidable dynamics* will appear in a formal system description or in a formal system model, (i) if it possible to generate *self-reference*; (ii) if it is possible to differentiate *program-data duality*; (iii) if the formal system, at a *minimum*, has the *potential* to access an *infinite computational medium,* and, (iv) if the formal system has the capacity to implement or instantiate *negation.* In 'sufficiently complex' formal systems, *truth* and *proof* are forever separated by a *gap* kept open by potential or instantiated *undecidability* [27].

'Sufficient complexity' for *formal reductive incompleteness* to appear in the context of 'bottom-up' reductive Logic employed in a Reductive Formal System Model (RFSM $= F_R$) composed in a reductive scientific Theory, T_1 ($T_{F_R} = T_1$), will probably involve *undecidability*, (i) generated by *self-reference*; (ii) an available media, which at least in principle, is capable of *infinite computation*, (iii) the possibility of differentiating *program-data duality*; and (iv) the capacity to instantiate *negation.* As in abstract formal systems, there will be an *inevitable logical gulf* inhabited by *undecidable reductive dynamics* and *formal reductive incompleteness* separating reductive *logical truth* from reductive *logical proof*. The separation and gap in *reductive Logic*, precipitated by *formal reductive incompleteness*, must be differentiated from the separation and gap precipitated by the inevitable challenge posed by the distance between what has been or can be *theoretically conceived* and what can be *experimentally demonstrated*. Reductive *theoretical truth* may therefore be supported by reductive Logic or/and by reductive experimental evidence—the *gap* separating *truth* from *proof* in reductive science, therefore, will be *dual.* While incompleteness and undecidable dynamics can influence reductive Logic, creating a separation and gap between truth and proof in Logic, there is also a separation and gap between truth and proof dependent upon the distance between stated *reductive theoretical truth* and the available *experimental evidence* supporting reductive *experimental proof* wherein Science must enter into dialogue with Nature.

'Sufficiently complex' Reductive Scientific Statements, (1) stated in experimentally falsifiable hypotheses; or (2) stated as notations from direct observation; or; (3) stated in the form of scientific narrative and story-telling; or, (4) stated in the form of multiple applied mathematical forms; or; (5) stated in the form of any computational model or digital information representation, fulfilling the definition of a UTM; are *all* scientific representations susceptible to *polarizing debates* wherein there can reside theoretical *contradictions* and potential *self-referencing paradoxes* which *may* herald the presence of *undecidable reductive dynamics* and *formal reductive incompleteness.* It is therefore important to do everything possible to reveal this significant property of reductive Logic and to spell-out its implications. Undecidable reductive dynamics and formal reductive incompleteness *set a limit* on the use of formal reductive Logic but undecidable reductive dynamics and formal reductive incompleteness also instantiate a potential for *incompleteness driven novelty generation,* which may play a significant role in creative 'reductive thought'. The implications of reductive incompleteness include the possibility of creatively reconstructing the

Reductive Scientific Paradigm so that its successor encompasses reductive incompleteness associated with 'bottom-up' reductive thought in a *Meta-Reductive Scientific Paradigm* capable of exploring the role in Nature of *naturally self-organized evolutionary incompleteness.*

Hypothetically, *natural evolutionary undecidable dynamics* and *natural instantiations of incompleteness in evolution*, may play a significant role in defining complex *boundaries* and *limits* in natural systems and may, as well, play an important role in *natural incompleteness driven novelty generation* in naturally evolving systems.

The next essential step in Gödel's proof involves the Self-Reference Lemma. We must turn this Lemma into an understandable reductive conception.

7 The Self-reference Lemma and Reductive Logic

Gödel first uses the Self-Reference Lemma, in relation to any formal mathematical system, employing first-order logic (first-order predicate calculus) and containing Peano's axioms of arithmetic. The Lemma states:

> Let $\mathcal{Q}(\ \mathcal{x})$ be an arbitrary formula of formal system $\mathcal{F}$ with only one free variable. Then there is a sentence (formula without free variables) $\mathcal{W}$ such that: $\mathcal{F} \vdash \mathcal{W} \leftrightarrow \mathcal{Q}\ulcorner\mathcal{W}\urcorner$

The Self-Reference Lemma can be restated in a form allowing its relation to reductive Logic to be stated more clearly:

> Let $\mathcal{Q}(\ \mathcal{x})$ be an arbitrary formula of the mathematized and recursive reductive formal system model F_R with only one free variable. Then there is a sentence (formula without free variables) $\mathcal{W}$ such that: $F_R \vdash \mathcal{W} \leftrightarrow \mathcal{Q}\ulcorner\mathcal{W}\urcorner$

Prokopenko notes that "this lemma is sometimes called the Fixed-point lemma or the Diagonalization lemma…The Self-reference lemma establishes that for any formula $\mathcal{Q}(\ \mathcal{x})$ that describes a property of a numeral, there exists a sentence $\mathcal{W}$ that is logically equivalent to the sentence $\mathcal{Q}\ulcorner\mathcal{W}\urcorner$. The arithmetical formula $\mathcal{Q}(\ \mathcal{x})$ describes a property of its argument, e.g., a numeral $\mathcal{x}$, and hence the expression $\mathcal{Q}\ulcorner\mathcal{W}\urcorner$ describes a property of the numeral $\mathcal{W}$. This is the numeral of the Gödel number of the formula $\mathcal{W}$ itself. Since the formula $\mathcal{W}$ is logically equivalent to the formula $\mathcal{Q}\ulcorner\mathcal{W}\urcorner$, one can say that the formula $\mathcal{W}$ is referring to a property of itself (being an argument of the righthand side).

Strictly speaking…the lemma only provides a (provable) material equivalence between $\mathcal{W}$ and $\mathcal{Q}\ulcorner\mathcal{W}\urcorner$, and one should not claim 'any sort of sameness of meaning'" [27].

In both Gödel's *recursive formal system* $\mathcal{F}$, and in our *recursive, mathematized, reductive formal system model,* F_R, it *is* possible to compose arithmetical and mathematical *self-referencing* statements. In the *recursive, mathematized, reductive formal system model,* F_R, and *also* in a related scientific Theory T_{F_R}, a *self-referencing statement* in *Logic* and the specific associated *meaning* of the statement, can be translated into *linguistic* and *narrative* form.

For instance, the theoretical statement of *reductive epiphenomenalism of consciousness* [32] can be stated as $\mathcal{W}$. The statement $\mathcal{W}$ *postulates* or *means* that: "Formal reductive Logic can fully *reduce* Mind to Brain; *reducing* Mind fully to an *epiphenomenon* of Brain and more fundamental physical processes." The sentence $\mathcal{W}$ is logically and materially equivalent to the sentence $\mathcal{Q}\ulcorner\mathcal{W}\urcorner$. This is derived by applying the Self-reference lemma, which establishes that for any formula $\mathcal{Q}(\ \mathcal{x})$ describing a *property of a numeral* $\mathcal{x}$, there exists a *sentence* $\mathcal{W}$ that is *logically and materially equivalent* to the sentence $\mathcal{Q}\ulcorner\mathcal{W}\urcorner$ describing a *property of a numeral* $\ulcorner\mathcal{W}\urcorner$. Therefore, a *property* of the *epiphenomenal statement,* $\mathcal{W}$, can be captured by the arithmetical formula $\mathcal{Q}\ulcorner\mathcal{W}\urcorner$, which describes a *property* of its argument, e.g., a *numeral* $\ulcorner\mathcal{W}\urcorner$. Hence, the expression $\mathcal{Q}\ulcorner\mathcal{W}\urcorner$ describing a property of the statement $\mathcal{W}$ is the *numeral* of the Gödel number of the formula $\mathcal{W}$ itself. Since the formula $\mathcal{W}$ is logically and materially equivalent to the formula $\mathcal{Q}\ulcorner\mathcal{W}\urcorner$, one can say that the formula $\mathcal{W}$ is referring to a property of itself (being an argument of the righthand side). Thus the *epiphenomenal* statement $\mathcal{W}$: "Formal reductive Logic can fully *reduce* Mind to Brain; *reducing* Mind fully to an *epiphenomena* of Brain and more fundamental physical processes", is *logically and materially equivalent* to the statement $\mathcal{Q}\ulcorner\mathcal{W}\urcorner$: "This statement of *reductive epiphenomenalism* says of itself that *reductive epiphenomenalism* has as one of its properties a *numeral* that is the Gödel number of the *reductive epiphenomenal* statement itself."

As in Gödel's argument and the fine-grained examination of Prokopenko et al., one should *not* claim 'any sort of sameness of meaning'. However, the logical and material equivalence prepares us for what comes next. Self-reference in the reductive Logic of a statement within a Reductive Formal System Model *does not imply* there will *necessarily* be found, contradiction and self-referencing paradox threatening to unravel the usefulness of the underlying Logic. It does however set the stage for this development.

The next step in the proof involves the *provability predicate.*

8 The Provability Predicate and Reductive Logic

The *provability predicate,* $Provable_F(x)$, captures the property of the statement x that it is provable in the formal system F. The *reductive provability predicate,* $Provable_{F_R}(x)$, captures the property of the reductive statement x that it is provable in the Reductive Formal System Model, F_R.

Now let the formula, $Proof_{F_R}(y, x)$ strongly represent a binary relationship in which y is the Gödel number of a *proof* of the formula in F_R with the Gödel number x (Prokopenko notes that it is always recursively and algorithmically decidable, whether a given sequence of formulas y constitutes a *proof* of a given sentence $x,$ in conformity with the rules of the *mathematized, recursive, Formal System Model,* and in our case, the *Reductive Formal System Model,* F_R).

The property of being *provable* in F_R can then be stated in the form: $\exists y Provable_{F_R}(y, x)$. This can then be abbreviated as Provable $F_R(x)$.

9 Negating the Provability Predicate in Reductive Logic

Gödel's final step in proving the First Incompleteness Theorem involves *a second* application of the Self-Reference Lemma to the *negated provability predicate.* Translating this step into the rules of the *mathematized, recursive, Reductive Formal System Model,* F_R, produces:

$$F_R \vdash \mathcal{W} \leftrightarrow \neg Provable_{F_R}(\ulcorner \mathcal{W} \urcorner) \tag{23}$$

This statement formally demonstrates that the system F_R can derive a statement: $\mathcal{W}$ is *true* if and only if $\mathcal{W}$ is *not provable* in the system F_R. Gödel, in his two Formal Incompleteness Theorems, and Prokopenko et al. in their discussion of Gödel's work, and now, also in this application in the context of Reductive Science and reductive Logic, we can go on to say: *if* the *mathematized, recursive, Reductive Formal System Model,* F_R, is *consistent*, then it is possible to show that a *true* sentence $\mathcal{W}$ is *neither provable nor disprovable* in F_R, thus proving that the *mathematized, recursive, Reductive Formal System Model,* F_R, must be *incomplete.* There are *true* statements in F_R that are *unprovable* and *undecidable.* It is once again important to state that the sentence $\mathcal{W}$ can be composed as a *well-formed formula*, wff, in the system F_R.

Kurt Gödel wrote less formally in his proof of the two Incompleteness Theorems that what we "have before us is a proposition that says about itself that it is not provable." What we have before *us* is a *well-formed* and *true* proposition that says about itself that it is *not provable* in the formal system F_R.

10 Strong and Weak Representation in F_R

The above demonstrates the presence of *undecidability* in F_R. We can say that a *mathematized, recursive, Reductive Formal System Model,* F_R, is *decidable* if the set of its theorems is *strongly representable* in F_R *itself*, where there is some formula P(x) of F_R such that:

$$F_R \vdash P(\ulcorner \mathcal{W} \urcorner) \text{ whenever } F_R \vdash \mathcal{W} \text{ and } F_R \vdash \neg P(\ulcorner \mathcal{W} \urcorner) \text{ whenever } F_R \nvdash \mathcal{W} \tag{24}$$

For a *weakly representable* set of theorems only the first half of the above statement need apply ($F_R \vdash P(\ulcorner \mathcal{W} \urcorner)$ whenever $F_R \vdash \mathcal{W}$). This defines *semi-decidability* where *negations* are not necessarily 'attributable' to *non-derivable* formulas. However, it is still possible to construct within the system F_R, a Gödel sentence V^P relative to P(x). This can then be stated:

$$F_R \vdash V^P \leftrightarrow \neg P\left(\ulcorner V^P \urcorner\right) \tag{25}$$

A *contradiction* must follow. Therefore, at least for this sentence *strong representability* does not hold and therefore F_R must be *undecidable*.

Following the lead of Prokopenko et al. [50] (Prokopenko, p. 7/23), it is important to clarify that crucially, the Gödel sentence V^P can be constructed as $V(\ulcorner V(x) \urcorner)$ for some wff V(x) with one free variable, and so the main expression clearly states…for the *mathematized, recursive, Reductive Formal System Model,* F_R:

$$F_R \vdash V \ulcorner V(x) \urcorner) \leftrightarrow \neg P(\ulcorner V(\ulcorner V(x) \urcorner) \urcorner) \tag{26}$$

In discussing the related formula associated with a *recursive formal system* Prokopenko et al. concludes [29]:

$$F \vdash V \ulcorner V(x) \urcorner) \leftrightarrow \neg P(\ulcorner V(\ulcorner V(x) \urcorner) \urcorner) \tag{27}$$

This formula would also hold for a *Reductive Formal System Model,* F_R:

$$F_R \vdash V \ulcorner V(x) \urcorner) \leftrightarrow \neg P(\ulcorner V(\ulcorner V(x) \urcorner) \urcorner) \tag{28}$$

Prokopenko et al. state that this evaluation makes it explicit that the Self-Reference Lemma or Diagonalization Lemma *is used twice* in Gödel's proof. The Self-Reference Lemma or Diagonalization Lemma *is also used twice* in the related formula associated with the *mathematized, recursive, Reductive Formal System Model,* F_R. Self-reference and diagonalization are used first *inside* and then *outside* the representative predicate P(x), which is inserted between the two applications of self-reference and diagonalization.

We can apply this proof to the construction of complex reductive theoretical statements. Failure to find any effective logical means of resolving self-referencing paradoxes in reductive theory, suggests it is necessary to consider whether there may be *no* satisfactory *exit* from the paradoxes stated by certain strong reductive arguments—they may be as *true in logic* as reductive science is capable of stating and the contradiction and paradoxical threat to reductive logical consistency, is *real*. This realization indicates specific 'unresolvable, self-referencing, paradoxical, reductive theoretical statements' may indeed be *complex undecidable reductive propositions*; at a *minimum* metaphorically related to Kurt Gödel's two formal incompleteness theorems but at a *maximum* clear evidence of *formal reductive incompleteness.*

For instance, the *reductive epiphenomenalism of consciousness* statement can be reformulated in conformity with Gödel's paradoxical statement: "This true statement cannot be proven". *Reductive epiphenomenalism* can be stated: "*This true* reductive epiphenomenal *statement*, composed logically to be about a conscious mind, erases from existence the same conscious mind, and says of itself, it *cannot be proven*". The self-referencing form of *reductive epiphenomenalism of consciousness* reveals the *undecidable* nature of this scientific and logically well-defended, well-formed reductive argument: "*This statement* of *reductive epiphenomenalism of consciousness*, composed by conscious, mindful, intentional human beings with brains, *says of itself* that a *property* of the reductive epiphenomenal statement *is* that the conscious, mindful, intentional human beings with brains, who composed the strong reductive statement, can be *fully reduced to brain, body* and more fundamental physical processes, leaving intentional mind and consciousness as epiphenomenal, empty, meaningless, unnecessary conceptions that can be erased from the universe as real phenomena: *Reductive epiphenomenalism* has as one of its properties, the *paradoxical contradiction* that *it is and is not* composed by brain, mind and human consciousness, and therefore the reductive epiphenomenal statement *cannot be proven.*"

The reductive statement of eliminative materialism provides a further example. *Eliminative materialism* can be stated: "*This* logically *true* eliminative materialist *statement*, composed by a conscious human mind, reduces the same conscious human mind to a determined and fundamental physical brain, eliminating the existence of the mind and consciousness that made the eliminative materialist statement in the first place, thus in paradoxical contradiction of itself and in response to the threat of logical inconsistency, the eliminative materialist statement must say of itself, it *cannot be proven*".

11 Proposition XI: Consistency and the Second Theorem

In the final section of his paper, Gödel states: "**Proposition XI**: If *c* be a given recursive, consistent class of *formulae*, then the *propositional formula* which states that *c* is consistent is not *c-provable*; in particular, the consistency of P is unprovable in P, it being assumed that P is consistent (if not, of course, every statement is provable)." [18].

The Liar paradox focuses on *truth*: "This statement is *false*", which makes it impossible to avoid the paradox [15]. Gödel's choice of paradoxical sentence focused on *proof*: "This statement is *unprovable*", which made it possible to state in a *well-formed sentence* a *truth* but also to *avoid proof* and the *paradox* and thus *avoid* logical *inconsistency.*

If the formal arithmetic of Peano and the proof of the *first* incompleteness theorem are *consistent*, which means that only *true* statements can be *proven* in the system, then Gödel's chosen well-formed statement: ("This statement is *unprovable*"), *must be true.* If Gödel's statement were *false* then it would be possible to *prove* it to be *false* but that would be *contrary* or *contradictory* to the *consistency* of the system. To be thoroughly paradoxical, Gödel's statement *cannot be proven,* for that would demonstrate exactly the opposite of what the statement asserts, which is that it is *unprovable.*

In the *second* incompleteness theorem, Gödel shows that in an abstract formal system of sufficient complexity, if one attempts to prove the consistency of the formal system from *inside* the formal system itself, then the whole argument determining consistency could be formalized and the statement: "This statement is unprovable", in its formal version, would thus be *proven*, and this leads to an immediate *contradiction*. The attempt to *prove consistency* from *inside* the system demonstrates the *inconsistency of the system,* rather than proving the consistency [16].

In sketching out the details of the *second* incompleteness theorem, Gödel generalized his findings to the axiom system of set theory M and to that of classical mathematics A "and here to it yields the result that there is no consistency proof for M or for A, which could be formalized for M or for A respectively, it being assumed that M or A are consistent. It must be expressly noted that Proposition XI represents no contradiction of the formalistic standpoint of Hilbert. For this standpoint presupposes only that the existence of a consistency proof effected by finite means, and there might conceivably be finite proofs which **cannot** be stated in P or in M or in A" [19]. Hilbert was angry and very upset that, by using the very formalism he had sought, his "grand scheme for securing the foundation of mathematics had been shown to be impossible" [19]. However, when convinced of the importance of Gödel's advance, the mathematics community began the difficult task of trying to understand Gödel's proof and its implications.

11.1 Proposition XI (Reductive Science Version)

If *c* be a given recursive, consistent class of *reductive formulae*, then the *propositional reductive formula* which states that *c* is consistent is not *c-provable*; in particular, the consistency of F_R is unprovable in F_R, it being assumed that F_R is consistent (if not, of course, then every reductive statement is provable)."

We can now generalize Gödel's findings to 'bottom-up' reductive Logic ($\mathcal{L}_\mathcal{R}$), Reductive Science (RS), the Reductive Scientific Paradigm (RP) and the Reductive Scientific Narrative (RSN) where Proposition XI (Reductive Science Version) yields the result that there is *no consistency proof* for 'bottom-up' reductive Logic ($\mathcal{L}_\mathcal{R}$), a Reductive Formal System Model (F_R), Reductive Science (RS), the Reductive Scientific Paradigm (RP) and the Reductive Scientific Narrative (RSN), which could be formalized for any of these respectively, it being assumed that they are each consistent in their application of reductive Logic.

Among the implications of *formal reductive incompleteness* is the possibility that what is *not* finitely *provable* in $\mathcal{L}_\mathcal{R}$, F_R, RS, RP and the RSN, might be *finitely provable* in a related form of reductive Logic, adapted Reductive Formal System Model, altered structure for Reductive Science, modified Reductive Paradigm or transformed Reductive Scientific Narrative, in each case using altered axioms or rules.

I have done my best to prove that Gödel's two incompleteness theorems can indeed be applied to formal 'bottom-up' reductive Logic employed in a Reductive Formal System Model, F_R, and further to prove that this application of incompleteness will follow and flow throughout every scientific context where formal 'bottom-up' reductive Logic is used in the Reductive Scientific Paradigm or the Reductive Scientific Narrative.

It follows from Gödel's comment about "finite proofs which **cannot** be stated in P"; for any particular example of a *formally undecidable reductive statement* declaring *formal reductive incompleteness* found in the Reductive Paradigm, there *must* be, *novel adapted* Reductive Formal System Models as well *as novel adapted* Meta-Reductive Scientific Paradigms in which modified reductive Logic, the elements of an adapted Reductive Formal System and the frame of an altered Meta-Reductive Paradigm, will have been *transformed* in such a fashion that it becomes possible to *prove* the previously undecidable statement, by spelling-out a finite statement in the adapted Logic, System and Paradigm.

When "bottom-up" reductive Logic and the RFSM, $\mathcal{F}_{\mathcal{R}}$, are used to model the 'simplified scientific situation' concerning a participatory scientist and a selected 'semi-isolated system of interest', they can create a three-level, evolved, conservatively emergent hierarchy of complexity; in which *each* of the *three* 'stacked' RFSM's, $\mathcal{F}_{\mathcal{R}_{(1-3)}}$ used to describe the three-level hierarchy, can be shown to contain *undecidable reductive propositions*, $u_{\mathcal{R}_{(1-3)}}$declaring the *recurrent formal reductive incompleteness* of the three-level hierarchical model ($i_{\mathcal{R}_{(1-3)}}$).

Consequently, in conformity with Gödel's first incompleteness theorems it becomes possible to create a definition of *sufficient complexity* associated with *formal reductive incompleteness*, in the reductive scientific context. The simplified context of the *three* 'stacked' RFSM's, $\mathcal{F}_{\mathcal{R}_{(1-3)}}$ used to describe the three-level hierarchy are sufficiently complex to be associated with formal incompleteness. More complex models will also be capable of undecidable dynamics.

In conformity with Gödel's second incompleteness theorem, the necessity of *meta-consideration* in determining *reductive logical consistency* can also be explained in the reductive scientific context. The abstract representation and generality of the 'simplified scientific situation' described above, implies a need to adapt *all* RFSM of sufficient complexity using rigorous reductive Logic, such that they incorporate within their core structures *undecidable reductive propositions* ($u_{\mathcal{R}}$) and *formal reductive incompleteness* ($i_{\mathcal{R}}$).

Additional implications of formal reductive incompleteness impacting any RFSM of sufficient complexity can then be spelled-out. These implications include, first, the possibility that an *alternative*, *adapted,* RFSM, can translate an *undecidable reductive proposition* into a *decidable reductive proposition*; and, second, *any* such constructed, alternative RFSM will inevitability reveal its own version of $u_{\mathcal{R}}$and $i_{\mathcal{R}}$. Further, formal reductive incompleteness and its implications can be *generalized* and carried into *any* and *every* aspect of 'reductive thought'; *wherever* 'bottom-up' reductive Logic is employed in Reductive Natural Science. Therefore, *meta-consideration* of conceivable *meta-constructions*, which alter the entire Reductive Scientific Paradigm (RP), offer a *novel incompleteness dependent definition*, for narrow intra-domain *adaptation* of a 'part' of natural science, or a global *revolution* involving the 'whole' Reductive Scientific Paradigm (RP).

12 Implications for the Reductive Scientific Paradigm

Among the significant conclusions and implications of this work are the following:

1. The Theory T_1 (F_R) describing the simplified conservatively reductive example of the three-level hierarchical model of Complexity and the complicated scientific situation composed within a mathematized and recursive *Reductive Formal System Model* (F_R) are *sufficiently complex* to demonstrate *undecidable dynamics* and *formal incompleteness*;
2. By analogy with Gödel's Formal Incompleteness Theorems we also see that even in this simple reductive model involving Theory T_1 (F_R), Formal System F_R, and the conservatively reductive three-level hierarchy of complexity; the situation still presents us with a potential for *self-referencing statements and propositions,* in a situation where *program/data duality can be differentiated* in a logical framework *capable of negation,* existing in a mathematical and computational medium capable of *infinite possible universal computation.*
3. The situation can be interpreted as implementing *program/data duality*—such that the abstraction of Theory T_1 (F_R), Formal System F_R*,* and the three-level hierarchy of complexity, become the abstract *program* while the implementation of specific processes of change defined by particular propositions and statements and wff's within the sequence of progression and evolution in the abstract system become the *data*;
4. *Mathematization* of Theory T_1 (F_R) and the Formal System F_R serve to explicitly reveal a potential for undecidable dynamics and incompleteness. *Mathematization* also reveals a three-part mathematical *equivalence*; wherein *Formal System models with their pattern of undecidability and Formal Incompleteness*, are completely *equivalent* and *translatable* into *Dynamical System Models* with an associated equivalent pattern of Dynamical undecidability and Dynamical Incompleteness. These two models, can then be *equivalently translated* again, into an *Information Theoretical System Model* based on an abstract Universal Turing Machine (UTM). This three-system equivalence has previously been explored in a course-grained examination, by Casti [5], and then very recently, in a fine-grained demonstration by Prokopenko et al. [27]. The three equivalent system models share a similar pattern of undecidable dynamics and incompleteness. This three-system equivalence can be further linked to more complex system models, derived in the study of non-linear systems, Chaos Theory, Cellular Automata, the Science of Complexity and Complex Adaptive Systems.
5. Through *arithmetization*, the fine-grained nature of undecidable dynamics becomes evident. At a minimum, it must be determined whether or not the undecidable dynamics found in equivalent system models is associated with or may closely approximate undecidable evolutionary dynamics in natural evolving systems and hierarchically organized complexity. The presence and exploration of undecidable dynamics in natural settings may demand re-defining phase transition, complementarity, and emergence, where these phenomena are found in association with undecidable dynamics within the evolution of systems and

hierarchically organized complexity. This may reveal significant and pervasive patterns of natural evolutionary undecidable dynamics and evolutionary incompleteness.

6. Contradiction, paradox and self-referencing paradox in all aspects of Scientific Theory construction and Formal System construction, should be *embraced* rather than avoided. This essential refocusing of scientific attention may open a window on how to effectively compose in a scientific theory an *undecidable reductive proposition*, indicating *formal reductive incompleteness* and how to explore through theoretical and experimental means, the possibly of natural evolutionary undecidable dynamics.
7. In conformity with Kurt Gödel's *second* Formal Incompleteness Theorem; formal reductive incompleteness and undecidable dynamics; any future attempt to demonstrate the *consistency* of any Reductive Theory $T\ (F_R)$ and any Reductive Formal System F_R, will ultimately need to be addressed through *meta-theoretical-consideration.*
8. The space of meta-reductive-theoretical-consideration will eventually lead to consideration of adapted forms of reductive Logic and meta-construction of Meta-Reductive Paradigms. In adjacent possible *adapted* Paradigms, *adapted* premises, assumptions and forms of formalized scientific logic, may allow previously *undecidable* reductive propositions (such as *reductive epiphenomenalism of consciousness*) composed in the original, historic and modern 'bottom-up' logic of the Reductive Paradigm, to be *decided* in the reformulated and adapted domain of an imagined *Meta-Reductive Paradigm.* Such a Meta-Paradigm might be composed specifically in order to deal with an undecidable proposition located in a $T\ (F_R)$ statement and it might encompass 'bottom-up' reductive Logic in an adapted form of meta-reductive Logic.
9. Consideration of adapted Meta-Reductive-Paradigms may reveal that there exist many *Meta-Reductive Paradigms,* which are capable of solving many previously unsolved scientific problems (such as the incommensurability of Relativity Theory and Quantum Physics or the relentless residue of Descartes Mind/Body Dualism). A Meta-Reductive Paradigm may be capable of addressing many remaining reductive scientific anomalies (such as objective demonstrations of anomalous subjective transpersonal experiences of consciousness).
10. However, we can also expect that any *Meta-Reductive Paradigm* and adapted meta-reductive Logic; will ultimately and inevitably reveal its own particular pattern of formal, dynamical and informational, undecidable dynamics and incompleteness.

Reductive formal incompleteness represents a kind of scientific 'unfinished description' specifically related to formal reductive Logic [25]. It is worthwhile beginning a search for other significant inter-related kinds of 'unfinished description' in relation to reductive natural science.

Reductive formal incompleteness introduces a novel way to work toward a deeper consilience of the Reductive Natural Sciences and the Complexity Sciences. As Reductive Natural Science and the Complexity Sciences explore the implications of formal reductive incompleteness there may be imagined adjacent possible Meta-Reductive Scientific Paradigms in which adapted reductive Logic more closely approximates the natural Logic of Nature [26]. In such a novel Paradigm, the vast evolved complexity of the human body/brain/mind and consciousness might finally be encompassed fully within meta-reductive natural science and fully within the evolutionary, self-organized, emergent, complex adaptive products of Nature.

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