International Perspectives on the Teaching and Learning of Mathematical Modelling

Frederick Koon Shing Leung
Gloria Ann Stillman
Gabriele Kaiser
Ka Lok Wong *Editors*

Mathematical Modelling Education in East and West

Springer

International Perspectives on the Teaching and Learning of Mathematical Modelling

This book series will publish various books from different theoretical perspectives around the world focusing on Teaching and Learning of Mathematical Modelling at Secondary and Tertiary level. The proceedings of the biennial conference called ICTMA, organised by the ICMI affiliated Study Group ICTMA (International Community of Teachers of Mathematical Modelling and Applications) will also be published in this series. These proceedings display the worldwide state-of-the-art in this field and will be of interest for a wider audience than the conference participants. ICTMA is a worldwide unique group, in which not only mathematics educators aiming for education at school level are included but also applied mathematicians interested in teaching and learning modelling at tertiary level are represented. ICTMA discusses all aspects related to Teaching and Learning of Mathematical Modelling at Secondary and Tertiary Level, e.g. usage of technology in modelling, psychological aspects of modelling and its teaching, modelling competencies, modelling examples and courses, teacher education and teacher education courses.

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Series Preface

Applications and modelling and their learning and teaching in school and university have become a prominent topic for many decades now in view of the growing worldwide relevance of the usage of mathematics in science, technology and everyday life. There is consensus that modelling should play an important role in mathematics education, and the situation in schools and university is slowly changing to include real-world aspects, frequently with modelling as real world problem solving, in several educational jurisdictions. Given the worldwide continuing shortage of students who are interested in mathematics and science, it is essential to discuss changes of mathematics education in school and tertiary education towards the inclusion of real world examples and the competencies to use mathematics to solve real world problems.

This innovative book series established by Springer *International Perspectives on the Teaching and Learning of Mathematical Modelling*, aims at promoting academic discussion on the teaching and learning of mathematical modelling at various educational levels all over the world. The series will publish books from different theoretical perspectives from around the world dealing with Teaching and Learning of Mathematical Modelling in Schooling and at Tertiary level. This series will also enable the *International Community of Teachers of Mathematical Modelling and Applications* (ICTMA), an International Commission on Mathematical Instruction affiliated Study Group, to publish books arising from its biennial conference series. ICTMA is a unique worldwide educational research group where not only mathematics educators dealing with education at school level are included but also applied mathematicians interested in teaching and learning modelling at tertiary level are represented as well. Six of these books published by Springer have already appeared.

The planned books display the worldwide state-of-the-art in this field, most recent educational research results and new theoretical developments and will be of interest for a wide audience. Themes dealt with in the books focus on the teaching and learning of mathematical modelling in schooling from the early years and at tertiary level including the usage of technology in modelling, psychological, social, historical and cultural aspects of modelling and its teaching, learning and assessment, modelling competencies, curricular aspects, teacher education and teacher education courses. The book series aims to support the discussion on mathematical modelling and its teaching internationally and will promote the teaching and learning of mathematical modelling and research of this field all over the world in schools and universities.

The series is supported by an editorial board of internationally well-known scholars, who bring in their long experience in the field as well as their expertise to this series. The members of the editorial board are: Maria Salett Biembengut (Brazil), Werner Blum (Germany), Helen Doerr (USA), Peter Galbraith (Australia), Toshikazu Ikeda (Japan), Mogens Niss (Denmark), and Jinxing Xie (China).

We hope this book series will inspire readers in the present and the future to promote the teaching and learning of mathematical modelling all over the world.

Hamburg, Germany
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Part I
Introduction

Chapter 1
Mathematical Modelling Education in the Cultural Contexts of West and East

Frederick Koon Shing Leung, Gloria Ann Stillman, Gabriele Kaiser, and Ka Lok Wong

Abstract The title of this book is *Mathematical Modelling Education in West and East*, arising from the ICTMA-19 conference with the same theme. It is argued that since both mathematics itself and mathematics education are human products, and solving problems in real-life context is at the heart of mathematical modelling and its applications, mathematical modelling, and its teaching and learning should be considered in their cultural contexts. Hence, consideration of issues about mathematical modelling in West and East will bring out the richness of mathematical modelling education. In this regard, the hosting of ICTMA-19 in Hong Kong, a meeting point of Western and Eastern cultures, has special significance for the discussion on mathematical modelling. After an introduction of the theme, the classification of the chapters of the book and structure of the book are explained.

Keywords Mathematical modelling education · Pedagogical issues · Assessment · ICTMA-19 · Confucian Heritage Culture (CHC) · International Mathematical Modelling Challenge (IMMC)

1.1 Introduction

Mathematics is often perceived as universal truth (Ernest 2009), and as a corollary, principles of mathematics education should be applicable irrespective of the culture and tradition students are situated in. Mathematical modelling education, as a branch

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of mathematics education, is perceived likewise. Based on this premise, a book on mathematical modelling should cover topics on mathematics modelling activities applicable to everywhere in the globe. However, research in mathematics education in the past three decades has shown that both mathematics itself and mathematics education are human products (Bishop 1988; Lerman 2000; Ellis and Berry 2005). As such, it will be very valuable for scholars from different cultural traditions to gather together in an international conference and share experiences on mathematical modelling education arising out of their own cultural tradition. In fact, it can be argued that since solving problems in real-life context is at the heart of mathematical modelling and its applications (Blomhøj and Carreira 2009), it is all the more important for mathematical modelling and its teaching and learning to be considered in its cultural context. Consideration of mathematical modelling in its cultural context, and sharing and contrasting research and practices from different cultures, will bring out the richness of mathematical modelling education. How boring it would be if mathematical modelling were done in the same way in all parts of the world!

The theme of this book, and of the ICTMA-19 conference held in Hong Kong SAR, China (https://www.ictma19.org/), from which the chapters of this book are derived, is *Mathematical Modelling Education in West and East*. China, a major country in the East, has fostered a unique tradition of mathematical education, which has profound influence on its neighbouring countries such as Japan and Korea throughout its history (Martzloff 1997). Mathematics education in China is deeply rooted in its profound culture established in a 5000-year history of civilization, particularly Confucianism, and the related so-called examination culture (Leung 2006) and textual culture (Cherniack 1994). As educators and policy makers around the world have come to realize the importance of learning from experiences of other countries under a different culture, the success of students in China and more generally in countries under the influence of the Chinese culture (referred to as Confucian Heritage Culture or CHC) (Biggs 1996) in international studies of mathematics achievement (Mullis et al. 2016; OECD 2016) has drawn attention from mathematicians and mathematical educators and researchers all over the world. Educators and policy makers are particularly interested in issues relating to the curriculum and the teaching and learning of mathematics, including the issue of integration of mathematical modelling in the teaching and learning of mathematics.

There was a strong emphasis on algorithm and computation in the mathematics tradition in China (Martzloff 1997), as illustrated in the *Nine Chapters of Mathematical Art* or *Jiu Zhang Suan Shu* (composed from tenth to second century BCE) (Straffin 1998), and this emphasis has now seen a renaissance in mathematical modelling and applications when China has made significant progress with economy, science and technology. In recent decades, teaching and learning of mathematical modelling as well as contests in mathematical modelling have been flourishing at different levels of education in Greater China including Mainland China, Hong Kong, Macau and Taiwan. Courses in mathematical modelling have become elective or core courses in universities or vocational colleges in China. At the secondary school level, in the forthcoming National High School Mathematics Curriculum in China (MOE 2017), mathematical modelling is included as one of the core mathematical competencies for students and is a compulsory requirement in the curriculum, with designated classroom teaching hours. Partly because of this emphasis, teachers and researchers in China today become keener to learn from their western colleagues in the teaching of mathematical modelling and its applications. And as pointed out above, researchers and educators in the West are also interested to learn about what is happening in the East in terms of the role played by mathematical modelling in mathematics education. Hence, dialogue and communication between colleagues from across the globe will provide new impetus and resources for mathematical modelling education and its research in both West and East.

Alongside this increased attention to mathematical modelling, a number of competitions related to mathematical modelling at various levels of the education system have been developed. It is admitted that mathematics education is not about competition. In fact, some scholars denounce competition as having a negative effect on mathematics teaching and learning. However, it should also be admitted that in certain areas of mathematics education, competition does have its role in promoting the learning of mathematics, unless it is overemphasized (Yao et al. 2012). Take the International Mathematics Olympic (IMO) as an example. It has played an important part in inculcating mathematics geniuses, but if the IMO type of competitions is promoted in a universal scale, as is alleged of being done in China, it may pose harmful consequences. As for mathematical modelling competition, it is argued that if managed well, it will exert a positive effect on mathematical modelling education, encouraging youngsters to solve complex modelling examples, which in turn will contribute to mathematics teaching and learning more generally. Moreover, international mathematical modelling competitions have a further effect of promoting the interflow of ideas (and friendship) through interaction of the students participating in the competitions.

In recent years, Hong Kong, the venue for ICTMA-19, has been paying increasing attention to the role that mathematical modelling and its applications play in the teaching and learning of mathematics, and one of the means through which this is achieved is via mathematical modelling competitions. For example, an interschool contest in mathematical modelling for secondary school students has been running for more than 10 years, and in recent years, it is merged with another newly established competition, International Mathematical Modelling Challenge (IMMC) in Hong Kong. With the hosting of ICTMA-19 in Hong Kong, mathematicians and mathematics educators and researchers from across the globe have been able to get to know more about the development in the promotion of mathematical modelling in Hong Kong and in the Greater China region more generally. Chinese educators and researchers are also afforded the opportunity to learn and connect with the international community of teachers and researchers involved in mathematical modelling and its applications.

With a history of more than 150 years, Hong Kong is unique for blending the eastern and western cultures. Hong Kong has played an important role in not only ushering in western culture and education to China but also in introducing China to the West. With the increasing awareness and efforts in mathematical modelling education all over the world, Hong Kong will continue to be a super-connector between West and East.

As mentioned above, this book is a result of the ICTMA-19 conference held in Hong Kong and is a means for providing dialogue and communication for researchers and educators from both West and East across the globe who are interested in sharing new ideas on modelling teaching and practices, inside and outside the classroom. Unfortunately but perhaps not unexpectedly, not too many of the papers presented at ICTMA-19 addressed the conference theme of Mathematical Modelling Education in West and East, similar to the situation in many other education conferences. However, the very fact that ICTMA-19 was held in Hong Kong sits very well with the theme of the conference, and has encouraged attendance of participants from the Asian region. In fact, compared to the past few ICTMA conferences, both the number of participants from the Asian region and the number of papers presented by participants of Asian origin are higher. And ICTMA-19 is only the second ICTMA conference being held in Asia (the last time was ICTMA-10 held in Beijing in the year 2001), in an international metropolis and educational and cultural hub known to be the meeting point of Western and Eastern cultures.

The chapters in this book come from some of the papers presented at ICTMA-19. At ICTMA-19, papers were roughly grouped under different themes such as Teacher Education; Teaching Cases at Primary and Secondary Levels; The Process of Modelling; Technology Use in Modelling; Teaching Methods; Students' Performance in Mathematical Modelling; Teachers' Knowledge; Teaching Cases in Higher Education; Cognition, Metacognition and Attitudes; Social and Cultural Influence on Mathematical Modelling; Task Design; Context and Strategy; Curriculum; etc. Given this wide classification, a more clear-cut classification is needed for the book. Instead, considering the nature of the content of the papers submitted for this book, we can easily find that a substantial number of papers addressed the standard issues of the nature of mathematical modelling, and issues addressing the pedagogy and assessment of mathematical modelling, and so the first three sections of the book cover theoretical issues, pedagogical issues and assessment issues respectively. There are a number of chapters on experiences of teaching practices in mathematical modelling, with quite a number of papers reporting some innovative teaching approaches, and these constitute the next two sections of the book. Some further examples on teaching mathematical modelling are provided in the following section, and there are also a number of chapters reporting experiences on mathematical modelling at the tertiary level. The last section covers miscellaneous topics on mathematical modelling. The chapters within each section are then arranged in alphabetical order.

1.2 Theoretical Issues

A number of theoretical issues in mathematical modelling are covered in the first section of the book.

Ang (Chap. 2) explores and explicates the role of computational thinking, one of the key skill sets for the future, in mathematical modelling, and examines the relationship between computational thinking and mathematical modelling. Examples from different modelling approaches are provided to contextualize the relationship, and to demonstrate that mathematical modelling may serve as a platform for the practice and development of computational thinking.

Fisher (Chap. 3) argues for the importance of enabling secondary school students to build models for analysing complex systems problems in order to increase their understanding of nonlinear feedback systems they will encounter as professionals and citizens in the future. Examples of the types of system models normally outside the reach of the secondary school students are provided, and their advantages for enhancing students' ability in analysis of real-world mathematical problems, as well as the use of technologies in solving these problems, are discussed.

Lewis (Chap. 4) establishes a theory for facilitating modelling tasks as a bridge between modelling as content and modelling as vehicle. An example of how one teacher vacillates between nurturing students' development of modelling as content and targeting curricular objectives through formalization of desired mathematical content as vehicle is used to illustrate how the teacher navigates between these two epistemological approaches to develop students' mathematical modelling capacity.

Orey and *Rosa* (Chap. 5) argue that the combination of local (emic) and global (etic) approaches in ethnomodelling research contributes to a holistic understanding of mathematics. Local knowledge is essential for an intuitive and empathic understanding of mathematical ideas and procedures, while global dialogical knowledge is essential for the achievement of cross-cultural communication. Acquisition of both local and global knowledge is a goal of ethnomodelling research, which should be conducted through respect, appreciation, dialogue, and interaction.

Rosa and *Orey* (Chap. 6) argue that ethnomodelling can aid in recording cultural-historical forms of mathematical practices developed by members of distinct cultural groups and bring in cultural perspectives to the mathematical modelling process. Insubordination triggered by ethnomodelling may evoke a sense of disturbance that causes conscious review of rules and regulations endemic to many curricula contexts. This process enables educators to use positive deviance to develop pedagogical actions that deal with content often disconnected from the reality of the students.

Sevinç (Chap. 7) provides a theoretical discussion on the epistemological content of self-regulated and collaborative model development. Utilizing Piaget's theory of cognitive development as the foundation for the "models-and-modelling" perspective, a genre of activities called model-eliciting activities are produced. It is argued that Piaget's reflective abstraction and series of successive approximations support the cyclic and self-regulatory nature of model development, which occurs as a series of assimilations, accommodations, and (dis)equilibrium.

1.3 Pedagogical Issues

This section of the book includes chapters related to pedagogical issues in mathematical modelling.

Ay and *Ostkirchen* (Chap. 8) present the pilot study of a project entitled Diversity in Modelling (DiMo+) which analyses how 15-year-old students in Germany handled mathematical modelling tasks. Different patterns of action in modelling are found among students of different social backgrounds. It is argued that compared to many other countries, educational success in Germany is strongly determined by social background. This is especially the case for success in mathematical modelling, where modelling tasks involve authentic use of extra-mathematical content.

Borromeo Ferri (Chap. 9) argues that teacher education in mathematical modelling is necessary so that modelling lessons can be realized in schools. The historical development of teacher education in mathematical modelling is then discussed, and an empirical study on measuring teacher competencies for mathematical modelling is presented. This is followed by the presentation of a case study on the views of university educators after teaching a mathematical modelling course as to what school teachers need to know in modelling.

Ferrando, *Segura* and *Pla-Castells* (Chap. 10) report a study in which 224 Spanish pre-service primary school teachers analysed students' written solution plan of a sequence of modelling tasks involving estimations. The results show that there is a relation between the solution plan used by the students and the characteristics of the context of the real estimation task. Conclusions regarding the characterization of this kind of modelling tasks and the potential use of this sequence of tasks to promote problem solving flexibility are then derived.

Geiger, *Galbraith* and *Niss* (Chap. 11) report the interim findings of a national project in Australia that aims to promote effective teaching and learning practices in mathematical modelling through attention to implemented anticipation. From the findings, a Design and Implementation Framework for Modelling Tasks (DIFMT) is generated. The study suggests that specific pedagogical practices can act as enablers of students' attempts to appropriate the process of mathematical modelling.

Guerrero-Ortiz (Chap. 12) reports a study on the relationships between the mathematical modelling processes adopted by pre-service teachers while designing modelling tasks and the knowledge in relation to content, technology, and pedagogy. A way to integrate modelling and Technological Pedagogical Content Knowledge (TPACK) into an analysis framework is demonstrated, which deepens the current understanding of teachers' knowledge and development of resources to support the integration of modelling and technology as a part of teaching practice.

Hartmann and *Schukajlow* (Chap. 13) examine whether students are more interested in and feel more enjoyment and less boredom while solving real-world problems outside than inside the classroom. Results of the study indicate that location does not influence the development of students' interest and emotions. The authors argue for the importance of authentic problems for students to develop interest and emotions.

Hearne (Chap. 14) explores the use of mathematical modelling to enhance grade 6 learners' understanding of fractions. It is found that learners' understanding improves as effective connections are made between and within their intra-mathematical and extra-mathematical knowledge, and they benefit by connecting symbols and their referents and procedures and their underlying concepts rather than focusing on the surface features of Arabic notation.

Huang, *Lu* and *Xu* (Chap. 15) employ a qualitative text analysis approach to analyse the mathematics curricular syllabi or standards at primary, middle and high school levels in China in order to investigate the historical development of mathematical modelling in the country. A number of interesting observations are made, e.g., the term "modelling" might not appear in syllabus, but the idea of mathematical modelling rooted in the tradition of "solving real-world problem" has been in existence for a long time.

Schmitz and *Schukajlow* (Chap. 16) study the role of pictures in solving mathematical modelling tasks through assessing the picture-specific utility value and modelling performance of upper secondary school students. The picture-specific utility value reflects the perceived usefulness of a picture for understanding the problem, students assign a lower utility value to the pictures that contain additional superfluous numerical information. However, no significant differences in the students' modelling performance are found.

1.4 Assessment Issues

Issues related to assessment in mathematical modelling are covered in this section of the book.

Alagoz and *Ekici* (Chap. 17) validate a mathematical modelling assessment with the input of content expert from multiple disciplines in building, defining, and clarifying the interdisciplinary competencies involved in the modelling tasks. The validation process involves scoring, interpretation and uses, and consequences of interdisciplinary mathematical modelling assessment results. Confirmatory factor analysis indicates construct validity for an assessment with two higher-order factors indicating conceptual and procedural dimensions of interdisciplinary learning enacted by mathematical modelling.

Frenken (Chap. 18) presents the construction of a test instrument for assessing metacognitive knowledge of mathematical modelling based on a theoretical definition of the term "metacognitive knowledge" and its domain-specific connection to mathematical modelling. The scalability and possible reduction of items of the instrument are analysed, and the item construction and evaluation process is described.

Göksen-Zayim, *Pik*, *Dekker* and *van Boxtel* (Chap. 19) explore the mathematical modelling proficiency in both primary school and lower secondary school in the Netherlands. Two modelling tasks on three difficulty levels are administered, and it is found that learners encounter difficulties when constructing a meaningful representation of the described modelling problem or may even fail to understand the problem. Representation problems are qualitatively analysed and are shown to be partially related to learners' language problems.

Wang (Chap. 20) investigates the mathematics modelling competency of pre-service mathematics teachers in 4 universities in China. A scoring framework of the mathematical modelling steps in solving a modelling item is used, and a questionnaire on modelling competition experience is administered. The results show that there is correlation between the modelling competition experience of student teachers and their modelling competency.

Wess, *Klock*, *Siller* and *Greefrath* (Chap. 21) present a theory-based development of a structural model and an associated test instrument to measure the competence of teachers in their skills and abilities for teaching mathematical modelling. The extent to which the proposed conceptualization of the structural model can be empirically confirmed is discussed, and insights into the test instrument are presented and results of the structural equation analysis of the model are presented.

1.5 Teaching Practice

This section of the book includes chapters on teaching practices in mathematical modelling.

Czocher and *Hardison* (Chap. 22) present a theoretically coherent methodological approach for understanding the situation-specific attributes students find relevant in mathematical modelling tasks, and when students' situation-specific meanings for inscriptions change while engaged in modelling. The utility of this approach is illustrated by analysing the modelling activities of a purposefully selected undergraduate student.

Hansen (Chap. 23) analyses the procedural choices and assessments the pre-service teachers let their pupils make and how they facilitate critical thinking during a practice period. It is found that although the pre-service teachers often emphasize mathematical exploration, they tend to offer specific tasks to assist pupils with the exploration, and pupils are not often given the opportunity to narrow down the mathematical modelling problem and decide how to collect and represent data.

Ikeda and *Stephens* (Chap. 24) survey pre-service mathematics teachers on the kinds of educational effects gained when addressing a task from the perspective that mathematical modelling can be used to enrich students' knowledge both in the real world and in mathematics. The results suggest that pre-service teachers are able to appreciate that modelling can not only enrich students' ability to solve real-world problems, but also deepen their ability to develop further mathematics.

Vargas and *Jara* (Chap. 25), in order to identify the implicit and explicit features in the practices of teachers in mathematical modelling, design a questionnaire consisting of two categories which emerge from a theoretical analysis using an onto-semiotic approach: epistemic and didactic. The questionnaire is administered to 30 ninth-grade mathematics teachers in Bogotá, Colombia who have extensive experience in teaching mathematical modelling. The data are collected using the Google Docs platform and analysed in relation to the theoretical framework.

Yvain-Prébiski (Chap. 26) presents an epistemological study to investigate the possibilities of giving students the responsibility for mathematical work that makes it possible to make an extra-mathematical situation accessible through mathematical treatment. A situation for teaching and learning mathematical modelling based on an adaptation of a professional modelling problem is designed, implemented and analysed.

1.6 Innovative Teaching Approaches

This section of the book reports some innovative teaching approaches in the teaching of mathematical modelling.

Brown (Chap. 27) investigates teacher noticing and novice modellers' developing conceptions of noticing during a primary school mathematical modelling task through teachers observing Year 3/4 students attempting the task. It is argued that to achieve success in solving real-world tasks, students must notice what is relevant and decide how to act on this to progress their solution, and teachers must also discern what is relevant and nurture student capacity to notice.

Buchholtz (Chap. 28) reviews findings on mobile learning with math trails and presents the results of a study on digital support of the mathematical modelling processes of 11th graders when doing math trails. It is argued that math trails contain tasks that promote essential elements of mathematical modelling such as mathematising, and the fact that math trails are more and more supported by digital media affects students' motivation and achievements.

Burkhardt (Chap. 29), based on his 55 years of experience as a researcher in mathematical modelling education, introduces some core concepts in mathematical modelling, and then focuses on the design strategies and tactics that are learned in the projects that he has been involved in, including the roles for technology. The difficulties of achieving improvement on a large scale are discussed, based on specific design issues in teaching modelling, and elements of a way forward are outlined.

Garfunkel, *Niss* and *Brown* (Chap. 30) contrast the opportunities for mathematical modelling offered to students in their normal classroom versus extra-curricular events in terms of the support available from a more knowledgeable other. Such support within the classroom is usually provided by the classroom teacher, while support for extra-curricular modelling opportunities is sometimes non-existent. Using the International Mathematical Modelling Challenge as an example, it is argued that the learning environment of such challenges is conducive to student engagement with mathematical modelling.

Jung and *Lee* (Chap. 31) explore the integration of group creativity into mathematical modelling in a ninth-grade class, grounded in a sociocultural perspective.

Findings from lesson observation and interviews with participants indicate that group creativity contributed to simplifying the situation and elaborating models, and to get a more elaborated model, group composition reflecting cognitive diversity and teacher's guide for interactions based on mathematical grounds should be emphasized.

Kawakami and *Mineno* (Chap. 32) examine ninth-grade students' data-based modelling to estimate previous and unknown Japanese populations. The results show that the data-based modelling approach can be used to construct, validate, and revise various models while flexibly combining mathematical, statistical, and contextual approaches generated by using data from real-world contexts. It is argued that data-based modelling can be a pedagogically dynamic and flexible approach for balancing the development of generic modelling proficiency and the teaching of mathematics and statistics through real-world contexts.

Manzini and *Mhakure* (Chap. 33) explore the implications of using mathematical modelling as a framework for the teaching and learning of mathematical concepts such as proportional reasoning in some under-resourced schools in low socio-economic areas of South Africa. The results show that the initial apprehension that students experienced when exposed for the first time to a model-eliciting activity is soon transformed into a diverse range of creative mathematical approaches, when they learn that the activity is open-ended by default.

Mhakure and *Jakobsen* (Chap. 34) investigate the mathematical thinking style of Grade 11 students in two schools from low socio-economic areas in South Africa when they are working on a modelling task involving a real-world problem on geometrical constructions. It is found that although students are able to find solutions to the scaffolded questions, they have problems with identifying the key mathematical concepts required during the mathematization process and the assumptions required to solve the modelling task.

Passarella (Chap. 35) presents a teaching case in a primary school class on multiplication as iterated sum during regular mathematics lessons, where the researcher designs a model-eliciting sequence with the aim of bringing out formal mathematical concepts from students. It is argued that the implementation of model-eliciting activities can foster emergent modelling, i.e., the students' attitude to discover and (re-)create new mathematical concepts.

Sokolowski (Chap. 36) reports a study of 21 high school students in a mathematical modelling activity involving the topic of the Fundamental Theorem of Calculus (FTC) that utilizes scientific reasoning to support the learning of mathematics concepts, based on research on and recommendations about designing effective exploratory STEM modelling activities. The students' responses show positive effects of this activity in understanding FTC.

Tangkawsakul and *Makanong* (Chap. 37), following a context-based approach, report the design of some mathematical modelling activities which emphasize authentic situations that are closely related to the real life of ninth-grade students. The aim is to encourage students to integrate mathematical knowledge, skills and processes in the creation of mathematical models to understand and solve problems. It is found that most of the students engage in mathematical modelling processes with their friends during the activities, which allows them to use and practically connect mathematics with real situations and problems encountered during their daily lives.

Zhou, *Li*, *He* and *Li* (Chap. 38) explore how to infiltrate mathematical modelling in calculus teaching (such as including definition introduction, theorem application and practice training) from the perspective of teachers. Three examples are presented in detail as illustrations. It is argued that integrating mathematical modelling into a calculus course teaching is an effective way to cultivate students' innovative and practical abilities.

1.7 Examples on Mathematical Modelling

This section of the book provides some further examples on teaching mathematical modelling.

De Bock, *Deprez* and *Laeremans* (Chap. 39) argue that instead of taking examples and contexts exclusively from physics or other natural sciences in learning about mathematical applications and modelling, applications from economics, business, or finance in secondary school mathematics should be more utilized. To study the role of such applications, all Proceedings of past ICTMA conferences are scrutinized. It is found that economic applications are indeed not well represented, however a positive trend is revealed since ICTMA12, the first ICTMA whose conference theme explicitly refers to economics.

Ekici and *Alagoz* (Chap. 40) report on design-based research experiments that extend the modelling of circular motion to advanced periodic orbits from a series of trigonometric functions. Inquiry-based orbital modelling allows students to experiment with modelling of periodic orbits with technology-rich tasks in interpreting the connections of periods and amplitudes of circular functions and the emergent patterns. The results show that learners experience coherence while interpreting, comparing, and validating their orbital models in circular, functional, and complex trigonometry with connections in between.

Greefrath and *Vos* (Chap. 41) discuss a variety of issues relating to the increasing use of digital tools and media in mathematical modelling tasks. A classification system for ICT-based mathematical modelling tasks is developed, and the classification is validated with three example tasks. A visual presentation based on the classification system enables the evaluation of qualities of a given ICT-based modelling task and can give insight into potential adaptations.

Kacerja, *Cyril*, *Gierdien*, *Herheim*, *Lilland* and *Smith* (Chap. 42) present a study in which Norwegian and South African prospective teachers discuss critical issues relating to a task on the mathematical model of the Body Mass Index. Four themes are identified, and the themes are discussed in relation to prior research on mathematical models in society and teacher education. The potential of such modelling examples to promote critical discussions about the role of mathematical models in society is argued.

Steffensen and *Kacerja (*Chap. 43) study how lower secondary school students reflect when using a Carbon Footprint Calculator (CFC) in their work with climate change in the mathematics classroom from a socio-critical modelling-perspective. The findings show that students reflect on various issues such as making sense of the use of CFC and global emissions. It is argued that CFCs have the potential to bring about critical reflections on mathematical models with the power to impact people's lives.

1.8 Issues at Tertiary Level

A number of issues on mathematical modelling at the tertiary level are covered in this section of the book.

Aragón and *Delgadillo* (Chap. 44) contrast the problems presented by a professor in an engineering course and the mathematical modelling project developed by students in the practical section of said course. It is argued that there is evidence of modelling competence in engineering being promoted and developed and that it is possible to consider it as a connector between the various training cores, identifying mathematical models that will allow us to understand and establish relationships between such training cores.

Durandt, *Blum* and *Lindl* (Chap. 45) report a study about the influence of two different teaching designs on the development of first-year engineering mathematics students' modelling competency. One is an independence-oriented teaching style, aiming at a balance between students' independent work and teacher's guidance, while the other is the more traditional teacher-guided style. The results show that all groups have significant learning progress, but the group taught according to the independence-oriented design has the biggest competency growth.

Julie (Chap. 46) reports a study in which a group of practising teachers in an introductory immersion course on mathematical modelling construct a model for funding and ranking of universities and present their model to other members of the participating cohort of teachers. Data analysis is anchored around the notions of internal and external reflections occurring during the interactions between the group who construct the model and their peers. The analysis renders four themes of which two are distinctly aligned to internal reflections, and the other two are an intertwinement between the external and external reflections.

Rogovchenko, *S.* (Chap. 47) analyses engineering students' written reports on a mathematical modelling assignment. A commognitive framework is used in the analysis of students' mathematical discourse in written solutions and oral discussions, and analysis of students' narratives indicates the development of exploratory routines in the process of solving mathematical modelling tasks. It is argued that teaching of mathematical modelling at the university does not only contribute to the development of mathematical competencies and motivates the interest to mathematics but also plays an important didactical role in promoting mathematical thinking of engineering students.

Rogovchenko, Y. (Chap. 48) reports a study on the extra-curriculum activities of biology undergraduates, focusing on the selection of mathematical modelling tasks with different levels of cognitive demand and the level of teacher's guidance during students' collaborative work on the tasks.

Spooner (Chap. 49) studies the experience of first-year university students in a mathematical modelling course. Using reflective thematic analysis, student interview data are inductively analysed to identify themes relating to their collective learning experiences. The results show that through guidance during lectures, students are able to have an independent modelling experience. To further enhance this, it is recommended that lecturers work through problems unfamiliar to themselves during lectures.

1.9 Other Subjects

A number of other subjects related to mathematical modelling are covered in this last section of the book.

Ärlebäck and *Frejd* (Chap. 50) report a study where upper secondary students devise and implement a plan for tackling a mathematical modelling question, and reflect on the aspects and factors that might have influenced their adopted strategy and results. The analysis focuses on students' reconstruction and categorization of the models, modelling strategies, and the variability that the activity elicits. The results show how the central statistical idea of variability is manifested in the models and strategies developed and implemented by the students.

Frejd and *Ärlebäck* (Chap. 51) analyse the 17 ICTMA books published to date and the books from ICME-6 and the 14th ICMI study in order to characterize the potential connections and synergies between statistics and mathematical modelling education. The results show synergies in terms of some identified themes on the teaching and learning of statistics and modelling. The context units analysed often provide suggestions for how to teach statistics using modelling approach, but seldom is the relationship between mathematical and statistical modelling from a theoretical point of view discussed.

Galbraith and *Fisher* (Chap. 52) provide illustrations on system dynamics modelling as a means of real-world problem solving relevant to secondary level and beyond. It is argued that national curricula around the world increasingly emphasize the importance of students being enabled to apply mathematics in the workplace, as citizens, and for private purposes. Examples of common structures (archetypes) are used to demonstrate application to problems made tractable by free online software.

Moutet (Chap. 53) reports a study on the teaching sequence for chemistry students in the last year of secondary school (grade 12) in France. The study aims to show how the extended Mathematical Working Space (extended MWS) theoretical framework can be used to analyse the tasks implemented during a few stages of a modelling cycle in a chemical problem. It is argued that the extended MWS theoretical framework makes it possible to study the multidisciplinary aspect of the different tasks that students perform on problems solving.

Tetaj (Chap. 54) describes an analytical scheme designed for investigating the mathematical discourse of biology tasks. The scheme is developed in the context of analysing tasks that are part of a fisheries management graduate-level course at a Norwegian university. Grounded in the commognitive perspective, the scheme focuses on different aspects of the tasks. The choice of the categories included in the scheme is justified and its use on one specific task is exemplified to illustrate the potential of analysis.

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Part II
Theoretical Issues

Chapter 2
Computational Thinking and Mathematical Modelling

Keng Cheng Ang

Abstract Computational thinking has been a subject of much discussion in education in recent times and is regarded by educators and policy makers as one of the key skill sets for the future. Many schools have introduced coding and programming to students, sometimes very early in their education years, in a bid to help them develop computational thinking. In this chapter, we explore and explicate the role of computational thinking in mathematical modelling, and examine the relationship between them. Examples from the different modelling approaches will be used to contextualize this relationship, and to demonstrate that mathematical modelling does indeed provide an excellent platform for the use, practice and development of computational thinking. In addition, these examples will also illustrate how computational thinking fits into mathematical modelling naturally in some modelling situations.

Keywords Computational thinking · Coding · Mathematical modelling · Programming · Simulation models

2.1 Introduction

There is a growing interest in computational thinking among educators and educational researchers, and its importance in K-12 education has been the subject of much discussion in recent years. The idea of computational thinking is gaining attention worldwide partly due to the perception and belief that the attitude and skill set involved are essential in tackling problems, and partly because of the widespread use of technology, and computing tools and devices for work and pleasure in the world

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today. Indeed, computational thinking is often seen as an important twenty-first century skill for "everyone" (Mohaghegh and McCauley 2016).

Because coding, or computer programming, is considered as one of the key skills required in computational thinking, various initiatives to teach coding have been taken in different parts of the world to reach out to students, and the general public. However, these are relatively new and recent initiatives, and their impact on problem solving in general has not been studied extensively (Denning 2017; Yasar 2018).

While the use of Information and Communication Technology (ICT) in mathematical modelling as well as in the teaching of mathematical modelling has been discussed extensively, the role of computational thinking in mathematical modelling is not as well studied. Indeed, being able to use a technological tool, no matter how sophisticated, does not equate to being able to formulate a problem or design a solution through a computational thinking process. The intent of this chapter, therefore, is to examine and explicate the relationship between computational thinking and mathematical modelling through the use of tested modelling tasks. Throughout the chapter, we will view mathematical modelling from the educational perspective, and the examples discussed will present and illustrate the different approaches to modelling that can be experienced in the classroom.

2.2 Computational Thinking

In his book, *Mindstorms: Children, Computers and Powerful Ideas*, published in 1980, Seymour Papert envisioned how the computing machine could help children learn and think in ways different from the traditional modes, the key idea being that learners construct knowledge with the help of computers. Further, one of the perspectives offered by Papert is the possibility of integrating what he suggests as "computational thinking" into everyday life (Papert 1980, p. 182).

Central to Papert's proposition was the use of "turtle geometry", which he describes as a computational style of geometry. In this environment, children can give simple commands to a "turtle" and make it draw geometrical shapes. So, instead of the computer teaching the user, the user is giving instructions to the computer to do something. As the shapes that one wishes to draw become repetitive or complex, one may then need to invoke the use of loops, iterations, mathematical formulae, procedures and sub-routines. Gradually, one begins to think in terms of steps and algorithms, and to solve problems systematically and in an organized manner.

Although there have been attempts to explain what computational thinking is, a precise and yet universally accepted definition has yet to emerge. Wing (2006), perhaps in an attempt to promote the study of computer science, suggests that "computational thinking is a fundamental skill for everyone" and that it can be used to solve any problem (p. 33). However, others argue that this may be over-selling computer science and raising expectations that cannot be met (Tedre and Denning 2016). Nonetheless, it seems fair to say that it is possible to recognize the different aspects and characteristics of computational thinking, and how these can be used or observed in the process of problem solving. Here, we adopt a more specific rather than generic view of computational thinking so that the discussion on its relationship with mathematical modelling may have a sharper focus.

For the purpose of the ensuing discussion, we shall adopt the notion that computational thinking is a process of designing, constructing and executing solutions to problems with a view to implementing them on the computer or using a computing tool. The key idea, therefore, is to think about problems in such a way that computers can help us solve the problems. In general, the skills or habits involved in this thinking process include the following:

- gathering important information to scope the problem and discarding non-essential parts or components (abstraction);
- studying and analysing the problem to see if there are trends or repeated sequences that may fit some known or familiar solution method (pattern recognition);
- breaking a large, complex problem into smaller parts so that these may be solved more effectively or efficiently (decomposition); and
- developing a set of step-by-step instructions that lead to a solution (algorithm).

The above four problem solving skills or approaches have been accepted by the community as the four "cornerstones" of computational thinking (ISTE 2011; Weintrop et al. 2016). However, it would seem that these ideas are not entirely new and are in fact what one would have normally observed as characteristics of mathematical problem solving. One might then ask, what is so special about computational thinking that makes it different from other forms of "thinking", such as mathematical thinking? In addition, how are these four skills, habits, or characteristics linked to computational thinking? One possible answer could lie in the one skill that is commonly taught in computer science courses—computer programming, or coding.

2.2.1 Habits Developed Through Coding Exercises

It is widely believed that one effective way of developing the skills stated above would be through learning computer programming or coding, and through the practice of solving problems that require some form of coding (Ho and Ang 2015). It is perhaps this belief that has led to growing interest in coding schools and classes, both formal and informal, in many parts of the world. Coding lessons are available freely at websites such as code.org, Hour of Code, Code Academy and FreeCodeCamp. In some European countries such as England, Greece and Estonia, programming is included in the school curricula as a compulsory subject and children are exposed to coding at a young age (see Mannila et al. 2014).

Certain useful and critical habits are gradually developed through the process of problem-solving with coding. Typically, one needs to think through the solution process in a logical and systematic manner, and develop an algorithm. The coder may make use of a flowchart to visualize the flow of the process. Sometimes, there is a need to break a big problem into smaller parts, and employ a "divide and conquer" strategy in solving the problem. This is equivalent to simplifying or decomposing a problem, and constructing procedures and sub-procedures in the code.

To write code, one has to follow and obey the syntax of the language used, and keep to certain rules. This is, in fact, a form and practice of abstraction since only the most important and relevant pieces of information will be extracted and used, just like in the model formulation phase in mathematical modelling. In addition, coding requires one to use variables as representations of factors involved in a problem. Quite often, the code will involve iterating through loops, or managing and manipulating data sets. There is also the opportunity or need to think numerically or in terms of actual numerical instances in solving the problem, especially when empirical data sets are involved. Such exercises help one develop a sense of pattern and pattern recognition in tackling modelling problems.

However, does coding necessarily lead to computational thinking? If computational thinking is seen to be a habit of mind, then coding is part of the strategy used to develop such a habit. In other words, while necessary, it may not be sufficient. That is, simply being able to write code does not mean that one will be able to solve a problem using a computational method. Computational thinking involves analysing a problem, examining the context, studying available data, simplifying the situation, designing an algorithm and finally, writing the code, if applicable, during implementation.

In the next section, we will discuss three examples of modelling tasks. The approach used in each case is one that involves a computational strategy, either in the model or in the solution method. These examples show that indeed, those aspects of computational thinking discussed earlier do manifest themselves in many mathematical modelling activities and tasks.

2.3 Examples

2.3.1 Example 1: From Data to Model

In this first example, we discuss how publicly available data on the outbreak of a contagious disease could be used to construct a model for its spread. Although the problem is not new or current, it provides a rich context for a discussion on the influence of computational thinking in constructing mathematical models. Here is the problem statement.

The SARS epidemic

In 2003, a deadly and contagious disease called the Severe Acute Respiratory Syndrome, or SARS, descended upon the world. Some countries in the Asia–Pacific region, in particular, were heavily hit and Singapore was one of them.

Table 2.1 Number of individuals infected with SARS during the 2003 outbreak in Singapore (Heng and Lim 2003)

Day (t)	Number (x)	Day (t)	Number (x)	Day (t)	Number (x)	Day (t)	Number (x)	Day (t)	Number (x)
0	1	15	25	29	101	43	163	57	202
1	2	16	26	30	103	44	168	58	203
2	2	17	26	31	105	45	170	59	204
3	2	18	32	32	105	46	175	60	204
4	3	19	44	33	110	47	179	61	204
5	3	20	59	34	111	48	184	62	205
6	3	21	69	35	116	49	187	63	205
7	3	22	74	36	118	50	188	64	205
8	5	23	82	37	124	51	193	65	205
9	6	24	84	38	130	52	193	66	205
10	7	25	89	39	138	53	193	67	205
11	10	26	90	40	150	54	195	68	205
12	13	27	92	41	153	55	197	69	205
13	19	28	97	42	157	56	199	70	206
14	23								

During the 2003 SARS outbreak in Singapore, 33 lives were lost within a span of about 70 days.

Using the data shown in Table 2.1, construct a mathematical model to describe the outbreak of the SARS epidemic.

One of the modelling approaches that can be used in this case is to examine the data carefully, and see if we can find an existing model that may be suitable. It is perhaps more convenient to study the data set visually and so the first step would be to plot a graph. Obviously, this would be a graph showing how the number of infected individuals varies with time, as shown in Fig. 2.1.

From the graph in Fig. 2.1, it is evident that the number of infected individuals increases slowly at first, and then rapidly from about Day 15–55 before slowing down again towards the end of the epidemic episode. One could recognize that this is generally how a sigmoid curve would behave and a suitable function that could be used to represent this behaviour would be the logistic function.

From a population dynamics perspective, it is known that in such a compartmentalized model can be represented by a logistic equation in an "S-I" epidemic model, where "S" and "I" represent the susceptible and infected individuals in the population, respectively. Using this model, we may construct the equation

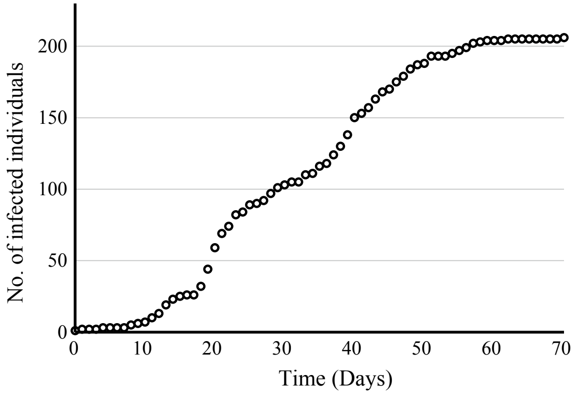

Fig. 2.1 A plot of the data showing the number of individuals infected with SARS

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \beta x y$$

where x and y are the number of infected and susceptible individuals at time, t, and β is a constant. Further, if we assume a closed community of N individuals, then, $+y = N$. The equation may be rewritten as

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{N}\right)$$

where the constant k represents the rate of transmission. Solving the differential equation with the initial condition, $x(0) = x_0$ yields the solution,

$$x(t) = \frac{Nx_0}{x_0 + (N - x_0)e^{-kt}},$$

which may be rewritten as

$$x(t) = \frac{N}{1 + \left(\frac{N}{x_0} - 1\right)e^{-kt}}.$$

The parameter k may be estimated from the dataset in a variety of ways. We could, for instance, define an "average error" between the data points and the model, and use the method of least squares or a computing tool, such as the *Solver Tool* in Microsoft Excel to minimize the error. For a description of this method, as well as explanation of how the Solver Tool is used to find estimates of k, the reader is referred to article where this example first appears (see Ang 2004).

Using this method with $x_0 = 1$ and $N = 206$, we obtain $k = 0.1686$. The graph of the model is plotted and compared with the data points in Fig. 2.2a. It can be seen that the model generally compares well with the actual data. However, there are certain parts with obvious deviations. In fact, by examining the figure carefully, we

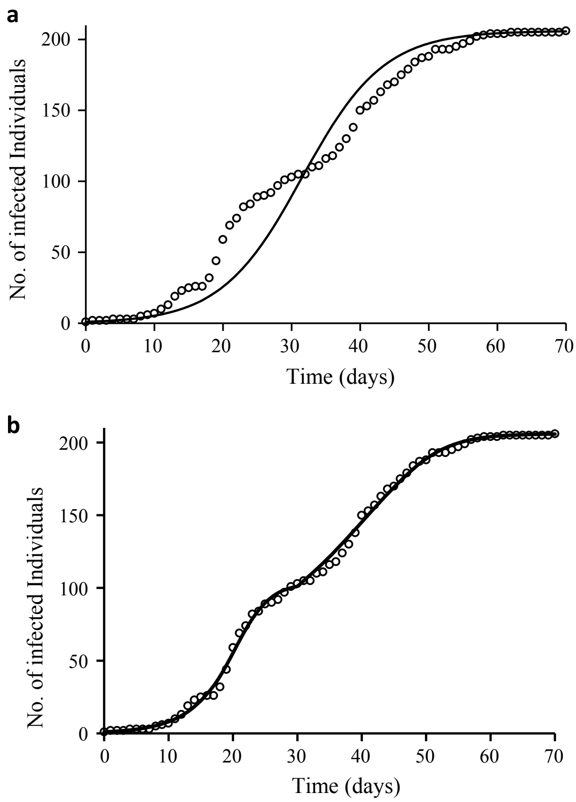

Fig. 2.2 Comparison of SARS data and logistic curve models: **a** SARS epidemic model using one logistic curve **b** SARS epidemic model using two logistic curves

observe that in fact two logistic curves may better describe the situation, as shown in Fig. 2.2b.

This example demonstrates that when dealing with data to construct an empirical model, the specific skill of observing and recognizing patterns in a dataset proves useful in formulating the mathematical problem. The resulting model is solved and refined using a computational method implemented on a computing tool such as an electronic spreadsheet. This habit of pattern recognition is one aspect of computational thinking that will further develop and expand a student's competencies in mathematical modelling.

2.3.2 Example 2: From Processes to Model

Consider the situation where a certain organization needs to hire a secretary by choosing the best person from a group of possible candidates through walk-in interviews. The specific conditions are described in the problem statement below.

The Secretary Problem

A company needs to hire the top-ranked candidate for a secretary's position under the following conditions.

- The total number of candidates, n, is known.
- The candidates are ranked with no ties.
- Candidates are interviewed sequentially and in a random order.
- *Relative* ranks of interviewed candidates are known.
- The candidate is either accepted or rejected right after the interview.
- Rejected candidates may not be recalled or accepted.

The task is to develop a strategy so that the best candidate is chosen.

In order to have a better grasp of the problem, one could construct a simulation of the hiring process. The steps involved can be written out as a flowchart as shown in Fig. 2.3.

The flowchart helps one to think through the process of the simulation in a systematic and organized manner, keeping track of the variables involved and going through the steps of the process. The flowchart therefore serves as a guide or algorithm for one to write the code. Using the flowchart, the code for simulating the hiring process based on the conditions of *The Secretary Problem* can be written. Coding often helps one understand the process even better, and in this case, it helps in providing a way for

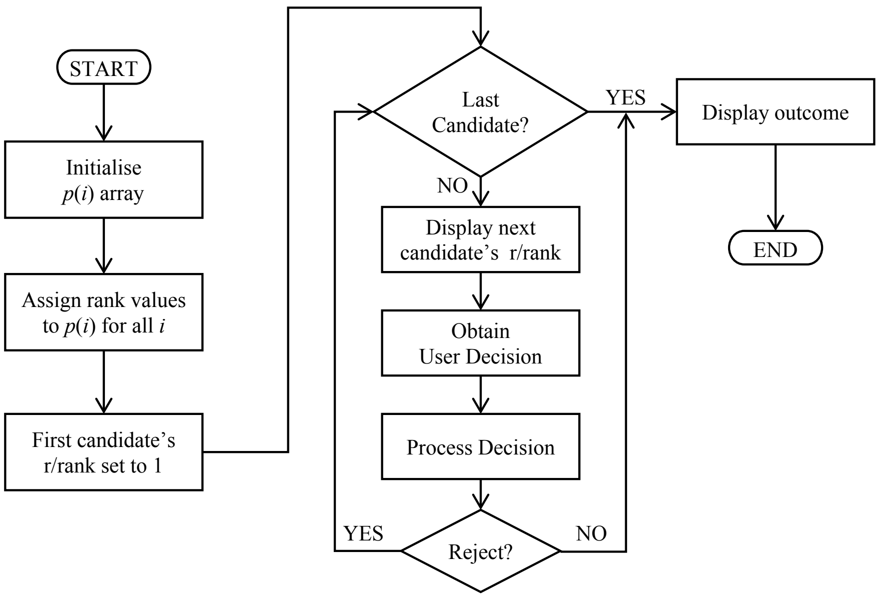

Fig. 2.3 Flowchart for the hiring process

one to think about what might be a good and consistent strategy to adopt. Randomly determining when to accept or reject a candidate would lead to nowhere.

With this simulation, the process is basically constructed for exploration. After a few simulation rounds, it is not hard to realize that one strategy would be that of always rejecting a certain number, say, k, of candidates, and then accepting the next one with a highest relative rank.

We can investigate this strategy by automating the hiring process with different values of k. For each value, we run the simulation many times and calculate the experimental probability of successfully picking the best candidate. A flowchart for this simulation is shown in Fig. 2.4.

Implementing the simulation using the steps detailed in the flowchart shown in Fig. 2.4, and running the simulations with 100 candidates and $k = 0$ to 99, and 10,000 trials for each k, the experimental probabilities can be found. These are plotted against k and compared with the theoretical probabilities in Fig. 2.5. The simulation results show that an optimal strategy would be to reject the first 37%, approximately, and then accept the next top candidate, which agrees with the theoretical optimal probability of $1/e$. The theoretical optimal probability has been derived and discussed by several other authors (e.g. see Ferguson 1989).

Running the simulation many times to obtain experimental probabilities to represent the actual probabilities is an application of the law of large numbers and a common approach in simulation models. The ability to think through the process and

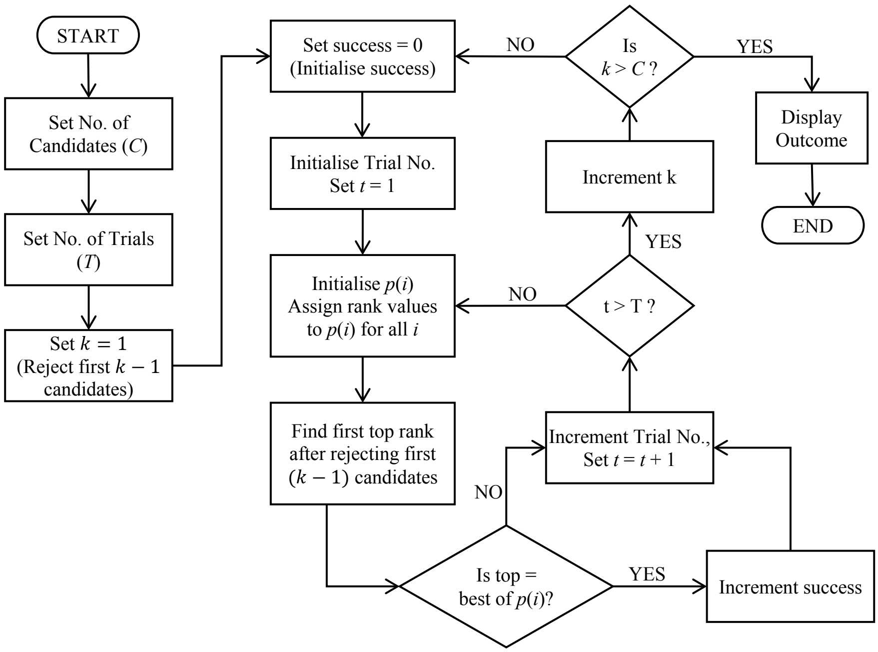

Fig. 2.4 Flowchart for simulating runs of hiring process for all possible values of k

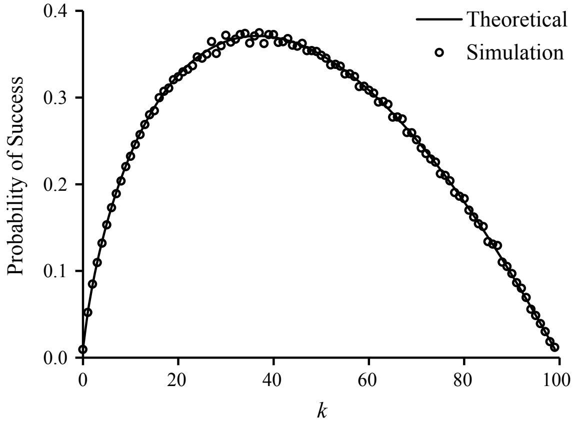

Fig. 2.5 Graph of experimental probability of success against k

to write the code for the simulation algorithm will certainly enhance one's ability to design, construct, and solve simulation models.

This example demonstrates how an algorithm for a simulation can be constructed by examining a process in detail, studying each step and the conditions for moving to the next step, and checking of the status of critical variables along the process. Flowcharts are commonly used by computer programmers to aid their coding process and in this case, the use of flowchart helps one think through the simulation model more systematically. Being systematic and abstracting only the essential information to formulate the simulation is another aspect of computational thinking that will also develop one's competency and acumen in mathematical modelling.

2.3.3 Example 3: From Decisions to Model

In this example, we discuss the problem of allocating funds to members of a department for the purpose of staff development. In particular, we consider the problem given below.

Resource Allocation Model

The Head of a department in an academic institution is allocated a fixed annual budget, based on "headcounts" (number of members in the department), to provide financial support to members of the department for staff development (such as attending or presenting at a conference) for that year. This could be, say, $\$x$ per person. Staff members will then indicate their staff development plans and apply for funding. However, because some may need more than $\$x$, while others may request for less (or do not plan to use the funds at all that year), the amount eventually allocated to each staff member will generally differ.

What would be a fair way of distributing the allocated budget while maximizing the utilization rate?

This is a real-life situation that may arise in many institutions and organizations; an annual budget is allocated for a purpose, and the expectation is to maximize utilization of the budget so that the funding authority will continue to fund and support the purpose. Should the utilization rate be low, the institution or organization risks having the budget reduced the following year as it may be deemed that a lower amount is actually required.

In this current situation, suppose at the beginning of the year, a budget of amount B is given and, obviously, the total funds allocated to staff must not exceed this amount. Suppose there are n requests for funding, and each amount requested is A_i where $i = 1, 2, \ldots, n$.

For simplicity, we make the following assumptions:

(i) each staff member can only make or submit one request for funding per year; and
(ii) the department head may approve up to the full amount requested.

Based on assumption (ii) above, suppose for each requested amount of A_i, the department head aims to approve $\mu_i A_i$ where $\mu_i \in (0, 1)$, then the objective would be to minimize the quantity,

$$X = B - \sum_{i=1}^{n} \mu_i A_i$$

subject to the condition, $\sum_{i=1}^{n} \mu_i A_i \le B$. In other words, what is required in this model is to find the set of μ_i for each i so that X is minimized.

In addition, it would not be unreasonable to take into consideration two other factors when deciding the amount of funding each staff member should be allocated. Firstly, the amount that one has been given the previous year should have some bearing on the amount that one ought to be allocated in the current year. Secondly, if the purpose of funding is for staff development, then the stages of academic career of the staff members should also play a role in the decision. Assuming that these are part of the objectives of the model, then we could impose the conditions that more support should be given to those staff members who:

(a) were allocated smaller amounts the previous year; and
(b) are junior in academic rank or are in more need of development.

If these two factors are to be considered in allocating the funds, then they must somehow be built into the model and be taken into account when determining all the μ_i. As a first step, a simple model would be to place each staff member into a

Table 2.2 Relative levels of support for staff members in different categories

Career stage (C)	Previous support (P)		
	Low (1)	Medium (2)	High (3)
(1) Junior	5	4	2
(2) Middle	4	3	1
(3) Senior	2	1	0

certain category or group for each of the two factors. For instance, for condition (a), the amount given to staff members in the previous year could be classified as "Low", "Medium" or "High" corresponding to the actual low, medium or high levels of financial support obtained. The known or given data would have to be the previous year's allocated amount to each staff member and based on these data, each staff member will be placed in the appropriate category.

To incorporate the second condition, we could also categorize the staff member based on their academic ranks. Here, one's academic rank is used as a proxy to represent one's need for professional development. For instance, a more junior member is probably more in need of such development and therefore ought to be better supported compared to a more senior member of staff. For simplicity, we could use three categories, "Junior", "Middle" and "Senior". For instance, in a typical university setting, a newly appointed Assistant Professor or Lecturer could be placed in the "Junior" category, while the "Senior" group would include tenured Full Professors.

Based on these simple, discrete categories, one could draw up a table or matrix with cells where the value in each cell indicates the relative level of support that a staff member emplaced in that cell should receive, with zero as the baseline. Here, we assume for a senior member of staff who had received a high level of support the previous year, the allocation this current time would be the baseline from which the rest will take reference. In other words, an application for funding from a member in this category should have the lowest value of μ_i. The other values in the cells are assigned, while arbitrarily in some sense, with the intention of satisfying conditions (a) and (b) above. An example of such a table is shown in Table 2.2.

Each staff member's application for funding will then be assigned a value based on where it is found in the table. Let this value be w_i. With this graduation of level of support assigned, the problem of finding all the μ_i can be simplified. In the simplest case, we adopt the following linear, stepwise model,

$$\mu_i = \mu + w_i \Delta\mu$$

where μ is a base value (between 0 and 1), and $\Delta\mu$ is a "step" from the base. The problem then reduces to finding $\Delta\mu$ that will minimize the quantity, X.

As an illustration, consider the following situation. Suppose a budget of $B =$ \$27,000 has been approved, and ten staff members of various appointment ranks have applied for funding for different amounts listed in the last column in Table 2.3. The total requested amounted to \$38,900, which the budgeted amount will not be able to meet. The fund given to each individual in the previous year is shown in

Table 2.3 Illustrative example of ten staff members and their applications for funding

No	Name	Career stage		Previous grant		w_i	A	μ_i	$\mu_i \times A$
		Rank	C	Amount ($)	P				
1	Aidel	Asst P	1	0	1	5	3990	0.79	3140.37
2	Bala	Assoc P	2	2950	3	1	3990	0.64	2543.27
3	Chen	Lecturer	1	3680	3	2	5230	0.67	3529.33
4	Dharna	Full P	3	2100	2	1	5230	0.64	3333.66
5	Emery	Assoc P	2	2820	3	1	3990	0.64	2543.27
6	Faharna	Asst P	1	950	1	5	1800	0.79	1416.71
7	Godfrey	Asst P	1	2250	2	4	1200	0.75	899.58
8	Haiyue	Full P	3	0	1	2	5230	0.67	3529.33
9	Ingham	Snr Lect	2	2410	2	3	3010	0.71	2143.83
10	Jiale	Assoc	2	1500	1	4	5230	0.75	3920.66
Total							38,900		27,000.00

Column "Previous Grant". The respective weights, w, are determined by the values of C and P.

Setting $\mu = 0.6$ (that is, everyone should get at least 60% of what is requested), we can proceed to use Excel's Solver Tool to find $\Delta\mu$ by minimizing X. In this case, it turns out that the optimal value for $\Delta\mu$ is 0.037. Using this value, we can then compute the allocated amount $\mu_i \times A$ for each application i as shown in Table 2.3.

This example demonstrates the need to reduce a complex problem, and hence the skill of decomposition shown in computational thinking. Decisions have to be made, and in this model, we consider logical factors and build rules into the model. In the process, we also make assumptions to simplify the situation, and to reduce to problem to a manageable size. Decomposition and tackling smaller bits to build a more complete model is a modelling skill students develop through this kind of computational thinking exercises.

2.4 Discussion

As can be seen in the examples described, modelling can be greatly enhanced if one possesses and is able to apply certain computational skills and a certain way of thinking such that these skills can be effectively applied to tackle the problems. In other words, the ability to think of problem solving strategies that make use of computational tools or programming constructs is a valuable modelling competency.

In Example 1, the use of data in the empirical model is evidently an opportunity for one to observe the pattern and determine if a known or existing model can be used to describe the disease outbreak. In addition, there is a need to first simplify the situation, and later refine the model. Certain important computational tools (such as the Solver

Tool in Excel) are used in the solution process. A computational thinker would likely study the data set, and explore ways of making use of it after observing the pattern. Of course, beyond that, one has to have some inter-disciplinary knowledge— in this case, knowledge in population dynamics or epidemics. To construct a model and realistically produce a plausible solution, however, such knowledge is necessary but not sufficient. Some computational skills are still required, as can be seen in the solution process in this example.

In Example 2, simulation models are developed, guided by step-by-step algorithms to first understand the problem situation and then to tackle the problem. Constructing the solution requires one to think computationally in terms of writing out the steps for the simulation, as well as the code for the simulation programs. Developing a step by step algorithm is a common practice in coding exercises or programming courses. The essential skills of identifying variables required in the problem, simplifying the process, recognizing the need to perform certain tasks in a certain sequence or order, and so on, are all part of computational thinking. When all these become a habit of mind, developing a simulation to model a process can be another useful and effective modelling competency.

In Example 3, some assumptions are first made to simplify a rather complex real-life problem. These assumptions also help in abstracting the real situation into a mathematical formulation, from which a model, which is essentially a decision-making model, can be constructed and subsequently solved. In a decision-making situation, a computational thinker would gather and consider the factors that would lead to the decision, turn them into variables and find a way to connect them to provide the necessary information to make the decision. Again, such situations arise quite often in coding exercises or computer programming problems, and over time, these exercises help one develop both the skill and the habit of systematically identifying and connecting variables in a real life problem. This aspect of computational thinking is well illustrated in this example.

Clearly, as discussed above, the skills involved in handling the modelling tasks in all three examples are not dissimilar to those that are closely related to computational thinking. To reiterate, these skills, which can be acquired through exercises such as coding or computer programming, serve to support the mathematical modelling process and provide additional tools for developing appropriate models. It is also important to point out that while skills can be taught and learnt in a short period of time, the habit of mind that makes one a computational thinker takes longer to develop.

Nevertheless, with additional computational skills, one could enhance one's competency and ability in tackling mathematical modelling tasks. At the same time, mathematical modelling tasks or situations provide an excellent platform for one to practise and apply one's computational skills and thinking.

2.5 Conclusion

In this chapter, we discuss three modelling tasks, and in each case, we use a computational approach to solve the problem posed. Using these examples, we identify the various computational skills that are useful in modelling and explain the computational thinking process that has led to the model. These skills are also developed through computational thinking tasks, such as coding exercises. It is clear that there is, therefore, a connection or link between computational thinking and certain approaches of mathematical modelling.

There are several questions that one could address to further examine the link between mathematical modelling and computational thinking. Some of these are listed below as further work that can be taken up by interested researchers.

- How do we "recognize" computational thinking? In other words, how can we tell when someone is "thinking computationally" when solving a problem? Does the use of a computational tool necessarily mean that computational thinking is part of the process?
- While coding is certainly a useful skill, how does it help in developing computational thinking and mathematical modelling competencies? Is there any way of determining the impact of coding on the development of computational thinking and competencies in mathematical modelling?
- If modelling activities are useful in developing computational thinking, how do we design activities targeted at doing that?
- How can we strategically and intentionally develop computational thinking through mathematical modelling?
- If or when we are able to design tasks that develop computational thinking through mathematical modelling, how do we detect and measure such development?

Given the digital world we now live in, computational thinking will remain an important and relevant concept in education for some time. Its usefulness and potential in the area of mathematical modelling has been demonstrated and explicated through the examples described in this chapter.

In conclusion, understanding the importance and relevance of the relationship between computational thinking and mathematical modelling could lead to better design of tasks in both computational thinking and mathematical modelling. Better and more meaningful tasks could in turn lead to enhanced learning and development of the relevant competencies and skills in both computational thinking and mathematical modelling.

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Chapter 3
Global Understanding of Complex Systems Problems Can Start in Pre-college Education

Diana M. Fisher

> *An invasion of armies can be resisted, but not an idea whose time has come.*
> Victor Hugo

Abstract In this chapter, an argument is made for the importance of enabling secondary school students to build models for analyzing complex systems problems, to increase understanding of the myriad nonlinear feedback systems they will encounter as professionals and citizens. Secondary school students in some schools in the USA have been building such models for over 20 years. A sequence of natural resource depletion models is presented to demonstrate the types of system models secondary school students can and have built. Advantages such activities have for enhancing the mathematical analysis of problems normally outside the reach of the secondary school curriculum are discussed. It is argued that the time is ripe for secondary school students to experience instruction which, using current technologies, can provide a wealth of applications rich, real-world, relevant problems.

Keywords System dynamics · Complex systems · Modelling · Secondary school students · Algebra · Technology

3.1 Introduction

> The conceptual basis of complex systems ideas reflects a dramatic change in perspective that is increasingly important for students to develop as it opens new intellectual horizons, new explanatory frameworks, and new methodologies that are becoming of central importance in scientific and professional environments. (Jacobson and Wilensky 2006, p. 12)

There has been a dramatic increase in the scope of applications of mathematics over the past few decades due mostly to an ability to create computer simulations,

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perform digital computations, and analyze large volumes of data (National Research Council of the National Academies 2013). Moreover, there has been a corresponding "explosion" over approximately the same time period in complex system science. Problems most commonly expressed in closed form, that required the assumptions of "smoothness" (in calculus), or independence (in statistics) are sharing space with problems that have multiple causes, interdependence, and nonlinearity (Bar-Yam 2012).

While facility with and creation of closed-form equations to represent functional relationships has served us well as guiding principles for secondary school courses in the past, it is no longer sufficient preparation for students in today's world. Emerging problems faced in business, science, engineering, politics, medicine, psychology, economics, management, and interdisciplinary pursuits require an understanding of complex, dynamic, systemic behavior (e.g., Galbraith 2010). For this purpose, an understanding of closed-form equation representation alone falls short. Technology provides new methods of observing and generating dynamic behavior, compressed in time, through which students can be prepared for future challenges in work contexts, and as responsible citizens. Examples of the serious issues facing our global community are described or displayed in the news every day.

One such "new" analytical method facilitated by the computer, system dynamics (SD), uses numeric solution of differential equations to enhance understanding of complex systems. This methodology was created by Jay Forrester at the Massachusetts Institute of Technology (MIT), in the mid-1950s, to model systemic problems. Recently, a free, web-based (basic) version of the software (Stella Online) has been created, allowing educators to bring this model building representation of dynamic systems into any web-accessible classroom.

This chapter will demonstrate, in a sequence of increasingly sophisticated models, how secondary school students can build (and have built, for over 20 years) models of complex, dynamic, feedback systems using the method of system dynamics.

3.2 Introduction to the SD Software

There are four primary icons that are used to create models using SD.

A **stock** (rectangle, see Fig. 3.1) is used to represent the main function variable whose value the modeler wants to track over time. It is an accumulator, an integral. A simple example of a stock variable might be the deer population in an ecosystem.

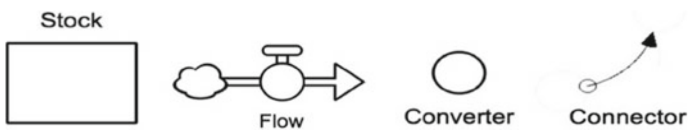


Fig. 3.1 Four main icons used to create SD models

Table 3.1 Recursive calculation of deer population

Year	Deer population (recursive calculation)
Initial deer population	30
Deer pop. after 1 year	30 – (0.1)30 + (0.3)30 = 30 + (0.2)30 = 36
Deer pop. after 2 years	36 + (0.2)36 = 43.2
…	…
Deer pop. after 26 years	2861.9 + (0.2)2861.9 = 3434.3
Recursive equation	$P_{t-1} + (0.2)P_{t-1} = P_t$

A **flow** (see Fig. 3.1) is a unidirectional[1] pipe with a valve that is drawn with the arrowhead either pointing toward the stock (increasing the stock value) or pointing away from the stock (decreasing the stock value). The combination of all flows attached to a stock represents the rate at which the stock value changes over time, hence represents the first derivative of that stock variable. For the previous deer population stock, an inflow might be designated deer births, while an outflow could be designated deer deaths.

A **converter** is a circular icon that holds additional parameters, variables, or simple arithmetic combinations of variables. For our previous example, a converter could hold the deer birth fraction.

A **connector** is a thinly shafted arrow that sends numerical information from one icon to another within the model. It shows dependencies of each variable upon parameters or other variables.

A simulation time unit, a fractional interval[2] of the time unit (DT) for updating model values, integration method (Euler or Runge–Kutta 4), and simulation duration can be specified by the modeler. The model engine uses recursive numerical calculations to update model values, recalculating each time step (DT).

Graphs and tables[3] can be produced to show the output of the model over the simulated time specified by the modeler.[2]

3.3 A Basic Deer Population Model

Consider an ecosystem containing 30 deer. Without a predator, we assume that deer normally have a 30% reproduction rate (birth fraction) and a 10% mortality rate (normal death fraction) each year. Assume that there is ample vegetation over the

[1]Bidirectional flows are permitted but are not used in any examples in this paper.

[2]The DT of the simulation software is like the "dt" of a calculus integral, or more accurately like a Riemann Sum or Simpson's Rule approximation of a calculus integral.

[3]Tables are not available in free Stella Online software, but one can read values from graph.

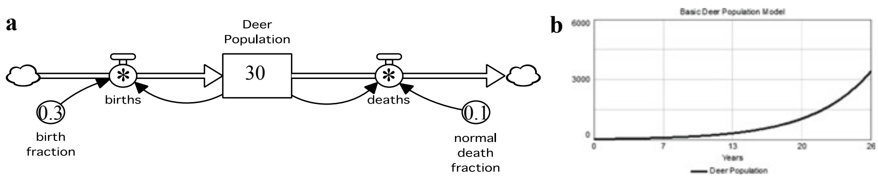


Fig. 3.2 **a** Equivalent Stella diagram representing this deer population, with icon values superimposed on each icon in the diagram, and **b** graphical output for the *Deer Population*

simulation time of 26 years. We want to track the number of deer in the environment over time.

Table 3.1 shows the recursive calculations that would occur within the simulation software given the data in the previous paragraph and, for simplicity, assume the time step (DT) is 1 year and the integration method is Euler.

After placing an icon in the modeling window, one double clicks on the icon and designates the value for that icon and its units in the panel that appears. So the stock, *Deer Population*, is set at 30 {deer}, *birth fraction* is set at 0.3 {deer/deer/year = 1/year}, *normal death fraction* is set at 0.1 {1/year}, *births* are defined as *Deer Population* * *birth fraction* {deer/year}, and *deaths* are defined as *Deer Population* * *normal death fraction* {deer/year}.

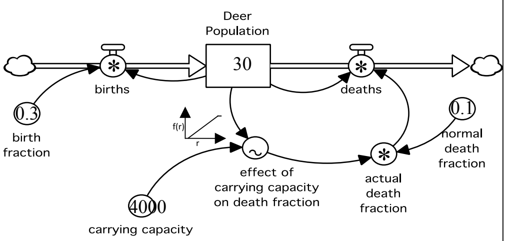
3.4 A More Realistic Deer Population Model

Resource availability places a constraint on any animal population. We will assume that the environment within which our deer are living can support at most a constant 4000 deer. The previous model can be modified by adding a component, *carrying capacity*, defined as 4000 {deer}. We will compare the current *Deer Population* to the *carrying capacity* each time step. As the *Deer Population* value grows toward the *carrying capacity* value, fewer resources are available per deer so the deer death rate begins to increase.

In the model, one uses a graphically defined converter, here labeled *effect of carrying capacity on death fraction*, to create the mechanism that increases the death rate as the *Deer Population* grows. See Fig. 3.3.

The *effect of carrying capacity on death fraction* usually has an independent variable axis definition represented as a ratio. Here, the independent variable is defined as *Deer Population/carrying capacity* (DP/CC). This ratio has domain values from 0 to 1. The vertical axis represents a unitless numeric value, designated as a multiplier (with a range from 1 to 3). As the *Deer Population* grows, the ratio DP/CC grows larger, moving the simulation to the right on this graphically defined function. As the simulation moves the DP/CC ratio to the right horizontally, the output value

a



b

$$P(t) = \frac{4000}{1 + \left[\frac{4000}{30} - 1\right] e^{-0.2t}}$$

Or

$$P_0 = 30$$

$$\frac{dP}{dt} = 0.3 * P - 0.1 * P * f(r)$$

$$\text{Where } f(r) = \begin{cases} 2 * r + 1, & 0 \le r \le 1 \\ 3, & r > 1 \end{cases}$$

r = Deer Population / carrying capacity

$$\text{So, } \frac{dP}{dt} = P(0.2 - 0.00005P)$$

Fig. 3.3 **a** *Deer Population* with fixed *carrying capacity* of 4000 deer. Defining values are superimposed on the appropriate icons. **b** An equivalent closed-form logistic representation and also calculus initial value problem representation for this scenario

(vertical axis value) is increasing, linearly (for this example).[4] This dimensionless output value is then multiplied by the *normal death fraction* value to create the value for the *actual death fraction* component. The *actual death fraction* is the value used to calculate *deaths* in the model. The *actual death fraction* grows larger and larger as the *Deer Population* approaches the *carrying capacity*, until, when *Deer Population* = *carrying capacity (*and the ratio DP/CC = 1), the multiplier is 3, so the *actual death fraction* becomes 0.3, matching the *birth fraction.* At this point, the model achieves steady state (or dynamic equilibrium).

3.5 Modeling a Real-World Scenario

Problem: In 1944, the US Coast Guard brought 29 reindeer to St. Matthew Island in the Bering Sea, about 200 miles off the coast of Alaska. They brought them as a potential supplement for their food supply. The US Coast Guard abruptly left St. Matthew's Island a few years later but left the reindeer behind. The reindeer ate mostly lichen on the island, which can take decades to regenerate. There were no reindeer predators on the island. In 1963, there were about 6000 reindeer on the island and the reindeer population collapsed so that only about 42 reindeer remained in 1965.

To replicate the behavior described in this scenario, we would not want to create a model with a constant carrying capacity but rather have the reindeer population depend upon a food source that is being depleted over time. Since it takes lichen so long to regenerate, we could model this food source as non-renewable.

We do not know how much original vegetation was on the island but we can estimate it using trial and error as we try to reproduce historical data. Reindeer

[4]The linear "effect of carrying capacity …" graphical shape, sketched as part of the definition by the modeler, is only one of many possible shape choices.

normally produce one offspring per year, so increasing the birth fraction to 40% may be a reasonable modification of our previous model.

Notice that the carrying capacity shown in Fig. 3.3a has been replaced by a stock of vegetation (lichen) (see Fig. 3.4a) which is consumed by the reindeer each year, and which takes so long to regenerate that it can be assumed to be a non-renewable resource. Hence, the *Vegetation* stock only has an outflow, *consumption* (by reindeer). Assume that each reindeer consumes one packet of *Vegetation* per year (the "needed" vegetation, we assume, reindeer must have to survive).

In preparation for designing the graphical converter that alters the death fraction based upon available vegetation, we need first to establish an appropriate ratio for the independent variable for this graphical definition. The death fraction value will be influenced by the ratio of *Vegetation/Reindeer Population* compared to *needed vegetation per reindeer. Needed vegetation per reindeer* is constant.

The graphical converter *effect of vegetation per reindeer on death fraction* will not change the normal death fraction—until the ratio of *Vegetation/Reindeer Population* drops below the *needed* … amount. Then, the *actual death fraction* value increases very quickly. Note that death fraction = 1/lifespan, so a larger death fraction indicates that the lifespan of the animal is reduced. A death fraction of 0.1 indicates a lifespan of 10 years. A death fraction of 1.5 = 15/10 means a lifespan of 10/15 or 2/3 of a year.

The graphical converter definition is shown in Fig. 3.4b.

Unfortunately, the output of the model shown in Fig. 3.4a does not replicate the desired historic output for the *Reindeer Population* as well as we would like. We need the population collapse to be more pronounced. Lack of food would not only influence the *actual death fraction* of the reindeer, but also influence the ability of the female reindeer (cows) to reproduce. Weak or starved cows do not come into heat.

Assume the ratio of *Vegetation/Reindeer Population* compared to *needed vegetation per reindeer* also influences the *birth fraction*. A new graphically defined

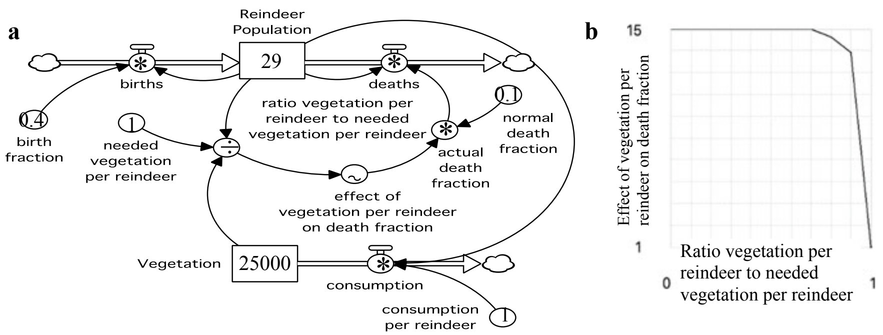


Fig. 3.4 **a** Current version of the SD model. **b** Graphical definition of how the current *Vegetation/Reindeer Population* compared to *needed vegetation per reindeer* affects the death fraction of the reindeer population

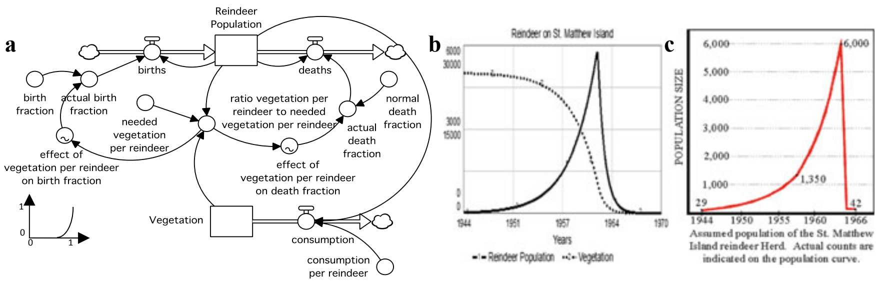


Fig. 3.5 **a** Final model for St. Matthew Island Reindeer Population. **b** Graph of *Reindeer Population* starts with 29 reindeer in 1944, shows 1500 in year 1957, and 94 reindeer at end of 1965. The herd reaches a maximum of 5751 reindeer in year 1962. **c** Historic reindeer population for St. Matthew Island. Retrieved from www.geo.arizona.edu

converter should reduce the birth rate dramatically if the Vegetation per Reindeer ratio falls below what is needed.

This model is not a perfect replication of the problem with the Reindeer Population on St. Matthew Island over the years 1944–1966 but it is a close approximation and something that secondary school mathematics students (ages 15+) can build and understand. The focus of SD model building is to design a stock/flow structure that includes the most important elements and (feedback) relationships of the systemic problem under study. As a guiding principle for the model design, the modeler will often have a behavior pattern she/he will want the model structure to replicate. Sometimes these behavior patterns are historical (as in the case of St. Matthew Island reindeer), or intuited from conversations with stakeholders of the problem. SD modeling is not designed to reproduce or capture specific data points. The focus is on behavior patterns over time and identifying characteristics of the problem that produce those patterns (allowing the modeler to test policies that might mitigate the problem).

Secondary school students have built models similar to Figs. 3.2a and 3.3a in regular (non-honors) algebra classes[5] and built original models (Figs. 3.4a, 3.5a and 3.6a) in a modeling class.[6]

[5]The models in Figs. 3.2 and 3.3 have been built by secondary school algebra students using guided discovery where students are then asked to modify and explain the model.

[6]Models of the types shown in Figs. 3.4a, 3.5a, and 3.6a (and other similar student original modeling projects) require additional time (10 weeks) for secondary school students (aged 15–18) to research, build, debug, and explain (by writing technical papers and giving presentations and/or producing posters).

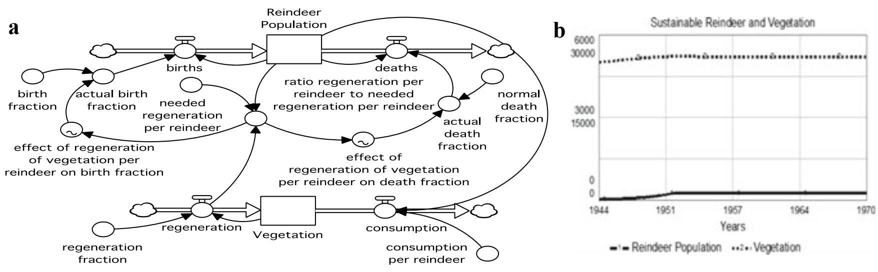


Fig. 3.6 **a** Model of a sustainable reindeer population (growing from 29 to about 265) and sustainable vegetation (growing from 25,000 to about 26,000 vegetation units) over 26 years, if reindeer are managed so they consume only the vegetation that can be regenerated each year. **b** Output of the model shown in part Fig. 3.6a

3.6 Can We Sustain the Reindeer Population?

We could modify Fig. 3.5a to allow *Vegetation* to be renewable. Such a modification would add an inflow to the *Vegetation* stock identified as *regeneration*, with some appropriate *regeneration fraction* per year (here 0.01). Here, the ratio (used by the two graphical converters) should be created to compare the regeneration of vegetation/reindeer to needed regeneration per reindeer per year.

3.7 Discussion

Interweaving SD model building representations with the traditional closed-form equation approach used in secondary school mathematics courses reinforces the basic characteristics of the functions studied in those courses. For example, taking an exponential function, as shown in Fig. 3.2a, the simplest closed-form equation representing the value of population over time would be $P = 100(1 + 0.3 - 0.1)^t = 100(1.2)^t$. While we simplify the closed-form equation, there can be advantages to keeping the growth rate separate from the decay rate (as demonstrated in Figs. 3.3a, 3.4a, 3.5a and 3.6a). Assessment questions regularly required students to move between closed-form equation and SD model representations, where possible. Similar closed form versus stock/flow representation arguments can be made for quadratic, logistic, and sinusoidal functions. As function equations become more complicated, the equivalent SD diagram can be very useful in displaying the problem structure, component relationships, and feedbacks that produce the characteristic behavior patterns.

Another advantage of having students build stock/flow models is that almost all of the interpretation of model behavior is done graphically. Students are expected to anticipate the model behavior, graphically, and then explain any discrepancies between their prediction and the model output. As we know, students' ability to correctly interpret graphs is problematic (Shah and Hoeffner 2002). Moreover, students are often asked to graph the flow behavior as well as the stock behavior on the same grid requiring that they use the flow behavior to explain the stock behavior. This is advantageous for two reasons: (1) Having students draw connections between flow behavior and stock behavior is a conceptual preparation for fundamental principles of calculus, at a level where the interpretation is clothed in application, repeatedly connected to the real world, without the overhead of complicated calculus notation, and (2) Calculus students have difficulty interpreting correlational graphs (Carlson et al. 2002). Carlson et al. (2002) indicate that understanding covariational reasoning, especially involving rates of change, is fundamental to "understanding major concepts of calculus… and for understanding models of dynamic function events" (p. 374). In addition to more practice interpreting graphs, these two additional attributes move students forward in mathematical understanding of dynamic phenomena while still at the early secondary school mathematics level of equation manipulation.

Now let's turn to the fact that a visual representation of the structure of a problem, using full words and phrases, allows other students and teachers to more easily understand the modeler's mental model, and hence to more naturally interact with the modeler to modify or enhance her/his model. This characteristic is valuable in helping to identify misconceptions a modeler may have about the problem being analyzed. (Models are built first to analyze a problem, but then to communicate the insights gained by the modeler after having built the model). So a more natural, visual, and descriptive representation of the problem is an aid to the modeler and the modeler's audience.

The visual nature of the model enhances not only communication but allows the modeler to move beyond what he/she would have been able to build using closed-form equations. The modeler can now represent more realistic situations (as is demonstrated by Figs. 3.3a, 3.4a, 3.5a and 3.6a). While the model shown in Fig. 3.6a represents a "first world" problem, it could be modified to represent a human population and food supply or potable water problem. SD models could involve topics of epidemiology, criminal justice, social services, causes of poverty, etc. Not only can students work on more sophisticated problems using scripted lessons, but the less abstract representation seems to increase their mathematical bandwidth of understanding, thereby allowing students to create a wide variety of original models, which they are able to make operational and explain[7] (Fisher 2011).

Students adapt much more quickly to using stock/flow representations than teachers do. Students are allowed to communicate with each other as they build SD models. One concern for teachers includes students' increased ability to ask questions for which teachers may not know the answers (Fisher 2016).

Finally, system dynamics modeling software environments support the ability to capture nonlinear relationships between variables—see many of the graphically

[7] A sample of over 20 secondary school student model diagrams, student technical papers explaining their models, and some student videos explaining their models can be found at: https://ccmodelingsystems.com.

defined components in Figs. 3.4a, 3.5a and 3.6a. Secondary school students have shown a facility with developing these nonlinear definitions for models, allowing them to cross the threshold to building models of complex, nonlinear, dynamic, feedback systems. Moreover, students can build SD models that capture material and information delays that occur in many real-world systems.

3.8 Conclusion

Arguably, the time has come for secondary school mathematics and science educators to supplement the emphasis given to the exclusive manipulation of closed-form equations, with approaches that teach students to understand and model the dynamic behaviors occurring everywhere in the world. To the extent that these abilities support students to become more active and competent global citizens, they will be better enabled to develop and test policies needed to tackle difficult problems. One of our current challenges is to intellectually fortify students, allowing them to retain hope that there is still time to effect the changes needed to secure human futures, personally and globally.

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Chapter 4
Theorizing 'Modelling as Bridge' Between Content and Vehicle

Stephen T. Lewis

Abstract In this paper, a theory for facilitating modelling tasks as a bridge between modelling as content and modelling as vehicle is established. Drawing on data from a micro-ethnographic study of one classroom, I outline one teachers' mathematical modelling practice when facilitating tasks. I examine how the teacher vacillated between elaborating on student mathematical ideas, nurturing the development of *modelling as content*, while simultaneously targeting curricular objectives through formalization of desired mathematical content in *modelling as vehicle*. Through her implementation, I identify how the teacher navigated between these two epistemological approaches to developing modelling capacity and use this analysis to establish a theory for *modelling as bridge* between content and vehicle.

Keywords Modelling as content · Modelling as vehicle · Discourse analysis · Scaffolding

4.1 Introduction

Achieving a balance of rigorous mathematics content, cultural competence and critical consciousnesses through mathematical modelling is a complex endeavour, yet an attainable goal that merits much scholarly attention (Anhalt et al. 2018). Cai et al. (2014) argued for additional research to help in characterizing the essential features of mathematical modelling, document what instruction looks like when modelling is facilitated in modelling contexts, determine what factors motivate task selection by teachers of modelling, and further to examine classroom discourse in mathematical modelling that supports modelling practice. This study addresses these epistemological, mathematical, and pedagogical issues in a theoretical nature by examining interactions that occur in mathematical modelling contexts. In particular, I explicate a teachers' particular view of mathematical modelling and consider how her sociomathematical modelling practice, or *sociomodelling practice* (Lewis 2018), resides

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as a bridge between *modelling as content* and *modelling as vehicle* (Julie and Mudaly 2007).

It is known that there are institutional, mathematical, cultural, and pedagogical barriers that exist that prevent the implementation and facilitation of modelling tasks with fidelity (Anhalt et al. 2018; Manouchehri et al. 2018), and further that there exists a need for additional scholarly reports on existing efforts aimed at improving specific pedagogical capacities among teachers (Manouchehri et al. 2018). Further, facilitating modelling cognition involves attention to cultural backgrounds, life experiences, intuitions, and prior mathematical knowledge that learners draw upon (Cai et al. 2014; Manouchehri and Lewis 2017; Orey and Rosa 2015; Rosa and Orey 2010). Yet, a productive method for bridging the gap between mathematics and the realities of learners within modelling contexts is still unknown (Anhalt et al. 2018). This study takes up the calls to action and examines the particularities of classroom practice when facilitating modelling tasks.

Two questions guided my research of classroom practice, (1) *How are social practices in mathematical modelling contexts developed through implementation of modelling tasks?* (2) *How does teacher scaffolding within these tasks promote development of modelling capacity in learners?* In this paper, I will focus primarily on the initial question and articulate how the teachers' practice in itself acted as a *bridge* between mathematical modelling as content and mathematical modelling as vehicle, which I characterize as *modelling as bridge.*

4.2 Background Literature

While various definitions of mathematical modelling may exist (Lesh and Fennewald 2010), agreed-upon processes in mathematical modelling involve assumption-making, mathematizing, parameterization, the selection of variables, simplification of a system, decision-making, and validation of the model (Blum and Leiss 2007; Lesh et al. 2000; Niss 2010; Niss et al. 2007). It is how these aspects of mathematical modelling emerge across definitions that one ought to consider. Regarding teacher practice in mathematical modelling, one cannot necessarily assume a teacher has a particular conception of mathematical modelling that necessarily coincides with one epistemological view of mathematical modelling, but rather that their definition is informed by their realities of teaching, background mathematical knowledge, knowledge of modelling, and that these definitional components emerge across their practice. Across the literature, two predominant epistemological views of modelling instruction exist, *modelling as a content*, and *modelling as a vehicle* (Galbraith et al. 2010; Julie and Mudaly 2007).

4.2.1 Teaching Mathematical Modelling

Galbraith et al. (2010) argue that *modelling as vehicle* focuses on developing prescribed mathematics and is used as a means for learning new concepts, procedures, ways to conjecture, or use context-driven justifications to solve problems. This they argue affords the use of modelling to service curricular needs of mathematics. Within *modelling as vehicle*, problems are pre-formulated and typically involve a particular real-life context. Through the use of these contextualized examples, the study of mathematics is motivated. The real-life situation is used primarily in this way, as a basis for abstraction.

Within *modelling as content*, mathematics is the outcome of the model development process (Galbraith et al. 2010; Julie and Mudaly 2007). These authors argue that *modelling as content* entails scrutiny, dissection, critique, extension and adaptation of existing models for the purpose of assessment and evaluation. *Modelling as content* is motivated by genuine real-world contexts and is anchored in the processes used for solving these problems. This view of modelling affords access to critical issues or inequities that people face through assumption-making to articulate a problem statement within the real world, and then develop models to address it.

While Julie and Mudaly (2007) argue that these epistemological views of mathematical modelling are not mutually exclusive, but rather a spectrum of presentation and dissemination, research in mathematical modelling has predominantly treated them as such. Tasks are most often characterized as *modelling as content* OR *modelling as vehicle*. However, in order to consider this spectrum at greater depth, a view of how tasks are facilitated in actual classrooms and by actual teachers needs to be established. In this way, I turned to discourse analysis (Bloome et al. 2010) to robustly analyse the implementation of modelling tasks and consider how practices pertaining to mathematical modelling emerged.

4.3 Methods

The overarching program of research aims to analyse interactions in mathematical modelling contexts, and focuses dually on teacher scaffolding mechanisms through reflexive discussion (Qualley 1997), as well as ways students interpret and respond to tasks drawing on their contextual and mathematical knowledge (Manouchehri and Lewis 2017). The data for this study comes from a micro-ethnographic study of an 11th-grade pre-calculus class at a private academy in the Midwestern United States that occurred during the 2017–2018 academic year. The teacher of this class was in her 11th year of teaching and had a focus on implementation of mathematical modelling tasks on a regular basis over the course of the year. She indicated that aside from her major curricular goals of teaching concepts in pre-calculus, she wanted to help her students understand the applicability of mathematics, and viewed modelling and applications as the means to accomplish these objectives.

In observing her teaching, I attempted to characterize those sociomodelling practices which established activity within mathematical modelling from both a local and dialogic perspective thus establishing cultural models for enacting mathematical modelling.

Prior to my formal data collection, I had attended the participating teacher's classes over the previous two years observing her instruction and developing a friendship as well as a professional relationship with the teacher. Targeted data collection for this study involved video and audio recording subsequent lessons across a 12-week period during the 2017–2018 academic year. Daily event mapping (Hennessy et al. 2016) was used to identify broadly what transpired across teaching sessions. These event maps were drawn from both video data and detailed fieldnotes transcribed during the class itself. Particular modelling events were selected, transcribed into utterances and analysed using line-by-line discourse analysis (Bloome et al. 2010). Key components of the teacher's mathematical modelling practice emerged across these observations and coded according to their social functions. The frequency of these codes occurrence was compared and those that appeared most often were deemed significant. They were then validated by the teacher through reflective discussions. Subsequent and previous class sessions were also observed to look for intercontextual links to these key events, and evolution of practice was considered. This triangulation of observations and reflective interviews establishes validity of interpreted results (Bloome et al. 2010). In order to establish reliability of the coding process and data, multiple coders examined the data and feedback were provided on particular components of practice, and their social functions. This was done so that the reported data accurately reflected the practice of the teacher and her intent.

4.4 Results

Analysis of data examined four deliberate modelling and application experiences: *The Answer Is*, a problem-posing task where students generated increasingly complex problems whose answer was $73.13. *Measure the Height*, a trigonometric geometry problem where students determined the height of selected objects using three different methods, and then results were compared and validated. S*hortest Path to the Lunchroom*, a task where students determined which buildings on campus yielded the shortest path to the cafeteria using both physical measurement and an aerial map of the campus. Lastly, *Trig Whips*, a task where students investigated rotational motion and angular velocity by linking arms and moved in a circle and determining the rate that the outward most person would have to move to maintain a solid chain.

Analysis of these sessions revealed that the teacher vacillated constantly between building modelling capacities, or the skills needed for effective modelling practice, and linking these capacities to targeted mathematical concepts of her curriculum. This *vacillation* was made apparent through the enactment of components of her practice. Twenty-three interrelated but distinct components were identified across these sessions[1] that targeted the advancement of mathematics, two of these components, in particular, and their interrelationship made the teachers' view of modelling transparent—*elaborating student mathematics* and *formalizing curricular mathematics*. Elaborating student mathematics marked instances where the teacher acknowledges the conceptual tools that the learners drew upon and supported their elaboration in mathematical ways. This involved attending to student views of particular problems and building on THEIR solutions mathematically, not those linked to a desired curricular outcome.

While the teacher's long-term goals involved developing modelling capacity in learners, she was also concerned with formalizing the curricular concepts mandated by the course. *Formalizing curricular mathematics* involved enculturating students into mathematical concepts through direct instruction or problem-solving as opposed to modelling. When she recognized the connection between contextual problems and her overarching content goals, she was more likely to include modelling in her instruction. During these instances, the teacher would start with modelling or contextual problem-solving, allow students to explore their ideas and intuitions, then link student solutions or ideas to deliberate mathematics during discussion. Often this involved selecting particular student solutions of which to expand.

Figures 4.1 and 4.2 document the vacillation between elaborating student mathematics and formalizing curricular mathematics across those specific tasks. Across all

	Event Phases	**Elaborating Student Mathematics**	**Formalizing Curricular Mathematics**
The Answer is Task	Introduction of problem posing task	x	
	Discussion of Task Interpretation	x	
	Establishing Solution Criteria	x	
	Student work on problem posing activity	x	
	Teacher Monitoring and Group Scaffolding	x	
	Sharing Solutions	x	
	Elaboration of Solutions	x	
	Teacher Summary of Task and Solutions	x	
Measure the Height Task	Introduction of Task and Demonstration of Inclinometer		x
	Establishing solution criteria	x	
	Student work to determine height of three objects in three different ways	x	
	Teacher monitoring and supporting solution strategies	x	
	Debrief discussion of measure the height task	x	
	Sharing and discussion of solutions	x	
	Formalizing Sine, Cosine, and Tangent as way of determining height		x
	Students practice using Sine, Cosine, and Tangent on exercises		x
	Validation of measurements for task using Sine, Cosine, and Tangent	x	x

Fig. 4.1 Elaborating and formalizing interrelationship: *The Answer is* and *Measure the Height*

[1] Specific components of teacher practice are elaborated in detail in Lewis (2018).

	Event Phases	**Elaborating Student Mathematics**	**Formalizing Curricular Mathematics**
Shortest Path to the Lunchroom Task	Shortest path task pre-discussion and task overview		
	Teacher monitoring of student solutions	x	
	Shortest path debrief discussion with whole class	x	
	Connecting shortest path to calculating triangle concepts		x
	Determining contextual angle measurements with protractor		x
	Comparing measurements to calculated solutions	x	x
	Problem posing exercise using aerial map of campus	x	
	Summary of shortest path task and solution outcomes		x
	Sharing strategies used for determining length of triangles	x	
	Validating measurements by remeasuring physical distances (error analysis)	x	
	Comparing calculated and collected values of distances	x	x
	Averaging measured distances and comparing calculated outcomes		x
	Teacher introduces Law of Sines Calculator Activity		x
	Law of Sines Exploration		x
Trig Whips Task	Introduction of angular speed and task overview		x
	Teacher monitoring and supporting student work on task	x	
	Trig whip debrief discussion and sharing of solutions	x	
	Formalizing angular speed (introduction of formulas)	x	x
	Applying angular speed to trig whips task		x
	Derivation of angular speed from linear speed and arc length		x
	Teacher monitoring of using angular speed on trig whip inquiry activity	x	x
	Trig Whip extension overview	x	x
	Angular speed and extension wrap-up discussion	x	x
	Angular speed practice exercises		x

Fig 4.2 Elaborating and formalizing interrelationship: *Shortest Path* and *Trig Whips*

problem-solving tasks, the teacher began the discussion by elaborating on students' mathematical ideas, and then transitioning into formalization of those ideas in some capacity. Formalization involved connecting the discussion to the mathematical and curricular goals of the lesson. In this way, the teacher was able to anchor the development of her intended curricular goals grounding these formalizations on students' mathematical ideas. Drawing on learners' ideas, intuitions, and solutions, she was able to portray them as *a* means for solving the problem, not *the* means. Thus, she was able to maintain the validity of learners' solutions along with ideas in the modelling process, while at the same time, advance her own curricular agenda. Some instances were dominated by formalization, while others elaboration, but consistent across encounters was a deliberate dance between facilitating tasks as modelling and content coupled with modelling as vehicle bridging the gap between these two epistemological views of modelling instruction.

4.5 Discussion and Conclusions

While the literature outlines and distinguishes between the epistemological views of modelling as content and modelling as vehicle, the teacher's view of modelling acted as a *bridge* between *modelling as content* and *modelling as vehicle*. On the one hand, the teacher adopted a holistic view of modelling as the interaction between mathematics and real life (Kaiser 2007). On the other hand, she also supported the organization of social practices to establish arguments and support their decisions, adopting a socio-epistemological perspective (Cantoral et al. 2018). In this way, the curricular objectives alone did not drive her teaching process of mathematical modelling, but are bridged with the stance that learners should be equipped to consider real-world problems through the lens of their own experiences and draw on conceptual tools to support the development of well-conceived solutions. In this way, she was able to reconcile both goals of advancing curricular knowledge and supporting learners' real-world problem-solving and decision-making.

Analysis further revealed that in establishing her sociomodelling practice, the teacher relied on her professional vision and pedagogical resources (Goodwin 1994; Schoenfeld 2009) when planning for facilitating modelling tasks. Professional vision is characterized as those particular ways that members of a group examine events of interest and affords a means to notice and interpret actions, and further is the driving force for what transpires across interactions around a professional agenda. Resources are considered to encompass both the conceptual and physical tools at one's disposal.

Figure 4.3 reflects the relationship between the teacher's sociomodelling practice, components of that practice, and its relationship to her epistemological view of

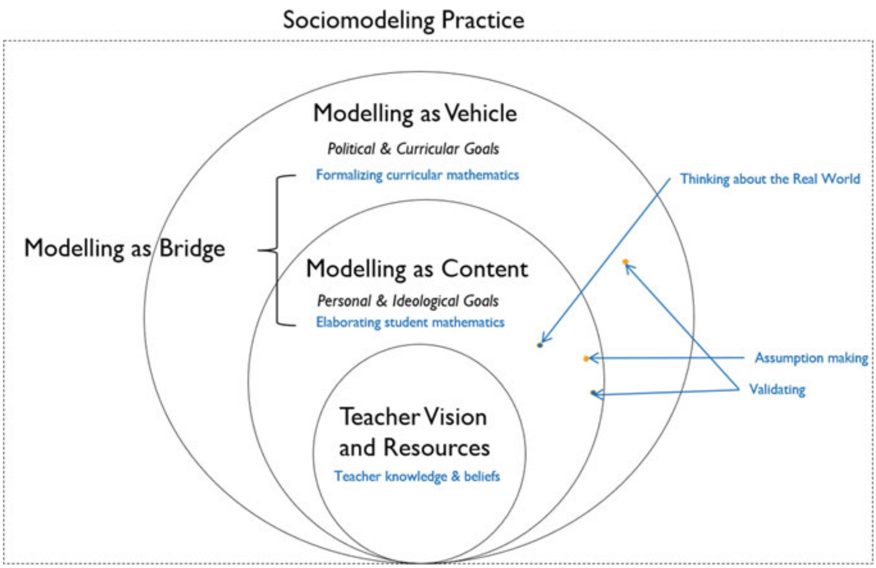

Fig. 4.3 Modelling as bridge

teaching mathematical modelling. At the heart of sociomodelling practice is professional vision, informed by her contextual knowledge, mathematical knowledge and her beliefs about practice. With the teacher's agenda in developing modelling capacity in her students, there exists evidence of both modelling as content and modelling as vehicle, and components of her practice emerged in each phase of the modelling process. *Elaborating student mathematics* drawing on their experiences is intertwined with *modelling as content* from a definitional standpoint, and *formalizing curricular mathematics* to *modelling as vehicle*. In this capacity, her overarching practice was observed to vacillate between these two epistemological views of mathematical modelling linking the development of student mathematical ideas to her curricular objectives. This phenomenon I characterize as *Modelling as Bridge*, as the teacher attempts to bridge these two epistemological views of mathematical modelling in and across her instruction.

In contemplating the significance of these findings, it was only in those instances where the teacher recognized a link between her curricular goals and contextual problems that mathematical modelling was initiated. It was her vision that afforded modelling tasks sensitive to learners' ideas while keeping an eye on curricular objectives. The teacher had to reconcile her short-term instructional goals of mastering particular mathematical concepts and her long-term goals of developing modelling capacity.

Anhalt et al. (2018) argued that teachers need to become fluent with the nature of the mathematical modelling cycle as an approach to solving open-ended problems in familiar contexts (p. 558) and further to promote creativity in solutions should resist steering learners' towards pre-determined approaches but rather support their own thinking (p. 560). However, generic talk about implementation of mathematical modelling tasks or even showing examples of modelling is necessary, but not sufficient. Rather the skill of facilitating modelling tasks demand a bridge between curricular mathematics and learner unique solutions need to be further developed. In this sense, one could revisit curricular resources and elaborate on productive methods of facilitation through the development of curriculum guides that highlight how to promote both modelling and mathematics. Further, these guides need to find productive ways to expand on multiple solution paths and how each path might target a particular curricular or contextual outcome. Without guidance of this nature, it is not likely that the breadth of our research in mathematical modelling and tasks will come to fruition in classroom settings.

The ways in which teachers vacillate between elaboration of learners' ideas and formalization of mathematical concepts to accomplish curricular needs strengthen the argument for facilitating modelling tasks as *modelling as bridge.* More studies of this nature ought to be facilitated in order to gain a better conception in the particularities of these types of mathematical modelling practices.

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Chapter 5
Glocalization of Cultural Mathematical Practices Through Ethnomodelling

Daniel Clark Orey and Milton Rosa

Abstract In this chapter, the authors share how the acquisition of both *local* (emic) and *global* (etic) knowledge forms an alternative goal for the implementation of ethnomodelling research. Local knowledge is essential for an intuitive and empathic understanding of mathematical ideas and procedures developed throughout history. Global dialogical knowledge is essential for the achievement of cross-cultural communication that demands standard analytical units and categories. It is used for conducting ethnomodelling research that applies both local and global knowledges through respect, appreciation, dialogue, and interaction. Our main objective is to share the combination of local and global approaches in which ethnomodelling research looks at how diverse ideas and procedures contribute to the acquisition of a holistic understanding of mathematics.

Keywords Ethnomodelling · Glocalization · Local · Global · Cultural practices · Cultural approaches

5.1 Initial Considerations

A challenge for investigators is to develop methodological procedures that assist in perceiving or understanding what is often deemed as *culturally bound* mathematical ideas, procedures, and practices that have been developed by members of distinct cultural groups without letting their culture interfere with the cultural background of these individuals. In this context, as it is often based on philosophy, assumptions, and values that are strongly influenced by colonization, modern Western civilization, and technological advancement. Thus, it is necessary to deconstruct the notion that mathematical ideas and procedures are uniquely European in origin.

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There are investigators who believe that mathematical practices are unique to Western science and are the same for all cultures, and that their techniques are equally applicable to everywhere. At the same time, many non-Western cultures developed their own interpretation of the *local* environment (*emic* approach) opposed to the outsiders' *global* interpretation (*etic* approach) of that specific cultural group (Rosa and Orey 2019). An important goal of research in ethnomodelling is to understand and explain existing variations found in diverse mathematical ideas, procedures, and practices that often vary due to influences of history, context, culture, race, ethnicity, gender, sexual orientation, and other *sociocultural traits*.[1]

This context enables ethnomodelling to become a tool in the translation of daily experiences that apply mathematical ideas and practices within a culture. It is also a fluid and dynamic research tool that incorporates both culturally universal and culturally specific mathematical practices that often lead to new discoveries and innovative pedagogical actions in the development of inclusive approaches in mathematics education.

5.2 Developing Three Cultural Approaches to Ethnomodelling

When working with ethnomodelling, it is possible to identify at least three approaches that have come to help us investigate, study, and further understand mathematical ideas, procedures, and practices developed by the members of any given cultural group.

Global (*etic*) refers to the outsiders' view on beliefs, customs, and scientific and mathematical knowledge developed by the members of distinct cultural groups. Globalization has reinforced the utilitarian approach to school mathematics, as well, it has helped to globalize pervasive mathematical ideologies, including the view that mathematics is the same for everyone everywhere. Particularly, traditional school mathematics is a culturally homogenizing force, a critical filter for status, a perpetuator of mistaken illusions of certainty, as well an instrument of power (Rosa and Orey 2015). In this approach, comparativist researchers have attempted to describe differences among cultures, and thus are considered, for better or worse, *culturally universal* (Sue and Sue 2003).

Local (*emic*) refers to the insiders' view of their own culture (*how we do this*), customs, beliefs, and scientific and mathematical knowledge. Local knowledge is important because it has been tested and validated within the local context (Cheng

[1] Sociocultural traits are considered as socially learned system of beliefs, values, traditions, symbols, and meanings that members of a particular cultural group develop throughout history. These traits identify members of a specific culture because they are deposits of knowledge, experiences, actions, attitudes, hierarchies, religion, notions of time, roles, spatial relationships, universe concepts, and artifacts developed by the members of distinct cultural groups from generation to generation through consistent acknowledgment and valorization of individuals and their collective efforts (Samovar and Porter 2000).

2005), and creates a framework from which members of cultural groups are able to understand and interpret the world around them (Barber 2004). In this approach, members of distinct cultural groups describe their culture in its own terms and values (Rosa and Orey 2013). Currently, many investigators recognize the importance of local perspectives to the development of scientific and mathematical knowledge. They are considered as *culturally specific* individuals (Sue and Sue 2003).

Glocalization (*emic-etic/dialogical*) represents a cultural dynamism between two or more cultures in continuous and ongoing interaction between globalization and localization. It offers a perspective that both approaches (emic-etic) are elements of the same phenomenon (Kloos 2000). It involves blending, mixing, and adapting two processes in which one component addresses the local, as well the outside system of values and practices (Khondker 2004). In a *glocalized* society, members of distinct cultural groups must be "empowered to act globally in their own local environment" (D'Ambrosio 2006, p. 76). Therefore, it is necessary to work with different cultural environments and, acting as ethnographers, describe mathematical ideas and practices of other peoples. As well, it is crucial to give meaning to both these mathematical findings (D'Ambrosio 2006).

It is necessary for us to first focus on the local knowledge and then integrate global influences in order to create individual and collective views rooted primarily in local experiences and contexts. This approach is equipped with a glocal knowledge that creates a sort of localized globalization (Cheng 2005). It goes deeper than traditional multicultural views of mathematical practices. In this regard, ethnomodelling allows researchers to move beyond what is in danger of being relegated to the *curious* or *exotic* findings and focuses on creating deeper understanding toward how members of distinct cultural groups actually use, or came to use, mathematics to solve their own problems within their own local communities.

For example, can researchers agree with imposed cultural universalities (global) of mathematical knowledge or they might take on techniques, procedures, and practices of its cultural relativism? Thus, researchers seeking to link universal (global) and community-specific (local) approaches face the classic dilemma of scientific goals conflicting with investigations in ethnomodelling (Rosa and Orey 2019). Yet, both local and global approaches are often perceived as incommensurable paradigms. We beg to differ. While these approaches are often thought of as creating conflicting dichotomies, in the context of ethnomodelling they are considered complementary viewpoints.

However, rather than posing a dilemma, the use of both approaches in ethnomodelling research deepens our understanding of cross-cultural scientific and mathematical investigations (Rosa and Orey 2013). Since these two approaches are complementary, it is possible to delineate forms of synergy between the local and global aspects of mathematical knowledge through the development of ethnomodelling research. Hence, a combined local–global approach requires researchers who first attain local knowledge developed by the members of distinct cultural groups. This approach may allow them to become familiar with relevant cultural differences in diverse sociocultural settings (Rosa and Orey 2015).

Similarly, the resurgence of debates regarding cultural diversity has also renewed the classic *global/local* or *emic/etic* debate since investigators need to comprehend how to build scientific generalizations while trying to understand the role of sociocultural diversity. Yet, attending to the unique mathematical interpretations developed in each cultural group challenges fundamental goals of mathematics in which the main objective is to build a theoretical basis that helps to describe the development of mathematical practices in academic ways through ethnomodelling (Rosa and Orey 2019).

In this context, local observations seek to understand culture from the perspective of internal dynamics and relationships as influenced within a specific cultural group. A global approach is a cross-cultural contrast or comparative perspective that seeks to comprehend or explain different cultures from a worldview best described as from the outside (Rosa and Orey 2013). Local worldviews clarify intrinsic cultural distinctions while the global viewpoints seek objectivity as an outside observer across cultures (Anderson 2007). This local approach helps to examine native principles of classification and conceptualization from within each cultural system. Local knowledge and interpretations are essential to emic analyses.

In this regard, it is from the viewpoint of the participants that will convey messages about sociological and behavioral dimensions for the understanding of cultural contexts. Therefore, it is important to highlight that "what is emphasized in this approach is human self-determination and self-reflection" (Helfrich 1999, p. 133). A global analysis has a cross-cultural approach. Many etic-oriented investigators have examined the question of a cross-cultural perception so that their observations are taken in accordance with externally derived criteria. This context enables for comparisons of multiple cultures where "both the objects and the standards of comparison must be equivalent across cultures" (Helfrich 1999, p. 132).

Accordingly, in the conduction of ethnomodelling research, the cultural, social, linguistic, political, religious, and ethnic affiliations are integrated into a unified holistic solution. This approach allows for a deeper examination of ethnomathematical ways from what has been in the past considered as a study of the *strange*, *simplistic*, *curious*, or *exotic* mathematical ideas and procedures developed by the many *others*. In this manner, the intended mathematical practices are given a stake in the overall process and not just its mere result, or traditional competitive comparisons related to: *this is nice, but we do it better*.

5.3 Tree Trunk Cubing: An Example of a Glocal Ethnomodel

Because mathematics is a culturally bound social construct, the authors define ethnomodelling as the study of mathematical phenomena within a culture, which brings the cultural aspects of mathematics into the modelling process. The objective of this ethnomodel is to share the combination of local and global approaches where ethnomodelling research takes ethnomathematics beyond the study of the *exotic* or *curious* to look at how diverse ideas and procedures contribute to the acquisition of a holistic understanding of mathematical practices developed in distinct contexts.

Some ethnomodelling investigations have revealed sophisticated mathematical practices that include geometric principles in craft work, architectural concepts, and practices in the activities of many native and indigenous peoples, local, and vernacular cultures. These mathematical practices are related to diverse numeric relations found in measuring, calculation, games, divination, navigation, astronomy, and modelling, as well in a wide variety of mathematical procedures and techniques found in cultural artifacts.

In investigations conducted in Brazil, it was proposed that the elaboration of mathematics activities related to the determination of the volume of tree trunks with participants of this movement (Knijnik 2006; Rosa and Orey 2019). These activities were related to the method of *cubagem* (cubing) *of the tree trunk*, which is a traditional cultural practice used by the members of the *Landless Peoples' Movement* (*Movimento dos Sem Terra–MST*), which allows them to estimate the volume of a tree trunk in their settlements (occupation sites).

In this regard, wood cubing processes are associated with the sociocultural environment of members of this cultural group. Cubing is used by these members to determine how many cubic meters of woodcutters use in the construction of sheds, houses, and animal shelters (Knijnik 1993). In this context, one of the *MST* members provided an example in which she used a tree trunk found on the ground to determine her method in determining the volume of a log (Knijnik 1996), which we consider as an emic ethnomodel:

> To begin with, we mark here in the middle of the tree trunk, because it is thicker there at the end of the log and it is thinner at this end of the trunk. So, the middle of the trunk gives, more or less, its average. Now, I take a string and I go around it. Now, I have the measurement of the trunk outline at its middle. Then, I fold the string into four parts. After that, I measure it to see how many centimeters are there. It is 37 cm. Now, I multiply 37 by 37, which gives 1369. Then, I measure the length of the tree trunk. It is 1 meters and 64 centimeters. Now, I multiply the length of the log by 1369. It's 199874 cubic centimeters of wood. It's the same as doing side times side times length.

This emic mathematical knowledge can be represented by an etic ethnomodel applied in the cubing procedure used to estimate the volume of a given tree trunk (Amorim et al. 2007).

(a) First, it is necessary to estimate the center point of the tree trunk, so that the diameter is taken at half the length of the log (Fig. 5.1).

(b) From this point, by using a string, the perimeter of the tree trunk (circumference) is determined (Fig. 5.2).

(c) Then, the string related to the perimeter that was previously determined is folded into four equal parts (Fig. 5.3), which gives: $2\pi\ r = 4\ sides$ or $2\pi\ r = 4s$.

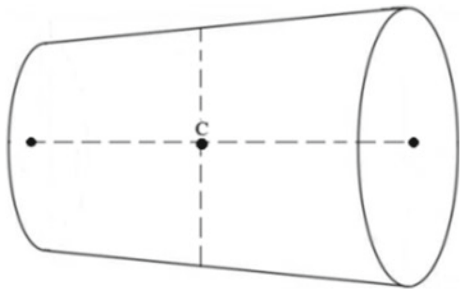

Fig. 5.1 Estimation of the center point of the tree trunk

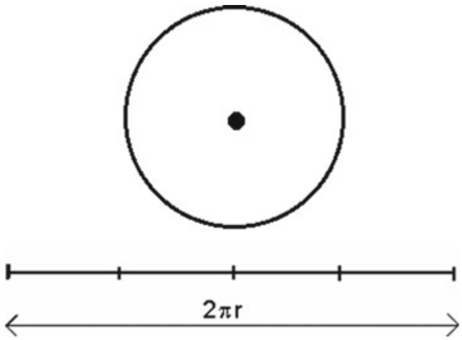

Fig. 5.2 Determination of the perimeter of the tree trunk

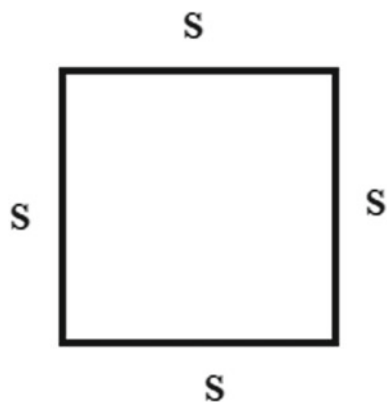

Fig. 5.3 Division of the string into four equal parts

$$2\pi r = 4s$$

$$s = \left(\frac{2\pi r}{4}\right)$$

$$s = \left(\frac{\pi r}{2}\right)$$

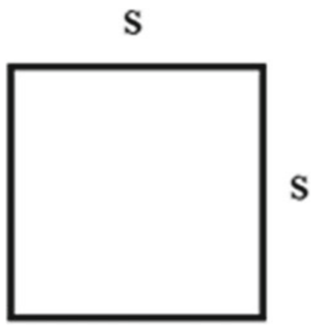

Fig. 5.4 Squaring the quarter of the string

(d) Then, the measure of the quarter of the string (circumference) is squared (Fig. 5.4).

$$A = \left(\frac{\pi}{2}\right)^2$$

(e) And, finally, the value of the quarter of the string (circumference) is multiplied by the height of the tree trunk in order to obtain the volume in cubic meters (m^3) of the wood. The volume is calculated as if the log was a cylinder.

In the glocal (dialogical) ethnomodel shown in Fig. 5.5, members of *MST* approximate the truncated cone (tree trunk) by a cylinder. This approximation is given as perimeter by determining the average between the perimeters of the smallest and the largest bases of the tree trunk.

The minor difference at the top of the tree trunk is compensated by the major difference at its bottom. By dividing the string into four parts and raising it to the square, the members of *MST* then calculate the area of a square by transforming the circle into a square (Fig. 5.6).

Although the perimeters are the same, the areas are different (Fig. 5.7).

Subsequently, the volume of a square prism is calculated by multiplying its area of the base by its height. The volume calculated in this way is relatively accurate if the shape of the tree trunk approaches a cylinder (Fig. 5.8).

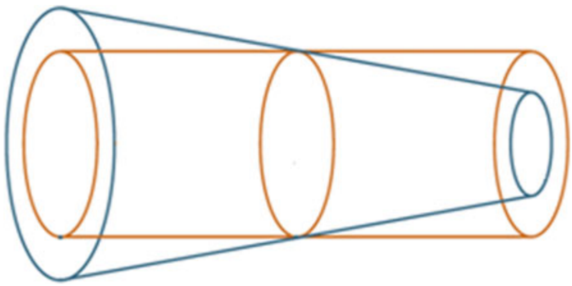

Fig. 5.5 Approximation of the truncated cone to a cylinder

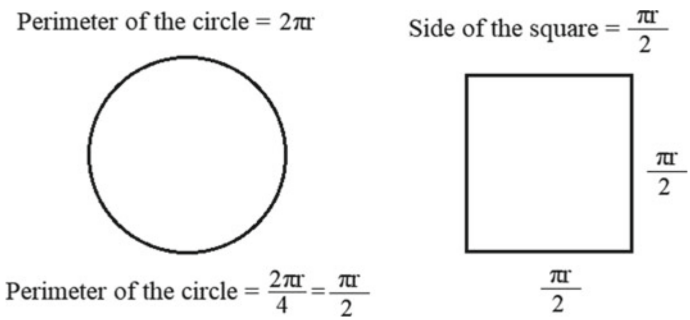

Fig. 5.6 Transforming the circle into a square

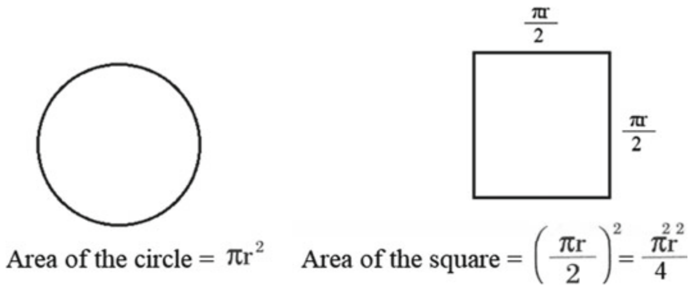

Fig. 5.7 Different areas, same perimeters

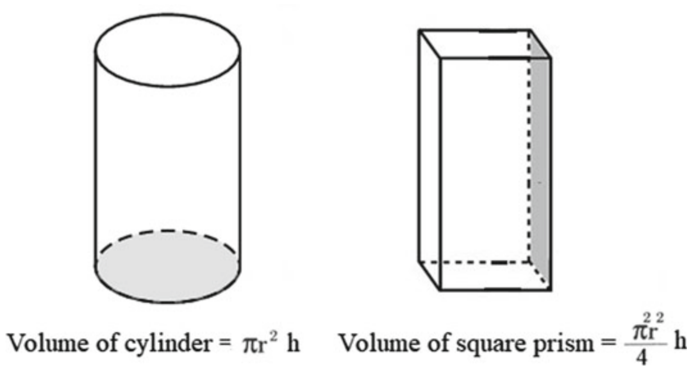

Fig. 5.8 Volume of a square prism

This method used to determine the volume of a tree trunk basically consists of two steps. In the first step, a tree trunk (essentially a cylinder) was identified through a mathematization process in which its circumference coincides with the middle part of the tree trunk. In the second step, a tree trunk (again a cylinder) was identified as a square prism whose side measurement is equal to a quarter of the perimeter of the cylinder base in this mathematization process. This method of cubing wood (*cubagem*) finds the volume of the trunk as the volume of a square prism whose side of the base was obtained by determining the fourth part of its circumference, which corresponds to the base of the cylinder, and was obtained through an ethnomodelling process, that is, as part of the elaboration of a dialogical ethnomodel of the tree trunk.

The representation of this cultural practice can be explained by the ethnomodel that transforms the trunk of tree into a cylinder. The authors believe that the emic approach, such as found in this example, may be considered an attempt to understand, translate, discover, and describe a mathematical system used by this specific cultural group, in its own terms, and by identifying its units and structural procedures, whereas an etic approach is primarily concerned with characteristics pertaining to academic mathematics.

The particular type of mathematical knowledge used and developed by *MST* members consists of socially learned and transmitted mathematical practices, which are represented in the elaboration of ethnomodels taken from sociocultural systems that are part of their own daily life. In the glocal (dialogical) approach, the emic observation sought to understand the *cubação of the tree trunk* from the perspective of the internal cultural dynamics and relations of this movement with the environment in which they lived while an etic approach provided a cross-cultural contrast, employing comparative perspectives with the use of academic mathematical concepts.

5.4 Final Considerations

Across human history, members of many different cultural groups have come into close contact often through colonization, conquest, and/or trade. In some cases, these cultural encounters sought for a mutual understanding in terms of the culture to which one belongs as well in terms of the specificity of cultural knowledge pertaining to the cultures encounters (Iser 1994). As a "result of these encounters, no culture can call itself static and definitive" (D'Ambrosio 2006, p. 76). It is necessary to present alternative approaches to hegemonic views of *globalization* (etic-outsiders) by arguing for a contextualization guided by *localization* (emic-insiders).

In this context, ethnomodelling can be seen through the lens of glocalization, which provides an approach that looks at ethnomathematics as expressions of glocal (dialogic) relations between local and global procedures and practices. This dialogue provides the development of *glocal mathematical knowledge* that has the potential to generate empowering synergies between localization and globalization. This process enables to conceive ways to articulate mathematical knowledge in more inclusive and synergistic modes.

Glocal (dialogic) approaches help us to create synergistic spaces of interdependent, reflexive, and co-arising relationships between global and local processes (Kloos 2000) for the development of glocal mathematical knowledge. However, it is necessary that global mathematical practices adapt themselves to local cultures and vice versa. According to D'Ambrosio (1998), contact of local knowledge with other external knowledge systems provokes the development of cultural dynamism.

In this regard, intense cultural dynamics caused by globalization may produce innovative mathematical models, perspectives, ideas, and thinking developed in diverse contexts (D'Ambrosio 2006). Similarly, glocal mathematical knowledge helps us realize how objectivity and subjectivity, global and local, transcendental and cultural, universal and specific, and Western and non-Western can peacefully coexist side-by-side (Robertson 1995), and indeed can support each other in the development of new mathematical ideas, procedures, and practices.

Therefore, if we look at glocalization as a useful tool for creating dialogue and a curriculum for local and global knowledge systems, we obtain a better understanding of the challenges and potential benefits of this dialogue. *By* using ethnomodelling to describe the relation between these two interdependent and mutually constitutive approaches, we help individuals explain how members of distinct cultural groups experience their world in multi-scalar sociocultural terms, and to connect local communities to play important roles in developing and sustaining global mathematical practices (Rosa and Orey 2019).

In this context, the term glocalization is a process by which a culture easily absorbs foreign (outside) ideas and/or the best practices that meld those with their own points of view, needs, and traditions without the loss of ancient practices or self-esteem. This approach provides a voice and context for understanding the ethnomodelling process, how the group identity is constructed, and how processes of globalization and localization work in tandem to create innovative scientific and mathematical knowledge through the development of unique cultural forms.

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Chapter 6
Positive Deviance in the Pedagogical Action of Ethnomodelling

Milton Rosa and Daniel Clark Orey

Abstract An impasse in mathematics education is related to its often lack of acknowledgment of local mathematical practices in its research theoretical basis. Pedagogical action of ethnomodelling can aid in recording cultural-historical forms of mathematical procedures and practices developed by members of distinct cultural groups. Ethnomodelling adds cultural perspectives to the mathematical modelling process without attempting to replace academic mathematics during the development of this process. Hence, *insubordination* triggered by ethnomodelling is *creative* and often evokes a sense of disturbance that causes conscious review of rules and regulations endemic to many curricula contexts. This process enables educators and investigators to use *positive deviance* to develop pedagogical actions that deal with content often disconnected from the reality of the students.

Keywords Ethnomodelling · Ethnomodels · Mathematization · Method of *cubação* · Pedagogical action · Positive deviance

6.1 Initial Remarks

Ethnomodelling is a form of pedagogical action that offers a contrast to traditional academic curricula by challenging the view that members of local and/or distinct cultural groups only develop exotic and/or simplistic mathematical ideas, procedures,

[1]The concepts of *creative insubordination* (Crowson and Morris 1982), *responsible subversion* (Hutchison 1990), or *positive deviance* (Zetlin et al. 1990) are equivalent as they relate to the adaptability of rules and regulations in order to achieve the welfare of the members of distinct cultural groups. However, there are subtle conceptual differences that must be discussed during the development of investigations.

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techniques, and practices. According to Rosa and Orey (2017), the development of ethnomodelling is related to the concept of *positive deviance*[1] (Zeitlin et al. 1990) as it relates to the flexibility of rules and regulations that helps these members to achieve a deeper understanding of their mathematical thinking and reasoning.

The concept of positive deviance refers to the practices of researchers and educators that, in an insubordinate, creative, subversive, and responsible way, and with discernment, are opposed to educational practices that no longer make any real pedagogical sense, especially in regard to the educational bureaucracies and traditions of public policy (Rosa and Orey 2017). This approach refers to actions assumed in relation to norms and institutional rules that aim at better commitments to the needs of students who compose the school population.

Researchers and educators who are able to create innovative pedagogical alternatives, such as ethnomodelling, are conscious in relation to the achievement of better results for the common good of the community that is constituted by their colleagues, students, parents, and school administrators. In this context, these professionals can be candidates for positive change in their communities. This action is often in opposition and, generally, represents a challenge to established authorities and long-held school traditions, even if they are related to, or cause unintentional exclusion and/or discriminatory school policies.

For example, a wide variety of mathematical procedures and techniques challenge *primitivist*[2] views held by members of distinct cultural groups as ideas that possess mathematical knowledge they use to explain the world around them, navigate, and create remarkable architectural monuments, and to solve problems faced in their communities. Positive deviance also challenges epistemological stereotypes most damaging to these members. Thus, a sense of positive deviance becomes an important source for adaptive transformational capacities by members of distinct cultural groups that produce non-conformist actions. Its main objective is to modify these norms and rules by applying inclusion, innovation, creativity, and adaptability through ethnomodelling (Rosa and Orey 2017).

Positive deviance means that researchers and educators gain awareness about when, how, and why to act against established procedures or guidelines that are unjust, racist, homophobic, or unfavorable to any member of a school community. This means that individuals who are *positively deviant* because they assume that members of distinct cultural groups are unfinished human beings who take criticality, creativity, responsibility, and curiosity as the foundation of an ongoing and transformative process in the production of mathematical knowledge (D'Ambrosio and Lopes 2015).

Ethnomodelling can be considered both a positive and deviant pedagogical action because it causes a certain disruption to the existing order in academic mathematics by encouraging and developing the study of the mathematical ideas, procedures,

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[2]Primitivism refers to cultures believed to lack cultural, social, technological, or economic sophistication or development. Historically, primitivism has been used to justify conquering the members of *other* cultural groups. In cultural terms, primitivism means a deficiency in those qualities that have been used historically in the Western world as indicators of so-called civilized cultures (Rhodes 1995).

techniques, and strategies found locally, which includes diverse mathematical practices that are in accordance with the *emic*[3] perceptions of the members of distinct cultural groups (Rosa and Orey 2017).

Through ethnomodelling, it is possible to recognize divergent ways, as well as value the diverse modes that mathematical knowledge is produced by other cultures and environments (Rosa and Orey 2015). It is necessary to reclaim contributions of the conquered, minority, or marginalized peoples in the development of mathematical knowledge through the elaboration of ethnomodels in the ethnomodelling process. Ethnomodelling generates a new respect for diverse forms of mathematical knowledge and assists members of distinct cultural groups in resolving ethical dilemmas involved in these investigations.

Thus, positive deviance can be triggered by initiating a *disturbance* that causes a review of school mathematical knowledge by increasing the potential for growth and the emergence of new opportunities for the discussion of the nature of the mathematics curriculum. For example, Rosa and Orey (2015) affirm that positive deviance contributes to the confrontation of taboos toward assumptions suggesting that mathematics is a field of study without traditions and cultural roots. Mathematical knowledge is acquired through unequal cultural interactions and conflicts, which reflects the dynamics of the cultural encounters.

6.2 Aspects of Positive Deviance in Ethnomodelling Research

Researchers in ethnomodelling have revealed cultural influences in the evolution of mathematical knowledge through the study of real-life contexts. This approach helps the analyses of mathematical ideas, procedures, and practices developed locally by offering innovative views about the nature of this knowledge (Orey 2000). This context enables a posture of positive deviance to be developed because the traditional trajectory of learning, norms, and rules applied in academic mathematics are often found to be inconsistent with the mathematical knowledge developed in terms of the local realities, customs, and needs of the learners and their realities. Investigations in mathematics education have often ignored connections between academic mathematical knowledge and the practices developed locally by the members of distinct cultural groups.

However, in order to reduce the gap between theoretical and practical mathematical knowledge, there is a need for both researchers and educators to look for possible connections between mathematical practices developed in diverse cultural

[3]The emic and etic approaches were developed by Pike (1967) from a distinction in linguistics between *phonemic* and *phonetic*. In their original meanings, phonemics refers to examination of sounds for their meaning-bearing roles in a particular language, while phonetics denotes study on universal sounds covering all languages.

contexts and mathematical modelling. The positive deviance aspect of ethnomodelling recognizes both the uniqueness and diverse perspectives of members of distinct cultural groups by emphasizing the relevance of emic knowledge systems (Rosa and Orey 2015). Unique combinations of geography, climate, language, religion, politics, economic, and social, cultural, and environmental contexts influenced the way members of distinct cultures counted, ordered, patterned, measured, and mathematized and modelled their own realities.

In this regard, Lloyd (2011) suggests the development of actions that search for creative and innovative solutions to these challenges because research on these practices can be regarded as a form of resistance toward the imposition of academic mathematical knowledge and as pedagogical actions, which value the development of local mathematical knowledge. According to D'Ambrosio (2011), members of distinct cultural groups, in their search for transcendence and survival, develop explanations for problems they face, as well as they collect information that makes for the creation of their own myths and mysteries, which help them to explain their sociocultural and natural environments by developing cultural artifacts. Ongoing investigations in ethnomodelling describe the ideas and procedures implicit in locally developed mathematical practices.

Material representations of reality (artifacts), which are organized in the form of spirituality, language, procedures, strategies, and techniques, are observable and can be interpreted by the use of codes and symbols, which are created through the development of mental images that are shared by these members through the use of diverse artifacts that help them to constitute their cultural background (Orey 2000). Mathematical artifacts are first generated by the members of distinct cultural groups who try to both cope and deal with natural, social, economic, political, and sociocultural environments in order to solve problems, and to explain and understand mathematical facts and phenomena that occur in their day-to-day life (D'Ambrosio 2011).

In this regard, Rosa and Orey (2017) emphasize the importance of community for schools, as it seeks to connect academic mathematical practices to mathematics developed locally. It is necessary that the development of school curriculum be designed to value and promote the valorization of local knowledge and practices developed by communities who integrate school contexts. This perspective provides a necessary balance to mathematics curriculum since it integrates cultural components into the mathematical modelling process. This approach aims at the humanization of mathematics through the elaboration of contextualized activities by applying the pedagogical action of ethnomodelling.

6.3 Land Demarcation: An Example of a Positive Deviance Ethnomodel

Ethnomodelling proposes a dialogue between local and academic approaches to the construction of mathematical knowledge through cultural dynamism. The development of ethnomodelling processes increases the potential for continual growth in the debate related to the nature of mathematical modelling and how it relates to culture. This process enables the development of concepts for *positive deviance* that offers a basis for decision-making processes in the elaboration of diverse ethnomodels. The acknowledgment of local mathematical knowledge as well as its implications for social justice, cultural empowerment, and political transformation encourages debate about the true nature of mathematics as it relates to culture and society by analyzing cultural artifacts (Rosa and Orey 2017).

In this context, Rosa and Orey (2013) discussed an example of land demarcation used by members of the *Landless People's Movement* (*Movimento dos Sem Terra—MST*) in Southern Brazil. This "demarcation activity examined the method of *cubação* of the plots, which is a traditional mathematical practice applied by the participants of this movement" (p. 80) Thus, Rosa and Orey (2019) argue that the daily necessities of these movement members caused them to capture the procedures of these techniques, showing that, despite their low level of schooling, they were able to develop procedures and techniques related to the methods of *cubação* of land, which is one of the tools they used to solve problems related to the measurement of land with irregular shapes by applying distinct methods to determine this area.

This method met the specific needs of the members of this movement because they applied it to determine land areas related to the delimitation of planting sectors as well as to demarcate the plot of land of each family in the settlement (Knijnik 1996). The access to a plot of land and to live and produce on it makes the practices of measuring the land to be a central activity of the members of this movement, mainly because of the importance placed on sustainability and planning of agricultural production (Rosa and Orey 2019). The validation of this method within agricultural communities and settlements results from the development of informal agreements of signification that results from a cumulative process of generation, intellectual organization, social organization, and diffusion of this knowledge.

For example, mathematical practices investigated in the study conducted by Knijnik (1993) consisted of a method that was called by her students in the classroom as *Adão's Method*. This chapter presents a further development of this method, which was briefly described by Rosa and Orey (2013). In this context, Adão, one of the members of *MST* movement, explained how to determine his method that transforms the shape of an irregular quadrilateral into a rectangle (Knijnik 1993):

> Well folks, this is the most common formula that is used on the countryside, up there on the farm, right? And, let's assume that I am the owner of a crop and I lent this *frame* here to a friend *to mow* and I told him that I will pay three thousand by the *fourth*. Then, he mowed this land and he even *passed the rope* himself to find its area. Then, he measured this *wall*

> here, 90 metres, the other, 152 metres, 114 metres, 124 meters. Did you notice that there is no *wall*, no base, and no height that has the same measure, right? The two landmarks that are *lying down* are the *bases* and the *heights* are those that are *standing up*. Ok, so, I did the following here, right: I added the two bases and divided the sum by 2. I found 138. So, the base is 138 here and 138 there, understood? So, I have here the two heights, 114 plus 90. I found 204 and divided it by 2, 102, right? So, now we just need to multiply the base times height, Ok? I think the answer is 14076 square metres, right? This is the area that he *mowed*.

According to Rosa and Orey (2019), it is important to state that, during his narrative, Adão used jargons that are specific expressions locally relevant to the members of *MST* cultural group, such as:

(a) *Walls* (*paredes*) that mean the landmarks of the land.
(b) *Frame* (quadro) that means the area of a land with a quadrilateral shape.
(c) *To mow* (*carpir*) means to clean or to prepare the land for planting.
(d) *Fourth* (*quarta*) that means an area measurement used in the Brazilian rural context, which is equivalent to a quarter of a Paulista bushel used in the state of São Paulo, Brazil, that measures 24,200 m^2.
(e) *Pass the rope* (*passar a corda*) means to measure the land by using a rope (p. 21).

These terms are the jargons used by these members to describe the procedures of the development of their local mathematical practices (Rosa and Orey 2019). Table 6.1 shows Adão's method of determining an area of a land with irregular shape.

The representation of this mathematical practice can be explained by the following etic ethnomodel procedures: (a) transform the shape of the irregular quadrilateral in a rectangle the area of which can be determined through the application of the area formula, (b) determine the dimensions of the rectangle by calculating the average of the two opposite sides of the irregular quadrilateral, and (c) determine the area of the rectangle by applying the formula: $A = b \times h$.

According to Rosa and Orey (2013), the "mathematical knowledge of the landless can be represented by a model that transforms the shape of the given land into a rectangle of 138 m × 102 m with an area of 14,076 m^2" (p. 81). Figure 6.1 shows

Table 6.1 Adão's method of determining an area of a land with irregular shape (Adapted from Knijnik 1993, p. 24)

Adão's explanation (Emic knowledge)	Academic explanation (Etic knowledge)
This is a piece of land with four walls	This is a convex quadrilateral
First, we add two of the opposite walls and divide them by two	First, we find the average of two opposite sides
Second, we add the other two opposite sides and also divide them by two	Second, we find the average of the other two opposite sides
Third, we multiply the first obtained number by the second one	Third, we determine the product of the two average numbers previously determined
That is the *cubação* of the land	This is the area of the rectangle whose sides are the average of the two pairs of opposite sides of the convex quadrilateral

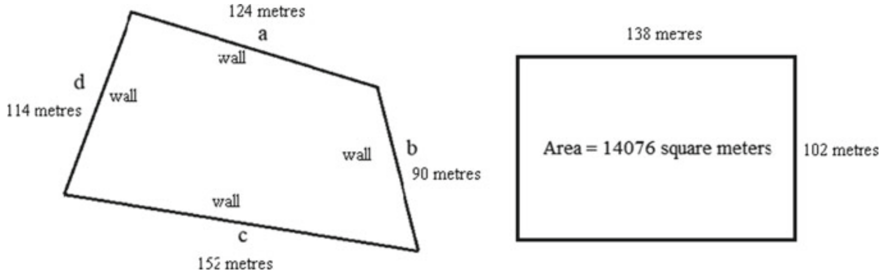

Fig. 6.1 Representation of an ethnomodel that transforms an irregular quadrilateral into a rectangle (Adapted from Rosa and Orey 2013, p. 81)

a representation of an ethnomodel that transforms an irregular quadrilateral into a rectangle.

Thus, this mathematical knowledge can be represented by the elaboration of an ethnomodel that transforms the shape of the given land into a rectangle of 138 m × 102 m with an area of 14,076 m^2.

$$\text{Area} = \left(\frac{a+c}{2}\right) \times \left(\frac{b+d}{2}\right)$$

$$\text{Area} = \left(\frac{124+152}{2}\right) + \left(\frac{90+114}{2}\right)$$

$$\text{Area} = \left(\frac{276}{2}\right) \times \left(\frac{204}{2}\right)$$

$$\text{Area} = 138 \times 102$$

$$\text{Area} = 14{,}076\,\text{m}^2$$

Indeed, it is relevant here to state that there is historical evidence that the method of *cubação* in which a quadrilateral is transformed into a rectangle was used with the purpose of land taxation in Ptolemaic and Roman periods, as well in ancient Egypt (Peet 1970). This method is also used in Chile and Nepal, and in the Brazilian states of Bahia, Pernambuco, Rio Grande do Norte, Rio Grande do Sul, São Paulo, and Sergipe (Silva 2012). It is also important to state that positive deviance related to the development of this method was orally transmitted and diffused to *MST* family members by their ancestors across generations.

In this context, Knijnik (1996) has affirmed that the method used by Adão is a mathematical practice that rural workers in Southern Brazil employ in order to transform irregular figures into regular ones. This method demonstrates procedures that rural workers in this distinct cultural group employ in order to transform figures with irregular shapes that represent their land into squares and rectangles, which are well known geometric figures. The choice of the quadrilateral geometric shape used by these members is due to the fact that it is the one that is similar to the configurations of the agricultural areas in Southern Brazil.

This method is used to calculate the total area of a region, after its occupation, in order to calculate the amount of money needed to be paid or received for the cleaning work of the property or for the preparation of the land for planting as well as the demarcation of areas to be cultivated, to plan, and to delimitate areas for the construction of houses and shelters for animals. It is important to emphasize that this method can be related to the development of the positive deviance concept in the teaching and learning process in regards to the use of local techniques in solving problems faced by members of distinct cultural groups in their daily lives.

According to Rosa and Orey (2019), mathematical knowledge involved in this local method is also related to productive activities that members of this specific cultural group performed in their daily routines. For example, the need for the development of *cubação of land* with irregular shapes was in accordance with its accessibility depending on its topology and the quality of desired agricultural products. In the study conducted by Knijnik (1993), it was proposed for the elaboration of curricular mathematics activities related to the demarcation of land with participants of this movement. These activities were related to the method of *cubação of land*, which is a traditional mathematical practice applied by participants of this specific cultural group to measure and determine the area of the land in their settlements.

It is important to emphasize that positive deviance in the context of ethnomodelling research refers to behavioral, cultural, political, economic, environmental, and social changes premised on the observation that when members of distinct cultural groups confront similar challenges they employ uncommon, yet successful mathematical ideas, procedures, and strategies that enable them to find solutions to the problems they face in their own communities (Rosa and Orey 2017).

Consequently, D'Ambrosio (2011) discusses how cultural artifacts provide necessary material tools that help in the development of clothing, shelters, navigation and defense, and transportation, which have come to assist members of distinct cultural groups to solve daily problems by using their own scientific and mathematical techniques and strategies. These artifacts are considered as tools, devices, and *instruments of observation.*

This is one concrete example of how it is possible to apply local mathematical ideas in the context of teaching mathematical content. It is important, as well as enjoyable to seek the construction of effective bridges between the method of *cubação* of land and academic mathematics. This context is an example of why we can state that this approach can be used as a pedagogical action in the mathematics classrooms in order to help students to (re)discover mensuration relationships by developing ethnomodels.

6.4 Final Remarks

The example shared in this chapter has enabled the use of positive deviance in conducting research in ethnomodelling. For example, mathematical thinking is developed and used in distinct sociocultural contexts with specific needs and ways of life. Thus, it is important to analyze the relation between culture and mathematics by questioning the predominant view that mainstream mathematics is *culture-neutral*. It is also necessary that both researchers and educators are willing to, indeed, be supported in taking risks associated with the decision of exploring local mathematical knowledge in the formal mathematics curriculum. For example, D'Ambrosio (2006) affirms that one important pedagogical action for the development of mathematical modelling is related to the transformation of mathematics into a living knowledge that integrates real situations through questionings, analysis, and critical reflection of phenomena that occur in the everyday life of the students.

This approach can be understood as a fight against the dehumanizing effects of bureaucratic authority that occurs during the conducting of research and investigations related to ethnomathematics as a program. By developing systematic studies by using ethnomodelling, it is possible to comprehend new contexts and perhaps skills that allow us to observe mathematical phenomena on more inclusive and broader wavelength (Rosa and Orey 2017). Thus, ethnomodelling can be considered as the study of mathematical phenomena within a culture, and it differs from the traditional conception that considers it as the foundations of one kind of mathematics that is constant and applicable to everyone and everywhere. Mathematics then becomes a social construct because it is culturally bound.

This chapter discussed concepts of positive deviance from the perspective of ethnomodelling. This specific form of pedagogical action helps students to overcome the use of disassociated techniques and formulas often blindly memorized. As well, it allows them to develop strategies and techniques in order to give access to diverse mathematical representations in a new formative dimension of the mathematical nature. This pedagogical action transcends physical environments in order to welcome knowledge and procedures developed in the diverse sociocultural contexts of students (Rosa and Orey 2015). This approach recognizes that it is in the school community itself that researchers and educators can easily find didactic elements of mathematical content necessary in the development of mathematics curriculum (D'Ambrosio 2006).

In the context of ethnomodelling, positive deviance can be considered as a tool to combat the dehumanizing effects of curricular and bureaucratic authority by decolonizing mathematical ideas, procedures, and practices in a search for peace. For example, Rosa and Orey (2017) argue that the objective of this deviance is to ensure that curricular bureaucracies do not disservice students when public policies and institutional procedures have no real connections within the school communities. In this regard, positive deviance aims to reduce prejudice, inequity, and harm due to disconnections between mathematical knowledge as practiced in the academy and its practical use in everyday life. Ethnomodelling may then lead us to new viewpoints in the development of mathematical modelling process in order to improve cultural sensitivity in the development of teaching practices.

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Chapter 7
Models-and-Modelling Perspective Through the Eyes of Jean Piaget

Serife Sevinç

Abstract Models-and-modelling perspective has produced a genre of activities called model-eliciting activities. Many researchers addressed American pragmatists for the evolvement of the central premises of the models-and-modelling perspective. In this chapter, I focused mainly on Piaget's theory of cognitive development as a foundation of this modelling perspective and employed document analysis that incorporated thematic analysis. In-depth analysis of Piaget's ideas and models-and-modelling literature indicated that Piaget's reflective abstraction and series of successive approximations supported the cyclic and self-regulatory nature of the model development that occurred as a series of assimilations, accommodations, and (dis)equilibrium, in Piagetian terms. Thus, this chapter provided a theoretical discussion on the epistemological content of self-regulated and collaborative model development through the eyes of Jean Piaget.

Keywords Models-and-modelling perspective · Model-eliciting activities · Accommodation · Self-regulation · Social interaction · Document analysis

7.1 Introduction

Models-and-modelling perspective is one of the modelling perspectives and categorized under contextual modelling (Kaiser and Sriraman 2006). This perspective is centered on model-eliciting activities in which students develop a model as a solution to the given problem and through which the models are elicited. These activities are constructed using the following six design principles:

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1. *reality or personal meaningfulness principle*—involving a realistic and meaningful problem situation,
2. *model construction principle*—requiring the construction of a model that express students' mathematical interpretation,
3. *construct share-ability and reusability principle*—requiring the construction of a model that would be shareable with other people and applicable in similar situations,
4. *model documentation principle*—encouraging students to document their evolving ways of thinking,
5. *self-assessment principle*—asking for self-evaluation and validation of the model, and
6. *effective prototype principle*—providing an effective prototype for students' future modelling processes (Lesh, Hoover et al. 2000).

Model development in these activities requires students to go through several cycles of expressing, testing, and revising the model. More specifically, these modelling cycles involve an understanding of the problem situation, developing a mathematical model as a solution to the given problem, expressing the model via some form of representations such as tables, graphs, and equations, testing the usefulness of the model and revising/refining the model if needed (Lesh and Lehrer 2003).

Since model-eliciting activities aim to motivate students to work in groups and develop a mathematical model that is based on their own resources and competencies rather than authoritative directions (Lesh et al. 2003; Zawojewski et al. 2003), it was seen in "the realm of psychological concept development" and as possessing "promising aspects associated with both socio-cultural theories and theories of situated cognition" (Kaiser and Sriraman 2006, p. 306-307). Furthermore, a number of researchers investigating models-and-modelling perspective have addressed the theoretical underpinnings of this perspective (e.g. Lesh and Doerr 2003; Lesh and Lehrer 2003; Lesh and English 2005; Lesh and Sriraman 2005). According to these researchers, this perspective evolved primarily out of Piaget, Vygotsky, and the American pragmatists such as William James, Charles Sanders Peirce, Oliver Wendell Holmes, George Herbert Mead, and John Dewey (English et al. 2008; Kaiser and Sriraman 2006). In this regard, models-and-modelling perspective presuppositions have been summarized as follows:

- Conceptual systems are human constructs, and so they are fundamentally social in nature (Dewey and Mead);
- The meanings of these constructs tend to be distributed across a variety of representational media ranging from spoken and written language, to diagrams and graphs, to concrete models, to experience-based metaphors (Pierce);
- Knowledge is organized around experience at least as much as around abstractions—and the ways of thinking which are needed to make sense of realistically complex decision making situations nearly always must integrate ideas from more than a single discipline, or textbook topic area, or grand theory (Dewey);
- The "worlds of experience" that humans need to understand and explain are not static. Because they are, in large part, the products of human creativity, they are continually

> changing, and so are the knowledge needs of the humans who create them (James) (Kaiser and Sriraman 2006, p. 306).

Albeit their discussions addressed the theoretical essences, they have drawn a general rather than a detailed picture (e.g. Lesh and Harel 2003; Lesh and Lehrer 2003). In this chapter, I intended to provide a theoretical discussion of Piagetian roots serving as a foundation of the models-and-modelling perspective. Specifically, I addressed the following research question: *What are the Piagetian roots of the models-and-modelling perspective?*

7.2 Methods

For this theoretical investigation, I employed document analysis that incorporated thematic analysis as a method of qualitative inquiry. Document analysis is a systematic and analytical review of printed and/or electronic documents (Bowen 2009). As Merriam and Tisdell (2016) stated, "[*d*]ocuments of all types can help the researcher uncover meaning, develop understanding, and discover insights relevant to the research problem" (p. 189). Document analysis starts with selecting and organizing the data that can be "excerpts, quotations, or entire passages" (Bowen 2009, p. 28) and follows by the analysis of the data that often takes place as content analysis and/or thematic analysis. Thus, for this investigation, I first carried out a comprehensive reading of the selected documents, including the books of Piaget and his followers, to understand Piaget's ideas in depth. I also examined models-and-modelling literature to identify fundamental features of this perspective and to articulate the features that are rooted in Piaget's theory of genetic epistemology and cognitive development. Although there are a variety of modelling perspectives (Kaiser and Sriraman 2006), the scope of this investigation was limited to the models-and-modelling perspective, and therefore the related literature was included in the data corpus.

To make sense with the size of the data corpus, chapter-sized documents (i.e. book chapters and/or journal manuscripts) were identified as the units of the document analysis. Thus, the data corpus involved 182 book chapters on Piaget's theory of cognitive development, and 104 book chapters and 21 journal manuscripts on models-and-modelling perspective. All the documents in the data corpus were read several times. The first round of reading resulted in the selection of the relevant passages and recording them as block quotations into MAXQDA (VERBI Software 2017), a qualitative data analysis software, for coding. The index pages of the books and keywords for the manuscripts were utilized to identify the relevant passages. The unit of analysis for coding varied from a single sentence to a paragraph or an entire document. In other words, in the second round of selective reading, any meaningful unit indicating a characteristic of the models-and-modelling perspective was identified as a segment to attribute a related code. Then, I grouped the codes into broader categories through focused coding and then into themes through theoretical

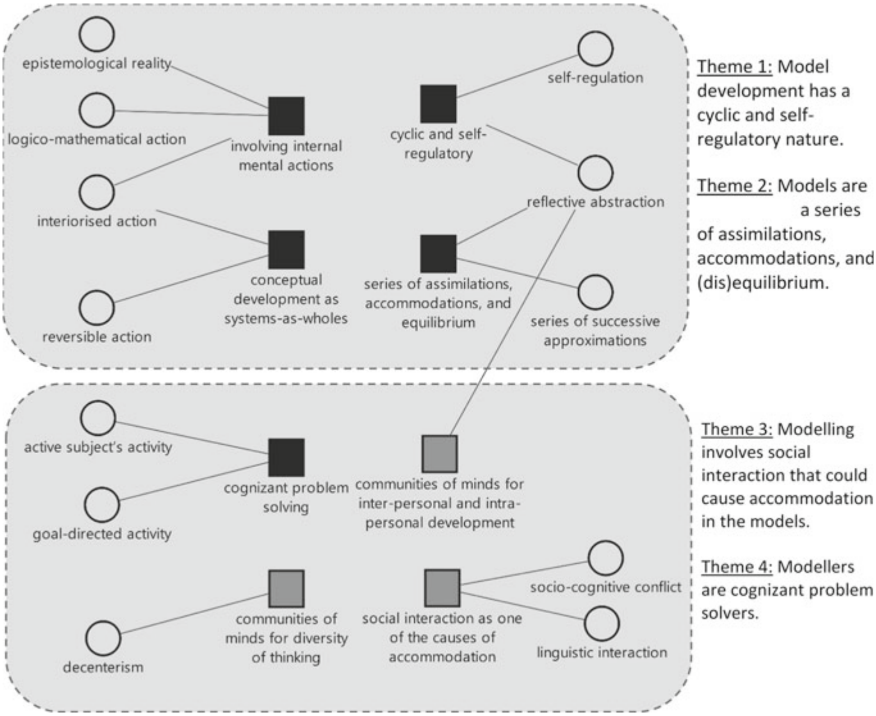

Fig. 7.1 Coding path of the thematic analysis

coding (Saldaña 2009). This pattern recognition process is the core analytical act of the researcher in the thematic analysis (Bowen 2009). Figure 7.1 below shows a simplified version of the thematic analysis path from codes to categories and themes.

In this thematic analysis path, circles and squares represent the codes derived from Piaget's theory and the models-and-modelling perspective, respectively. I used the black-filled squares to indicate the cognitive aspects and the gray-filled squares to designate both the cognitive and social aspects of the models-and-modelling perspective. As seen in Fig. 7.1, this thematization process resulted in four fundamental ideas of the models-and-modelling perspective, all of which were found to be based on Piaget's theory of cognitive development. To triangulate the themes and codes, I consulted with another researcher who had extensive research experience in both Piaget's theory and the models-and-modelling perspective. We discussed the themes and the codes that produced the themes and resolved the disagreements through a series of meetings.

7.3 Piagetian Roots of the Models-and-Modelling Perspective

According to the models-and-modelling perspective, a model informs teachers and researchers about students' cognitive processes that are elicited through their group work in model-eliciting activities (English et al. 2008), indicating that the modelling process involves both cognitive and social aspects. During the thematic analysis, I articulated the following aspects of the model development:

1. Model development has a cyclic and self-regulatory nature.
2. Models are the result of a series of assimilations, accommodations, and (dis)equilibrium.
3. Modelling involves social interaction that could cause accommodation in the models.
4. Modellers are cognizant problem solvers.

Considering Piaget's assertion of experience and social transmission as two of the four basic factors (along with maturation and equilibrium) in explaining children's knowledge formation (Piaget 1964), these aspects were found primarily based on Piaget's theory of genetic epistemology and cognitive development. Thus, the following sections present the results of the document analysis; that is, historically, philosophically, and theoretically important ideas in the models-and-modelling perspective, through the eyes of Jean Piaget.

7.3.1 Modelling as a Series of Assimilations and Accommodations

This section presents the first two aspects listed above; that is, model development is a cyclic and self-regulatory process and involves a series of assimilations, accommodations, and (dis)equilibrium. Lesh and Lehrer (2003) pointed out Piaget's view that complex conceptual systems are formed using primitive-level systems. However, to reach a complex level system, simply gathering lower-level systems is not sufficient. Instead, it requires evaluating the fitness of a variety of systems and reorganizing them through reflective abstraction (Piaget 1977/2001). This continuous self-evaluation has roots in Piaget's notion of reflective abstraction (Piaget 1970). Piaget (1985) explained reflective abstraction as follows:

> Reflective abstraction includes two indissociable activities. One is "reflecting" or projecting onto a higher level something borrowed from a lower level… The other is more or less conscious "reflexion" in the sense of cognitive reconstruction or reorganization of what is transferred. (p. 29)

Both ways of reflective abstraction are related to cognitive development involving "reorganization" of concepts and "reconstruction" of the ones that were already present in the conceptual structure. Similarly, in the models-and-modelling perspective, conceptual changes occur when an existing conceptual structure fails to fit a new experience (Lesh and Lehrer 2003). Students start model building with intuitive and primitive level conceptual systems and then develop new systems:

> …, emergent properties at higher-level systems evolve from (and are reflectively abstracted from) systems of interactions at more primitive/concrete/enactive/intuitive levels; and, these conceptual reorganizations occur mainly when models fail to fit the experiences they are intended to describe, explain, or predict. (Lesh and Lehrer 2003, p. 120)

When problem solvers realize the inadequacy of their models, they feel the perturbation or disequilibrium, in Piagetian terms, which motivates them to continue to revise and refine their models until they overcome the disequilibrium. During this process, problem solvers experience a series of what Piaget called assimilation, accommodation, disequilibrium, and equilibrium in successive modelling cycles. As Inhelder et al. (1974) explained, "Piaget holds that objects can only be known by a series of successive approximations constructed by the subject through his various activities" (p. 6). Piaget's "series of successive approximations" parallel with the series of modelling cycles of expressing, testing, revising, and refining, and the activities ensuring the continuity of modelling cycles rely on the *self-assessment principle* of the model-eliciting activities. Lesh and Doerr (2003) expressed the importance of this principle as follows:

> … students themselves must be able to judge the relative usefulness of alternative ways of thinking. Otherwise, the problem solvers have no way to know that they must go beyond their initial primitive ways of thinking; and, they also have no way of judging the strengths and weaknesses of alternative ways of thinking—so that productive characteristics of alternative ways of thinking can be sorted out and combined. (p. 18)

Thus, it is not surprising to articulate that continuous modelling cycles are aligned with the continuous knowledge development proposed by genetic epistemologists:

> For genetic epistemologists, knowledge results from continuous construction, since in each act of understanding, some degree of invention is involved, in development, the passage from one stage to next is always characterized by the formation of new structures which did not exist before, either in the external world or in the subject's mind. The central problem of genetic epistemology concerns the mechanism of this construction of novelties which creates the need for the explanatory factors which we call reflective abstraction and self-regulation. (Piaget 1970, p. 77)

Likewise, the model development process includes expressing ideas, testing, and revising them based on the feedback from the continuous self-evaluations. Therefore, models-and-modelling perspective views the modelling process as interacting, self-regulating, and continually adapting (Lesh, Lovitts et al. 2000), and perceives the problem solvers as evolving, self-regulating, and adapting organisms (Lesh and Lovitts 2000). Von Glasersfeld (1995) affirmed the cyclic and self-regulatory nature of thought development through assimilation and accommodation: "As Piaget himself has occasionally suggested, action schemes are like feedback loops, because their inherent dual mechanism of assimilation and accommodation make them self-regulations and therefore circular in that specific sense" (p. 73). Thus, the cyclic and self-regulatory nature of the modelling process, as well as the continuity of modelling cycles during model construction, have foundations in Piaget's theory of cognitive development.

7.3.2 Modelling Involves Social Interaction as Well as Cognition

This section presents Piagetian foundations of the latter two aspects of the models-and-modelling perspective that were drawn from the thematic analysis. These two aspects consider the modelling as involving social interaction that could cause accommodation and the modellers as cognizant problem solvers.

Smith (1996), a Piagetian researcher, argued that "social experience is stated to be necessary—but not sufficient—for intellectual development from cradle to the grave" (p. 110). Not only his followers but also Piaget, himself, addressed the role of social interaction in cognitive development even though his theory has often been criticized for the lack of a specific social component. As von Glasersfeld (1995) stated, "Piaget has stressed many times that the most frequent cause of accommodation is interaction, and especially linguistic interaction, with others" (p. 66). Doise (1985) also pointed out that social interaction is a cause of "socio-cognitive conflict" and subsequent accommodation that is a component of the modelling process, as mentioned in the above section.

Similarly, Brown et al. (1996), Piagetian researchers, explained the role of social interaction in producing perturbations: "The conflict arising from group disagreement creates disequilibrium and the resulting adjustment to this state is a primary cause of cognitive development" (p. 146). Piaget (1962) emphasized that not only hearing other people's ideas but also understanding the differences among ideas can trigger accommodation:

> … if an individual A mistakenly believes that an individual B thinks the way A does, and if he does not manage to understand the difference between the two points of view, this is, to be sure, social behavior in the sense that there is contract between the two, but I call such behavior unadapted from the point of view of intellectual co-operation. (p. 8)

Piaget called the communication that builds intellectual cooperation as a socialized speech that involves cognitive position-taking. This position-taking, which entails understanding and evaluating each other's perspectives, is also a key element in modelling cycles because collaborative work on model-eliciting activities allows for entertaining different perspectives, selecting the most appropriate and the most useful ideas, eliminating irrelevant ideas, and combining the relevant ones (Lesh and Carmona 2003; Lesh and Yoon 2004). As Zawojewski et al. (2003, p. 342) stated, "by placing our discussion of small group work in the context of model-eliciting activities, we can focus on understanding the processes that lead to the potential for mathematical power in collaboration." This mathematical power is enhanced by peer interaction during the work of understanding each other's perspective because each student brings his/her own potential. Furthermore, thinking from different perspectives and/or being challenged by others and other situations serve for the *construct share-ability and reusability principle* of model-eliciting activities. To produce models that are shareable with other people and applicable in other situations, modellers need to consider different perspectives and different situations, which could be achieved by welcoming group members' ideas and taking them into account during the model development. Thus, Piaget's views about the role of individuals' communication with others in cognitive development set a foundation for small-group model development in the models-and-modelling perspective.

Given the centrality of language in human communications, Piaget commented that "one of the ways in which environment influences cognitive development is through language" (Inhelder et al. 1974, p. 17). Language has a central role in cognitive development because it is the tool used by "the mind of the thinker," in not only interpersonal communication (with others) but also one's intrapersonal communication. On the one hand, the power of the small-group work in the models-and-modelling perspective signifies the role of language in one's interpersonal communication with others. On the other hand, language as a tool in one's intrapersonal communication indicates that the modelling process demands a cognizant problem solver who experiences a series of accommodations toward a particular goal given in the real-life modelling problem. Piaget (1959) also mentioned that cognitive development is a conscious goal-directed activity, and the language plays a significant role in that activity:

> Directed thought is conscious, i.e. it pursues an aim which is present to the mind of the thinker; it is intelligent, which means that it is adapted to reality and tries to influence it; it admits of being true or false (empirically or logically true), and it can be communicated by language. (p. 43)

Similarly, the models-and-modelling perspective describes mathematical models by emphasizing the purposeful nature of problem-solving:

> Mathematical models are conceptual systems that are: (a) expressed for some specific purpose (which John Dewey referred to as an "end-in-view"), and (b) expressed using some (and usually several) representational media. (Lesh and Lehrer 2003, pp. 111–112)

Therefore, the language mediates the modelling process toward a particular goal, and the use of language both depends on and is limited with the cognizant modellers. Because of the primary role of the subject, Piaget's theory has been characterized as "the child's theory of mind," referring to children as active seekers of knowledge through constructing ideas within their social world (Brockmeier 1996). Von Glasersfeld also described Piaget's conceptualization of knowledge as "the organization of the experiential world, not the discovery of ontological reality" (p. 18), in which the organization involves a series of modifications and transformations of the cognizant subject.

Language plays an essential role both in Piaget's cognitive development and in the models-and-modelling perspective. It is because, on the one hand, it functions as a purposeful tool of the cognizant problem solver in model development as mentioned above, and on the other hand, it functions as a tool used to express the models that are sharable and re-usable. Models-and-modelling researchers described the models as conceptual systems that are developed and expressed through media (Lesh and Doerr 2003; Lesh and Lehrer 2003). The media used to express a model could be in the form of spoken language, written language, diagrams, graphs, or any concrete models, or any way of expressing ideas. Language is thus one of the most important resources for collaborative model development, model documentation, and model share-ability and reusability (Lesh, Hoover et al. 2000).

7.4 Conclusion

This chapter provided a document analysis indicating that Piaget's theory of cognitive development provided a robust foundation for the cognitive and social aspects of the modelling process. Specifically, during the model development, problem solvers work in groups, interact and communicate with others, and experience a series of assimilations, accommodations, and equilibrium. Modellers create communities of mind that invite different perspectives to the model development process. Not only interaction with others but also self-assessment and self-regulation aspects of the modelling contribute to cognitive development. Thus, this chapter provided a theoretical discussion on the epistemological content of self-regulated and collaborative model development in the models-and-modelling perspective through the eyes of Jean Piaget.

I consider this chapter as one of the steps in deepening the epistemological understanding of the models-and-modelling perspective, and therefore found worthwhile to inquire about the links between this modelling perspective and Piaget's theory of cognitive development. Such investigations are important to understand the theoretical orientation of modelling-based research that is strongly related to educational psychology. However, this investigation was limited to accessible documents that articulated Piaget's theory of cognitive development. Therefore, incorporating other theories such as Vygotsky's socio-cultural theory of development and carrying out a comparative analysis would extend the epistemological understanding of the models-and-modelling perspective. While this can be one path for future research, another one can be articulating the epistemological roots of other modelling perspectives, which will open a gateway to the exchange of knowledge within the community of mathematical modelling researchers from East to West.

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Part III
Pedagogical Issues

Chapter 8
Influence of Social Background on Mathematical Modelling—The DiMo⁺ Project

Ilja Ay and Friederike Ostkirchen

Abstract Educational success in Germany is—compared to many other countries—strongly determined by social background. Therefore, teachers and education systems need to consider these social disparities. Naturally differentiating tasks may help to create learning environments, where students themselves decide on the difficulty of their approach and benefit from on their individual level. Especially mathematical modelling tasks have a strong potential considering their authentic use of extra-mathematical content. Still, there seem to be different patterns of action in modelling among students of different social backgrounds. The aim of this pilot study of the project *Diversity in Modelling* (DiMo$^+$) is to analyse individual characteristics of 15-year-old students in terms of social background and show how their handling of modelling tasks differs. This chapter presents the operationalization of social background and first video analyses.

Keywords Mathematical modelling · Mathematical performance · Natural differentiation · PISA · Qualitative content analysis · Social background

8.1 Introduction

PISA and many other studies have shown relations between mathematical performance and the social background of children (e.g. Organisation for Economic Co-operation and Development [OECD] 2013). Comparing German children from more and less privileged parental homes the *Institute for Quality Development in Education* (IQB) found a discrepancy of three years between those two groups regarding their mathematical competencies (Pant et al. 2013). In his international meta-study, Hattie (2008) found that socio-economic status plays an important role in students'

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learning ($d = 0.52$). Since the effect size, *d,* is higher than 0.4, Hattie refers to socio-economic status as being within the "zone of desired effects", which represents the zone of greatest importance. Thus, students' school success depends on the income, education level and profession of their parents. Considering these findings and that "families from different social classes are not equally equipped to support their children's learning" (Calarco 2014, p. 25), it stands to reason that the OECD (2016) concludes that the "Socio-economic heterogeneity in student populations poses another major challenge for teachers and education systems" (p. 63).

Hence, the educational system needs to create conditions that reduce social disparities, to give all children opportunities to benefit from their school education. One way is to differentiate in mathematical teaching such that it is manageable for teachers. Therefore, tasks are crucial that can be solved "by using different processes or strategies but also [...] allowing for students at different stages [...] to benefit" (Small 2017, p. 7) from them. These tasks transfer the responsibility and opportunity of differentiation to the students naturally since they decide on the complexity of their approach (Krauthausen 2018). Modelling tasks might fulfil these characteristics.

Yet, it is little studied how social background influences students' modelling of real-world problems and thereby, partly contradictory findings emerge. While Cooper and Dunne (2000) point out that socio-economic status is more strongly related to the resolution rates of items with realistic content than to the resolution rates of purely mathematical items, Schuchart et al. (2015) could not confirm these findings. Concerning the handling of modelling tasks, according to Cooper and Dunne (2000) students from less privileged homes tend to get stuck in context and thus overlook the mathematical core of the task. These students seem to overemphasize everyday experiences. Whereas according to Leufer (2016) these students tend to use official methods (e.g. formulas) to resolve uncertainties. Thus, they are more likely to overemphasize the mathematical context. What the studies have in common is that these students tend to make wrong decisions, with regard to the intended scope of use of mathematics, when processing modelling tasks.

The investigation being reported focusses on this issue to connect—and to contribute to—the debate on social background and mathematical modelling. First, the concept of social background and its determination will be presented. Afterwards, mathematical modelling and the modelling task used will be explained with regard to the German Educational Standards (KMK 2003). This chapter aims to illustrate a statistically safeguarded operationalization of social background and presents connections between education, occupation and wealth of parental homes within the given sample. Furthermore, first tendencies will be presented of how students with different social backgrounds handle mathematical modelling tasks.

8.2 Theoretical Background

According to Bourdieu's habitus (1984), social background summarizes opportunities, restrictions, preferences and aversions, which were internalized during childhood. Hence, certain resources and values are firmly anchored in identities and, thus, lead to various ways of thinking. As a consequence, typical approaches and behaviours during modelling processes may occur. Identifying these could thus lead to new ways of dealing with social, cultural and economic diversity. However, social background is a broad theoretical concept (OECD 2016), which cannot be queried comprehensively. Hence, it will be reduced to the quantifiable PISA *Index of Economic, Social and Cultural Status* [ESCS] (OECD 2017).

8.2.1 ESCS—Index for Economic, Social and Cultural Status

The ESCS is a composite score built by three indicators (OECD 2017): (1) The *International Socio-Economic Index of Occupational Status* (ISEI), (2) the *Index for the parental education in years of schooling* (PARED) and (3) the *Index of home possessions* (HOMEPOS). Figure 8.1 visualizes the computation of the ESCS through the example of Linda (see also Sect. 8.4.2).

Linda's father is a medical doctor. To compare his occupation with others, it is transferred into the ISEI measure. The ISEI captures the socio-economic status of an occupation by putting it on a one-dimensional hierarchical scale (Ganzeboom et al. 1992). This measure focusses on knowledge, expertise and income and is scaled from 10 (e.g. kitchen helper) up to 89. Linda is attributed 89 as *highest ISEI* of her parents (HISEI) since her mother is a nurse (ISEI: 48). Secondly, in order to find indicators for the education level of her parents, "education programmes and related qualifications" (OECD et al. 2015, p. 9) are scaled. For every country, school education and vocational training are coded differently into the PARED by estimating the parental number of years of schooling (OECD 2016). Since one of Linda's parents holds a university degree, her PARED yields 18 in Germany (OECD 2017, p. 435). Thirdly, the students are asked about several home possessions. It is believed that data about household possessions "capture wealth better than income, because they

Fig. 8.1 Computation of ESCS through the example of Linda (based on OECD 2017, p. 340)

reflect a more stable source of wealth" (OECD 2005, p. 283). The HOMEPOS is an index for the equipment of the parental home. Included are cultural, educational and other home possessions (OECD 2017). On this basis, the person parameters can be estimated on a one-dimensional dichotomous Rasch Model to obtain a metrical measure for the home possessions (Warm 1989). After standardizing the indicators HISEI, PARED and HOMEPOS, so that the population has average scores of zero and standard deviations of one, the ESCS can be constructed via *principal component analysis* [PCA] (e.g. Izenman 2008). Overall, the ESCS "is judged to be a valid and comprehensive index of social background" (Ehmke and Siegle 2005, p. 1).

The research aims to combine these thematic areas and find new ways of dealing with this diversity. As pointed out in Sect. 8.1, modelling tasks have the potential to address this diversity, therefore, mathematical modelling will be presented briefly.

8.2.2 Mathematical Modelling

As one of six general competencies described by the German Educational Standards (KMK 2003), mathematical modelling requires students to translate a situation into mathematical terms, structures and relations, to work within the respective mathematical model and to interpret and check results with respect to the corresponding situation. Modelling tasks are reality-related and initiate these activities. Additionally, good modelling tasks should contain relevant problems for students' daily life or future. They should include authentic use of extra-mathematical context and mathematics in the particular situation and, furthermore, should be open, that is to say, allow multiple possible solutions (e.g. Maaß 2010).

An example of a modelling task used in the study is the *Fire brigade Task* (see Fig. 8.2). The decisive factor in this modelling task is, that there is more information given than necessary to solve the task. Students have to decide, which information is imported for the solution. The task can be solved, for instance, with Pythagoras' Theorem.

8.3 Design and Method

The pilot study of the DiMo$^+$ project explores the following research questions:

Q1: Can the ESCS be constructed statistically safeguarded within this sample?
Q2: How does the handling of modelling tasks among students differ in terms of social background?

This study was conducted in the summer of 2019. Sixty-four students, as well as their parents, participated in the survey. The participants from two different high schools were, on average, 15.3 years old. Moreover, both, the students and their parents completed questionnaires, querying indicators of their social background (see

Fire-brigade

In 2004, the Munich fire-brigade got a new fire engine with a turn-ladder. Using the cage at the end of the ladder, the fire-brigade can rescue people from great heights. According to the official rules, while rescuing people, the track has to maintain a distance of at least 12 metres from the burning house.

The technical data of the engine

Engine model:	Daimler Chrysler AG Econic 18/28 LL – Diesel
Construction year:	2004
Power:	205 kw (279 PS)
Capacity:	6374 cm³
Dimensions of the engine:	length 10m; width 2.5 m; high 3.19 m
Dimensions of the ladder:	30 m length
Weight of unloaded engine:	15540 kg
Total weight:	18000 kg

From which maximum height can the Munich fire-brigade rescue persons with this engine?

Fig. 8.2 Modelling problem *Fire brigade* (according to Schukajlow et al. 2015)

Fig. 8.1). From this sample, eight students participated in a video study. Four students were selected according to their ESCS, whereby two were from the upper quartile, one from the middle quartiles and one from the lower quartile. These students were free to choose a partner and then, four pairs were filmed while solving the modelling task *Fire brigade* (see Fig. 8.2). The basis for the data evaluation is Mayring's (2014) qualitative content analysis. Here, the deductive category system for the analysis of the processes is based mainly on the following modelling sub-competencies according to Blum and Leiss (2007): understanding, simplifying/structuring, mathematizing, working mathematically, interpretation, validation and presenting.

8.4 Results

The following subsections will present the determination of the ESCS in the conducted study and, subsequently, first quantitative and qualitative results.

8.4.1 Determining the ESCS

For measuring the HOMEPOS the students were asked about 17 possessions in their home environment, based on PISA and IQB studies (among others OECD 2017). Conducting reliability analysis and a PCA, the most important independent factors were extracted. The Kaiser–Meyer–Olkin Test (KMO = 0.56), the significant Bartlett's Test of Sphericity ($p < 0.001$) and MSA > 0.5 for each variable indicate

Table 8.1 Factor loadings and reliability [Cronbach's alpha (1951)]

	Hisei	Pared	Homepos	reliability
Factor loadings [this survey]	0.89	0.80	0.75	0.75
Factor loadings [Germany] (OECD 2017, p. 340)	0.83	0.81	0.74	0.70

a good factor analysis (Hartas 2010). After removing five critical items, the person parameters of the *home possessions* were estimated for every participant (Warm 1989). Andersen's (1973) Likelihood Ratio Test evidenced the validity of the Rasch Model (LR value = 19.01, $p = 0.061$). For determining the *estimated number of years of schooling* (PARED), the parents were asked about their school education and their vocational training. The professions of the parents were queried in the student's as well as in the parent's questionnaire by asking them to describe the parental professions in detail. This double-check helped coding the professions into the socio-economic status ISEI (Ganzeboom 2010). Comparing 48 coded descriptions of occupations the *Intra-Class-Correlation* (0.973) indicated excellent reliability (Koo and Li 2016). Thus, the highest socio-economic status of the parents (HISEI) can be interpreted purposefully and statistically safeguarded. In this population it yielded approximately 54 on average, which, for example, corresponds to trade brokers or police inspectors (Ganzeboom 2010).

Finally, the *Index for Economic, Social and Cultural Status* (ESCS) was determined via PCA of the three z-standardized variables (Izenman 2008). This analysis retained one factor, including all three components, which accounted for 66.8% of the total variance. The factor loadings were close to each other and thus, of similar importance for the construction of the ESCS and they, furthermore, deviated only slightly from the loadings of the German population (see Table 8.1). The determined standardized ESCS scores lie within a range of −3.07 and 1.54. ESCS scores higher than 0.74 were assigned to the upper quartile and scores lower than −0.79 to the lower quartile.

Comparing those variables in this survey, the HISEI correlated strongly with the PARED ($r = 0.61$, < 0.001; Cohen 1988) and the HOMEPOS (=0.50, <0.001). There was also a moderate correlation between the HOMEPOS and the PARED ($r = 0.33$, $p < 0.001$). Thus, students—whose parents work in skilled occupations—had on average more cultural and educational possessions in their home environment and were wealthier. Most of these socio-economically advantaged students (highest 25% of the population) had highly educated parents with 87% completing at least university level tertiary education.

8.4.2 Video Analysis

Eight students participated in the video study. All of them read the task, identified what they considered to be important information, mathematized their real model

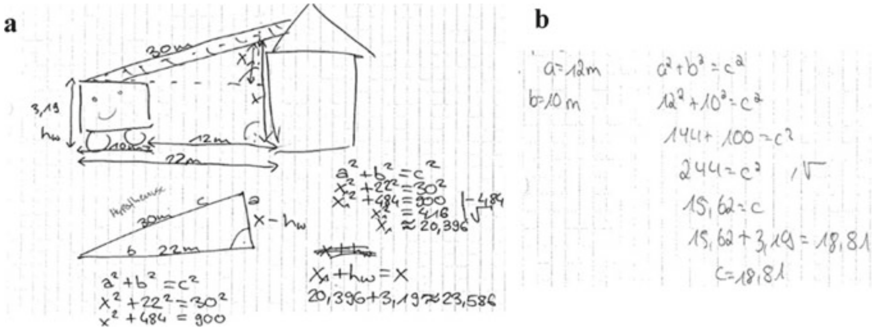

Fig. 8.3 Student solutions: **a** Christine and Linda. **b** Mark

using Pythagoras' Theorem and worked mathematically. Two student pairs drew a real-world sketch (real model) as well as an inner-mathematical sketch (mathematical model). These students developed an adequate mathematical model by setting up the term of Pythagoras' Theorem correctly and considering the height of the fire brigade vehicle (see Fig. 8.3a). The other students did not draw a sketch and additionally developed a wrong mathematical model (see Fig. 8.3b). Exemplarily, two solutions will be presented briefly.

Christine (ESCS: 1.52) and Linda (ESCS: 1.37) approached the problem after reading the given information by selecting important data such as height, length and width of the vehicle and the length of the ladder. Their first sketch plotted elements of the real world (such as the house, the ladder and the fire vehicle) based on Pythagoras' Theorem. By drawing the sketch, they discussed which length was unknown and that the height of the vehicle had to be added in the end. Afterwards, they identified the right angle and confirmed their decision to use Pythagoras' Theorem. During this mathematization, they also converted the information into an inner-mathematical sketch of a triangle to identify the sides and the hypotenuse. The following fragment of the transcript shows a validation of their solution.

Christine: About 20 Point …
Linda: 396.
Christine: Okay, great. And well … eh … shall we just write down the answer? Linda: Yeah. Let's do it.
Christine: Well it … wait. May I have a look?
Linda: Ah! Plus the height of the vehicle.

They validated, that they have calculated the height of the house without the vehicle height and, therefore, summed up their solutions.

After reading the task, Mark (ESCS: −1.72) immediately started a process of mathematization by deciding to use Pythagoras' Theorem. He did not draw a sketch, rather he developed the term of Pythagoras' Theorem by using the formula. Without any discernible considerations, he indicated values for the formula. It may be assumed that he has not developed an adequate situation model of the real-world problem.

Afterwards, he explained his solution to a classmate (ESCS: 0.12) and noticed his mistake:

> I did it completely stupid, because … OK, I notice my mistake. I have looked wrong, and instead, the length of the ladder would have been 10 m and I have not seen, that the length of the ladder is 30 m. (Mark)

His classmate groaned and they consensually decided to take the wrong solution anyway.

8.5 Discussion and Conclusion

First quantitative analysis of the questionnaires shows, that the socio-economic status can be determined from descriptions of occupations, with good inter-rater reliability. Thereby, querying parents and students turned out to be useful to extract the most important information. Also, the ESCS could be constructed statistically safeguarded via PCA, whereas the factor loadings of the three components HISEI, PARED and HOMEPOS are similar to the German populations' (see Table 8.1) and, summarized in a single factor, accounted for 66.8% of the total variance. For this population, it could be shown, that parental occupations and home possessions are strongly connected. Following Zhu (2018), the results show that most parents with high skilled occupations are highly educated as well. Also, students from socio-economically advantaged families have better access to cultural and educational resources in their home, as stated by Calarco (2014).

Within this sample, exploring students' processes showed that students who did not draw a sketch developed a wrong mathematical model (see also Rellensmann et al. 2017). Regarding modelling sub-competencies, these students were more likely to begin a process of mathematization after reading the task instead of communicating a spatial idea of the problem. The analysis suggests that these were more often students from less privileged parental homes. Therefore, findings, that children from less privileged homes tend to argue non-formally referring to their everyday life (Cooper and Dunne 2000), have not become apparent (yet). Rather, only students from more privileged homes drew real-world sketches as well as inner-mathematical sketches. The findings rather support the assumption that students from less privileged homes were more likely to look for the "right" formula to solve the task compared to their more privileged peers. Considering these results, it may rather be presumed that less privileged students tend to overemphasize the inner-mathematical context, as Leufer (2016, p. 242) pointed out.

However, some limitations of the study need to be considered. Within this pilot study, analyses of four modelling processes only allow the drawing of tendencies so interpretations of preliminary results should be viewed with great caution. Moreover, there is no evidence that the patterns of action found can be traced back to social background. Therefore, a standardized mathematics achievement test should be carried out, to take the impact of achievement into account. Also, a larger sample allows better monitoring of gender-specific patterns of action. Further, this requires a stringent selection of the sample with pairs as homogeneous as possible in terms of social background. For future surveys, students should be given possible partners with similar ESCS and similar mathematics achievement from which they can then choose. That way the advantages of free choice of partners can be combined with the advantages of homogeneous groups and the impact of achievement can be controlled.

Nevertheless, this study shows the need to address social disparities in mathematical modelling. First analyses suggest that there could be different patterns of action in modelling among students of different social backgrounds. Analysing these patterns could help to reduce social disparities in mathematics education, such as through targeted interventions and awareness-raising among teachers. To give one example, Hoadley (2007), among others, suggests being more explicit about evaluation criteria to promote the participation of all students. To explore connections between performance, social background and the way of dealing with modelling tasks, taking also natural differentiation into consideration, more investigations are necessary and will be presented in the future.

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Chapter 9
Mandatory Mathematical Modelling in School: What Do We Want the Teachers to Know?

Rita Borromeo Ferri

Abstract Defining mathematical modelling as mandatory content within school curricula is taking place in many countries around the world. Teacher education in modelling is necessary so that modelling lessons can be realized in schools. Within the international discussion, one finds best practice examples of teacher education in modelling, which differ concerning regional, national and cultural aspects. What do we want the teachers to know? This chapter sheds light on this difficult question. The aim is mainly to present the historical development of teacher education in mathematical modelling. In addition, an empirical study on measuring teacher competencies for mathematical modelling is presented. This is followed by a case study, which gives insight into the views of university educators after teaching a mathematical modelling course and their opinion as to what teachers need to know.

Keywords Assessment · Comparative study · Historical overview · Modelling course · Modelling teaching competencies · Teacher education

9.1 Introduction

Mathematical modelling in the sense of Pollak (1969), that means linking mathematics to real world situations and problems, is presently a strong and internationally well-recognized research field in mathematics education. This becomes evident through various international groups and research programs such as the International Conference on the Teaching of Mathematical Modelling and Applications (ICTMA), the Congress of European Research in Mathematics Education (CERME) or the International Congress on Mathematical Education (ICME). Also, it is evident through regional meetings, for example, in Latin America, the modelling and technology strand within the framework of the Latin American Meeting of Educational Mathematics (RELMEin Spanish).

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Furthermore, this is clearly a result of the strong research discussion, modelling is seen as a necessary practice for learning mathematics and it helps to prepare students for their daily and professional lives (see, e.g., Hernandez-Martinez and Vos 2018). Thus, in some countries earlier (e.g. Germany) and in some more recently (e.g. South Korea), educational policy makers have decided to implement mathematical modelling as a mandatory part of the school mathematics curriculum. This represents a great success but at the same time presents a challenge. Far more than before it becomes clear that we need teachers who are able to teach quality mathematical modelling with professionalism. What do we want the teachers to know?

This chapter tries to shed light on this difficult question. At first, the historical development on teacher education in mathematical modelling is discussed. This overview shows that it took a long time before today for teacher education in mathematical modelling, and competencies for teaching mathematical modelling, to become an explicit focus for research.

But what is meant by teaching competencies for mathematical modelling? There is currently no general characterization for the term "teaching competencies for mathematical modelling". However, based on current studies (Borromeo Ferri 2019; Klock et al. 2019), the competencies for teaching mathematical modelling would include Pedagogical Content Knowledge (PCK) (e.g. Ball et al. 2005) for modelling. These are expressed as dimensions or facets in current models, for example, knowledge about the modelling cycle, diagnostics, assessment, knowledge about modelling tasks and their development, planning and conducting modelling lessons or teacher interventions. Furthermore, teaching competencies for modelling are not the same as modelling competencies. According to Blomhøj and Jensen Højgaard (2007), modelling competency is the ability to construct and to use mathematical models by carrying out appropriate steps, as well as to analyse or to compare given models. Without going into discussion about modeling competency at this point, it quickly becomes clear that modeling competency is a part of the teaching competencies to be acquired.

In Sect. 9.2, an empirical study is shown as an example of a current research emphasis, in which the teaching competencies for mathematical modelling were assessed before and after a university course. The development and the evaluation of university courses and workshops for teaching modelling have been dealt with in the past 20 years within the modelling discussion. However, there was little knowledge gained about how and whether one can measure teacher competencies for modelling. The empirical study presented in this chapter is therefore, amongst other things, a way of showing which teacher competencies for mathematical modelling are seen as necessary and which instruments can be used to measure these competencies.

In addition to the measurability of teacher competencies for modelling, there is the fundamental question of how teaching at university should be designed so that teachers are appropriately educated for the implementation of quality modelling in school. Therefore, when an overview of research in mathematical modelling teacher education is presented in the second section of this chapter, the term "teacher education" is clarified in terms of the chapter focus. Many years ago, Allen (1940) indicated that terms like teacher education and teacher training refer to the policies and procedures designed to equip (prospective) teachers with the knowledge, attitudes, behaviours and skills required to perform their tasks effectively in the classroom, school and wider community. This kind of definition is used by the author as a basis by specifying this for mathematical modelling:

> Teacher education or training in mathematical modelling means to equip (prospective) teachers with knowledge and competencies that they require to teach modelling qualitatively and effectively in school, which include Pedagogical Content Knowledge for modelling.

If prospective teachers at university are to be adequately trained to teach mathematical modelling, then also good lecturers are needed. So far, there is not much empirical evidence about the views of lecturers, what prospective teachers and educators of the educators should be able to do, and how they subjectively assess such outcomes after their modelling course. This important aspect rounds off the chapter, in which a case study with university educators from Germany, Spain and Japan is presented. This qualitative study gives insight into the level of achievement of some competencies for teaching mathematical modelling, which the university educators consider to be particularly important. Finally, to end the chapter, an outlook on further research in teacher education in mathematical modelling is presented.

9.2 Historical Overview—Teacher Education in Mathematical Modelling

Learning and teaching go hand in hand, and thus one can assume that research on teacher education should always and automatically be a part of both from the beginning. In order to confirm this hypothesis to a certain degree for the field of mathematical modelling education, an analysis of ICTMA proceedings in particular (but also of journals like *ZDM*) was conducted. The goal was to find studies that deal with research on developed courses and research on the development of courses for teaching mathematical modelling for prospective and practicing teachers. Furthermore, studies were sought that show, based on reliable test instruments, the degree of learning success of the teachers after a course in mathematical modelling.

Because research on teacher professionalism in modelling up to the present day is a long-term development process, achievements are made transparent by considering three time periods. The first before the year 2000, the second from then until 2019, while the third offers an outlook for possible future research topics in this field. The justification for the three time periods resulted from the literature search. Research on teacher education or on developed and evaluated modelling courses was not really in focus until about the year 2000. From the year 2000, professionalization of teachers generally came to the fore in education policy in most countries of the world. Many international comparative studies focusing primarily on mathematics have benefited from this. This mainstream focus affected various areas of mathematics education as well as mathematical modelling. This development continues to this day, but in 2019, new knowledge regarding the measurability of teacher competencies for mathematical modelling can be gained. The time periods are described in more detail below, as far as this is possible for reasons of space.

Period 1 (before year 2000): In contrast to the topic of modelling competency, which has been a continuing research area until today (e.g. Kaiser 2006), research on teacher education and professionalism was rarely to be seen before year 2000. Therefore, it is not comparable in terms of the number of empirical studies that can be found on modelling competency. Before 2000, many empirical studies showed insights into the effects of using strategies when teaching modelling, which had a positive effect on the modelling competency of learners (e.g. Mevarech and Kramarski 1997). These aspects illustrate that while there have been studies on teaching modelling, little attention has been paid as to how the associated approaches are to be taught to the teachers who implement them. For a long time, the focus relied heavily on the learners' perspective and the teachers' involvement was more implicit. In terms of the effect chain, "teaching competencies=>quality teaching=> student learning" (Borromeo Ferri 2018), until the year 2000 we were more at the end than the beginning.

Period 2 (since year 2000): The development of teacher education as a research field in modelling since the year 2000 was strongly influenced by the fact that teacher education received more attention in general through large-scale studies. Through the findings of international comparative studies on teacher professionalism, in particular TEDS-M (e.g. Blömeke et al. 2011) the national German study, COACTIV (Kunter et al. 2013), and also the meta-analysis from Hattie (2009), teacher education became more important both in general and for educational policy in the sense of "Teacher matters most!". In turn, this leads to deeper thinking about what teacher professionalism means specifically for mathematical modelling.

From the beginning of 2000, the importance of teacher education increased—somewhat slowly—but one landmark was set by the ICTMA 10 book (Ye et al. 2003). Zhonghong et al. (2003) presented modelling courses which they conducted for preservice teachers. They made clear that the overall goal was to encourage the teachers to solve real-life problems and to understand what mathematical modelling means. Furthermore, a contribution from Holmquist and Lingefjard (2003) showed that prospective teachers could acquire modelling competency through modelling activities. However, solving modelling tasks is not the only competency that teachers need for teaching modelling at school. The types of modelling courses mentioned above often did not make the connection to practice, which is of great importance. With practice, the author means conducting modelling at school with learners and reflecting on their own teaching.

Between 2000 and 2019, the field became much broader, featuring several ideas for modelling courses and empirical research in teacher education. Since 2005, many modelling courses for pre- and in-service teachers have become visible in the literature, and thus, the list of cited researchers in this time period is incomplete to a degree. Pragmatically one can subdivide the developed teaching approaches into three categories: (1) modelling days/weeks; (2) modelling courses at university/training courses; (3) distance/online courses. In the following, they are described in a more detailed way.

Modelling days/weeks normally connect theory with practice. The goal of modelling days is that pre-service teachers who are prepared through university courses to solve complex modelling problems, then coach students in modelling in school for three or more days. The school component not only focuses on solving complex modelling problems, but also in particular on other aspects, such as teacher interventions, which are introduced, deepened and discussed using, for example, videos (e.g. Blomhoj and Hoff Kjeldsen 2006; Borromeo Ferri 2018; Bracke 2004; Kaiser and Schwarz 2010).

The second approach, modelling courses, also mostly connects theory with practice. The difference from the modelling days/weeks is that the prospective teachers do not supervise a group of learners for several days in school. As part of learning about the teaching competencies, the prospective teachers are required to develop a modelling problem in groups, to plan and carry out corresponding lessons in school, and to present and reflect on their results. In the literature, one can find many different ways that these courses are structured and also where the focus lies (e.g. Borromeo Ferri 2018; Borromeo Ferri and Blum 2010; Huincahue et al. 2018; Schorr and Lesh 2003).

The planning and implementation of digital learning in many educational areas is currently a much discussed and researched topic. Especially when educational institutions are closed due to crises, digital learning is the saviour for home schooling. However, digital learning is also used when teachers cannot personally attend training programs or when a topic is not part of the curriculum in teacher training. Distance learning through e-learning tutorials allows people to continue education. E-learning courses specifically for learning and teaching of mathematical modelling for school purposes are still rare, but they exist and are important for those teachers who have no other possibility for being educated in modelling, but wish to do so (Biembengut and Faria 2001; Maaß and Gurlitt 2011; Orey and Rosa 2018).

In addition to the development and evaluation of these courses for teachers,many empirical research studies concerning teachers' roles within learning and teaching of mathematical modelling have been conducted. The results of these studies offer the opportunity to integrate them into courses for mathematical modelling teacher education over the time. Aspects like teacher interventions and scaffolding (e.g. Leiß 2007; Stender and Kaiser 2015), formative assessment (e.g. Besser et al. 2013), teacher noticing (e.g. Galbraith 2015), relevance of multiple solutions (e.g. Schukajlow and Krug 2014), teachers' beliefs, aspects of quality teaching and technology (e.g. Blum 2015; Brown 2017; Greefrath et al. 2018) etcetera are seen as relevant content in addition to the central competency of solving modelling problems. So, the list is very long if all achievements are to be included. Although many teaching materials and modelling problems are available now, for teachers, there are still barriers to teach modelling, because in their view, they do not have the right materials. Furthermore aspects of time and limited knowledge about assessing mathematical modelling from a teachers' view are challenges as well (Borromeo Ferri and Blum 2014). We, as educators of the educators, still have to take this into account.

9.3 Teaching Competencies for Mathematical Modelling and Their Measurement

Papers addressing the conceptualization and measurement of teacher competencies for mathematical modelling have been recently published (Borromeo Ferri 2019; Klock et al. 2019). This section gives a brief overview of how one model for teacher competencies for mathematical modelling was developed. See Borromeo Ferri (2018) for more details. The development of a test instrument based on this model, geared to measuring the PCK of prospective teachers in an intervention study is described.

The author started developing and conducting modelling courses for prospective and in-service teachers in the year 2004. At that point, the main questions were, if and how future teachers (for all school types) can be prepared in tertiary courses for teaching modelling at school, and in particular, what contents and methods would be appropriate. Additionally, a focus was to investigate how prospective teachers' processes of learning and understanding develop during such courses, to identify the main difficulties and problems, and investigate how progress can be observed.

The question "what do teachers need to know?" should be answered through a long-term process approached through design-based research (DBR) (Collins 1990), so that finally a suitable structure, specific content and several teaching methods can support the development of a course for pre- and in-service teachers (Borromeo Ferri 2018). Such a modelling course taught all over the world (e.g. Turkey, USA, Spain, Chile, etc.) can enrich teaching through the development of cultural perspectives. In 2010, a first model for teaching competencies was conceptualized (Borromeo Ferri and Blum 2010) which mainly reflected the structure of the modelling course being described. The model was further modified to its current form as shown in Fig. 9.1 (Borromeo Ferri 2014, 2018; Borromeo Ferri and Blum 2010). In the meantime, further research groups have developed other approaches, which include facets of teacher professionalism in mathematical modelling (e.g. Klock et al. 2019).

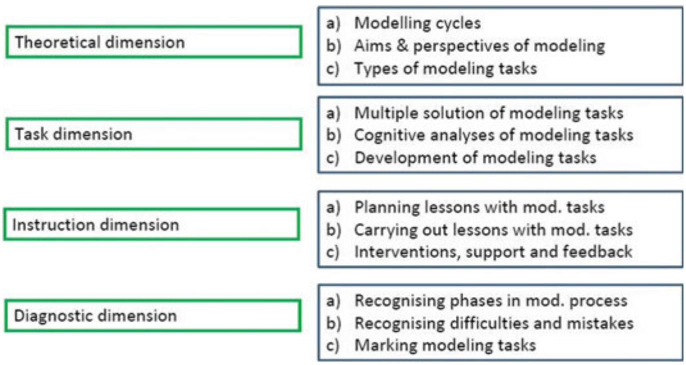

Fig. 9.1 Model for teaching competencies for mathematical modelling PCK (Borromeo Ferri 2018)

Using a design-based research approach over several years, with continuous evaluation of the course, and through the written learning diaries of the prospective teachers, it became evident on a qualitative level that teaching competencies increased with respect to the four dimensions as shown in Fig. 9.1. At a certain point, however, the question arose as to how these teaching competencies could be measured empirically, in order to assess the effect of the modelling course. This represents a challenge when recognizing the complexity of the model with its four dimensions and several sub-facets.

Within the first approach of test development and evaluation, most of the items were open ended. However, the decision to use a multiple choice format with closed items was finally made for measuring declarative and conceptual knowledge in a balanced and economic way across all dimensions. Concretely, one has to choose a correct answer according to a given statement, or for example, on the basis of the analysis of a written dialogue produced by learners while modelling. Options varied between two or five possibilities, depending on the item—respondents were required to select the right answer, or to write down the correct response for the question being asked. The answer format offers the possibility for dichotomous coding, which means 0 for incorrect and 1 for correct answer. The final test version was rated and discussed with experts during construction, and then piloted intensively with several cohorts of prospective primary, secondary and high school teachers in their fourth semester at university. Thus, a reliable test instrument was ready for use. In order to make items more transparent, some examples are shown. Within the theoretical dimensions for example, the testing of declarative knowledge with 26 items was in the foreground. Two of them are presented in Fig. 9.2. Within the scale "instruction dimension", 14 items cover declarative and conceptual knowledge. Two items of this scale are shown in Fig. 9.3. For more details concerning the test instrument with examples of items, see Borromeo Ferri (2019).

Here, your theoretical knowledge is asked. Choose, if the statement is right (yes) or wrong (no):

	yes	no
The basis of mathematical modelling are problems from real life	☐	☐
The "Complexity" describes one criterion of a modelling problem	☐	☐

Fig. 9.2 Example items of the scale "theoretical dimension" (Borromeo Ferri 2019, p. 1157)

Here, your knowledge about teaching modelling is asked. Choose, if the statement is right (yes) or wrong (no):

	yes	no
The introduction of modelling activities works with over-determined problems	☐	☐
Responsive interventions lead back to the teacher	☐	☐

Fig. 9.3 Example items of the scale "instruction dimension" (Borromeo Ferri 2019, p. 1157)

In the following, results of an intervention study with $N = 66$ prospective secondary, high school and vocational teachers in their third year at university are presented. In order to measure the increase of teaching competency, a pre- and post-test design was used, where the modelling course was the treatment. This modelling course had four blocked sessions each of three hours, taught by the author, based on the PCK for modelling shown in the model as shown in Fig. 9.1. Between the third and fourth session, the prospective teachers taught a modelling lesson in school and observed learners during modelling activities.The pre-test was administered at the beginning of the first session, and the post-test at the end of the fourth and last day of the course. Before starting with the modelling course, the participants had only limited knowledge about mathematical modelling, acquired from one lecture in their first semester. In Table 9.1, an overview of the number of items per scale (teaching dimension) and the corresponding Cronbach's Alpha measure is given. Following the well-known rule of thumb for Cronbach's Alpha (e.g. George and Mallery 2003), the reliabilities of the scales are acceptable—with 0.69 for the instruction dimension close to the notional value of 0.70.

For analysing the data, a *t*-test was carried out for related samples. The four scales formed by the sum score were examined in a pre-post comparison. Cohen's *d* was calculated as the effect measure. Looking at Fig. 9.4, the values of the x-axis are the mean values in comparison and on the y-axis are the four dimensions.

Table 9.1 Number of items per scales (four teaching dimensions) and Cronbach's Alpha

	Teaching dimensions			
	Theoretical	Task	Instruction	Diagnostic
Number of test items	26	11	14	14
Cronbach's Alpha	0.86	0.71	0.69	0.71

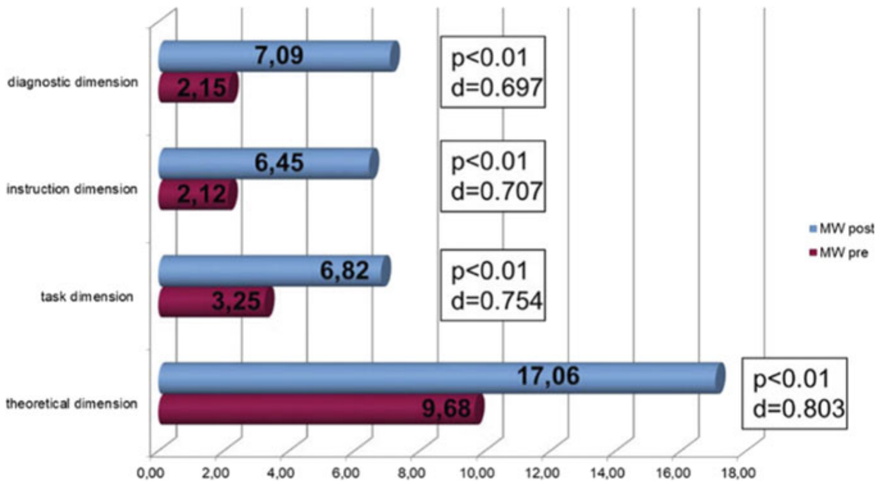

Fig. 9.4 Results of the intervention study for teaching competencies

When looking at the means in Fig. 9.4, one can see already visual differences for pre- and post-tests. These are confirmed by the statistical calculations. The Cohen's *d* statistic for each dimension shows a strong effect. Thus, the difference from pre- to post-test was significant in all four dimensions and interestingly most of all in the theoretical dimension.

The test results are encouraging, however, of great importance is that modelling courses with all their activities are being conducted in mathematics teacher education in many international locations. This test instrument helps to make more visible, the possibility that teacher competencies can be measured. Other test instruments exist (e.g. Klock et al. 2019). These test instruments can be used for several aims, for example, by teacher educators in order to evaluate their teaching or for offering (prospective) teachers some kind of additional certificate—because such modelling courses are mostly not in the curriculum for teacher education. Such a certificate could be seen as a further and necessary qualification, and finally, it could be a good argumentation base for policy makers.

Nevertheless, what results can be expected, when the test instrument presented in this chapter is used in other modelling courses? "You get what you test!" The test instrument can, in principle, be used worldwide, but it will certainly work best, if the content of the modelling courses is very close to the one presented here—with the four dimensions of teaching competencies covered. Taking this as a challenge, a comparative study was conducted with Germany, Japan and Spain. The teacher educators were offered the course slides including all materials, which were transferred, modified and implemented to fit the possibilities and circumstances in their countries.

The further focus in this chapter will not lie in the presentation of the quantitative results of the prospective teachers from Germany, Japan and Spain, but on qualitative results of a case study on the views and challenges of university teacher educators in these three countries. Regarding the question "what do we want the teachers to know?", it is important to think about what is possible in our teacher education in mathematical modelling right now in several countries, especially when modelling is mandated in the school curriculum.

9.4 Views of the Educators of the Educators for Teaching Modelling

Due to the fact that altogether only four teacher educators participated, (because they taught the courses in their countries), the presented insights are results of a small qualitative case study and thus limited concerning generalization, particularly in regard to cultural comparisons. However, the written responses from four experienced teacher educators, one each from Japan and Germany and two from Spain, offer interesting insights and subjective views, on the questionnaire developed by the author. These may be used to plan further empirical studies. The questionnaire consisted of 19 open and closed items, which the teacher educators answered in a written form after teaching the modelling course in their countries. The following sections give an idea of the broad range of information that was included:

(a) Educator background (e.g. since when they taught modelling courses),
(b) Course preparation (e.g. if they used only author's slides or other material),
(c) Course conduct/conditions (e.g. number of course lessons, time restrictions for teaching modelling),
(d) Participants/feedback/testing (e.g. opinion of active involvement of participants),
(e) Teachers' knowledge—educators' of educators knowledge (e.g. opinion which knowledge, competencies are needed).

The data were analysed according to the principles of grounded theory (Strauss and Corbin 1998), a social science approach for the systematic processing of primarily qualitative data with the aim of generating theory. In order to come to a theory, the procedure is to use open, axial and selective coding for all data. The starting point of open coding is reading the texts and marking text passages using short, concise and comparatively abstract concepts (codes) that characterize the content of the respective text passage. Axial coding is about working out the context and conditions that make it possible to identify actions or omissions, strategies, routines and their consequences in their respective social frameworks. Selective coding is becoming increasingly compact, and key categories are being worked out.

The core purpose of the analysis was firstly to identify the views concerning teachers' necessary knowledge, and secondly to access university educators' knowledge. However, feedback from all parts of the questionnaire was included in the analysis for these purposes. In the questionnaire, the educators were asked to answer in addition, two central questions regarding their focus:

(1) "What do you think teachers need to know (which competencies do they need) for teaching modelling?", and
(2) "Which knowledge/competencies should the educators of the educators possess for teaching how to teach modelling?"

When thinking about what knowledge and skills university educators must have in order to be able to train teachers in teaching modelling, there are already assumptions from theory. One assumption is, for example, to offer a clear structure of the course with aims and goals; another is to be able to offer a balance between theory and practice in the course for participants (Lesh and Doerr 2003).

At first, the comparison of questions (1) and (2) above revealed that the differences between the knowledge and competencies that the teacher educators and prospective teachers should have were not great. It was argued of course that the teacher educators especially should have knowledge and experience of how to teach a modelling course effectively with appropriate methods. The focus is on good preparation for a modeling course, which is clear from the following quote: "What I have to have with me is a well-prepared script for each session and give my students all the support they need to believe that they will be able to do mathematical modelling in their future classes."

Besides this aspect, both teacher educators and (prospective) teachers, should have theoretical knowledge of modelling, a central point being that teachers should be able to create their own modelling problems. This competency is seen as very important within all responses, because in doing this, a person will understand what modelling means, and how to implement this in the lesson. For example, one of the educators wrote: "I think that one of the essential competencies is the ability to develop suitable modelling problems for students".

To compress the outcome statements to a more abstract level, which is the goal of grounded theory, the result of the analysis can best be visualized in a causal chain as a theoretical approach as shown in Fig. 9.5. The analysis showed that many of the competencies required for teaching mathematical modelling at university or in training courses correspond to what is needed by the teachers they are educating. The intuitive idea that teachers teach modelling better if they are properly trained is obvious, but the analysis made it increasingly clear that the quality of the training for future teachers depends on the personal and external conditions of the teacher educators. In the following, Fig. 9.5 is explained more in detail.

Personal conditions refer to the individual conditions of the teacher educator, which can influence the qualitative training and education of teachers. Not everyone who trains teachers for teaching modelling is also a researcher in the field and therefore has a lot of background knowledge. Therefore, aspects such as professional background and previous knowledge of mathematical modelling education play a role. Educators with a strong mathematical background can focus differently on a course development than a teacher who has been in practice for a long time. There is also the point, which should not be underestimated, as to whether the educators are required to give a course, or whether they like to do it out of interest. For this reason, affective characteristics in relation to modeling are included. General experience with teacher training and further education gives the teacher educator an advantage in their individual planning and implementation. Finally, the teacher educator's own further training for teaching modelling should also be noted. This can be by way of reading the literature or participating in a webinar or e-learning course.

External conditions include factors that the teacher educator can only influence to a limited extent for the implementation of her/his course. Notably, the time available for a course differs within a country, but also between countries. Mostly, this depends

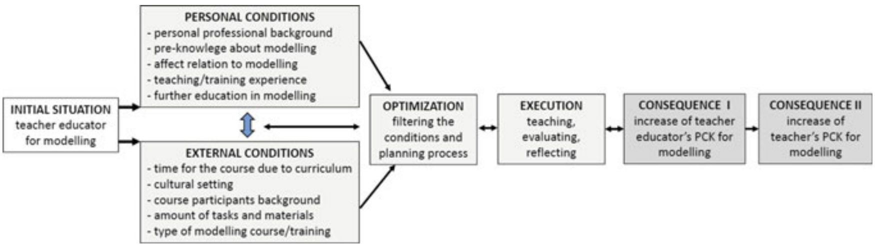

Fig. 9.5 Conditions for qualitatively taught modelling courses through teacher educators

on whether modelling is a part of the teacher education curriculum or not, and how important modelling is seen to be for teacher training. The cultural setting can influence the teaching methods on the one hand and the focus of the course on the other. The background of the participants can also be different and must be adjusted according to the situation. Dealing with teachers who have been teaching for 20 years requires a different approach from that taken with young prospective teachers. Another aspect to consider is the extent to which a university is equipped with books and materials for mathematical modelling so that course participants can use them. Finally, the question is what type of modelling course is needed by the teacher educator, who then requires preparation accordingly. Personal and external conditions are in constant interaction.

The *optimization* phase then follows, in which the conditions are "filtered". This means that the teacher educator uses the best results of this interaction as a basis to plan the course. Then follows an *executed plan*, which should always include evaluation and reflection. This leads to *consequence I*—namely that the teacher educator increases their own competencies through progressive iterative interactions between execution and optimization. The professionalization of the teacher educator ultimately benefits the teachers—as *consequence II*—with an increase in teaching competencies for mathematical modelling.

9.5 Summary and Outlook

Mathematical modelling is becoming a mandated part of the school curriculum in more and more countries across the world. This requires that teacher education in mathematical modelling starts at university. The historical overview shows that since the year 2000, teacher education in mathematical modelling has come more into focus through large-scale international assessment studies in teacher education and other types of courses. There has been a corresponding increase in research. Finally, models for teaching competencies for mathematical modelling could be developed. Existing test formats can be used to show that teaching competencies for modelling can be improved significantly by participating in a modelling course. Such test successes are particularly important as an argument for the importance of mathematical modelling in the current educational policy debate for STEM. They show that teachers and learners can obtain enough background to create realistic and desirable interdisciplinary lessons.

Following the historical timeline from Sect. 9.1, at this point time period 3 is described by presenting interesting research questions, which can form a basis for work in the coming years on teacher education in mathematical modelling. A central research question could be, whether teaching competencies have an effect on the quality of teaching of modelling in the classroom. From an empirical perspective, this is not an easy task. To actually measure empirically whether the modelling competence of learners has increased due to better teaching competence of the teacher in a given course requires good test instruments and strict test conditions. Adding a control group (i.e. teachers who did not take part in a modelling course) would make the differences visible.

A crucial point is still the question of the assessment of modelling processes of the students in all school classes. There, we need much more research, for example, what evaluating competence means, and how this form of evaluating competence can be conceptualized and measured with teachers. A first approach can be found by Strauch and Borromeo Ferri (in press).

To sum up, the knowledge and the power of teacher competencies should be used for further research with the goal of increasing teacher professionalism in mathematical modelling. What do we want the teachers to know? Although it is a difficult question, this contribution aims to give some first answers. There is a consensus that our mathematical modelling teachers need to get the best education or training from a likewise well-trained educator. It has become clear that a combination of external and personal influences on teacher educators ultimately leads to teachers who are well trained to teach mathematical modelling. Additionally, teachers should be motivated to recognize how great modelling lessons can change their minds and also those of their learners.

Let me end with a citation of Socrates:

> "I cannot teach anybody anything, I can only make them think". (Socrates)

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Chapter 10
Analysis of the Relationship Between Context and Solution Plan in Modelling Tasks Involving Estimations

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Abstract In this chapter we analyse students' written solution plan of a sequence of modelling tasks involving estimations. Our research objective is to statistically analyse whether there is a relation between the solution plan and the characteristics of the context of the real estimation task. From previous work, we have identified some task variables that are directly related to the context. In this study we have designed a sequence of modelling tasks and we have analysed the productions of $N = 224$ Spanish pre-service primary school teachers. The results show that there is a relationship between the variables of the task and the solution plan used by the students in each case. From the results of this study, we derive conclusions regarding the characterisation of this kind of modelling task and the potential use of this sequence to promote problem solving flexibility.

Keywords Context variables · Estimation · Modelling · Pre-service teacher · Task variables · Solution plan

10.1 Introduction

Different researchers have shown that the approach of modelling tasks based on real contexts make mathematics meaningful and motivating for students (Blum 2011; Kaiser and Sriraman 2006). In this work we use problems that focus on making estimates of a large number of elements enclosed in a bounded area, such as knowing the number of people who fit in a public square. These real estimation tasks are contextualised problems that students can solve by introducing modelling elements

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(Albarracín and Gorgorió 2014). It has been shown that students often have difficulties when they have to consider aspects of the real context that are described in the task statement (Greer 1993). Indeed, students' written productions when faced with modelling tasks are a key source of information for understanding aspects linked to the teaching and learning processes of mathematics.

In previous studies we have analysed students' written productions when faced with solving tasks consisting of obtaining a reasoned estimate of the number of elements that can fit into a rectangular region and we have observed that students' productions are diverse (Ferrando et al. 2017; Gallart et al. 2017). Although these problems are similar and ask the same question (How many can fit?), the diversity of student solutions has led to the question of whether different contexts in which the number of elements in a rectangular region must be estimated promote one type of solution (a model or a strategy) more than another. In order to characterise the different contexts of these problems, five context variables have been considered in this research: size of the region, size of the elements, shape of the elements, arrangement of the elements and shape of the region. These context variables have been used to design a sequence of four problems with different real contexts. A total of $N = 224$ students have solved this sequence and their productions have been categorised in four different solution plans. A mixture of analysis combining qualitative and quantitative techniques is the key to answer the research question: Is there a relationship between the context of the problem and the solution plan proposed by the student in this type of modelling task?

10.2 Theoretical Framework

A particularity of real context problems is that they often do not contain all the information needed to obtain a solution. These problems, known as problems with missing information, foster skills such as estimation, considered important and useful for students (Ärlebäck 2009). Ärlebäck (2009) defines Fermi problems as open, non-standard problems requiring the students to make assumptions about the problem situation and to estimate relevant quantities before engaging in, often, simple calculations. Certainly, there is a strong connection between the Fermi problem solving process and the work developed during the modelling cycle for the development of a mathematical model (Ärlebäck 2009; Borromeo Ferri 2006).

The development and creation of mathematical models intended to describe or abstractly represent a given phenomenon or reality is a complex process (Blum, 2011). Following the definition proposed by Lesh and Harel (2003), a mathematical model is a system formed by mathematical concepts, symbolic representations of reality, relations, regularities or schemes, as well as the procedures, mathematical or not, associated with its use. In Achmetli et al. (2019), the authors established three ways to differentiate solutions of a real context problem. The first one is to fix different assumptions when solving real-world problems with vague conditions and generally leads to different results. The second one results from applying different mathematical strategies to solve a problem and generally leads to the same mathematical result. Finally, the third one is the combination of the two above. In previous work, based on this third way and Lesh and Harel's model definition, we have developed the notion of a solution plan for the analysis of student productions when they solve modelling tasks (Gallart et al. 2017). The solution plan has been the key to identify differences between the productions of students with experience in modelling with respect to novices.

The solution plan is formed by two components (that are dependent on each other): an *emerging model* that corresponds to the conceptual component of the model, and a *solution strategy* that corresponds to the procedural component of the model. In the following we will give a complete definition of these components for Fermi problems. In order to delimit our research, we focus on problems involving estimation of the number of elements in a bounded enclosure.

The emerging model refers to the initial model that includes different assumptions related to the configuration and the distribution in the space (e.g. a carpark) of the elements (e.g. cars) whose number must be estimated. Indeed, when we have to obtain a reasoned estimate of the number of objects that fit into a bounded enclosure (e.g. a porch), the first step is to fix the distribution of the objects (e.g. people) in space. One way to do this is to assume that the elements are arranged in rows and columns; this leads us to reduce the initial problem (of areas) to a problem of lengths. This *one-dimensional emerging model* corresponds to what Albarracín and Gorgorió (2014) call a "*grid distribution model*". Otherwise, the elements can be distributed directly on the surface and this necessarily implies that the solver will argue from the estimated value of the total area using two possible strategies that will be described later. This configuration corresponds to a *two-dimensional emerging model*.

Once an emerging model has been set, it is necessary to use some strategy to obtain the estimated number of elements. The most elementary—but the least efficient—strategy is the direct count. Another way is to argue from the space (area or length) occupied by an element and get the result by dividing the total area (or length) by the area (or length) occupied by an element. This corresponds to the *base unit procedure* established in Gallart et al. (2017). Finally, it is possible to argue from density, estimating the number of elements in a given unit of area (or length) and multiplying this value by the total number of units of area (or length). Different combinations of emerging model and strategy produce different solution plans. Section 10.3 presents a categorisation of the solution plans based on the productions of the students who participated in this experience.

When we present tasks to students it is important to identify which elements of the task can influence the solving process. Kilpatrick (1978) studied and classified the characteristics of a task as possible values of what he called "task variables". Following the definition established in the book edited by Goldin and McClintock (1984):

> Task variable will mean any characteristic of problem tasks which assumes a particular value from a set of possible values. A task variable may thus be numerical (e.g., the number of words in a problem) or classificatory (e.g., problem content area). (Kulm 1984, p. 16)

Kilpatrick (1978) included three classifications of these variables: context variable, format variable and structure variable. In the present work we will focus on *context variable* understood as the physical characteristics of the real context of the task. Since we are interested in those tasks involving an estimation of the number of elements in a bounded enclosure, we identify five context variables: the size of the elements, the size of the area, the elements' shape, the distribution of the elements and the shape of the enclosure.

These variables allow designing a sequence of this type of task in which some values of the variables change. Through this sequence it is possible to analyse whether the values of the context variables foster the choice of a solution plan. A positive answer will allow us to study in further work whether this sequence promotes students to change their solution plan from one problem to another, that is, what Elia et al. (2009) call inter-task flexibility. Therefore, the present work is a first step towards a systematic investigation of flexibility in the framework of real context problem solving.

10.3 Method

In this section we will describe the methodological design of the experience in three parts: description of the sample of students who participated in this experience, justification of the design of tasks and the procedures used, and the data analysis.

10.3.1 Sample

The experience was developed throughout the academic years 2017–2018 and 2018–2019. The $N = 224$ participants were students in their last year of the Degree in Primary School Education at the University of Valencia (Spain). This is an incidental sample that includes 25% of the total population: prospective teachers in the last year of their formation in the biggest Faculty of Education of the region of Valencia. The choice of conducting the research with prospective teachers is based on the fact that subject-related teacher competencies have a strong influence on students' performance (Baumer et al. 2010).

10.3.2 Procedure and Tasks

First, we designed a sequence of problems that request students to describe a solution plan to obtain a reasoned estimate of the number of elements in a bounded enclosure. The criteria for the design were the following: the sequence should include four problems; all problems consist of obtaining an estimate of a number big enough that it cannot be effectively solved by counting; all problems are contextualised in the immediate student environment.

Since our objective is to identify if there is a relationship between the context variables of the tasks and the solution plans, it is important to clearly identify the variables and the possible values they can take. As we have already remarked in the theoretical framework, when a task consisting of estimating the number of elements in a bounded enclosure is posed, the following *context variables* and their corresponding values are identified:

- size of the elements: big (more than 1 m^2), medium (between 1 cm^2 and 1 m^2) or small (less than 1cm^2);
- size of the area: big (about 100^2 m^2), medium (about 10^2 m^2) or small (about 1^2 m^2);
- shape of the elements: homogeneous or heterogeneous;
- distribution of the elements: there's a regular pattern, there's no regular pattern;
- shape of the enclosure: the enclosure can be a simple shape (rectangular, triangular, …) or it can be the combination of different simple shapes.

Since we only want to set four tasks, we will fix some values of the context variables and some will not be considered. The variable "shape of the enclosure" has been set as rectangular in all the problems of the sequence. Aware that this implies a limitation of the scope of the study, this decision has been taken to simplify students' calculations and to be able to observe, in this case, the influence of the other variables in the choice of the solution plans. In Table 10.1 we present the combinations considered with respect to the sizes of the elements and of the area. We have shaded the problems where there is homogeneity in the shape of the elements, and we have used bold letters in those in which the elements are arranged following a regular pattern. The four problems are:

P1-People. How many students can stand on the faculty porch when it rains?
P2-Tiles. How many tiles are there between the education faculty building and the gym?
P3-Grass. How many blades of grass are there in this space?
P4-Cars. How many cars can fit in the faculty parking?

Table 10.1 Combinations of context variables considered in the sequence design

Area size	Element size			
	Less than 1 cm^2	1 cm^2 to 1 m^2		More than 1 m^2
About 1 m^2	P3	XXX		XXX
About 10^2 m^2	XXX	P1	**P2**	XXX
About 100^2 m^2	XXX	XXX		**P4**

Note A shaded cell indicates homogeneity of the elements in the problem. Bold indicates elements are in a regular pattern

During the experience, we provided each student with the written problem statements and a small image for each one. We allowed a half page blank space for each problem so that the students could write down their solution plan. One of the researchers was present during the experience at each group classroom. The students worked individually in the usual classroom. The working session lasted 45 min. During the first 10 min, it was explained to the participants that they were going to face a sequence of four tasks. The following aspects were emphasised: in each problem they should raise a possible solution plan indicating the measures they would need to obtain the estimation; the work should be done individually; they should explain their procedures in written form and may use drawings or diagrams; and, finally, they were not expected to obtain a solution but rather only to explain how to get the requested estimate.

The experience included a second part in which the students, in groups, had to choose one of the solution plans proposed in the first part and, taking data in situ, they had to carry out the strategies. In this second part of the experience, the mathematical work, interpretation and validation phase were dealt with. However, for the present study we will focus only on the data collected during the first part because our aim is to study the relationship between solution plans and context variables.

10.3.3 Data Analysis

The data analysis has two phases: first a qualitative analysis was done and then we conducted the quantitative one. In this section we will first describe the criteria and the procedures of the qualitative analysis. The collected data were qualitatively analysed at the end of each academic year. Following Van der Zee and Rech (2018), we consider that interpretation of qualitative data depends on the stances adopted by the researchers before the analysis. Therefore, in order to ensure use of fixed criteria for the qualitative analysis of the productions, we split the codification process into two phases. For the 2017/18 academic year productions, one researcher made a first analysis followed by a revision of the other two researchers and by discussion in case of discrepancies. For the 2018/19 academic year productions, the analysis was done directly by pairs. We have classified students' productions in five categories. We illustrate some categories with transcriptions of students' answers to *P1-People,* How many students can stand on the faculty porch when it rains?

Incomplete resolution: is the one where not enough detail to obtain the estimate is given as shown in this student example:

We need to know the size of the porch as a whole. We would have to measure the width and the length to be able to obtain the total square metres.

Counting: In this case, students just propose a direct exhaustive counting procedure to get the estimation.

Linearisation: This corresponds to the productions that propose a one-dimensional emerging model. For example, a student wrote:

> This problem can be solved by using the width and length of the porch. Once we have these measures, we take another, which corresponds to the measure of a person. Imagine that a person occupies about a half metre, with this measure we can know how many people fit in each row and multiply by the number of rows that can be made in total. Example: in each vertical row → 30 people. 30 × 120 rows → 3600 people.

In this case, the student assumed that people stand up in rows and columns, thus we consider that he based the resolution on a one-dimensional emerging model. Regarding the strategy, the student used a base unit procedure. Nevertheless, in the present work, all the written productions based on a one-dimensional emerging model will be in the same category regardless of the strategy used.

2D-Base unit: This corresponds to the productions based on a two-dimensional emerging model. In all the cases, students proposed, from one side, to obtain the total area of the rectangle, and from the other side, to obtain the estimate by dividing this area by the area occupied by an element. For example, a student wrote:

> First of all, with the measurements of width and length, I would calculate the space inside. Then I would calculate the space occupied by one person. Finally, you would get the number of people by dividing the total measure by the measure of one person.

2D-Density: this corresponds to the productions based on a two-dimensional emerging model. In all the cases, students proposed, from one side, to obtain the total area of the rectangle, and from the other side, to obtain the estimate by multiplying this area by the estimated density. A student example is:

> To begin with I would measure the width and length of the covered porch, then I would change to square metres. Thirdly, I would measure several times how many people fit in a square metre. Then I would take an average and multiply it by the square metres of the covered porch.

Once the qualitative analysis of the students' productions has been carried out, we proceed to count the number of productions in each category for each problem. From the contingency table, we perform an inferential statistical analysis to determine whether there is a significant relationship between the categories identified in the solution plans and the problem variables. In the following section we will show the results of these analyses.

10.4 Results

Table 10.2 contains the absolute frequency and the percentage of use for each solution plan for each problem.

In order to determine whether there is a statistical relationship between the context variables of the problems and the categorised solution plans proposed by the students, we have performed an inferential analysis based on the Chi-Square Test for independence ($df = 12$, $N = 896$). We have assumed as null hypothesis that there is no relationship between the context variables of the problems and the categorised

Table 10.2 Classifications and frequency of student productions for each problem ($n = 224$)

Task	Incomplete	Counting	Linearisation	2D-base unit	2D-density
P1 People	34 (15.2%)	1 (0.4%)	28 (12.5%)	110 (49.1%)	51 (22.8%)
P2 Tiles	34 (15.2%)	6 (2.7%)	92 (41.1%)	71 (31.7%)	21 (9.3%)
P3 Grass	44 (19.6%)	2 (0.9%)	15 (6.7%)	67 (29.9%)	96 (42.9%)
P4 Cars	28 (12.5%)	4 (1.8%)	31 (13.9%)	160 (71.4%)	1 (0.4%)
Total	140 (15.6%)	13 (1.5%)	166 (18.5%)	408 (45.5%)	169 (18.9%)

solution plans. We fix $\alpha = 0.001$, and the test gives us a result for $\chi^2 = 269.92$ and p-value $= 0.000$ that led us to reject the null hypothesis. Since Chi-Square Test for independence may not provide a reliable guide to measure the strength of the statistical relationship between the variables, we used a Cramers's V (see Acock and Stavig 1979). A value of Cramer's V close to 1 means that the relationship between the variables is very strong and if it is close to 0, the relationship is very weak. In this case $V = 0.31$, which is considered a medium to large effect (Leppink 2019).

10.5 Discussion and Conclusions

Inferential analysis indicates that there is a statistically significant and moderate relationship between the context variables of the tasks and the components that have enabled us to categorise the solution plans: emerging models and strategies. Based on the DISUM model of the modelling process (Blum 2011), in this experience students are working on constructing the situation model, simplifying/structuring the situation model in order to obtain the real model of the problem and, finally, mathematising. In the second step of this process the identification of variables of interest is crucial (see, for instance Houston 2007). This process is deeply related to construction of the emerging model.

From Table 10.2 we can infer some effects from the four context variables considered in the sequence design (as shown in Table 10.1) in the construction of the emerging model and, particularly in the identification of variables. The regularity in the *distribution pattern of the elements* whose number is being estimated increases the occurrence of one-dimensional emerging models (i.e. linearisation): we observe that this happens when the regular arrangement pattern of the elements is evident (as in the case of the *P2-Tiles* problem). Indeed, although the percentages of one-dimensional and two-dimensional (i.e. base unit and density) emerging models are equal in this problem, the proportion of one-dimensional emerging models is significantly higher than in the other problems. However, in problem *P4-Cars* only 14% of productions are based on a one-dimensional emerging model. In fact, although the distribution of elements could be regular, it is not as evident an assumption for students as in problem *P2-Tiles*: in the parking there are large empty spaces and the sizes of vehicles are irregular. Perhaps, in this case the students do not consider that the value of the variable "distribution" can be a "regular pattern". On the contrary, the irregular distribution of the elements fosters, as we see in problems *P1-People* and *P3-Grass*, emerging two-dimensional models.

In two-dimensional emerging models there is another context variable that is identified by the students and, therefore, leads to particular solution strategies: the *size of the elements*. This variable will influence the mathematisation phase of the modelling cycle. Indeed, in the problem *P4-Cars* we find a significant proportion of solution plans that include the strategy that corresponds to the base unit procedure. In fact, considering the average size of a vehicle, it is more natural to argue from its dimensions (in this case, the estimated area) than from the number of vehicles that fit into a given area. Moreover, the high proportion of solution plans that include the density strategy in the *P3-Grass* problem confirms that it is a more natural strategy than the base-unit when the size of the elements is small.

Nevertheless, in order to confirm whether relative size of the elements also has effects on the strategies associated with the one-dimensional model, it would be convenient to analyse in more detail the productions categorised here as "Linearisation" or even to design an alternative sequence with problems involving lengths and not areas (e.g. estimating the number of students needed to surround the perimeter of the yard or finding the number of cars parked along an avenue). This, together with the fact that we have only focused on the first phases of the modelling process, is a limitation of the present work.

Although there is a significant relationship between the context variables of the problem and the categorised solution plan, in almost half of the analysed productions the students posed a two-dimensional emerging model associated with the base unit strategy. This, together with a high ratio of incomplete solutions, leads us to suggest a possible reason might be students' flexibility, that means: to what extent students know different solutions for this type of modelling task and are able to adapt them according to the context? This requires further investigation.

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Chapter 11
Generating a Design and Implementation Framework for Mathematical Modelling Tasks Through Researcher-Teacher Collaboration

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Abstract How to support student in applying the mathematical modelling (MM) process is an ongoing line of research enquiry. This chapter outlines interim findings from an Australian national project that aims to promote effective teaching and learning practices in MM through attention to implemented anticipation. This effort gained focus through attention to the generation of a Design and Implementation Framework for Modelling Tasks (DIFMT). The DIFMT was the result of collaboration between teachers and researchers aimed at the effective design and implementation of MM tasks in upper secondary classrooms. The study suggests that specific pedagogical practices can act as enablers of students' attempts to appropriate the process of MM.

11.1 Introduction

In keeping with a number of countries, Australia has been stressing the importance of equipping students to apply their mathematics in real-world settings (e.g., ACARA 2015). Such abilities are necessary for (1) successful participation in other school subjects where the use or interpretation of models is important; (2) gaining access to mathematics, science, technology and engineering (STEM) careers or other professions based on applied mathematics (e.g., economics); and (3) for informed participation in personal, civic and work life. In this chapter we outline our efforts to

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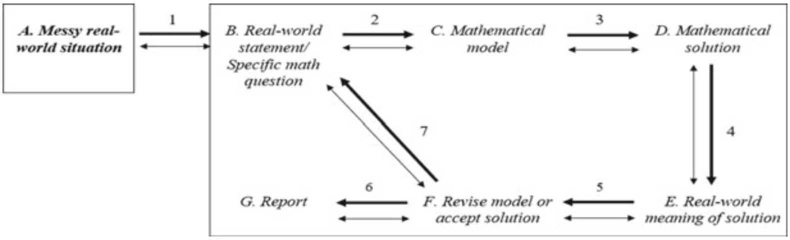
address these needs within a curriculum context in which mathematical modelling (MM) is a mandatory element of mathematics assessment within the final years of schooling (Years 11 and 12). Despite the requirement that Years 11 and 12 students engage with MM, experience and expertise in instruction for this element of the curriculum is varied—from very capable designers of MM tasks through to novices. Even among those that were capable task designers, we found a dearth of expertise in the implementation of MM activities. Our response to this theory/practice gap has been to work in collaboration with teachers to develop effective principles for instruction embedded in a Design and Implementation Framework for Modelling Tasks (DIFMT) within a nationally funded project. Central to the development of this framework was an understanding that the capacity to anticipate, is an essential meta-cognitive facility in both the deployment of the modelling process by students and teachers' capability with its instruction. Consequently, the aims of the project are to:

(i) describe the nature of anticipatory metacognition and identify and describe the enablers necessary for students to translate real-world situations into successful mathematical models;
(ii) design modelling tasks that support the development of students' anticipatory metacognition, and/or allow for the identification of issues that are problematic for that development;
(iii) develop, trial, and refine teaching practices that support the growth of students' anticipatory metacognition while working on effective modelling tasks.

In the section which follow, we focus on the theoretical perspectives that underpin the DIFMT and describe other enablers of MM which emerged when teachers attempted to align their instructional practices with this framework. Evidence for the efficacy of these enablers are drawn from teachers' commentaries on their implementation of tasks.

11.2 The Nature of Mathematical Modelling

Given the plethora of interpretations within the field of modelling in education we provide clarification of our meaning of the term. Consistent with statements in the opening paragraph, we are concerned to nurture qualities that enable students to apply mathematics to solve problems in domains outside itself (see Niss et al. 2007, p. 4). In the following we outline sequential stages in the modelling process; as an analytical reconstruction of a modelling/problem-solving process, remembering it is neither a lock/step approach, nor a detailing of moves made by individual modellers. In the diagrammatic representation below (Fig. 11.1a), the heavy clockwise arrows (1–7) depict the modelling process as a problem-solving activity, connecting stages (A–G). The double headed arrows indicate that in pursuing a solution there will be intermediate transitioning/revisiting, within and between any of the stages. This will a



b

1. *Identify a (real-world) problem* 2. *Specify related mathematical question(s)* 3. *Formulate a mathematical model to address the question (involves making assumptions, choosing variables, estimating magnitudes of inputs etc.)* 4. *Solve the mathematics*	5. *Interpret the mathematical results in terms of theirreal-world meanings* 6. *Make a judgment as to the adequacy of the solution to theoriginal problem(s)* 7. *Either report success with recommendations, or makeadjustments and try for a better solution*

Fig. 11.1 **a** Representation of the modelling cycle (Galbraith 2013), **b** Transitions between modelling phases (Galbraith 2013)

include meta-cognitive and anticipatory activity. (These arrows are incomplete for clarity—they potentially connect any of the stages).

11.3 Anticipatory Metacognition

Implemented anticipation, as formulated by Niss (2010), is a process by which students anticipate and carry out within the act of modelling: (a) actions that they perceive as potentially useful context-wise and mathematically in subsequent steps; and (b) decision making that brings those steps to fruition. *Implemented anticipation* is central to a modeller's ability to mathematise and to undertake the mathematical processes entailed, and then complete a modelling problem successfully.

The term *anticipatory metacognition* describes an associated construct that also includes the additional capabilities of 'modelling oriented noticing' and strategic planning, e.g. with regard to seeking and gathering information and data and deciding whether to involve statistical analyses of the data collected. This applies before and during a modelling experience. It represents the capacity to recognise possible avenues to pursue during the modelling process when engaging with an unstructured real-world problem by taking cues from progress made in other contexts and situations. Both require an ability to think forward and are applicable to learners and teachers.

For teachers it represents thinking along the lines "Where, in the modelling process, will this group of students be likely to encounter obstacles? And what can/should I do to help them move forward?" It involves reflecting on student thinking as intermediary to the problem itself. Resulting prompts direct students to use the modelling process to resolve an impasse, rather than giving direct hints as to the solution itself.

11.4 Anticipation and Modelling

Because modelling proceeds through ideal–typical stages, an attribute for success is the ability to look forward and to anticipate what may be needed at a later point in the process, requiring that the modellers project themselves into subsequent modelling steps before taking them; and implement such anticipation throughout the modelling process (Niss 2010; Niss, Martin 2017; Jankvist and Niss 2020).

Implemented anticipation as an essential component of *anticipatory metacognition* pertains to all necessary steps in the modelling process: pre-mathematisation (e.g., posing questions, assumptions, simplifications), mathematisation, mathematical treatment, interpretation, and model evaluation. This capability is significant for individual modellers, but also for teachers and mentors, who seek to promote the development of modelling abilities in their students. Examples are listed below:

- Anticipating features that are essential in mathematising a feasible problem from the real situation being currently considered; anticipating mathematical representations and mathematical questions that, from previous experience, or present analysis, seem likely to be effective when forming a mathematical model.
- Thinking forward about the utility of the selected mathematisation and the resulting model to provide a mathematical solution to the questions posed.
- Thinking forward to identify related problems and refinements that are suggested by progress. Some of these may not have been thought of at the outset of the problem.

11.5 Enablers of Implemented Anticipation

Enablers of implemented anticipation, developed previously (Niss 2010), were directed specifically at features central to developing individual modeller capabilities. Their European origins paid attention to contexts where the worth of modelling could not be taken for granted, for example, where only pure mathematics is considered an approved subject for study by the education system, or by students. Australia has a history within which applied mathematics has occupied an accepted role. However, ways in which respective preferences (e.g., pure versus applied) impact on teaching and learning remain a continuing influence. In theoretical terms these are impacted by considerations of socio-mathematical norms (e.g., Yackel and Cobb 1996) and didactical contracts (e.g., Brousseau 2002). Bearing in mind the Australian context, adaptations of Niss' original modelling enablers (ME) have been developed and an

Table 11.1 Niss' enablers adapted for Australian contexts

ME1: (Adapted for Australia): Students believe that the inclusion of modelling activities is a valid component of mathematical coursework and assessment
ME2: Students possess mathematical knowledge able to support modelling activities (e.g., possess mathematical knowledge and skills, and ability to manage abstraction) **ME3**: (Additional): Students possess an understanding of a systematic modelling process that includes successive stages from problem question to model evaluation **ME4**: Students are capable of using their mathematical knowledge when modelling. (This implies a core understanding of and engagement with the modelling process (Formulate, Solve, Interpret, Evaluate) so that the right questions can be asked and pursued systematically) **ME5**: Students have perseverance and confidence in their mathematical capabilities (e.g., continue to follow through, or try new directions within a problem if necessary)

Table 11.2 Implementation enablers

IE1: The mathematical demand of problem tasks does not exceed the mathematical capabilities of the student group
IE2: Problem tasks are introduced so as to engage the students fully with the task context, while ensuring that goal of the task is understood **IE3**: Assistance provided during modelling sessions (measured responsiveness) is geared to helping students use the modelling process to reach a solution, rather than treat a problem as an individual exercise **IE4**: Students are encouraged/required to organise and report their work using headings/sections consistent with the modelling process **IE5**: Productive forms of collaborative activity are used to enhance and hold to account the quality of on-task progress. Effective use of digital technologies. Students' interest in a problem

additional enabler, to do with knowledge of the modelling process, has been added to the original set of modelling enablers—ME3 (Table 11.1).

In terms of the project, the centrality of effective implementation means that *teaching (or implementation) enablers* (identification and description) have been added to the originals that were directed at enhancing the modelling process itself. See Table 11.2. In reviewing the developing enablers framework, after initial classroom observations, we became aware of factors, that while not exercising a gatekeeping role, could facilitate (or not) the success of modelling activities. We have designated them Catalytic Enablers (IE5).

11.6 Approach to Developing the DIFMT

The project has been conducted over a three year-period. Data for this chapter are drawn from the engagement of three teachers from different schools and one class of their students per year (Years 9–11). The project coincided with a time of curriculum revision which included new course content and greater scrutiny of assessment practices, including a component devoted to MM. Two of the teachers had extensive prior experience in developing and implementing modelling tasks, while the third had only superficial familiarity.

The research design was based on an iterative process of design-implement-reflect as the basis for researcher/teacher collaboration in developing the DIFMT. This process was effected through three whole-day researcher/teacher meetings and two classroom observation visits per year. Classroom visits took place between researcher/teacher meetings. The purpose of researcher/teacher meetings was to: develop MM tasks; plan for their implementation in classroom; reflect upon the design of tasks and their implementation after each successive round of implementation; draft and refine the DIFMT. Classroom observation visits were conducted to generate data related to the effectiveness of: tasks, for specific classroom conditions; and teachers' approaches to task implementation. Initial tasks and advice on implementation was provided by researchers, with teachers becoming increasingly involved, moving towards autonomy, in the development of principles for the design of tasks and their implementation—leading to the drafting and successive refinement of the DIFMT as the project unfolded [for detail of this approach see Geiger et al. (2018)].

Data collection methods included video-recorded classroom observations of small groups of students during observation visits, teacher pre- and post-lesson interviews, student post-lesson interviews and student video-stimulated recall sessions following each visit. Students who were likely to articulate their approaches to a task clearly and without a sense of reserve were invited to participate in both video and interview sessions on the basis of teacher advice.

11.7 The DIFMT

In this section we provide an outline of the DIFMT. Word limit prevents a full discussion of its development; thus, the purpose of the following description is to provide the reader with sufficient background to link the DIFMT to implementation enablers for which we provided illustrative excerpts.

The DIFMT consists of three overarching structural dimensions—*Principles for modelling task design*, *Pedagogical architecture*, and *Completion* under which sit defining elements and their descriptions. While this chapter focuses on the Pedagogical Architecture dimension of the framework, a condensed version of the whole is presented in Table 11.3.

The dimensions and defining elements of the DIFMT are aligned with the *implementation enablers*. For example, IE1, which relates to the articulation of students' mathematical capabilities and the embedded challenge within a problem, is an important element of *task design*. The students' introduction to a problem (IE2) requires careful attention during the *pre-engagement/initial problem presentation* phase. The type of assistance students should receive when engaged with a problem (IE3) is captured in the *body of the lesson* descriptors. Responses to a problem will need to be reported in a structured manner (IE4), as outlined in the *completion* element of the DIFMT. Productive collaboration (IE5) is seen as a catalytic enabler and is also included in the *body of the lesson* descriptors.

11.8 Emergent Enablers

During task implementation, using the DIFMT as a guide, other aspects that promoted or constrained students' attempts at 'modelling problems' emerged. These

Table 11.3 Integrated modelling task and pedagogy framework

Principles for modelling task design	
Nature of problem	Problems must be open-ended and involve both intra- and extra-mathematical information
Relevance and motivation	There is some genuine link with the real world of the students
Accessibility	It is possible to identify and specify mathematically tractable questions from a general problem statement
Feasibility of approach	Formulation of a solution process is feasible, involving (a) the use of mathematics available to students, (b) the making of necessary assumptions, and (c) the assembly of necessary data
Feasibility of outcome	Solution of the mathematics for a basic problem is possible for the students, together with interpretation
Didactical flexibility	The problem may be structured into sequential questions that retain the integrity of the real situation
Pedagogical architecture	
Pre-engagement	Understand of the modelling process and its application—illustrate what the modelling process. Support materials include a modelling process diagram
Modelling process review	Reviewing pre-engagement as required
Initial problem presentation	• Teacher provides brief general description of the problem scenario • Students organised into small groups and provided with time to read the task description and ask questions of clarification • Students in groups discuss how to approach the problem (including defining a mathematical question?) and report back to whole class via a group representative • Teacher orchestrates discussion of mathematical question(s) towards consensus • Students in groups consider assumptions and variables relevant to the agreed mathematical question. Outcomes reported back to whole class by a group representative • Teacher synthesises/prioritises students' initial assumptions and variables sufficient to begin modelling process for an initial model (As students gain experience teacher scaffolding in this section can be greatly reduced and perhaps eliminated)

(continued)

Table 11.3 (continued)

Principles for modelling task design		
Body of Lesson	Students • Proceed in groups to create model, solve, interpret, etc. in terms of their mathematical question. • Engage in productive student–student collaboration. • Identify and make use of technology where applicable (e.g., source relevant information, check calculations and/or generate solutions) • Develop a report of their progress in terms of the stages of the modelling process (e.g., formulate, solve, interpret, evaluate)	Teachers • Help bring to student consciousness those things that are implicit • Activate teacher meta-meta cognition: (a) How will the students be interpreting what I as a teacher am doing/saying at this point? (b) What should the students be asking themselves at this point in the modelling process? • Structure mathematical questions that promote a viable solution pathway • Support students with making progress through the modelling process • Anticipate where students might have problems, e.g., interpreting the problem • Employ measured responsiveness—rather than providing specific advice about the problem, students should be prompted to think about where they are in the modelling process • Encourage the use of tools (digital or other) • Support student progressive development of a report (e.g., guidelines on report writing)
Completion		
Present findings and summary	• A representative from each group shares their findings with justification. Findings should be reported in a succinct fashion (e.g., 3–4 min video) • Teachers/students ask questions of clarification or to test arguments	
Report	• Students communicate their findings via a succinct, coherent, systematic report. The report must make use of appropriate mathematical language • Teacher checks for the validity of the solution and supporting justification	

included: actions related to teachers' personal engagement with a modelling task and its implementation; influences upon the teaching/learning environment (e.g., socio-mathematical norms and/or aspects of the didactical contract); and teachers' own anticipatory actions. We now present illustrative examples of such emergent enablers—supported through references to teachers' comments recorded during interviews that followed task implementation sessions.

11.8.1 Core Teaching Enabler: Utilising the Modelling Process

It became apparent that teachers' thorough understanding of both the modelling process and the detail of any modelling problems they implemented was fundamental to their students' success in modelling. Teacher A was adamant that the modelling process must be understood by teachers themselves if instruction was to be effective.

> Teacher A: [Teachers need to] go through the framework. Not just the problem but the process itself.
>
> Teacher B comments on the importance they placed on developing a thorough personal understanding of a problem before implementing it in their classroom.
>
> Teacher B: It was actually quite challenging for me to figure out exactly what I would do. I spent a fair bit of time researching.

11.8.2 Learning/Teaching Environment

The degree to which teachers took advantage of opportunities to engage their students with modelling tasks was influenced by their perception of factors that shaped classroom socio-mathematical mathematical norms and/or the didactical contract. For example, teachers perceived both opportunities and constraints related to their state-wide curriculum context. This perception inhibited or provided encouragement for how often they were prepared to implement tasks. Comments by Teacher C indicate he saw the demands of a new syllabus as limiting his opportunity to engage students with modelling activities because of expectations about developing student mastery of content objectives in a limited period of time. This was despite a strong emphasis in the syllabus on mathematical modelling.

> Teacher C: We don't do [modelling] as much as we used to…because we just don't have time. The new syllabuses just don't allow that sort of stuff.
>
> Teacher B, working within the same curriculum context, saw no such impediment.
>
> Teacher B: I think it's a good task for Year Ten because we do all that volume and money exchange too, there's a little bit of that… It's good for Methods [Year 11] and General Maths [Year 11].

These differing commentaries on opportunities to implement modelling tasks point to in-school expectations about which aspects of mathematics should be prioritised—in this case, fluency with mathematical techniques versus open-ended mathematical learning experiences in the form of modelling tasks. How the influences of curriculum requirements are perceived can become manifest as school specific socio-mathematical norms and the didactical contract that, in turn, trickle down to student expectations of what should take place during mathematics instruction—their interpretation of the didactical contract. Thus, such influences can act as enablers or dis-enablers of student opportunity to engage with modelling tasks. Another interesting observation was that some of the teachers tended to scaffold students' work rather tightly by teaching them what to do and how to do it, thus extending traditional mathematics teacher behaviour to contexts where this is likely to impede students' independent modelling work—thus another potential dis-enabler.

11.8.3 Teacher Anticipatory Capability

Also emergent from classroom observations was the importance of teachers' own anticipatory capabilities as these related to looking forward into a lesson to where students might experience difficulties or blockages. This form of anticipation enabled teachers to plan for how to scaffold students' modelling efforts in a measured but effective fashion. For example, Teacher A anticipated that some students might find challenge in the selection of essential information from a larger list.

> Teacher A: It will be interesting to see if they can pick out that information from the table that's there. I think that will be a stumbling point for some of them …And they might be seeking a little bit of clarification there.

> Teacher A did not see this challenge as a negative experience for students but rather an enabler of their development as modellers provided adequate support was in place—thus reinforcing the important role of their own anticipatory capability.

> Teacher A: I think that students need a bit of struggle and challenge…but with bringing them back together and just getting that clarification before we go on, I think then they'll be right, and they'll run with it.

11.9 Conclusion

This chapter reports on interim findings from a national project, conducted in Australia, that aims to promote the effectiveness of both teaching and learning in mathematical modelling through a focus on teachers' and students' anticipatory capabilities. Both teacher and student practices, as syntheses of previous scholarly work or observed during initial implementation phases of the project, are represented in the form of the DIFMT—developed in an iterative fashion as a collaboration between teachers and researchers. Identifying other enablers or dis-enablers of students' opportunities to learn to model is ongoing. These include factors such as teachers' preparatory practices before engaging students with modelling, socio-mathematical norms and the didactical contract, and the development of teachers' own anticipatory capabilities. Our future work, within this study, will continue to focus on the identification of enabling factors, related to both students and teachers, that promote or inhibit students' efforts to employ mathematical modelling effectively when solving real-world problems and in particular those that impact on the pre-mathematisation and mathematisation phases of the modelling process.

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Chapter 12
Pre-service Mathematics Teachers' Technological Pedagogical Content Knowledge: The Case of Modelling

C. Guerrero-Ortiz

Abstract In this chapter, the process of modelling task design for teaching mathematics in digital environments developed by secondary school mathematics pre-service teachers is examined. A way to visualise and integrate modelling and Technological Pedagogical Content Knowledge (TPACK) into an analysis framework is demonstrated to describe pre-service teachers' knowledge using an empirical study. This is followed by a qualitative case study highlighting the relationships that emerged between the modelling processes adopted by pre-service teachers while designing a task and their knowledge in relation to content, technology, and pedagogy. Findings yielded by this investigation deepen current understanding of pre-service teachers' knowledge and development of resources to support the integration of modelling and technology as a part of teaching practice.

Keywords Technology · Mathematical modelling · Technological pedagogical content knowledge · Simulation · Dynamic geometry software

12.1 Introduction

In recent years, use of digital tools in all life domains has increased dramatically, transforming the way in which we carry out daily activities (Santos-Trigo 2019). This shift is evident in the school context as well, necessitating that teachers provide instruction using different types of technologies for different purposes (Santos-Trigo 2020). To expand the opportunities for mathematical modelling in the classroom, it is essential that teachers know how to take advantage of modern technology to promote learning processes. In particular, when teaching modelling, teachers need to identify the types of tasks that promote modelling activity (Maaß 2010) and the processes that can be developed (Borromeo Ferri 2006). They also need to be able to modify modelling lessons in line with anticipated student difficulties (Borromeo Ferri 2018). The role of the teacher for teaching mathematics in environments permeated by

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technologies has been explored from different perspectives (Drijvers et al. 2010). Thus, the aim of the research reported in this chapter is to gain a better understanding of pre-service teachers' technological pedagogical content knowledge (TPACK), and to enhance the learning of mathematical modelling in digital learning environments. To meet this aim the research question guiding this study is: What elements of mathematics pre-service teachers' TPACK are present when designing modelling tasks?

TPACK is the knowledge that teachers must possess in order to effectively use technology while teaching (Koehler and Mishra 2009). In particular, for the case of mathematics, Guerrero (2010) observed that this involves the management of technology, instruction, and pedagogical knowledge. She thus identified some key components of TPACK for the use of technology in the mathematics classroom, namely Conception and Use of Technology, Technology-Based Mathematics Instruction, Management and Depth, and Breadth of Mathematics Content. More recently, Koh (2019) examined the type of knowledge needed to support the design of technology-integrated lessons for mathematical inquiry with authentic problems. However, although some aspects associated to teacher's knowledge in the case of modelling and technology with teachers in service have been explored (Brown 2017), studies into pre-service teachers' TPACK in the case of mathematical modelling are less prevalent. This gap is addressed in the present study. For this purpose, mathematical modelling is understood as a process involving repeated transitions between reality and mathematics (Borromeo Ferri 2018). Special attention is paid to modelling as an instrument for analysing the processes that future teachers adopt when designing a modelling task. This is shown with an example of how the TPACK framework and the modelling cycle can be integrated to study the knowledge of pre-service teachers of mathematics.

12.2 Modelling, Technology, and Teacher Knowledge

In pertinent literature, some elements related to the use of technology that teachers must take into account for the teaching of mathematics are defined, including types and uses of digital tools, class management in a digital environment, affordances and constraints, digital tools as mediators of learning, and beliefs about the use of technology (Guerrero 2010; Santos-Trigo 2019). However, there is little empirical evidence with respect to the affordances and constrains of using technology in modelling activities, and some questions remain unanswered (Borromeo 2018), even though the benefits of introducing modelling and technology in pre-service teacher education are widely recognised (Villareal et al. 2018).

In mathematics classrooms, modelling has been positioned from two different but complementary perspectives—modelling as a content to be taught, and modelling as a means of learning and developing mathematical skills. Thus, modelling can be considered as a didactic strategy or as a mathematical practice. In any case, it has become part of many national curricula, and teachers are responsible for its implementation. In extant studies, development of the competences that a teacher must possess to teach modelling effectively has been addressed through four dimensions (Borromeo Ferri and Blum 2010): *theoretical dimension*, *tasks dimension*, *instruction dimension* and *diagnostic dimension*. The first dimension considers knowledge about modelling cycles, goals/perspectives for modelling, and types of modelling tasks. The second dimension involves the ability to solve, analyse, and create modelling tasks. The third relates to the ability to plan and execute modelling lessons and knowledge of appropriate interventions during the pupils' modelling processes, while the fourth dimension includes the ability for recognising phases in the modelling process, as well as recognising student difficulties and mistakes. Modelling task design is one of the key factors of teacher's knowledge, and different aspects of this knowledge become stronger when creating modelling tasks (Guerrero-Ortiz 2019). In the case of modelling task design in technological environments, Geiger (2017) showed how relationships between student, teacher, task and digital tools become dynamic, requiring the teacher the ability to adapt the task according to the students' solution processes. In this research, the changes to the modelling cycle introduced by the technology (Greefrath et al. 2018) are considered as having a potential impact to keep in mind in modelling task design.

In their study on the influence of technology on modelling processes, Greefrath et al. (2018) considered, in addition to the real world and the mathematical world, a technological world. These authors point out that digital tools are used after mathematical expressions have been translated into a language that is understood by the tool, and then after working in the technological world the results offered by technology are translated back into mathematical language. Moreover, they identified the potential of the use of technological tools in the modelling cycle as a means of better understanding the problem through simplification and mathematisation. Specifically focusing on the case of a dynamic geometry software, Greefrath and Siller (2017) characterised the uses of digital tools when students work on modelling tasks as drawing, visualising, constructing, measuring, experimenting, calculating, and researching. Other researchers have shown how pre-service teachers use technology to find and filter information, and how technology influences the process of mathematical problem-solving and solution validation (Villareal et al. 2018). Although these studies shed light on the interaction of technology and modelling processes, in the present investigation, modelling tasks design is examined from the teacher's knowledge perspective, focusing on the modelling process developed by pre-service teachers when designing a teaching task and on their knowledge about technology, pedagogy, and mathematical content. For this purpose the notion of TPACK is now introduced.

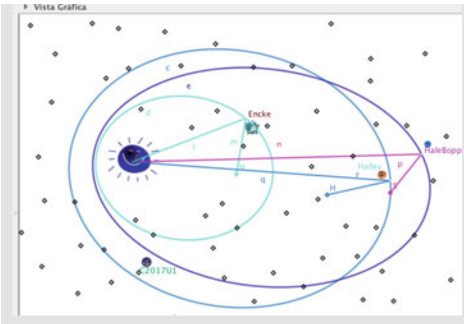
The definitions presented here are based on more general descriptions of TPACK framework offered by Koehler and Mishra (2009), which were refined in the light of the findings put forth by Guerrero (2010) and Koh (2019) to specifically relate them to the teaching of mathematics. *Pedagogical Content Knowledge* (PCK) considers aspects related to the learning of mathematics, such as students' conceptions and what may be challenging or interesting for them. It also includes the knowledge teachers must possess to plan a lesson, along with the consideration of students' previous knowledge, errors and difficulties, different representations of objects and characteristics of teaching tasks. Knowledge of the content (CK) involves knowing mathematical concepts and their definitions, sequencing or nesting of mathematical concepts, proof, demonstration and approaches to the development and generation of mathematical knowledge (Ball et al. 2008). For analysis development, the term EMK has been introduced to refer to extra-mathematical knowledge. *Technological Content Knowledge* (TCK) involves the knowledge and mastery of a variety of technological tools (TK) that can be used to process information, represent and manipulate mathematical objects, solve problems, and interpret and communicate results (Santos-Trigo and Moreno-Armella 2016). It also involves decision-making regarding the ways in which content can be addressed depending on the advantages and disadvantages imposed by the tools. *Technological Pedagogical Knowledge* (TPK) includes knowledge of the pedagogical affordances and constraints of the tools, such as the implications that different tools have for the design and strategies for teaching mathematics in digital environments. The teacher must know which technologies are best suited for learning, and how teaching is modified depending on the choice of technological tools (Santos-Trigo and Moreno-Armella 2016). He/she must also be aware of the ways in which students perform actions such as exploring or building a mathematical object or even how to save information (Koehler and Mishra 2009). TPACK subdomains will be explored in mathematics pre-service teacher's modelling task design.

12.3 Method and Context

To know what elements of TPACK are present when designing modelling tasks, pre-service mathematics teachers were required to design a task for teaching in secondary school (aimed at students aged 15–16 years). This was the final product of a course, where participants should integrate their content and pedagogy knowledge by creating a modelling task. The mathematical content should be freely chosen and related to the curriculum. The participants were in a course, at the third of four years of study, where the author was the teacher. All had a general understanding of problem-solving and modelling strategies and were at an intermediate level in terms of practical experience with the dynamic geometry software, Geogebra, used for completing the task. Moreover, for about 1.5 months, the participants worked in small groups (2–3 members) on the modelling task design, after which each group made oral presentations on their progress. At the end of the course, fifteen groups' written reports were received, along with an electronic file containing the modelling task. In the reports, each group described how the real situation chosen by them was studied, simplified, and adapted to become a teaching task. The lesson plan and a priori analysis of the expected students' answers were also included. In the following section, the work of one group is analysed. This group consisted of three participants, two of them very interested in exploring different forms of teaching, and the other one proficient in the use of Geogebra. In order to better identify the content knowledge, tasks where the mathematical concept to be taught was clearly defined and related to the dynamic configuration given in a simulation were chosen. This was the selection criterion for choosing this group. Written reports, electronic files, and researcher's file notes were the data sources that were subjected to content analysis (Bardin 1986) by the author. For data analysis, the design stages were characterised according to the phases of the modelling cycle of Greefrath et al. (2018). Then, the content of selected paragraphs from the written report was coded according to the subdomains of TPACK (Koh 2019). The analysis of the record resulting from the Geogebra construction processes complemented this coding. In the analysis, focus is given to the initial design formulation, without considering its implementation (Guerrero-Ortiz 2019). Finally, to ensure reliability, the coded and interpretations were discussed and refined in a seminar with a group of researchers.

12.4 Analysis of Task Design

The task analysed here was intended for introducing the concept of ellipse to secondary students. The learning objective was to recognise the ellipse as a geometric locus, and according to the definition, the ellipse is the locus of all points on a plane, such that the sum of the distances to two other fixed points, called foci, is constant. The task was presented in its entirety in the Geogebra environment (Fig. 12.1), where the movement of comets is observed, without initially showing the graph of their respective trajectories. Hence, those that find the solution should communicate their findings, along with their reasoning regarding the movement of the corresponding point (Item 1). In Item 2 (Fig. 12.1), the objective was directed to measuring the distances between individual comets and the Sun, as well as a fixed point, or "focus". Intending to introduce the definition of ellipse as a geometric locus, students should explore when the sum of different distances is constant.



Questions raised:

1. Describe the movement of the comet around the Sun. What geometric locus is it?
2. Is there a relationship between its distance to the Sun and to the focus?

Fig. 12.1 *Movements of Comets.* Task designed by pre-service mathematics teachers

12.4.1 Elements of TPACK Emerging in Task Design

In this section, elements of TPACK that emerged in the task design carried out by secondary school pre-service mathematics teachers are identified. When analysing the written report to obtain information of the design process, their knowledge of physics (EMK) was evident, as the pre-service teachers correctly concluded that the movement of comets is governed by Kepler's laws of planetary motion. The first law states that planets move in elliptical orbits around the sun, while the second law states that the radius vector that connects the planets and the sun sweeps equal areas in equal time periods. In other words, when the planet is furthest from the sun, its speed is minimal, but it is greatest when it is at the point closest to the sun. On the other hand, the definition of ellipse as a geometric locus was recognised in the activity intended for students (PCK). Therefore, the pre-service teachers' knowledge of physics and mathematics is associated to CK and was put into practice when designing the activity in the Geogebra environment. During this process, CK underwent a transformation, emerging as TCK. In relation to this subdomain, in the analysis of Geogebra's record of the construction, three tool domain levels (TK) were identified: basic, intermediate, and technical. Participants that are at the basic level of proficiency recognise the Geogebra interface, know how to use dependent and non-dependent basic objects, and are able to modify properties of the graphical interface. In particular, the participants who designed this task used the *dot* object to represent comets, the Sun, and stars. They used *point on object* to fix the movement on an elliptical path, where the sun is represented by one of the foci. The use of the tool to *measure lengths* was also observed. At the intermediate level of proficiency, the use of buttons to control the animations of several objects simultaneously is expected. In addition, to visually simulate the non-constant speed of movement of the points when they are in different positions on the elliptical path, participants modified *the properties* of the point to *define speed as an inverse* of *the distance* of a moving point on the elliptical path around another fixed point.

Next, when analysing the activity intended for secondary students, it can be observed that the objective and the questions denote that the focus is placed on the exploration of the ellipse as a geometric locus to later relate it to its algebraic representation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. On the other hand, pre-service teachers' knowledge of physics, such as an approximation of Kepler's third law—which postulates that the square of the orbital period T of a planet is directly proportional to the cube of the semi-major axis r of its orbit, $T^2 = k\, r^3$—while used for the construction of the dynamic configuration, was not demonstrated to the students. In other words, models pertaining directly to physics were integrated into the construction of a dynamic simulation, but the learning objective for students was restricted to the introduction of mathematical concepts. The next paragraph, extracted from the pre-service teachers' group report, confirms this observation:

> To simplify the task, we focus on the ellipse [...]. In other words, Kepler's second law was only used in the creation of Geogebra [dynamic configuration] to show that comets move faster near the Sun.

Table 12.1 Classification of the elements observed in the task design

In relation to the simulation	In relation to the expected activity from the students
Content Knowledge Model of Kepler's laws Ellipse definition **Technological Content Knowledge** Drawing an ellipse given two foci Point on object Measure distances Control Defining the velocity of the movement of a point	**Content Knowledge** Ellipse definition **Technological Pedagogical Knowledge** Use of a simulation to introduce mathematical concepts

In relation to the TPK, this task can be pedagogically interesting for secondary students as it would motivate them to visualise different types of movement through technology. However, it is necessary to identify contexts in which such activities would be cognitively enriching for them. Technology, in this task, was used solely as a means to simulate a physical phenomenon and explore related mathematical concepts. Table 12.1 summarises the elements associated with the TPACK subdomains that were identified in the task. The left column shows elements referring to the use of software tools, along with the physical and mathematical concepts identified in the pre-service teachers' task design. The right column shows elements identified in the activity intended for secondary school students.

When this group of pre-service teachers designed a task aimed at introducing secondary school students to the concept of ellipse as a geometric locus in technological environments, elements of their TPACK were evidenced, as previously explained. In order to complement the previous analysis, now is described how the work on the task design highlights their progression through the modelling cycle (Greefrath et al. 2018). Where the CK and EMK could be expanded when the pre-service teachers investigated published literature and online sources to gain insight into the phenomenon to be studied. This strategy allowed them to develop a model of the situation that was subsequently associated with Kepler's laws and the concept of ellipse (CK). With the definitions of these concepts as models from the fields of mathematics and physics, the group built a dynamic configuration to simulate the movement of comets, thereby transitioning to the world of technology. In the world of technology, future teachers experimented with the dynamic configuration to anticipate the actions of students. In this process, aspects related to their TCK, such as the use of software tools to experiment, recognise, describe, measure, and test elements related to the ellipse, were enhanced. In Fig. 12.2, the continuous line shows the path traced by this group in the design process in relation to the modelling phases. Using a dotted line, the activity intended for students is identified within the same cycle, revealing intentions in the promoted activity (Fig. 12.3).

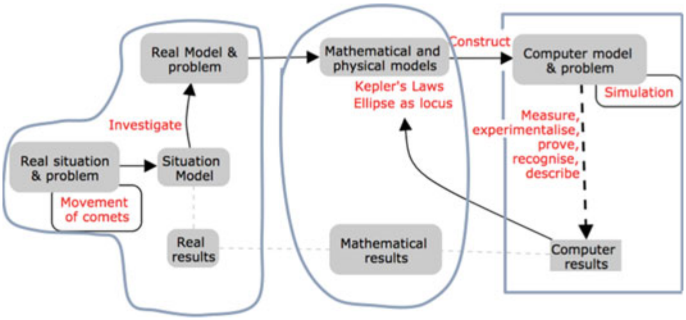
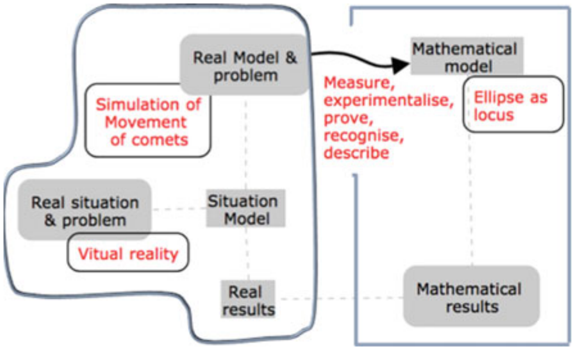


Fig. 12.2 Modelling cycle followed by the pre-service teachers when designing the task

Fig. 12.3 Proposed activity for students



As shown in Figs. 12.2 and 12.3 since the task was designed to be solved in a simulation environment, the activity intended for students starts from a situation created in the virtual world, and the proposed activities are fully developed in the same virtual environment. Pre-service teachers described this as follows:

> We plan that students trace the points associated with comets in order to realise that they follow a fixed path, governed by the relationship between the point representing the comet and the sum of the distance from this point to each focus. Once students conjecture about these aspects, the teacher will explain the definitions and equations.

12.5 Discussion and Conclusion

The extended modelling cycle of Greefrath et al. (2018) allows one to systematise the analysis of mathematical, pedagogical, and technological knowledge of pre-service mathematics teachers when designing a modelling task. In adopting this strategy, it was possible to elucidate how elements of their TPACK are present in the design process. Moreover, pre-service teachers' pedagogical intentions were expressed through the activities intended for secondary students when they expect students to measure the distances between planets and the sun, recognise and describe the locus, experiment when measuring different varying distances and realise that the sum is constant; something they can mathematically prove. Detailed analysis supported on the modelling cycle showed which phases of the design process encouraged pre-service teachers to use different types of knowledge (see a summary in Table 12.2). The findings revealed that, in exploring the situation chosen by them, knowledge about planetary motion emerged (CK, EMK). Then, in the construction of dynamic simulation (TPK), models from physics and mathematics were used as

Table 12.2 Modelling cycle phases and TPACK subdomains

Modelling processes	TPACK	Description
Real situation	TK—Use of technology to search for relevant information EMK—Extra-mathematical knowledge associated with planetary motion PCK—Choosing a situation to motivate secondary school students	Exploration of several real-life situations takes place, one of which is selected for teaching
Situation model	EMK—Extra-mathematical knowledge associated with planetary motion	A mental representation of the situation is formed depending on the PCK
Real model	PCK—Considerations about understanding the situation, such as simplifications, idealisations, and teaching objectives	PCK is built from idealisations influenced by teaching objectives
Mathematical and physical model	CK—Identification of relations between mathematical objects and objects from physics PCK—The mathematical model is built according to the curriculum and teaching objectives	Constituted by the configuration of mathematical objects that represent elements of the situation
Computer model	TCK—Construction of a geometric configuration using the ellipse object; defining the velocity of the movement of a point on object, use of *point on object* and *control* tools TPK—Using a simulation to motivate student learning	Comprises the geometric configuration that simulates the situation
Computer results	TCK/TPK—Measure distance between points for students to conjecture about the ellipse definition	The work with the simulation could allow students a better understanding of the ellipse definition

tools (TCK). However, in the activities intended for students, the focus was restricted to the concept of ellipse as a geometric locus (TPK).

The main contribution of this study stems from the analysis of the modelling process performed by pre-service teachers when designing the task and the activity intended for the students. When pre-service teachers designed the tasks, their mathematical and extra-mathematical knowledge was enhanced, particularly in relation to drawing upon knowledge of other sciences, as was previously shown by Guerrero-Ortiz (2019). In this case, the physics models remained hidden behind the dynamic configuration proposed for the work secondary students were expected to perform, and the objective was to motivate the learners to explore the concept of ellipse through a simulated situation in virtual reality. This observation confirms the need to rethink the learning scenarios included in prospective teachers' training programs to allow them to learn in an interdisciplinary way and better understand how the use of technology influences content design and teaching (Santos-Trigo and Moreno-Armella 2016). The relationships between the modelling phases and TPACK subdomains reveal those elements of the teacher's knowledge that should be improved when designing modelling tasks in digital learning environments. Therefore, it can be a tool for teacher training. Further research with a broader range of data to deepen in other characteristics of TPACK that could appear in different modelling tasks design is required. It is also necessary to explore what secondary students learn about modelling with this type of task, which was a limitation of this study. Finally, further empirical research needs to be developed in order to contrast the subdomains of TPACK stemmed from the design and those that can be seen in the pre-service mathematics teachers' practice.

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Chapter 13
Interest and Emotions While Solving Real-World Problems Inside and Outside the Classroom

Luisa-Marie Hartmann and Stanislaw Schukajlow

Abstract Solving real-world (or modelling) problems outside the classroom can link students' real lives with mathematics on the basis of an authentic experience with the subject matter. This may trigger students' interest and positive emotions and diminish their negative emotions. In the present study, we examined whether students are more interested in and feel more enjoyment and less boredom while solving real-world problems outside than inside the classroom. To answer these research questions, students ($N = 43$) were randomly assigned to two groups, an outside group and an inside group. Our results indicate that location does not influence the development of students' interest and emotions. We hypothesise the importance of authentic problems for students' development of interest and emotions and suggest to examine this hypothesis in future studies.

Keywords Real-world problems · Math trail · Teaching methods · Interest · Enjoyment · Boredom

13.1 Introduction

Interest and emotions are important for students' learning. However, students tend to feel more boredom than enjoyment in mathematic classes (Goetz and Hall 2014), they are often not interested in mathematics, and their interest in mathematics even tends to decrease from grades 5 to 10 (Pekrun et al. 2007). What are possible reasons for these findings? Although mathematics is a part of our everyday lives (Niss 1994) and mathematical knowledge fosters the understanding and development of aspects of diverse extra-mathematical areas (e.g., medicine, pharmacy, architecture, security

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of online banking, or email encryption), students often do not recognise the connection between mathematics and reality. They perceive real-world problems in their lessons as artificial and do not link their everyday or future lives to the contents of mathematics lessons. As a result, students might not be interested in mathematics and might thus feel bored in or fail to enjoy their mathematic classes. In order to strengthen the connection between the real world and mathematics, school tasks sometimes include photographs of real-world objects or videos of real-world situations. However, the typical mathematic class takes place inside the classroom. Given that no one said that mathematics classes must take place inside the classroom, we hypothesised that solving real-world problems outside the classroom as offered by a math trail might motivate students more than solving the same problems inside their regular classrooms. Following these considerations, we aimed to investigate the effects of this teaching method on students' interest and emotions.

13.2 Theoretical Background

13.2.1 Interest, Enjoyment, and Boredom

Interest describes a relationship of a person (e.g., a student) and an object or activity (e.g., solving a mathematical problem) (Hidi and Renninger 2006). Theories of interest have distinguished between situational and individual interest. If the student enjoys solving the problem and values the problem, he or she will experience high situational interest. This type of interest can be triggered by environmental stimuli and can fluctuate from moment to moment (interest as a 'state') (Hidi and Renninger 2006). However, if this situational interest is maintained over time, it can change into individual interest (interest as a 'trait'). Students with a high level of individual interest look for mathematics in their environment, solve mathematical problems in their free time, and discuss mathematical problems with other people (Schukajlow et al. 2017). In the present study, we focussed on task-specific interest (i.e., situational interest) because of its importance for the early stage of interest development. According to the theory of interest, learning environments that provide meaningful activities that have personal significance can trigger students' interest (Hidi and Renninger 2006). Students might perceive solving real-world problems with an authentic experience with the subject matter of their everyday life as a meaningful activity with personal significance, and therefore, this activity might improve students' situational interest.

The construct of interest and the construct of emotions are closely related to each other. Emotions can be described as a complex, multi-dimensional construct that comprises motivational, expressive, physiological, and cognitive parts (Pekrun 2006). In the present study, we focussed on the emotions of enjoyment and boredom because these emotions are two of the most frequently reported emotions in the context of learning (Pekrun et al. 2002). According to the *control-value theory of* *achievement emotions*, an emotion can be activating or deactivating and have a positive or negative valence. For example, enjoyment is a positive-activating emotion. If students enjoy a situation, they will want to continue task processing and will feel happy. Boredom is a negative-deactivating emotion. If students are bored, they will not want to continue task processing and will not like the situation (Pekrun 2006). In the *control-value theory of achievement emotions*, enjoyment occurs with high control appraisals (e.g., high perceived competence in solving a problem) and high value appraisals (e.g., the perceived importance of a learning activity) (Pekrun 2006). Boredom occurs with too high or too low control appraisals and low value appraisals (Pekrun 2006). Control appraisals are too high, for example, if the presented task is too easy for the student, and they are too low, for example, if the presented task is too difficult. Value appraisals are low, for example, if students do not consider task processing to be important for them. An authentic experience with the subject matter while solving real-world problems might improve students' perceived importance of task processing and therefore their perceived value. Thereby this activity might affect their enjoyment and boredom.

13.2.2 Real-World Problems in the Context of a Math Trail

Real-world (or modelling) problems require demanding transfer processes between reality and mathematics. Students begin to solve real-world problems by constructing a model of the situation in the real world. Then they translate this model into a mathematical model and switch from the real world to the mathematical world. After that, calculations can be made in the mathematical world, and the mathematical results have to be interpreted and validated with respect to reality.

Real-world problems are usually complex, open-ended, and authentic (Maaß 2006). The authenticity of a problem can be determined by the presented context or the learning environment. The present study focusses on authentic learning environments because increasing authenticity can strengthen the relation of a problem to the real world (Vos 2015). An example of a real-world problem is *The Climbing Frame* task.

In Fig. 13.1, we present a real-world problem that can be offered to students in the classroom. To solve a problem with missing information (also called a Fermi-Problem), students must notice the missing information and make realistic assumptions, including identifying and supplementing the missing quantities (Krawitz et al. 2018). Photographs or videos can be helpful for estimating the missing information and can make the relation between the problem and the real world more obvious. However, real-world problems can be offered not only in the classroom but also outside. Kleine et al. (2012) suggested that working on real-world problems outside the classroom is more motivating than working with photographs or videos in the classroom. A possible explanation could be that the learning environment outside offers an authentic experience with objects in students' environments, and students therefore perceive that the processing of the task is more valuable.

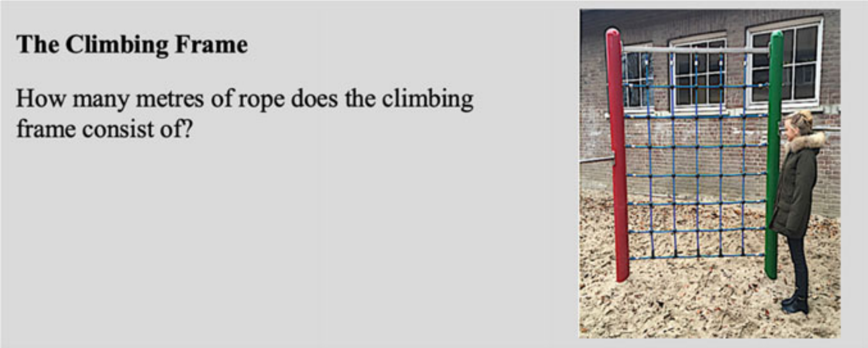

Fig. 13.1 Real-world problem The Climbing Frame

Prior research investigated real-world problems outside the classroom as part of a math trail. Math trails are out-of-classroom activities. In such a trail, students can solve mathematical problems that refer to real objects. Students discover these real problems in their environment as they follow a planned route (Cahyono and Ludwig 2017). This learning environment offers an authentic experience with the subject matter (Buchholtz and Armbrust 2018). Cahyono and Ludwig (2017) showed that students were interested in solving real-world problems along a math trail outside the classroom with the help of the MathCityMap-App,[1] and a study by Buchholtz and Armbrust (2018) revealed that students enjoyed solving real-world problems on a math trail outside the classroom. However, to the best of our knowledge, neither of these studies compared the effects of solving real-world problems inside versus outside the classroom on students' interest or emotions.

13.2.3 Research Questions

To help close this research gap, we aimed to address the following research questions:

(1) Are students who solve real-world problems outside the classroom on a math trail more interested in solving these problems than students who solve the same real-world problems inside the classroom?
(2) Do students who solve real-world problems outside the classroom on a math trail feel more enjoyment and less boredom than students who solve the same real-world problems inside the classroom?

On the basis of the importance of the learning environment for triggering students' interest (Hidi and Renninger 2006) and results from empirical research by Cahyono

[1]The MathCityMap-App is a project from the IDMI of the Goethe-University in Frankfurt. It provides the opportunity to develop interesting tasks concerning objects in reality and to solve them in the form of a math trail (Cahyono and Ludwig 2017).

and Ludwig (2017), we expected that students who solved the real-world problems outside the classroom on a math trail would be more interested in solving the problems than students who solved the same real-world problems inside the classroom. Concerning students' emotions and based on results from empirical research by Buchholtz and Armbrust (2018), we expected that students who solved the real-world problems outside the classroom on a math trail would feel more enjoyment and less boredom than students who solved the same real-world problems inside the classroom because solving problems outside a classroom might improve the value of task processing, which is important for enjoyment and boredom.

13.3 Methodology

13.3.1 Participants and Procedure

To answer these research questions, 50 sixth graders [74% female, 26% male, average age: 11.38 years (SD = 0.49)] from a German middle school took part in this study. The students had no prior experience in solving real-world problems with missing information. On the basis of pretest results, students from each class were randomly assigned to two groups with the same number of students in each group such that the average age, interest in mathematics, ratio of males and females, and average achievement level in mathematics did not differ. The students solved six problems that referred to their school environment in groups of four to five and were given 60 min to finish the tasks (10 min each). Afterwards, they completed a questionnaire about their task-specific interest, enjoyment, and boredom.

The experimental group solved six real-world problems outside the classroom by measuring directly on the object in the real-world. The MathCityMap-App is used in the group to locate the objects in the school environment. As real-world problems solved inside the classroom usually contain photos of the real-world object, the control group solved the same six real-world problems inside the classroom with photos or videos. The problems were presented to the experimental group in the app,[2] whereas the control group used print-outs that were left on tables. In addition, a photo of the object and a hint about the size of the object were located on each table in the classroom. The tables in the classroom were arranged in a learning circle. During task processing, students could fall back on three staged hints. The experimental group could access them in the app, whereas in the classrooms, they were presented on flash cards on the different tables. After task processing, the experimental group entered their result in the app and received direct feedback on its correctness. The students in the classroom compared their results with the result on a flash card. Both groups could then read one solution to the problem—the experimental group in the app and the control group on the flash cards.

[2]For an impression of how the real-world problems were presented in the app, see https://mathcitymap.eu/de/portal/#!/trail/891164.

Table 13.1 Items used to assess task-specific interest, enjoyment, and boredom

Scale	Item
Task-specific interest	Task processing was exciting
	I am already curious about further tasks
	I would like to work on such tasks more often
Enjoyment	I enjoyed task processing
	I was happy during task processing
	Task processing was great fun for me
Boredom	Task processing was boring
	I got so bored during task processing that I had trouble remaining alert
	I did not want to continue my work because it was so boring

13.3.2 Measures

To measure interest, enjoyment, and boredom, we used well-evaluated 5-point Likert scales ranging from 1 (not at all true) to 5 (completely true). Interest was measured with three self-developed items based on a well-evaluated scale used in prior studies (Frenzel et al. 2012) (see Table 13.1). The scale for task-specific interest achieved good reliability (*Cronbach's* α = .88). To measure enjoyment and boredom, we used items from the well-evaluated Achievement Emotions Questionnaire (Pekrun et al. 2011). Each scale included three items (see Table 13.1). The Cronbach's alpha reliabilities were 0.88 for enjoyment and 0.69 for boredom.

13.3.3 Data Analysis

To test the results for significance, we used t-tests for independent samples. We excluded three students with missing values (two students from the experimental group and one student from the control group) and four students with outliers (two students from each group) to avoid distorting the results. Thus, the number of students was reduced to $N = 21$ in the experimental group and to $N = 22$ in the control group.

13.4 Results

13.4.1 Task-Specific Interest

We expected that students who solved the six real-world problems outside the classroom on a math trail would be more interested in the tasks than students who solved the same problems inside the classroom. Table 13.2 presents students' task-specific

Table 13.2 Values for students' task-specific interest

Location		Task-specific interest	
	N	*M*	SD
Outside (EG)	21	4.25	0.92
Inside (CG)	22	4.14	0.65

interest while solving real-world problems inside and outside the classroom.

Both the experimental and control groups reported high task-specific interest. The statistical analysis revealed that contrary to our expectations, students experienced the same level of task-specific interest while solving the real-world problems inside and outside the classroom ($t(43) = 0.46$, $p = 0.646$) and that the location had only a small effect on students' task-specific interest ($d_{\text{Cohen}} = 0.138$).

13.4.2 Enjoyment and Boredom

For students' enjoyment and boredom, we expected that students who solved the real-world problems outside the classroom on a math trail would feel more enjoyment and less boredom than students who solved the same real-world problems inside the classroom. The descriptive statistics concerning students' enjoyment and boredom are presented in Table 13.3 and revealed a high level of enjoyment and low level of boredom in both groups.

Contrary to our expectations, students' enjoyment during task processing did not differ between the experimental and control groups ($t(48) = 0.49$, $p = 0.627$) and the location had only a small effect on students' enjoyment ($d_{\text{Cohen}} = 0.145$). Students' boredom during task processing did not differ between the groups either ($t(47) = -0.67$, $p = 0.491$) and the location also had a small effect on students' boredom ($d_{\text{Cohen}} = 0.210$). Hence, students experienced the same level of enjoyment and boredom while solving real-world problems inside and outside the classroom.

Table 13.3 Values for students' enjoyment and boredom

Location		Enjoyment		Boredom	
	N	*M*	SD	*M*	SD
Outside (EG)	21	3.99	0.94	1.40	0.45
Inside (CG)	22	3.85	0.99	1.51	0.59

13.5 Discussion

13.5.1 Task-Specific Interest

In this chapter, we aimed to analyse how solving real-world problems outside the classroom would affect students' task-specific interest. On the basis of theoretical considerations from interest theory (Hidi and Renninger 2006) and prior research that indicated that students are interested in solving real-world problems outside the classroom (Cahyono and Ludwig 2017), we expected that students would be more interested in solving real-world problems outside the classroom than inside. However, our analysis did not confirm this hypothesis. Students experienced the same level of task-specific interest no matter whether they worked on it inside or outside the classroom. One possible explanation for these results could be that the problems were similar concerning their reference to objects in students' school environment. Although the objects were presented on photographs in the control group, students may have perceived the problems inside the classroom as authentic problems.

13.5.2 Enjoyment and Boredom

Additionally, we aimed to analyse how solving real-world problems outside the classroom would affect students' enjoyment and boredom. On the basis of theoretical considerations from the *control-value theory of achievement emotions* (Pekrun 2006) and prior research that indicated that students enjoy solving real-world problems from a math trail outside the classroom (Buchholtz and Armbrust 2018), we expected that students who solved the problems outside the classroom would experience more enjoyment and less boredom than students who solved the problems inside the classroom. Contrary to our expectations, our analysis revealed that students experienced the same level of enjoyment and boredom while solving real-world problems inside and outside the classroom. Hence, our analysis did not confirm our hypothesis. One possible explanation for these results could be that due to the similarity of the problems, the two processing situations were accompanied by high control and value appraisals. Both processing situations might be accompanied by high control appraisals due to the staged hints and therefore the adaption of the tasks to students' competences. High value appraisals may have been enhanced by the authentic and realistic contexts and the significance of the problems for the students' lives and group work.

13.6 Strengths and Limitations

As we aimed to investigate the effects of working on real-world problems outside the classroom, we posed identical real-world problems with high levels of authenticity in the experimental and control groups. Therefore, interest and emotions may have been influenced by the newness of the problem type in both groups. Further, the experimental group used digital technology, which can also influence students' interest and emotions, whereas control group worked with print-outs in the classroom. The reason for combining "outside" group with technology and "inside" group with print-outs was the external validity of teaching methods and our intention about drawing practical implications from our study. We do not think that the digital technology (MathCityMap-App) decreased the positive effects of working on the problems outside the classroom because digital technology was found to be the prevalent source of students' enjoyment of task processing (Cahyono and Ludwig 2017). The real-world problems in our study were characterised by their relation to students' school environment. However, the results have to be validated for other types of real-world problems. In our study, students worked in small groups of four to five students because group work was found to be preferable for solving real-world problems (Schukajlow et al. 2012). However, the clustering effects could have affected our results because the students in each small group may have influenced each other's perceptions of interest and emotions. Finally, due to the small sample size, our results have to be interpreted with caution.

13.7 Conclusion and Summary

Working on real-world problems as part of a math trail can give students the opportunity to perceive the connection between their world and mathematics. This can offer an authentic experience with the subject matter and might thereby trigger positive emotions and interest. As interest and emotions have a high impact on students' learning (Schukajlow et al. 2017), one of the main aims of mathematics classes should be to foster students' interest and positive emotions and to diminish their negative emotions. Therefore, the aim of this work was to examine whether students are more interested in and experience more enjoyment and less boredom while solving real-world problems outside the classroom than students who solve the problems inside the classroom.

Our findings can contribute to a better understanding of the role that authentic learning environments (e.g., outside the classroom) play in the context of solving real-world problems. Overall, our results indicate that students have high interest and experience high enjoyment and little boredom while solving real-world problems, whether the problems are solved outside on a math trail or inside the classroom. We conclude that it is not the learning environment outside the classroom that is important for the development of students' interest and emotions. We hypothesise that the authentic problem type referring to students' school environment is important for the development of students' interest and emotions. This hypothesis should be investigated in future studies.

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Chapter 14
Learners Developing Understanding of Fractions via Modelling

Lindy Hearne and Dirk Wessels

Abstract Fractions are a notoriously difficult area to master. The use of a fraction as an operator is one of the least commonly accessed sub-constructs of fractions. We explore the use of mathematical modelling to enhance Grade 6 learners' understanding of this sub-construct. Learners' understanding improves as effective connections are made between and within their intra-mathematical and extra-mathematical knowledge. The quality of connections made during the task differed between groups. We conclude that learners benefit by connecting symbols and their referents and procedures and their underlying concepts rather than focusing on the surface features of Arabic notation.

Keywords Modelling · Understanding · Fractions · Mathematics education · Semiotic approach

14.1 Introduction

Mathematical modelling has been developed as a vehicle for teaching mathematics for understanding (Blum and Niss 1991). Though research into the use of modelling in primary school mathematics has been gaining traction in the last ten years, it is still in its early stages. Stohlmann and Albarracín (2016) recommend research on developing, implementing, and assessing Modelling Eliciting Activities (MEAs) at the primary school level:

> For representational and conceptual competence future research can expand on the [mathematical] content that has been studied. The content of … [amongst others] fractions… can be explored as to how modelling can enable students to develop conceptual understanding through different representations (p. 6).

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Poor sequencing (Aliustaoglu et al. 2018; Bayaga and Bossé 2018; Newstead and Murray 1998), unmonitored informal misconceptions about fractions (Newstead and Murray 1998), and a limited variety of problems have been identified as reasons children find it difficult to learn fractions (Bayaga and Bossé 2018; Newstead and Murray 1998). Modelling allows for use of learner's informal knowledge and enlarges the variety of activities that learners are exposed to using their fractional knowledge. Modelling is thus, theoretically, a viable tool for the learning of this content area.

Though fractions are one of the most important content areas in this age group (Aliustaoglu et al. 2018), very little mathematical modelling research has been conducted in this content area in this age group. Few MEAs have been developed (for one such study see Shahbari and Daher 2013) in the content area of fractions. Furthermore, though these studies have reported on the capacity of modelling to increase conceptual understanding, there is still a theoretical gap for exploring how that understanding comes about. "Teachers need deeper knowledge of the comprehension processes during the solution of reality-based tasks so that they can also emphasize linguistic and contextual aspects and provide targeted help to students (Leiss et al. 2019, p. 1)." During the modelling process, learners are afforded the opportunity to make connections between their representations. In understanding fractions, it is important that learners make connections (Hiebert 1985) between the symbols and their referents, between the procedures and their underlying concepts, and between solutions and their reasonableness in both the real world and in relation to other mathematics that is known.

Mathematical modelling utilises "tasks that require an effective linkage and translation between extra-mathematical context and intra-mathematical content (Leiss et al. 2019, p. 2)". This requires learners to display "the ability and willingness to perform these translation processes" (Leiss et al. 2019, p. 2), in order to be competent modellers. This translation is a pragmatic process requiring higher levels of integration by learners.

14.2 Fractions

According to Hiebert (1985), deep learning of fractions happens at two sites, one of form and one of understanding. Fractions in the intermediate phase in many South African schools are taught via a developmentally graded procedural system. This establishes a strong foundation at the site of "form". To encourage application, fractions have associated word problems. Few, if any, of these word problems are non-routine and there is little or no exploratory learning. Considering the possible interrelationship between understanding the fraction concept and solving word problems involving operations between fractions (Aliustaoglu et al. 2018), this approach may not be optimal for establishing fractions at the site of "understanding".

> However, some researchers recognize that contextualizing fraction learning in real-world problems helps to demonstrate the semantic structure of fractions and leads to greater learning… a number of factors can contribute to students gaining deeper semantic understanding of fractions. (Bayaga and Bossé 2018, p. 1)

The five main sub-constructs in fractions include part-whole, measure, ratio, operator and quotient (Aliustaoglu et al. 2018; Bayaga and Bossé 2018; Shahbari and Daher 2013). Many important big ideas, or important concepts, are found within these sub-constructs. For many researchers, these subconstructs form part of the semantic boundary of fractions. That is, they are included in what fractions are collectively understood to represent. However, the fraction as an operator, also known as multiplying a fraction with a natural number (Aliustaoglu et al. 2018) or finding the fraction of a set (Newstead and Murray 1998), has been found to be the least commonly accessed subconstruct of fractions (Aliustaoglu et al. 2018; Newstead and Murray 1998).

14.3 Modelling and Fractions as an Operator

If we are to utilise learner's real-world knowledge, Modelling Eliciting Activities (MEAs) should incorporate topics that learners are exposed to in their everyday lives. All the learners in the class where the study reported in this chapter took place are from a seaside town and four of them are avid surfers. The topic was thus designed to incorporate their real-world knowledge, hoping to provide opportunities for arguing with fractions as well as solving the sub-construct. The following *Surfboards to Rent* MEA was designed:

Surfboards to rent

Mr Pieters has started a small business renting out boards at the beach. He plans on renting out stand-up paddle boards (SUPs), long boards and surf boards. He has a 4×4 with a tow hitch and has approached a business to design a trailer to store and transport the boards. The designers can build a trailer with a rack for a maximum of 24 boards.

Mr P has already bought the long boards and the SUPs. He knows one third of the trailer will be filled with SUPs and a sixth will be filled with long boards. He is currently trying to decide which smaller boards to buy.

He's done some research and knows that mini-mals are a good idea for people who are new to surfing, so he's considering three different size mini-mals. He's found a manufacturer, who will give him a good deal on the mini-mals. They have sent him a table of the board lengths that are recommended for different surfer weights. Mr P is considering the following three mini-mals:

Board (min-mal)	Length	Surfer weight (kg)
A	6′3″	45–64
B	6′6″	63–80
C	6′10″	80–91

Mr P obviously wants to rent out as many boards as possible. He's come to you for advice about which mini-mals he should buy. How many of each size mini-mal should he buy?

In this chapter, we do not focus on the whole modelling process, as modelling in this instance was used as a vehicle (see Mudaly and Julie 2007) to further learners' understanding of fractions as an operator. Rather we analyse in a more atomistic manner, focussing on mathematising as we are interested in the connections and translations made as learners link their mathematical knowledge to their extra-mathematical knowledge. Blomhøj and Højgaard Jensen (2007) concluded that "a balance between the holistic approach and the atomistic approach is necessary when considering the design of an entire educational programme aiming at [among other things] developing the students' mathematical modelling competence. Neither of the two approaches alone is adequate (p. 137)".

14.4 The Study

14.4.1 Data Collection

Six Grade 6 learners participated in the study. The sessions were video-taped and audio-recorded, and written representations were collected. Audio recordings were transcribed and mathematisation analysed according to semiotic categories.

The study took place during the COVID-19 pandemic. The final session occurred on the Monday just after the first cases of COVID-19 were identified in South Africa. A graded set of problem-solving activities was presented to the learners who worked in pairs. The pairs were randomly assigned with learners picking one of six playing cards at the door of the classroom. Group one consisted of Alan and Ann, group two of Byron and Brett and group three of Cam and Cindy (names have been changed to protect learners' privacy).

All the learners had worked with arithmetic fraction procedures, initially using pre-partitioned drawings and then learning the Arabic procedures. The target sub-construct, fraction of a set, has been covered formally in the classroom. All of the subjects have learned to multiply both mixed numbers and fractions procedurally. However, despite having completed six formal exercises and two assessments which incorporated fractions as an operator, within their last 6 months at school, not all the learners who participated in this study were able to solve the following problem:

There are 18 Smarties in a small box of Smarties.

(a) How many Smarties is one third of the box?
(b) How many Smarties is two thirds of the box?
(c) How many Smarties in one sixth of the box?
(d) How many Smarties in two sixths of the box?
(e) How many Smarties in four sixths of the box?

Three of the learners, Cindy, Cam and Brett solved *a* and *b* but only one of the learners, Cam, could solve all of the parts. As $1/3 \times 18 = 6$ and $1/6 \times 18 = 3$, it was postulated that the iterative nature of the solutions to these questions increased the cognitive load of the task contributing to the breakdown of understanding. This sub-construct of fractions was then targeted for additional support using a modelling approach. The rationale for a modelling approach was that increased interaction with the same set of data, in more depth, would allow time for the learners to stabilise this sub-construct of fractions.

The learners are familiar with problem-solving activities but are novices to more holistic and complex real-world problems. The problem statement asked them to give good advice to a new business owner (see *Surfboards to rent* above). As this was exploratory, an implicit approach to modelling (see Schukajlow et al. 2018) was employed. The learners were encouraged to reapply their findings to the situation once they had solved their mathematical models.

In the session following the modelling task, learners were presented with the problem-solving set again to ascertain stabilisation of the sub-construct.

14.4.2 Results and Analysis

All the learners initially struggled with the complexity of the problem statement. Their problem-solving experience is usually a short problem with no superfluous information, though they are used to some ambiguity. After some discussion, one of the learners, Cindy, began to utilise a strategy of circling important information, and this was observed and adopted by the other learners.

Each of the groups approached the task utilising a predominantly different representational modality. Group one, Alan and Ann, used a rectangular area model which required some reconstruction as they progressed (see Fig. 14.1). This was schematic in nature and was continually used for reasoning. Alan and Ann worked interactively through the solution process.

Group two, Byron and Brett, started with a representation of 24 boards but abandoned this approach and did not reason with their numerical representation. Instead, they predominantly used skip counting (linguistic number facts) to solve the MEA (see Fig. 14.2). They did not work interactively but rather in parallel. Both learners managed to find the individual quantities but were unable to expose them in relation

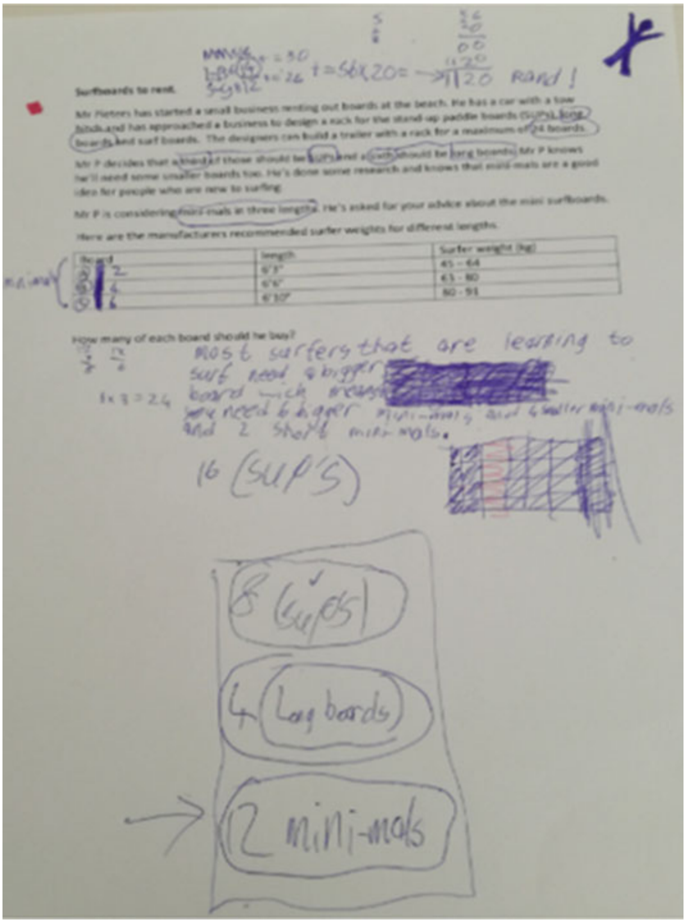

Fig. 14.1 Alan's written representation

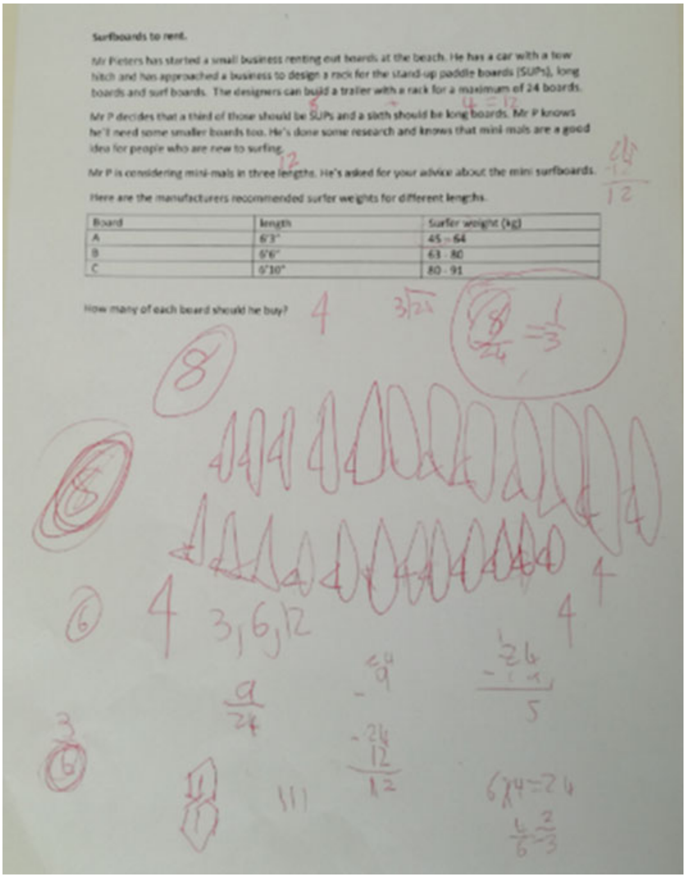

Fig. 14.2 Brett's written representation

to the situational model. This required intervention from the teacher to link it back to the real situation. They then wrote each quantity above the part of the problem statement it related to.

Group three, Cindy and Cam, used Arabic numerals to solve the MEA (see Fig. 14.3). Cam is the expert peer who had solved the entire problem-solving sequence. He had explained the problem-solving set using both Arabic procedures and a schematic representation. Cam solved the MEA using what he called "reverse simplification". Cindy observed his writing but waited for him to complete his calculations before asking him what she should write. Though Cam showed good translation between his mathematical and real-world knowledge, Cindy did not. Once they had discussed solving for 1/3 of the set, Cam began to help Cindy solve for 1/6 of the set. The following interaction followed Cindy writing 1/6 = 4/24:

Cindy: Reduce it now?

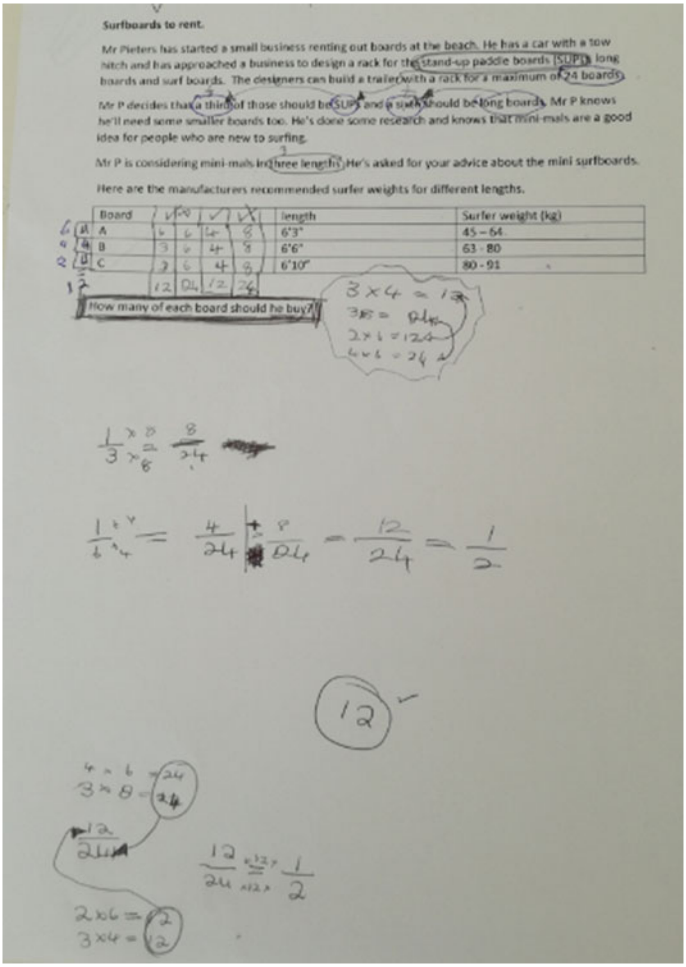

Fig. 14.3 Cindy's written representation

Cam: No but, but you want it into twenty-four because that's it that's how much… the maximum, 24 boards.

Cindy needed Cam to pinpoint each step of the Arabic procedure, telling her where to write each Arabic numeral. She continually wanted to reduce each fraction and was unsure of both the procedure and what it revealed. When reminded that "everyone needs to understand", she used the time to "learn" what to say. The arrows on her page are a strategy to remember the mathematisation process if she was asked.

Only Alan and Ann used their real-world knowledge of surfers and surfing to determine the quantity of each mini-mal. They discussed their findings in the light of who would be most likely to rent the boards. They decided it would be learner surfers who were adults. They did, however, include some smaller boards, in case there were families.

Group three, Cam and Cindy, initially apportioned them mathematically. 12 minimals, three types, so 4 of each. They expressed frustration when asked to go back to reason with their answer.

Cam: 4, 4, 4 is the right answer. Why are you saying it can be something else?

After some discussion around what it would mean to give Mr P good advice, they reassessed their answer. They reasoned that as long boards and SUPs are already large boards, they would recommend predominantly smaller mini-mals so there was a greater variety in sizes.

Group two, Brett and Byron, figured out that there would be space for 12 mini-mals but did not make any recommendations about the number of each mini-mal.

In the follow-up lesson, only four learners participated in the problem-solving task. One, Byron, was absent due to the looming COVID-19 pandemic. The learners were not paired up with the same partners. Cam, who had solved the problem set in the first lesson, completed a different modelling task.

Both Ann and Alan were able to solve the problem set correctly. They answered the questions without discussion, writing them in Arabic notation but without calculations or a diagram. However, they were reluctant to present their solutions. The other two participants, Brett and Cindy, from group 2 and 3, respectively, solved all the parts of the problem set, but incorrectly. Cindy initially agreed with Brett's answers. After some discussion between the learners, Cindy and Brett modified their answers, agreeing with the solutions shared by Alan and Ann.

14.4.3 Analysis

We focus on Alan and Ann's developing understanding of fractions as an operator afforded by their mathematising of the MEA. Alan and Ann showed effective translation between fractions, 1/3, division and their multiplication facts, showing some connection between fractions, division and multiplication. These, it could be argued, are the underlying concepts of the process of using fractions as an operator.

Alan: I'm… three of those.. um.. this is hard.
Ann: 1/3, what is 24 divided by 3, that's…8. [Alan writes $8 \times 3 = 24$.]

They also made connections between their schematic representation and other mathematics, for example:

Ann: So 8. 1, 2, 3, 4, 5, 6, 7, 8. [Ann indicating the two left columns, Alan filling in the two rows.]
Ann: Cool, cool, we got 8 is SUPs.

They also translated between their extra-mathematical and intra-mathematical knowledge, making connections between their representation and the real world.

In solving for 1/6, Ann made an error in accessing her multiplication facts:

Ann: And what's 24 divided by 6. I'm guessing it would be 3, yeah, 3.
Ann: Okay, and then 6th, what's 6ths…3.
Alan: Wow, this is hectic, we made a breakthrough, guys. We're getting it, just don't look 'cause that's wrong!
Ann: Well, 24 divided by 6 is 3.
Alan: Huh!

Ann, however, responded to her partner's check. Alan was engaged in the process and not just following instructions.

Ann: Oh, sorry it's just, it's not, so it's 4, then that means that we must put in another 4.
Alan: I need teacher to help us a bit hey?

Alan was uneasy and hoped for clarification from the teacher. Ann, however, carried on with her process, linking her other mathematics to the schematic representation. She then utilised her results to effectively link the left over squares to number of mini-mals.

Ann: Because we have to fill in this row. [Alan colours the row in.]
Ann: Then there's 12 less, so that's 12 mini-mals.

14.5 Discussion

Though both Ann and Alan have had formal instruction using fractional procedures for over a year, they do not use Arabic notation of fractions in their mathematising. Rather, they make sense of the problem using an area model used in an array and their multiplication facts. As drawing a schematic diagram has been encouraged in weekly problem-solving classes, this may indicate their implicit understanding of the problem-solving classroom's contract.

In generating a diagram, Ann and Alan were not immediately able to produce the accurate array; their representation originally had three rows, for reasoning with thirds. However, they adapted this during their mathematising and removed the final column, after checking that there were only 24 blocks to represent the 24 boards. Thus, their partitioning of the rectangular array required adaption. This indicates an accessing of relative magnitude. In this dialogue, we saw Ann make several connections between her multiplication facts and the total number of boards. She indicated these, and Alan tracked this on their diagrammatic scheme. From there, she made a connection between the total number of boards and the number of SUPs. Interestingly, not all learners were able to interpret their results in this context.

When Ann accessed her number facts incorrectly, she utilised the feedback resolving this conflict correctly. She then connected her multiplication facts more automatically to both the total number of long boards and their diagrammatic representation, indicating some generalisation of her process. She was able to use her diagram to reason, linking the left over squares back to the problem statement and solving for the required number of mini-mals. The approach, it could be argued, may not benefit her ability to use Arabic notation in finding a fraction of a set; but her developing connections between important aspects of this concept are evident. This supports the aim of modelling to connect the mathematical and extra mathematical world.

Alan is an avid surfer. This played a role in his development of the situational model in order to mathematise. Alan and Ann did not spontaneously apply their findings back to the real situation. The teacher encouraged them to think about the advice they would give Mr P, reminding them that he had come to them for advice because he needed help choosing the boards. The task then further engaged Alan's knowledge base of surfing. He used his outside world knowledge to contribute to the real-world application in order to make good recommendations as to the number of each minimal Mr P should buy. This is evident in their reasoning for the distribution of 2, 4, and 6 to board *A*, *B*, and *C*, respectively. They had a discussion around who would hire boards, and concluded that it would be novice surfers, as most surfers adapt to their own boards. They also utilised the knowledge that beginners need bigger boards as they are more stable on the water.

Alan and Ann both solved with fractions and argued with fractions effectively. Solving with fractions allowed them to make effective connections between their mathematics using a schematic diagram and other mathematics (their linguistic number facts). It is likely that arguing with fractions allowed them to also establish connections between the usefulness of the fraction of a set and their real world. The modelling task made a clear difference in Alan and Ann's ability to use the subconstruct of the fraction as an operator. In contrast, both Cindy and Brett were still unable to independently solve the problem-solving set.

14.6 Conclusion

Utilising an implicit modelling approach, with learners being encouraged to reassess their answers in response to the problem statement, was effective for two of the three groups. Though not spontaneous, learners used their knowledge to argue with fractions, and not just solve fractions. This allowed for translation of, and connections between, their intra-mathematical and extra-mathematical knowledge.

Use of a MEA in the content area of fractions shows various benefits to learners' ability to solve fractions as an operator. The quality of reasoning within the modelling task indicated the benefit to their developing understanding of fraction of a set. The quality of engagement of an individual with the model eliciting activity played a role in their growth of understanding. Connecting symbols and their referents and procedures and their underlying concepts appears to be of more benefit than figurative involvement with surface features of Arabic system (as in the case of Cindy) or with number facts (as in the case of Brett).

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Chapter 15
The Historical Development of Mathematical Modelling in Mathematics Curricular Standards/Syllabi in China

Jian Huang, Xiaoli Lu, and Binyan Xu

Abstract This study investigates the historical development of mathematical modelling(MM) in China's curricula. It employs a qualitative text analysis approach to analyse the mathematics curricular syllabi/standards (MCSs) at primary, middle and high school levels in China. It was found that (1) the term "modelling" was not seen in the MCSs before 1996, but the MM thinking rooted in "solving real-world problems" has been in existence for a long time; (2) the MM cycle has developed from a four-step cycle to a seven-step cycle which is consistent with the cycles described in international literature; (3) the MCSs for high school have more requirements for students than those for middle school, but they both lack requirements in students' affective aspects; (4) the 2017 edition of the high school mathematics curricular standard puts more emphasis on the connection between the mathematical world and the real world than the 2013 edition.

Keywords Mathematical modelling · Curricular syllabi · Standards · Qualitative text analysis · China · Modelling cycle

15.1 Theory

15.1.1 Different Perspectives on Mathematical Modelling

Mathematical modelling(MM) has been central to mathematical education during the last 40 years. Though there is no consistently accepted definition, the understanding of what modelling entails will not vary greatly from field to field (Blum

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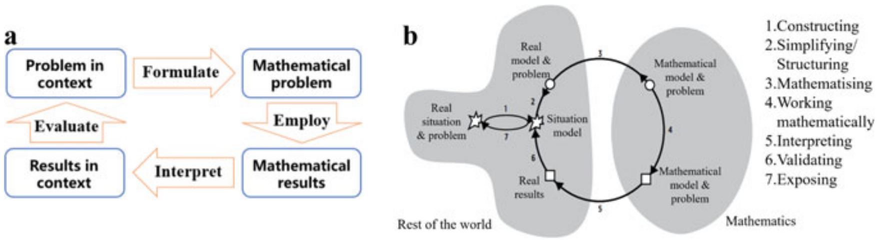

Fig. 15.1 **a** Four-step modelling cycle**b** Seven-step modelling cycle (Blum and Leiß 2007)

et al. 2007; Lesh and Fennewald 2010). Modelling is viewed as the link between the "two faces" of mathematics, namely its grounding in reality, and the development of abstract formal structures (Greer 1997).

When considered as a mapping process from the real world to the mathematical world, MM has a typical four-step cycle (See Fig. 15.1a). However, by paying attention to the changes in the psychological state of participants in the process of MM, Blum proposed a five-step cycle (Maaß 2006). Furthermore, by subdividing the objective existence form and subjective understanding form of the real situation into two states, we get the seven-step cycle proposed by Blum and Leiß (2007) (see Fig. 15.1b). In this cycle, the modelling process consists of six states and seven stages.

MM research in the field of education has gradually displayed many different international perspectives. Kaiser (2017) categorizes the latest perspectives on MM in schools. She includes: realistic or applied modelling (e.g. Kaiser and Schwarz 2006), epistemological or theoretical modelling (e.g. Garcia et al. 2006), educational modelling (e.g. Blum 2011), contextual modelling or model eliciting perspective (e.g. Kaiser 2014), socio-critical and socio-cultural modelling (e.g. Barbosa 2006) and cognitive modelling as metaperspective (e.g. Stillman 2011). Proponents of realistic or applied modelling emphasize pragmatism and believe that the purpose of modelling is to apply mathematics rather than to develop mathematics. Educational modelling comprises two facets: (1) didactical modelling and (2) conceptual modelling. Didactical modelling emphasizes that students' focus should be on developing various modelling competencies, while conceptual modelling followers believe that the teaching of modelling should serve the learning of mathematical concepts. Cognitive modelling as a metaperspective is more concerned with the changes in cognition and emotion that occur in the students' MM processes.

15.1.2 Mathematical Modelling of Curricular Standards in Different Countries

There have been attempts during at least the last four decades to attribute a sizable place and role to models and modelling in different mathematics curricula and in different contexts of teaching and learning (Niss 2012). Modelling competencies play an essential role in many national curricula, showing the relevance of MM at a broad international level (Kaiser 2014). The German mathematics education standards issued at the end of 2003 named MM ability as one of the six major mathematical abilities that students should develop. In 2010, the *Common Core State Standards for Mathematics* issued by the USA regard MM as a route to solve problems. MM was listed as a basic mathematical activity in the draft opinions on high school MCSs issued by the Australian Curriculum, Assessment and Reporting Authority in 2010. The current Swedish curriculum standards state that one of the aims of education is to develop students' ability to design and use mathematical models and critically evaluate conditions, opportunities and limitations of different models (Ärlebäck 2009). In the newly issued national curricular standards for high school mathematics in China, MM has been recognized as one of the six core competencies students should develop during school mathematics (MOE 2018). When compared to Western countries such as Germany and the USA however, there is little information on the state of MM in China. Both theoretical and empirical studies are needed.

Using the seven-step modelling cycle as a basis, this study defines MM as: "Understanding and building a real model in the face of a real situation, translating the real model into a mathematical problem, building a mathematical model and solving the mathematical problem using mathematical methods, then interpreting and checking the mathematical solutions according to the real situation, and finally validating the rationality of the model (Xu 2013)".

What then is the course of development of "mathematical modelling" in Chinese mathematics curricular standards? Specifically, we investigated the research questions: how is mathematical modelling described and what is required in the mathematical modelling process from the curricular standards of mathematics in China from 1902 to 2018?

15.2 Method

15.2.1 Research Objects

The curricular documents analysed are mathematics curricular syllabi/standards (MCSs) in China from 1902 to 2018. There are 24 primary mathematics syllabi and 43 secondary mathematics syllabi, published in *Mathematics Volume of the Collection of Primary and Secondary Curricular Standards/Syllabi in 20th Century in China*, from 1902 to 2000. Since 2000, there have been four national MCSs, two for Grade 1 to Grade 9 students (compulsory school)—2001 version and 2011 version, and the other two for Grade 10 to Grade 12 students (senior high school)—2003 version and 2017 version. In sum, there are 71 MCSs (Table 15.1).

Table 15.1 Research objects

Year	Text	Code	Pages
1902–2000	Mathematics Volume of the Collection of Primary and Secondary Curricular Standards/Syllabi in 20th Century in China		685
2001	Mathematics curriculum standards for full-time compulsory education (Experimental version)	"01 Compulsory"	102
2003	High school mathematics curriculum standards (Experimental version)	"03 High School"	122
2011	Mathematics curriculum standards for compulsory education (2011 Edition)	"11 Compulsory"	132
2018	High school mathematics curriculum standards (2017 Edition)	"17 High School"	180

15.2.2 Qualitative Text Analysis

The main research method is text analysis. Kuckartz (2014) divided coded text analysis into three categories: thematic qualitative text analysis which focuses on identifying, systematizing and analysing topics and sub-topics and how they are related; evaluative qualitative text analysis which involves assessing, classifying and evaluating content; and type-building text analysis which aims to differentiate rather than develop a general theory. In order to get closer to the specific meaning of the text, the coding process mainly adopts the thematic qualitative text analysis method.

15.2.2.1 Text Filtering

First, we screened all texts, sorted out and recorded the relevant paragraphs. A requirement for inclusion was that the extracted paragraphs contain the terms "model" (mo-xing) or "modelling" (jian-mo). We then checked that the paragraphs from the two researchers complemented each other, and finally, we made sure that no paragraphs related to MM in the texts were missed. After that, the excerpts were filtered twice: firstly, the paragraphs from the appendix section of the syllabus were removed; secondly, we removed any paragraphs that did not conform to MM definitions. In these paragraphs, models did not refer to MM but rather to geometric or physical objects. For instance, the text mentioning "geometric object model (cube or cuboid)" in the 1952 syllabus was not included. Finally, we found 128 paragraphs that met the requirements to be encoded. The coding framework was constructed by combining inductive and deductive methods.

15.2.2.2 Coding and Analysis

The first step was developing the main thematic categories and performing the first coding process. Using the "two faces" of MM, two major topics of "mathematics and modelling" and "reality and modelling" were determined. The former uses mathematics to do something and the latter connects mathematics with reality. In addition, it is important to include emotions in research on the teaching and learning of modelling, but there is a significant lack of papers that investigated emotions (Cai and Xu 2016; Schukajlow et al. 2018). Based on this, we regard "emotional attitude" as the third topic. Two researchers conducted a first-step double-blind coding of all data according to these three main topics, with consistency of more than 95% (Xu and Zhang 2005).

The second step was further developing the main thematic categories and performing the coding process. Fifty per cent of the data under each main topic was randomly selected for preliminary classification. The two researchers then used induction to code the selected data back to back. The final coding system is shown in Table 15.2.

The MCSs emphasize two functions of mathematical knowledge in MM. These fall under the main thematic category "mathematics and modelling" and include two categories: (1) applying knowledge to a model or modelling and (2) applying knowledge during model solving. The former emphasizes the role of mathematical knowledge in MM, while the latter emphasizes the role of mathematical knowledge in the model solving process. In addition, the concept of using MM activities to promote the understanding and learning of mathematical knowledge is also mentioned in the MCSs. Thus, under this topic, three secondary codes were obtained.

Under the thematic category "reality and modelling", the description in the paragraphs has a good correspondence with the steps described in the seven-step modelling cycle. We therefore constructed four secondary codes according to the seven modelling steps of Blum. These excluded the steps of modelling and mathematical solving, and they fall under another main thematic category. We combined interpreting and validating, encoding these as one secondary category. In addition, we found many expressions similar to "use of modelling ideas to solve practical problems", which we categorized as "solve practical problems".

For the eight sub-categories coded in the first two topics, only two codes are not included in the seven-step cycle. These are "promoting mathematical learning" and "solve practical problems", which reflect the idea of teaching and learning MM in the curriculum.

The sub-categories under the topic of "emotional attitude" were directly encoded by the inductive classification method to obtain "increase interest" and "improve attitude".

The third step was to code all of the data according to the elaborate category system. Two researchers coded all the paragraphs in a double-blind fashion. The consistency of this two-person coding was more than 90%. Where there were differences, the two coders reached consensus by discussion. The final total number of codes is shown in Table 15.3.

Table 15.2 Coding system

Category	Code	Description	Example (translated texts)
C1: Mathematics and modelling	C11: Mathematical models	Using existing mathematical models or using mathematical knowledge to build mathematical models	Using these functions to develop models; selecting proper function models; constructing models
	C12: Mathematical solving	Using mathematical knowledge to get a solution	Solving the models; calculating and getting the solution
	C13: Promoting mathematical learning	Enhancing the understanding of mathematical knowledge, acquisition of skills, et cetera	Developing mathematical knowledge; acquiring necessary knowledge and skills (through modelling)
C2: Reality and modelling	C21: Understanding	Understanding the real problems	Finding proper objects to study from a mathematical perspective
	C22: Simplifying/Structuring	The given real problem is simplified in order to build a real model or posing mathematical problems	Expressing the problem with mathematical language; translating to mathematical problems; mathematizing
	C23: Interpreting/Validating	Interpreting the mathematical results and validating	Improving the model; verifying the solutions (in real situation); reflecting on the modelling
	C24: Apply it to practice	Further apply the model results to the actual	Interpretation and application; explaining economic phenomenon
	C25: Solve practical problems	Use of modelling ideas to solve practical problems	Dealing with realistic problems; solving real-world problems
C3: Emotional attitude	C31: Increase interest	Conducive to the improvement of interest	Inspiring students' interest in learning mathematics

(continued)

Table 15.2 (continued)

Category	Code	Description	Example (translated texts)
	C32: Improve attitude	Improve students' views on mathematics	Gaining good experience in affect; feeling the value of the application of mathematical theory

Table 15.3 Number of codes

Curriculum	Number of codes	Curriculum	Number of codes
Before 1996	0	2001 compulsory education	18
1996 senior high school	4	2003 senior high school	114
2000 junior high school	4	2011 compulsory education	18
2000 senior high school	5	2017 senior high school	147

15.3 Results

15.3.1 Evolution of Mathematical Modelling

The terms "mathematical modelling" and "mathematical model" have not been in the Chinese MCSs for long (see Table 15.3). The latest high school curriculum (1996), *Full-time general high school mathematics syllabus (for trial use)*, changed the sentence "… make students better understand and master knowledge, learn to use mathematics knowledge to solve simple real-world problems" (p. 605) found in the previous edition of the syllabus to "… make students better master the basic knowledge, enhance the awareness of using mathematics and be able to use mathematical model to solve some real-world problems" (p. 644). This shows the evolution of MM from the long-standing "solving [of] real-world problems".

Before 2000 (1996–2000), MM seldom appeared in the syllabi and was not clearly defined, but the description of its process had initially been formed. When compared to the four-step cycle, it contained the process "convert real problems into mathematical models and then solve to get mathematical results", but neither "interpreting" nor "validating". In the twenty-first century (2001–2017), the incomplete four-step cycle is gradually moving towards a seven-step cycle.

"01 Compulsory" further developed the MM process, pointing out that students should be allowed the experience of "abstract[ing] the real-world problem into a mathematical model and then explaining and applying it" (MOE 2001, p. 61). The MM process at this point already included "interpretation, application and extension", but lacked the "validating" step. "03 High School" promulgated in 2003 highlighted the importance of cultivating MM capabilities. In terms of the number of codes, there are 114 codes in "03 High School", which is almost 100 more than those in the

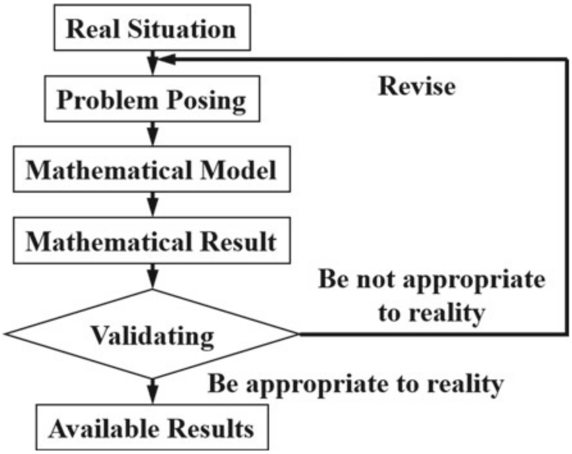

Fig. 15.2 "03 High School" mathematical modelling framework diagram (MOE 2003, p. 41)

previous version. From the text itself, "03 High School" gave the framework diagram of the MM process for the first time (Fig. 15.2). The diagram is a complete MM cycle and is in line with the five-step cycle (with "problem posing" and "validating" being added).

In the "11 Compulsory", "model thinking" became one of the ten key terms (MOE 2012, p. 5). The MM process described in the "11 Compulsory" conformed to the framework diagram given by "03 High School". The description of the MM process in the newly issued "17 High School" was upgraded from a five-step cycle to seven-step cycle. It describes the process of MM as "discover problems in realistic situations from a mathematical perspective, pose problems, analyse problems, construct models, determine parameters, calculate and solve, verify results, improve models and finally solve the realistic problems" (MOE 2018, p. 35). In comparison to "03 High School", the transition process from the real situation to model construction has become clear and complete. In particular, the process of "finding problems from a mathematical perspective" corresponds to developing a "situation model" in the seven-step modelling cycle.

15.3.2 Requirements of Mathematical Modelling

15.3.2.1 Comparison Between High School and Compulsory Education

When comparing the coding data of high schools ("03 High School" and "17 High School") in MCSs since the twenty-first century with the coding data of compulsory education developed for Grades 1–9 ("01 Compulsory" and "11 Compulsory"), it can be seen that the number of codes in high schools is much higher than those of compulsory education (see Fig. 15.3). MM is mentioned more frequently in the MCSs of high schools, and the corresponding teaching requirements are higher.

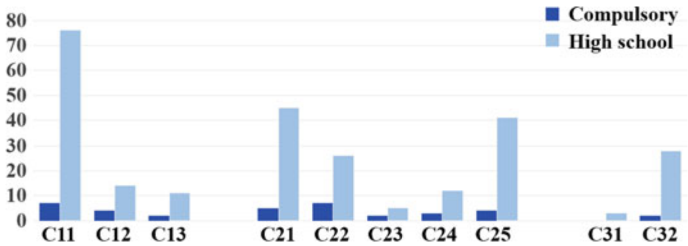

Fig. 15.3 Data comparison between high school and compulsory education

The same conclusion can be drawn from specific texts. Throughout the two MCSs of compulsory education, the low requirements for MM can be seen by their descriptions in terms such as "experiencing" and "realizing". "01 Compulsory" requires that students "experience the process of abstracting a real problem into mathematical models, interpreting and applying them…" (MOE 2001, p. 1). Although "11 Compulsory" emphasizes model thinking, it is not listed as the most important training goal. Students are only required to "experience model thinking".

The high school MCSs for MM requirements are more stringent. "03 High School" requires students to "propose ways to solve problems, establish proper mathematical models and then try to solve the problems" (MOE 2003, p. 88–89). "17 High School" specifies the teaching objectives of MM, requiring that "through learning high school mathematics, students can express the real-world consciously with mathematical language, discover and propose problems, make sense of the connections between mathematics and reality, learn how to use mathematical models to solve real problems…" (MOE 2018, pp. 5–6).

15.3.2.2 Comparison of Two Versions of Curricular Standards in High School

The requirements for MM in the Chinese MCSs are mainly placed in the high school segment, so are there any differences in the descriptions in the two versions of the high school curriculum standards? When comparing the coded data (Fig. 15.4), it is obvious that the prevalence of the three major topics is significantly different. Under the topic of "mathematics and modelling", there is little difference in the number of codes found between the two. Under the topic of "reality and modelling", the number of "17 High Schools" is significantly higher than that of "03 High School", but under the topic of "emotional attitude", "03 High School" has more codes.

Looking at the codes in the high school curricula, it is apparent that there are fewer affective aspects included in the "17 High School". The number of affective codes decreased from 23 in the previous version to 8. This is probably because affect and attitudes were one of the three basic curricular ideas of the 2003 version. The previous version proposes a three-dimensional teaching goal as its basis namely "knowledge and skills", "processes and methods" and "emotional attitudes and values". As a

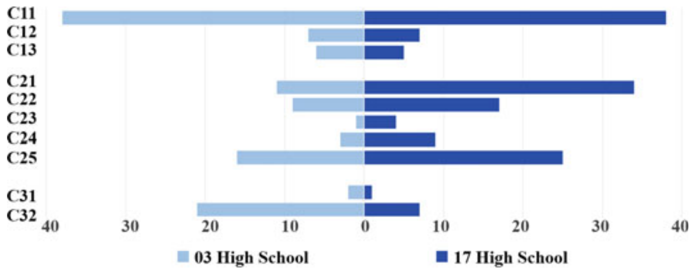

Fig. 15.4 Comparison of coding data between “03 high school” and “17 high school”

result, almost every “03 High School” curricular requirement is connected with affect and attitudes. In contrast, “17 High School” emphasizes six core mathematical competencies, and affect is not one of these. There are, therefore, fewer descriptions of affective aspects and a greater focus on MM competencies, especially on reality and modelling. The prevalence of codes under the category reality and modelling has increased significantly to 89 (in the 2017 version) from 40 (in the 2003 version). This is consistent with its curricular idea which emphasizes the connections between mathematics and reality.

15.4 Discussion

Throughout history Chinese mathematics education has a cultural background strongly highlighting practicality. The core problems in ancient Chinese mathematical works are a variety of “shu (术)” (i.e. give a general solution to a certain type of problem).“Shu” is a mathematical model. In MCSs, the 1923 syllabus requires students to learn to “solve real problems in their own life”; the 1951 syllabus more specifically proposes to “train students to use mathematics familiarly to solve various real problems in daily life”. The Chinese perspective on MM would thus be the pragmatist view. China introduced the term “modelling” in 1996, conforming internationally and continuing the tradition of Chinese mathematics education.

The findings of this chapter have the following implications for understanding MM in the intended MCSs in China: (1) MM seems to be more demanding in the curriculum for high school mathematics than for middle school mathematics, (2) the characteristics of connections with reality of MM have been recognized and the curriculum emphasizes mathematical foundations in the promotion of MM and (3) affective aspects, such as students’ interest, are no longer prioritized. These characteristics are consistent with the common understanding of mathematics education in China, well recognized for its emphases on the mathematical contents and students’ performance rather than their interest in learning (e.g. Leung 2001).

The description of MM in Chinese MCSs is biased towards the “realistic or applied modelling” perspective similar to Kaiser and Schwarz (2006). It emphasizes MM as important in applying mathematical knowledge, and the coding data also supports this view. The MCSs, however, do not pay enough attention to the function of MM "promoting mathematical learning". "Conceptual modelling" needs to be emphasized in future practice (cf. Blum 2011).

In recent years, MCSs for MM have become more and more demanding. The "interpreting" and "validating" steps in the modelling cycle, however, need more attention compared with Kaiser (2007). Analysis of Chinese MM education highlights the need for comprehensive development of all aspects of MM. This study has focused on the intended mathematics curriculum. Further research is needed to gain insight into the state of MM in the enacted mathematical curriculum in China.

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Chapter 16
Pictures in Modelling Problems: Does Numerical Information Make a Difference?

Marcus Schmitz and Stanislaw Schukajlow

Abstract Pictures are an important part of everyday life, and they often accompany modelling tasks. However, we do not know much about the role of pictures in modelling. To address this research gap, we randomly assigned students to three groups. In the experimental groups, in addition to the text, the problems included useful or superfluous numerical information in pictures, whereas the pictures that went with the problems in the control group did not include any numerical information. We assessed the picture-specific utility value and modelling performance of 110 students in upper secondary school. The picture-specific utility value reflects the perceived usefulness of a picture for understanding the problem. Students assigned a lower utility value to the pictures that contained additional superfluous numerical information. However, we did not find differences in the students' modelling performance.

Keywords Pictures · Cognitive load theory (CLT) · Text-picture comprehension · Utility value

16.1 Introduction

Improving students' ability to solve real-world problems by using mathematics is an important goal of mathematics education; thus, modelling competence is part of school curricula all over the world (Niss et al. 2007). In order to strengthen the extent to which modelling problems are linked to the real world, modelling problems often include pictures. In addition, being able to deal with the combination of pictures and text is important for professional and everyday life. Despite the importance of

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pictures for modelling, there is a large deficit in research on the effects of pictures in modelling tasks and on the processing of tasks that include pictures.

Multimedia theories such as the integrated model of text and picture comprehension (Schnotz 2014) have suggested that text–picture design influences mental processing and learning effects. One prerequisite for supporting students' understanding of problems is that the students notice the usefulness of pictures that accompany the problems. Prior studies have yielded inconsistent results concerning whether students perceive pictures as useful while solving real-world problems (Böckmann and Schukajlow 2018; Dewolf et al. 2015). Moreover, we did not find any research on the effects of different types of numerical information in pictures on the usefulness of pictures or mathematical performance.

On the basis of these considerations, we aimed to gain more knowledge about the role of additional numerical information in pictures on modelling performance and the perceived usefulness of additional numerical information for understanding the task. When we refer to 'additional numerical information' in this study, we mean additional drawn information contained in pictures that is also described in the text. For example, this additional numerical information may refer to distances with the given length drawn in the picture.

16.2 Theoretical Framework

16.2.1 Pictures in Modelling Problems

At the core of mathematical modelling, there is a demanding process by which information must be translated between the real world and mathematics. There are several activities that are part of the solution process that are often described in a cycle that begins with the student's understanding of the real-world situation and ends with the validation of the results (e.g. Blum and Leiß 2007). More specifically, students need to construct a model of the situation that they will then simplify and idealise before constructing a mathematical model. At the end of the solution process, students need to interpret and validate their results.

In order to strengthen the extent to which modelling problems are linked to the real world, modelling tasks that are presented in the classroom should, and often do, include text and pictures. We assume that pictures can support certain modelling activities and thus influence students' modelling performance. For example, certain pictures can be particularly helpful for understanding and creating a model of the situation. The extraction of the necessary information from the text represents a potential barrier for students when they solve modelling tasks. Furthermore, superfluous information in the text increases the difficulty of the task. Pictures can potentially help students organise information, simplify the situational model, and mathematise the information.

Kite

Lucas got a new kite as a birthday present. The kite is 1m in length and 50cm in width. Lucas flies the kite with his friend Susan (see picture). They are standing at a distance of 80m from each other. The kite‘s string has a length of 100m. Susan is right under the kite and 20m from the sea.

How high is the kite flying at this moment?

Fig. 16.1 *Kite Task* with no additional numerical information in the picture

Pictures used in combination with text can serve different functions. Elia and Philippou (2004) developed a taxonomy of pictures for problem solving. There are *decorative* pictures that are irrelevant to the contents of the corresponding text. The picture does not refer to events or information in the text. Pictures with a *representational* function "represent the whole or a part of the content of the problem" (Elia and Philippou 2004, p. 328). *Informational* pictures present information that is essential for solving the modelling problem. All pictures used in this study have a representational function. In our study, we used photos as the pictures because they are closely connected to reality. Such realistic pictures are two-dimensional simulations of objects from a specific perspective with a great deal of potential to support mental model construction (Schnotz and Cade 2014). Figure 16.1 shows an example of a modelling task used in our study.

Comprehension of the *Kite Task* results in a model of the situation that includes Lucas and Susan, a kite, a piece of string, and the positions of the two people and the kite. To calculate the desired height of the kite, students can use the Pythagorean theorem and add an estimate of Lucas' height. The picture in the task can help the problem-solver organize the information and construct a model of the situation. In the modelling process, the picture can be used as an easily accessible external representation of the situation.

16.2.2 Text and Picture Comprehension

Several studies have shown that students generally learn more deeply from text when it is combined with pictures than from text alone (Mayer 2005). Models such as the cognitive theory of multimedia learning (Mayer 2005) or the integrated model of text and picture comprehension (Schnotz 2014) describe this positive multimedia effect. They assume that a multimedia effect occurs only under certain conditions. One assumption is that the text and the picture can only be processed into a joint mental model if they are closely semantically connected. This conforms to the *coherence condition.* According to the *contiguity condition*, the text and the picture can only contribute to the construction of a joint mental model if they are presented closely together in space or time.

Furthermore, in multimedia theories, working memory plays a central role and determines to a large extent whether the multimedia presentation leads to a positive learning effect through the optimal use of working memory or whether it hinders learning through overloading. The cognitive load theory (CLT) (Sweller 1994) describes the different loads on working memory. It is integrated in multimedia theories and builds the basis for the effectiveness of design of pictures (Mayer 2005). CTL distinguishes between the cognitive structures in long-term and working memory. Long-term memory can store large amounts of information in *schemas.* Schemas refer to cognitive structures that incorporate multiple elements into a single element with a specific function. Schemas can be retrieved from long-term memory into limited working memory in which all conscious cognitive processing occurs. Thus, working memory can perform complex cognitive activities despite its limited capacity by retrieving these schemas. CLT therefore represents learning as the process of acquiring schemas.

According to CLT, there are three types of cognitive load on working memory that occur during the processing of new and already stored information: intrinsic, extraneous and germane cognitive load. *Intrinsic load* describes the load on working memory caused by the complexity and difficulty of the learning content. Intrinsic load is characterised by the number of interacting learning elements kept in working memory for processing. The amount of load depends on the learners' individual level of expertise since the number of processed elements depends on the schemas stored in long-term memory. Thus, all instruction has an inherent difficulty associated with it, and this inherent difficulty, which produces intrinsic load, cannot be altered by an instructor.

The manner in which learning material is designed can also produce cognitive load. When such load is unnecessary and thereby interferes with building schemas, it is referred to as *extraneous load*. Thus, extraneous cognitive load is generated by the manner in which information is presented to learners and is under the control of instructional design.

The third source of cognitive load is *germane* cognitive load. Whereas extraneous cognitive load interferes with learning, germane cognitive load enhances learning. So germane load is related to information and activities that foster processes of schema construction and automation. Thus, when pictures support modelling activities such as understanding or structuring, they produce germane cognitive load.

A central assumption of CLT is that the three types of cognitive load can be accumulated into the total cognitive load. If this total cognitive load exceeds the capacities of working memory, learning cannot occur (Sweller 1994). This hypothesis is only valid if the intrinsic load is sufficiently complex. A high intrinsic load combined with a high extraneous load can lead to an overload of working memory resources and prevent germane load. However, if the learning content (intrinsic load) is very low, an unfavorable design style (extraneous load) will not lead to an overload of working memory. These ideas must be considered when designing learning material and are therefore also important for the use of pictures in modelling tasks.

16.2.3 Picture-Specific Utility Value

The expectancy-value theory links expectancies and personal values and describes utility (or extrinsic) value as one of four components of values that influence task performance, task choice and motivation (Eccles 1983). A task's utility value refers to the importance of a task or its parts (e.g. pictures) for extrinsic indicators of success such as an accurate solution, grades, or career. In this study, we analysed utility value of pictures for understanding modelling problems. We investigated different types of pictures and the picture-specific utility value to determine whether the pictures facilitated students' understanding of the modelling problems and thus supported the solution process.

A positive relation between values and students' performance was confirmed for problems with and without a connection to reality (Schukajlow 2017). Further, students usually realise that decorative pictures are less helpful for understanding and solving problems than pictures with representative or essential functions (Böckmann and Schukajlow 2018). Otherwise, students often do not use information from representative pictures in problems (Dewolf et al. 2015) or essential pictures in arithmetic word problems (Elia and Philippou 2004) for their solution process.

The extent to which additional numerical information in pictures influences picture-specific utility value or modelling performance has not yet been investigated, and thus, we aimed to address this research gap in the present study.

16.2.4 Research Questions

We conducted this study to address the following research questions:

(1) How do students rate the utility value of representative pictures that contain additional useful or superfluous numerical information?
(2) How does additional useful or superfluous numerical information in representative pictures affect students' modelling performance?

Prior research has shown that students rate the utility value of representative pictures higher than pictures with a decorative function. Thus, we expected that students would assign a higher utility value to pictures with additional useful numerical information and would assign a lower utility value to pictures with additional superfluous numerical information than to pictures with no additional numerical information.

The integrated model of text and picture comprehension (Schnotz 2014) describes the concept that the positive multimedia effect depends on the text–picture design. We expected that pictures with additional useful numerical information would result in higher modelling performance and pictures with additional superfluous numerical information would result in lower modelling performance compared with pictures without additional information.

16.3 Method

16.3.1 Design

One hunderd and ten students from five upper secondary schools in Grades 9 and 10 (mean age = 15.26, SD = 0.89; 47.8% female) participated in the study. The students in each class were randomly assigned to one of three groups: a control group with no additional numerical information in the pictures (CG), an experimental group with additional useful numerical information in the pictures (EG-U), and an experimental group with additional superfluous numerical information in the pictures (EG-S). Students first estimated the picture-specific utility value for understanding problems that described six modelling tasks in a questionnaire. The instructions in the questionnaire were: 'Read each problem carefully and then answer some questions. **You do not have to solve the problems!**' Then students read each problem and answered the question about utility value. After completing the questionnaire, the students solved the tasks.

In the present study, we used six modelling problems on the topic of the Pythagorean theorem. The tasks were developed and tested in prior studies (Böckmann and Schukajlow 2018; Schukajlow 2017). Unlike in the prior studies, all tasks included representational pictures in all three groups. The pictures used in this study represent visually key mathematical elements of the situation (e.g. spatial geometrical structure). In this study, the pictures representing the tasks differed across the three groups in the additional numerical information given in them. In the experimental groups, in addition to the text, the problems included useful or superfluous numerical information in the pictures, whereas for the control group, the pictures that accompanied the problems did not include any additional numerical information. The pictures from a sample problem (i.e. the *Kite Task*) with three different types of additional numerical information are shown in Fig. 16.2.

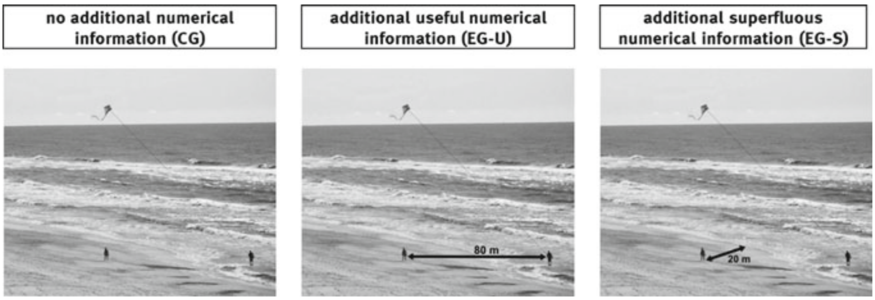

Fig. 16.2 Different pictures of the *Kite Task* with no additional numerical information (CG), additional useful numerical information (EG-U) and additional superfluous numerical information (EG-S)

16.3.2 Utility Value

To measure the utility value of pictures with different kinds of additional numerical information for each modelling problem, we used the statement ‘The picture helps me to understand the problem’. The students rated the item on a five-point Likert scale (1 = not at all true; 5 = completely true). Cronbach’s alpha as a measure of reliability for the picture-specific utility value was satisfactory (0.73).

16.3.3 Modelling Performance

To assess students’ modelling performance, we estimated the accuracy of their solutions to the problems on a three-point scale. Students achieved 0 points for a task if they used an incorrect mathematical model. If students used a partially accurate mathematical model, they received 1 point. Students received 2 points for their modelling performance if their mathematical model was completely accurate. Figure 16.3 shows an exemplary solution for the *Kite Task* that received a score of 2 points. After calculating the leg, the student added 1.65 m because of the height of Lucas who is holding the kite. This is why we gave the solution 2 points for modelling performance.

To test the inter-rater reliability of the codingprocedure, more than 15% of the solutions were coded by two members of the research team. The inter-rater reliability resulted in a good match between the two coders (Cohen’s Kappa = 0.98).

16.4 Results

The comparison of school grades in mathematics with an ANOVA indicated that the experimental groups and the control group did not differ in their mathematical abilities, $F(2, 105) = 1.57$, $p = 0.212$. To compare the groups, we calculated arithmetic means for utility value and modelling. A one-way ANOVA with a post-hoc Bonferroni correction was used to analyse group differences.

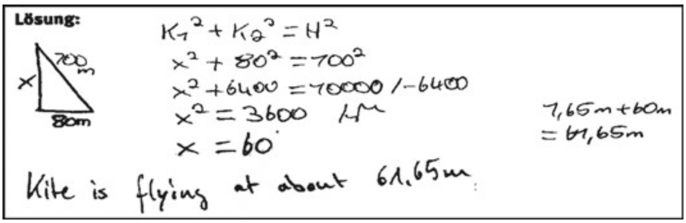

Fig. 16.3 Exemplary student solution for the kite task with 2 points for modelling performance

Table 16.1 Means and standard deviations for picture-specific utility value

Additional numerical information in pictures		
Superfluous (EG-S)	Useful (EG-U)	None (CG)
3.03 (0.70)	3.65 (0.59)	3.5 (0.73)

16.4.1 Picture-Specific Utility Value

To answer the first research question, we compared the utility value for the students who rated the pictures that contained additional superfluous numerical information (EG-S), additional useful numerical information (EG-U), and no additional numerical information (CG). Table 16.1 shows that the utility value means differed across the three groups.

As expected, students gave the lowest utility value ratings to the pictures that contained superfluous additional numerical information and the highest to the pictures that contained useful additional numerical information. There were significant differences in picture-specific utility value between the three groups, $F(2, 107) = 8.41$, $p < 0.001$. A post-hoc analysis using a t-test confirmed significant differences between the EG-S and CG ($t(71) = 2.83$, $p = 0.006$, $d = 0.66$) and between the EG-U and EG-S, $t(71) = 4.11$, $p < 0.001$, $d = 0.96$. No significant difference was found between the EG-U and CG, $t(72) = -0.97$, $p = 0.337$, $d = 0.23$.

16.4.2 Modelling Performance

The second research question referred to the comparison of modelling performance in the three groups. Table 16.2 shows that the means and standard deviations of the modelling performance scores differed across the three groups.

In contrast to our expectations, students' modelling performance in the EG-S and EG-U was close to each other and slightly higher than in the control group. The ANOVA showed that there was no statistically significant difference in the three groups' modelling performances, $F(2, 105) = 1.43$, $p = 0.244$.

Table 16.2 Means and standard deviations for modelling performance

Additional numerical information in pictures		
Superfluous (EG-S)	Useful (EG-U)	None (CG)
6.30 (2.57)	5.97 (2.24)	5.02 (3.05)

16.5 Discussion

16.5.1 Additional Useful Numerical Information

According to the integrated model of text and picture comprehension (Schnotz 2014), the conditions needed to create a positive multimedia effect are that the text and the picture are semantically connected to each. In this study, we expected that the coherence between the text and the picture and thus the picture-specific utility value would be higher in the EG-U and that a stronger multimedia effect would increase the modelling performance results. However, against our expectations, students assigned similar utility value to the pictures that contained additional useful numerical information (EG-U) and the pictures that did not contain additional numerical information (CG) with respect to understanding the task. A similar finding was revealed for modelling performance. One reason for these results might be that the difference in coherence of text and pictures between the EG-U and the control group was too small in our study.

16.5.2 Additional Superfluous Numerical Information

In line with our expectations, the EG-S showed the lowest utility value in this study. Furthermore, we expected that the additional superfluous information in the picture would increase the extraneous cognitive load and overload working memory for some students and decrease their modelling performance. Contrary to what we expected, the EG-S showed the highest modelling performance, even though it did not differ significantly from the other groups.

It is possible that the superfluous information may have led the students to study the pictures more intensively, thereby supporting their overall understanding of the situation. According to this view, recognising that the information was superfluous would be an example of one kind of cognitive load required to understand the learning material. Thus, the pictures that contained the additional superfluous information would result in an increased germane cognitive load, which could have a positive effect on learning and would explain the slight increase in modelling performance in the EG-S.

16.5.3 Overall Discussion and Implications

The results provide initial indications of the effect of different types of numerical information in pictures that accompany modelling tasks. Students assign higher utility value to pictures that provide additional useful numerical information than to pictures with additional superfluous numerical information. However, a higher perceived utility value of pictures with additional useful numerical information did not result in an increase in modelling performance. One possible explanation for this result is that assigning numerical information to the appropriate object in the picture might not be the main barrier to solving modelling problems. Other modelling activities such as noticing that information is missing from the problem or making assumptions about missing information were found to prevent students from finding realistic solutions and solving modelling tasks (Krawitz et al. 2018).

Pictures with representative function can be designed differently. In our study, we selected pictures as visual representation of the described situation in the text. Moreover, it does not include any extraneous information (such as dogs, boats, etc.) that might distract problem solvers from constructing mathematical models. The results might be different for other implementations of representational pictures.

The results of our study offer initial implications for the design of pictures in modelling tasks. The findings on utility value indicate that students noticed pictures while solving modelling problems. Thus, additional numerical information that is included in the pictures can influence the modelling process to a considerable extent. We therefore suggest that teachers should think about designing pictures in modelling tasks and prepare them conscientiously.

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Part IV
Assessment Issues

Chapter 17
Validity of Mathematical Modelling Assessments for Interdisciplinary Learning

Cigdem Alagoz and Celil Ekici

Abstract Teaching mathematical modelling produces interdisciplinary learning outcomes that can be measured with formative assessments. Building, defining, and clarifying the interdisciplinary competencies involved in the modelling performance assessment tasks require the input of content experts from multiple disciplines. These interdisciplinary perspectives create the foundation for a valid modelling assessment before administering and interpreting its results. The validation process involves scoring, interpretation and uses, and consequences of interdisciplinary mathematical modelling assessment results. Confirmatory factor analysis indicated construct validity for a mathematical modelling assessment with two higher order factors indicating conceptual and procedural dimensions of interdisciplinary learning enacted by mathematical modelling.

Keywords Validity · Assessing mathematical modelling · Assessment validation · Complex learning · Interdisciplinary learning

17.1 Introduction

Mathematical modelling is a research-based teaching practice with interdisciplinary collaboration for K-20 mathematics, science, and technology education advocated around the USA, Europe, and the globe (Andresen 2009; Blum 2015; Borromeo Ferri 2013; NGACBP & CCSSO 2010; NGSS 2013). As a cognitively demanding activity, the mathematical modelling involves non-mathematical competencies and extra-mathematical knowledge (Blum 2015). Interdisciplinary learning tends to cross traditional boundaries between disciplines building on a fact, quality, or condition that brings two or more academic fields (Roth 2014). Interdisciplinary mathematics

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education taps into mathematical modelling as a process to generate conditions motivating crossing boundaries between disciplines to bring together relevant disciplinary perspectives for a common modelling purpose (Ärlebäck and Albarracín 2019; Williams and Roth 2019). Mathematical modelling can be implemented as a prototype for interdisciplinary mathematics education demanding a coordination and assessment of interdisciplinary complex learning outcomes (Borromeo Ferri and Mousoulides 2017). Michelsen (2015) highlights that the mathematical modelling can involve both mathematical and science competencies. Intradisciplinary and interdisciplinary complex learning outcomes are emergent during the process of mathematical modelling (Zawojewski et al. 2008). The defining challenge in mathematical modelling assessment is how to account for learning through connections within and across disciplines as students engage in mathematical modelling tasks.

Boaler (2001) called for new theories of learning to account for the nature of the complex learning occurring during mathematical modelling. We need to recognize that if complex learning outcomes during modelling are to be assessed well, then an assessment that conveys this complexity is needed. We believe that the interdisciplinary practices of mathematical modelling should be an area of research requiring attention in developing valid measurements and assessments of mathematical modelling competencies accounting for their use in practice across disciplines. Here, we present an argument-based approach towards establishing validity of mathematical modelling assessments for interdisciplinary learning.

17.2 Problem and Background

Teaching mathematical modelling allows us to teach certain learning outcomes that are not readily available in traditional instruction (Boaler 2001). The measurement of the mathematical modelling skill set requires non-traditional measurement tools to inform our teaching theory and practice. Mathematical modelling as an interdisciplinary activity system (Williams and Roth 2019) offers a driving goal to bring perspectives from different disciplines for learners to confront, interpret, and process new understandings, practices, and motives in response to a modelling situation. Building on our previous work with mathematical modelling for STEM Education (Ekici et al. 2018), the validity of mathematical modelling assessments is considered here for interdisciplinary learning. For example, when modelling the population dynamics of the queen conch, which is known to be a tasty, nutritious, and overfished sea snail in the Gulf of Mexico, this mathematical modelling activity requires knowledge of fisheries, biology, sustainability, cultural habits, et cetera. The assessments should properly align with the interdisciplinary learning with mathematical modelling. The validity of mathematical modelling assessments requires interdisciplinary perspectives involving different content experts and relevant community members. This validity claim can be evaluated by analysing the psychometric properties of a modelling measure. When interdisciplinary competencies are heavily involved in mathematical modelling performance tasks, their aligned assessment requires content expert input from multiple disciplines in building, defining, and clarifying the interdisciplinary competencies involved. This interdisciplinary input from multiple disciplines creates the foundation for a valid assessment before its administration, and interpretation of its results. In the USA, mathematical modelling is integrated into standards for teaching and learning secondary mathematics and science education. We investigated the validity of a standardized mathematical performance assessment towards measuring mathematical modelling performance with the interpretation and use of performance traits with affordances and challenges for interdisciplinary learning. In this study, we examined the validation process for assessments towards accounting for interdisciplinary learning with mathematical modelling.

17.3 Theoretical Framework

While there are multiple frameworks for validity, there has been a shift in the theories developed since the 1950s. Earlier theories dating back to the 1950s used many types of validity such as correlational, content, and construct. During the 1980s, the source of evidence to establish validity became more of a concern (Messick 1989). The validity theories of that era focused on various areas such as content, response process, internal structure, relation to other variables, use of scores, and consequences. Kane (1992) takes a scientific approach to validity where assumptions are first made and then evaluated resulting in hypotheses, and their analyses. From this perspective, every assessment has a claim or an argument about a competence measure. Interpretations and uses of scores are only valid when appropriate evidence is provided (Kane 1992). Kane's (1992, 2013) argument-based approach to validity is adopted here. This approach considers scoring, generalization, extrapolation, decision, and use.

17.3.1 Argument-Based Approach to Validity

The argument-based approach to validation involves two kinds of arguments. An interpretation/use argument (IUA) explicates the reasoning behind the proposed interpretations and uses of test scores (Kane 2013) and articulates clearly what is being claimed. The validity argument provides an evaluation of the IUA. Once developed, it provides a framework for test development and validation, offering criteria for evaluating the proposed interpretations and use. The validation of a score interpretation involves an investigation of whether the scores mean what they are supposed to mean, and the interpretation is said to be valid if the claims inherent in the interpretation are supported by appropriate evidence. Establishing validity involves a hypothesis about a specific interpretation or decision focused on a specific construct and a collection of evidence to support or refute the hypothesis about the targeted competencies.

Our approach to providing a validity framework for a mathematical modelling assessment involves identifying and addressing the potential uses and interpretations of the scores. The following inferences are identified as related to a mathematical modelling assessment. Interpretation and use arguments most commonly include inferences and assumptions from scoring, generalization, extrapolation, theory-based inferences, and score uses.

17.3.1.1 Scoring Inferences

Using a unidimensional and continuous scoring method to measure the mathematical modelling achievement starts with the assumption that mathematical modelling achievement is a unidimensional construct. The scoring process that places students on a unidimensional continuum is based on the claim that we can order students' mathematical modelling skills on a unidimensional continuum. If we consider the outcome as a dichotomy, we claim that students either have mathematical modelling skill or not, rather than having this skill to some degree. Consequently, these inherent assumptions carried in scoring processes have implications for the construct validity of mathematical modelling.

17.3.1.2 Inferences of Score Uses: Consequential Validity

After the scoring, the consequences of using the scores provide consequential validity evidence. If the scores determine a students' achievement in mathematics, introduction of non-mathematical skills in mathematical modelling tasks poses problems to the validity. On the other hand, avoiding the multidimensional, and interdisciplinary measurement due to its complexity has detrimental consequences in our instructional practices such as resorting to teaching simplified skills which do not prepare our students for real life. The use of simpler unidimensional and multiple-choice measurement for mathematical modelling could have unexpected and unintended consequences that challenge their validity.

17.3.1.3 Theory-Based Inferences: Construct Validity

A mathematical modelling construct is multidimensional. There are various theories about the components and measurement of the modelling construct (Zöttl et al. 2011; Hankeln et al. 2019). Learning outcomes that are expected to be taught and to be learned during mathematical modelling instruction are informed by the theory-based definition of the mathematical modelling construct. Dimensions of this construct can be explored and confirmed with a measurement administration and analysis of the results. One goal of the measurement is to evaluate the multidimensionality of interdisciplinary mathematics learning as operationalized with a mathematical modelling rubric.

17.3.1.4 Generalization and Extrapolation

One important aspect of validity of an assessment is established by the degree that assessment can be generalized. Learning outcomes of a mathematical modelling instruction are not restricted to a certain level of mathematical knowledge. One modelling problem can be approached and solved with the use of different mathematical and science knowledge levels. Consistency of the construct across ages and grades can be evaluated to provide evidence for the use of the scores.

17.4 Method and Data

17.4.1 Setting and Participants

Forty-one in-service secondary mathematics and science teachers participated in modelling and assessment workshops conducted by the chapter authors during the Summer of 2017 and 2018. Directed by the second author, teachers had been collaborating in developing and implementing interdisciplinary projects on locally relevant issues since 2015. This research problem emerged from the need for valid assessments that can measure interdisciplinary learning through modelling for STEM education. The workshop activities focused first on mathematical modelling as a common interdisciplinary practice for integrated STEM learning and then on valid assessment of mathematical modelling for interdisciplinary learning. In-service STEM teachers engaged in the modelling process with locally relevant problems such as the queen conch population. In total, 28 teachers completed mathematical modelling assessment tasks. An assessment and validity workshop facilitated theoretical and practical training on measuring and assessing mathematical modelling and interdisciplinary learning. As a measure of modelling performance, we adopted a commonly used rubric developed by the New York Performance Standards Consortium (2016). The teachers analysed and discussed the assessment results with the rubrics providing a hands-on training of content and processes involved in interdisciplinary mathematical modelling practice. Participants revisited the modelling cycle again working with interdisciplinary pairs and provided feedback. GAIMME's modelling cycle (Garfunkel and Montgomery 2016, p. 31) was given to reflect on modelling phases and components during the assessment task such as "defining the problem, making assumptions, defining variables, getting a solution, assessing the model" (p. 197). Our research study employed mixed methods using quantitative analysis for reliability and construct validity of measurements, qualitative methods for interpretation and use arguments on validity.

Our design of the professional development for modelling assessment and validity was informed by Blum and Borromeo Ferri (2009). We attended to the following four dimensions for STEM teachers' pedagogical content knowledge (PCK) on mathematical modelling and assessment: (1) theoretical dimension (modelling cycles, phases, assumptions, interdisciplinary perspectives), (2) task analysis (multiple solutions, connections), (3) instructional dimension (culturally responsive pedagogies; anticipated interventions), and (4) a diagnostic dimension (students' difficulties).

17.5 Results

17.5.1 Scoring Inferences

Scoring of a complex formative assessment is achieved in multiple steps. The scoring process involves multiple raters from different backgrounds. Training was provided on scoring with the performance assessment rubric. The rubric served as a standard setting tool. Multiple raters scored each paper and inter-rater reliability was acceptable for three factors, while it was low for communication. The rubric identified the dimensions, and provided explanation for these dimensions. Scoring is undertaken with a cognitive diagnostic classification model and diagnostic feedback is provided from the scoring. This scoring has provided validity evidence consistent with the formative nature and instructional purpose of the assessment.

17.5.2 Inferences of Score Uses: Consequential Validity

Since the purpose of the mathematical modelling assessment was instructional, it was designed as a tool to support teaching mathematical modelling. While taking the assessment, participants assumed the role of students, teachers, and raters. Teachers scored papers in interdisciplinary teams composed of secondary mathematics and science teachers. This allowed them to experience the whole assessment process from multiple perspectives.

The assessment results for science teachers had direct consequences on their instructional practices in science classes. For example, in response to the modelling assessment, several participants were observed to have made scientifically unsound assumptions with their constants and the key variables related to queen conches. Science teachers paid more attention to assumptions made helping the interdisciplinary team to identify key variables and constants drawing from their scientific knowledge, local policies and practices impacting the queen conch population such as their harvesting age and abundance rates such as 25 adults per acre for ecological self-sustenance. The mathematics teachers were benefiting from the contextually relevant scientific knowledge science teachers were bringing. On the other hand, we observed science teachers struggled with *the mathematizing phase* with the exponential or logistic growth models; for example, one science teacher noted: "It is not clear to me how this activity fosters reasoning and proof skills".

Table 17.1 Factor loadings from confirmatory factor analysis (CFA) of mathematical modelling

	Conceptual factors	Procedural factors
Problem solving	0.340	**0.709**
Reasoning, justification, proof	**0.576**	0.483
Communications	**0.983**	0.178
Connections	**0.742**	0.204
Representations	0.124	**0.992**

When asked to evaluate the measurement process and its consequences, teachers focused their comments on the rubric. The descriptions of performance indicators provided by the rubric are accepted as the standard by teachers. This highlighted the importance of vetting the rubric by the practitioners thoroughly before adoption allowing them to consider its use and consequences.

17.5.3 Theory-Based Inferences: Construct Validity

The construct is hypothesised to be multidimensional. Multidimensionality analysis is provided as construct validity evidence for the use of mathematical modelling scores. Maximum likelihood extraction is used with Varimax rotation. A two-factor model adequately fits to the observed data with $\chi^2 = 1.4$ and "$df = 1$", $p = 0.229$.

From the factor analysis as seen in Table 17.1, *reasoning, justification, and proof*, *communications*, and *connections* are clustered together which we interpreted as referring mainly to *conceptual* competencies in interdisciplinary mathematical modelling. Secondly, *problem solving* and *representations* are clustered together aligning with *procedural* competencies during interdisciplinary mathematical modelling. We found that interdisciplinary mathematical modelling assessment tasks have balanced conceptual and procedural factors each of which are dominated by *communication* and *representations,* respectively. The CFA measurement model depicts the factors and their clusters as higher order factors as shown in Fig. 17.1. These results show that this construct is valid for interdisciplinary modelling assessment.

17.5.4 Generalization and Extrapolation

Mathematical modelling assessments have their own set of disciplinary and interdisciplinary competencies and skills. The pattern emerging among teacher reflections collected after the validity session indicated that there are three leading factors of the mathematical modelling assessment works for interdisciplinary learning: "communication", "representation", and "connections", aligning with their overall CFA

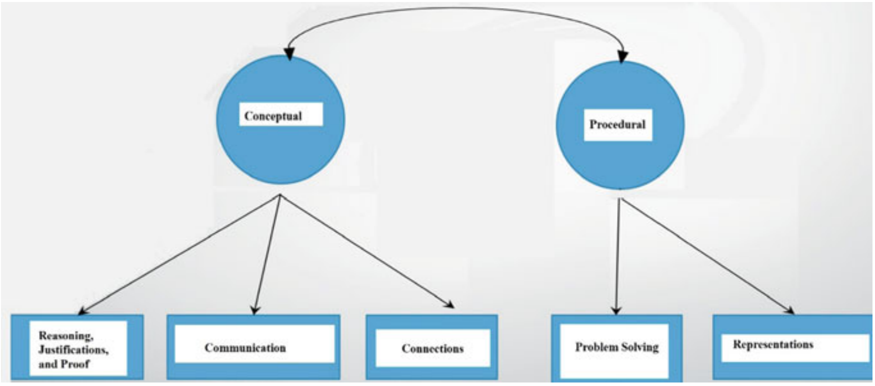

Fig. 17.1 Path diagram for mathematical modelling as measured by the existing rubric

factor loadings. Reflections indicated that science teachers found this rubric working best in assessing interdisciplinary modelling competencies in communication and connections. While they agreed on representations as critical in interdisciplinary modelling assessment, there was a lack of consensus between science and mathematics teachers' understandings and practices with representations. Mathematics teachers initially struggled on the *connection* and *communication* competencies. They benefited from working with science teachers and interdisciplinary experts providing the support with interdisciplinary connections, building on locally relevant justifications of their assumptions from local scientific perspectives.

Based on the assessment results, the generalization and extrapolation require interdisciplinary participants to reflect on how they can adopt this, what they can do next at their grade levels and across grade levels. This critical adoption process helps interdisciplinary teachers to generate plans for adopting the mathematical modelling assessment for their practice. The consistency of the construct across ages and grades can be evaluated after teachers adopt it and implement it in their practice to provide evidence for the use of scores. Building on modelling assessments as case studies for interdisciplinary learning, this process can create new claims to be tested to expand mathematically, scientifically, and contextually relevant knowledge and practices aligning their use and interpretations across mathematics and the sciences.

17.6 Discussion

Communication is a critical performance element for interdisciplinary learning. Teachers in this study realized that interpretation of the model brings back the problem into its original context. Discussing the problem with interdisciplinary teachers allows teachers to question their earlier assumptions.

Based on their background, teachers were able to identify intra-disciplinary connections and multiple viable solutions when they were analysing "the task dimension" of the mathematical modelling assessment. One of the modelling assessments was based on the population dynamics modelling for the queen conch population using different harvesting schemes. Depending on how the problem was formulated, modelling "the changes in the proportion" of harvested queen conch reduced the problem into a quadratic model. This re-framing decision for mathematizing made the model more accessible. Otherwise, science teachers and middle school mathematics teachers experienced problems with the logistic model in its exponential and continuous form. "Doing the math" phase impacted "how the problem is framed" mathematically. Ekici and Plyley (2019) demonstrate that mathematical modelling tasks in growth modelling for lionfish or the conch population can be mathematically framed with discrete, continuous, and stochastic models to generate alternative pathways for the mathematizing phase towards building intra-disciplinary connections with implications on the intradisciplinary learning outcomes from the modelling process.

Cultural context with its motives (Roth and Williams 2016) shapes the interpretations and how to set up the problem. To better differentiate, teachers were asked first to work in pairs on culturally relevant mathematical modelling tasks, such as lionfish and queen conch population modelling. Motives were compared in modelling the harvesting schemes for conch and lionfish. For conch population, modellers wanted to keep the population alive for long-term sustainability of a desirable population as a part of the livelihood of the ecological system. On the other hand, the targeted harvesting goal for lionfish population was set to eradicate this invasive species due to their threat to the ecological balance, directly or indirectly, with their high rate of reproduction and growth, their voracious feeding capacity and lack of predators. In addition, in modelling the conch population, participants are expected to examine different conch harvesting scenarios, revising the growth functions, and harvesting at different rates.

17.7 Conclusions

In collaborative interdisciplinary modelling projects, there are critical roles for interdisciplinary content experts in the mathematical modelling assessment design and validation. The mathematical modelling problems with science and engineering contexts benefit heavily from the rich contextual discussion provided by the science and engineering educators in evaluating the assumptions and interpretation of the targeted common modelling competencies from their disciplinary perspectives. Towards making mathematical modelling more culturally and socially responsive, the learning community should be inclusive of relevant interdisciplinary perspectives supported by science, technology and engineering teachers, and community partners who can be involved in and out of the classroom during the mathematical modelling process. We need to reconsider the modelling assessments to be more inclusive to perspectives from relevant disciplines towards interdisciplinary learning. Construct validity essentially starts with an articulation process in determining what to assess in terms of mathematical and interdisciplinary learning outcomes informing the construct design of complex learning assessment, interpreting the meaning of the performance construct and the scores. This process should be informed by multiple disciplinary standpoints within and across disciplines. The assessment validation for interdisciplinary learning with modelling requires teacher collaboration to interpret, use, extrapolate, and generalize the results for a coordinated mathematical modelling community of practice across subjects in schools.

Using CFA, we identified that there are two high-order clusters for interdisciplinary mathematical modelling—the *conceptual dimension* and the *procedural dimension.* This result aligns with the two-dimensional model for the sub-competencies identified by Hankeln et al. (2019). The *conceptual dimension* refers to reasoning justification, connections, and communication, concurring with the combined interpreting and validating sub-competencies observed by Hankeln et al. In contrast, the *procedural dimension* as identified here refers to problem solving and representations, aligning with Hankeln et al.'s (2019) second dimension that combines simplifying and mathematizing. Their model fits well for two but better with four dimensions.

In establishing validity, assumptions are not trivial in setting up the model with its interdisciplinary and intra-disciplinary connections as assumptions, problem posing, and formulation are often critical parts impacting on how the content is enacted during the mathematical modelling process (Galbraith and Stillman 2001). The validation of the assumptions in a mathematical modelling assessment requires interdisciplinary collaboration.

17.7.1 Future Directions

This study presents an approach to facilitating a standard-setting like process for mathematical modelling assessments. The modelling assessments should align and validate interdisciplinary and intradisciplinary perspectives utilized in authentic mathematical modelling assessment tasks. We suggest involving interdisciplinary groups of teachers as users and experts in learning outcomes in the validation process for developing and revising the assessments for mathematical modelling.

There is a clear need for differentiation and consensus building in interdisciplinary mathematical modelling assessments. With the mathematical modelling assessments, relevant domain analysis should be performed by mathematics, science teachers, and faculty and content experts.

Different scoring guidelines, according to their purposes, should be established to ensure validity. The same scoring or assessment rubrics should not be used interchangeably for formative and summative assessments. This has implications for the construct related dimensions of assessments. The consequential validity helps to articulate for interdisciplinary practitioners the relevant information emerging from mathematical modelling assessments so as to inform their practice in mathematics and science classes at different levels. Mathematical modelling assessments must be flexible to account for multiplicity of solutions depending on the assumptions made, how the problem is formulated, and how the theoretical framework is used for its mathematization with discrete, continuous, stochastic, or deterministic methods.

Broader questions we pose requiring further investigations are how to value, evaluate, and validate interdisciplinary learning outcomes with mathematical modelling with targeted assessments. By positing mathematical modelling as an interdisciplinary practice across natural and mathematical sciences, collaborative research should address how a series of modelling assessments can be designed to examine the conditions for the transfer of interdisciplinary learning within and across disciplines. As mathematical modelling plays a pivotal role in cultivating interdisciplinary learning through collaboration, valid and reliable assessments are required to measure its potential for intra-disciplinary learning by tapping into mathematizing from multiple disciplinary perspectives.

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Chapter 18
Measuring Students' Metacognitive Knowledge of Mathematical Modelling

Lena Frenken

Abstract The support of modelling in school is a common issue in investigations and in the relevant literature on modelling competence. In this chapter, research is presented on constructing a test instrument for assessing metacognitive knowledge of modelling. Based on a theoretical definition of the term "metacognitive knowledge" and its domain-specific connection to mathematical modelling, a large number of items were developed. The scalability and possible reduction of items are analysed in this chapter. The process of item construction and evaluation is described in detail. With the help of a one-parameter Rasch analysis, it can be deduced that a selection of items is suitable for measuring at least some aspects of metacognitive knowledge of mathematical modelling.

Keywords Metacognition · Metacognitive knowledge · Item development · Assessment · Measurement · Rasch analysis

18.1 Introduction

Metacognition is—among other aspects such as sub-competencies or one's own attitude towards modelling—important for a successful holistic modelling process (Kaiser 2007; Tanner and Jones 1995). Furthermore, studies have shown that digital tools enrich and change modelling at school, for example, through using appropriate tools or presenting the problems in a more realistic way (Molina-Toro et al. 2019). Nevertheless, the question of how technology should be used effectively for modelling at school has not yet been answered satisfactorily (English et al. 2016) partly due to ongoing developments of technology. Especially the imparting of metacognitive knowledge, as a selected subcategory of metacognition, could be considered as a viable possibility for promoting students' modelling competencies,

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while learning in a computer-based learning environment, because of the associated self-regulated working formats (Veenman 2007). Furthermore, the correlation between modelling sub-competencies and metacognitive knowledge has not yet been investigated (Hankeln et al. 2019). These aspects are intended to be investigated in the project *Modi—Modelling digitally,* which comprises the conceptualization and development of a test instrument on metacognitive knowledge of mathematical modelling, among other things.

Initially, a detailed description of the term metacognitive knowledge including the establishment of a connection to mathematical modelling is presented in this chapter. In addition, the test construction and its analysis are described so that conclusions can be drawn about the use of the test instrument and the implications of the results for modelling.

18.2 Theoretical Background

In order to conceptualize a test instrument that assesses students' metacognitive knowledge of mathematical modelling, it is necessary to first understand its underlying concepts. It is well known that metacognition, which is the paramount concept of metacognitive knowledge, is fuzzy (e.g. Flavell 1981; Schoenfeld 1987), often used, but often criticized as well, due to a lack of precise definitions. However, in this chapter, similar terms are delimited in contrast to metacognitive knowledge, and a construct definition that can be used for research in mathematics education is targeted as a result.

18.2.1 Metacognitive Knowledge

Following a fundamental definition of metacognition, the term can generally be understood as "knowledge and cognition about cognitive phenomena" (Flavell 1979, p. 906). A differentiation between levels of cognition makes clear that metacognition is part of cognition itself, whereby cognitive processes entail defining objects of other cognitive processes (Nelson and Narens 1990). For example, solving a system of linear equations involves a cognitive process and in contrast, answering the question of how well someone solves such a system initiates a cognitive process about the previous cognitive processes. To answer the latter question, proceeding metacognitive activities mainly affect the answer. Thus, different cognitive levels become evident. Because metacognition is still a broad field and the explicit contents are not obvious, a partition of the concept into two to four theoretically considered, interacting aspects is undertaken by several authors (e.g. Flavell 1979; Scott and Levy 2013; Vorhölter 2018). For the Modi project, the division into metacognitive knowledge and metacognitive skills is fundamental. To divide these terms, a clear distinction is needed. In doing so, the most important aspect is to differentiate between the knowledge that individuals accumulate through various situations over time about cognitive processes and the actions that individuals undertake to regulate cognitive processes. Metacognitive knowledge means, for example, being aware of the fact that the Gaussian algorithm is an appropriate strategy for solving a system of linear equations. In contrast, the decision to execute the Gaussian algorithm and not the addition method for solving a given system of linear equations can be seen as a performed action and therefore constitutes a metacognitive strategy, because it regulates the cognitive process of solving it. At this point, a question arises as to the differentiation between cognitive strategies and metacognitive strategies, but is beyond the scope of this chapter. Hence, here the author refers only to the distinction between the two levels that were used to clarify the differences between cognition and metacognition. Further, knowledge is verifiable, and thus can be rated as wrong or right (Bolisani and Bratianu 2018). Taking a more detailed look at metacognitive knowledge, Flavell (1979) considers three facets of the knowledge about influencing factors in cognitive processes: person, task and strategy. The personal factors can again be divided into knowledge or beliefs about *inter* individual differences, *intra* individual differences and universals that are generated from the experienced differences. Thus, an example (again about solving a system of linear equations) of the person category would be that there still is a lack of understanding on the Gaussian algorithm—with regard to oneself, to the person sitting next to oneself or to the whole class. Knowledge about the possible number of solutions when solving a system of linear equations, ranging between none, one or infinitely many, constitutes an example of the task variables. Knowledge about the Gaussian algorithm as an appropriate strategy was already mentioned regarding clarifying the differentiation between metacognitive knowledge and metacognitive strategies and can be related to the strategy variables. Furthermore, knowledge about the aims of the Gaussian algorithm can be mentioned as an example of this category.

Summarizing, metacognitive knowledge in this study is used as the generic term for verifiable, domain-specific knowledge about the factors that affect cognitive processes, which can be considered as relating to knowledge about the involved person(s), about the tasks to solve and about appropriate strategies, including their aims and objectives.

18.2.2 Metacognitive Knowledge of Mathematical Modelling

Mathematical modelling processes are executed as cognitive activities (Blum and Leiss 2007). Accordingly, taking into consideration the domain-specific characteristics (Veenman 2007), cognitive processes and thinking about mathematical modelling can be seen as metacognitive activities of mathematical modelling. The modelling processes therefore become the object level (Nelson and Narens 1990). Specifying this basis with regard to the focus of the chapter, metacognitive knowledge can also be defined by referring to mathematical modelling. Thus, following the summary in Sect. 18.2.1, metacognitive knowledge of mathematical modelling is used as the term relating to knowledge that affects the execution of modelling processes, and can be considered as divided into the categories *person, task and strategy*. It should also be taken into account that knowledge can be assumed as verifiable. The personal variables can be interpreted as knowledge about difficulties encountered during the modelling process, accumulated through learning about oneself and one's problems as a modeller (intra-individual differences), compared to problems of other modellers such as classmates (inter-individual differences). The task variables contain knowledge about properties and characteristics, in this case, of modelling tasks. This includes information about the design of modelling tasks, as well as about possible structures of solutions and modelling processes. Underlying the two facets already described, the strategy variables refer to the objectives behind appropriate strategies on the one hand and knowledge of a repertoire of useful strategies on the other hand. Referring to modelling, knowledge about a solution plan, different reading strategies, searching for an analogy, making a drawing or verifying the solution by comparing it with known sizes, can all be mentioned as appropriate strategies during the process of solving reality-based problems (e.g. Schukajlow et al. 2015; Stillman 2004; Vorhölter 2018). Summarizing, metacognitive knowledge of mathematical modelling is part of a competency that includes memorizing facts about different strategies, properties of modelling and potential difficulties during the process.

Though Cohors-Fresenborg et al. (2010) state that procedural metacognition is the aspect of metacognition that is important for modelling, previous investigations have shown that some aspects of the above-mentioned definition of metacognitive knowledge are crucial for a successful modelling process as well or could at least influence it positively. For example, important relations between knowledge about different models such as the real or the mathematical model and their setting, by identifying misconceptions related to difficulties or errors, were provable (Maaß 2007). Furthermore, it was shown that the awareness of different strategies and their aims is a basis for decision-making when working on real-world problems (Stillman and Galbraith 1998). Nevertheless, a lack of investigations on the structure of metacognitive knowledge (of mathematical modelling) is conspicuous. Finally, the question of an existing correlation of metacognitive knowledge and modelling competence has been raised (Hankeln et al. 2019) and could help fill the gap in investigations on the influence of metacognitive aspects of students' modelling processes (Vorhölter et al. 2019).

18.3 Method

Because no quantitative test instrument on metacognitive knowledge about mathematical modelling has yet been constructed (Vorhölter 2018), this chapter addresses the evaluation of newly developed items. Therefore, an attempt is made to answer the following question:

Is it possible to measure metacognitive knowledge of mathematical modelling as a latent construct?

The author assumes that not all of the constructed items statistically fit the global test instrument, although this is the requirement for investigating student differences or developments concerning the metacognitive knowledge of mathematical modelling. However, several steps with regard to ensuring the quality of items are undertaken. These various elements are presented in the following section, whereby the focus is laid on student performance in five different classes and an analysis of them.

18.3.1 Item Construction and Data Collection

With respect to the research question posed, several steps were undertaken. First of all, the items were formulated and designed on the basis of the theoretical concepts on the structure and contents of metacognitive knowledge, whereby the aim was to create items which can be rated as either wrong or right. Afterwards, the items were given to experts, with the instruction to fill in the test and mark all critical aspects. The specialists, who all work at the Institute of Education in Mathematics and Computer Science at the University of Muenster, conducting research on either test construction or mathematical modelling, gave detailed advice. Especially, discussions about the coding of items and the scales used were included. The content validity was also ensured in this way. On this basis, items could be reformulated in a first round and the items for the next step—the testing in classes—were selected. The distribution of selected items across the categories *strategy, task and person* is shown in Table 18.1.

As can be seen from Table 18.1, in total, 39 items were to be included. Because the processing time for solving a test with all items in one lesson of 45 min was expected to be too long for 15-year-old students, two different versions of the test

Table18.1 Overview of the items and their distribution across the variables

Version A		**Version B**
Introduction		
Personal data		
Strategy repertoire I (6 Items, short answer)	2 anchor items	Strategy repertoire II (6 Items, short answer)
Person category (7 Items, Combined-Single-Choice)		
Task category (8 Items, Combined-Single-Choice)		
Strategy aims (14 Items, Combined-Single-Choice)		

were developed. Each version consists of 35 items, which may not seem like a large reduction, but it should be taken into account that the quantity of the most time-intensive items was indeed reduced. Furthermore, a qualitative study was conducted within the research context of a master thesis, which aimed to identify the difficulties students encountered while answering the test. On this basis, the results of the statistical analysis can be reconsidered. Before the results of the statistical analysis are presented, an overview of item examples is provided. The test was administered to five classes, at two schools, of 15-year-old students, whereby 115 students participated in total.

18.3.2 Item Examples

The description of selected items follows the structure of the test, which is shown in Table 18.1. Before the metacognitive knowledge is assessed, the test starts with an introduction and a short query on personal data. Afterwards, one item block on the strategy variable is used to assess the repertoire. The strategy variable had to be divided into two parts in the test, because in previous investigations on strategy knowledge, there were criticisms that the assessed strategies were mentioned explicitly (e.g. Pintrich et al. 1993).

Items with the following expected and as correctly rated strategies were designed: solution plan, searching for an analogy, making a drawing, verifying the solution by comparing with known sizes and reading strategies. As shown in Fig. 18.1, each of the items on strategy repertoire consists of a modelling task, a dialog between two

2.1 Record Nail
In order to present the name of his restaurant, Mr. Nail set up an oversized steel nail in front of it. He took a picture of this nail. It is about 7 m long and has a diameter of about 22 cm. A commercial carpenter's nail is only about 20 cm long and has a diameter of 6 mm. Now the nail shall be moved. The unloading crane of the truck available for transporting the nail is 12 metres long and can lift a maximum of 1.5 t. Is it possible to move the nail with this truck?
(Adapted from Drüke-Noe et al. 2012)

Timur: So much text. Did you read everything?
Pascal: Yes, I'm done. But I don't know exactly what is important and what is not.

What would you advise Timur and Pascal? *(You don't need to solve the nail task.)*

Fig. 18.1 Example item for the strategy repertoire students who are faced with a difficulty, and the question as to what advice the student would give. Answers to the shown example that are rated correctly contain reading strategies, for example, marking important information or reading the task again. This block of items was positioned at the beginning of the test, because the participating students became familiar with various modelling tasks in this way. Therefore, the term *modelling task* could be used in the following items. Item 3.2, shown in Fig. 18.2, is an example of assessing the person variables. Because beliefs dominate this category in Flavell's considerations (1979), the inter- and intra-individual parts were not assessed. Nevertheless, universal difficulties during the modelling process were assumed to represent the knowledge gained from the two categories; therefore, one block of items was constructed, introduced by the request to think about oneself and one's classmates as modellers. During the expert discussions, the proposal was made to defuse rigid categories of true and false, by using a four-point Likert scale for the assessment (True—possibly true—possibly false—false). This still offers the possibility of conducting a one-parameter Rasch analysis to ensure the quality criterion of scalability, by interpreting the categories *true* and *possibly true* as one and the categories of *possibly false* and *false* as the alternative. Assuming that the personal variables can be statistically ascertained as part of metacognitive knowledge includes the characteristics of being verifiable. But, because the explanations of metacognitive knowledge also often include the term *beliefs*, the need for a different allocation within the map of modelling competence can be assumed, and therefore, the Likert scale was included as an alternative. An answer was thus coded as correct, when the student, for example, ticked the box for true or possibly true, while rating a correct statement and vice versa.

Concerning the assessment of the task variable—item 4.1 shown in Fig. 18.2 belongs to this category—the two categories of true and false were used directly, and the content of statements is based on properties and characteristics of modelling tasks, following a few items from the test instrument of Klock and Wess (2018). In the second part of assessing the strategy variables, the aims of the above-mentioned

3.2 When information gaps have been discovered, many find it difficult, …	True	Possibly true	Possibly false	False
… to do targeted research filling those gaps.	✓	✓		
… to estimate missing sizes.	✓	✓		
… to ask the teacher to fill those gaps.			✓	✓

4.1 Modelling tasks …	True	False
… may contain unimportant information.	✓	
… may contain too little information.	✓	
… contain exactly the information that is needed.		✓

5.5 Using a solution plan in modelling tasks is suitable for …	True	False
… doing the calculation correctly.		✓
… structuring the process of solving.	✓	
… receiving an accurate result directly.		✓

Fig. 18.2 Example items on the categories *person* (3.2), *task* (4.1) and *strategy aims* (5.5)

strategies are formulated in different statements. With item 5.5 in Fig. 18.2, an example of this category is provided as well. The chosen format of combined single-choice items reduces the probability of guessing and includes one complete item only being coded as correct, if the three accompanying statements are ticked in the right combination of (possibly) true and (possibly) false.

18.4 Results of the Quantitative Analysis

To evaluate the test instrument, the data were scaled using a one-parameter Rasch model (Rasch 1960) utilizing the software ConQuest (Wu et al. 2007). After the requirement of normal distribution of the WLE-scores was checked, items with a discrimination index under 0.2 were excluded as in PISA (OECD 2012), so that 27 items remained. The other 12 items have to be reformulated or excluded totally. On the basis of the discrimination index, the problem of assessing the category *person* becomes clear, because almost all of the items in this section had to be skipped and only three remained. The reliabilities are satisfactory; the item separation reliability amounts to 0.983 and the EAP/PV reliability, which can be compared to Cronbach's Alpha, is 0.641. The item fit statistics, which constitute another criterion for ensuring the test quality, range between 0.82 and 1.13 for the unweighted mean square and between 0.94 and 1.1 for the weighted mean square. Following Bond and Fox (2007) and PISA (2012), these values indicate a high level of quality. A further analysis of the item difficulties shows a floor effect, which means that the items were generally too difficult for the participating students. Finally, an Andersen test was conducted with a result of $p = 0.28$, which leads to the conclusion that the items do indeed measure the one-dimensional construct of metacognitive knowledge of mathematical modelling (Andersen 1973).

18.5 Summary and Discussion

This chapter focused on deducing a definition of metacognitive knowledge of mathematical modelling aiming at developing a test instrument that measures the associated aspects of this term. Items were constructed on that basis and a pilot study conducted to revise the quality criteria. It was found that the test instrument is usable for a comparison of groups, and measures metacognitive knowledge of mathematical modelling (Bond and Fox 2007; Boone 2016). Incidentally, the differentiation between measurement using a standardized, scalable test instrument, and assessment using a test instrument that, for example, does not distinguish between item difficulties, must be mentioned (Mislevy 2017). Therefore, the evolved test instrument also enables measuring parts of the competence of mathematical modelling and is an addition to existing test instruments that measure or assess other aspects, such as sub-competencies (Hankeln et al. 2019) or metacognitive strategies (Vorhölter 2018).

Nevertheless, the unsatisfactory results in the dichotomous assessment of the person category show that—despite the many investigations on the structure of modelling competence and considerations on what aspects belong to it—the holistic competence is still not fully determined. The assumption that the described aspects of the person category belong to facets like self-efficacy or motivation, which influence modelling processes and have to be assessed with other methods, can legitimately be made. To complete a series of test instruments on facets of modelling (Kaiser 2007), further analyses using the developed four-point Likert scale, and adding motivational factors or the facet of self-efficacy, should be conducted. The three remaining items on the personal variable will be excluded, and in further studies, strategy knowledge and task knowledge remain as measured aspects of metacognitive knowledge of mathematical modelling. Because the categories of metacognitive knowledge were considered theoretically (Flavell 1979) and not examined empirically, these new assumptions about the structure and content of metacognitive knowledge should be verified in other studies and different domains. However, sufficient statistical evidence is already on hand, and at least aspects of metacognitive knowledge of mathematical modelling can be measured with the instrument (Andersen 1973; Boone 2016). The test instrument will be enhanced by consulting the results of a qualitative study on difficulties encountered during the test processing. Beside the results concerning the suitability of the test instrument, the students' limited metacognitive knowledge of mathematical modelling reveals the importance of, and need to create, learning environments for mathematical modelling, as well as integrating them into schooling more often and intensively.

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Chapter 19
Mathematical Modelling in Dutch Lower Secondary Education: An Explorative Study Zooming in on Conceptualization

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Abstract In the Netherlands, mathematical modelling has become a major subject in the higher secondary education curriculum. However, it is absent from the greater part of lower secondary education. To improve the vertical coherence in the curriculum, this study explores the mathematical modelling proficiency in both primary school and lower secondary school. Additionally, this study also gains insight into the difficulties that students encounter while solving modelling tasks. The study includes two modelling tasks on three difficulty levels for 248 learners ranging from 11 to 15 years old. At each level, learners encounter difficulties when constructing a meaningful representation of the described modelling problem or may even fail to understand the problem. These representation problems are qualitatively analysed and are shown to be partially related to learners' language problems.

Keywords Conceptualisation · Context · Language proficiency · Mathematisation · Modelling tasks · Task-based interviews

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19.1 Introduction

Mathematical modelling is a major subject among the activities associated with mathematical thinking and has received more attention in recent years, including in the Netherlands. Mathematical modelling became a new component of the examination programmes for the five-year stream (HAVO) and the six-year stream (VWO) in 2015. However, mathematical modelling is absent in the greater part of lower secondary education in the Netherlands.

In the field of mathematical modelling, various representations of the modelling cycle exist. According to Blum and Leiß (2005), the modelling process begins with understanding the real situation and problem, resulting in a situation model. Then, the given situation has to be simplified, structured and made more precise, which results in a real model (Blum and Leiß 2005; Blum and Borromeo Ferri 2009). In the modelling cycle of Perrenet and Zwaneveld (2012), these first two parts of the process are taken together as the conceptualization phase, followed by mathematizing, solving, interpreting and validating. Plath and Leiß (2018) emphasize the importance of the conceptualization phase and use this as the basis for all subsequent decisions (see also Blum and Leiβ 2005; Borromeo Ferri 2006; Leiβ et al. 2019). Therefore, in this chapter, we will focus especially on the difficulties that students encounter in the conceptualization phase.

Assumption making is one of the modelling competencies used to understand a real problem and to set up a model (Maaβ 2006). Galbraith and Stillman (2001) emphasized the role of assumption making as an underrated aspect of successful modelling activity. Seino (2005) argued that assumptions are “the bridge” that connects the real world and the mathematical world. While the ability of novice modellers to make assumptions is rather weak (Chan et al. 2012), it hardly receives attention in the Dutch mathematics curriculum. Therefore, it is important to examine students’ difficulties related to assumption making, especially in lower secondary education.

Usually, modelling problems in context-rich assignments are offered to learners through texts. One of the first obstacles students may encounter is reading and interpreting text. In secondary school, being able to read a problem is a decisive factor in solving a problem (Korhonen et al. 2012). The language used at school often forms an obstacle to learning mathematics (Van Eerde and Hajer 2009). Language proficiency may play a different role in every phase of the modelling cycle. In the conceptualization phase, the student has to be able to understand the text in which the problem is posed to translate it into a conceptual model. Plath and Leiβ (2018) pointed out that the linguistic features of understanding and solving mathematical modelling tasks have not been thoroughly examined. Therefore, this study will also investigate the role of language comprehension in the conceptualization phase.

To improve vertical coherence in the curriculum, more insight is needed into the modelling ability of students in lower secondary education and the difficulties they encounter while solving modelling tasks. Therefore, this study explores two research questions: How do Dutch lower secondary students perform on context-rich mathematical modelling tasks? Which problems do they encounter in the conceptualization phase?

19.2 Method

To answer these questions, we developed two modelling tasks and two mathematical core assignments for three age groups and conducted task-based interviews.

19.2.1 Participants

The participants in the study were 73 students from Grade 6 (age 11–12), 116 students from Grade 8 (age 13–14) and 59 students from Grade 10 (age 15–16). In the Netherlands, Grade 6 is the final year of primary school, Grade 8 is part of lower secondary education and Grade 10 is part of upper secondary education. In total, four primary schools and four secondary schools with seven classes located in an urban environment participated in this research (see Table 19.1).

Schools A, B and C were primary schools and schools D, E, F and G were secondary schools. All schools were located in an urban environment. In schools B, D and E, most students were raised bilingually with different parental languages. School G had a more mixed population. The other schools, A, C and F, have more homogeneous populations whose first language is mainly Dutch.

The teacher of each class selected two students, one with strong language proficiency and one with weak language proficiency, with whom we performed task-based interviews. These teachers had taught these students for over a year. Task-based interviews were performed with 26 learners (see Table 19.2).

Table 19.1 Number of students per task, grade and school

School	A	B	C	D	E	F	G	Total
Task 1, Grade 6		16	23					39
Task 1, Grade 8						39	19	58
Task 1, Grade 10				22				22
Task 2, Grade 6	21			13				34
Task 2, Grade 8					17	41		58
Task 2, Grade 10					11		26	37
Total per school	21	16	23	35	28	80	45	248

Table 19.2 Number of interviewees per task and grade

Grade	Task 1	Task 2	Total
6	4	4	8
8	6	4	10
10	4	4	8
Total	14	12	26

19.2.2 Modelling Tasks

We designed two paper-and-pencil mathematical modelling tasks in a rich context with three difficulty levels. The first level was Grade 6, the second level Grade 8 and the third level Grade 10. The complexity increased with each level, such as by adding more data to the process (*Task 1*) or by providing a context that is further from students' daily experiences (*Task 2*). Furthermore, the modelling tasks were developed according to the design principles of Galbraith (2006) and were improved using feedback from two primary school teachers, three secondary school teachers and an independent mathematics education researcher.

Task 1

You want an iPad for your birthday. That is why your mother asks you to investigate the prices of iPad Pros. Figure 1 shows the two different sizes of the iPad Pro in inches. In many English-speaking countries, an inch is used as a measure of length.

Imagine that your mother travels the world for her work. She is able to buy an iPad for you in one of the countries she is visiting. She only does this if it is cheaper than in the Netherlands. Next week she has to go to San Francisco. That is in the USA, where they use the American dollar. Then, she travels to Singapore. That is in Asia. In Singapore, they use the Singapore dollar. The values of the various currencies against the euro can be found in Table 1. The prices of the various iPads are shown in Table 2.

Advise your mother where the best place is to buy the iPad. It is important that you also explain which format you choose and why. Explain to your mother how you came to your decision.

Task 2

Just before the holiday you organize a dance party in this classroom for the children in your grade. There will be 32 children at the party. There are a few tables and chairs and a few more closets.

1. Try to calculate if there is enough space to dance.
2. Make a map of the classroom during the party and give the dimensions.

The first modelling task consisted of an algebraic problem. In this task, all the information needed for the student to solve the task was given. Consequently, this task contained a longer text to read. The student had to discern the information relevant to construct a model. We used a single best answer question format, which is comparable to problems in mathematics textbooks and the Dutch national examinations. The task concerned a pupil who needs an iPad for school. Her mother travels the world for her work and would be able to buy an iPad in one of the countries she is visiting. The question for Grade 6 students was to calculate where the iPad is the least expensive. *Task 1* shows the shortened version of the task for Grade 6. We left out the tables showing the currencies from different countries, the iPad prices in the different countries and an image of an iPad. Grade 8 students also had to account for the Value-Added Tax, and Grade 10 students also had to calculate the import taxes.

The second task concerned geometry. The problem description was stated as an open-ended question. The task concerned the organization of a dance party. Grade 6 students had to organize a dance party in the classroom for the students in their grade, as shown in *Task 2*. The original version of this task also contained a picture of dancing children in a classroom. Grade 8 students had to organize a dance party in the school canteen and Grade 10 students had to complete the same assignment for the music hall. Students needed to calculate the dancing space for the appropriate number of party-goers and make a map of the party, including the dimensions. This second task had missing information that required students to make spatial and numerical assumptions.

19.2.3 Mathematical Core Assignment

We designed a mathematical core assignment focusing on the mathematical content without any context to identify pure mathematical problems. The mathematical core assignment of the first task focused on currency calculations, percentages and reading abilities. The students in Grades 8 and 10 had to solve an additional question with a percentage calculation. For all grades, the table showing the currencies in the different countries was given. The core assignment of task two asked for the meaning of the word area, applications of the metric system and the area calculation. The students in Grades 8 and 10 had to solve a second question regarding calculating an area and a third question for which they had to draw a 0.5 dm^2 area.

19.2.4 Task-Based Interview

We conducted semi-structured interviews with 26 students. We prepared ten main questions and, depending on the given answers, the interviewer asked clarification questions. The questions that were posed focused on the understanding of the task, text comprehension, word problems, problems the students encountered, outcomes and the approach taken, focusing on the different ways of solving the task. Examples of the questions asked include the following: Can you explain in your own words what you had to do? Are there words that you did not know, or are there sentences that you did not understand? How did you perform the task?

19.2.5 Procedure

The students had to construct their answers individually. After the modelling task was handed in, the core assignment was given. Most of the students finished both assignments in 30 min. The interviews were conducted at school directly after the assignments.

19.2.6 Analysis

19.2.6.1 Analysis of Student Work

All student answers were scored using an answer model. In addition, we highlighted (parts of) the answers that could inform us of the problems that the students encountered. Because all tasks had different total scores, we calculated the percentages of the points obtained for each student and task. A portion of the student answers were scored by a second rater ($n = 37$). A Cohen's kappa of $\kappa = 0.73$ indicated sufficient inter-rater agreement. Linear mixed model analyses were conducted in SPSS to account for the hierarchical structure of the data. In the first step of the analysis, a three-level null model (model 0) was estimated without explanatory variables. This baseline model was used to determine the variance within and between Task 1 and Task 2. In the next step (model 1), the explanatory variables, the mathematical core assignment scores, were added and the interaction between the task and mathematical core assignment (MCA). In the second step of the analysis (model 2), we included grade and the interaction between the task and grade. We ultimately excluded the school level due to the small numbers. The correlations between the scores for the modelling task and the mathematical core assignment were calculated.

19.2.6.2 Analysis of Task-Based Interviews

The interviews with each student lasted from 6 to 20 min. The audio recordings of the interviews were transcribed. We used the modelling cycle of Perrenet and Zwaneveld (2012) as a tool to analyse the students' answers (see also Kaiser et al. 2006). First, we coded the data in terms of the modelling activities of conceptualization, mathematization and solving, interpretation, validation, reflection (on the modelling process) and iteration (to improve the model). Assumption making was also added to this coding scheme. Second, we coded the problems that students encountered. Next to difficulties with understanding words and sentences (which we asked about during the interview), we used open coding with an ongoing formulation and refinement of the categories.

19.3 Results

In this section, we report the results of the paper-and-pencil modelling tasks and the results of the task-based interviews.

19.3.1 Results of the Modelling Tasks

Table 19.3 reports the average percentages of the correct answers given for Task 1, Task2 and the respective mathematical core assignments (MCA 1 and MCA 2). The students generally did not perform well on the modelling tasks, although the standard deviations indicated some variation. In Grade 6, students failed to earn half the number of points possible on Task 1 and the corresponding mathematical core assignment (MCA 1), while students from Grades 8 and 10 performed better on these tasks. In contrast, the multilevel analysis showed that Grade 6 students performed better on Task 2 than Grade 8 students ($p = 0.02$). The same effect could not be shown for Grade 6 students versus Grade 10 students ($p = 0.07$).

In each grade, the learners who performed well at the mathematical core assignment also performed better at the modelling task ($p < 0.005$). For each additional point on the mathematical core assignment, the score on the modelling task was 0.277 points higher ($p < 0.002$).

A remarkable finding is the better performance on Task 2 of Grade 6 students compared with the performance of students in Grades 8 and 10. It is possible that this group's surprisingly better performance on the second task can be attributed to the physical surrounding in which the problem of the Grade 6 students was situated (the classroom) while the problem of the Grade 8 students was the canteen of their school. Galbraith and Stillman (2001) have mentioned the significant importance of students' physical experience with the context. Therefore, we more closely examined students'

Table 19.3 Means and standard deviation on the tasks per Grade

Grade	Task 1		MCA 1		Task 2		MCA 2	
	M	SD	M	SD	M	SD	M	SD
6	41.9	29.3	46.6	19.2	59.1	20.0	48.5	26.1
8	57.1	30.3	70.7	16.6	40.1	21.4	43.4	24.3
10	57.6	24.2	82.5	15.9	47.6	19.3	45.9	24.0

drawings and noticed differences in the quality and detail of the drawings that students created during Task 2. As in Rellensman et al. (2017), we found situational and mathematical drawings but also drawings where learners experienced problems with reducing three-dimensional objects to a two-dimensional map, as well as drawings that were too abstract and have lost too much detail for the student to successfully continue with the modelling problem. The latter type of drawings occurred more often in the higher grades. The students in Grade 6 focused more on the details than the students in Grades 8 and 10.

Furthermore, we asked in MCA 2 for the meaning of the word area. Remarkably, most of the students simply provided the formula of length multiplied by the width instead of offering an explanation. Finally, in Task 1, a frequently occurring mistake was that students multiplied instead of divided in currency calculations. This mistake is related to students' understanding of the context and mathematical knowledge. In Task 2, most of the students encountered difficulties with calculating the dancing space, and in the mathematical core assignment, it appears that they had difficulty using the metric system.

19.3.2 Results of the Task-Based Interviews

The task-based interviews showed that most of the interviewed students enjoyed solving the given tasks, but they also found it difficult to make assumptions and solve the task. In addition, they indicated that they had not performed a similar task before.

Contrary to our expectations, the data did not show a substantial difference between the students with a strong language proficiency and those with a weak language proficiency. The interviews, however, illustrated that for some students, language was an important obstacle. In those cases, the learners failed to construct a meaningful representation of the described situation. The transition from reality, presented by the text, to a conceptual model stopped halfway. In all grades, most of the students repeatedly re-read the text of the modelling task and learners at each level encountered difficulties in constructing a meaningful representation of the described modelling problem, sometimes even failing to understand the problem. These representation problems were partially related to language problems. Most of the students were sufficiently able to restate the problem in their own words. They mostly agreed that the text did not contain any difficult words or sentences. Nevertheless, they still had their own interpretations and associations of the context. For example, there were students who drew a map of a classroom party for Task 2 in which the tables were in groups in the middle of the classroom instead of creating an empty dance floor. The following conversation between the researcher and a student shows how the student construed the meaning of a dance party.

Researcher: Why have you drawn the classroom in this way?
Student: Because it has to be, right?

Researcher: Have you ever been to a class party?
Student: No, not really a class party.
Researcher: What was it then?
Student: Just to a Christmas dinner in class, but not really a class party.

According to Dewolf et al. (2011) and Galbraith and Stillman (2001), the context provided in the task exerts an important effect on the interpretation and, thus, also on the solution. This student associated the context with something he recognized (a Christmas party). Although his knowledge of mathematics was sufficient, the student nonetheless failed.

These problems with students' interpretation of the context occurred in the conceptualization phase. Although some students did not reach a solution, we found that most of them were sufficiently able to explain what the task asked for. They became stuck when they had to formulate this concept mathematically. For this group of students, there seemed to be a barrier between the conceptual model and the mathematical model.

19.4 Conclusion and Discussion

In this chapter, we examined the performance of Dutch lower secondary students on context-rich mathematical modelling tasks. We compared their performance with the performance of Grade 6 (primary school) and Grade 10 (upper secondary school) students. We found that overall, students did not perform well. In Grade 8, on average, students earned 57% of the total points for task 1 and 40% of the total points for task 2. Although the tasks were assessed by different teachers, the tasks may have been too difficult. The mathematical core assignments showed that mathematical knowledge is indispensable for solving modelling tasks. Moreover, these students had not received any education focused on mathematical modelling or on making assumptions. The standard deviations indicated substantial variation in student performance. When introducing mathematical modelling in lower secondary education, it is important that teachers cater to students' different learning needs or use collaborative learning tasks in which students can learn from one another.

Our second research question focused on the problems that students encounter, particularly during the conceptualization phase. From the data, we found four types of problems: the inability to simplify, structure and make the problem story more precise; problems of context; the inability to make correct interpretations; and the lack of mathematical direction shown by making overly abstract drawings.

Many of the students encountered problems in translating the real problem to the conceptual model, in the conceptualization phase of the modelling cycle. These findings are in line with previous studies showing that students experience difficulties with reading the problem (Korhonen et al. 2012) and making assumptions (Chan et al. 2012). In all grades, most of the students repeatedly re-read the text of the modelling task, and students encountered difficulties when constructing a meaningful representation of the described modelling problem.

We found that most of the students were able to retell the problem in their own words but were unable to sufficiently solve the problem. The problem for most students seemed to arise at the end of the conceptualization phase. The conceptualization phase (Perrenet and Zwaneveld 2012) consists of the first two steps (from real situation to situation model and then from situation model to real model) of the modelling cycle of Blum and Leiβ (2005). Understanding the problem is the first step, and most of the students were successful at that stage. The second step is to simplify, structure and make the problem more precise, which is where most students became stuck. Assumption making was also a part of this difficulty.

Every student interpreted the given problem in his or her own way. In some cases, these interpretations, caused by a limited or incorrect understanding of the keywords in the problem description (e.g. dance party), led to difficulties in making correct assumptions and affected their solution of the problem. Thinking aloud would be a good addition to gain more insight into students' difficulties and interpretations. In Task 1, all the needed information was given, unlike in Task 2. For Task 2, we found that Grade 8 students experienced more problems than Grade 6 students, and we also found differences in their assumption making and drawings. This study supports the findings for students aged 13–14 years old from Kaiser and Maaβ (2007), that "strong students choose more challenging models while weaker students prefer simpler ways to achieve their final solutions" (p. 104). Students from Grades 8 and 10 tended more towards abstract drawings and models, so they experienced more difficulties in solving the problems than the Grade 6 students who kept their drawings and models fairly simple. We found that the transition from reality, presented by the text, towards a conceptual model often stopped halfway.

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Chapter 20
Investigation of the Mathematics Modelling Competency of Mathematics Undergraduate Student Teachers

Yangyu Wang

Abstract This study investigated the mathematical modelling competencies of junior mathematics student teachers (n = 273) in four universities in Jiangsu, Zhejiang, and Shanghai in China, using a scoring framework of the mathematical modelling steps. "Peeling a pineapple" was selected as the item for the modelling competency test. The study also used a questionnaire on modelling competition experience. The results show the performance of the student teachers, the differences between genders and between different types of universities, and revealed the correlation between the modelling competition experience of student teachers and their modelling competency.

Keywords Modelling competency · Modelling steps · Mathematics student teachers · Modelling competition experience · Mathematical modelling

20.1 Introduction

As one of the core competencies of mathematics (Cai and Xu 2016), mathematical modelling competency is an important part of mathematics education. The Chinese version of the *Guidelines for Assessment and Instruction in Mathematical Modelling Education* has drawn attention from the mathematics education community (Liang 2017), and mathematical modelling will become a compulsory part of the high school mathematics curriculum in China (YZZ 2017). However, determining how to teach mathematical modelling remains a challenge for teachers since students from grade 9 to 11 have a relatively weak competency in mathematics modelling (Ludwig and Xu 2010). The competency of the students largely depends on the modelling competency of mathematics teachers. It is essential to investigate and understand the mathematical modelling competency of mathematics student teachers and to enhance it, as they will be teaching mathematics in the future.

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20.2 Theoretical Framework

There are several perspectives from which to evaluate modelling competency worldwide (Kaiser and Brand 2015). Blum and Leiss (2007) have defined “mathematical modelling competency”:

> as the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation, as well as the ability to analyze or compare given models by investigating the assumptions being made, checking properties and scope of a given model etc. (p. 12)

Modelling competency is reflected in the modelling process. This study referred to the steps of mathematical modelling by Garfunkel (2016) and defined the scoring framework of mathematical modelling steps to determine modelling competency. The modelling steps for scoring are as follows:

- Step 1: Nothing is written or only a result is presented.
- Step 2: Variable is identified, and an assumption is made, but it is unreasonable.
- Step 3: Variable is identified, and a reasonable assumption is made, but the mathematical solution is inaccurate.
- Step 4: Variable is identified, an assumption is made, and the mathematical solution is accurately given, but the model is unverified.
- Step 5: Variable is identified, an assumption is made, the mathematical solution is performed, and the model is validated.

In step 1, there is no modelling component (before defining the problem), while step 2 features an unreasonable assumption (defining an unreasonable problem situation), both of these are common in the modelling cycle in the real world before being perfected in the mathematical world (Blum and Leiss 2007). In step 3, a reasonable situation is defined, but an accurate mathematical solution was not offered. Step 4 defines variables and offers a reasonable assumption as well as an accurate solution, but the model is not validated. Step 5 validates the model in the modelling cycle. In this process, steps 3 to 5 are equivalent to the mathematics world in the modelling cycle.

This theoretical framework was used to study the modelling competency of mathematics student teachers, and the following research questions arose:

1. What is the status of the modelling steps reached by the student teachers?
2. Are there differences in the mathematical modelling competency among students based on their school and gender?
3. What is the correlation between the modelling competency of student teachers and their modelling competition experience?

Fig. 20.1 A pineapple is being peeled

20.3 Study Design

20.3.1 Task Design

20.3.1.1 Real Situation

This study offered the real situation of "peeling a pineapple" (Ludwig and Xu 2010) as shown in Fig. 20.1.

> **Peeling a pineapple**
> In China, April is pineapple season. When a customer buys a pineapple, the vendor peels it for them. This is an artistic practice, as the peels leave the fruit in nice spirals. We probably take this for granted, but as a mathematician or mathematics teacher, please consider the following: why does the vendor peel the pineapple in this way? Please explain it mathematically.

20.3.1.2 The Solution to the Real Situation

A possible solution to *Peeling a pineapple* is as follows:

Suppose the black seeds are connected in a rhombus (Fig. 20.2).

Suppose $\angle ABD = \theta\left(0 < \theta < \frac{\pi}{2}\right)$ and $AB = a$, then $BD = 2a\cos\theta$ and $AC = 2a\sin\theta$.

For horizontal peeling, there are a total of $2l$ rows, and each row has a length of $2ah\cos\theta$, so the total length of the peeled fruit is $z = 4ahl\cos\theta$.

For longitudinal peeling, there are a total of $2h$ columns, and each column has a length of $2ah\sin\theta$, so the total length of the peeled fruit is $z = 4ahl\sin\theta$.

For diagonal peeling, there are a total of h diagonal lines, each diagonal line with length of $2al$, so the total length of the peeled fruit is $z = 2ahl$.

When $0 < \theta < \frac{\pi}{6}$, $AC < AB < BD$; when $\frac{\pi}{6} < \theta < \frac{\pi}{4}$, $AB < AC < BD$; when $\frac{\pi}{4} < \theta < \frac{\pi}{3}$, $AB < BD < AC$; when $\frac{\pi}{3} < \theta < \frac{\pi}{2}$, $BD < AB < AC$.

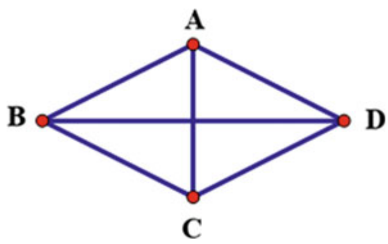

Fig. 20.2 Suppose the black seeds are connected in a rhombus $ABCD$

So, AB is at a minimum when $\frac{\pi}{6} < \theta < \frac{\pi}{3}$, which means that the least pineapple is removed when being peeled diagonally. (See Ludwig and Xu, 2010, for a fuller solution.)

20.3.1.3 Questionnaire

The following instruction was given in a questionnaire after introducing the above-mentioned situation: Please discuss your level of experience with mathematical modelling competitions. Three options were presented: have not participated, have participated but have not won, and have participated and won. The modelling competition could be at the school, provincial, national, or even international level.

20.3.2 Sample

This study selected undergraduate junior mathematics student teachers from four universities in northern and central Jiangsu, southern Zhejiang, and Shanghai. In particular, two first-tier universities in Shanghai and Jiangsu and two second-tier universities in Jiangsu and Zhejiang were involved.

A total of 285 test papers and questionnaires were distributed in the test, of which 273 were valid. Amongst the valid test papers, the sample distribution is as follows (see Table 20.1).

Table 20.1 Survey sample distribution

School Type	Region	Boys	Girls	Total	
First-tier University	Shanghai	22	48	70	144
	Jiangsu	18	56	74	
Second-tier University	Jiangsu	20	60	80	129
	Zhejiang	16	33	49	
Total		76	197	273	

20.3.3 Test Analysis

Based on the rating criterion for mathematical modelling competency (see p. 2), the solutions given by the student teachers were carefully rated and classified.

We encoded 273 solutions based on the modelling steps and their performance. To improve the reliability of the encoded data, about 15% of the participants' test papers from different schools were collected and sampled, and they were independently coded by two researchers. At the beginning, the coding consistency of the test was about 80%, so the two researchers had to reach a consensus on inconsistent coding. This procedure was repeated several times before the coding consistency reached about 90% in the consistency test.

20.4 Results

20.4.1 Performance of the Student Teachers

A small number of the mathematics student teachers (see Fig. 20.3 in which 40.3% of the student teachers stopped at step 1 or 2) could not turn real-world models into mathematical models, whereas a majority of them (see Fig. 20.3 in which 59.7% of the student teachers reached steps 3, 4 and 5) could transform real-world models into mathematical models. Once the models were accurately transformed, most of the student teachers (63.2%, as shown in Fig. 20.3) could solve problems and obtain accurate mathematics solutions. Step 2 marks a key indicator in evaluating the mathematical modelling competency of the student teachers because some of them might have difficulties turning real-world models into mathematical models.

In addition, most of the mathematics student teachers were unable to reach step 5, which means they could not test the rationality of a solution in the real world or

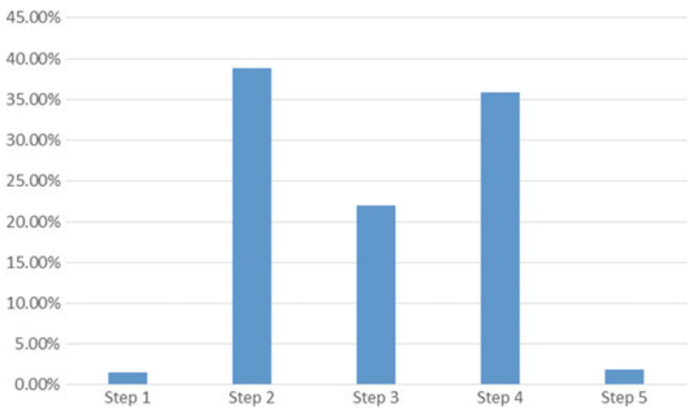

Fig. 20.3 Percentage of student teachers who reached step x

in real scenarios. This may be related to the approaches they used for mathematical solutions.

20.4.2 *University and Gender Differences*

20.4.2.1 Differences in Terms of Type of University

There was a significant difference in the mathematical modelling competency of the mathematics undergraduate student teachers from different types of universities ($p < 0.01$). Students from first-tier universities completed more steps (3.17 steps, on average) than students from second-tier universities (2.77 steps, on average). Furthermore, there was no significant difference ($p > 0.05$) in the mathematical modelling competency of students from the same university tier in different regions. Meanwhile, those who were able to reach modelling step 5 were all from first-tier universities.

20.4.2.2 Differences in Terms of Gender

There was no significant difference in mathematical modelling competency between males and females ($p > 0.05$). It was found that 63.96% of the females reached the mathematics world, whereas only 48.68% of the males, which was far lower than the percentage for females who were able to do so. By contrast, more males stayed in the real world than females (see Fig. 20.4). The data showed that more than half of the males remained in the real world, indicating that compared with females, males were less capable of "peeling a pineapple" in a mathematical model.

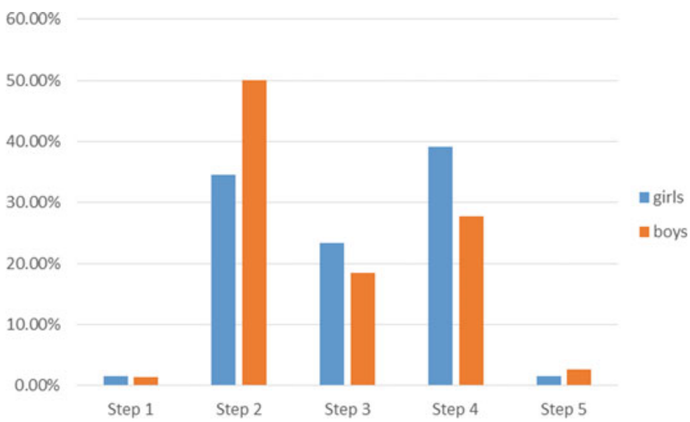

Fig. 20.4 Percentage of females and males who reached step x

Table 20.2 Correlation between modelling competition experience and modelling step in terms of type of university

University	N	Correlation
First-tier University	144	0.292**
Second-tier University	129	− 0.089
Total	273	0.197**

Table 20.3 Correlation between modelling competition experience and modelling step in terms of gender

Gender	N	Correlation
Female	197	0.227**
Male	76	0.123
Total	273	0.197**

20.4.3 Correlation Between Modelling Step Reached and Modelling Competition Experience

Although there is a difference in the modelling steps reached of student teachers with different modelling competition experiences ($p < 0.01$), the correlation is low (see Table 20.2 or 20.3, in which the correlation is 0.197), which means that student teachers who had no modelling competition experience still had the potential to improve their modelling competency. In addition, the correlation between modelling competition experience and the mathematical modelling step reached of first-tier university students (see Table 20.2, in which the correlation is 0.292) is higher than that of second-tier university students (see Table 20.2, in which the correlation is −0.089). The correlation between modelling experience and the mathematical modelling step of females (see Table 20.3, in which the correlation is 0.227) is higher than that of males (see Table 20.3, in which the correlation is 0.123), and the student teachers who could reach mathematical modelling step 5 all had modelling competition experience.

20.5 Conclusions and Outlook

The results show that the modelling difficulty the student teachers encountered was the transformation of problems in the real world to mathematical models, which is consistent with the results of a study on grade 9 to 11 students by Ludwig and Xu (2010). There may be a correlation between students' weak mathematics competency and that of their teachers, which will hopefully be the topic of follow-up studies; in particular, since mathematical modelling is about to be carried out in teaching in the high school in China. The promotion of modelling competencies amongst learners requires qualified teachers (Blum 2015). Therefore, improving the mathematical modelling competencies of teachers or student teachers is important, which is also a topic to be studied in the future.

The data also show that the modelling steps have a correlation with modelling competition experience, supporting the findings of my previous research (Wang 2018). However, the correlation is low, indicating that student teachers without modelling experience still have the potential to improve their modelling competency. In addition, modelling experience is positively correlated with the modelling competency of student teachers from first-tier universities, whereas it is negatively correlated with the modelling competency of student teachers from second-tier universities. In an interview with the student teachers, it was found that those from second-tier universities won more awards in school or provincial competitions and fewer awards in national or international competitions. However, student teachers from first-tier universities had more awards in national or international competitions, which could have played a role in the results of the test. One of the first-tier universities studied is, in fact, reforming its mathematical modelling curriculum, providing inspiration for follow-up studies. Meanwhile, as student teachers from first- and second-tier universities may have a gap in mathematics knowledge when they enter university, which could have an impact on their modelling competency, this is also a topic worth studying in the future.

This study uses and expands the framework by Garfunkel (2016). The framework is also related to the modelling cycle in the real world and the mathematical world in Blum's work (2007). As a combination of the two frameworks, the theoretical framework of this study needs to be further improved. Meanwhile, the real situation used is an early test from Ludwig and Xu (2010). Although China still uses this method to peel pineapples, the manner of peeling pineapples in the real world has changed substantially. Perhaps a new test can be developed for subsequent studies. Furthermore, this study only targets universities located in the Yangtze River Delta of China, a highly developed region in China. Conducting research in the central and western regions of China in the future would be desirable.

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Chapter 21
Measuring Professional Competence for the Teaching of Mathematical Modelling

Raphael Wess, Heiner Klock, Hans-Stefan Siller, and Gilbert Greefrath

Abstract Teaching mathematical modelling is a cognitively challenging activity for (prospective) teachers. Thus, teacher education requires a detailed analysis of professional competence for teaching mathematical modelling. To measure this competence, theoretical models that accurately describe the requirements placed upon teachers are needed, as well as appropriate evaluation tools that adequately capture skills and abilities in this field. This is where the present study comes in, contributing to the teaching of mathematical modelling through the theory-based development of a structural model and an associated test instrument. In particular, this chapter discusses to what extent the proposed conceptualisation of the structural model can be empirically confirmed. To this end, insights into the test instrument are presented, as well as results of the structural equation analysis of the model.

Keywords Mathematical modelling · Professional competence · Teacher education · Model development · Test development · Structural equation analysis

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21.1 Introduction

In recent years, many empirical studies have dealt with the questions of how modelling can be taught in school (e.g. Blomhøj 2019), how modelling (sub-) competencies can be assessed (e.g. Haines and Crouch 2001) or how modelling can be integrated into university teacher education (e.g. Borromeo Ferri and Blum 2010). However, some questions in the field of mathematical modelling remain open, for example:

- To what extent can prospective teachers be prepared by university courses?
- To what extent can selected contents and methods contribute to the promotion of teachers' competences?

To answer these and other questions, teacher education requires a detailed analysis of teacher competences and a detailed analysis of professional competence for the teaching of mathematical modelling. In this context, we understand competences as context-specific cognitive dispositions for achievement that relate functionally to specific situations and demands in specific domains (Klieme et al. 2008). Accordingly, current professionalisation efforts are not only limited to the acquisition of theoretical knowledge but also include its application in concrete situations.

Now that the global professional competence of (prospective) mathematics teachers has been comprehensively structured, operationalised and measured in various large-scale studies (e.g. Baumert and Kunter 2013; Blömeke et al. 2014), the question arises of a local, purposeful *modelling-specific* arrangement of these competences. This chapter presents the theoretical derivation of a structural model of professional competence for teaching of mathematical modelling. Furthermore, empirical results on the quality of the model and the selected test instrument are presented.

21.2 Theoretical Frame

In addition to good modelling tasks, which form the necessary basis for productive modelling processes, the promotion of modelling competences among learners requires specific competences of teachers (Blum 2015)—especially given their important role in the context of teaching-learning processes (Hattie 2009). Building on Shulman (1986), a distinction in the aspect of teacher professional knowledge is made between content knowledge, pedagogical content knowledge and pedagogical-psychological knowledge. In this chapter, the concretisation of a structural model of professional competence relating to the imparted competence of "mathematical modelling" is carried out by using the competence model of the COACTIV-Study (Baumert and Kunter 2013).

In the COACTIV-Model, professional competence is composed of the superordinate aspects of beliefs/values/goals, motivational orientations, self-regulation

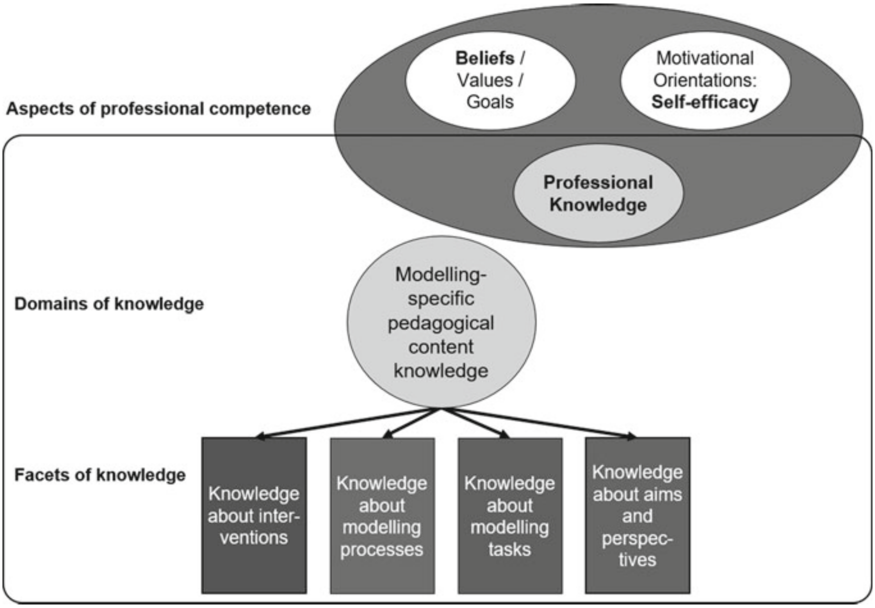

Fig. 21.1 Structural model of professional competence for the teaching of mathematical modelling

and professional knowledge. Professional knowledge is in turn subdivided into the following domains of competence: content knowledge, pedagogical content knowledge, pedagogical-psychological knowledge, organisational knowledge and consulting knowledge.

Regarding the necessary professional competences for the teaching of mathematical modelling (cf. Fig. 21.1), in addition to beliefs/values/goals and motivational orientations, the pedagogical content knowledge, as a part of the professional knowledge, in particular is characterised by modelling-specific contents. In contrast, other aspects and domains like pedagogical-psychological knowledge and self-regulatory skills tend not to contain any clear modelling-specific aspects and are therefore not considered more closely.

21.2.1 Professional Knowledge

The interpretation of the *modelling-specific pedagogical content knowledge* is based on the four theoretically derived competency dimensions for the promotion of modelling competences among learners according to Borromeo Ferri and Blum (2010): the theoretical dimension, the task dimension, the instruction dimension and the diagnostic dimension. Each of these dimensions is concretised by facets of

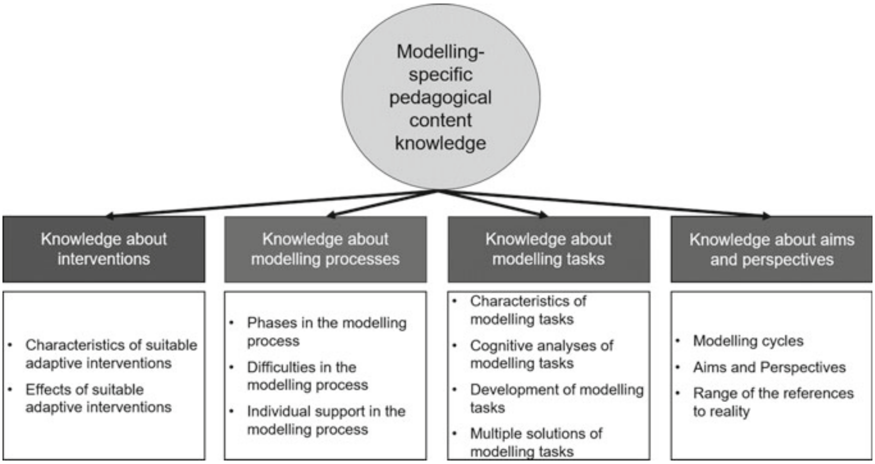

Fig. 21.2 Modelling-specific pedagogical content knowledge

knowledge and abilities, which relate both to declarative and procedural aspects of the knowledge of (prospective) teachers.

Accordingly, we subdivide the modelling-specific pedagogical content knowledge—following the COACTIV-Model (Baumert and Kunter 2013)—besides the facets of *knowledge about interventions*, *knowledge about modelling processes* and *knowledge about modelling tasks* additionally into *knowledge about aims and perspectives* of mathematical modelling. These competence facets were developed with selected aspects of the competency dimensions mentioned above (cf. Fig. 21.2). All aspects are also mentioned in overview articles on mathematical modelling (e.g. Blum 2015).

According to Borromeo Ferri and Blum (2010), knowledge about interventions represents a facet of teaching knowledge that is important for adequate support of modelling processes. In addition, the teaching of mathematical modelling results in a different teacher role, which is associated with new demands. With the definition of adaptive teacher interventions following the principle of minimal help and a taxonomy of teacher support, characteristics of suitable interventions could be determined and used to assess assistance in the modelling process (Leiss and Wiegand 2005). Good interventions in mathematical modelling processes are therefore oriented to students' solution process and are minimal as well as independence preserving. These interventions are specific to the field of mathematical modelling, since they are intended to promote independent work by learners and metacognitive competences. They are determined by the openness of the tasks and the confrontation with a multitude of different solutions.

Knowledge about modelling processes is characterised by specific diagnostic knowledge. In particular, teachers need skills to identify and document progress and difficulties in students' learning process. In the diagnostics of modelling processes, for example, the focus is on identifying the modelling phase in which the learners are currently working (Borromeo Ferri and Blum 2010). Furthermore, in numerous studies, difficulties occurring in the modelling process have been assigned to the modelling phases in which they appear (e.g. Galbraith and Stillman 2006). Thus, focused diagnostics are made possible in this way, the aim of which is the identification of opportunities to provide individual support for the learner's modelling process. In this context, Brunner et al. (2013) show the high relevance of teachers' diagnostic skills for the learning process of students.

The knowledge and skills to analyse, process and develop modelling tasks represent facets of a task-related competence dimension that forms the basis for productive modelling processes among learners (Borromeo Ferri and Blum 2010). The comprehensive classification scheme for categorising and analysing modelling tasks according to Maaß (2010), in conjunction with the explanations on task design according to Czocher (2017), provides a theoretical foundation for the facets mentioned here, in particular the criteria-based development of modelling tasks with focus on reference to reality, relevance, authenticity and openness. These facets form "the interface between student and teacher activities in the mathematics classroom" (Neubrand et al. 2013, p. 127) and thus represent an indicator for the teaching dimension of cognitive activation.

The facet of aims and perspectives consists of selected aspects of theoretical background knowledge. On the one hand, knowledge about modelling cycles and their suitability for various purposes is described, for example as a metacognitive strategy for learners or as a diagnostic tool for teachers. On the other hand, different perspectives of research on mathematical modelling are illustrated (Kaiser and Sriraman 2006), for example modelling as vehicle to learn mathematics and to serve other curricular needs (Julie and Mudaly 2007). In addition, teachers should be aware of the corresponding goals of mathematical modelling in teaching and of the varying relevance of reality references for learners.

21.2.2 Beliefs

The COACTIV-Study defines beliefs as "psychologically held understandings and assumptions about phenomena or objects of the world that are felt to be true, have both implicit and explicit aspects, and influence people's interactions with the world" (Voss et al. 2013, p. 250). Furthermore, Woolfolk Hoy et al. (2006) distinguish between epistemological beliefs and beliefs on teaching and learning mathematics. Epistemological beliefs can be operationalised in the following aspects: the formalism aspect, the application aspect, the process aspect and the schema aspect (Rösken and Törner 2010). Due to the reality reference of modelling tasks, a reference to the application aspect appears to be suitable. This aspect describes the relevance of mathematics in the world, which is why positive beliefs about mathematical modelling represent perspectives that give modelling a meaning in everyday life and work. In contrast, beliefs on teaching and learning mathematics include views on teaching objectives and teaching method preferences as well as classroom and group management. They are operationalised by statements that give mathematical modelling a justified place in mathematics teaching. Both facets of beliefs can be understood within the framework of the antagonistic epistemology of behaviourism and constructivism. Transmissive beliefs go hand in hand with the view that learning is the absorption of knowledge and the reinforcement of positive behaviour. In contrast, constructivist beliefs see the learner as an active participant in the learning process who constructs his knowledge individually (Voss et al. 2013). The constructivist beliefs go hand in hand with the self-reliant and cooperative handling of realistic, authentic and thus situationally connected modelling tasks. For this reason, constructivist beliefs of teachers are normatively regarded as positive for high competences in teaching mathematical modelling (Blömeke et al. 2014).

21.2.3 Self-Efficacy

As part of the motivational orientations, the self-efficacy of (prospective) teachers is regarded as an empirically founded characteristic of professional competence. Tschannen-Moran and Woolfolk Hoy (2001, p. 783) define the concept of self-efficacy as follows: "A teachers' efficacy belief is a judgement of his or her capabilities to bring about desired outcomes of student engagement and learning, even among those students who may be difficult or unmotivated". Self-efficacy can be related to concrete teacher competences and is suitable for recording ideas about one's own abilities in the field of teaching mathematical modelling. As already mentioned, knowledge about modelling processes from a theoretical perspective as a diagnostic component of modelling-specific pedagogical content knowledge has a strong influence on students' learning processes (Brunner et al. 2013). Accordingly, it forms a decisive facet of competence for teaching mathematical modelling. For this reason, our structural model operationalises self-efficacy by assessing the (prospective) teachers' own ability to diagnose the performance potential of learners in the modelling process. We assume that the diagnostic requirements for the teacher differ depending on the modelling phase in which the learners work. Thus, the self-efficacy of the (prospective) teachers can also be differentiated according to the phase. Furthermore, scaling analyses indicate that a distinction can be made between phases specific to the modelling process (simplifying, mathematising, interpreting, validating) and unspecific ones (working mathematically).

21.3 Empirical Validation of the Structural Model

For an empirical examination of the conceptualised structural model of professional competence for teaching mathematical modelling, the following research questions arise:

1. To what extent can the proposed conceptualisation of the structure of professional competence for the teaching of mathematical modelling be empirically confirmed?
2. To what extent are there connections between beliefs, motivational orientations, and modelling-specific pedagogical content knowledge?

To answer these questions, the described structural model was evaluated on the basis of data from 349 prospective teachers for secondary schools at the German universities of Münster, Koblenz-Landau and Duisburg-Essen. In this context, a test instrument was developed (Klock and Wess 2018) that operationalises the described four facets of the modelling-specific pedagogical content knowledge over a total of 64 dichotomous test items in multiple and combined single-choice formats. The items in the facets of knowledge about modelling processes and knowledge about interventions relate to modelling tasks, which are supplemented by text vignettes on the concrete modelling processes of learners (cf. Fig. 21.3).

The items of beliefs (16 items) and self-efficacy (19 items) for mathematical modelling were collected using a five-point Likert scale (from 1 = "strongly disagree"

7.1 Traffic Jam (Grade 9)

It is the start of the summer holidays and there are many traffic jams. Chris is on holiday in Germany and has been stuck in a 20 km traffic jam for 6 h. It is hot and she is longing for a drink. Although there are rumours that the Red Cross is coming around with a small lorry distributing water, she has received nothing so far. How long will the Red Cross need to provide everyone with water?

STUDENT 1: We should actually know how many cars are in the traffic jam.
STUDENT 2: Huh? Right!
STUDENT 1: How should we calculate how long it takes? Many things are missing in the task!
STUDENT 3: Yes, and we do not know how long it takes for each car.
STUDENT 2: Such a stupid task.
STUDENT 1: We can devide the 20 km by the 6 hours, then we know how fast the small lorry has to be.
STUDENT 3: Exactly! We did not get any further information anyway.

Diagnose students' difficulty working on the task in **this situation**. Please check **one** box.	
The Students …	
... have problems making assumptions.	☐
... draw a wrong conclusion from their mathematical result.	☐
... have problems understanding the context.	☐
... use an inappropriate mathematical model.	☐

Fig. 21.3 Test item assessing knowledge about modelling processes on the basis of the traffic jam task (cf. Maaß and Gurlitt 2010)

Table 21.1 Characteristics for belief and self-efficacy scales

Construct	Scale	Number of Items	Cronbach's Alpha
Beliefs	Constructivist Orientations	12	0.83
	Transmissive Orientations	4	0.65
Self-Efficacy	Modelling	11	0.88
	Working Mathematically	8	0.84

Table 21.2 Dichotomous Rasch models for knowledge scales

Scale	Number of Items	EAP Reliability	Andersen Test	Pt.-Bis. Corr
Interventions	19	0.71	0.061	> 0.30
Modelling Processes	18	0.74	0.072	
Modelling Tasks	17	0.81	0.086	
Aims and Perspectives	10	0.70	0.058	

to 5 = "strongly agree"). These scales were checked on the basis of a confirmatory factor analysis and show a Cronbach's α of at least 0.65 (see Table 21.1), which can still be described as acceptable.

The dichotomous items were scaled using Rasch models and the scales in this context were checked for sufficiency. Using the eRm package (Mair and Hatzinger 2007) of the software R, item difficulties were estimated on the basis of the solutions of the tasks, and person ability parameters were estimated on the basis of the performance of the interviewees. Various scale characteristics were calculated to assess the scalability (see Table 21.2). The EAP reliabilities (comparable to Cronbach's α) are above 0.70 and are therefore acceptable. The Andersen tests for assessing the model fit are all not significant and therefore point to a fit of the Rasch models. Furthermore, the point-biserial correlations of the items are all greater than 0.30 and thus also of acceptable quality.

21.4 Results

The conceptualised model was verified by structural equation analysis using the SPSS extension AMOS. Since it was not possible to load the items directly onto the latent variables due to the small sample size ($N = 349$), the standardised sum scores or the person ability parameters were used. In view of the fit indices (cf. Figure 21.4), the model specified in this way has a very good global fit with the data set (Hu and Bentler 1998). Empirically, significant correlations of medium practical relevance between self-efficacy and beliefs in mathematical modelling ($r = 0.57, p < 0.01$), as well as between self-efficacy and scores in modelling-specific pedagogical content knowledge ($r = 0.53$, $p < 0.01$), can be demonstrated. Also, a significant

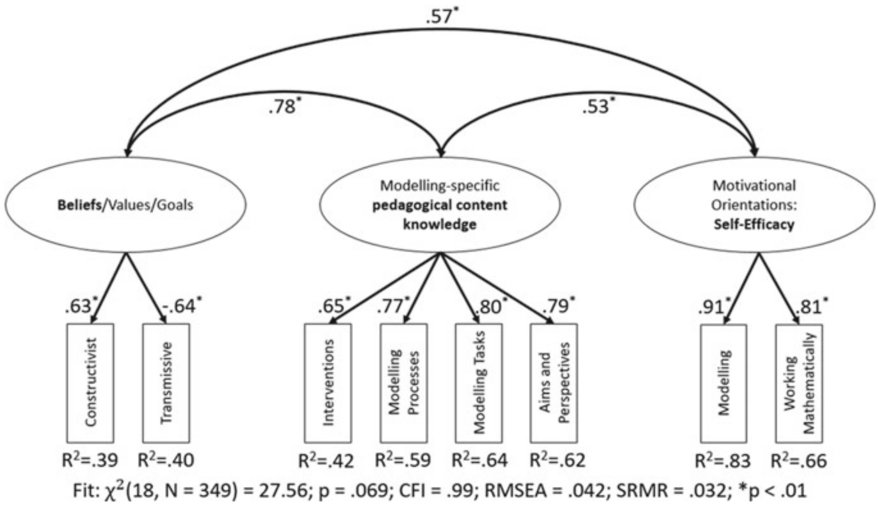

Fig. 21.4 Structural equation analysis of the model

correlation of medium-to-high practical relevance between beliefs and pedagogical content knowledge can be identified ($r = 0.78$, $p < 0.01$). In addition, all scales show significant loadings with high effect sizes.

21.5 Discussion

The results of the empirical validation confirm the basic structure of the model of professional competence for the teaching of mathematical modelling in the theoretically conceptualised form. In particular, the high correlation between the beliefs and the facets of the modelling-specific professional knowledge is in line with the findings of expertise research (Baumert and Kunter 2013). However, the scale of transmissive beliefs in mathematical modelling showed little reliability. One reason for this could be the low number of items. In follow-up studies, it would therefore be desirable to increase this number. The quantitative research approach also reveals further limitations. In particular, the use of dichotomous test items in the Rasch model leads in a way to a normative setting of true and false statements. Especially with modelling problems this is a challenge, which leads to the fact that many interesting examples could not be used because they could not be clearly classified into true and false. However, more sophisticated scales, such as Likert scales, are not suitable for measuring *knowledge*, so this limitation must be dealt with. For this reason, qualitative additions, such as the analysis of modelling tasks created by (prospective) teachers in vivo, represent a necessary and vital starting point for future studies. Furthermore, due to the unavailability of comparative tests, the discriminatory and convergent validity cannot be conclusively assessed. However, the good model fit indicates structural validity. In this context, it should be borne in mind that the results of the study merely represent an empirical foundation of the structures under consideration for prospective teachers at the participating German universities. Thus, further work on the examination for practising teachers on the one hand and in international contexts on the other is still outstanding—perhaps there is a German tradition of teaching modelling that cannot be generalised. Furthermore, the theoretically derived and empirically verified structural model does not fully describe the professional competence for the teaching of mathematical modelling but only *modelling-specific* aspects. In addition to pedagogical content knowledge, the teacher must have well-founded pedagogical-psychological knowledge, for example about organising and monitoring group work, as well as content knowledge in order to be able to adequately carry out modelling processes. It would therefore be necessary to capture facets of pedagogical-psychological knowledge and the modelling competence of (prospective) teachers with suitable instruments in order to comprehensively describe professional competence for teaching mathematical modelling. However, this would have led to a considerable increase in the test period, so the additional survey of these domains was initially dispensed with.

21.6 Conclusion and Outlook

Using the example of teaching mathematical modelling, it could be shown that professional competences of teachers can be concretised in order to evaluate the associated knowledge and skill facets. The finding that the conceptualised, modelling-specific competences can be recorded in an empirical and structurally valid manner indicates added value for further research on teaching mathematical modelling, since, for example, a wide variety of university courses can be evaluated more precisely and thus given a more differentiated assessment. It also seems sensible to apply this approach to other competences (e.g. problem-solving), because the exact description of such specific professional competences is what enables them to be systematically promoted within the framework of university courses and practical teacher training. Modelling competency as modelling-specific content knowledge was not captured in the context of this study. Against the background of general professional competence, especially for secondary school teachers, the COACTIV-Study demonstrates a close connection between content knowledge and pedagogical content knowledge. Whether this connection can also be reproduced in the field of professional competence for the teaching of mathematical modelling is a question for future studies.

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Part V
Teaching Practice

Chapter 22
Attending to Quantities Through the Modelling Space

Jennifer A. Czocher and Hamilton L. Hardison

Abstract Understanding students' modelling processes is critical for informing facilitator interventions. More specifically, it is important for facilitators to understand the situation-specific attributes students find relevant in modelling tasks, if and how these are manifested in their inscriptions, and when students' situation-specific meanings for inscriptions change while engaged in modelling. In this chapter, we present a theoretically coherent methodological approach for attending to the aforementioned features. Our approach foregrounds the quantities projected by students when engaged in modelling, as well as attends to the situation-specific quantitative referents for their mathematical inscriptions. We illustrate the utility of this approach by analysing the modelling activities of a purposefully selected undergraduate student and consider implications for future research.

Keywords Qualitative · Quantities · Post-secondary · Modelling space · Facilitator intervention · Cognition · Representations

22.1 Introduction

Mathematical modelling pedagogies obtain optimal learning outcomes when students work out their own solutions (Kaiser 2017), which means that students need help not only in identifying *that* their models may be inadequate, but also support in revising them appropriately. Model revision is an under explored and under conceptualised topic. This is partly due to the myriad methodological questions that surround systematic inquiry into how and why students choose to revise their models (or to take up or ignore facilitators' suggestions, e.g. Stender and Kaiser 2015). Important questions remain like: What changes does (or might) a student make to her model? Why?

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Do the changes she makes meet her goals? Under what conditions do facilitator interventions occasion model changes? And ultimately, *what qualifies as a change anyway?* It is only by addressing this last question that we can uncover ways to help students make meaningful changes, culminating in student-centred teacher training on facilitating mathematical modelling tasks.

Our broad research objective is to produce a theoretically and methodologically coherent means for documenting changes to a model—changes both to external presentations and to the student-specific meanings they may carry. In this chapter, we adopt a semiotic perspective (Kehle and Lester 2003) to argue that tracing model evolution entails attending to the quantities a student projects onto a situation, the relationships conceived among these quantities, the inscriptions produced along with their quantitative referents, as well as the modifications in these aforementioned elements. We illustrate our argument using a detailed case study approach (Ragin 2004) to explore interactions between inscriptions and quantitative reasoning. Our contribution is theoretical and methodological: we document our methodological approach and report on some insights regarding facilitating students' revisions to their models as they work on modelling tasks.

22.2 Relevant Theoretical Constructs

We view mathematical modelling as a cognitive and iterative process. Often, the process is conceptualised as a series of phases of cognitive activity (Kaiser 2017; Maaß 2006) where student decisions made during each phase contribute to the dynamic evolution of the model. In essence, we elaborate on the *systematising* and *mathematising* phases through a semiotic lens to garner insight into the ways models could change. Following Kehle and Lester's (2003) application of Peircean semiotics to mathematical modelling, modelling can be seen as a process of unification among a sign, a referent (the object the sign stands for), and an interpretant. Thus, mathematising a situation involves generating mathematical expressions and assigning situationally relevant meanings compatible with the modeller's physical theory. In this view, the meanings of symbols within an equation are not inherent, but must be constructed by the modeller and inferred by an observer (e.g. a facilitator or interviewer). The mathematising phase depends on how the modeller coordinates knowledge about the real-world entities and relationships in the scenario she identifies as relevant (or not) with her anticipation of the mathematical concepts and signs that will appropriately signify them. Scholars from physics education have conceptualised the coordination as follows. Systematising occurs through coordination of physical theory, which is a "representational system in which two sets of signs coexist: the mathematical signs and the linguistic ones" (Greca and Moreira 2002, p. 107), with a *mathematical model*, taken to be a deductively articulated axiomatic system and attendant mathematical concepts. Statements of physical theories are about simplified and idealised physical systems, termed physical models, not the real-world scenario itself. We use the term *representation* to refer to an outward expression of an individual's mathematical model and *inscription* to refer specifically to the expression's written form. These distinctions allow for precision in talking about how components of a mathematical model (a mathematical conceptual system with cognitive links to a real-world system) may change independently (or in coordination) with one another.

Attending to meaning-making processes merits elaborating the role of quantities as interpretants. According to Thompson (2011), quantities are mental constructs, not characteristics of objects in the world. Individuals conceive of quantities via a *quantification*, which is "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bilinear, or multi-linear) with its unit" (Thompson 2011, p. 37). One can conceive of various instantiations of the object, with each instantiation manifesting different extents of the relevant attribute, and coordinate these instantiations with a value. We operationalise quantification as the set of operations an individual can enact on a particular attribute (Hardison 2019). These mental acts may become quite familiar or nearly automatic if one has much experience in the context; however, the quantification process is generally nontrivial (Thompson 2012). An attribute's quantification is idiosyncratic because two individuals may not enact the same mental operations on a given attribute; thus, they may conceive of a specified quantity differently (Steffe and Olive 2010). Thus, quantities are not synonymous with variables nor with the objects they quantify. And, returning to mathematising, it can now be operationalised as conceiving and representing the relationships among the quantities involved. We leverage Sherin's (2001) theory of symbolic forms to explain how both mathematical and quantitative meaning can be associated with equations. A symbolic form consists of a template and an idea to be expressed in the equation. For example, $\square + \square = \square$ expresses a "parts-of-a-whole" relationship, where each box is a placeholder for a (potentially different) quantity. Familiarity with symbolic forms helps individuals "know" which equations to use in a given situation.

22.3 Methodology

The theoretical perspectives outlined above distinguish among the quantities an individual projects onto a situation, operations (quantitative or numerical) enacted on these quantities or their values, and the representations (inscriptions as well as utterances) she uses. The distinction is necessary in order to increase understanding of, and respond to, a student's evolving conceptual system. Quantities and operations are conceptual entities, whereas inscriptions and utterances are observables. From the researcher perspective, quantities and operations can only be inferred through the observables generated by a student. Given the quantities that a student projects into a particular situation and the operations available to the student, we refer to the set of (conceptual) mathematical models a student might generate within a given modelling task as *the modelling space*. We view the modelling space as the set of mathematical relationships that act via composition on the situationally relevant quantities available to the student. For example, suppose that during a modelling task about a projectile subject only to gravity, we are able to infer that a modeller has introduced the quantities *initial height above ground*, *time elapsed*, *current height above ground*, *mass of the object*, and *initial velocity*. We represent the modeller's available quantities as the sequence (TIME, HT_I, HT, MASS, V_I). His modelling space would be all of the mathematical combinations of those quantities meaningful to him. For example, his modelling space may contain $h = h_0 - v_0 \cdot t$ to relate HT to TIME, where the symbols correspond to experts' conventions. However, his modelling space would not (yet) contain the element $h = h_0 - v_0 t - \frac{1}{2} g t^2$ because he had not yet introduced gravity as a situationally relevant quantity.

In our analysis, we sought a means to trace changes to models relevant to *systematising* and *mathematising*, namely: (1) introduction or modification of inscriptions, (2) introduction of quantities, and (3) shifts in meaning of inscriptions due to shifts in the roles of quantities. Therefore, we developed procedures attending to these three phenomena through retrospective analyses of task-based interviews (described below). For a given student working on a given task, we first catalogued all inscriptions that he introduced and documented any modifications he made to them throughout the session. We did so by attending to the spatial and temporal organisation of inscriptions on his paper. We judged his *mathematical representation* to have changed if either the system of signs comprising the representation changed (e.g. introducing a symbolic equation for a quadratic relationship after working with a graph) or a new inscription was created in a different location on the page (Czocher and Hardison 2019). To identify substantive changes to the meaning for a given representation, we considered (a) whether there was evidence to infer that information or meaning was distributed to the representation or removed from it, (b) whether the student modified an inscription, or (c) whether he modified an inscription in a way suggestive of transporting meaning to, or from, another representation. Second, we sought to identify the quantities the student projected onto the scenario. We analysed records of the interviews and identified situational attributes to which the student attended during the course of the session. By *situational attributes*, we mean we were able to infer a situational referent for the attribute within the scenario (e.g. a tree's height). Generic attributes for which we were unable to infer situational referents (e.g. height, without indicating height of *what*) were not considered situational attributes. Additionally, we searched for *evidence of* the student engaging in mental operations suggestive of a conceived measurement process for each attribute. Through inductive and iterative analysis, we obtained a set of 8 observable criteria (see Table 22.1) to use as indicators of a student projecting situationally relevant quantities onto a scenario. The criteria are not mutually exclusive. For example, specifying a unit of measure (QC6) may co-occur with observing variation in an attribute (QC1). Three independent coders systematically applied these criteria via constant comparison to the interview records. Disagreements among coders were resolved through seeking consensus as to whether there was evidence that at least one of the criteria was met for a quantity. The result was a list of quantities we could infer the student projected

Table 22.1 Descriptions of quantification criteria (QC)

QC	Description	Example
1	Discussing variation of a situational attribute	"The horizontal distance in terms of how far he is from the monkey, that's the one variable that I am ultimately gonna have."
2	Substituting, assuming, or deducing a numerical value for a symbol with a situational referent	"You are going to be taking away gravity which is 9.8 metres per second squared." "Say he's standing 30 ft away."
3	Expressing a desire to measure a situational attribute (e.g. "if I knew")	"I need to know the angle he's going to fire at."
4	Interpreting the value in context	"In this particular case it would be 40 ft." "… it's going to be moving upward at a linear rate, it's going to be moving down at 10 m/s^2 so that effect is going to cause a parabola …"
5	Specifying a situational reference object (e.g. line or point from which to measure; situational 0)	"Anytime you know how far away he is from the monkey … so all that matters is how far he goes away …"
6	Specifying a (potentially non-standard) unit of measure for a situational attribute	"Yeah, so it's still going to be negative 10 for every metres per second squared."
7	Explicitly expressing a quantitative relationship, a dependence or causal relationship among already-introduced quantities, describing one quantity in terms of other quantities	"So if you can have those two as variables (height and distance) in a system to get from there to whatever angle he needs"
8	Nominalising an attribute via verbally labelling, symbolically labelling/indicating, implicitly describing its relation to other quantities	[Draws the tree diagram and labels the horizontal distance between the vet and the dart as x and the height of the tree as y and the angle the vet makes with the top of the tree as θ]

while modelling (Table 22.2). The quantities projected onto the situation formed the basis for our conception of the student's modelling space on that task.

Finally, we used the quantities to conceptualise the student's modelling space and trace the evolution of her model. For this analysis, we considered two sets of instances: those in which we could infer quantitative situational referents for symbols constituting the inscriptions and instances in which we were unable to do so. We next sought instances in which the student's activities indicated the quantitative situational referent of a given inscription changed during the interview. We used these three categories of instances to develop conjectures about the student's quantification of some situational attributes which (from our perspective) supported or constrained her modelling process.

Table 22.2 Potential quantities projected by Merik

Quantity	Type	Time	Description
ANG_{STR}	Angle	2:08	Measure of angle gun is aimed relative to the horizontal, for straight path
$DIST_{VET/TREE}$	Length	2:09	Horizontal distance from vet to the tree/under the monkey
$HT_{MKY/GUN}$	Length	2:10	Height of the monkey relative to the vet's gun
$VVEL_{DART-I}$	Rate	2:47	Initial vertical velocity of the dart
ACC_{DART}	Rate	3:20	(Vertical) acceleration of dart
$HT_{GUN/GRD}$	Length	3:35	Height of gun (or vet) relative to ground
$HT_{TREE/GRD}$	Length	4:13	Height of the tree
$DIST_{VET/MKY}$	Length	4:36	Length of the straight path from the vet's gun to the monkey
ANG_{PAR}	Angle	6:04	Measure of angle gun is aimed relative to the horizontal, for parabolic path
$IVEL_{DART}$	Rate	11:37	Initial linear velocity of the dart
HT_{DART}	Length	15:35	Height of the dart
TIME	Time	16:08	Elapsed time
$ANG_{VET,3D}$	Angle	24:38	Measure of the plane angle formed by a designated axis and the line through the tree and veterinarian in 3-space
$HVEL_{DART-I}$	Rate	25.42	Initial horizontal velocity of the dart

22.4 Theory-Building Case Presentation and Analysis

We conducted a series of task-based interviews with participants ranging from middle grades to advanced undergraduates. The tasks ranged from simple word problems to applications to more complex problems where participants needed to make simplifying assumptions about the scenario. The purpose of our retrospective analysis of these data was methodological, specifically, developing a theoretically coherent procedure for tracing the evolution of a student's model throughout an interview. Here, we share the work of Merik, who was a non-traditional student. He returned to university after working in concrete industry management and in the automotive industry to pursue a mathematics degree with an education minor. He had completed courses through Integral Calculus and was taking Vector Calculus. Merik was asked to think aloud as he addressed the *Shoot The Monkey Task* in any manner that would be satisfying to him:

Shoot The Monkey Task

A wildlife veterinarian is trying to hit a monkey in a tree with a tranquillising dart. The monkey and the veterinarian can change their positions. Create scenarios where the veterinarian aims the tranquillising dart to shoot the monkey.

We assumed that Merik's interpretations of the task and his work could differ from ours. We provisionally accepted his work, without actively teaching, leading, or removing ambiguity (Goldin 2000). Follow-up questions and interventions aimed to clarify or document his thinking. We purposefully selected Merik to illustrate the need to explicitly analyse the role of quantification in mathematical modelling exactly because his work on the *Shoot The Monkey Task* embodies the phenomenon of interest "operating in a microcosm" (Walton 1992, p. 122). Because Merik was articulate and a capable mathematics student, his work is ideal as an illustrative case. He described his mathematical thinking and introduced many different inscriptions, quantities, and mathematical representations indicating that it would be possible to closely examine changes in his mathematical and contextual knowledge about the situation. We see his work on the task as a "meaningful but complex configuration" of the theories we elaborated above, "not as homogeneous observations drawn at random from a pool of equally plausible selections" (Ragin 2004, p. 125).

Merik created a total of 11 distinct representations. From Table 22.2, we observe that he rapidly introduced quantities, projecting more than half of the quantities of his cumulative modelling space in the first six minutes. This is consistent with previous research positing that identification of (ir)relevant quantities and variables occurs early in the modelling process (e.g., Blum and Leiss 2007). Many were not necessary to achieve a normative solution but provide evidence of the richness of his conceptions of the scenario. We infer that for Merik, introducing one quantity supported the projection of related quantities, perhaps due to prior scholastic experiences, such as when he introduced ANG1_{GUN}, DIST_{VET} and HT_{MKY} by projecting them onto an inscription representing a right triangle. There was not a one-to-one correspondence between symbols and quantities, because we could not infer situation-specific quantities for some symbols.

In the following, we offer an illustration of how attending to quantities, inscriptions, and situational referents enabled a detailed characterization of the evolution of Merik's modelling process and provide insight into why attending to these aspects inter-dependently is necessary. At around 9 mins 30 s into the session, Merik indicated he was seeking a quadratic equation. He explained his goal was to generate the equation such that the monkey would be located along the path that the dart would travel. Although Merik had not yet produced inscriptions resembling a quadratic equation, we interpret Merik's goal as indicating an implicit symbolic form. To gain insights into the situation-specific meanings Merik might hold for this quadratic symbolic form, the interviewer asked, "What variables and parameters would be present in your equation?" Merik immediately inscribed $f(x) = Ax^2 + Bx + C$. In Sherin's (2001) notation, Merik's inscription fits the template: $\square = \square \cdot \square^2 + \square \cdot \square + \square$. Although a quadratic function was relevant for Merik, we were initially unable to infer that he had imbued the symbolic form with situation-specific meaning to the task at hand. Later, Merik began to describe his meaning for the symbols in $f(x)$, "I know that my A is -10," indicating attention to gravity. Then, he indicated that B "would be whatever the initial velocity is, which I don't have." Merik later substituted particular values for $f(x)$ and x, representing specific instantiations of HT_{MKY} and DIST_{VET}, respectively. Although Merik indicated $C = 0$, he did not indicate a situation-specific quantitative referent for C. Having substituted particular values for x, $f(x)$, and A, Merik solved the equation obtaining a particular value for B, which he indicated was the initial velocity of the dart. At this point, we infer Merik's meaning for the quadratic symbolic form shifted from an inscription absent any situation-specific meaning to an inscription with situation-specific quantitative referents. We symbolise the situation-specific quantitative meaning we infer Merik attributed to $f(x)$ as: $\text{HT}_{\text{MKY}} = \text{ACCEL}_{\text{DART}} \cdot (\text{DIST}_{\text{VET}})^2 + \text{IVEL}_{\text{DART}} \cdot \text{DIST}_{\text{VET}} + \square$. Merik treated Ax^2 as a placeholder for the effects of gravity and Bx as a placeholder for effects of the initial velocity of the dart. Because Merik indicated x referred to DIST_{VET}, we hypothesise that at this point during the interview Merik intended the quadratic equation to represent the flight path of the dart. However, from our perspective, Merik's quantitative referents for A and B were suggestive of a parabola with a temporal component (i.e. x referred to elapsed time).

Later in the interview, we inferred Merik's quantitative referent for x shifted. At times, the symbol x explicitly referred to the veterinarian's distance from the tree; at other times x implicitly referred to elapsed conceptual time. Although we viewed Merik's quadratic template as a viable foundation for a mathematical model of the situation, we see evidence of competing meanings for the constituent inscriptions. For Merik, the quadratic equation referred to, at different times, a purely spatial parabola (i.e., the flight path of the dart) and a parabola with a temporal component. We hypothesise these competing meanings are one factor that may have prevented Merik from using the symbolic form to achieve a satisfactory conclusion, from his perspective and from ours, to the modelling task. A second factor that may have impeded Merik's progress in The *Shoot The Monkey Task* is related to attributes that did not satisfy any of the quantification criteria in Table 22.1. We did not find evidence that Merik attended to some quantities useful for achieving a normative quadratic solution. Although Merik conceived the dart's initial velocity with a horizontal component, we found no indication during the interview that Merik considered the dart's horizontal distance travelled at arbitrary moments in elapsed conceptual time. Merik considered the parabolic path the dart would travel, yet he did not indicate conceiving of the vertical and horizontal distances travelled co-variationally (see Carlson et al. 2002). Had Merik considered elapsed time and conceived the parabolic path in a covariational sense, we hypothesise he may have made greater progress towards a satisfactory solution.

22.5 Implications

Our theoretical and methodological considerations have resulted in three types of changes to models that should be considered: to meanings, to inscriptions, and to quantitative referents. Through our analysis of Merik's modelling activities, we identified two factors that may have impeded his progress in solving The *Shoot The Monkey Task*. First, there were competing mathematical and kinematic meanings for Merik's quadratic symbolic form. A single inscription (e.g. x) could refer to different quantitative referents (e.g. the veterinarian's distance from the tree or elapsed conceptual time). Although Merik's template choice and inscription were normatively correct, his meanings for these representations were not always compatible with expert conventions or consistent. Understanding students' meanings for inscriptions should be an ongoing pursuit through probing for situationally specific referents for students' representations. Second, some quantities relevant to achieving a normative solution were absent (from our perspective). To interpret significance of the second observation, we employ the modelling space construct. The set of models a student might produce in a given modelling task is dependent upon, and constrained by, the quantities a student introduces into the situation at hand. Merik may not have achieved a satisfactory solution precisely because his modelling space during the interview did not contain a quantity associated with the dart's horizontal distance travelled. To be clear, we are not asserting that a variable was missing from Merik's equation. Rather, Merik may not have been able to take up the facilitators' suggestions about how to resolve the competing meanings for his symbolic form because this quantity—which could have played a supporting role establishing a covariational relationship—was missing. If a facilitator thinks that a satisfactory solution is outside the student's modelling space, she can intervene in ways to engender consideration of the missing quantity. Had the interviewer drawn Merik's attention to the dart's horizontal distance travelled at intermediate times, Merik may have projected the quantity into the situation; in turn, his modelling space would have expanded to perhaps include a satisfactory solution.

Understanding meanings for students' inscriptions should be an ongoing pursuit from the facilitator's perspective. Facilitators could ask probing questions to elicit the situationally specific referents for students' representations, rather than assuming that the student has quantified an attribute in the same way as the facilitator. By carefully attending to the quantities students project into situations, facilitators can imagine the models a student is capable of producing and whether a satisfactory model might be among them. If a facilitator thinks that a satisfactory solution is outside the student's modelling space, she can intervene in ways to engender consideration of the missing quantity. In some cases, it may be necessary to support the student in quantification of the situational attribute within the specific task context, rather than asking directly about a missing variable. Future research is needed to understand how these implications might be adapted for teachers of modelling in whole-class settings.

There is immense generative potential to the methodology proposed here and the modelling space construct. They move the field closer to being able to systematically trace changes in a mathematical model—how they are precipitated, ways they change, and documenting how students may potentially respond to scaffolding. Attending to these issues is incremental but paves the way to making recommendations to teachers that are grounded in students' conceptual systems.

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Chapter 23
Characteristic Elements Influencing the Practices of Mathematics Teachers Developing the Modelling Process in Ninth Grade

Blanca Cecilia Fulano Vargas and Nelson Enrique Barrios Jara

Abstract The goal of this study is to identify the implicit and explicit features in the practices of teachers of mathematical modelling. Specifically, we investigate the characteristic aspects affecting the practices of teachers in public schools in Bogotá, Colombia, developing modelling in the ninth grade. To do this, a questionnaire was designed, considering two categories, which emerged from a theoretical analysis using an onto-semiotic approach: epistemic and didactic. The study was carried out with thirty mathematics teachers who had extensive experience in teaching mathematical modelling in ninth grade. The data were collected using the Google Docs platform and analysed in relation to the theoretical framework.

Keywords Mathematical modelling practices · Epistemic aspects · Didactic suitability · Onto-semiotic approach · Teachers

23.1 Introduction

This chapter presents an exploratory study to determine the elements that characterize the mathematical modelling practices of teachers who teach mathematics in the ninth grade of compulsory secondary education in Colombia. According to Frejd (2014), mathematical modelling "is … considered as a bridge between the mathematics learned and taught in schools and the mathematics used at the workplace as well as in society" (p. 5). Furthermore, Biembengut and Hein (1999) recognize the leading role of the teacher in students' experiences of modelling. Teachers, according to their knowledge, thematic contents and their institutional reality, choose contexts

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or phenomena that give meaning to the teaching of mathematical modelling; therefore, the teacher is responsible for choosing what type of mathematical modelling experiences are promoted in his/her practice. In this regard, the problem is to establish what are the characteristic elements that influence the practices of mathematics teachers when developing the modelling process in the ninth grade. The goal that guides the study is to identify the characteristic elements that affect the practices of mathematics teachers when developing modelling in the ninth grade in the district schools of Bogotá.

In this chapter, to fulfil our goal, the theoretical references from the onto-semiotic approach will be presented initially followed by epistemic aspects of mathematical modelling and the different elements of didactic suitability (Godino et al. 2016). The method used in the study to establish the elements that influence the practices of mathematics teachers in the city of Bogotá is then outlined. Finally, the analysis and discussion of results are presented.

23.2 Onto-Semiotic Approach: Didactic Suitability

According to Godino et al. (2016, p. 2), the notion of *didactic suitability* of an instructional process is defined as the coherent and systemic process articulated in six facets/aspects of didactical knowledge: epistemic, cognitive, affective, interactional, mediational and ecological. Figure 23.1 shows the suitability facets, their components (e.g. attitudes, affects, motivations, beliefs and values for the affective facet) and basic didactic suitability criteria (e.g. implication for the affective facet–student involvement in the study process). The model shown describes the implicit

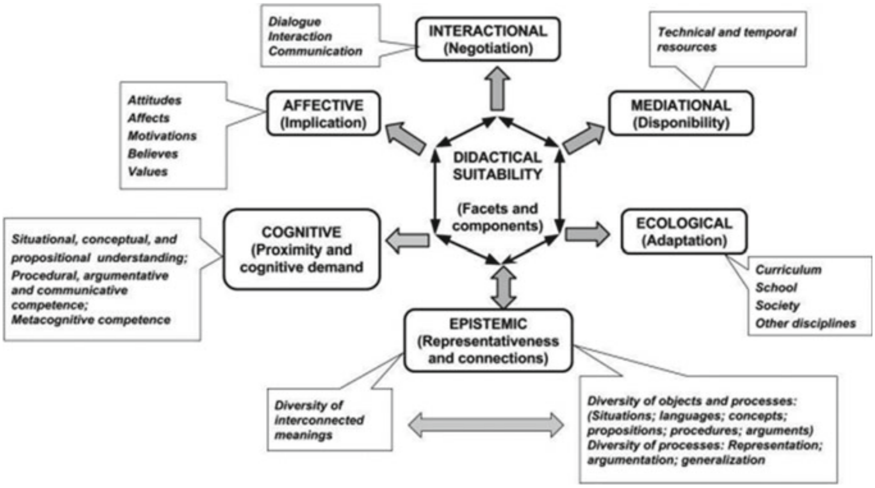

Fig. 23.1 Facets, components and basic didactical suitability criteria (Godino et al. 2016, p. 3)

aspects of the teacher in the face of decision-making in his/her teaching practices. This is how the cognitive aspect is related to the ways students learn, previous knowledge, and class objectives. The affective aspect is related to the motivational factors that it anticipates are mobilized in the classroom. The interactional aspect recognizes the student as the centre of learning, which leads the teacher to countless interrelations within the classroom among all subjects. The mediation aspect implies the use of technical human resources, technological elements and time. The ecological aspect encourages the interrelationships between society, curriculum and school. The epistemological aspect is didactic-mathematical knowledge about mathematical modelling. All these aspects are not isolated but are interrelated and are relevant to assess the teacher's instructional process.

Following this theoretical framework, we will analyse and synthesize epistemic aspects of mathematical modelling in Sect. 23.2.1 and didactical suitability criteria for mathematical modelling in Sect. 23.2.2 in the light of relevant literature.

23.2.1 Epistemic Aspects of Mathematical Modelling

In the first instance, *epistemic suitability* is related to mathematical modelling constructs, in this sense, authors such as Blomhøj (2019), Borromeo Ferri (2006), Kaiser and Sriraman (2006) and Stillman and Brown (2014), among others, present the theoretical considerations regarding: what is mathematical modelling, what is a model, what the modelling cycle consists of, and what are the contexts and representations.

The mathematical modelling process (MM), in the second instance, is considered as the scientific activity in mathematics, which involves obtaining models of the sciences (Biembengut and Hein 2004). On the other hand, it is considered as a didactic strategy. According to Villa et al. (2008), "MM is the production of a mathematical model based on a problem or phenomenon in the real world, it demands a period of time on the part of the model and requires some mathematical knowledge" (p. 2). In this sense, it could be said that mathematical modelling implies research from two focuses: from sciences other than mathematics and from education.

Regarding what is a mathematical model, Biembengut and Hein (1999) recognize that a mathematical model is a phenomenon or problem situation which "is a set of symbols and mathematical relationships that represents, in some way, the phenomenon in question" (p. 106). Blum and Niss (1991) consider that the mathematical model is "a triple (S, M, R), consisting of a real problem situation (S), a collection of mathematical entities (M) and a relation (R) between the objects and relations of S and objects and relations of M" (p. 39). These two concepts are closely related to scientific practice.

Faced with the epistemic concept of mathematical modelling, Biembengut and Hein (2004, p. 108) recognize that a teacher is able to implement mathematical modelling in two ways: the first, allows developing programmatic content from mathematical models applied to the various areas of knowledge and the second guides students to do modelling work. In contrast, Skovsmose (1994) affirms that "the mathematics put into social life, in the classroom, includes communication for the development of democratic discussions" (p. 19). Mathematical modelling becomes a strategy that enables the understanding of a mathematical concept from relationships and meanings. At the same time, Blomhøj (2019, p. 38) recognizes that the modelling process involves solving mathematical problems, immersed in a real context and includes problems of an extra-mathematical nature. Therefore, when talking about mathematical modelling, contexts make sense.

According to Kaiser and Schwarz (2006), "students acquire competencies that enable them to solve real mathematics problems including problems in everyday life, in our environment and in the sciences" (p. 196). From their theoretical framework, the OECD (2017) recognizes different contexts "relating to the self, family and peer groups (personal), to the community (local and national), and to life across the world (global) and applications are: health and disease, natural resources, environmental quality, hazards, and the frontiers of science and technology" (p. 80). It can be concluded that the variety of intra-mathematical and extra-mathematical contexts are inherent and are put into play when making decisions by the teacher when designing their teaching strategies.

On the other hand, authors such as Pollak (1979), Berry and Davies (1996), Geiger (2011), and Blum (2015) consider that the modelling process is cyclical. For the present study, the seven modelling stages by Blum (2015, p. 76) are considered: 1. Constructing, 2. Simplifying/structuring, 3. Mathematizing, 4. Working mathematically, 5. Interpreting, 6. Validating, and 7. Exposing.

Finally, Dan and Xie (2011. p. 460) recognize that the objective of the modelling work is aimed at strengthening creative thinking on the part of the modeller, by establishing existing relationships between the parts of the object, manipulating those relationships and creating new mathematical objects. This objective allows expanding the use of tools, signs, symbols and representations in the face of cognitive exercise in the construction of new schemes. According to D'Amore (1999), "semiotic representations can be discursive (natural language, in formal language) or non-discursive (figures, graphs, diagrams, tables)" (p. 273).

23.2.2 Didactical Suitability Criteria for Mathematical Modelling

Secondly, in the theoretical framework, from the onto-semiotic approach, an important role is the treating of the suitability criteria of an instructional process such as mathematical modelling for cognitive, affective, interactive, mediational and ecological facets of didactical knowledge.

The *cognitive suitability* corresponds to the degree of adequacy of the objectives. Breda et al. (2017) assert that the tasks must present a high cognitive demand (through generalization, intra-mathematical connections, conjectures, etc.), which implies in the practice decision-making taking into account the possibilities of identifying conflicts and negotiating meanings.

About the *affective suitability*, Beltrán-Pellicer and Godino (2020) recognize that this refers to the degree of involvement of students in the class; therefore, interest and motivation have a major role in the teaching-learning process. This suitability criterion is regulated by the emotional component, interest, personal commitment, tolerance of failure, feelings of self-esteem and aversion. These are fundamental for the development of mathematical modelling.

Regarding the *interactive suitability*, Perrin (1999) recognizes that the teacher's tasks are linked to the management of the interaction between the students and the mathematical knowledge that underlies the mathematical problem, that is, that the teacher when building a didactical proposal takes into account the different interactions and the analysis of these allows decision-making to develop mathematical modelling.

Mediational suitability is evidenced in terms of both operational and discursive practice and they take place in the configuration and selection of the means and resources with meaning necessary for the development of the activity and its instructional complements. According to UNESCO (2015), for "any educational resource including curriculum maps, course materials, study books, streaming videos, multimedia applications, podcasts and any material that has been designed for teaching and learning" (p. 5), it is important to have in mind the teaching-learning process since it allows the development of the modelling process.

With respect to *ecological suitability*, Font et al. (2010, p. 9) point out that the teacher must propose possibilities, be able to recognize internal and external elements, social relations with mathematics and establish links with other disciplines and with the daily life of the students. Ecological suitability implies that the teacher recognizes the curriculum both from the exogenous elements given by academic organizations of a global nature, the Ministry of National Education of Colombia and the territorial entities; and the endogenous elements given by the guidelines of the Institutional Educational Project, the area plans and the dynamics that are generated inside the school.

23.3 The Study

Thirty mathematics teachers (20 female and 10 male) teaching in ninth grade from public schools in Fontibón-Bogotá, Colombia, participated in the study presented in this chapter. All teachers had more than eight years' teaching experience; 21 (69.93%) were graduates in mathematics and 9 (29.97%) were engineers. In addition, 5 (16.65%) had a specialization and the other 25 (83.35%) had a master's degree. Finally, all teachers had knowledge about theories and experience in the area of teaching mathematical modelling in ninth grade.

To achieve the goal of the study, namely, to identify the characteristic elements that affect the practices of mathematics teachers when developing modelling in the ninth

Table 23.1 Categories, operative definition and number of statements in questionnaire

Category	Operative definition	No. of statements
Epistemic	Teacher's conceptions: mathematical model, mathematical modelling, modelling cycle, use of the representations and contexts	9
Didactic	Contents organizations, objectives, motivation, sources and classroom interactions	33

grade in the district schools of Bogotá. A questionnaire was prepared consisting of 42 statements using Likert scales, with five grading levels to identify the degree of agreement or disagreement with each statement. Structurally, the questionnaire provided demographic data and aspects related to mathematical modelling. The questionnaire was validated by two research experts in the area of mathematical modelling at the international level: Martha Isabel Fandiño Pinilla, a mathematician researching in mathematics education and Tulio R. Amaya de Armas, a researcher in innovation and didactics.

The questionnaire was consolidated into a Google Docs form and subsequently sent to each teacher by email. Each time a teacher answered the questionnaire, it automatically recorded his/her responses, which facilitated data analysis. The instrument is divided into two categories that emerged from the analysis and synthesis of the theoretical framework: the epistemic and didactic categories (see Sect. 23.2). Table 23.1 shows the description of each category.

23.4 Analysis and Discussion of Responses

The analysis of results was carried out by means of a descriptive scope, interpreting each statement according to the teachers' responses, which allowed consolidating the information for each category, and then contrasting it with the theoretical references. The results are demonstrative of the implicit and explicit elements in the mathematical modelling practices carried out by teachers of public schools in Bogotá, Colombia.

23.4.1 Epistemic Category

Related to the teachers' positions on mathematical modelling, 53.28% (16 out of 30) considered that mathematical modelling is inherent to scientific activity. Of this percentage, 60% (10) corresponds to teachers with a master's degree, while being consistent with the results obtained regarding the conception of mathematical model.

As for what is a mathematical model, it was evident that 86.66% (26 out of 30) of the teachers conceived of a model as a relationship between certain mathematical objects with a situation or phenomenon of a non-mathematical nature. The other

13.33% (4) of the teachers acknowledged that the model was associated with different representation systems. It was observed that 35% (9) of the teachers with master's degrees and more experience took the first perspective, while in the case of teachers with specialization only one had this conception that relates to the concept of Blum and Niss's (1991) mathematical model.

When asked: What is mathematical modelling? 26.64% (8) of the teachers took into account that mathematics is a tool that helps solve real problems, while 39.96% (12) acknowledged that mathematical modelling is similar to a scientific practice. One-third of the teachers did not recognize mathematical modelling as a didactic strategy that helps organize the teaching-learning process. In this sense, this finding is related to the position of Biembengut and Hein (2004) who point out that teachers develop programmatic content from mathematical models applying it to various areas of knowledge.

Regarding the conception of the modelling cycle, teachers prioritized three stages, simplifying, interpreting and validating, which gave didactic organization to their practices, aimed at the development of these skills in students. In this sense, 100% (30) of the teachers recognized validation more frequently, which corresponds to the sixth stage proposed by Blum (2015) for the modelling cycle.

About the representations, 46.62% (14) of the teachers prioritized symbolic representations, followed by dynamic representations (33.33%), since they implemented the use of GeoGebra and Excel. On the other hand, 13.33% (4) preferred that students use Cartesian representations and lastly, 6.66% (2) prioritized the use of tabular representation. This confirms the epistemic coherence of the teacher by recognizing the modelling process from a scientific and eminently symbolic position.

Concerning contexts, 86.58% (28) of teachers agreed with using intra-mathematical contexts, for example, the contexts that come from algebra and geometry. The rest used extra-mathematical contexts, for example, experiments. According to what was proposed by Kaiser and Sriraman (2006), teachers can be considered to relate the modelling process to different types of situations. All teachers with a master's degree applied mathematical modelling into intra-mathematical contexts, which is consistent with the epistemic stance on the scientific nature of a mathematical model.

Regarding the use of representation systems, two-thirds of the teachers used discursive representations, that is, symbolic and those corresponding to computational dynamics, while the other third used non-discursive representations, such as figures, graphs, diagrams and tables.

23.4.2 Didactic Category

Concerning the didactic category, the most important findings are related to the cognitive, affective, interactional, mediational and ecological suitability.

Analysing the *cognitive suitability*, it was found that the objectives set by 39.96% (12) of the teachers were directed at developing abilities for identification of regularities and patterns in the students. Regarding the validation of the model, one-third of the teachers prioritized this objective, requiring students to check values that relate the variables. Lastly, 26.64% (8) of the teachers intended for the student to carry out prediction processes based on the proposed model, in order to estimate different behaviours of the problem.

About *emotional suitability*, all teachers promoted values such as responsibility, discipline and perseverance in their classroom, seeking that students recognize the usefulness of models in everyday life and strengthen positive attitudes towards working with mathematical models that arouse interest and challenge. Such practices are consistent with the approach of Beltrán-Pellicer and Godino (2020) since teachers take into account in their teaching strategies for the promotion of interest and motivation in the mathematical modelling process.

Regarding the *suitability of interaction*, the teachers confirmed that they questioned students and promoted group work establishing different types of interrelation. However, the teachers acknowledged that, due to the high number of students in the classroom (on average 40), they could not guarantee the participation of all students. In this sense, the teacher's tasks are due to the interaction between students and mathematical knowledge, as suggested by Perrin (1999). Of the 30 teachers in the study, 19 (63.27%) prioritized interaction between students and 11 (36.73%) prioritized teacher-student interaction.

In relation to *mediational suitability*, according to UNESCO (2015), teachers consider in their didactic organization the use of technologies. With respect to this, 26.64% (8) of the teachers used GeoGebra, 19.98% (6) used Excel; on the other hand, 26.64% (8) used school texts, given that the educational authorities provide some schools with guide books. Concrete material was used by 16.65% (5) of teachers when proposing models to students in the development of geometry and only 9.99% (3) of the teachers carried out experiments to develop mathematical modelling in the classroom.

With reference to *ecological suitability*, all teachers developed the curriculum corresponding to ninth grade algebra, according to the guidelines and standards proposed by the Ministry of National Education of Colombia (MEN 2006), prioritizing linear, quadratic and cubic models. This finding confirms the position of Biembengut and Hein (1999), who point out that teachers develop content from mathematical models applying it to various areas of knowledge.

23.5 Conclusion

When analysing the epistemic suitability of teaching practices in mathematical modelling at Grade 9 level in Colombia, we found a model conception framed within a predominantly scientific practice. This is in accordance with the use of symbolic representations. These two elements make the teacher's didactics go in the line of monitoring the programmatic content and there is little assumption of creative and challenging postures. This conception greatly influences decision-making regarding the ecological suitability of mathematical modelling practices since teachers adhere to curricular guidelines and standards in teaching linear, quadratic, and exponential mathematical models. It is important to emphasize that a small percentage of teachers recognize mathematical modelling as a didactic strategy that allows students to create their models based on their real-world situations that have to do with extra-mathematical contexts.

As an implication of the previous results, it is advocated that the teachers consider the didactic references that involve the suitability of epistemic, cognitive, affective, interactional, mediational and ecological aspects for the teaching of mathematical modelling, which allows students to develop the modelling process. However, it is necessary for teachers to delve into the design of practices that strengthen the interrelation of suitability criteria.

It is important to consider the feedback of the teachers' practices, for which the creation of accompanying strategies and collaborative work among peers and between teachers with greater training is suggested, since these processes allow innovative practices and contributions that enhance and enrich extra-mathematical contexts. Likewise, the need to establish a mathematical modelling teaching network in Bogotá is determined, which allows feedback on the way modelling is being taught and conducting research processes in the classroom as well as serving as an organ to innovate in modelling practices. Such implications could be piloted in other countries where a similar state of affairs with respect to the teaching of mathematical modelling in secondary schooling is evident.

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Chapter 24
Pre-service Teachers' Facilitations for Pupils' Independency in Modelling Processes

Ragnhild Hansen

Abstract Recently modelling and applications were included in the revised mathematics curriculum for the Norwegian grade levels 1–10. The focus on mathematical modelling in primary grade education is a challenge, because of limited experience with modelling at this educational level. This chapter is based on the study of documents written by primary grade pre-service teachers, containing their reflections on modelling activities they had implemented during a practice period. From this content, we studied what procedural choices and assessments the pre-service teachers let the pupils make and how they facilitated their critical thinking. We found that pre-service teachers often emphasised mathematical exploration, but that they tended to offer specific tasks to assist pupils with this. Pupils were not often given the opportunity to narrow the modelling problem and decide how to collect and represent data.

Keywords Independent modelling processes · Critical thinking · Pupils' inquiries · Pre-service teachers' scaffoldings · Primary grade · Document analysis

24.1 Introduction

Educational research involving critical perspectives on modelling often focuses on model applications in society, but the complexity of these models causes them to be difficult to introduce at lower grade levels. In the perspective referred to as critical mathematics inquiry (CMI) proposed by Greenstein and Russo (2019), mathematical learning situations are perceived to contribute to critical mathematics education as long as the pedagogy is considered democratic. Democratic pedagogy is understood as inviting students to think mathematically in equitable classroom discussions, where students' inquiries are pursued and valued. As such, the CMI-perspective implies that the societal fruitfulness of educational modelling mostly depends on the pedagogical process and that models to be critically examined can

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be based on situations inside a classroom, as well as outside a school community. Critical thinking is central in the CMI-perspective and is also a prerequisite for well-founded judgements when operating stages in modelling processes. Considering collective work with models as communities of inquiry (Lipman 2003) supports the assertion that critical thinking can be fostered through educational modelling. According to Lipman (2003), a pedagogical goal of critical thinking is to equip students to think for themselves, and self-regulated modellers are considered central in promoting modelling competency (Kaiser and Brand 2015). Particularly for the lower school levels, we have found little research on how teachers can support pupils' independency and critical thinking in modelling processes. Based on this background, we have formulated the following research question: *How do pre-service teachers support primary grade pupils to make their own choices and judgements, and raise inquiries, during modelling processes?* Results from our investigation of this question are discussed with reference to the CMI-perspective, at end of the chapter.

Our study was included in the first cycle of an extensive research project based on Educational Design Research (EDR) (Akker et al. 2006). In EDR, researchers and practitioners continually can change conditions of the research process, with the aim to improve upcoming research cycles. By constructing interventions of increasing workability and effectiveness, EDR can contribute to the relevance of educational research and improve educational practice (Akker et al. 2006). Therefore, EDR could be effective to support implementations of new themes from curriculums, such as introducing modelling in compulsory education.

24.2 Literature Review and Theoretical Framework

It is not obvious what skills primary grade teachers should possess to complete productive modelling lessons, nor how to provide them with such capabilities. When introducing modelling to Grade 3 pupils and their teachers, English and Watters (2005) recommended the teachers make the children familiar with reading data tables and working collaboratively before starting the modelling process. They found that pupils' ability to distinguish between practical knowledge of the modelling context and scientific input data was important to work successfully with problems. This also enabled pupils to make predictions based on patterns in the data. Paolucci and Wessels' (2017) study of pre-service teachers' abilities to design appropriate modelling problems for Grades 1–3 showed that the pre-service teachers struggled with how to let pupils represent the problem context mathematically, as well as asking strategic questions to assist pupils in progressing towards solutions. In contrast, the pre-service teachers were proficient at creating modelling problems that contained contexts relevant to the pupils, and they often created problems which had feasible solution methods (p. 337). Ng (2018), who studied how experienced secondary mathematics teachers designed, promoted and assessed pupils' modelling processes, found one challenge to be how teachers intervened in the process and wanted to steer modellers towards specific mathematical outcomes instead of listening to their discussion (p. 295). The teachers also tended towards selecting a mathematical learning outcome from the curriculum as the starting point for the modelling process, for then to adapt a real-world context to this outcome. A positive feature was that in designing the tasks, the teachers referred to real-world contexts that were meaningful and relevant to the pupils, but generally they struggled to help pupils to progress on different steps of the modelling process.

Raising inquiries is a substantial part of critical thinking (Lipman 2003). Lindfors (1999) characterises inquiries as "language acts", through which they can be recognised. In inquiry acts, one attempts to engage another to help him go beyond his present understanding (p. 4). She distinguished between information seeking and wondering inquiry acts, the latter as open and playful, involving engagement in a process for its own sake, dealing with the imagined, uncertain or ambiguous (p. 40). A type of discourse, that has been demonstrated particularly to influence mathematics classrooms, is IRF-structured communication (e.g. Attard et al. 2018). In its prototypical form, it consists of an initiation (I) (usually a question or test put forward by the teacher) a response (R) to this initiation by the pupils, and evaluative feedback (F) from the teacher. This communication form has been associated with the "exercise paradigm" in mathematics (Mellin-Olsen 1996). Studying pre-service teachers' modelling processes, Barbosa (2007) detected two teacher discourses; *directive*, where teachers respond readily to questions, correct errors and provide direction for students' work, and *open*, where teachers attempt to provoke a reactive pattern by forming questions based on students' utterances. He claimed that the first type of discourse was not fitting for "authentic" modelling experiences (p. 239).

24.3 Method

Data for investigating the research question was the content of 14 documents written by groups of second-year pre-service teachers preparing to teach at Grades 1–7 in Norwegian compulsory school. In the documents, the pre-service teachers had reflected on the accomplishment of an assignment which asked them to implement modelling during two weeks of practice teaching. The modelling assignment and practice teaching were both part of a university college mathematics didactics course of 15 ECTS. Altogether there were 47 students divided between two separate college classes. Each document was written by 3–4 pre-service teachers from the same college class who conducted their practice at the same primary school. Before practicum, the classes had received three lectures on educational modelling, and the pre-service teachers had reflected on modelling literature before performing self-selected modelling activities in randomly selected groups. In an earlier course at the college, they had been taught basic statistics.

In accordance with EDR (Akkers et al. 2006), course teachers and researchers cooperated on including pedagogical guidelines into the compulsory assignment. This was to assist the pre-service teachers with facilitating pupils' independency and critical thinking in the modelling processes the pre-service teachers were supposed to implement. As a result, the pre-service teachers were encouraged to reflect on how they could act according to pupils' participation in (1) choosing the context for the modelling problem, (2) limiting it (making it approachable) and (3) collecting data to solve it. They were also supposed to reflect on (4) what solution methods one could expect pupils to use, (5) how they would encourage them to use their own formal or informal representations and (6) how they expected pupils to argue concerning model parameters. Further, they were supported to think about (7) how pupils could compare and verify their model results, (8) relate to error sources and (9) present their results (poster, group presentation, etc.). The pre-service teachers were also asked to reflect on how they had (10a) supervised the pupils and (10b) facilitated communication in the modelling process. The idea was to investigate the first part of the research question by studying how the pre-service teachers supported pupils' modelling processes according to these guidelines. This approach is further explained in Sect. 24.3.1.

To operationalise the second part of the research question, how the pupils were supported when raising inquiries, we analysed one of the documents using theories on classroom discourse from Attard et al. (2018), Lindfors (1999), Mellin-Olsen (1996) and Barbosa's (2007) studies of discourses detected in pre-service teachers' supervision of modelling processes. Document excerpts were selected strategically, according to whether they contained information relevant to investigate the research question.

24.3.1 Analytical Framework

A review of the 14 documents revealed that the students had followed the pedagogical guidelines only to some extent. We therefore introduced an analytical framework (Fig. 24.1) based on a grounded theory approach to investigate pupil involvement during the modelling processes. From a randomly selected subset of the documents, we created 10 hypotheses to be used as a basis for later interpretation of the remaining documents. These hypotheses were inspired by the pedagogical guidelines (1)–(10) and the literature in Sect. 24.2. In particular, we wanted to examine pre-service teachers' skills in selecting contexts for the modelling problems, how they facilitated pupils to represent contexts and if they tried to steer the processes. The findings were to be compared with corresponding results from Ng (2018) and Paolucci and Wessels (2017). Items H9–H10 were considered useful to discuss whether critical discussions of model results were emphasised (Barbosa 2009). We also wanted to investigate the applicability of the pedagogy accompanying the CMI-perspective. This approach resulted in the hypotheses in Fig. 24.1. The corresponding findings are analysed in Sect. 24.4.

To give more detailed information on how the analytical framework in Fig. 24.1 was induced, we present excerpts from two pre-service teachers' documents:

> The first thing we did, was to find out what the pupils' interests were. To find out of this, we had a conversation with the pupils, where everybody got the opportunity to say something they could imagine working with. In this conversation, it appeared that one pupil had got a new school bag and was very interested in talking about this. This [the school bag theme]

H1. Pupils participated in *selecting a context* from the real world that could serve as starting point for designing a modelling problem

H2. Pupils participated in discussing how to *limit a context* to a modelling problem that was possible to investigate

H3. Pupils had the main responsibility to *collect data* to investigate the modelling problem

H4. When collecting the data, pupils decided how to *represent* them

H5. Student teachers offered the pupils *mathematical tasks* that were meant to assist them in performing the modelling activity

H6. The student teachers emphasized correct calculations by use of methods or algorithms *already known*

H7. One of the aims with the mathematization (modelling process) was to *explore* a mathematical or statistical concept, strategy or idea

H8. Pupils were encouraged to discuss *different mathematical solution methods* for the problem

H9. Pupils were encouraged to discuss *error sources* that could have been present in the modelling process

H10. The document explicitly describes that pupils were encouraged to *present and discuss* their modelling results in class

Fig. 24.1 Framework to trace how pre-service teachers facilitated pupils' independency in modelling processes

> was also engaging the whole class. We quickly found that this was what we should work with, since we wanted the pupils to participate in designing the teaching program. After many suggestions, we agreed with the pupils that we should weigh their school bags every day and find differences and similarities from day to day. This was the start of our modelling project. (Grade 3)
>
> The tasks that were given to the pupils was to make a cardboard miniature version of the school using scaling. […] The pupils got a review of the concept "scaling" on a PowerPoint before they went out to carry out measurements. (Grade 7)

The first excerpt indicates that the pupils, to a large extent, had participated in choosing a real-world context (weight of school bags) that could serve as the starting point for designing a modelling problem. This seems not to have been the case for the context with the cardboard model in the second excerpt. From investigations like this, we developed H1 as one category. Similar interpretations were accomplished for the other categories in Fig. 24.1.

The 14 documents were then coded according to the framework in Fig. 24.1 by answering either "yes" or "no" to each hypothesis (for countability we used the number "1" if the answer was yes and "0" if it was no). This analysis relates to two interpretation levels; pre-service teachers' interpretations of the classroom-situations, and our interpretations of the replications and reflections described by the pre-service teachers. The pre-service teachers addressed pupils' performance of the modelling tasks, as well as their own supervision of the pupils. To exemplify, they often reproduced excerpts from classroom dialogues that had taken place. After having coded the documents we collected the results in tables.

The general potential of the analytical framework in Fig. 24.1 should be further analysed. For this study, it contributed to an overview of the documented modelling processes, so that the first part of the research question could be answered.

24.4 Findings and Analysis

Table 24.1 describes the activities that were implemented in practice teaching by the seven pre-service teacher groups that constituted one of the two college classes.

The students described the work with these activities as "mathematical modelling" or "modelling". Table 24.1 shows that all the pre-service teacher groups generated modelling contexts from pupils' nearby communities (weighing of bags, representations of birthdays, etc.). This finding is consistent with the findings by Ng (2018) and Paolucci and Wessels (2017) concerning teachers' proficiency in choosing pupil-relevant contexts for modelling problems. Applying the framework in Fig. 24.1 to the content of the documents describing the students' reflections, after they had implemented the activities in Table 24.1, gave the results in Table 24.2. This table is an overview of pupils' opportunities to make their own choices and judgements at different stages of the seven modelling processes. According to this table, H1 is zero for most of the activities. This shows that even if the societal contexts in Table 24.1 can be considered as relevant to the pupils, the pre-service teachers often ignored pupils' contributions in selecting them. From Table 24.2, we further notice that in many cases where H3 equals one, the value of H2 or H4 (or both) is zero. That H3 equals one, means that the pupils often were given the main responsibility to practically collect data. Despite that the pupils often were assigned the role as data-collectors, they achieved limited experience with making the modelling problem approachable (H2 often zero) and deciding how one could register or represent the collected data (H4 often zero). We came to this conclusion because many documents contained attachments presented as empty tables or diagrams the pupils were supposed to complete when collecting data. This finding can be compared to Paolucci and Wessels (2017) reporting that PSTs had difficulties with developing problems which required students to create a mathematical representation of the context. In

Table 24.1 Modelling activities implemented by seven groups of student teachers during a practice period

	Modelling activities	Grade
1	Distribution and representation of pupils' birthdays in different ways on the yearly quarters	1
2	Registration of colours of pupils' sweaters. Use of this information to discuss what colours to expect on the sweaters next week	2
3	Weighing of school bags every day during a week. Use of this information to predict weight of school bags for next week	3
4	Measurements of height differences on earth. (Pupils decided to measure the highest and lowest points and calculated the vertical difference between these points.)	4
5	Competition on shortest time spent for collecting most garbage over given distance. Questions about how much to collect to win other distances	5
6	Dropping of ball from various heights. Use of this information to make a model for bouncing height as function of drop height	5
7	Creation of cardboard model of the school	7

Table 24.2 Results from applying the analytical framework in Fig. 24.1 on the documents describing the seven modelling processes in Table 24.1

Activity	1	2	3	4	5	6	7
H1	0	0	1	0	0	(b)	0
H2	0	0	1	1	0	0	0
H3	0	1	1	1	1	1	1
H4	0	0	0	(a)	0	1	(a)
H5	1	0	0	1	1	1	0
H6	0	0	0	0	0	1	0
H7	1	1	1	1	0	1	1
H8	1	0	0	1	0	0	0
H9	0	0	0	0	1	0	0
H10	0	1	0	0	0	0	0

Numbering 1: the answer to the hypothesis was "yes", 0: the answer was "no", (a) not discussed in document, (b) hypothesis not relevant

cases where the pre-service teachers had decided how to perform the data collection, some groups reflected critically in the document on this in retrospect. An example can be found in the document describing activity number seven in Table 24.1. Here the pre-service teachers had equipped the pupils with a measuring wheel to map the circumference of the school. At the end of the document, they wrote, "The pupils should have gained more possibilities to find their own solution methods to the task, instead of, for example, us, deciding that they should walk outside and measure [the circumference with a measuring wheel]". Here, the pre-service teachers reflect on not having supported the pupils to make independent decisions about what they could have measured at school, and what measurement techniques they could have used. The pre-service teachers were aware of having addressed these stages (which can be associated with H2 and H5) by prescribing a procedure.

In about half of the activities, the pupils had to answer some sort of mathematical task (H5 equals one in four out of seven cases). According to Barbosa (2007), this can be interpreted as directive instructions. It can also be interpreted as mathematical steering of the modelling process (Ng 2018, p. 295) . Despite this administration, the pre-service teachers did not necessarily emphasise correct calculations or known algorithms (H6 is mostly zero). Instead, the aim with the modelling task was to explore a mathematical concept, algorithm or idea (H7 is often one). For example, in the sixth modelling problem, pupils were to use Excel to create a bar graph and compare median heights found by different groups to explore this concept. The content of many documents showed that the pre-service teachers were aware of the importance of letting pupils experience different solution methods. In our view, a few groups succeeded with this (H8). Critical reflections on error sources like measurement uncertainties were not present in many documents (H9 is often zero). Finally, few documents explicitly indicated that the pupils presented and discussed their modelling results in class (H10) which, according to Barbosa (2009), is central for reflexive discussions to appear. We applied the framework in Fig. 24.1 to the other college class, also consisting of seven pre-service teacher groups. The tendency that pupils were given the main responsibility to practically collect data persisted. As for the first class, pupils were left little independence in how to register data, find contexts for the modelling problem, limiting it and presenting results in class, but they continued to be afforded several opportunities to explore mathematical concepts. A little more often than for the first class, small tasks, known algorithms and different mathematical solution methods were offered, and the pre-service teachers were a little more attentive to error sources.

To investigate the second part of the research question, we arbitrarily selected the document describing the third activity in Table 24.1 to study the quality of the discourse in classroom dialogues, which the pre-service teachers had referred. To illustrate, we present an excerpt where they refer to communication that had taken place in the period after the pupils had registered the weights of their bags:

> The student [*a student from the practice teaching group*] then asked what days the bags had been the heaviest and the lightest, and what pupils thought could be the reason for this. A pupil answered that "the bag perhaps was heaviest on Monday, because then we got homework books and we brought swimsuits" [*…students here describe similar utterings from the pupils…*] Another pupil thought that "my bag perhaps was lighter on Wednesday, because I had eaten my lunch when we weighed the bag". Then the pupils worked two and two together and answered the tasks on the last page of the questioning-scheme they were working with (see att. 3). Finally, the student had a summary of what the pupils had answered on the various questions.

The question raised in the first line can be interpreted as inviting the pupils to explore a situation, in this case, why the bags were heavier or lighter on some days. Because the pre-service teacher is requesting the pupils to think about reasons for what they explore, the question can be comprehended as inviting critical thinking. The pre-service teachers recount that the pupils used the wording "the bag perhaps was heaviest…" and "my bag perhaps was lighter…" when responding to the question. The inclusion of the word "perhaps" indicates that the pupils expressed uncertainty. One possible interpretation of this is that these expressions were wondering inquiry acts (Lindfors 1999), suggesting that pupils wished to explore the situation with the weight of the bags more carefully. The following lines of this transcript (the last three lines) do not indicate that the pre-service teachers at this moment went into explorative dialogues with the pupils. Instead, they reported that the pupils started to work in pairs with a questioning-scheme followed by a summary led by themselves. By reading the scheme, we found the questions referred to which days the bags were heavier and lighter, what things were in the bag on these days, and if the pupils could detect a connection between the number of things in the bag and its weight. This situation can be related to the findings of Paolucci and Wessels (2017) who reported that pre-service teachers had difficulties with generating appropriate sequential scaffolding for modelling processes, and it can be compared to directive discourse approaches (Barbosa 2007). Further reading of the document revealed that later in the process the pre-service teachers represented pupils' data in a weight versus day Excel-diagram and performed a class-discussion about why the weights had varied during the week. According to the pre-service teachers' document, not all pupils could interpret a bar graph, and this was one goal with the activity. We notice that, even if the pupils were encouraged to independently collect the data (detect the weight of their bag every day), the pre-service teachers decided how to finally represent them (the Excel-diagram). This shows that the pre-service teachers struggled with how they could support pupils to make a mathematical representation of the context themselves (Paolucci and Wessels 2017). Instead, they steered the process (Ng 2018) towards the interpretation of the Excel-diagram.

The beginning five lines of the above excerpt can be characterised as an IRF-structured dialogue (the pre-service teacher asks a question, to which the pupils respond). Pupils are mostly justifying ("…because then we got homework books…"). In analysing the 14 documents, we often detected similar examples, containing a mixture of open-ended questions in the frame of an IRF-structured dialogue. This could have been due to pre-service teachers' earlier experiences with the exercise paradigm (Mellin-Olsen 1996) and typical discourses they had experienced in their own mathematics classrooms (Attard et al. 2018; Lindfors 1999).

We now consider what the pre-service teachers appraised to be the critical aspect of this modelling process:

> The pupils were critical to the model. When we asked them if they believed that the results would be the same for the next week, the pupils answered, as mentioned, "no" and argued about why they thought that the weight would be different next week. We interpret their answers as they had reflected on the results and found connections between the weight and the content of the school bags.

By questioning the results for the next week, the first lines of this transcript show that these pre-service teachers considered model critique to be connected to predictions. The last sentence shows that their interpretation of the quality of pupils' predictions was related to the context (the weights) not patterns in the data (English and Watters 2005). They did not reflect on this experiment as being theoretically ill-defined for making prognoses. A similar situation was detected for the second modelling activity in Table 24.1. Here the pre-service teachers asked the pupils about what possible sweater colour combinations they could wear next week. This problem would require logging the colours of all sweaters of the pupils. By reviewing the rest of the documents, we did not find evidence that any of the pre-service teacher groups in some way theoretically considered the validity of prognoses based on statistical data.

24.5 Discussion and Conclusion

This study was the first of several cycles in an EDR project directed towards implementation of modelling in primary grades. The student teachers were novices to mathematical modelling, and to some degree unfamiliar with college mathematics. Still, they were asked to experiment with modelling in a practicum period. They succeeded in finding modelling contexts that were familiar to the pupils, but often without including the pupils in this activity. In light of the CMI-perspective, familiarity of the modelling contexts is important at lower grade levels, because this facilitates democratic discussions. The democratic aspect could have been increased by including pupils in the process of selecting the contexts. Other efforts that could have supported the CMI-perspective would have been to more actively include pupils in limiting modelling contexts to become applicable problems, judge what data one could collect, and how to represent these mathematically, and emphasise critical reflections towards different solution methods. Since many of the pre-service teachers' questions were characterised as open, they supported the inquiry part of CMI, but the IRF structure appeared to have dominated the discourse. We found that even if students' supervision often was directive and teacher-centred, the modelling activity was still centred around exploration of a mathematical or statistical concept or method. This is inspiring and shows the importance of letting novice teacher education students gain experience with how to facilitate for explorative dialogues in modelling processes. For the forthcoming cycle of EDR, researchers and practitioners need to discuss how to support pre-service teachers to become more flexible in progressing pupils' modelling processes.

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Chapter 25
Investigating Pre-service Teachers' Experiences with the "A4 Paper Format" Modelling Task

Toshikazu Ikeda and Max Stephens

Abstract Our research question asks what kinds of educational effects are gained for a group of pre-service mathematics teachers when we address the *A4 paper format task* from the perspective that mathematical modelling can be used to enrich students' knowledge both in the real world and in mathematics. Around 60% of the pre-service teachers perceived that they could enrich their knowledge both in the real world and in mathematics, while around 30% were able to anticipate connections to their actual teaching in future. This suggests that pre-service teachers are able to appreciate these dual aims of modelling, that is, modelling can not only enrich students' ability to solve real-world problems but also deepen their ability to develop further mathematics.

Keywords Mathematical modelling · Pre-service education · Paper (DIN) formats · Mathematical knowledge · Prescriptive modelling · Descriptive modelling

25.1 Aim and Research Question

Teacher education concerning modelling has been an important issue at the international level (Borromeo Ferri 2018; Stillman and Brown 2019). This study focused on a group of second-year pre-service mathematics teachers in a Japanese education university who were taking a first course in mathematics education. The *A4 paper format task*, illustrated as one of the prescriptive modelling tasks (Niss 2015), was treated from the perspective that mathematical modelling may also be effective to point to new mathematical knowledge (Blum and Niss 1991; Ikeda and Stephens 2020).

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This study aims to understand pre-service teachers' views on mathematical modelling teaching. Instead of using direct questionnaires or individual interviews, we first allowed a group of pre-service teachers to participate in mathematical modelling activities and then analysed their evaluations and reflections. Our study design might provide a more objective insight into the true views of pre-service teachers. Our research question was: What kinds of educational effects are gained for these pre-service mathematics teachers when we treat the task, *A4 paper format*, from a perspective that mathematical modelling can be used to enrich students' knowledge both in a real world and in mathematics? We analysed the value pre-service teachers derived from this task and have drawn some conclusions about how the same task might be used with regular high school students.

25.2 Perspective of Teaching Modelling

Constructing mathematical knowledge starting from modelling has been discussed as one of the perspectives of teaching modelling. In the international classification of modelling perspectives by Blum and Niss (1991), "Prompting a mathematics learning argument" was identified. Under this perspective, our framework comes into line with Gravemeijer's idea (1999) presenting a "model-of/model-for shift" where both the model and the modelling facilitate reflection. However, in this framework, mathematization is restricted to two types: horizontal and vertical. Our framework treats multiple mathematizations (in our terminology, translations) from one world into another world and focuses on the comparisons and contradictions between competing perspectives among plural worlds (Ikeda and Stephens 2017). The principle that underpins our framework is that mathematics can be abstracted or concretized repeatedly from one world to another. Further, our focus is on comparisons and contradictions between competing perspectives that will promote the enrichment of both mathematical knowledge and modelling. The intention is to deepen students' knowledge both in the real world and in mathematics by connecting and integrating the outcomes constructed in each world.

25.3 Explanation and Justification of the *A4 Paper Format Task*

The *A-paper (DIN) format* is in widespread use in our world. Almost everyone is familiar with the common paper sizes (A3 and A4) used in current office photocopiers. However, people may be unaware of their exact measurements and may even be surprised to see how some measurements are replicated in the next size up or down, as shown in Fig. 25.1, which displays the measurements in millimetres for three common A-formats. Some would also be familiar with the enlargement/reduction

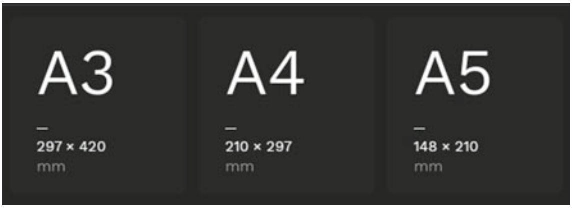

Fig. 25.1 Measurements for three A-paper formats

capabilities of modern photocopiers linking A3, A4, and A5 formats in which each can be enlarged or reduced onto the next size without any cut-offs or margins. This feature is a *defining* property of the family of *A-paper (DIN) formats.*

As Niss (2015) reminds us, the *A-paper (DIN) formats* system is composed of the following three properties: (P1) each sheet of paper is rectangular, (P2) the area of the largest sheet in the system is 1 m^2, and (P3) if any sheet of paper in the system is bisected across a mid-point transversal between the two longest sides, each half sheet is also in the system and is similar to the original one, that is, its side proportions remain the same. Properties (P1) and (P3) are fundamental, for example, to explain why an A4 sheet can be enlarged without margins to an A3 format simply by using the 141% A4 → A3 command.

These mathematical/geometrical features are fundamental to the experimental teaching episode described in this paper. What makes the *A4 paper format task* especially interesting is the opportunity it provides for students to access four different worlds. The first is *a real world*; the second is *a concrete operational world* where students are to fold the A4 paper; the third is *a geometric operational world*, which allows them to draw/investigate the geometrical figure; the fourth is *a symbolic operational world* which represents the phenomena by a numerical/algebraic formula. An important goal is to make students understand that, for all A formats, the ratio of the sides of rectangle has to be 1 to $\sqrt{2}$, namely "a silver ratio." By checking the actual measurements of an A4 paper format, they will confront questions such as "Why are the length and width of the A4 paper format 210 mm and 297 mm, respectively, and how does this relate to the ratio 1 to $\sqrt{2}$?" This contradiction between the actual data in the real world and the ideal value in a symbolic operational world requires them to develop further mathematics. Concrete and geometric operational methods in addition to the symbolic operational method are introduced to explain that $\sqrt{2}$ cannot be represented in a fractional notation. The second question is how to represent the irrational number by a sequence of rational number approximations. A rational sequence that converges to $\sqrt{2}$ is 3/2, 7/5, 17/12, 41/29, 99/70, …, which can be introduced by investigating the concrete operational and geometric operational methods to connect with the actual A4 format of 210 mm × 297 mm. Three times the numerator and denominator of the fifth term 99/70 is 297/210, which corresponds to the actual measurements of an A4 format. This interpretation is somewhat different from the method of *A-paper (DIN) format* as explained by Niss (2015).

25.4 Experimental Teaching Design

25.4.1 Settings on the Teaching of Modelling for Pre-service Teachers

One author, who was the classroom researcher, conducted a case study with 37 pre-service teachers of mathematics in the second year of a Japanese education university. Pre-service teachers first participate in completing a *A4 Paper format* modelling activity. Organized experimental teaching was conducted for 90 min by having pre-service teachers engage with this task on December 4, 2018. We assumed that pre-service teachers were already familiar with; (1) $\sqrt{2}$ as an irrational number, which is considered in secondary school using *reductio ad absurdum* and (2) how to execute the Euclidean algorithm by using numerical and geometrical methods. However, we did not assume that pre-service teachers knew that the irrationality of $\sqrt{2}$ can be explained with concrete operational and geometric operational methods and that a rational number sequence (Euclid's algorithm) converging to $\sqrt{2}$ can be developed by investigating the concrete operational and geometric operational methods. These features were integral to the design of the experimental teaching.

25.4.2 Task Design and Time Sequence in Experimental Teaching

The design and time sequence of the experimental teaching are shown in Table 25.1. In Parts 1–3 of the experimental teaching, A4 paper was shown to the pre-service teachers who were asked to investigate the ratio between the lengths of the shorter and longer sides. The essential property of the A4 paper format (and other associated A formats) is: if the paper is bisected across a mid-point transversal between the two longer sides, each half sheet is similar to the original one. This allowed pre-service teachers to set a formula such as $1{:}x = x/2{:}1$, to show that the ratio of the sides is 1: $\sqrt{2}$. By asking how this property of a paper system is applied in the real world, pre-service teachers could consider and clarify the unique utility of the A4 paper system.

Using the actual lengths of the shorter and longer sides of A4 paper as 210 and 297 mm, an important outcome of Part 3 was to show using a calculator that the ratio 1.414285714… is close to the value of $\sqrt{2}$. But is it possible to relate the actual dimensions of the shorter and longer sides of the A4 paper format as 210 and 297 mm more meaningfully to the ideal value of $\sqrt{2}$ in a symbolic operational world? This is the question to be investigated in Parts 4–6 of the experimental lesson.

In Part 4 of the lesson, pre-service teachers carried out a geometric investigation on a representation of a generalized A-format sheet (i.e. the so-called "silver-ratio rectangle") to show that, by covering the rectangle with the different kinds of squares

Table 25.1 Task design and time sequence

Time	Activities	What pre-service teachers are doing	What the classroom researcher is doing
15 min (1)	Solving the original problem	Analysing the properties of A4 paper	Asking what is the ratio between the lengths of the shorter and longer sides to clarify the properties of A4 paper
8 min (2)	Appreciating the solution	Appreciating the utility of A4 paper	Asking why the A4 paper system is applied in the real world and sharing ideas among pre-service teachers
7 min (3)	Sharing the contradiction	Sharing contradictions between the modelling results and the real situation	Setting the situation so that pre-service teachers can realize that the real ratio of the actual two lengths of A4 paper is very close but not equal to the value of $\sqrt{2}$
10 min (4)	Clarifying the additional problem	Understanding that the problem is to show the similarity between two rectangles	Introducing the geometric interpretation of the fact that $\sqrt{2}$ is an irrational number and focusing on the problem that pre-service teachers need to tackle
25 min (5)	Solving and explaining the problem	Explaining that $\sqrt{2}$ is not represented in a fractional notation	Asking pre-service teachers how to explain the similarity between two rectangles and letting share the ideas among them
15 min (6)	Elucidating the nature of the contradiction	Investigating the rational number sequence converging to $\sqrt{2}$	Guiding pre-service teachers to investigate how to make successive approximations to an irrational number by a rational number
10 min (7)	Reflection	Pre-service teachers writing what they learned from the lesson	Confirming two points of views: first, what pre-service teachers have learned from today's teaching and second, where and what can they apply of this in the future

iterated forever as shown in Fig. 25.2, $\sqrt{2}$ must be an irrational number. Group activities were used in this stage. By confirming inductively that the number of same sized squares is sequenced in a particular pattern, such as in "1, 2, 2, 2, ….," students' focus was directed to how to explain this fact in a geometric operational world. By discussing the similarity of several rectangles in Fig. 25.2, it was concluded for pre-service teachers that it is enough to show the similarity between rectangle ABCD

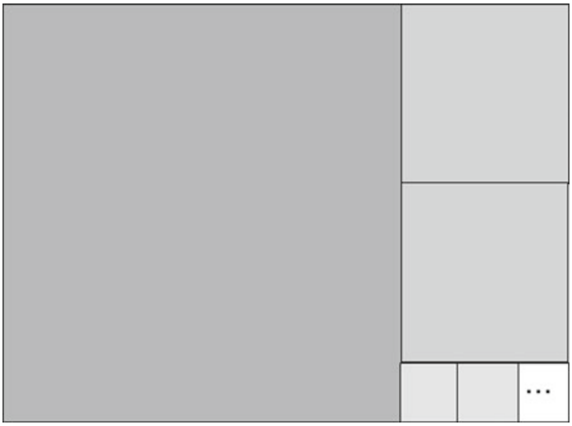

Fig. 25.2 Iterated geometric pattern

and rectangle EFGC in Fig. 25.3. These concrete and geometric operational methods were used to explain that $\sqrt{2}$ cannot be represented in fractional form.

In Part 5 of the lesson, the results from the iterated geometric figures in Fig. 25.2 were further extended by asking pre-service teachers: "Is it possible to make rectangles so that the ratio between the shorter and longer sides can be expressed in a fractional notation closer to $\sqrt{2}$?" Several rectangles were derived by the pre-service teachers and a sequence of rectangles was constructed that converge to the silver ratio rectangle as shown in Fig. 25.4. This question was intended to show students how to make successive approximations to an irrational number by a rational number. From the first four rectangles shown in Fig. 25.4, a series of ratios was derived as "3/2, 7/5, 17/12, 41/29."

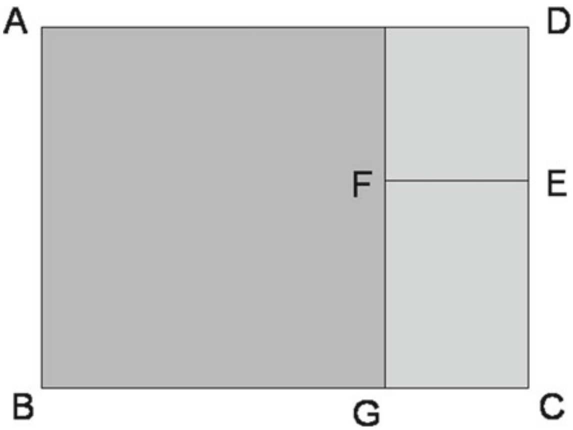

Fig. 25.3 Similarity of two rectangles

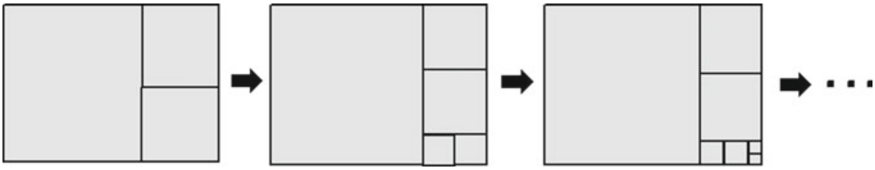

Fig. 25.4 A series of rectangles that converge to the silver ratio rectangle

Given the difficulty of further investigating these ratios visually, students were asked to consider the rule inductively. This is the focus of Part 6 of the lesson, where a rational number sequence can be developed, which in fact conforms to Euclid's method. The generating rule of the sequence was explained by one of the pre-service teachers as follows: "The next denominator is equal to the sum of the previous numerator and the previous denominator, and the next numerator is equal to the sum of the previous denominator and the next denominator." Only one student successfully explained the generating rule, and this allowed all pre-service teachers to consider the sequence "3/2, 7/5, 17/12, 41/29, 99/70, …." At this point, one of the pre-service teachers observed that three times the numerator and denominator of the fifth term "99/70" is "297/210," which exactly matches the lengths of the longer and shorter sides of an A4 paper format. This was the critical finding of Part 6, illustrating the mathematical significance of the actual dimensions of the A4 paper format. (Notice that 297/210 is not as close an approximation to the value of $\sqrt{2}$ as the sixth term of Euclid's sequence, which is 239/169.) Part 6 of the lesson built on the concrete and geometric operational methods explored earlier in the lesson, and would not have made sense without those preceding explorations. Part 6 linked together the previous investigations involving four different worlds.

25.4.3 Comparing This Approach to A4 Paper Format with Niss (2015)

Niss (2015) presented an *A-Paper (DIN) formats* task as an example of prescriptive modelling, which contrasts with descriptive modelling. For Niss, the general term is directed under the properties (P1), (P2), and (P3) by using a numerical method, such as recursion and induction, as follows; "$L_n = 100/[2\hat{}\{(2n - 1)/4\}]$ cm, $S_n = 100/[2\hat{}\{(2n + 1)/4\}]$ cm; L_n means the longer side of nth sheet and S_n means the shorter side of nth sheet." As a result, A4 paper is calculated as L_4: 29.73 cm and S_4: 21.02 cm. In this paper, the following three points are different from the approach by Niss; (1) not presenting the property (P2) "the area of the largest sheet in the system is 1 m^2," (2) focusing on the sequence of rectangles which converge to the silver ratio rectangle, and (3) using concrete operational and geometric operational methods in addition to symbolic operational methods.

25.5 Analysis of Pre-service Teachers' Reflections

25.5.1 Design of a New Analytical Tool to Assess Pre-service Teachers' Reflections

Pre-service teachers were asked to reflect and write down what kinds of educational effects they gained from the lesson. In Part 7 of the lesson, the following questions were used: firstly, what did you learn from today's lesson? secondly, what can you apply from what you learned today and where would you apply it?

Two coding categories were developed to identify the differences among pre-service teachers' writings about what they have learned from the experimental teaching. The first category was concerned with pre-service teachers' perceptions of modelling. Did pre-service teachers perceive the mathematical analysis involved in their modelling as enriching their knowledge in the real world or did they perceive modelling as enriching their knowledge in mathematics as a result of solving a real-world problem? If the pre-service teacher's writing was concerned with both viewpoints, it was assumed that this pre-service teacher could enrich his or her knowledge both in the real world and in mathematics, that is, in plural worlds. When assessing pre-service teachers' writings, we applied these two different but not mutually exclusive points of view. The second category was concerned with identifying whether students could anticipate connections to a high-school mathematics lesson. For this viewpoint, an assessment was made about how pre-service teachers were prepared to explore this kind of activity in their future teaching. Further, we recorded whether students added any consequent notes for a lesson. Table 25.2 shows the types of category criteria and sample comments from students. One author and a Japanese associate performed this coding. The pre-service teachers' responses were coded individually by the two coders and then discussed to resolve any discrepancies in the coding. The percentage of inter-coder agreement was 81.1%.

Written responses in Part 7 of the lesson were coded in two categories: The first concerns their perceptions of modelling, while the second concerns their anticipated connections to a school mathematics lesson. Examples of pre-service teachers' writings for two categories are as follows. Coding comments are given in italics.

> Student S: I was surprised to notice that it is possible to consider the device hidden in A4 papers and the problem of rational and irrational numbers by applying the contents learned in junior high school mathematics, such as geometric similarity and square roots. I was unsure whether or not we could apply mathematics contents such as geometric similarity in a real world. However, I understand mathematics can be applied in the real world from this kind of application. I think it is good for junior high school students to learn this kind of application and realize the fun of mathematics, as well as to develop various ways of mathematical thinking. [*Enriching knowledge both in the real world and in mathematics as well as the willingness to treat this activity at school are evident, but there is no description about interaction among plural worlds.*]

> Student F: I learned with interest because it is easy to understand the essence of $\sqrt{2}$ by using things in the real world. It is hard to concretize irrational numbers such as $\sqrt{2}$; however, it

Table 25.2 Categories of pre-service students' writings about their experimental teaching

Type of category criterion	Sample comment
Category 1: Generated perceptions of modelling 1-1: Enriching real-world knowledge 1-2: Enriching mathematical knowledge 1-3: Interactions among plural worlds Category 2: Connection to a school mathematics lesson 2-1: Willingness to treat this activity 2-2: Notes in a lesson	1-1: Mathematics is used in everyday life, as in A4 papers 1-2: Further mathematics is developed in ways such as proving why $\sqrt{2}$ is unrepresentable in fractional notation by using the concrete operational method or the geometric operational method 1-3: I think it is important to consider multi-directionally, as we experienced that the irrationality of $\sqrt{2}$ can be explained by the concrete, the geometric, and the symbolic operational methods 2-1: I would like to address this problem with high school students because I had a meaningful experience that revealed to me that mathematical knowledge can be greatly enriched by using A4 papers 2-2: If I approach this activity in junior high school, I will need to pay attention to how to deal with the idea of contra-position, because students have not yet studied it

becomes possible to deepen our mathematical knowledge by using the material in the real world [Here only *enriching knowledge in mathematics is evident*].

25.5.2 Generated Perceptions of Modelling

First, pre-service teachers' writings were assigned to Category 1: generated perceptions of modelling. The number and percentage of pre-service teachers who belong to each sub-category are shown in Table 25.3. Here, 1-1 means "enriching their knowledge in the real world," 1-2 means "enriching their knowledge in mathematics," and 1-3 means "describing the interactions among plural worlds." Thirty-six pre-service teachers (97.3%) wrote that they could enrich their knowledge in mathematics by tackling the *A4 format task*. On the other hand, 22 pre-service teachers (59.5%) wrote that they could enrich their knowledge in the real world by tackling the *A4 format task*. Further, only 15 pre-service teachers (40.5%) pointed to the interactions among plural worlds as meaningful. Next, in Table 25.4 we analysed the relations between 1-1, 1-2, and 1-3. No pre-service teacher wrote only 1-1, whereas

Table 25.3 Result of generated perceptions of modelling

	1-1	1-2	1-3
Number of pre-service teachers	22 (59.5%)	36 (97.3%)	15 (40.5%)

Table 25.4 Detailed result of generated perceptions of modelling

	1-1 and 1-2	1-1 and 1-2 and 1-3	1-1 not 1-2	1-2 not 1-1	1-1 and 1-3 not 1-2	1-2 and 1-3 not 1-1
Number (%)	22 (59.5%)	8 (21.6%)	0 (0%)	14 (37.8%)	0 (0%)	7 (18.9%)

14 (37.8%) of the pre-service teachers wrote only 1-2. These pre-service teachers tended to value mathematical knowledge rather than knowledge in the real world. However, 22 (59.5%) pre-service teachers wrote 1-1 and 1-2. Around 60% of the pre-service teachers appreciated mathematics can be enriched thanks to real-world problem solving; and a real world can be enriched by mathematics. Only eight pre-service teachers wrote 1-1, 1-2, and 1-3 (21.6%), showing that it is difficult for pre-service teachers to reflect and write that modelling enriches their knowledge both in mathematics and in the real world by pointing to meaningful interactions among plural worlds. The following student's writing included 1-1, 1-2, and 1-3:

> Student K: In the real world, irrational numbers such as $\sqrt{2}$ are not used because it is impossible to measure them. However, the idea of $\sqrt{2}$ is applied in systems of paper formats, such as A4 paper. Although the relation between the A4 paper format and the irrational number is not found at a glance, the relation between them is gradually found out by using the Euclidian Algorithm, the geometric operation, and so on. From this result, we can understand that a phenomenon can be interpreted in a variety of ways, so our mathematical knowledge is also expanded … Today's teaching begun from a question about one thing, and then … further questioning this answer. I think that our knowledge will grow by considering "why." [*Enriching knowledge both in the real world and in mathematics, interaction among plural worlds*, *but no anticipated connection to a school mathematics lesson.*]

25.5.3 Anticipated Connections to Future Teaching

Thirty-seven pre-service teachers' writings were assigned to Category 2: anticipated connection to a school mathematics lesson. The number and percentage of pre-service teachers who belong to each sub-category is shown in Table 25.5, where 2-1 means "willingness to treat this activity" and 2-2 means "additional notes in a lesson." Twenty pre-service teachers (54.1%) wrote down that they wanted to use this type of activity in the classroom, and 15 pre-service teachers (40.5%) wrote that it is necessary to pay attention to anticipated students' difficulties in a practical teaching situation. Next, in Table 25.6, we analysed the relations between 2-1 and 2-2.

Nine pre-service teachers wrote only 2-1 (24.3%), whereas four pre-service teachers wrote only 2-2 (10.8%). Eleven pre-service teachers wrote both 2-1 and

Table 25.5 Result of connections to a school mathematics lesson

	2-1	2-2
Number of pre-service teachers	20 (54.1%)	15 (40.5%)

Table 25.6 Possible connections to a school mathematics lesson

	2-1 and 2-2	2-1 not 2-2	2-2 not 2-1
Number (%)	11 (29.7%)	9 (24.3%)	4 (10.8%)

2-2 (29.7%). There were four pre-service teachers who wrote 2-1 and 2-2 in addition to 1-1, 1-2 and 1-3. Here is one example:

> Student N: It is amazing for me that A4 paper is interpreted not only as the outcome using the idea of geometric similarity, but also as the means to examine the irrationality of $\sqrt{2}$. The format of A4 paper is ideally made up as 1: $\sqrt{2}$. However, it is impossible to make an accurate value in a real world. I had goose bumps to get the actual data of A4 paper as the rational sequence converged to $\sqrt{2}$. It is interesting to consider the irrationality of $\sqrt{2}$ using concrete operational and geometric operational methods, not simply relying on the numerical method called *reductio ad absurdum* commonly taught at high school…… It is new for me to be able to prove the irrationality of $\sqrt{2}$ visually by using the similarity of geometric figures. I want to do teaching like this so that students can apply a variety of their ideas.

Student N's comment demonstrates interaction among plural worlds, showing how this student's knowledge has been enriched both in the real world and in mathematics. Also evident is this student's ability to make a clear connection to a school mathematics lesson.

25.6 Discussion and Conclusions

All pre-service teachers perceived that they could enrich their knowledge in mathematics through this activity. Nearly 60% of pre-service teachers perceived that they could enrich their knowledge both in the real world and in mathematics. However, this paper does not examine how pre-service teachers shifted their perceptions of modelling as a result of the experimental lesson. This should be examined in future. Further, pre-service teachers were motivated to apply their experiences of mathematical modelling in a classroom in the next mathematics education course. In addition, the *A4 Paper format* mathematical modelling task illustrates very clearly how teachers can build on contradictions between the modelling results and the real situation.

Finally, the categories of pre-service students' reflections on the experimental teaching episode provide some inspiration for future research, such as how pre-service teachers' perceptions of the value of, and relevance of, mathematical modelling to their future teaching change over time. Experimental teaching activities, such as those reported in this study, which are designed from a modelling and mathematical perspective, enable students to bridge between a concrete operational world and a mathematical (symbolic and geometrical) world. We envisage that, having participated in this experimental session, no pre-service teacher could in future observe the 141% A4 → A3 command and simply think that 141% was an arbitrary ratio, unrelated to all other A-paper formats.

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Chapter 26
Didactical Adaptation of Professional Practice of Modelling: A Case Study

Sonia Yvain-Prébiski

Abstract In this chapter, the aim is to study the possibilities of giving to students the responsibility for mathematical work that makes it possible to make an extra-mathematical situation accessible through mathematical treatment. I briefly present the elements of a first epistemological study of researchers' modelling practices. I show how I used it to design, implement and analyse a situation for teaching and learning mathematical modelling, based on an adaptation of a professional modelling problem on tree growth prediction.

Keywords Mathematization · Modelling cycle · Modelling practices · Problem-solving · Transposition · Horizontal mathematization · Phase of questions and answers

26.1 Introduction

Research, especially at ICTMA (Blum 2015), shows the importance of the learning and teaching of mathematical modelling development in secondary school but also highlights hindrances concerning mainly the conception and implementation of modelling activities in classrooms. In France, the modelling of extra-mathematical situations has been part of the curriculum since 2016 (Ministère 2015). But in most cases, the choices necessary for the mathematical treatment of such a situation are not the responsibility of the students. The research question under consideration is how to make the devolution (Brousseau and Warfield 2014) to secondary students (11–18 years old) of mathematization work necessary to make a situation rooted in reality accessible to a mathematical treatment. Following the French tradition on the roles of epistemology in didactics (Artigue et al. 2019), I assume that an epistemological study of researchers' modelling practices can enrich this didactic work. After

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specifying the theoretical framework of the research, I briefly present the results of the epistemological study (Yvain-Prébiski 2018). I then show how I have used them to provide elements of an answer to the research question posed in this chapter.

26.2 Theoretical Background

I focus on the teaching and learning of mathematical modelling based on extra-mathematical situations with a specific interest in the devolution to secondary students (11–18 years old) of the work of mathematization necessary to make a situation rooted in reality accessible to mathematical treatment. Thus, within the theoretical framework of Realistic Mathematics Education (RME), I consider the distinction introduced by Treffers (1978) and Freudenthal (1991) between *horizontal mathematization* which "leads from the world of life to the world of symbols" and *vertical mathematization*, as work within the mathematical system itself.

> Treffers, in his thesis of 1978, distinguished horizontal and vertical mathematising […]: Horizontal mathematising, which makes a problem field accessible to mathematical treatment (mathematical in the narrow formal sense) versus vertical mathematising, which effects the more or less sophisticated mathematical processing. (Freudenthal 1991, p. 40)

In the line of the French didactic tradition in mathematics (Artigue et al. 2019), the research methodology used for the didactic analyses is didactic engineering (Artigue 2015), an essential characteristic of which is based on the comparison between a priori and a posteriori analyses of the didactic situations concerned.

26.3 Epistemological Considerations

The specificity of this research is to support a didactic work on an epistemological study of the contemporary practices of researchers using mathematics in modelling work. The objective is to identify in the discourse of interviewees some invariant practices allowing the construction of indicators likely to attest that a horizontal mathematization is at stake. In this section, I report on elements that emerge from this epistemological study, on the one hand from the literature, and on the other hand from a study of researchers' actual modelling practices.

26.3.1 Evidence from the Literature Review

I followed Israel (1996), who defines a mathematical model as "a piece of mathematics applied to a piece of reality" and specifies that "a single model not only describes different real situations, but this same piece of reality can also be represented by different models" (p. 11). By crossing this point of view with RME, horizontal mathematization seems relevant to explore educational issues related to the research question. It led me to define different forms of horizontal mathematization that seem relevant to characterize this type of work, namely: choosing a piece of reality to question in order to answer the problem; identifying and choosing the relevant aspects of the piece of reality (context elements, attributes); and relating together the chosen aspects in order to construct a mathematical model. In addition, following Chabot and Roux (2011), I added another form of horizontal mathematization: quantification which refers to the association of some aspects of reality to quantities (essentially consisting in measuring).

26.3.2 Main Findings of the Study of Researchers' Modelling Practices

In the educational perspective of studying what can be transposed from researchers' actual modelling practices, I led a study of researchers' modelling practices. I conducted, transcribed and analysed interviews with researchers using mathematical modelling in the context of life sciences (Yvain 2017). The main findings are the identification of three invariant features in the practices of researchers which contribute to the transformation of reality to mathematical solvable problems: (a) *simplifying the problem and selecting a piece of reality*; it supposes to identify relevant variables and choose relevant relations between the selected variables by anticipating the mathematical treatment that these choices induce; (b) *choosing a model among those known* by the researcher in order to initiate vertical mathematization, at the risk of having to refine or reject the initial model later; and (c) *quantifying in order to compare the "real data" with the results obtained* within the model. This contemporary epistemological study has helped me to better identify the form and role of horizontal mathematization in a mathematical modelling activity. It also allowed me to develop a diagram of the modelling cycle including the dialectical relationships between horizontal and vertical mathematization, which I will detail in the next section.

26.4 Towards an Enriched Modelling Diagram

The work of Maaß (2006) underlines on the one hand the diversity of schema proposals to illustrate the modelling process in the literature, and on the other hand that this diversity of proposals is essentially correlated to the learning objectives and

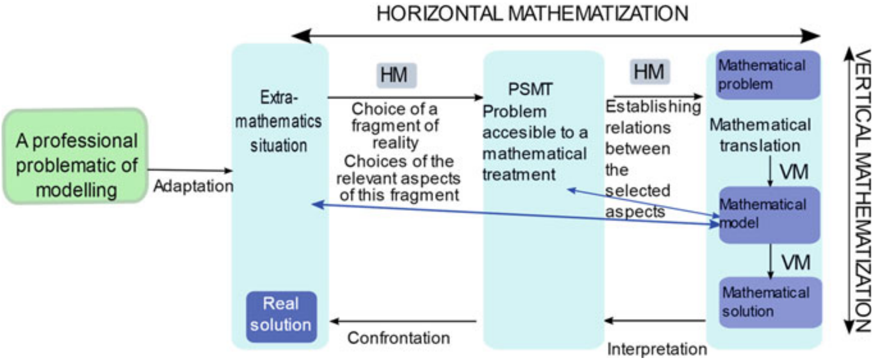

Fig. 26.1 Diagram describing the modelling process. Adapted from Blum and Leiss (2007)

the choice of skills to be developed /targeted by the authors. Taking this into consideration, I conducted a study to develop a diagram of the modelling cycle including the dialectical relationships between horizontal and vertical mathematization.

To do this, I relied on the classification work of Borromeo Ferri (2006) included in Rodriguez's thesis work (2007) and on the work of Blum and Leiss (2007). From this study and the results of the previous epistemological study (Yvain-Prébiski 2018), I developed a diagram describing the modelling process (based on the modelling cycle of Blum and Leiss 2007) taking into account the two aspects of mathematization and the invariant elements identified in the research modellers' practices (Yvain 2017). This diagram (see Fig. 26.1) begins with an adaptation of a professional modelling problematic which leads to a statement rooted in reality. To address this problem, the students need to choose a fragment of reality and the relevant aspects. This issue becomes a problem accessible to mathematical treatment, and students have to establish relations between the selected aspects to make this as a mathematical problem and then to build a mathematical model. I use two line segments with arrowheads on either end showing that students can begin the solving by choosing a known mathematical model and then testing it. That highlights the dialectic relation between the horizontal and vertical mathematizations. In this chapter, I focus my analysis on the part of the diagram which can highlight what happens between steps 2 and 3 of the Blum and Leiss diagram (2007) (see Fig. 26.2).

26.5 Design of the Modelling Situation

Based on the results presented in Sects. 26.3 and 26.4, I have developed a methodology to design, implement and analyse a teaching and learning situation for mathematical modelling. The objective is to foster the devolution of horizontal mathematization to students. I have characterized the problems likely to promote the learning

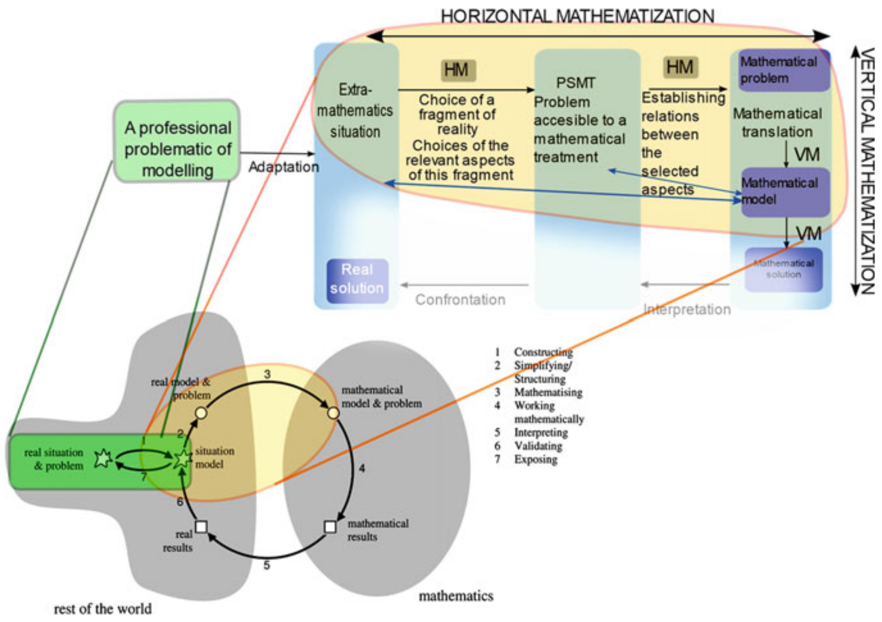

Fig. 26.2 Part of the diagram used for the analyses

of this mathematization. I call such situations "Realistic Fictions conceived as Adaptations of a Professional Modelling Problem" (FRAPPM). They should lead students to reflect on the system to be modelled and to bring them to become conscious of (a) the necessity to develop a model to solve a problem, (b) the necessity to make choices to mathematically address the problem, (c) the importance of the question set to them during the development of the model and (d) that the work behind the development of the model requires mathematical thinking within the model chosen to answer the questions. I have chosen to adapt a historically and epistemologically important modelling problem in life sciences: plant growth modelling (Varenne 2007). It is about predicting the growth of a tree based on information about its first years of growth given by diagrams:

The Tree Botanists from a Botanical Garden have discovered an exotic tree. To study this new species, the botanists have sketched the tree every year since 2013. The botanists want to build a greenhouse to protect it. They believe it will have reached its full size by 2023.

To help them predict how the tree will be in 2023.

I have chosen didactic variables and their values in order to promote the devolution of horizontal mathematization to students: *V1*: *2D versus 3D diagrams*: I have chosen 2D diagrams to provide a sufficiently realistic framework while allowing a modelling activity from the 6th to the 12th grade, *V2*: *The number of diagrams*: I have proposed three of them. This number is sufficient to allow students to make choices about the growth rules of the different elements of the tree, *V3*: *The shape of the tree* (symmetrical versus asymmetrical): I have chosen asymmetrical growth to encourage students to question the forecasting of the tree's growth. The shape chosen does not resemble that of a known (or easily recognizable) tree to avoid a possible search on a search engine (or other) on the growth of the potentially recognized tree, *V4*: *The number of new branches appearing each year*: I have chosen to make two or three new branches appear to quickly encourage students to make choices about the tree's growth, *V5*: *The lengths of the trunks and branches*: they were chosen to question a possible choice of a regular growth model and *V6*: *To give a scale or not*: I have chosen to give a scale to allow measurements and instrumented information to be taken from the drawings. With these choices, simplifying assumptions and identification of relevant variables that influence the actual situation are necessary to consider a mathematical treatment of the given problem. The choice of an initial growth model can be a known mathematical model (e.g. proportionality) that can allow students to work on vertical mathematization in order to shed light on the problem even if it means refining or rejecting the chosen model by reconsidering their first choices. Providing a scale and allowing students to take instrumented information could encourage students to choose a model and then compare the results in the model to their knowledge of how trees grow in the real world.

26.6 Implementation of the Modelling Situation

To help teachers implement in their classrooms the situation of *The Tree* presented in Sect. 26.5, I worked with a group of teachers in a professional development programme on mathematical modelling. This programme called ResCo (collaborative problem-solving) is proposed by a group of the IREM of Montpellier (Research Institute for Teaching of Mathematics). In this group (existing for more than ten years) some researchers and teachers work collaboratively. Each school year, this programme offers a collaborative project for volunteer teachers with their classes from grade 6 to the end of high school. The scenario includes five sessions (one hour per week). Groups of three classes are formed and all classes interact using an online forum regulated by the ResCo group (see Modeste and Yvain 2018 for further details of the programme). One of its particularities is a question-and-answer session designed to begin the resolution of the problem. This first step of the scenario requires teachers to devote the first session to getting students to ask questions about the problem, to send them to the two other classes in their group, via the ResCo online forum, and then to devote the second session to trying to answer questions from other classes. The aim is to have the students ask themselves questions about the different possible choices that would allow them to deal with the problem mathematically. It is during this phase of questions and answers that, on the one hand, the relevant questions for solving the problem will emerge and, on the other hand, different possible modelling choices will appear. The questions received lead to discussions that allow students to become aware of the need to make choices to deal mathematically with the problem, particularly around the identification of relevant magnitudes. In the third session, the students discover and discuss the answers to the other classes. Between the second and third sessions, based on the questions and answers submitted on the forum, the ResCo group develops a "relaunched realistic fiction". It is addressed to all classes during this third session, in order to set modelling choices to allow further collaboration in solving a common mathematical problem. The intentions of the group are to make visible to students the need to make choices to solve the problem. During the fourth session, the students continue the research of this same mathematical problem, resulting from the modelling choices set by the "relaunched realistic fiction". During the last session, teachers are invited to carry out an assessment with their students to close the session. The ResCo group uses all student productions posted on the forum to produce an assessment of the concepts, mathematical skills and heuristic skills that the problem has implemented as well as elements of a mathematical solution to the problem. In this chapter, I will focus my analyses during the phase of questions and answers which contributes to making the situation accessible to mathematical treatment.

26.7 A Priori Categorization for Analyses

To analyse the horizontal mathematical work, I am interested in the question–answer pairs produced by the students. To do this, I define a priori categorization for the questions and another one for the answers.

26.7.1 A Priori Categorization of Questions

I categorize a priori the students' questions by using three indicators based on the forms of horizontal mathematization. The first one Q_{mod} concerns question showing the *search for a model* to address the proposed situation. It could highlight a work of horizontal mathematization insofar as the research of the model induces the student to make preliminary choices of one or more fragments of reality and some of their aspects. This indicator essentially reflects the interconnection between horizontal and vertical mathematizations when moving from the extra-mathematical situation to mathematical model, represented by the double arrows on the diagram of the modelling process (Fig. 26.1). The second indicator Q_{mag} concerns *questions about identification of magnitudes* relevant to consider mathematical processing in order to develop a mathematical model. The third Q_{cont} concerns *questions relating to the choice of context elements* to be taken into account for mathematical processing. The Q_{mag} and Q_{cont} indicators highlight the horizontal mathematical work represented on the first horizontal axis of the modelling process diagram (see Fig. 26.1). The Q_{cont} indicator raises questions about the relevance of the chosen fragment of reality in relation to the choices of context elements to be taken into account. The Q_{mag} indicator concerns the identification of relevant magnitudes when moving from "the real situation" to a problem accessible to a mathematical treatment ("PSMT") and the relationship between the selected magnitudes (moving from the "PSMT" to the mathematical problem).

26.7.2 A Priori Categorization of Answers

The characteristics of the *FRAPPM* (see Sect. 26.5) allow students to make choices either from reality, or from mathematical processing associated with taking measurements or from their knowledge of existing models that they plan to test. Therefore, I defined three indicators: the first one $R_{\text{“Real”}}$ for the choice of a model or relevant magnitudes based on considerations rooted in the real context of realistic fiction. The second one $R_{\text{“A priori”}}$ for choices based on a model known to the student for the purpose of testing it or for choices of a magnitude, made without considering the real context and without further justification. And the last one $R_{\text{“Math”}}$ for choices of a model or relevant quantities made from mathematical work or based on considerations made on the statement's diagrams. These kinds of answers would show that from the work of horizontal mathematization (production of the question) students can enter into vertical mathematization work (that would highlight the back and forth between the two aspects of mathematization) (see Fig. 26.1). A response of this type bears traces of the transposition of an expert's practice insofar as it can highlight that either the horizontal mathematization work triggers a vertical mathematization work, or that the work of horizontal mathematization is interconnected with that of vertical mathematization in the sense that the choices are made in anticipation of the feasibility or complexity of the mathematical processing that would result. These indicators for the response development phase should make it possible to: better understand how realistic fiction and the response phase lead students to make choices when considering mathematical treatment of the problem, highlight a possible transposition of the expert practices previously defined and highlight a possible devolution of horizontal mathematization to students.

26.8 Results

About 2000 students participated in the experiment. Questions and answers were collected and analysed qualitatively (using language markers) and quantitatively in relation to the above-mentioned indicators. Of the questions, 36% were in category Q_{mod}, 39% in category Q_{mag} and 25% in category Q_{cont}. For each category of questions, the results for the answers are shown in Table 26.1 and some examples of question–answer pairs are given in Table 26.2.

The analysis showed that students made simplifying hypotheses about elements of reality to consider a mathematical treatment, by selecting one or more fragments of reality by focusing on branches, greenhouse, leaves, et cetera. They identified relevant variables that influence the real situation (e.g. number of branches) as well as variables or context elements that are not relevant to the problem to be solved (e.g. fertilizer, leaf, tree height) and chose relevant relationships between the selected variables by using mathematical frameworks (e.g. functional, proportional, geometric). The choice of didactic variables (the diagrams of the tree, the asymmetrical shape of the tree, the number of new branches, the lengths of the trunk and branches, the scale) allowed students to ask authentic questions about the magnitudes to identify and relate in order to consider a mathematical treatment of the problem. The sufficiently realistic framework of the fiction encouraged reflection on the contextual elements to

Table 26.1 Results for the answers for each category of questions

	$R_{\text{“Real”}}$ (%)	$R_{\text{“A priori”}}$ (%)	$R_{\text{“Maths”}}$ (%)
Q_{mod}	26	35	39
Q_{mag}	27	36	37
Q_{cont}	70	27	3

Table 26.2 Examples of question–answer pairs

Question–answer pair	Example
Q_{mod} $R_{\text{“Maths”}}$	Is there proportionality between years and tree size? Size of the trunk, height, width of the branch, number of branches? (Q_{mod}): Using the 3 cm scale on the drawing = 1 m in reality, we can measure the tree each year, and we see that the coefficient to move from one year to another is not constant, so there is no proportionality ($R_{\text{“Maths”}}$)
Q_{mod} $R_{\text{“Real”}}$	Do trees evolve in a proportional way? (Q_{mod}): Given the natural context, trees do not evolve in a proportional way, but for the resolution we will consider that they do ($R_{\text{“Real”}}$)
Q_{mag} $R_{\text{“Real”}}$	Should we only consider the height or also the width? (Q_{mag}): Both must be taken into account, as the greenhouse will have to have a roof (height) and walls (width) ($R_{\text{“Real”}}$)
Q_{cont} $R_{\text{“A priori”}}$	Does exposure to the sun or watering have anything to do with tree size? (Q_{cont}): We think not, otherwise the statement would contain more information on this subject ($R_{\text{“A priori”}}$)

be taken into account, leading students to realize that some information about the real context was not necessarily useful in solving the problem or that it was preferable to neglect it in order not to make more complex the construction of the model. The majority of students tried to test a known model, the proportionality model, which they sometimes rejected by comparing the results obtained in this model with what they know about tree growth in the real world, leading them to reconsider their choice. The scale opened up the possibility of carrying out instrumented measurements, allowing students to quantify certain quantities and compare them with real data.

26.9 Discussions and Outlook

The main results highlight that the characteristics of a "*FRAPPM*" with an initial phase of questions-answers between peers have encouraged the devolution to students of the horizontal mathematization work. Analyses of the question–answer pairs show traces of this devolution. These analyses show that students make choices to make the extra-mathematical situation accessible for mathematical treatment. Through the indicators developed for the analyses, based on the main findings of the study of researchers' modelling practices (Yvain 2017), we can understand how students make these choices: by simplifying hypotheses about elements of reality to consider mathematical processing, by identifying relevant magnitudes and by questioning the influence of external parameters. Or, they made explicitly choices based on mathematical work, essentially based on measurements to verify after calculations, whether a model can be chosen. They also tried to test this model by comparing the results obtained in this model with real data. The back and forth between the horizontal and vertical aspects of mathematization, pointed out several times in the analyses, support the claim that horizontal and vertical mathematizations are interconnected in a mathematical modelling activity of this type in the classroom (for more details, see Yvain-Prébiski 2018). The question-and-answer phase makes it possible to bring to life in class the horizontal axis of the diagram of the modelling process developed for the study. However, what still remains open for future research is to consider the impact of this initial phase (question–answer) to the whole process of modelling and to study how students' responsibility was really involved in these activities and to what extent this issue was identified by teachers in a way that allows a full devolution of the learning issue (Yvain-Prébiski and Chesnais 2019).

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Part VI
Innovative Teaching Approaches

Chapter 27
An Opportunity for Noticing by Students and Teachers

Jill P. Brown

Abstract Mathematical modelling allows teachers to teach in engaging ways and students to become increasingly confident in working with challenging mathematical tasks, yet it remains less common in the primary years of schooling. To achieve success in solving real-world tasks, students must notice what is relevant, and decide how to act on this to progress their solution. Teachers must also discern what is relevant and nurture student capacity to notice. This chapter investigates teacher noticing and novice modellers' developing conceptions of noticing during a primary school modelling task. In the study, 62-Year 3/4 students attempted *The Packing Task*, observed by 13 teachers.

Keywords Challenging tasks · Mathematisation · Pre-mathematisation · Productive noticing · Primary school · Real-world

27.1 Modelling in Primary Schooling

In mathematical modelling, sense-making opportunities abound (see e.g. Brown 2017) as complex problems are presented to learners who then engage in decision making. Learner choice is important in mathematical sense-making. "Sense-making in mathematics classrooms is enhanced through less teacher structuring and learners using their own informal methods" (Biccard 2018, p. 8). With regard to modelling in primary school, more research is needed (English 2003; Stohlmann and Albarricín 2016). Also, English (2010) advocated modelling problems being integrated throughout the primary years. Concurring with this view to increase research and modelling activity, Stohlmann and Albarricín note, "mathematical modelling has mainly been emphasised at the secondary level, but for students to become more adept modellers the elementary grades need to be given more attention" (p. 1).

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English (2003) has long championed the use of mathematical modelling activities in primary school (and beyond) if "we intend our students to make modelling a way of life" (Brown and Stillman 2017, p. 354). English argues that evidence from her extensive research with primary school students, shows that modelling tasks are not too challenging, and students can successfully engage with such tasks regardless of their current mathematical capabilities (English 2003). More recently English noted, "introducing young children to mathematical modelling where they are challenged to mathematise problematic [real] situations … can cultivate their mathematical capacities" (English 2015, p. 104). An empirical study of three Year 6 classes utilising two different tasks, highlighted the importance of task design impacting on whether students saw the task as realistic (Brown 2013). Brown concluded that "tasks that required students to reflect … and make their thinking explicit can contribute to … students perceiving themselves as playing an important role in interpreting the real-world problem situation and relating it to the world of mathematics" (p. 304).

27.2 Productive Noticing in Modelling

Arcavi's (2003) ideas of students focussing on irrelevancies have been integral in the project, this study is part of, in encouraging students and teachers to articulate everything they notice in a particular image, task et cetera and then attending to which of these are mathematical and or relevant to the task being considered. Galbraith et al. (2017) note the importance of skilled 'noticing'. Choy's (2013) notion of productive mathematical noticing was extended to modellers, including student modellers, by Galbraith et al. (2017) as *productive Modelling Orientated Noticing* (pMON). Following Wenger and Wenger (2015), pMON needs to be nurtured in novices, and displayed by experts in mathematical modelling. Galbraith et al. argue that to develop as modellers, novice modellers must engage with modelling activities. Only through this activity will they develop conceptions of discerning 'noticing' as they select, develop, and communicate their modelling appropriately. From a pedagogical perspective, *productive Modelling Orientated Noticing* includes when the teacher is "monitoring and observing student decision making during modelling activity.… [and goes beyond noticing to include the essential] discernment of the relevance of what is noticed" (p. 74). From a modelling perspective, for the student modeller, productive noticing involves noticing what is relevant and what is not in a productive manner by acting on what has been discerned as relevant and rejecting or ignoring what is irrelevant.

To begin to model, novice modellers must become proficient in two processes, mathematisation and pre-mathematisation. Jankvist and Niss (2020) describe as pre-mathematisation, the processes of specification and idealisation, where modellers reduce the complexity of the messy real-world situation. This involves "making choices and assumptions concerning the features deemed significant to the modelling enterprise, thus reducing its complexity so as to make it tractable" (p. 469) to solution. Pre-mathematisation is critical to successful mathematisation. Hence, pMON plays a critical role as decision making regarding identification of significant features and making assumptions requires productive noticing. Mathematisation occurs when the modeller translates the idealised real-world problem into the mathematical world.

27.3 The Study

The research reported here is part of a four-year *Teacher as Learner Research* (TALR) project with one primary school in Australia. It focused on the development of *mathematical content knowledge* and *pedagogical content knowledge* by the teachers, and hence classroom practices to enhance student learning through teacher and student noticing. Key aspects of the project include productive teacher noticing and reflection, with an emphasis on students' mathematical reasoning and their collaboration to solve challenging tasks. This task reported here occurred in the fourth year of the project; hence, teachers had participated in many such demonstration lessons previously. The following research questions, related to the implementation of a modelling task, were the focus of the study reported here. RQ1: What developing conceptions of teacher noticing did this professional learning elicit? RQ2: What developing conceptions of modeller noticing did the modelling activity elicit in novice modellers?

27.3.1 Participants and Procedures

Sixty-two Year 3/4 students, aged 7–9 years old, worked on a modelling task observed by 13 teachers. *The Packing Task* was implemented during a 1-h class taught by the researcher. Students were asked to work with a partner, make a plan, then solve the task, and to keep a record of their mathematical thinking. To support teacher noticing during lesson observations, teachers used a researcher-designed recording sheet to focus on key mathematical ideas and language and what the students did and what the teacher (i.e. researcher) did. To focus on particular student pairs, the back of the recording sheet asked teachers to pay particular attention to the progress of three pairs throughout the lesson. Teachers were encouraged to ask students to clarify their thinking or approach to the task.

The Packing Task

How many rolls of paper are in one layer/box?	How many rolls of toilet paper are in the boxes?	The boxes are transported in this truck
		 How many boxes will fit in the truck? How many rolls of toilet paper is this?

When the task was implemented, students had limited formal school mathematics experiences with area and volume. They had some notions of area as covering, filling space, and had worked with arrays extensively. Students worked in self-selected pairs. Year 4 students were expected to have more mathematical knowledge and experiences of the notion of packing than Year 3 students, although still limited. The task was designed to include in-task scaffolding. Specifically, the parts focused on a single layer, the box, a vertical layer of boxes, and contents of the truck. The first step is to understand the situation, that is, that rolls and boxes are arranged in equal rows and in layers. The main requirement of the task is to translate between the mathematical world and the real world, specifically to determine how many boxes can fit in the truck. The dimensions of the boxes and the way they were packed relative to the dimensions of the truck had to be noticed and accounted for.

27.3.2 Data Sources and Analysis

Students' approaches to solving *The Packing Task* and teacher written reflections on these (both in-the-moment and at end of the day) were analysed. Data used to inform the qualitative analysis include student scripts, teacher observations during the lesson, reflection following post lesson discussion involving all teachers and the researcher, photographs of student scripts taken over the duration of task solution and the researcher's field notes. Thematic analysis (Guest et al. 2012) was conducted following coding of data to focus on what teachers and students noticed and acted on.

27.4 Findings

27.4.1 Teacher Noticing and Irrelevancies

Four themes emerged in the teacher observations. These related to teacher noticing with respect to irrelevancies, pre-mathematisation, mathematisation, and diagram use.

Noticing Irrelevancies. Irrelevancies such as colour or patterns on the wrapping paper need to be filtered out. Some pairs noticed and captured realistic aspects of the wrapped toilet roll pictorially (e.g. wrapping paper pushed into central cylinder and extending beyond the roll, see Fig. 27.1a). This did not necessarily hinder subsequent idealisation and mathematisation, but tended to be time consuming, thus reducing time to focus on task solution. Pair 25 used a careful pictorial representation to represent the initial situation recording irrelevancies and the truck (Fig. 27.1b, c), leaving no time for them to complete the last part of the task.

In the first part of the task, several teachers noticed students recording or discussing irrelevancies, for example, Teacher 12's [T12] observation of Pair 20 included, 'I see rows' and 'noticing patterns in colours etc.'. These students clearly saw the former as essential (they identified three rolls under the flap) and dismissed the latter as irrelevant but did not record this noticing. Teacher 17 noted that Pair 13 initially attended to colour, and Teacher 18 noticed Pair 23 suggested it was ice-cream in the box as well as attended to colour. When Pair 13 was trying to determine what proportion of the rolls the flap covered, T17 noticed this and recorded, 'students need hands on material to challenge ideas' however, she did not draw the students' attention to any of the available materials nearby. In contrast, Teacher 8 distinguished between essential aspects and irrelevancies, recording her general observations, 'Noticing the patterns on the paper – irrelevant. Some noticed a hidden roll.' Teacher 8 noticed that Pair 1, 'thought there could be 0 or 3 behind the cardboard flap' which clearly related to their assumption the box was full. She also noticed this pair's unrealistic consideration of mini-rolls under the flap. Furthermore, she noted that Pair 8, although not initially recognising the objects in the box, could visualise there were 12.

Noticing with respect to Pre-mathematisation. When Teacher 16 noticed Pair 6 struggling with *pre-mathematisation when trying to identify the number of rolls in the box*, specifically the importance of the number of layers, she acted. 'They weren't

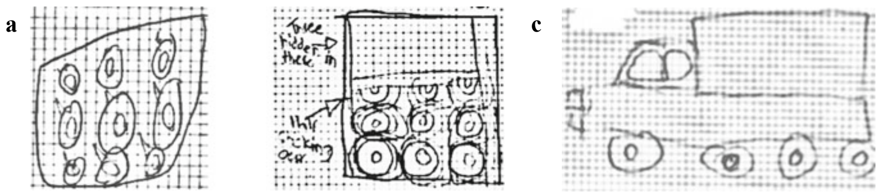

Fig. 27.1 Student focus on irrelevancies, **a** the roll, **b** the box, **c** the truck

drawing so I suggested they use unifix'. After seeing them create two layers of nine, she directed them to the photograph of the box to see 'if [their construction of unifix] looked like the picture'. In fact, one of the pair then closely observed the actual closed box that was in the room. Although Teacher 16 clearly noticed the students had not recognised the significant feature that all layers had the same number of rolls, it is unclear if her approach explicitly drew their attention to the lack of connection between their identification of 12 rolls in the top layers and their concrete model with nine. Similarly, when Teacher 17 noticed that Pair 13 had identified the number of layers of rolls in the box as being significant, but were unable to determine what this number was, she suggested they 'go to the box' which they did and returned having identified there were four layers. Teacher 12 noticed Pair 20 also initially did not recognise this essential element. There is no evidence of her acting on this, although the pair did 're-establish 12 as important' according to her notes.

When the focus was on packing the truck, Teacher 2 noticed that Pair 10 partitioned both side and back view of the truck into a four by four array, 'with no discussion as to why they did this' and furthermore 'added 16 + 16 to get 32' boxes in the truck. There is no evidence, this teacher probed or challenged the pairs' noticing. Teacher 16 noticed Pair 30 'working out the width and height of the toilet paper and trying to relate this to the size of the truck'. They recorded 'toilet paper = 12 cm long, 10 cm tall. Box = 36 cm width 40 cm tall'. Clearly, the pair was incorrectly assuming the rolls were oriented sideways, which conflicts with the photographs. Based on this, they identified two dimensions of the box. They later recorded 48 but there is no evidence where this came from or what it represented. Had Teacher 16 questioned their *pre-mathematisation*, they would have had an opportunity to recognise the diameter of the roll determined two dimensions of the box. This may have also led to their recognising their incorrect orientation of the rolls. Teacher 5 recorded his noticing of their inability to determine dimensions of the box based on the roll but no actions of his own to intervene to support resolving this.

Noticing with respect to Mathematisation. Other teachers noticed other difficulties as students mathematised the situation related to filling the box. Teacher 7, for example, noticed that although Pair 14 had identified 'the top layer had 12…and decided 4 rolls would fit vertically [the] total rolls would be 16'. She noted, 'when probed he said it was by counting by 4s'. Teacher 14 also noticed their mathematisation was actually of '4 toilet rolls vertical and 4 going across so he did $4 \times 4 =$ 16, 16 rolls.' Neither teacher intervened sufficiently for the pair to recognise their mathematisation was flawed.

Noticing with respect to Diagrams. Several teachers noticed student difficulties with drawing diagrams, or not considering a diagram would be helpful to represent and / or solve the problem. Teacher 16 noted Pair 2 were 'struggling to draw what's under', that is to represent the 3D box of rolls. When she noticed Pair 6 not drawing, she recommended the use of concrete materials. There is no evidence she made the same suggestion, or any other, to Pair 6. Teacher 12 noticed that this pair 'did not understand that diagrams do not have to be the exact labelled measurements'. There was no evidence that any teacher supported student noticing that a diagram may be helpful in representing or solving the task.

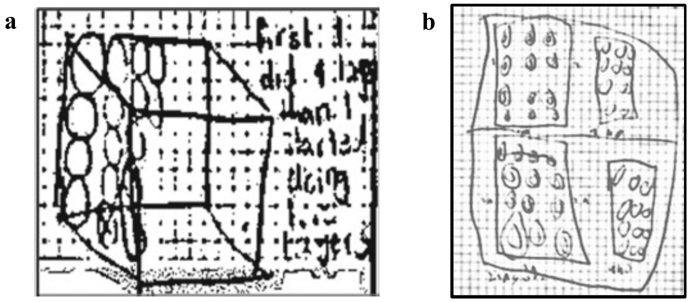

Fig. 27.2 Challenges representing the 3D situation **a** attempted 3D representation, **b** separate layers

Teacher 2 noticed Pair 10 struggled to represent the 3D box containing the rolls. In this instance, the students also noticed this and recorded their error, ‘First, I did 4 layers then I started doing five layers’ (Fig. 27.2a). Pair 10 did not continue their diagram but were successful in identifying 48 rolls in the box. There is no evidence of interaction between Teacher 2 and this pair. Pair 7 [no teacher observations] overcame the same challenge by representing the four layers side-by-side (Fig. 27.2b).

In conclusion, the four themes in the teacher observations are related to noticing with respect to irrelevancies, pre-mathematisation, mathematisation, and diagram use. Articulating observations can be an important part for a learner then recognising they are irrelevant. It is only when such articulations are time consuming and distracting students from the task at hand that a teacher should intervene—however this requires the teacher to also recognise the irrelevancies. When noticing that, for example, available concrete materials, or drawing a diagram, would support a particular student approach, teachers need to act to encourage students to notice and utilise such support. Furthermore, teachers should notice that some materials are more helpful than others. Similarly, teachers should challenge student thinking when unrealistic ideas are proposed, or there is a disconnect between student representations (concrete or diagrammatic) and reality.

27.4.2 Student Noticing

27.4.2.1 Number of Rolls Per Layer

When considering how many rolls in the top layer, only Pair 1 recorded evidence of making *assumptions* as they discussed and recorded ‘0 or 3’ hidden and so ‘maybe 9’ or ‘maybe 12’ in the top layer. Eventually they decided on 12 having explicitly considered and then assumed the box was full. The remaining pairs just took it for granted the box, and hence top layer was full (e.g. Pair 5: ‘if we open it [the flap] we will see three more’) and there was space for one additional row (e.g. Pair 21: ‘it would only fit 1 more row’).

All except Pair 3 recognised—not necessarily immediately—the *essential feature* that there must be the same number of rolls in each row (Pair 2: 'It's going by 3's'), and hence a total of 12 in the layer. Pair 3, whilst considering some rolls were hidden, could not agree if there were two or four hidden, failing to notice that each row had the same number of rolls.

When identifying the number of rolls in the top layer, there was little written recording of *mathematisations*. Only six pairs explicitly recorded their mathematisation, with two of these recording two alternative mathematisations using multiplication (Pair 31: $3 \times 4 = 12$, $4 \times 3 = 12$) and repeated addition (Pair 21: $3 + 3 + 3 + 3 = 12$, $4 + 4 + 4 = 12$). A third approach was repeated doubling (i.e. $3 + 3 = 6$, $6 + 6 = 12$). Another five pairs recorded a counting strategy (counting by threes or fours, counting on from nine, or counting all). Eighteen pairs recorded no mathematisation, clearly some of these 'saw 12' and had no need for mathematising.

27.4.2.2 The Box

Pair 26, for example, explicitly considered if the box was full or not, and then *assumed* it was full. They recorded, 'I counted by 12. So, if I was to fill the whole box it will be 48'. Of the 28 pairs who correctly identified 48 rolls in the box, 17 recorded their *mathematisation*. Fifteen of these focused on four layers of 12 although six of these used a doubling approach focusing on two layers and then four. The remaining two pairs mathematised beginning with the rolls in the additional three layers ($3 \times 12 = 36$, $12 + 36 = 48$). All of these 28 pairs noticed the need to ascertain the number of layers, and that each layer held the same number of rolls. The remaining three pairs did not notice the importance of the number of layers.

27.4.2.3 The Stack

When considering the stack of boxes, several pairs explicitly considered if there were hidden vertical layers behind the clearly visible one. For example, Pair 13 made the *assumption*, 'there are no boxes behind'. Two pairs made general statements indicating they considered the possibility, with Pair 2 noting, 'there might be more behind' and Pair 4, 'maybe there's more behind'. Teacher observation indicated at least three other pairs also considered if there were hidden boxes (pairs 11, 21, 26). Teacher 3 recorded Pair 11 as saying, 'there might be more than 12 [boxes]. Can't see behind'. Only one pair explicitly recorded their assumption of four vertical layers, noting 'behind there are three layers' (Pair 22).

From either the mathematisation or the result, four pairs proceeded on the assumption of multiple vertical layers. Three pairs correctly identified there would be 2304 rolls if there were four vertical layers of boxes. Pair 11, reported to Teacher 3, 'we did 48 rolls $\times$ 48 boxes' however, Pair 11 later erased their result of 2304 and wrote 576, in neither instance recording their mathematisation. Only Pair 22 recorded their mathematisation, '$48 \times 48 = 2304$'. Pair 16 assumed there were two vertical layers, as evident from their mathematisation, ‘we added 48, 24 times’ although their result for this mathematisation of 1152, was incorrect.

Of the 19 pairs assuming a single vertical layer, 13 successfully noted 576 rolls as being in the 12 boxes. Only eight of these correctly recorded their mathematisation (e.g. $12 \times 48 = 576$). A further four pairs correctly mathematised the problem but were unsuccessful in their calculation. The other two pairs incorrectly mathematised the problem. Seven pairs provided no evidence of a calculation and two others recorded unclear calculations.

27.4.2.4 Packing the Truck

When considering packing the truck, pre-mathematisation was problematic. Ten pairs *focused on the dimensions* of one or more of the roll, box, and truck. Pair 18 recognised the need to coordinate these but was unsuccessful in doing so. They did not recognise all three dimensions of the box were essential, not just the height of 40 cm which they used in their mathematisation to identify five layers of boxes, seven high, nor did they notice that the packing was 3D. Four pairs attempted to coordinate the dimensions of the roll and box, but only Pair 26 successfully mathematised all three dimensions of the box. Pairs 1 and 20 successfully mathematised one dimension of the box. Whilst Pair 30 appeared to consider all three dimensions, they failed to notice the orientation of the roll relative to the box. Three pairs merely focused on the roll and two others just on the truck. Pair 21 ignored the box as they unsuccessfully attempted to coordinate the roll dimensions with the truck.

Eight pairs *focussed directly on packing the truck*, ignoring the importance of the dimensions of any elements.[1] Four of these recognised the 3D nature of the packing and attended to both side and back views. Two pairs partitioned the truck as having the same number of layers left and back view and had the correct orientation of the box. Pair 10 (Fig. 27.3a) drew on the truck (picture supplied) showing four layers of boxes whilst Pair 16 had three layers and added ‘48’ to each box. Although recognising the number of layers must remain the same, irrespective of the view, neither showed any other measurement sense or other evidence in determining the number of layers. Both pairs stated the number of boxes as being those visible not recognising the need

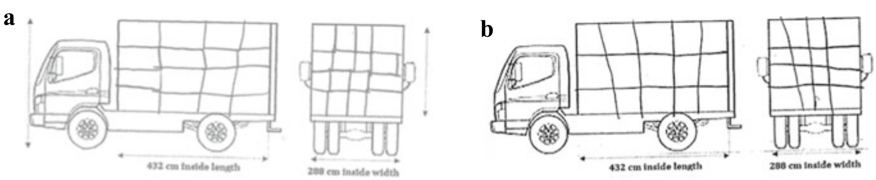

Fig. 27.3 Packing the truck with boxes **a** layers coordinated, **b** layers not coordinated

[1]Given roll dimensions, reasonable dimensions for a box were 36 by 48 by 40 cm high, for tight packing. Truck held 5 layers each layer 9 by 8 or 12 by 6 depending on box orientation.

to account for hidden boxes. The other two pairs were unable to coordinate the back and side views of the truck. Pair 8 (Fig. 27.3b) represented the left view having three layers, whilst the back view has four layers. Similarly, Pair 28 represented the left view having three layers (of 7), and the back view as five layers (of 4). Neither of these pairs recorded a result, or any calculation, for the total number of boxes. Three other pairs only accounted for one view when packing the truck. Pair 27, ignoring the boxes, attempted to determine directly how many rolls would pack into the truck. Of the remaining pairs, two focussed on irrelevancies, with Pair 11 drawing the roll and Pair 25 the truck, four pairs reported a number (24, 192, or 216) with no reasoning or mathematisation recorded. Eight pairs did not respond to the problem.

27.5 Discussion and Conclusion

In this study, some teachers did carefully draw some students' attention to their lack of productive noticing, but in many situations, this did not occur. Teachers need to notice when students are having difficulties mathematising and intervene, for example, asking students to specify what the problem they are attempting is, or how they anticipate their current actions will allow progress towards a solution. Importantly, teachers should be "monitoring and observing student decision making during modelling" (Galbraith et al. 2017, p. 85) discerning the relevance of what has been noticed. When significant features, for example, had not been noticed, teachers should ask students what they noticed, question if these were relevant, and, if relevant, if essential, that is, significant to what is being solved. These questions are appropriate across myriad tasks.

Post-task discussion allowed teacher observations to be shared and discussed. This appeared invaluable, in-the-moment. In a non-research situation, the value of teachers first solving such tasks cannot be underestimated, as this provides additional opportunities for *productive Modelling Orientated Noticing* (Galbraith et al. 2017) to focus on identifying assumptions, recognising key features, and opportunities for decision making (Jankvist and Niss 2020) in both pre-mathematisation and mathematisation. Cooperative planning provides critical opportunities for teachers to notice essential features themselves and plan appropriate teacher responses to instances where students do not attend to these (Stender and Kaiser 2015).

With regard to the students' developing conceptions of noticing, all students recognised that modelling involves decision making. Furthermore, they appreciated the necessity to make sense of the real-world context. In each part of *The Packing Task*, at least some pairs identified all the significant features. In the early parts of the task, almost all pairs did so. In the final, most challenging part of the task, student identification of all significant features was lower with most noticing only some significant features. Students who identified all significant features were unable to coordinate these in-the-moment to solve the task. Few students were identified as having explicitly made assumptions, although it can be inferred from scripts or observations that other pairs had done so, but this was not deemed as important enough by the students to record. Most pairs took it for granted the boxes and truck were full.

Clearly tasks such as *The Packing Task* address three of the "five overarching pedagogical meta-practices [for the primary years, namely] development of a productive disposition, emphasis on mathematical modelling, use of cognitively challenging tasks" (Dooley 2019, p. 3). The task addressed the call of English (2010, p. 295) to allow primary students to deal with a complex situation and allowed "for a diversity of solution approaches and enable[d] [all students] to participate in, and benefit from [the] experience". With experience, students' conceptions of modeller noticing will continue to evolve. The role of the teacher in developing these conceptions is critical. In order for student modellers to recognise the importance of assumptions, their teachers must also do so. As teacher noticing develops, teachers are better able to support student noticing, as teacher questioning is more likely to draw students' attention to productive noticing of what the teacher has also noticed.

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Chapter 28
Modelling and Mobile Learning with Math Trails

Nils Buchholtz

Abstract The chapter provides an orientation on the concept of mobile learning and how it can be pursued with mobile math trails. Math trails contain tasks that promote essential elements of mathematical modelling, such as mathematising. Research shows that, nowadays, math trails are more and more supported by digital media, and that this affects students' motivation and achievements. The chapter collects existing findings on mobile learning with math trails and expands the findings with the results of a study on digital support of the modelling processes of 11th graders when doing math trails.

Keywords Math trails · Mobile learning · Itinerary method · Mathematisation · Math & The City · Actionbound

28.1 Introduction

Math trails emphasise an extracurricular and playful approach to learning essential aspects of mathematical modelling, especially mathematising (Buchholtz 2017). In a math trail or a mathematical city walk, students work collaboratively on modelling tasks related to real objects in the school's or city's surroundings, moving outdoors from site to site, like in a rally (Blane and Clarke 1984; Shoaf et al. 2004). The tasks in math trails include estimating and measuring variables, setting relevant sizes up in mathematical models, and calculating and comparing sizes, areas, and volumes (Buchholtz 2017).

Math trails, in this form, have existed since the 1980s as an out-of-school leisure activity for families and persons interested in mathematics (Blane and Clarke 1984; Kaur 1990). The mathematical content in math trails can range from primary to

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https://doi.org/10.1007/978-3-030-66996-6_28

secondary level mathematics, as can the complexity and difficulty of the selected tasks. Although not a new idea, math trails have recently gained more attention in mathematics education research. Nowadays, math trails are included in school curricula and can be enhanced by the use of mobile technologies, such as smartphones and tablet PCs (Cahyono 2018; Fessakis et al. 2018; Ludwig and Jesberg 2015; Wijers et al. 2010). Digital tools, such as geolocation apps, response systems, sensors, dynamic geometry systems, and augmented reality applications, can be used on math trails to support the task-solving process (Buchholtz et al. 2019; Bokhove et al. 2018; Roschelle 2003). An additional advantage lies in the ability to adapt the trails to the students' learning requirements (e.g. including support videos or additional information) (Buchholtz et al. 2019).

In recent discussions on the use of digital media in education, these functions are often associated with *mobile learning* (Frohberg et al. 2009), a special form of *e-learning* that places emphasis on extra-curricular and informal learning with mobile devices, such as the type of learning facilitated by math trails (see Sect. 28.2.1). The question arises: to what extent can math trails enhance mobile learning, and what does mobile learning with math trails look like? Research findings on students' learning outcomes with regard to math trails are scarce, partially because math trails were originally invented as a leisure activity for people interested in mathematics (Shoaf et al. 2004) and have not yet been the subject of systematic mathematics education research. This chapter intends to contribute to filling this research gap by providing a collection of research findings on the use of mobile devices in math trails and by presenting findings from the project *Math & The City* (Buchholtz 2020), where math trails are used to give students their first experiences with modelling. It is, therefore, particularly interesting to see how digital devices are used by students to process the tasks.

28.2 Mobile Learning with Math Trails

28.2.1 Mobile Learning

Mobile learning is a comparatively young field in educational research. Earlier definitions of mobile learning included the involvement of mobile devices in the learning process and the physical mobility of the learners as central and necessary characteristics of this educational concept (O'Malley et al. 2005). More recent definitions of mobile learning highlight the importance of the personalisation of the learning content and its context-relatedness (de Witt 2013; Frohberg 2008; Frohberg et al. 2009). Thus, the notion of mobile learning is increasingly overcoming the boundaries between formal and informal learning contexts. Context-relatedness here means that the place and the situation in which the learning takes place, as well as the people with whom the learner studies (*context of being*), are utilized and have a significant relevance to the learning environment (*context of learning*) (Frohberg et al.

2009). Examples include digital museum guides or digitally enhanced expeditions in nature, such as app-based birdwatching. Math trails are well suited as prototypes of educational settings for mobile learning.

The context-relatedness can be acknowledged in both ways. The physical context of the learning environment has a clear relation to the learning content and co-determines it (Frohberg 2008). To solve the given modelling problems, the students need to explore the real objects and their characteristics on the math trail and take relevant measures or use mathematics to determine sizes that are not directly accessible (e.g. for tall buildings). In addition, the social context can be acknowledged by the collaborative learning element, in which situations, relationships and emotions can be linked with the learning experience in the environment (Frohberg 2008). Mobile end devices, with their location-independence, are ideal for extra-curricular learning environments and mediate between the physical and the social context, for example, when multimedia and interactive apps are used (Buchholtz et al. 2019). Math trails that are supported by digital media can contain elements of participation and gamification (Gurjanow and Ludwig 2017), meaning that the students take an active role in the learning process while working with the digital device. Furthermore, the learning is supported by technology in such a way that students get immediate feedback on their calculations after entering their results in the mobile device. Different apps can be used for designing math trails (e.g. www.actionbound.com, www.mathcitymap.eu or www.google.com/maps). *Actionbound*, which is used in the *Math & The City* project, is an app developed specifically for the field of media education for the creation of digital learning paths.

28.2.2 Mobile Math Trails

If an app-based learning path sequences several mathematical tasks, and geo-coordinates link them to different locations, we consider this a *mobile math trail*. Mobile math trails consist of tasks with respective sub-tasks composed of different mathematical concepts or different steps in the modelling process. The tasks are designed to ensure that the students have to carry out concrete measurements and identify required quantities of the real objects autonomously in groups (Fessakis et al. 2018; Ludwig and Jesberg 2015). The tasks vary in the degree to which assumptions must be made and are therefore particularly suitable for learning individual elements of the modelling process. They process mathematising by a meaningful assignment of determined or estimated variables into a mathematical model (Buchholtz 2017). All solutions to the tasks must be entered in the app, which provides immediate feedback on whether the solution was correct or incorrect. Figure 28.1 shows an example of a task that is part of an *Actionbound*-based math trail in Oslo.

The task requires the calculation of the volume of the water in a well-known Oslo fountain. It is embedded in the context that the fountain has to be filled with water every spring, because winters in Oslo are very long and have very low temperatures. Therefore, the city council needs to know the volume of water to be filled. The

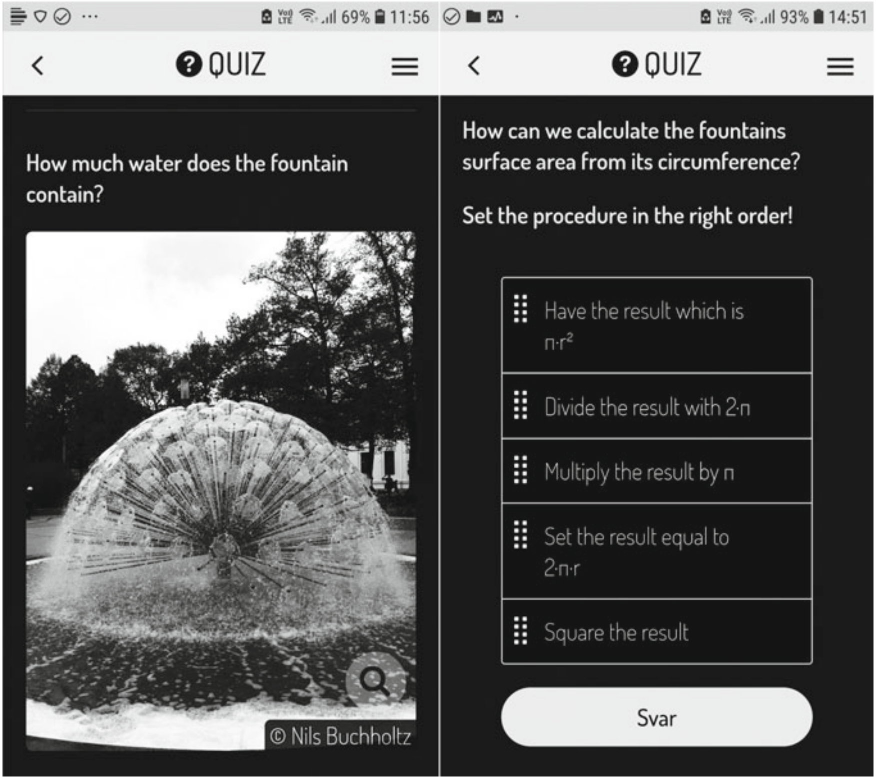

Fig. 28.1 The *Påfugl fountain task* on volume calculation

students have to consider how to determine the volume of water in the cylindrical fountain, as the diameter of the basin (7.6 m) is not accessible and therefore cannot be measured. As such, it must be determined by means of an appropriate mathematisation using the circumference of the fountain (24 m). For example, students could use footsteps as a non-standard unit and walk around the fountain, counting the number of steps, and afterwards multiply the length of a footstep accordingly. An additional sub-task in which the necessary steps of mathematisation have to be arranged to get from the circumference to the surface area serves as a scaffold in the digital medium, providing aid for the necessary steps in solving the task (Fig. 28.1, right side). Care must be taken to measure the stones on the inner edge of the fountain; otherwise, the results will vary considerably. With the corresponding water level of the fountain (0.36 m), the approximate volume of the water (16.5 m^3) can then be determined.

When creating a mobile math trail, the geo-coordinates of the tasks must be fixed so that the app can guide the students via the geo-localisation of the mobile device. In *Actionbound*, not only can tasks be displayed in text form, but the integration of external links, images, videos and audio recordings is also supported. It is easy to add explanatory videos, an interesting article or a mathematical sketch with relevant sizes, depending on which degree of difficulty is chosen for the modelling tasks and which forms of support are to be offered. In this manner, the tasks in the mobile math trails can be adjusted to the respective needs of the learning group, which creates added value over merely presenting the tasks in the paper and pencil format (Buchholtz et al. 2019).

28.2.3 Recent Research Findings on Mobile Math Trails

The research group around Matthias Ludwig develops a network of math trails around the world using the app *MathCityMap* (Ludwig and Jesberg 2015). Gurjanow and Ludwig (2017) investigated the influence of gamification elements on the motivation of German students. They examined whether the implementation of a reward system and a ranking in the mobile math trails positively influenced the intrinsic motivation of 25 participating students. They found that the reward system had no influence on the intrinsic motivation of the students, whereas the ranking system had a positive influence on intrinsic motivation, especially among male students. Other studies found effects of mobile math trails on students' achievement as well. In an explorative case study of four students in Greece, Fessakis et al. (2018) revealed that the digital map they used in their study (Google Maps) provided easier navigation and information on the communication and cooperation between different groups of students (Fessakis et al. 2018). In most cases, research on mobile math trails is concerned with their impact on student motivation or achievement, not so much with students' mobile learning while doing the math trail.

28.3 Findings from the Project Math & The City

28.3.1 Research Design and Approach

To analyse student's mobile learning on math trails, what happens during the trail, and how the mobile device is involved in the modelling process are of foremost interest. In this case, this means observing the students' interactions with the real objects connected to the different tasks on the trail, their use of the mobile device during the math trail, and their contextual modelling processes and strategies when estimating and taking measures.

Exploratory qualitative research methods are used in the *Math & The City* project to analyse students' mobile learning. The method of itinerary (*Méthode des Itinéraires*) was originally invented in sociology to collect and describe the subjective views of pedestrians in order to draw conclusions about city planning (Miaux et al. 2010; Petiteau and Pasquier 2001). Central to this method are city walks during which the researcher takes a passive role, guided by the participants, and interviews and records audio of the participants while a photographer walks behind and takes pictures at each change in direction or emotional change. The method is adapted here for the mobile math trails in order to gather information about the subjective use of mobile devices when working with tasks, which is a good indicator for mobile learning. Central to this adaptation is the video-recording produced by an action camera that the students wear on their body. In this way, the observation remains minimally invasive (for further details: Buchholtz 2020).

For the project, two math trails were developed in downtown Oslo. One math trail consists of five tasks on the topic of circle calculation (among them, the task from Fig. 28.1); another focuses on the topic of linear functions. After piloting the trails in summer 2019 (Buchholtz 2020), the trails were carried out with two school classes (11th grade) and their mathematics teachers in autumn 2019. For each math trail, five groups of three students were equipped with action cameras, a tape measure and an iPad on which the app was installed. In addition, the students were allowed to use their own smartphones, and they were responsible for recording the process at the individual stations of the math trail. All the necessary declarations of consent were obtained before the data collection, and the study was approved by the Norwegian data protection authority (NSD). The data evaluation is based on the qualitative content analysis (Mayring 2014). The data are still subject to evaluation, but in this chapter, the first results from the video recordings are presented.

28.3.2 Findings on Students' Mobile Learning When Modelling

In the recordings, we identified different phases of the modelling processes where the mobile device supported or scaffolded the activities of the students.

When following the math trail, the app guided the path from task to task. The app presented the tasks as soon as a location had been found. The students then had to understand the tasks on the iPad. The quantities that were relevant for the tasks were localised in the real object by shifting the attention between the digital presentation of the task (e.g. photo, sketch, text or video) and the respective object properties (see Fig. 28.2), often associated with deictic gestures. The students then made context-related assumptions about or simplifications of the real model (for example, if the form of the real objects differed from ideal mathematical forms that were presented). When working with the *Påfugl fountain task*, they discussed where and how to take the relevant measurements. Then, suitable methods of data retrieval were found (see Fig. 28.3). When mathematising, the students used their smartphones to take notes or look up relevant formulas (see Fig. 28.4).

The students also processed the task using their smartphones as calculators (Fig. 28.5). It was also possible for students to take a photo of the object or make an audio recording with their iPads and then upload the work in the app; for example, parts of the real objects that were relevant for the calculation or that had to be

Fig. 28.2 Students try to locate the task in the properties of the fountain

Fig. 28.3 Students take measures of the water height with the measuring tape

Fig. 28.4 Students looked up a formula to calculate the radius from the circumference

Fig. 28.5 Students used their smartphones as calculators and entered results into the app

compared in the tasks could be documented. The students could also upload sketches and calculations that they made on paper or on their smartphones as they worked with the tasks.

The entered results were automatically validated by the app with solution intervals or programmed correct responses so that the mobile device could give immediate automated feedback. The app then provided pre-programmed assistance in case of incorrect answers, and the students could use the feedback to look for errors in their solution strategy, their estimates or their measurements. If the solution was entered correctly, the app rated the result with points and sent the students to the next task.

28.4 Discussion

In contrast to regular modelling tasks in class, the contextualisation offered by the real objects seems to play a special role on math trails. In order to accomplish the tasks, students had to measure, scale, count, or estimate quantities and place them into a correct mathematical relation or reconstruct or calculate relevant but inaccessible quantities from measured quantities—the actual mathematising. Perhaps most importantly, the groups were able to directly validate the mathematical results against the real objects. Overall, we could identify three important areas where the mobile device supported the students' modelling processes: first, in task presentation and contextual support (variety of approaches to the task); second, as a technical aid in task processing and mathematisation (research tool and computer function); and third, in providing immediate feedback that motivated the students to find errors and validate their solutions. These findings show first insights into the use of mobile devices for modelling and math trails in general. A more systematic review of scientific findings on this topic should take place in the next few years because there is great potential for the use of mobile devices.

Mobile math trails enable educators to combine texts, images and animations, as well as audio recordings, construction plans, floor plans, or even technical drawings when designing modelling tasks, thus offering the students a differentiated access to mathematical concepts and an opportunity to relate the mathematics involved to different realistic contexts. Using different (and even dynamic) representations in task formulations can strengthen the networking between mathematical concepts. The possibility of additional "augmented reality" content—for example, in additional explanation videos—opens up possibilities of variation in task access and simultaneity of representations, thus linking the different levels of representations together.

Acknowledgements The project Math & The City is funded by DIKU, the Norwegian Agency for International Cooperation and Quality Enhancement in Higher Education.

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Chapter 29
Modelling in School Mathematics: Past Achievements—Current Challenges

Hugh Burkhardt

Abstract This chapter describes my 55 years of learning about teaching modelling, from initial explorations through a sequence of modelling projects at the Shell Centre. After introducing some core concepts, the chapter focuses on the design strategies and tactics that were learned in each project, including roles for technology. A discussion of specific design issues in teaching modelling leads into asking why improvements are so difficult to achieve on a large scale. Elements of a way forward are outlined.

Keywords Modelling · Tasks · Roles · Microworlds · Change · Assessment

29.1 Introduction

The Shell Centre for Mathematical Education was created in 1967 by the mathematics professors at the University of Nottingham, Heini Halberstam and George Hall—himself a pioneer in teaching modelling. Initially, a professional development centre, when I joined as director in 1976 we decided it should focus on research-linked development of materials to improve the teaching and learning of mathematics. This chapter describes this 'engineering research' (Burkhardt 2006) approach to teaching modelling at the Shell Centre since then.

To begin, I will clarify some assumptions that run throughout the chapter.

29.1.1 Models and Modelling

First the distinction in Fig. 29.1 between learned models, illustrating how mathematics has been used to understand real world situations and active modelling by students.

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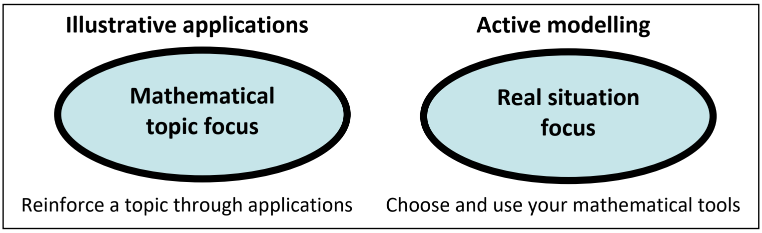

Fig. 29.1 Different purposes with different priorities

These activities are complementary—a knowledge of existing models of a wide range of practical situations is essential for creating models of situations that seem in some way similar. But it is far from enough to enable active modelling. Here, I shall be concentrate on active modelling by students of all ages, choosing and using tools from their mathematical toolkit to better understand practical situations, and how we can develop these capabilities in mathematics classrooms.

29.1.2 The Central Role of the Task

We believe that the tasks that students work on should provide opportunities for:

1. **Using good mathematics**, however simple.
2. **Cognitive demand** that requires 'productive struggle'—thinking not just imitating.
3. **Equity** so that all students should be able to engage with the task.
4. **Agency**—for students to feel the solution is their own, not the teacher's.
5. **Feedback** in the classroom—formative assessment that forwards learning.

More broadly, these are the five features of powerful classrooms (TRU) set out by Alan Schoenfeld, based on a series of research and development studies (Schoenfeld et al. 2016).

Figure 29.2 shows two modelling tasks that potentially have all these attributes and work well in classrooms. *Airplane turn-round* is a simple task that can be used to show students (and adults) what modelling is about. Presented with this problem in a mathematics classroom, students just add the numbers—"that's what you do in maths". When the teacher then asks "Is there any way you could do the turn-round more quickly?" students recognize that this is a different game where their life experience, common sense and imagination are needed.

Cats and Kittens is a richer and more complex task where strategy is the main challenge.

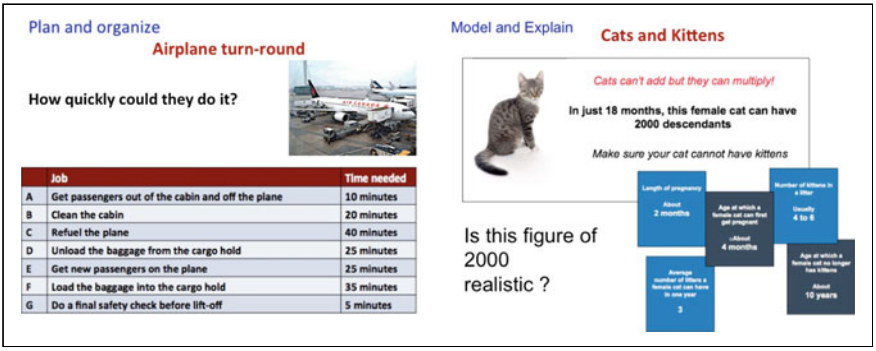

Fig. 29.2 Two modelling tasks

29.1.2.1 Types of Task

We have found it useful (Burkhardt and Swan 2017) to introduce students to a variety of types of modelling task, notably those which ask students to:

- **Plan and organize** Find a good solution, subject to constraints.
- **Design and make** Design an artefact or procedure and test it.
- **Model and explain** Invent models to explain the situation, make reasoned estimates.
- **Explore and discover** Find relationships, make predictions.
- **Interpret and translate** Deduce insights, translate representations.
- **Evaluate and improve** an argument, a plan or an artefact.

Since all these can make you more effective in facing life's challenges, a rich modelling curriculum should bring all of them in from time to time.

29.1.2.2 How Realistic Should the Tasks Be?

It is clear that few tasks that are presented in the classroom by the teacher are entirely realistic for the students' life in the real world. I devised (Burkhardt 1981) the following semi-serious classification:

A. **Action problems** affect students' own lives—e.g. situations involving money (What apps can I afford?) and risk (Should I worry about being killed by a terrorist?).
B. **Believable problems** are those that might arise in the future and do concern others—finance, risk, design, planning all come in here.
C. **Curious problems** are simply intriguing—Cats and Kittens, or the "Birthday Party Problem"—with 22 people at a party it is likely that two have the same birthday.

D. **Dubious problems** are just there to practice maths—the real world situation is purely cosmetic. “If it takes 5 men 7 hours to paint a fence, …”
E. **Educational problems** are fundamentally dubious but mathematically irresistible—real world examples of exponential growth, Fibonacci series and the Golden Ratio.

So to develop modelling capability in a way that is most valuable in later years, we have found it appropriate in developing materials to focus largely on Believable problems, with a sprinkling of the other types.

29.1.2.3 Task Difficulty

...is multi-dimensional. The difficulty of a task depends on various factors, notably its *complexity, unfamiliarity, technical demand* and *the length of the chain of autonomous reasoning* expected of the student. We have found it useful to distinguish:

- **Expert tasks** come in a form in which they might naturally arise; they involve all four aspects, so *must not be technically demanding*—there is a “few year gap” between the mathematical techniques that students can manage in imitative exercises and those they can choose and use in tackling non-routine complex problems. For problem solving, mathematical concepts and skills must be well absorbed and connected to other concepts and other applications so their relevance to the new context can be perceived.
- **Apprentice tasks** are expert tasks with scaffolding added. This reduces the complexity and the student autonomy. Apprentice tasks provide an important element in developing modelling skills—like climbing a mountain with a guide to develop mountaineering skills.
- **Novice tasks** are short items with mainly technical demand, so can be “up to grade”, including concepts and skills that have been taught and practised recently.

Thus each type of task has a different balance of sources of difficulty. This must be taken into account in choosing tasks that provide ‘productive struggle’ for all students. Thus, *Airplane turn-round*, where the technical demand is only simple addition, is challenging to students through its strategic demand: deciding how to organize an approach. *Cats and Kittens* requires students to devise a representation and a strategy that will handle the inherent geometric growth, summed over successive generations.

29.1.3 55 Years of Learning How to Teach Modelling

I will begin the story of how we and others have gradually learned more about teaching modelling with a very brief historical account, before turning to specific design issues.

29.1.4 *Early Explorations—the 1960s*

The unexpected launch by the Soviet Union of the first satellite led to soul-searching in the USA: "*The West is being left behind* …." The wave of 'Post-Sputnik mathematics reform' that followed was led by mathematicians and scientists, whose views on learning and teaching were intuitive rather than research based. Their core belief was that teachers need to understand the mathematics more deeply, going back to the foundations: set theory! In Birmingham from 1960 to 62, Peter Hilton and Brian Griffiths set up weekly 3-hour lectures for high school teachers on fundamentals of pure mathematics and how they relate to school arithmetic and algebra.

29.1.4.1 Personally

In 1962–64, almost by accident, I found myself faced with a challenge. After a couple of years, the organizer thought the topic should change to applied mathematics; my head of department asked me to run the course. At that time, the mathematics in the last two years of British schools included, along with algebra and calculus, Newtonian Mechanics of particles and rigid bodies in 2 dimensions—just as Newton had designed it 300 years earlier! So, in the first year, I reviewed in some depth the physics behind the highly stylized problems in the syllabus, like those in Fig. 29.3a. These were based on 12 learned models, with no active modelling.

I became so frustrated with 'perfectly light inextensible strings' running over 'perfectly smooth weightless pulleys' that I decided to make the next year's course focus on modelling. It opened with a practical workshop '*On falling off ladders*' that led the teachers to understand that, among the various instabilities, the 'bottom slipping away' in Fig. 29.3a was the most dangerous—but only when someone is climbing the ladder, which was not in the syllabus! I devised a version of the now-standard modelling process diagram, shown in Fig. 29.3b. (Note how it handles 'processes' and 'states'—an issue that persists to this day.)

I then started teaching modelling in the same way to undergraduates with problems like:

- *'On owning a car'* What is the best age (of the car!) to buy, then to sell a used car, so as to minimize the cost. The students were able to expose various myths.
- *'On walking in the rain'* so as to keep as dry as possible. Should I run?How fast?What if there is a wind?

29.1.4.2 Other Early Initiatives on Teaching Modelling

In England, Ron McLone and a few other 'explorers' in university mathematics departments were experimenting with modelling in their courses. In the US Henry Pollak, then Head of Mathematics and Statistics at 'Bell Labs', worked with various efforts to introduce modelling in schools.

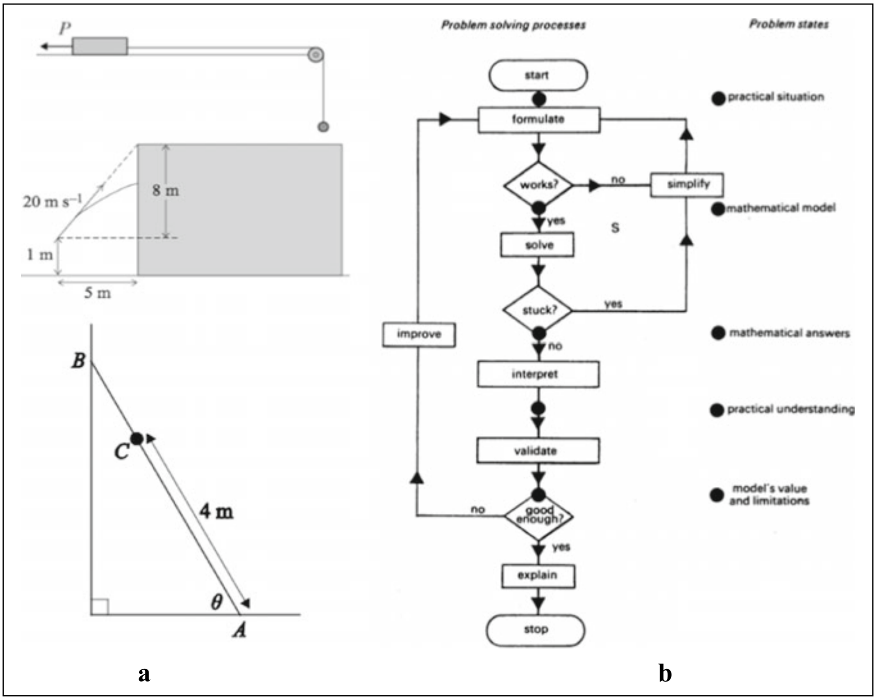

Fig. 29.3 **a** Stylized mechanics tasks, **b** Modelling diagram—1964

Special mention must be made of USMES: *Unified Science and Mathematics in Elementary Schools.* Led by Earle Lomon, an MIT physicist, the team at the Education Development Centre (EDC) developed a series of modules to support 6-week whole-class projects on practical topics including *Classroom design, Organizing school lunch, Welcoming a newcomer to the community* and *Kids' dessert preferences*—this one led some classes to develop factor analysis! Teacher support was through 'Teacher logs', each written and illustrated by the team with a teacher working on the project, and 'How to cards' describing useful techniques—practical, scientific and mathematical. USMES was wonderful for students but only outstanding teachers could handle this extended open problem solving in a productive way. This inspired me later 'to make USMES accessible to typical teachers' in the Shell Centre's *Numeracy through Problem Solving* project.

29.1.5 The Development of the Modelling Community—1970s and '80 S

In the following decade, many other people joined the modelling enterprise, which has gradually grown into the ICTMA community. I will mention only some key examples here. Niss and Blum (2020) give a more comprehensive survey of the development of modelling.

In the UK, George Hall in Nottingham developed a modelling course for undergraduates. Chris Ormell in Norwich developed a modelling-based mathematics course for humanities-focused high school students. David Burghes in Exeter founded the *Journal of Mathematical Modelling for Teachers*. Ian Huntley in Sheffield and Chris Haines in London were among the pioneers in the polytechnics, leading the first large-scale implementation of modelling in mathematics courses—and, for a long time, the only one. They were helped by an institutional factor. While each university devises its own courses, the courses in polytechnics were then approved by a central organization, CNAA. The CNAA mathematics panel decided that there should be a modelling course in all three years of any mathematics degree. Thus, modelling became institutionalized in a way that never happened in English universities, where innovation is easy but tends to be evanescent.

In the USA, inspired by Henry Pollak, groups arose around Boston and elsewhere. Sol Garfunkel began to develop COMAP from a university-focused collaboration towards the diverse achievements that were later recognized with an ISDDE Prize for Excellence in Design. Ed Silver, Richard Lesh and Helen Doerr developed schools of modelling with rather different foci on the relationship between modelling and mathematics. Max Bell, another pioneer, made modelling and applications the core of the UCSMP *Everyday Mathematics* curriculum, which is widely used in US elementary schools to this day.

In Denmark, Mogens Niss recognized a potential role for modelling in mathematics curricula as a student instructor in the 1960s, first in statistics then in microeconomics. In 1972, as the first mathematician at the new Roskilde University, where he and his colleagues were designing all the courses 'from scratch', Mogens started with a course on 'mathematical model building'. Modelling grew and flourished on that foundation, with Morten Bloemhoj and Thomas Jensen among the pioneers in the ongoing work.

In Germany, Werner Blum was inspired by working with Henry Pollak on the applications theme group for ICME-3 in Karslruhe, where Gabriele Kaiser became involved. An early problem was on the design of the income tax system. Again the group flourished with the work of Katja Maass, Rita Borromeo-Ferri and others.

In Australia, a strong strand of work on modelling grew through the work of Peter Galbraith and Gloria Stillman. In South Australia, John Gaffney, Vern Treilibs, Jeff Baxter and others developed modelling tasks for use in schools and in high-stakes examinations. Treilibs, on a year visit to the Shell Centre, performed a study of the formulation process that remains significant (Treilibs et al. 1980; Burkhardt 2017).

In the Netherlands, the work at what became the Freudenthal Institute deserves special mention. Led initially by Freudenthal and carried forward by Jan de Lange, the team took a complementary approach to modelling, seeing it as a route to concept development in mathematics through a process of successive abstraction and generalization from concrete situations with the concept 'hidden' at the centre. *Realistic Mathematics Education* was developed and later became the basis of *Mathematics in Context*, the US middle school curriculum, and later developments in the UK.

29.1.6 ICTMAs

A key long-term event in this period was when, in 1983, David Burghes launched the first International Conference on the Teaching of Modelling and Applications in Exeter. The title of my paper in the book that followed (Berry et al. 1984) was *Modelling in the Classroom: How can we get it to happen?*—a central concern still. Exeter '83 began the series of ICTMA conferences and publications leading to the Hong Kong ICTMA-19. This turned the diverse work of many contributors into a coherent community.

The other important connecting strand arose from Gabriele Kaiser's work as editor of *ZDM Mathematics Education*, in particular the special issues on modelling.

29.2 Developing Design-Focused R&D

My exploratory work on teaching modelling led to an invitation to move to Nottingham in 1976 as director of the Shell Centre for Mathematical Education, with the immodest 'brief': "To work to improve the teaching and learning of mathematics regionally, nationally and internationally",. I decided that this required:

- Focus on direct impact on practice in classrooms
- Recognizing that 'scale' can only be achieved through reproducible materials
- An engineering style research and development approach, with practical products
- A focus on design: strategic, tactical, technical (of which more later)
- A central role for modelling—for student motivation and real world usefulness.

These principles have informed the sequence of linked design research and development projects that have developed tools that support classroom teaching and learning, assessment both formative and summative, teacher professional development and systemic change.

The engineering research methodology we have developed, standard in other applied fields, embodies: input from prior research and development (ours and others'); imaginative design; and systematic iterative development through trials in increasingly realistic conditions, each revision based on rich and detailed feedback from classroom observation.

I shall describe some of these projects in brief to bring out the more general design principles that we developed, project by project.

29.2.1 Testing Strategic Skills (TSS 1980–85)

The strategic design (Burkhardt 2009) of this project set out to exploit the huge influence of high-stakes examinations on teachers choices of learning activities for their classrooms. WYTIWYG*: what you test is what you get* was obvious to teachers (and to me) though not then accepted by examination providers. WYTIWYG is now accepted as a fact though "tests worth teaching to" remain rare. Key features of TSS were gradual year-by-year change, with specific integrated support for the new area of learning. The first module was on non-routine problem solving tasks in pure mathematics; modelling skills came in with the second year's TSS module on translation skills, *The Language of Functions and Graphs* (Swan et al. 1985)—for which its lead designer, Malcolm Swan, was awarded the first annual prize for excellence in design of ISDDE, the International Society for Design and Development in Education (The 'Eddie').

The TSS design tactics were to:

- Introduce each year to the high-stakes examination one new task-type that is important but currently not assessed
- Offer schools well-engineered materials that exemplify the new task type, support the 3-weeks of new teaching involved, and in-school do-it-yourself professional development
- Give schools two-year's notice of the change, and remove content that involves 3-weeks teaching from the syllabus.

This approach was popular with teachers and students, whose performance in this new area, not surprisingly, improved substantially.

We learnt from TSS the power of these broader design strategies:

- *Gradual change* Plan the pace of change to answer: How big a change can typical teachers carry through successfully each year—given the support we can provide?
- *WYTIWYG* High-stakes examinations are powerful levers—for better or, usually, for worse. So work to turn the exams into "exams worth teaching to"
- *Alignment* Avoid mixed signals. Harmonize and link: policy documents, examinations, curriculum materials, and professional development
- *Materials-based professional development* This can increase the power of session-based PD, offering the leaders a level of support they know most teachers need—but they have never sought for themselves.

The tasks in Fig. 29.4 show the flavour of LFG.

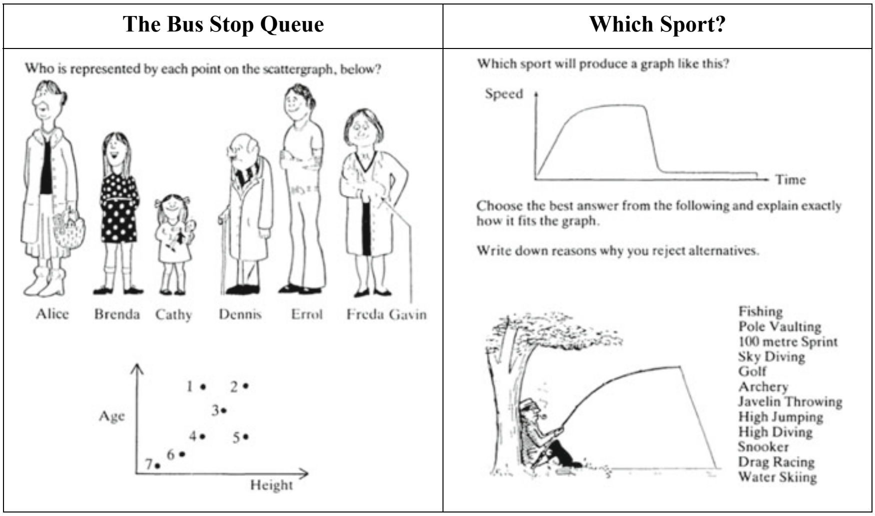

Fig. 29.4 Tasks from *The Language of Functions and Graphs*

29.2.2 Numeracy Through Problem Solving(NTPS 1984–88)

Modelling is the focus of NTPS with the strategic design goal of making teaching modelling accessible to typical mathematics teachers. The key tactical design elements are:

- 3-week group projects tackling practical problems
- Activity sequences led by student booklets, supported by a teacher's guide
- Ensuring final products from each group, evaluated by the class
- Assessment at three levels: Basic level during the project, with external exams assessing transfer to closely similar situations (Standard level) and more distant situations with similar structure (Extension level).

Five NTPS modules were developed on this basis (Shell Centre 1987–89):

- *Design a board game*—design, develop, construct, evaluate both board and rules.
- *Produce a quiz show*–create a TV style game show: choose the format, develop *fair* questions, run it with the rest of the class as an audience that later chooses the best show. Running the quizzes in real time is challenging for the teacher.
- *Plan a trip*– plan and carry through a class day-trip to another town.
- *Be a paper engineer*—design pop-up cards and boxes—develop skills and infer geometric principles
- *Be a shrewd chooser*—work our how to make well-informed consumer decisions, learn pitfalls, etc.

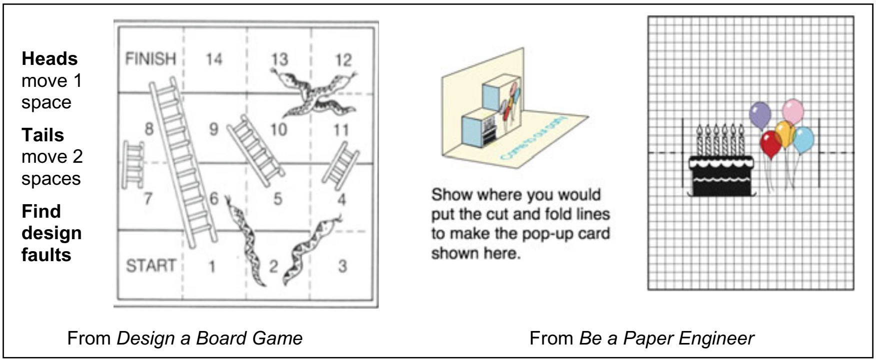

Fig. 29.5 Tasks from *Numeracy through Problem Solving*

The print materials are available from www.mathshell.com. The tasks in Fig. 29.5, from *Board Game* and *Paper Engineer*, give something of the 'flavour' of the work.

Each 3-week module is built on a sequence of four phases:

- *Explore the domain*—the materials supply examples for the students to analyse and criticize to build their understanding of what is involved—that a board game needs a board and a set of rules that makes for a competitive game that ends. For example, the task *Coin 'Snakes and Ladders* in Fig. 29.5 many weaknesses, some easier to find than others. Students take delight in finding these faults.
- *Design your product*–this involves exchanging ideas within the group, then converging on the principles of a design.
- *Construct your product*—this involves detailed design and construction.
- *Evaluate*—review and analyse the products from other groups, explaining your choices and reasons for them.

The response in the trial schools was enthusiastic among teachers and students of all levels of performance in mathematics—which did not always match their level on the modules.

The project's examinations were not part of the high-stakes system. So, in contrast to TSS, WYTIWYG did not operate and the take-up was limited. To influence the mainstream, an alternative GCSE option built around the modules was later created; the take up was substantial but still a minority of schools, probably reflecting the pedagogical challenges and the absence of new content—essential to keep the total difficulty within bounds.

The broader design tactics we learnt from NTPS included:

- The value of examples with faults—some ridiculous, some with deep mathematics—in motivating student engagement
- The power of examining a well-specified real world domain in detail
- Injecting essential points through the student materials—but only after the group should have noticed them (e.g. getting parental permissions for *Plan a Trip*).

29.3 The Modelling Potential of Technology

Modelling involves using mathematics to get deeper insight into situations in the real world outside the mathematics classroom, and evaluating how good a description each model provides. How do you bring enough of the real world into the classroom to make this work? Students' practical knowledge, written descriptions and video can play a part. But technology can offer a vivid way of bringing into the classroom a variety of simplified 'real situations' that help students develop the skills of active modelling. We call these apps investigative microworlds. Figure 29.6 shows some examples from Shell Centre projects.

29.3.1 Investigations on Teaching with Microcomputers as an Aid 1979–88

ITMA focused on the potential of a single computer to present a class with small investigative microworlds to help teachers develop modelling skills in their students. *Eureka* and *Bottles* in Fig. 29.6 both show graphs of water level against time. In *Bottles* a, steady stream of water flows into a vessel—the issue is how the shape of the graph relates to the shape of the container. *Eureka* offers a four-command programming language to construct, with suggestions from the class, various scenarios for a bath. The challenge comes in the next phase of the lesson, when only the graph is shown. What shape is the bottle? What was the scenario for the bath? The apps provide examples of increasing challenge to ensure productive struggle for all students. The graph in *Traffic* is similar, relating the position of cars against time to the overhead 'video'.

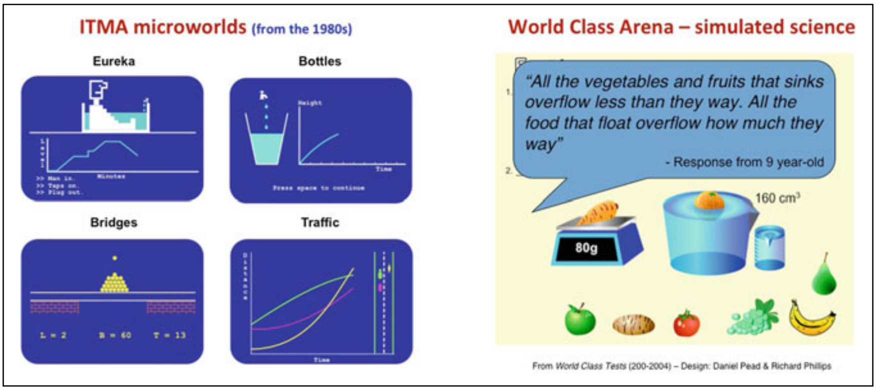

Fig. 29.6 Investigative microworlds

The other two examples in Fig. 29.6 involve a more complete modelling process. The students work on the computer, alone or in pairs, to collect data from the app and to build a model that explains it. In *Bridges,* they can vary the length, width and thickness of the plank bridge; the app then gives the weight it can support. The challenge is to organize this information and use the patterns revealed to construct a verbal, graphical or algebraic model. It turns out that the breaking weight is linear in the width (like two planks side by side so fairly obvious), quadratic in the thickness, and inversely proportional to the length. This provides a 'ramp' of difficulty, challenging students at all levels.

The microworld on the right is from the 1999–2005 *World Class Arena* project; students collect data on the water that overflows for objects of various weights, some of which sink while others float. Archimedes was said to have sorted out this, his principle, leaping out of his bath crying Eureka! There is evidence that the restricted universe of such simulations works better in enabling students to make scientific inferences than real experiments, where so much effort is involved in just getting the experiment to work!

29.3.2 Role Shifting and the 'Classroom Contract'

Brousseau (1997) formulated the concept of the 'classroom contract'—the understanding between teachers and their students as to who will do what, what roles each will play. In traditional mathematics classrooms, teachers mainly play the directive roles: they manage the classroom activities, eXplain new concept and skills with worked eXamples, and set eXercises that ask the students to imitate what they have been shown—the 3X model of 'demonstrate and practice' teaching. Modelling implies much more active roles for the students—managing their work through the modelling process, asking themselves questions along the way, and explaining what they have found.

A Shell Centre study of teachers and students working with the ITMA modules led to results with general implications for design (Phillips et al 1988). 17 teachers worked with one ITMA lesson a week for ten weeks, closely observed by the team. We observed changes in the interactions and identified over 30 roles played by teachers and/or students; we simplified them into six groups (Fig. 29.7).

Of these, only Resource needs explanation—a resource provides information, but only on demand. The ITMA study found that, using ITMA microworlds, teachers

Fig. 29.7 Classroom roles

Directive roles	Supportive roles
Manager	Counsellor
Explainer	Fellow Student
Task setter	Resource

spontaneously moved away from the directive roles in a way that allowed students to take them up. The app screen became a kind of 'teaching assistant', in task setting and in answering "what if" questions that arose in the class. Teachers moved into the supportive roles, as counsellor and fellow student, asking: What have you tried? What do you think? What's it doing there? We call this *role shifting*.

The broader design tactics we learnt from these projects include:

- Explicit design for role shifting is an essential element in modifying the traditional classroom contract for the development of non-routine mathematical skills and more generally, in meeting the five dimensions of TRU set out in 1.1
- Creating 'investigative microworlds' is a powerful tactical design tactic for changing the classroom contract in this way.

29.3.3 The Multiple Roles of Technology

We end this section by acknowledging the myriad other roles that computers can usefully play in a modelling curriculum. Computers of various levels of sophistication provide tools *for doing mathematics* and *for learning mathematics*:

- For *calculation*: from simple calculators through spreadsheets and computer algebra systems to sophisticated tools like *Mathematica*, technology is now central to the way calculation is done—except in too many mathematics classrooms
- For *investigation*: in addition to our examples, there are domain specific tools for dynamic geometry, graphing, data analysis and presentation
- For *communication*: the combination of computer power for analysis and for presentation, pioneered in exploratory data analysis, makes communicating mathematical results and insights much more effective.

Few of these tools are to be found in most mathematics classrooms. This partly reflects a mismatch of timescales—it takes a decade to design and develop the profoundly new curriculum that technology implies, while the technology itself changes every few years. Whatever the causes, the huge mismatch between school mathematics and mathematics in the 'real world' is a profound problem, which modelling can help cure. No one objects to the use of technology in tackling real world problems—or its use in other STEM subjects.

29.4 Formative Assessment in Modelling Lessons

Rich and timely feedback is central to improving the behaviour of any system. Hence the fifth in the TRU list of essential features of powerful classrooms with which I began: formative assessment. A more detailed account of the key role this plays is given in (Burkhardt and Schoenfeld 2019); here I shall look at what it means for teaching modelling and the design tactics that we developed for that purpose (Swan and Burkhardt 2014).

29.4.1 Mathematics Assessment Project(MAP 2010–15)

Most work on helping teachers to develop the skills of providing feedback to students in the course of their teaching has been based on extensive, and thus expensive, professional development. In MAP, we set out to explore how far teachers could achieve effective formative assessment lessons (FALs) in modelling when supported by specifically designed teaching materials. (These can be downloaded free from map.mathshell.org). The structure of each FAL is a sequence of individual, small group and whole class activities. I will illustrate the structure from the lesson on the *Matchsticks* task in Fig. 29.8.

In a prior lesson, each student tackles the unscaffolded modelling problem.

The teacher reviews (not scores) the work of the class and prepares qualitative feedback.

The teacher's reviewof the student work is supported by the *Common Issues table* for the problem. A powerful design tactic, this lists the challenges and misunderstandings that students are likely to have and suggests non-leading questions or prompts, as in Fig. 29.9.

In the main lesson

- Each student writes a response to the teacher's comments.
- *In pairs or small groups*, students work to produce and share a joint solution

The groups are directed to create a joint approach though explicit instructions:

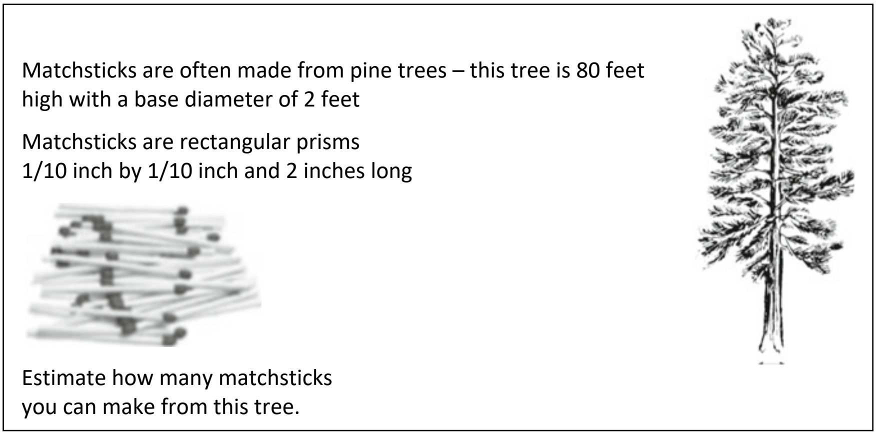

Fig. 29.8 The *Matchsticks* task

Common issues	Suggested questions and prompts
Has difficulty getting started	• What do you know? What do you need to find out? • How could you simplify the problem?
Ignores the units For example: The student calculates the volume of a matchstick in cubic inches and the volume of the tree trunk in cubic feet.	• What measurements are given? • Does your answer seem reasonable if you consider the size of a matchstick compared to the size of a pine tree?
Makes incorrect assumptions For example: The student multiplies the volume of the tree trunk in cubic feet by 12 and assumes this gives the volume of the tree trunk in cubic inches.	• Can you explain why you have multiplied by 12? • When you figure out a volume how many dimensions do you multiply together? How does this calculation effect how you convert the volume from cubic feet to cubic inches? • Can you describe the dimensions of the tree in inches? What do you notice?

Fig. 29.9 The start of the Common Issues table for *Matchsticks*

- *Share your method with your partner(s)—and your ideas for improving your individual solution.*
- *Together in your group, agree on the best method for completing the problem.*
- *Produce a poster, showing a joint solution to the problem.*
- *Write down any assumptions you have made. Check your work.*
- *Make sure that everyone in the group can explain the reasons for your chosen method, and describe any assumptions you have made.*

Students then review carefully designed 'sample student work', as in Fig. 29.10, comparing different approaches. This is a second design tactic worth noting. The students in pairs are asked to analyse each response and comment on strengths and weaknesses—specifically: *What has this student done correctly? What assumptions has (s)he made? How can (s)he improve the work?* Notice the shifting of the assessment role from the teacher to students.

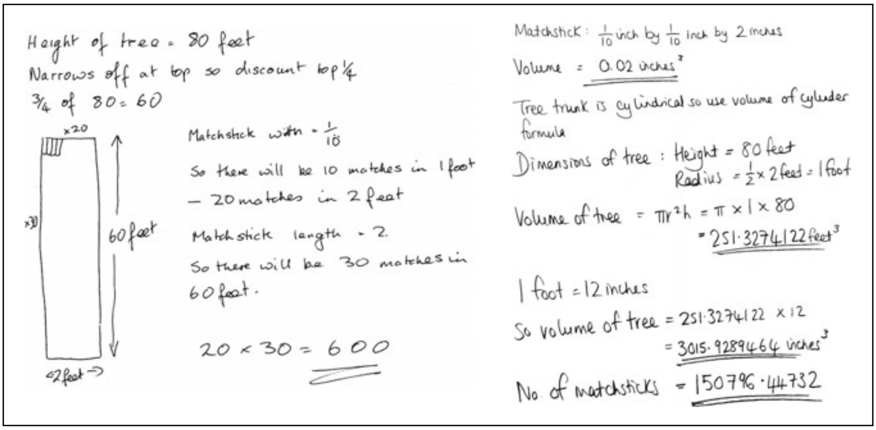

Fig. 29.10 Student responses to the *Matchsticks* task

Originally, there were, of course, calculation errors in the responses; we removed them because we observed in classroom trials that students focused only on this familiar kind of error, not noticing the strategic and tactical mistakes—in the first example in Fig. 29.10, the neglect of units (big matchsticks!) and using area instead of volume, and the subtler confusion of units in the much stronger second response (not to mention the 'spurious precision' that so often comes from thoughtlessly reading numbers off a calculator).

Then

- *The whole class* discusses the quality of each solution and the payoff of the mathematics.
- *Individual students* improve their solutions to the initial problem, again prompted with specific questions: *How did you check your method? What assumptions did you make? Is your method similar to one of the sample responses? What are the differences?*
- Finally, they write about what they have learned.

It is worth looking at this process in terms of the standard phases of modelling:

- Understanding the situation: "fitting matchsticks into the tree" is a volume-ratio problem
- Formulating a model for the tree: roughly a cylinder? a cone? ignore branches?
- Solving: Calculating the volumes (formulas provided). Getting the units right, relating feet^3 to inch^3. Computing the ratio, handling the big numbers. Choosing appropriate accuracy.
- Evaluating your answer: "Does this make sense?" "How can we improve it?" "Do we need to?"

We call this 'formative assessment'; where is the assessment?

- From the prior lessons responses, teachers get information on what students can do unaided.
- Teachers offer differentiated support to students, through questions, as and when this is needed—*differentiation through support* became a standard design tactic
- Students get constructive feedback via other students, and the teacher, as student work is discussed.
- Students act on feedback by improving their responses.
- Teachers get feedback on learning by comparing performances before and after.

29.5 Why is Large-Scale Improvement So Elusive?

Policy folk always say they want modelling in mathematics. Official mathematics curriculum documents in many countries include modelling as an essential element. I hope the reader may now see the basis for my claim that, as a community, we have learned how to enable typical teachers to teach modelling much more effectively. Yet

I believe that, if one dropped in on 100 randomly-chosen classrooms, you would be lucky to see any sign of active modelling by students. Thus it is fairly clear that we do not know how to lead school systems to make the changes needed for modelling to happen on a large scale. That seems to me still the central challenge. Can better design of materials alter this? If so, how?

29.5.1 Challenges of Change

In any school system, there are at least three important communities with the same declared priority for improving student learning and implementing the changes that this implies; however, each community has its own current day-by-day pressures that, in practice, turn the core priority into just a long-term aspiration. In a recent paper (Burkhardt 2019), I analyse these pressures in some detail and suggest ways forward, in the British context at least. I point to the contrast with the relationship in health care between the research, development, policy and practice communities. I suggest that a key is to move technical issues of teaching and learning out of the political domain into a long-term program of systematic research and development, designed to offer policy makers a range of well-engineered options for implementation. There are three strands that need ongoing support:

- Solution-focused 'engineering' research and development of tools and processes for teaching, learning and professional development
- Moving the balance of insight-focused research to support practice (Burkhardt and Schoenfeld 2003; Burkhardt 2016), and
- A structure to evaluate these innovations and advise government on a coherent implementation program.

The key is to recognize that, as in medicine, improvement will be gradual, based on well-engineered innovations. There is some interest in government at moving in this direction—though, as ever, they have other, more urgent things that occupy their attention!

29.5.2 Some Issues in Teaching Modelling

Great progress has been made across ICTMA on ways of teaching mathematical modelling. We know that a modelling curriculum needs standard elements including: regular work on worthwhile tasks; a variety of challenges at multiple levels; student discussion and reflection on alternative solutions and their implications. There remain a number of issues where more research is surely needed.

What else? What other elements should be included in a modelling curriculum? Should we teach modelling strategies explicitly, as in the Special Project unit on modelling that we designed for the Australian reSolve project (Wake et al. 2019)? Should we teach the subskills of model formulation (Treilibs et al. 1980; Burkhardt 2017)? and there are more.

'Grain size' What is the appropriate balance of lengths of modelling activities across: short intriguing tasks that stimulate 10–20 min of discussion; lessons like *Matchsticks* that take a few class periods; units that take a week or two like NTPS *Be a Paper Engineer.*

Balancing activity patterns What sequencing of individual, small group and class working best promotes both individual modelling skills and the ability to work in teams.

What about the classroom contract? When reasoning, not just answers, is the core goal, as in modelling, what variety of roles should the teachers and their students' experience? Should this new classroom contract be explicitly shared with the students? How far does it help in addressing the five dimensions of TRU?

Support for the teacher The teaching described in this chapter is a far cry from the 3X 'demonstrate and practice' model that most teachers around the world are used to. It involves tasks with a much longer 'reasoning length', with the teacher moving from directive to supportive roles. More profoundly, beliefs about mathematics as a set of procedural skills to be learned by students have to broaden to include their thinking about novel problems and constructing substantial chains of reasoning and explanation.

And finally:

How do we make these things a reality in every mathematics classroom?

29.6 "Maths Is Boring"—How Modelling Can Help Engagement

Most math lessons are not interesting to most students. Most maths classrooms often have *just one lesson genre*—the'same old, same old' *'XXX'*. In contrast 'English' (first language) lessons have many *genres*: students *write/read/analyse/critique* a wide variety of arguments/stories/poems/plays. Mathematics needs more lesson genres (Burkhardt and Swan 2017), based on modelling, that ask students to *interpret/critique/create/improve* models in design/planning/finance/risk and many other fields. 'Surprise and delight'for students should be a design goal. So I conclude this chapter by proposing a strategy that seems more novel than it is: *design dramas*. I begin with an assertion:

Every classroom is a theatre, each lesson a play.

Unlike some other subjects, mathematics lessons rarely 'fit the bill'. Usually, the math teacher is the director *and* the star. The students are the audience. The play begins with a long soliloquy, then it's over! Students just imitate what they have been shown, many times over. There is no dramatic tension—or surprise. No wonder "maths is boring" to so many.

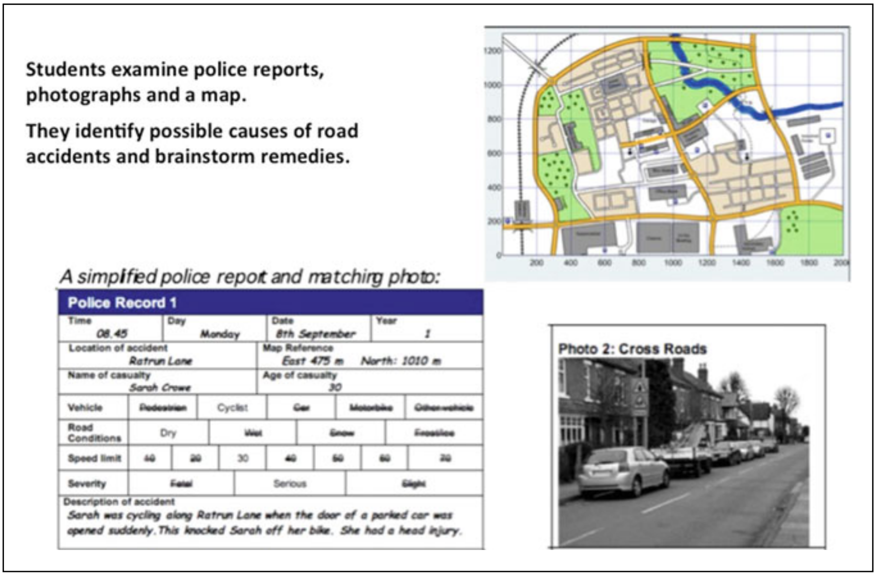

Fig. 29.11 Reducing road accidents—an investigation

Let us consider a different approach. A lesson begins with a mystery—a puzzle or a question. The students are detectives: they gather evidence, build theories. Arguments are presented. The student witnesses are cross-examined. The class is the jury.

So designing for drama supports'role shifting', moving students into high-level roles– consultant, designer, planner, teacher. The class teacher is still the director—but no longer the star. This is not new, for example, the *Numeracy through Problem Solving* modules are built around investigation. Making drama a normal part of mathematics lessons is a fine goal; modelling provides a route.

Software microworlds can support investigation. In *reducing road accidents*, for example, students investigate the data in the graphical database in Fig. 29.11, looking for patterns and making recommendations—often vividly presented—on how best to use the budget.

To summarize, this chapter has described a process of learning about the design of ways of teaching modelling, emphasizing the general design strategies and tactics that I and the Shell Centre team have developed through a sequence of projects over the years. These are described, with linked examples, in (Burkhardt and Pead 2020).

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Chapter 30
Opportunities for Modelling: An Extra-Curricular Challenge

Sol Garfunkel, Mogens Niss, and Jill P. Brown

Abstract Opportunities for mathematical modelling offered as part of normal classroom activity or via extra-curricular events. The environment in which modelling occurs varies and this includes variation in support available from a more knowledgeable other. Most common are opportunities for modelling within the usual classroom environment where support is provided by the classroom teacher. Less common, but increasing in number, are extra-curricular modelling opportunities. Support from a more knowledgeable other is non-existent in the International Mathematical Modelling Challenge. Success in The Challenge indicates the learning environment of such challenges is clearly conducive to student engagement with mathematical modelling. We can infer learning from the usual classroom environments is utilised by students as part of their successful modelling activity.

Keywords Extra-curricular challenge · Knowledgeable other · International mathematical modelling challenge · Modelling challenge · Real-world · Scaffolding

30.1 Introduction

It is well known that mathematical modelling is a critical aspect of mathematics (e.g. Niss et al. 2007) and increasingly included in curriculum documents (e.g. NGA 2010; VCAA 2015). Much has been written about the challenges of teachers including mathematical modelling in their teaching practice (e.g. Blum 2015; Niss et al. 2007;

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Stillman, Kaiser et al. 2013). Where mathematical modelling occurs, the degree of teacher control necessarily decreases as opportunities for student decision making increase. In "true" mathematical modelling, this control will be low. Naturally, when students (and teachers) first experience mathematical modelling, additional teacher support may be warranted. This is especially the case when either or both teacher and students are novice modellers. The lessening of control also relates to typical practices in a specific classroom. In other words, in classroom environments where students are expected to work collaboratively, make decisions, record and communicate their thinking, the distance from that environment to that needed for mathematical modelling is less than in a teacher-directed classroom environment (see Blum 2015).

Kaiser and Stender (2013, p. 279) posit the following overarching question and sub-questions,

> how complex authentic modelling problems put forward by the realistic or applied perspective on modelling can be integrated into mathematics education, what kind of learning environment is necessary, whether a change in the role of the teacher to a coach or mentor of the students is needed.

This chapter describes one such way: through extra-curricular challenge with the aim of allowing students to experience the power of mathematics to solve real-world problems.

30.2 How is Mathematical Modelling Implemented/Experienced?

In locations where students engage in modelling tasks, this occurs in one of three broad "environments". The first, and highly desirable, environment is when students engage with modelling activities in their "normal" classroom environment. This is usually only reported in the research literature; however, when it is also the subject of a research project (e.g. Blum 2015). Secondly, students engage in extra-curricular modelling experiences with or without the support of a more knowledgeable person.

30.2.1 Extra Curricula Modelling: Opportunities for Scaffolding

In some circumstances, students engage in mathematical modelling outside the normal classroom environment. Extra-mathematical modelling experiences include modelling days or weeks (e.g. Kaiser et al. 2013; Stender et al. 2017) and modelling challenges and are increasing in popularity. These experiences are described by Stillman, Brown et al. (2013) as activities where "their focus is on mathematical modelling activity, they are events that are external to normal classroom activity, they involve experts external to the school and are usually not at the student's school" (p. 218). In these environments, a tutor or mentor is present to support student modelling. In these circumstances, the level of support for student, external to task design, is intended to be low. However, the intended degree of support varies.

Kaiser and Stender (2013) describe the tutors in their modelling "weeks" as future teachers. Hence, "by this means the students were intensively supervised during their work on the modelling problems whilst on the other hand the future teachers had the opportunity both to gain practical experiences during the [modelling week] project" (p. 309). The students involved were Year 9 students working on one task for three days. The future teachers were prepared as tutors during a modelling seminar prior to the modelling week (Stender 2019). The seminar included working on all three modelling tasks the school student would be working on and learning about teacher interventions including scaffolding. Heuristic strategies were a key focus. The intention was that during the modelling activity, tutors would provide *motivational support* then *strategic help* using heuristic strategies (e.g. can you simplify the situation? Would a graph help?) before resorting to *content related strategic help* (e.g. have you considered an exponential model for your plot?), and finally if needed provide *content related help* (e.g. can you find an exponential function in the form $y = A.b^t + C$ for your data?). To maximising student learning and problem solving, interventions were planned and intended to be provided from least to most support. Tutors support for students was a key focus and the tutors had been "educated for adaptive support" (Stender et al. 2017).

Stillman, Brown et al. (2013) describe results from their study involving 70 Year 10–11 students from Australia and Singapore participating in the 2010 A.B. Paterson Mathematical Modelling Challenge in Australia. Students worked in mixed groups (i.e. Year level, country, gender) on a self-selected problem. In this Modelling Challenge, for students at the Year 10–11 level, the mentors began the two-day Challenge by presenting about

> the nature of modelling during which the modelling cycle is presented as a scaffold and students are given a short modelling task (e.g., optimum location for a hospital) to tackle …. [Subsequently, student teams] begin the process of choosing a situation of their own to model, posing a problem and generating questions to answer…. From this point on the mentor's role is mainly supervisory and to intervene as little as practicable. (p. 219)

In the modelling weeks, students are presented with a selection of tasks, whereas in the A.B. Paterson Mathematical Modelling Challenge, the Year 10–11 student teams were expected to pose their own modelling problem. Hence, the degree of decision making and the extent of support expected to be provided by the tutor/mentor between these two examples of extra-curricular modelling activity varies. In both cases illustrated, student feedback was positive. Kaiser and Stender (2013) reported that student participation in modelling weeks may impact positively not only on mathematical skills but on attitude towards mathematics. Results from the study by Stillman, Brown et al. (2013) found that the Challenge was "considered by most [students] as inherently valuable as a learning experience about modelling and application of mathematics in real situations" (p. 226).

These examples illustrate the extent of differences in mathematical modelling where there are opportunities for different degrees or type of support from a more knowledgeable other. In one example, the opportunities for support were carefully planned reading to be implemented in the moment as needed. In the other example, the support was limited to ensuring students had a similar understanding of what modelling is, were introduced to a modelling cycle diagram that could be used to scaffold their own modelling, and had been supported in an introductory problem to begin working as a collaborative group. Other differences related to the background of the students (one state versus two countries), participation guidelines (higher achieving versus interested in real-world problems) and choice of problem as described previously.

30.2.2 Extra Curricula Modelling: No Opportunities for Scaffolding

A third circumstance of modelling environment exists where no teacher, tutor or mentor (i.e. no knowledgeable other) is involved to support the students' mathematical modelling. These include the China Undergraduate Mathematical contest in Modelling (CUMCM) (beginning 1992), and in the USA, several contests organised by the Consortium for Mathematics and its Applications (COMAP). These include the Mathematical Contest in Modelling (MCM) (beginning 1985) and from 1999, the Interdisciplinary Contest in Modelling (ICM), and the High School Mathematical Contest in Modelling (HiMCM). In each of these, teams can use any inanimate data source—but all sources must be referenced appropriately—but no help from people other than team members.

The CUMCM (see Jiang et al. 2007) involves teams of three undergraduate students in China. They have three days to solve one of two modelling problems. Teams can only discuss the problem with each other, "not their advisor or anyone else, except members of their own team" (Jiang et al. 2007, p. 168). In 2019, over 42,900 teams participated. COMAP (see https://www.comap.com/undergraduate/contests/) describes the MCM as "a contest where teams of undergraduates use mathematical modelling to present their solutions to real world problems". Teams of three select one of six available problems to work on for four days. The ICM is a subset of the MCM where the teams select one of three specified problems. In 2019, 14,108 teams and 11,262 teams participated in the MCM and ICM, respectively. In the HiMCM, teams of up to 4 high school students work on one of two problems for three days. In 2019, 868 teams participated. The overwhelming majority of participants in these contests are from the USA and China, even though they are open to others around the world. The large numbers of participants in these contests is clear evidence of students valuing mathematical modelling and participating in mathematical modelling.

30.3 The International Mathematical Modelling Challenge

The International Mathematical Modelling Challenge (IMMC or IM^2C) for high school students was established in 2014 by collaboration between COMAP (Bedford, MA, USA) and NeoUnion ESC (Hong Kong SAR, China). The purpose was to foster and further high school students' interest and competence in mathematical modelling by offering them challenging problems to work on in teams. The intention is to promote the teaching of mathematical modelling and applications at all educational levels for all students. The idea is that, both students and their teachers need to experience the power of mathematics to help better understand, analyse and solve real-world problems outside of mathematics itself—and to do so in realistic contexts, that is in a collaborative team, with access to any digital tools including the internet they need. The IMMC allows students to do this directly, and their teachers vicariously via their students. Teams are awarded one of the following designations: Outstanding, Meritorious, Honourable Mention, Successful Participant. For further details see the website www.immchallenge.org.

Every year a new modelling problem is posed. The challenge is meant to more closely mirror real experiences with mathematics, as teams of up to four students are permitted to use any inanimate resources and are given an extended time period to do their work. The teams have up to five days to work on the problem and submit the report of their solution. The problems in the first five years of the IMMC, 2015–2019, have seen a diversity of contexts. In brief, the tasks were:

- *Movie scheduling*—design a model for the effective filming and production of a motion picture.
- *World record insurance*—design a model to effectively plan for expected pay outs for world record-breaking performances at a track and field meet warranting a prize, make recommendations regarding insurance to cover such an outcome from both the organiser and insurance company's perspective.
- *International meeting*—Design, and test, an algorithm, to allow international attendees to determine the best location for a face to face meeting
- *Best hospital*—Devise a model, including mortality as one factor, to measure the quality of a hospital, so that when an individual has a non-emergency you can advise them which of several hospitals they should select to attend.
- *Earth's carrying capacity*—Develop a model that identifies the earth's carrying capacity for human life under current conditions, propose how this carrying capacity can be raised accounting for perceived or anticipated future conditions.

30.3.1 Global Participation

With the intention of being genuinely international, teams can select a five-day time frame within a longer time window that best suits them. In the IMMC, students do not choose from a selection of problems, all teams attempt the same problem. There

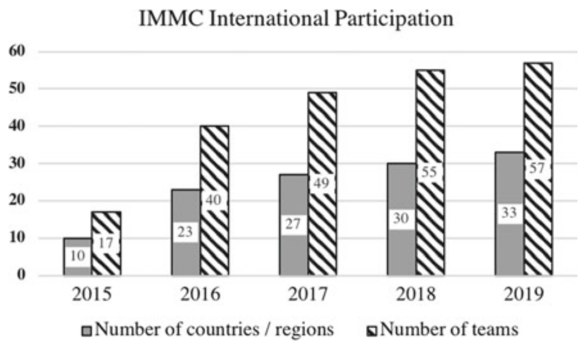

Fig. 30.1 Numbers of participating countries/regions and teams 2015–2019

is no cost to enter the IMMC. Figure 30.1 shows the increasing participation by both counties/regions and by the number of teams. If we assume that the teams all had four members, the number of participants has risen from 68, in the first year of The Challenge, to 228 in 2019. As each country or region can only have two teams, the increase per year is slowing at this global level. To see the real participation and impact of the IMMC, we need to look at the local level, as will by illustrated by the case of Australia.

30.3.2 Local Participation: The Case of Australia

How each country or region selects the two teams each year varies. In some countries, other challenges or competitions are used to select the two teams. In other countries, including Australia, teams participate in The Challenge in local dates set, for example, by the Australian IMMC Advisory Group. These submissions are then judged by the Australian Judging panel and the two deemed best are submitted to the international expert panel for judging. In Australia, Fig. 30.2 shows how interest has grown exponentially in the four years of participation. In the four years of participation by Australian students, 2016–2019, roughly half of the registered teams actually

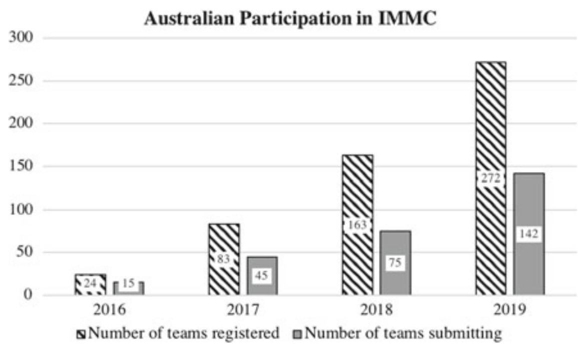

Fig. 30.2 Australian participation in the IMMC

submit their solution to The Challenge. In terms of participation by gender, 40–46% of students registering for The Challenge were female, and of the students who submit 44–46% of those were female.

30.4 The 2019 Challenge

In 2015, Niss drew our attention to prescriptive modelling (see also Brown 2019). In *descriptive modelling*, the purpose of the modelling activity is to *come to grips with aspects of an existing reality*—we want to capture, understand, explain, or predict aspects of the world as it exists before us. In *prescriptive modelling,* we want to *create or shape reality*, which will then eventually change aspects of the world. In the 2019 IMMC problem, the main task clearly involved descriptive modelling as student teams needed to understand the world as it exists today. They needed to determine what are the key factors impacting the carrying capacity of the earth and how these factors interacted with each other. In the latter part of the task, prescriptive modelling occurred as students made recommendations as to what "should be" along with associated actions to achieve increased carrying capacity.

In 2019, 57 teams from 33 countries submitted solutions through the national selection procedure to the following problem:

> **What is the Earth's *Carrying Capacity* for Human life?** 1. Identify and analyze the major factors that you consider crucial to limiting the Earth's carrying capacity for human life under current conditions. 2. Use mathematical modelling to determine the current carrying capacity of the Earth for human life under today's conditions and technology. 3. What can mankind realistically do to raise the carrying capacity of the Earth for human life in perceived or anticipated future conditions? What would those conditions be?

The Expert Panel chose five solutions as Meritorious. The teams were from St. Paul's Co-Educational College, Hong Kong (SAR) China; Brisbane Boys College, Australia; Manurewa High, Auckland, New Zealand; Il Liceum Ogólnokształcące z Oddziałami Dwujęzycznymi, Warszawa, Poland; and Utrechts Stedelijk Gymnasium, Utrecht, The Netherlands. The teams presented their solutions at ICTMA19 in Hong Kong (See Fig. 30.3).

The Expert Panel [Stolwijk 2019] had this to say about the solution papers:

> The IM^{2}C judges wish to congratulate all students who took part in the 2019 IM^{2}C. The judges were impressed by the efforts of all participating teams, the mathematics shown in the solutions and the high quality of the final submissions. All of the 57 papers submitted (from 33 different countries/regions) showed great creativity in working on the Challenge. While the Expert Panel judges see only two papers from each participating country or region, we recognise that many more students participate in the Challenge. It is exciting to know that so many students are engaging in, and successfully completing, this mathematical modelling

Fig. 30.3 The 2019 IM^{2}C meritorious teams, their advisors, and contest officials

opportunity. We encourage students to continue to form teams, compete to represent their country or region and aspire to be named among the top teams at the international level Expert Panel judging.

This year's problem was quite a different type of problem than in the former four IM^{2}C in that it was fairly open ended. In many cases, teams found it difficult to find the appropriate mathematics to apply. Students performed analysis on data mined from available Web sources, leading to a variety of descriptions of "the Earth's capacity for human life under today's conditions and technology", but not necessarily leading to a useful mathematical model to address the actual problem. Many teams concentrated their efforts on collecting, analysing, and (sometimes) recalculating the data on annual supply rates and annual per capita demand rates of various resources. At that point, teams needed to analyse and look across these data, but many limited their mathematical model of carrying capacity to a very simple one that did not take into account that different critical resources are not independent. Thus, one of the main discriminators of "better papers" was inclusion of some interdependence of the resources into the mathematical model, as well as posing and solving some kind of optimisation problem for such a model.

30.4.1 Meritorious Solutions

From the 57 papers judged, five were considered to be Meritorious. These were Team 2019006 Hong Kong, Team 2019018 Australia, Team 2019029 New Zealand, Team 2019031 Poland and Team 2019046 The Netherlands. The five countries/regions represented in this year's top category exemplify the growing international impact of IM^{2}C. Most of these five teams considered 3 or 4 main factors, together with additional factors. All five teams included food and water as two of their main factors. Other main factors included shelter, land, carbon emissions and oxygen. Additional factors considered included, roads, hospitals and schools (2019046) and nitrogen and phosphorus emissions, material and water footprints, % of population living in poverty, education, and access to electricity (2019031). An explanation of the factors and their interdependence was critical in the 2019 problem. For example, some teams considered space which was then considered in terms of both living requirements

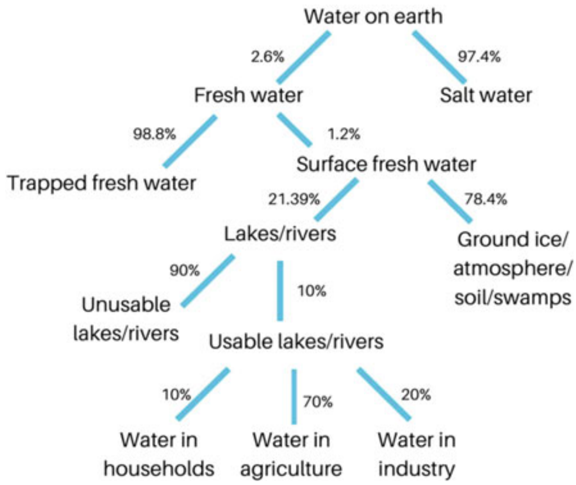

Fig. 30.4 Consideration of the factor: water by team 2019029 (New Zealand)

and food production. Figure 30.4 illustrates how the New Zealand Team carefully considered the various sources of water and its use.

The meritorious teams were careful to justify their choices and use of information—facts, figures and graphs—sourced from the Internet, and also referenced the source of the information used. A common approach was to initially consider each factor individually, determine the carrying capacity based on this factor and then the smallest such carrying capacity became the actual carrying capacity as determined by that limiting factor as all of that resource was used up.

The mathematics used in the 2019 Challenge was fairly simple in most cases. Teams spent significant time determining what factors were the most important to consider. Each factor was then considered, independently at least initially, to determine the population based on that factor (e.g. what population could be sustained by the available water given its multiple uses). Subsequent factors were considered, and the minimum predicted population taken as the sustainable value. Generally, factors were interdependent, and this had to be addressed.

Carbon emissions were identified as the limiting factor by 2019006 and 2019031, whereas food was the limiting factor identified by 2019029. Team 2019018 broke the world into 13 regions, "to allow for greater differences in quantity of food, shelter and water around the Earth" and this team determined differing limiting factors for each region. Team 2019046 used area as their main unit of analysis, that is, to determine how much is required to produce food and oxygen and provide shelter. They determined water availability was more than sufficient, thus, land was the limiting factor in their model.

With regard to digital technology use, of the top five teams, all clearly used the Internet to source information related to the problem. With respect to other digital technology, it was not very clear what was used, or how. Team 2019031 from Poland used the programming language Python "to apply a system dynamic model" although the code was not illustrated nor included. Team 2019018 from Australia used extensive spreadsheet calculations as they divided the world into 13 regions and determined current land and water usage for a variety of purposes and hence the limiting factor and carrying capacity for that region. Hong Kong Team 2019006 set up and solved simple equations as they determined that food consumption and carbon emissions, for example, were linearly related to population.

30.4.2 Detailed Consideration of the Solution from the Netherlands Team

For Team 2019046 from The Netherlands, *assumptions* included a world without conflict nor natural disasters nor epidemics, allowing the full capacity of the Earth to be reached. In addition, the majority of work is done from home thus reducing unnecessary use of land to accommodate workplaces, climate change is halted, and the current average use of resources remains. The *critical factors* the team determined were food, water, oxygen, shelter, energy and other variables (services—hospitals and schools, and roads).

With regard to *food*, they noted the different eating habits (food type and average intake of calories) around the globe. They mathematised this as needing to determine the area required to produce the number of kilograms per capita per year. To simplify the situation, they considered only some major food sources (e.g. sugar (beet and cane), fruits, vegetables, cereals and meat). For animals used for meat, milk or eggs, they also considered the need to feed these animals. They calculated the average number of hectares required to feed people for each continent and overall for the World.

For *water*, they considered all freshwater sources excluding ice caps, glaciers, and permanent snow. Noting the Water cycle natural allows water to be renewed every 40 days, and they found water was a smaller problem than they expected. Water was not the limiting factor for their model.

For *oxygen*, the team considered two sources: algae and trees. For these they considered the area (ha) of the earth covered by each and how much was produced. Finally, they determined the number of people that could be supported by oxygen produced from algae plus the number of people that could be supported by oxygen produced from trees.

For *shelter*, the team ascertained the habitable area of the Earth. They mathematised this as, average space per person is the average house size divided by the average number of households. They noted that multi-storey dwellings reduce the area needed. They observed that some uninhabitable parts can be reclaimed as is already occurring in The Netherlands.

With regard to *energy*, only indefinitely sustainable sources through the use of solar panels, wind turbines, and hydroelectric power stations were considered. For each, they determined how much energy was produced and how much area was needed.

They also considered the unused surface of the Earth that could be developed to source more energy.

Keeping in mind, they wanted to keep services as simple as possible they calculated the area needed for major services, that is, schools and hospitals. Roads were seen as essential for transport and the area of these was determined. Clearly, is the Earth's surface is used for these services it cannot be used to produce food or house people.

Other than for water, the team thus determined the area of the Earth's surface available for each of the selected factors and how many people this could support. The Team found that it was oxygen that determined the carrying capacity. This capacity was between 8 and 10 billion people. Note that at the time of the challenge, according to the world population clock (https://worldpopulationreview.com) the population was approximately 7.7 billion, so this is certainly a reasonable result.

Changes that would allow the carrying capacity to be increased were stop eating meat, stop wasting food, increase non-meat food production using multilevel agriculture methods, increased use of multi-storey flats for shelter; Lower oxygen ratio; grow more algae to increase oxygen production; shift schooling online and hence remove school buildings, reduce need for roads as less travel needed, consider travel alternatives to road-based methods, and increase land reclamation.

In the end, making all of these adjustments and fully optimising the carrying capacity, the team claim the carrying capacity could increase to 17–6—23.6 billion people that is an increase of around 114–137%! However, the team did not stay in the mathematical world as they comment, "Now that all the mathematical questions have been answered, one question still remains; do we really want this to happen?" They note that although the solution may be mathematically possible, it is not a desirable way to live. Furthermore, it would be very cramped living conditions and we humans value our space. Whist they have determined the potential carrying capacity, they did not factor in the happiness level of people living this.

30.5 Final Words

The modelling problems presented in this chapter are certainly what Kaiser and Stender (2013) would classify as complex authentic problems. We argue that modelling challenges are one such approach for successful integration of modelling in mathematics education. In such challenges, a different learning environment necessarily exists compared to that in the usual mathematics classroom; however, this environment and the role of "knowledgeable others" vary. The increasing participation in such events is evidence that small groups of students can adapt their usual learning environment (e.g. Stender et al. 2017; Stillman, Brown et al. 2013) to that of a small group, extended time on a single task and variable additional input.

Clearly the learning environment of modelling challenge sees student groups drawing on the experiences in "normal" classroom environments. In the IMMC environment, there is no "knowledgeable other" beyond the team members themselves.

However, although the teacher is absent in-the-moment, the skills and knowledge and ways of working have already been "taken up" by the students, who are able to draw on these learning to successfully work "teacher-less" during the challenge. Clearly any team of students who successfully completes IMMC has demonstrated both modelling competencies (e.g. Niss et al. 2007) and metacognitive modelling competencies (e.g. Stillman 1998).

Thus, the environment, be it a normal classroom, extra-curricular activity or modelling challenge, by its very nature, determines the role of the knowledgeable other, from teacher to mentor or coach to no-human help during modelling. However, there is no doubt that previous teacher advice and interventions experienced by the students in their regular mathematics classrooms were drawn on by the student teams as they worked during the modelling challenge to successfully solve the task. Being self-sufficient in activities such as the IMMC, must, at least in part, be attributed to the collective previous teaching and learning experiences of the team members. Finally, as the International Mathematical Modelling Challenge is truly a case where mathematical modelling occurs in both the East and the West and student teams from many locations around the world solve the same real-world problem each year without scaffolding from knowledgeable others, it is evident that student teams around the world are able to draw on their experiences and in the moment work 'sans' a "knowledgeable other" to solve the complex mathematical modelling tasks. We can infer that the classroom environments of the students, include situations where students work collaboratively, make decisions, understand the importance of communicating their thinking not only to each other during task solving but also to others about the final solution rather than being solely in a teacher directed classroom environment (see Blum 2015).

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Chapter 31
How Mathematical Modelling Can Be Promoted by Facilitating Group Creativity

Hye-Yun Jung and Kyeong-Hwa Lee

Abstract This study approaches mathematical modelling and creativity grounded in a sociocultural perspective. We examine an exploration of integrating group creativity into mathematical modelling in ninth grade class. Data were collected from lesson observation and interviews with participants. Findings indicated that group creativity contributed to simplifying the situation and elaborating models and that to get a more elaborated model, group composition reflecting cognitive diversity and teacher's guide for interactions based on mathematical grounds should be emphasized. Different types of interaction and creative synergies for group creativity along with modelling stage and different effects on modelling according to the emerged group creativity are described.

Keywords Mathematical modelling · Sociocultural psychology · Group creativity · Interaction · creative synergy · Role play

31.1 Introduction

Mathematical modelling has been regarded as important within mathematics education during last few decades (English 2006). Mathematical modelling activity, however, is rarely integrated into everyday classroom, and the difficulty of modelling activity for students is often attributed to this situation (Blum and Borromeo Ferri 2009). Although many studies suggested alternatives such as using technology, this still remains unsolved and as a meaningful challenge for researchers. In this study, we also challenge this. For supporting students and promoting mathematical modelling in everyday classroom, we focus on the sociocultural nature of mathematical modelling that has not been concerned (Lesh and Doerr, 2012).

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One of the distinctive characteristics of mathematical modelling activity is that it is performed in groups (Vorhölter 2018). Group creativity, which is also based on sociocultural psychology, develops in group activity and gives group members opportunities for idea sharing, productive conflict, and critical argumentation (Glăveau 2018). Because mathematical modelling is a group activity, group creativity can be also developed in mathematical modelling and group members can have opportunities to share and review their ideas about modelling activity which group creativity provides.

With this perspective, we propose group creativity as a pedagogical strategy to promote mathematical modelling. We explore how to facilitate group creativity through mathematical modelling and how to promote modelling activity through group creativity in a school context with grade 9 students. The research questions are as follows. (1) At each stage of modelling, which types of interaction and creative synergies are developed? (2) What effects of group creativity on modelling are observed?

31.2 Theoretical Background

In this section, we present the main issue of this study. First, we confirm the characteristics of mathematical modelling as a group activity. Second, special aspects, the definition of group creativity and the possibility of group creativity as a pedagogical strategy for promoting mathematical modelling, that are important for this study are reviewed.

31.2.1 Mathematical Modelling as a Group Activity

Although the definition of mathematical modelling is debated, in general, it is regarded as a cyclic and recursive process that requires understanding a real world, simplifying the situation, and creating a model (Blum and Borromeo Ferri 2009). Mathematics modelling has inherently sociocultural nature (Lesh et al. 2003). Previous studies also concern with its sociocultural aspects in which models are often developed by groups and the development of models often involves social functions (Lesh & Doerr 2012). Noting that mathematical modelling is a group activity, researchers have attempted to interpret multiple aspects of it that are distinguished from individual aspects. For example, Vorhölter (2018) pointed out that modelling tasks are usually performed in groups and that, for successful modelling, students have to share their knowledge, plan together, monitor and explain themselves to each other. English (2006) also indicated that students communicate with each other in the process of developing models, including evaluating models and presenting other models. Seen from a sociocultural perspective, these mathematical modelling activities are inextricably connected with sociocultural effects.

31.2.2 Group Creativity as a Pedagogical Strategy

Although the definition of group creativity varies from researcher to researcher, many studies (e.g. Glăveau 2018) present common key concepts in the definition: sharing ideas through 'interaction' and the 'creative synergies' resulting from it. In contrast to individual creativity, group creativity emphasizes the process rather than the output (Zhou & Luo 2012). Based on these, we define group creativity in this study as follows: the process or outcome of having creative synergies as thoughts presented by group members are shared through interactions within a group.

Interaction is an idea sharing process in a social context (Glăveau 2018). According to the manner of sharing, it is divided into three key types (Jung and Lee 2019): (1) mutually complementary interaction, or the cumulative sharing of diverse thoughts; (2) conflict-based interaction, or the confrontation caused by inconsistency of thoughts; and (3) metacognitive interaction, or the critical thinking process that evaluates or validates thoughts. Diversity and conflict are interrelated, in that high levels of diversity among members can cause conflicts (Kurtzberg and Amabile 2000–2001). Moreover, a full tapping of the shared ideas would require additional process, such as evaluation and elaboration (Sawyer 2012).

Creative synergy involves more creative problem solving or extended knowledge composition compared to individual creativity (Levenson 2011). According to previous studies (e.g. Levenson 2011), creative synergy can be divided into four types: (1) group fluency, which produces additional solutions by following up other members' ideas; (2) group flexibility, which produces solutions based on different strategies; (3) group originality, which produces original thoughts as thoughts are built based on previous ideas; and (4) group elaboration, which critically reviews others' thoughts following up other members' ideas. The key feature of creative synergy that distinguishes it from individual creativity is that members' solutions come from others' suggestions (Levenson 2011). Figure 31.1 depicts the development process of group creativity. In sum, as various thoughts are shared, combined and extended through three types of interaction within a group, the result becomes greater than the sum of each individual's creativity (Baruah & Paulus 2009).

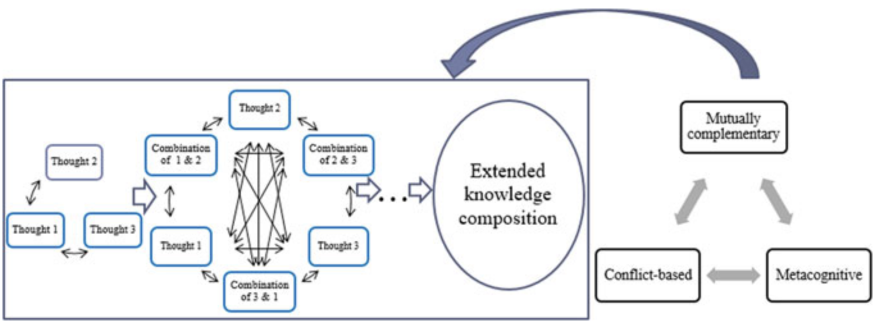

Fig. 31.1 Development process of group creativity

Table 31.1 Group composition and students' roles

Group	Thought presenter	Conflict inducer	Thought evaluator
2	S2	S1, S3	S4
5	S8	S5, S7	S6

As a pedagogical strategy, group creativity reinforces iterative review within a group through three types of cyclical interactions. By giving opportunities for students to critique and rethink problem solving process, students can share, combine and extend their thoughts within a group (Sawyer 2012). To be a pedagogical strategy, group creativity requires open, complex and ill-structured tasks rather than routine tasks (Levenson 2011). With this respect, modelling task, of which features are similar to that of group creativity (English 2006), also provides students with participating in the development process of group creativity that allows students to critique and rethink solving process. Sum up, group creativity can be used as a pedagogical strategy for promoting mathematical modelling.

31.3 Method

31.3.1 Participants and Research Context

Twenty grade nine students and a teacher with three years' teaching experience participated in this study. The students were divided into five groups of four. Each group was formed considering students' cognitive diversity and friendships (Baruah and Paulus 2009). To reflect the cognitive diversity, students clarified their thinking style and attitude toward new ideas and participated in group discussions in different ways according to the roles assigned to them based on those data: Thought presenter, Conflict inducer and Thought evaluator (see Table 31.1). The three roles induced mutually complementary, conflict-based and metacognitive interaction, respectively. This was the first time for them to participate in role playing. In this study, we focus on two groups: Group 5 with the most active interaction and Group 2 without.

31.3.2 Instructional Design and Mathematical Modelling Task

Classes were held three times, and each was 45 min long. Students spoke freely and used smartphones to search for information. The students had the initiative in their activities. The teacher guided the students to the interaction. One of the researchers observed the entire course of the lesson as an observer. The task was as follows.

Find the Best Snacks Last winter, Yejin went to America and made an American friend named Yeony, a famous YouTuber who uploads snack reviews. After returning to Korea, Yejin received an e-mail from Yeony.

'I want to upload a video on YouTube to introduce Korean snacks. I would be very happy if you would recommend two snacks according to the following criteria:

1. Avoid fattening snacks;
2. Avoid snacks with unhealthy ingredients; and
3. Avoid expensive snacks compared to quantity as much as possible.'

After checking the e-mail, Yejin asked you to choose snacks to recommend to Yeony. Based on the above criteria, analyse five snacks given to you and then write a letter recommending two of those to Yeony with valid reasons.

31.3.3 Data Collection and Analysis

All lessons were videotaped, audio-recorded and transcribed. The students' worksheets and the researcher's field notes were collected. Semi-structured interviews with participants were conducted after lessons and then audio-recorded and transcribed. The students are represented as Table 31.1, and the teacher and researcher are represented by T and R, respectively.

For the data analysis, after checking all collected data, we selected the case of three types of interaction through winnowing. The unit of analysis was the observed types of interaction at each stage of modelling. Interactions and creative synergies were analysed and categorized based on the theoretical background and the characteristics of the content. In order to increase the validity and reliability of the analysis, an intensive description was presented, and to analyse from various aspects, participants' interviews for the case were added. In addition, peer debriefing and member checks were done.

31.4 Results

After understanding the problematic situation, the students simplified it. They chose the three most important factors affecting the selection of snacks and found the figures of the snacks for the three chosen factors. Subsequently, students suggested and examined three mathematical models. Afterwards, they chose one of them and selected two snacks using the chosen model that integrated the figures. This study suggests that group creativity developed in mathematical modelling might contribute to the mathematical modelling such as the elaborate analysis of various factors embedded in the situation and the model refinement. In this section, we analyse group creativity and its effects developed in the process of simplifying and deriving mathematical model.

31.4.1 Group Creativity and Its Effects Developed at Simplification Stage

1. S5: The price. We can't buy snacks if we don't know price.
2. S6: I agree.
3. S7: No, the calorie. The most important thing is how much energy, the source of our strength, you can get when you eat snack.
4. S5: If I have only 1 dollar and the snack is 2 dollars, then we can't buy it.
5. S7: We should choose factors that have a wide gap between snacks.
6. S6: Why should we do that?
7. S7: If the difference is similar, all snacks will be the same. But, if we choose those factors,
8. S5: All snacks are different.
9. T: Think about Yeony's criteria.
10. S8: Teacher told us to think about Yeony's criteria.
11. S5: I got it! So, we have to check the price.
12. S6: The price of the snacks is similar.
13. S7: First is fat. Sodium corresponds to Yeony's second criterion. Price per gram for Yeony's third criterion.

In Group 5, three types of interaction occurred. At first, S5 suggested the price as the key factor (1), and S6 supported this (2). However, S7, a conflict inducer, intentionally disagreed with S5 and suggested another factor, calorie (3). In this case, the criterion to select each factor was from their real-world experiences, and S5 and S7 disagreed with each other (4). This is a conflict-based interaction. Students argued with their subjective experiences, not objective reasons. As the conflict was not resolved, members discussed another criterion, 'the size of difference in figures between snacks' (i.e., group fluency, group flexibility). S7 suggested this (5), and S6, a thought evaluator, asked a question about clear reasons for the criterion (6). Following S6, S5 and S7 elaborated on the meaning of it (7, 8). Following previous thoughts, they added thoughts complementarily (i.e., mutually complementary interaction and group fluency). As the students were not concerned about task context, Yeony's criteria, the teacher guided them to check it (9) and the students did (10). Using Yeony's criteria, members checked the aforementioned factors again (11, 12) and chose three factors based on the cumulative thoughts (13). As the criterion changed, the 'price' factor was evaluated differently (i.e. group flexibility). Based on the cumulative thoughts, the members checked factors repeatedly (i.e. metacognitive interaction). As a result, three criteria were integrated, and members selected three

Table 31.2 Group creativity and its effects observed at simplification stage

Group	Group creativity		Effects
	Interactions	Creative synergies	
2	Mutually complementary	Group fluency, group originality	Selecting the factor
5	Mutually complementary, conflict-based, metacognitive	Group fluency, group flexibility, group elaboration, group originality	Checking various factors Setting the elaborated factor selection criteria Selecting the appropriate factor

factors based on it (i.e. group elaboration): fat, sodium and price per gram. These factors are different from other groups' factors (i.e. group originality). This shows extended knowledge composition (Zhou and Luo 2012) and support rich evidence for the simplification.

In contrast to Group 5, there was only mutually complementary interaction in Group 2. At first, S3 presented the price as the key factor. In connection with this, S2 presented price per gram (i.e. group fluency). In this case, the criterion was the students' real-world experiences, which is the same as Group 5. However, the type of interaction that occurred is different. As a result, the students chose the three factors: price per gram, calorie and quantity. These are differentiated from other groups (i.e. group originality).

Table 31.2 shows the types of interaction, creative synergies and its effects observed at simplification stage in Groups 2 and 5. In the mathematical modelling process, group creativity emerged by combining three types of interaction can contribute to review the factors extensively and to refine criteria for selecting elaborated factors. By inducing repeated reviews, it serves as a catalyst for students to select the appropriate factors, while mutually complementary interaction alone contributes only to the expansion of factors.

31.4.2 Group Creativity and Its Effects Developed at Mathematical Model Stage

After simplifying, for each snack, the students calculated figures corresponding to each factor. Then, the students examined three models to integrate the figures for each snack.

14. S5: Add up figures and get the average of it.
15. S7: 5 points to the first, and 4 points to the second.
16. S5: No, add 1, 2, 2. Then, select the lowest average. For example, C is 1.7.
17. S7: C is 5. What? An average? Not addition?
18. S6: What do you mean?

19. S5: I got it! Not division but addition. 1 plus 2 plus 2. Is it right? (omitted)
20. T: If you give 100 weights only to fat and C is 1, 2, 2 (place),
21. S7: This means that, to the first place in the most important information,
22. S5: give 100 weights. And if you think that the price is not important, give 10 weights.

At first, S5 suggested 'the average of figures' (14). Following S5, S7 added an Example (15). Next, S5 disagreed with S7 and suggested ranks, not figures. (16). Following previous thoughts, the students had added models and disagreed with the added opinion (i.e., group fluency, mutually complementary and conflict-based interactions). As S7 raised a question about 'the average of figures' model (17), S6, a thought evaluator, asked a question about the difference between the models (18). Due to the question of S6, S5 identified that the mathematical meanings of the three models, the average of figures, the average of ranks and the sum of ranks, were the same (19). The thought evaluator, S6, provided an opportunity to review the shared models and the conflict inducers, S5 and S7, reviewed the model together rather than inducing additional conflict without logical grounds. Following the shared thoughts, they elaborated on the mathematical meanings of the models with reasonable justifications (i.e. metacognitive interaction, group elaboration). Afterwards, 'the sum of weights' was discussed. As the teacher gave an example (20), S7 and S5 completed the meaning of the model (21) and applied the model to a given situation (22). As in simplification stage, emerged group creativity is the catalyst of a model development, since students can text and refine ideas based on others' ideas.

In Group 2, unlike Group 5, there was only mutually complementary interaction and group fluency. At first, S2 presented 'the sum of weights' model. Following S2, S3 added a model, 'cost-effectiveness'. Following the model proposed by S3, S2 suggested a similar model, 'health'. Members did not review the validation and features of models.

Each group chose 'the sum of weights' model. Although the two groups chose the same model, the developed group creativity and its effects differed. Table 31.3 shows the types of interaction, creative synergies and its effects observed at mathematical model stage in Groups 2 and 5. The development of group creativity engages students in developing, understanding, modifying and using a model to make sense a context and to solve a modelling task. Table 31.3 also shows that although mutually complementary interaction can extend the number of sharing models, conflict-based

Table 31.3 Group creativity and its effects observed at mathematical model stage

Group	Group creativity		Effects
	Interactions	Creative synergies	
2	Mutually complementary	Group fluency	Sharing the model
5	Mutually complementary, conflict-based, metacognitive	Group fluency, group elaboration	Sharing the model Confirming the mathematical meaning of the shared models

and metacognitive interactions are needed to confirm the mathematical meaning of the shared models.

To sum up Tables 31.2 and 31.3, interactions, creative synergies and effects are linked. As more types of interaction occur, more types of creative synergies occur and more effects occur. For example, at simplification stage, while Group 2 just selected factors, Group 5 set elaborated criteria and selected the proper factors based on it. And, at mathematical model stage, while Group 2 just collected models, Group 5 checked the model's meaning. The teacher and the student mentioned effects of group creativity in modelling process as follows.

R: What effects do you think have occurred through the interaction for group creativity?

T: It helped in terms of expansion of students' thinking. There were many cases where they could see through the opinions of other friends what they had not thought of before.

R: How did activities in this class help you solve the given task?

S7: (when simplifying) We could choose the most reasonable information after checking the shared information.

31.5 Discussion and Conclusion

Using group creativity as a pedagogical strategy, we observed that majority of the students could better engage in each modelling stage. Focusing on the three types of interaction and four types of creative synergies, the major findings are as follows. First, according to the groups and the stages, different types of interaction and creative synergies and different effects were observed. Second, emerged group creativity promoted the modelling. According to the emerged group creativity, its effects on modelling were different.

Based on the significant differences in group creativity developed in the two groups, we can confirm that group creativity emerged by combing three types of interaction is needed for the learning and teaching of mathematical modelling. For developing model, it is needed for students to engage in multiple cycles of descriptions, interpretations, conjectures, and explanations that are iteratively refined while interacting with other students (Lesh et al. 2003). Group creativity, which is developed by the dynamic process of three types of interaction (Glăveau 2018), can support this. In particular, through conflict-based and metacognitive interactions, students engage in multiple interpretations and conjectures that are iteratively refined, and group elaboration occurred. By elaborating on the mathematical meaning of the shared model, students identified mathematical concepts contained in the model. Extended group creativity drove model refinement and knowledge creation (Zhou and Luo 2012). In this respect, Vorhölter (2018) also emphasized the capacity to monitor each other's work for modelling.

For the development of mathematical modelling and group creativity, previous studies (Baruah and Paulus 2009; Lesh et al. 2003) suggested some teaching method and environment. In addition to them, we suggest the followings for the learning and teaching of mathematical modelling through group creativity. At first, group composition reflecting cognitive diversity is needed. Appropriate role playing can enhance refinement in understanding situation and deriving model. In particular, the role of thought evaluator is critical. Unlike in Group 2, S6 performed her role well and this led to additional review. Second, teacher's guide for interactions based on mathematical grounds is needed. At simplification stage, similar to Lesh and Doerr (2012), students argued about a task based on their own experience not on mathematical grounds. The first criterion for the simplification was students' experience in each group and neither led to metacognitive interaction. In this case, to induce additional arguments, teacher's guidance is needed.

This study has the following implications. First, we present group creativity's particular contribution to the learning and teaching of mathematical modelling. As a pedagogical strategy, group creativity can promote mathematical modelling. Second, this study follows and extends previous studies (e.g. Lesh and Doerr 2012) presenting mathematical learning through modelling. As we can see in the results section, group creativity can support mathematical learning through modelling. In sum, group creativity can be developed through mathematical modelling activities, and group creativity can support mathematical modelling activities and extend mathematical knowledge.

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Chapter 32
Data-Based Modelling to Combine Mathematical, Statistical, and Contextual Approaches: Focusing on Ninth-Grade Students

Takashi Kawakami and Kosuke Mineno

Abstract This chapter examines ninth-grade students' data-based modelling to estimate previous and unknown Japanese populations. The results of the students' productions of group and individual models and their individual use of the group models demonstrated that the data-based modelling approach—which involves putting 'data' at the core of mathematical modelling—can be used to construct, validate, and revise various models while flexibly combining mathematical, statistical, and contextual approaches generated by using data from real-world contexts. Data-based modelling can be a pedagogically dynamic and flexible approach for balancing the development of generic modelling proficiency and the teaching of mathematics and statistics through real-world contexts.

Keywords Data-based modelling · Mathematics · Statistics · Real-world context · Population estimation · Boundary interactions

32.1 Introduction

With the advent of big data and the need to develop models to deal with uncertainty, several researchers in mathematical modelling education have emphasised the need to extend the concepts of models and modelling in mathematics education to the statistical domain (e.g. English and Watson 2018; Kawakami 2017). The mathematical modelling education community frequently promotes mathematical and real-world contextual approaches (Blum et al. 2007), whereas the statistics education community emphasises the use of statistical and contextual approaches in statistical modelling (Langrall et al. 2017). In this study, the data-based modelling

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approach—which involves putting 'data' at the core of mathematical modelling—contains characteristics of both mathematical and statistical modelling as it involves not only mathematical and contextual approaches, but also statistical approaches, in the modelling progress (English and Watson 2018). Furthermore, it can bridge the distance between learning mathematical modelling and statistics (Kawakami 2017, 2018). However, it remains poorly understood how, and to what extent, primary and secondary school students use data-based modelling to solve real-life problems (e.g. Engel and Kuntze 2011). This chapter illustrates ninth-grade students' use of mathematical, statistical, and contextual approaches in data-based modelling.

32.2 Combining Mathematical, Statistical, and Contextual Approaches Through Data-Based Modelling

Both mathematics and statistics education communities have increasingly emphasised the need for boundary interactions between mathematical modelling education and statistics education (English and Watson 2018; Langrall et al. 2017). The dual role of mathematical modelling—as an *end* and as a *means*—in mathematics education (Niss 2008) highlights the mutually beneficial relationship between learning mathematical modelling and statistics (Kawakami 2018). On the one hand, the real-world context of statistics can be employed as an end or content to learn generic modelling proficiency (competency and disposition). On the other hand, the modelling process can be a means to elicit and construct statistical concepts (i.e. distribution) to organise data, through constructing, evaluating, and revising models as sharable representations of a given system's structure (Hestenes 2010). Data-based modelling is a pedagogically dynamic and flexible approach and can be employed as either the former or the latter aspect (Kawakami 2017, 2018); this study focused on the former aspect.

Data plays key roles in developing models in both mathematical modelling and statistics (Engel and Kuntze 2011). Data are crucial in various aspects of modelling, such as model sources and references to test and validate models. Moreover, data are essential as a driving force for statistical inquiry involving problem definition, planning, data collection, data analysis, and conclusion (Wild and Pfannkuch 1999). Data have two epistemic characteristics: First, data are numbers with a real-world context (Cobb and Moore 1997). Second, data have a structure and variability (i.e. signal and noise) (Konold and Pollatsek 2002). Therefore, data are not only the objects of mathematisation to uncover or abstract the underlying mathematical and statistical structure of data in order to solve real-world problems but also the sources of mathematical and statistical knowledge, ideas, and concepts (Lesh et al. 2008; Wild and Pfannkuch 1999). Mathematics and statistics differ in terms of their ways of using data (e.g. Cobb and Moore 1997). On the one hand, mathematics mostly deals with operations regarding numerical values in data and abstractions, pattern identification and generalisations based on data, and generally adopts a deterministic view of the data and derived interpretations and conclusions. On the other hand, statistics always links data to real-world contexts and considers that there is no certainty in interpretations of data and their solutions due to data variability. Based on the aforementioned characteristics and handling of data, mathematical approaches (i.e. operations and reasoning that focus on invariant structures and signals in data), statistical approaches (i.e. operations and reasoning that focus on variability and noise in data), and contextual approaches (i.e. operations and reasoning that focus on real-world contexts beyond data) can be elicited in conjunction with data-based modelling, and this can help to observe, manipulate, and represent real-world data. Based on modelling epistemology, where components of several models are combined and reconstructed into a coherent model for inquiry purposes (e.g. Kawakami 2017), students can develop meaningful models by combining mathematical, statistical, and contextual approaches, thereby promoting generic modelling proficiency to utilise the characteristics of data through data-based modelling.

32.3 Methodology

This chapter addresses the following research question: How did the students use the three approaches in their models through data-based modelling? To investigate the question, we analysed group activities and post-lesson individual reports through a series of teaching experiments conducted in 2015. This study included four classes of ninth-grade students (73 males and 78 females aged 14–15 years) from a national junior high school in Japan. The experiments were coordinated by the classroom teacher, who is the co-author of this chapter. The students had previously learnt concepts in descriptive statistics (i.e. mean, median, mode, histogram, and relative frequency), probability (i.e. statistical probability and mathematical probability), and function (i.e. proportion, inverse proportion, and linear function). However, they had not learnt modelling with complex data. During the experiments, the students were asked to use calculators and not simulation software.

32.3.1 Design

32.3.1.1 Population Estimation

Population estimation is a crucial topic in statistics compared with demography because the future population has a considerable impact on the national policies and economy. Smith et al. (2001) stated that although experts use complex mathematical and statistical methods in population estimation, the following mathematical, statistical, and contextual approaches are the foundation of population estimation: (a) deriving an underlying data structure (i.e. population change pattern), (b) focusing on population change variability, and (c) considering variability sources, such as births, deaths, migration, and social circumstances.

In mathematics education, Galbraith (2015) cited population estimation as an authentic modelling example for real-world problem solving. Before World War II, in the Japanese textbook for tenth grade mathematics (Tyuto Gakko Kyokasho Kabushiki Kaisya 1944), population estimation was included in the 'Statistics and Probability' unit, which enabled the students to learn (a) decision making based on data and common sense, (b) evaluation and comparison of models based on assumptions, and (c) statistical probability (e.g. death probability). Therefore, population estimation is an excellent topic for learning data-based modelling; however, less empirical research has been conducted on population estimation in mathematics education at the lower secondary level.

32.3.1.2 Tasks

Population estimation tasks Population prediction is an important theme that affects not only the national economy but also policies. This year (2015) will be the census in Japan. What is the population of Japan by age group this year?

Task 1: In order to predict the population by age group in 2015, we would like to predict the population by age group in 2010 that can be verified. We collected Japanese population data by age group in 1995, 2000, and 2005 when the census was conducted from the website of the Ministry of Internal Affairs and Communications as in Table 32.1. Consider how to predict the population of Japan by age group in 2010 and compare the predicted and actual population data in 2010.

Task 2: Refer to findings from Task 1. Consider how to predict the population of Japan by age group in 2015.

The population estimation tasks comprised two tasks aimed to estimate the (then) upcoming 2015 census population of Japan. However, the word 'predict' was used

Table 32.1 Japanese population data by age group in 1995, 2000, and 2005 (Partially omitted)

Age	1995	2000	2005
0 – 4	6,001(thousand)	5,915	5,599
5 – 9	6,547	6,033	5,950
10 – 14	7,485	6,558	6,036
15 – 19	8,567	7,502	6,593
~	~	~	~
65 – 69	6,402	7,118	7,460
70 – 74	4,699	5,910	6,661
75 – 79	3,292	4,157	5,280
80 +	3,884	4,856	6,358
Total	125,571	126,927	127,766

instead of 'estimate' in the tasks and class to help students understand their goal. Because students may find model development difficult without scaffolding, a model development sequence through different situations (Lesh et al. 2003) was adapted for model development using different data. In Task 1, students conceived ideas of models for population estimation in 2010 based on the data, and in Task 2, they employed the model created in Task 1 to estimate the population in 2015. By estimating the population in 2010, the student models could be evaluated and improved compared to the actual population in 2010. Table 32.1 comprises data based on 5-year intervals and 5-year age groups (i.e. cohort: a group of people born in the same interval), and the students could understand patterns of population change through Table 32.1 and/or population pyramid graphs.

32.3.1.3 Teaching Experiments

All classes were taught two or three 50-min lessons. At the beginning of the first lesson, the teacher introduced the problem of the declining population of 14–15-year-old students in Japan and the subsequent decline in the number of university students, which would severely affect the management of universities and colleges. Numerous participating students aimed to acquire university education in the future, and thus, they considered population estimation to be necessary for individuals and society. The teacher then set the topic of the lessons as 'Predicting the population' and asked the following question: How do you predict population? The teacher presented Task 1 after the students provided their responses (e.g., 'collecting data on births and deaths' and 'collecting data on populations of each age'). The students were categorised into 40 groups (3–4 students in each group) and were asked to collaboratively complete Task 1, in order to advance engagement with real-world context (Brown 2017). However, the teacher distributed calculators and additional data and resources (e.g. population based on age groups, number of births, birth rate, mortality during 1990–2005, number of births based on mothers' age, and population pyramid graphs) on request. Each group summarised the designed group models for predicting population in 2010 on the provided whiteboards. During the second lesson (for one class, the second and third lessons), the group models were shared, compared, and evaluated to elicit different perspectives from the students for model creation in Task 2. The students observed and implemented the following aspects in group models: (a) prediction by focusing on a cohort, such as analysing a generation and diagonally arranged data in Table 32.1; (b) including percentage and not population difference; and (c) considering the real-world context of the data. For model validation, at the end of the class, the teacher asked the students to compare the population in 2010 that had been estimated using group models and the actual population in 2010. The students were curious to understand reasons for the matched or unmatched population values. Task 2 was an individual report assignment. In addition to finding the solutions and results of Task 2, the students were asked to prepare a report with answers to the following question: How did you implement the approach used for population prediction in 2010 (Task 1) to predict the population in 2015 (Task 2)?

32.3.2 Data Collection and Analysis

The analysed data comprised the group models in Task 1 (as documented on the whiteboards), individual models in Task 2 from the reports, and student responses to the question from the reports. To address the research question, group and individual models were categorised based on the approaches used by the students (i.e. mathematical, statistical, and contextual approaches). This study's mathematical approaches included mathematical operations and procedures with regard to data values, abstraction, pattern identification with functional perspective, generalisations from data, and deterministic perspectives regarding trends; its statistical approaches included the determination of operations and procedures with consideration of data variation and decision making under uncertainty; and its contextual approaches referred to the use of knowledge with regard to the real-world context of Japanese population and society. Moreover, the use of group models to create individual models was categorised according to frequently used terms and similar terms (e.g. 'apply', 'modify', and 'collect') observed in the student responses for understanding model development by the individual students.

32.4 Results

This section summarises the student productions of the group models from the whiteboards, individual models from the reports, and their individual use of the group models from the reports in order to illustrate findings regarding the addressed research question.

32.4.1 Student Productions of the Group and Individual Models

With regard to the results of models proposed for Task 1 by the 40 groups, mathematical approaches included focusing on patterns of change in cohort population differences, making generalised formulae to estimate population, calculating the rate of change in the cohort population, and calculating the rate of change in the total population for each 5-year interval. The statistical approach included the calculation of average cohort population differences to accommodate the variation in the data values. The contextual approaches included considering the causes of variability,

Table 32.2 Categories of group models in Task 1 ($N = 40$ groups) and individual models in Task 2 ($N = 151$)

Category	Characteristics	Percentage (%)	
		Task 1 %	Task 2 %
M1	Using only mathematical approaches	40	28
M2	Using both mathematical approaches and contextual approaches	10	9
M3	Using both mathematical approaches and statistical approaches	48	46
M4	Combining mathematical, statistical, and contextual approaches	2	17

such as the low birth rate, aging population in Japan, and immigration to Japan. For the individual final models in Task 2 by 151 students, in addition to the approaches used in the group model, the following approaches were used: calculating the change in birth and death rates (mathematical approach); identifying significant figures (statistical approach); comparing predicted and actual values (statistical approach); and considering the effect of natural disasters, such as earthquakes (contextual approach).

Table 32.2 presents the categories of group models in Task 1 and individual models in Task 2. For Category M1 in each case, the student groups or individual students used only mathematical approaches (as described above). They estimated population data for the years 2010 or 2015 based on patterns they obtained from changes in existing population data every five years (e.g. 'decreasing by about 5,000 thousand'). They did not explicitly mention data variation and argued that changes in population every 5 years are constant, so they were assigned to Category M1. For example, one student group estimated the population for 2010 by adding the differences between the decades 1995–2005 to the 2000 cohort population. Concerning Category M2 in each case, the student groups or individual students utilised both mathematical and contextual approaches (as described above). They interpreted the cause of the increase or decrease in the numerical value of the data based on the real-world context of Japanese population and society in addition to the characteristics of Category M1 (e.g. 'The decline in population is probably due to aging and the declining birth rate'). With regard to Category M3 in each case, the student groups or individual students used mathematical and statistical approaches (as described above). They estimated population data for 2010 or 2015 and calculated average and/or significant digits in order to accommodate data variation (e.g. 'The average was calculated because the variation in the value change was large'). They explicitly mentioned data variation in contrast to Category M1, so they were assigned to Category M3. The group students or individual students who constructed the models assigned to Category M4 combined mathematical, statistical, and contextual approaches (as described above). They interpreted the cause of the increase or decrease in the numerical value of the data and/or adjusted the estimates based on the real-world context of Japanese population and society in addition to the characteristics of Category M3.

Figure 32.1 provides a summary of a Category M4 model that was proposed by a

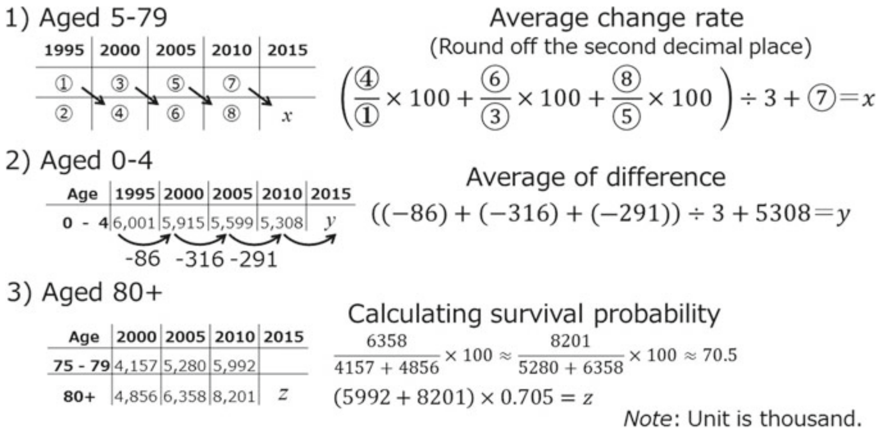

Fig. 32.1 An example of category M4 (Summary by authors)

student during Task 2. This student developed three models for each population group. For the age groups of 5–79 and 0–4 years, the student used an average rate of the change in cohort population and the average population difference, respectively. For each 5-year interval group in the 5–79 years group, the student calculated the average cohort change rate for 5 years, added the average to the 2010 cohort population (⑦ in Fig. 32.1), and estimated the cohort population 5 years later (x in Fig. 32.1). He generalised these procedures with a formula. For example, the formula for the 15–19 age group in 2010 was as follows:

$$\left(\frac{15-19\text{ age population in 2000}}{10-14\text{ age population in 1995}}\times 100+\frac{15-19\text{ age population in 2005}}{10-14\text{ age population in 2000}}\times 100\right.$$
$$\left.+\frac{15-19\text{ age population in 2010}}{10-14\text{ age population in 2005}}\times 100\right)\div 3$$
$$+\,(10-14\text{ age population in 2010})$$

For the age group of 0–4 years, the student calculated the mean because variation in data was evident. For the age group of higher than 80 years, the student considered medical care and aging society and calculated survival probability based on the following cogitation: ‘Medical care is advanced, and Japan is inclined to become an aging society. Thus, I considered another approach for prediction’. The mathematical approaches included abstracting the structure of cohort from the data and generalising it with a formula (especially for the age group 5–79 years), calculating the rate of change in each cohort population, and calculating the difference of population. The statistical approach included calculating the average cohort population differences to accommodate variation in data. Probability was calculated using mathematical and statistical approaches. Contextual approaches related to the Japanese society were used to develop alternative models.

Table 32.3 Categories of students' individual use of group models ($N = 151$)

Category	Characteristics	Percentage (%)
U1	Applying own group's model	24
U2	Modifying own group's model by	
(a)	Adding other data and models	19
(b)	Changing operations	31
(c)	Both (a) and (b)	16
U3	Correcting own group's model	3
U4	Changing own group's model	7

Regarding the results of the group models, 88% of the groups were assigned to Category M1 (40%) or Category M3 (48%); only 12% of the groups incorporated contextual approaches into their models. However, because this result was obtained after analysing group models that were summarised on whiteboards, students may have used contextual approaches in their model construction processes. On the other hand, for individual models, 74% of the students were assigned to Category M1 (28%) or Category M3 (46%); 26% of the students incorporated contextual approaches into their models, and 17% of the students combined mathematical, statistical, and contextual approaches.

32.4.2 *Students' Individual Use of the Group Models*

The use of group models by the students to develop individual models was classified into four categories (Table 32.3). The students applied, modified, collected, or changed their group models. Overall, 66% of the students modified their group models by using the following three methods: (a) adding other data and models, (b) changing operations, or (c) both. For example, in Category U2 (a), the students used the following approach: adding data, such as births, deaths, immigration, natural disasters, and so on; creating models for each age group by considering the differences in variability (e.g. age groups 0–4, 5–79, and higher than 80 years); and adding functional models by using and comparing the total population with the sum of the population of different age groups. On the other hand, Category U2 (b) included using aspects, such as the difference of change, rate of change, mean, and significant figures and using alternative data view. The students assigned to Category U2 explicitly developed mathematical, statistical, and contextual approaches. For Category U3, the students corrected errors, such as calculation errors, in their group's models, and for Category U4, they adopted the models proposed by other groups. Moreover, the students described the model features, including similarity, simplicity, generality, and shareability, in their individual use of the group models. One example of considering shareability was as follows: in his report, a student who was assigned to Category U2 (b) stated that he clarified the assumptions in his alternative model after reflecting on the fact that the meaning and intention of his group model was not shared properly with his classmates due to the ambiguity of his group model assumptions.

32.5 Discussion and Conclusion

This chapter addressed some aspects of the data-based modelling (i.e. estimating the Japanese population) performed by ninth-grade students. All the participating students developed individual models based on the group models (Tables 32.2 and 32.3). Moreover, 66% of the students improved their group models while pursuing the similarity, simplicity, generality, and shareability of their model. These findings may have resulted from the effect of the model development sequence (Lesh et al. 2003), which included model sharing and validation, and collaborative engagement with real-world contexts (Brown 2017) in lessons. Furthermore, mathematical, statistical, and contextual approaches were used flexibly and creatively in order to construct individual models (Fig. 32.1). The students also developed a mathematical and statistical understanding (English and Watson 2018) that reflected different concepts including the meaning and usage of proportions and mean. They combined and developed several mathematical, statistical, and contextual approaches by using data observation, manipulation, and representation while estimating the population. The aforementioned results suggest that a data-based modelling approach could also help students acquire the generic modelling proficiencies necessary for manipulating and using big data in order to obtain interdisciplinary solutions in the later grade levels, consequently in their adulthood. However, this study also revealed that students tended to experience difficulties when they attempted to incorporate contextual approaches into their models during the development of group and individual models (Table 32.2). This result validates the difficulty of incorporating contextual approaches into mathematical modelling practice (e.g. Brown 2017), thereby indicating that more research is necessary to examine the data-based modelling processes conducted by students using the collected data, including videos of student group work.

In conclusion, this study suggests that the data-based modelling approach can be used for constructing, validating, and revising various models while flexibly combining the mathematical, statistical, and contextual approaches generated by using data from the real-world context. Data-based modelling is a pedagogically dynamic and flexible approach, and it can be employed for various educational purposes, including the teaching of mathematical modelling with the aid of the real-world statistical context or statistics teaching by using mathematical modelling processes (Kawakami 2017, 2018). This study focused on the former purpose; the students developed not only mathematical and contextual approaches but also statistical approaches such as statistical probability (e.g. survival probability, as shown in Fig. 32.1). This finding provides a firm foundation for effectively teaching statistical and stochastic concepts with the aid of data-based modelling. Thus, the data-based modelling approach will help to resolve the challenge posed by Niss (2008) of balancing the development of modelling competencies and the teaching of mathematical and statistical contents using real-world contexts.

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Chapter 33
Implications of Using Mathematical Modelling as a Pedagogical Tool on the Mathematical Concepts of Proportions and Proportional Reasoning in a Non-prototypical Secondary Mathematics Classroom

Muzi Manzini and Duncan Mhakure

Abstract Consistent with international trends, mathematical modelling is heralded and has been documented as a construct that is imperative for the teaching and learning of mathematics in South African schools. The study discussed in this chapter explored the immediate implications of using mathematical modelling as a framework for the teaching and learning of powerful mathematical concepts such as proportional reasoning in South African schools located in under-resourced low socio-economic areas. Results show that the initial apprehension that students experienced when exposed for the first time to a model-eliciting activity was soon transformed into a diverse range of creative mathematical approaches, when they learned that the activity is open-ended by default.

Keywords Mathematical modelling · Models and modelling perspectives · Model-eliciting activities · Non-prototypical mathematics classroom · Powerful mathematical ideas · Proportional reasoning

33.1 Introduction

The teaching and learning of primary and secondary school mathematics in South Africa are informed by the National Curriculum and Assessment Policy Statements (DBE 2011). One of the aims of this curriculum is to ensure that students acquire and apply knowledge and skills in ways that are meaningful to their own lives and

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promote knowledge in local contexts while being sensitive to its global relevance. In this regard, mathematical modelling is prioritised as a key specific objective, deemed to be a set of imperative general principles that must apply across all primary and secondary grades. Currently, much international attention is being paid to mathematical modelling (Stohlmann 2017); in particular, to modelling as a problem-solving tool, or a resource for linking (through application) abstract classroom mathematics with real-world situations. Moreover, internationally there is also a revamped surge towards examining mathematical modelling as a pedagogical tool (Arseven 2015).

The study reported in this chapter seeks to explore the implications of the employment of mathematical modelling as a framework for the teaching and learning of powerful mathematical concepts in South African secondary schools in low socio-economic areas. These are schools located in communities that are characterised by poverty, inequality and related social ills that may manifest in (for instance) classroom disruption, student-teacher violence, etc. that is “non-prototypical mathematics classrooms”. For the purposes of this study we interpret “powerful mathematical concepts” in the same way as Skovsmose and Valero (2002); these speak to key ideas and processes geared at optimising the likelihood of success, in terms of student access to the opportunities of the twenty-first century and the fourth industrial revolution. These include, *inter alia,* quantitative literacy, and mathematical and statistical reasoning (e.g. proportional reasoning, reasoning using data, probabilistic sense-making, algebraic cognition, mathematical modelling, visual representations, problem-solving and -posing, etc.) (Moreno-Armella and Block 2002; Skovsmose and Valero 2002). This is especially critical in the context of delivering to students’ real-world mathematics that has meaningful and immediate implications for their own lives. More specifically, the study reported in this chapter sought to investigate the students’ mathematical struggles that acted as impasse factors when students are solving real-world problems on the concepts of proportions and proportional reasoning. Therefore, using the models and modelling perspective (MMP) theoretical framework, the following research question was investigated: What mathematical struggles are prominent as impasse factors when students are engaged in a model-eliciting activity involving the concepts of proportions and proportional reasoning?

33.2 Theoretical Framework

In line with our research context, this study used a theoretical framework underpinned by contextual modelling, the MMP. This framework has proven extremely valuable in studying the interaction between students, teachers, mathematics instruction, and the curriculum delivered. Thus, as in English (2003), the MMP was adopted in this study, because it has been shown to be a powerful conceptual framework for research on the development of interaction between students, curriculum resources, and instructional programmes, see also (Lesh and Lehrer 2003).

More specifically, for our context, the following points are what ultimately aligned the policy imperatives, the teaching and learning context, and the underlying theoretical framework. (1) MMP emphasises that thinking mathematically is about interpreting situations mathematically at least as much as it is about computing (Lesh and Doerr 2003). (2) MMP asserts that the development of elementary but powerful mathematical concepts and constructs (models) should be considered important goals of mathematics instruction (Lesh and Doerr 2003). (3) The MMP-related literature illustrates how modelling activities often lead to remarkable mathematical gains by students previously regarded as systematically disadvantaged, mathematically immature, or simply not gifted enough for such sophisticated and powerful forms of mathematical thinking (Lesh and Lehrer 2003; Lesh et al. 2003; Schorr and Lesh 2003; Kang and Noh 2012). (4) MMP draws on the design of activities that motivate students to develop the mathematics needed to make sense of meaningful situations (Stohlmann 2017).

The main problem-solving activity that students engaged with in this study is an example of what are called model-eliciting activities (MEAs). These activities are characterised by authenticity and meaningfulness of context, open-endedness, and the lack of structure typically associated with secondary school textbook word problems. More importantly, and as is characteristic of mathematics education research involving student-produced work, the emphasis was on the thinking and reasoning tools employed by students when engaged in non-routine activities. In this regard, the process that students engage in when solving MEAs is regarded as the main product or the fundamental aspect of the expected student-generated solution, as also explained by Lesh and Doerr (2003). Figure 33.1 below depicts the modelling cycle adapted from (Mooney and Swift 1999), and Table 33.1 explains the cycle in the context of a model-eliciting activity.

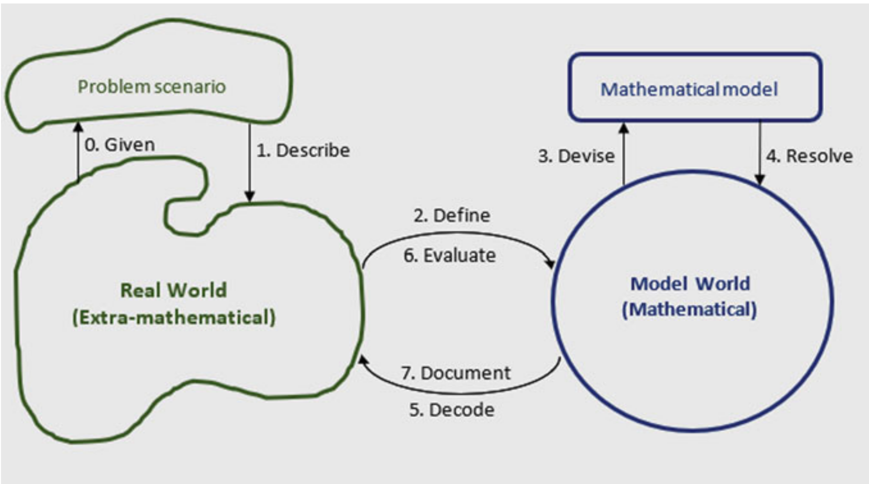

Fig. 33.1 The iterative mathematical modelling cycle

Table 33.1 A description of the steps involved in the MMP cycle, in the context of MEAs

Modelling stage	Description
0. Given a real-world problem scenario	A real-world context, a case study, is given to students. MEAs use real, practical problems, often sourced from print-media advertisements or articles, or other publications. These tend to have more *personal meaning* and are understandable to students
1. Describe the real-world problem	A clear and succinct description of the real-world problem must be stated. The modeller (the student, in this case) must also isolate and appreciate the practical features of the real-world scenario
2. Define the mathematical problem (Mathematise in simple language)	Once the real-world problem is fully understood and is well articulated, it must be translated into a clear, unambiguous mathematical problem; that is, the real-world scenario must be posed as an appropriate and related mathematical question (Pertamawati and Retnowati 2019)
3. Devise the model (Mathematise using Occam's razor)	Conception and formulation of a suitable mathematical model, i.e. the problem statement is now coded into some form of mathematical expression, equation, geometric construction, etc. This involves making assumptions, choosing suitable variables, and providing a sensible rationale for any decisions made in the *model construction* (Mousoulides et al. 2008; Mooney and Swift 1999)
4. Resolve the model	Using any suitable mathematical concept or construct, a solution to the mathematical model devised must be determined (Pertamawati and Retnowati 2019)
5. Decode the solution	Reinterpreting the model's solution in terms of the practical, real-world meaning. Does the solution make sense, in terms of the real-world problem? If so, what is the mathematical solution that tells us how to proceed (Stohlmann 2017)
6. Evaluate the model	This is about establishing the connection between the mathematical model (real result) and the real-world problem with the aim of validating and examining the mental representation from students and compare them to their initial assumptions on how to find the solution. This entails, in addition, checking assumptions made during the process of solving the real-world problem, and the acknowledgement and identification of the limitations of the proposed solutions (Borromeo Ferri 2006)

(continued)

Table 33.1 (continued)

Modelling stage	Description
7. Document the model solution	This stage requires a *documented final product,* either a written report or in oral form, outlining the problem-solving strategies and solution employed—that is, a report on the successful solution, or highlighting how further research could produce an improved solution. This must be a clear and concise explanation of how the client (the entity or person facing the real-world problem) should address the problem. In addition, where feasible this should also illustrate the *reusability* (if any) of the model in terms of its ability to help the client solve other problems of a similar nature (Galbraith and Holton 2018; Lesh and Doerr 2003)

As suggested in Galbraith and Holton (2018), for the purposes of the introductory MEA a cyclical model as depicted in Fig. 33.1 is most useful and applicable in practice if it is further augmented in a form that can guide or scaffold a systematised approach to individual problems. In this sense, the following list outlines the sequential steps of the process used by students progressing towards a group solution.

The actual process pursued by the students is analysed in the next section. In brief, the process followed by the students in converting the real-world problem into a mathematical model, from a MMP, is an important step for students engaged in a model-eliciting type of activity. It employs "Occam's razor", with the aim of emphasising to students the need to "avoid making things harder than they need to be" (Mooney and Swift 1999, p. 4). This encourages students to exclude all information and details not particularly useful for their purposes, or which cannot be processed given the constraints at hand. In turn, this permits the students to slice a real-world problem into manageable modules, and ameliorates the complexities associated with real-life problems (Mooney and Swift 1999).

33.3 Research Methodology

33.3.1 Participants and Procedure

A total of 129 (male and female) middle-secondary (Grade 10) mathematics students participated, aged 14–15 years, and from three schools. None of the students had any direct prior exposure to the mathematical modelling cycle process. The schools that participated in the study were in different locations but had similar socio-economic status. Since the study involved students who were minors, ethical clearance was sought from parents and guardians, the school, and the provincial department of education. The study took place during normal mathematics lessons, in the presence of both the researchers and the teachers. During the implementation of the model-eliciting activity, each student received his or her own copy of the activity and was given about five minutes to read and understand the task on their own. Subsequent to that, they were divided into groups of two or three, and worked on the MEA as a collective.

33.3.2 Data Collection and Analysis

Data was collected using audio recordings (and transcriptions thereof) of the student group sessions. In addition, all written student work and solutions were collected for documentary analysis. Each group submitted a brief report summarising the process and advising the client on the solution. Among other aims, the report was intended to motivate the students to articulate the assumptions made and processes required. The Galbraith and Holton (2018) process was adopted, as the MEA also served as an introduction to the modelling process for students with no previous experience of modelling. Depicted below is the underlying model-eliciting activity that was used in this study. The students' modelling activities were analysed within the modelling framework outlined in Table 33.1.

The modelling activity Thabo Mali, a keen sport supporter, is planning to host a group of friends on Saturday to watch a soccer derby match. He plans to serve them beef burgers at half time. Thabo knows his Gogo's recipe for making very juicy beef burger patties; however, the recipe only caters for 4–6 people, and he suspects he will have 6–9 friends visiting on Saturday. He would like advice on how he should adapt the burger patty ingredients to cater for all his friends. Write a report in which you advise Thabo on how to adjust the ingredients for:

(i) this coming Saturday, and
(ii) any other weekend.

Gogo's (Granny's) secret ingredients (4-6 servings)

(1) 3/2 kg beef
(2) 1 egg
(3) 3/4 cup dry bread crumbs
(4) 3 tablespoons milk
(5) 2 tablespoons Worcestershire sauce
(6) 1/8 teaspoon cayenne pepper
(7) 2 buds of garlic

33.4 Results and Discussion

In this section of the chapter, we use the MMP framework in the context of MEAs to describe, and analyse the interactions between students, teachers, mathematics instruction, and the curriculum delivered. These interactions are described and analysed within the eight modelling stages shown in Table 33.1.

(0) *given a real-world problem*: Initial indications were that being handed a task that did not look like what the students normally deal with in the mathematics classroom seemed to captivate them. The context of the task was reportedly familiar to the students, with one alleging "*That's my dad on sporting weekends*"; and thus, they immediately took interest in the activity.

(1) *describe the real-world problem*: In terms of describing the real-world problem, the task was designed to be as clear as possible, especially given that this was the students' first experience with a MEA. However, they were still required to extract and isolate the important features of the real-world problem in one the sentence and write them down, as a form of scaffolding for the next step. Here, it was clear that the students had started thinking about what exactly they were required to do.

(2) *define the mathematical problem (Mathematise in simple language)*: Beyond the puzzled facial expressions, they also started to ask questions: "*What does it mean to 'adjust' in mathematics*?"; "*So! what are we supposed to calculate, sir?"*—some students would seek clarification. Some description of the problem used phrases such as "Calculate how many more of each ingredient Thabo needs", "Increase the number of ingredients by how much?", etc. Although specifying the problem in simple English did not trouble the students much, evidently there was some disagreement within the groups as to how they would utilise their "answers" to advise the client.

(3) *devise the model (Mathematise using Occam's razor)*: This was clearly the first time the participants had been challenged with an open-ended type of mathematical activity, in which choice of approach depends on the drawing of individual assumptions about the real-world problem situation. The results show that initially the students experienced challenges in drawing these assumptions. As a result, initially some resorted to making quick guesses regarding the solution, without thinking deeply about what it would mean in reality.

(4) *resolve the model:* However, once they were clear that the structure of their solution would depend on them first making certain key assumptions and, that they could then devise a particular solution contingent on those assumptions—the students were very keen to communicate their assumptions. In the main, these were either client-focused or ingredient-availability-based, with the very clear aim of simplifying the calculations required. *Inter alia,* assumptions included: (i) Thabo (the client) must have enough food for everyone; (ii) Thabo must measure ingredients accurately; (iii) Thabo must know exactly how many (more) friends are coming; (iv) Thabo should get all the things he wants to buy in advance; (v) Thabo must organise the ingredients so that everyone gets one burger.

(5) *decode the solution*: This process of generating assumptions was facilitated by some directed student questions posed to the session facilitators, which indicated that the students were starting to engage critically with the task. These "what-if"-type questions included the following: (i) How do we know how many burgers per person Thabo will serve? (ii) Do all the friends need to receive the same amount of food? (iii) Can Thabo cut up any extras and share them equally? (iv) What if Thabo is broke, and cannot afford what he wants to buy?

(6) *evaluate the model*: In devising the mathematical model, there were struggles initially. Even when some of the student groups were clear on their situational assumptions, they struggled to agree among themselves on how to mathematise the task they had been given. Based on the simplifying assumptions, some students chose to make quick estimates of the required ingredient input amounts and, offered some justification for the decisions made. For example: "Multiply every ingredient by 3, because Thabo is expecting more than double the number of people catered for by Gogo's original recipe". These students were comparing the '4' from '4–6' to the '9' from '6–9'. There were a fair number of algebraic expressions generated, and also a few attempts at arithmetic and proportional reasoning.

(7) *document the model solution*: To solve the mathematical model, students were allowed to use any mathematics concept or construct they deemed appropriate, including proportions, reasoning and algebra. For example, Group A used proportions, i.e. a:b as c:d: given 4–6 and 6–9, 4:6 as 6:9 therefore 2:3 and 2:3; in other words, adapting from 4–6 to 6–9 simplifies to the equation *new amount* = *old amount* $\times$ *3/2*. Here is a transcription of the relevant conversation:

Teacher:	Does the answer make sense?
Group member 1:	Not sure, but I think in a way it does.
Teacher:	How so?
Group member 1:	Well sir, at least the recipe will be increased, since 3/2 is bigger than 1.
Group member 2:	Yes sir, there is a proportional increase—we are multiplying by an improper fraction.
Teacher:	So, how many eggs must Thabo use?
Group member 3:	1times 3/2! It's one and a half.
Group member 1:	Should we not round up, and say two eggs?

In some groups, the engagements showed members had a good grasp of the real-world interpretation of their solution, as well as the ability to evaluate the reasonable practicality of such a solution. Perhaps familiarity with the context of the MEA was critical in this regard. Two groups (B & C) reasoned as follows: (1) Reduce the "old" ingredients (4–6) to serve 1 person. (2) Increase the ingredients for 6–9 people. That is, when using the lower limits of the two ranges (4–6) and (6–9), first divide the original ingredient amount by 4; the result is an equivalent proportion for one person. Then multiply the answer by 6, i.e. *new amount* =(*old amount* $\div$ 4) $\times$ 6.

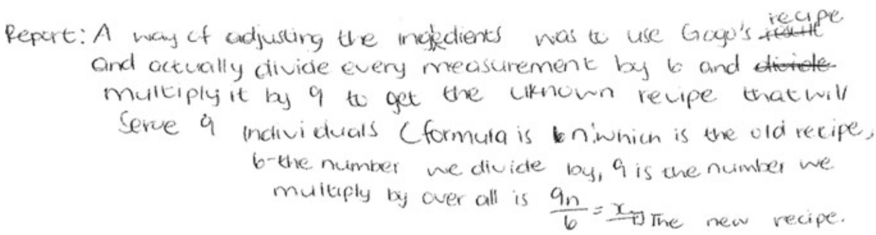
Report: A way of adjusting the ingredients was to use Gogo's recipe and actually divide every measurement by 6 and ~~divide~~ multiply it by 9 to get the unknown recipe that will serve 9 individuals (formula is n which is the old recipe, 6-the number we divide by, 9 is the number we multiply by over all is $\frac{9n}{6} = x$ → The new recipe.

Fig. 33.2 A groups' summary report of the modelling process

The study also showed that some students harbour misconceptions about handling fractions (simplifying or arithmetic operations), or about translating from spoken language to mathematical language. For example, in the group discussions some members were unsure whether "increase an ingredient by 3/2" translates to "add 3/2" or "multiply by 3/2". Part of the reason for this seems to stem from the students' undeveloped use of mathematical language; for instance, no student was able to articulate the calculated amount (3/2) as a "factor", which would have indicated from the outset that the answer must be a number used in finding a "product".

There was some apprehension evident regarding writing a report, as students in mathematics are not typically asked to document their answers. Initially, they were not sure what to write. However, once it was clear that there was no prescribed structure for the report, and that they simply needed to summarise what the client should do, they managed to turn in their documentation. (See the sampled report in Fig. 33.2.)

33.5 Conclusions

This chapter investigated the students' mathematical struggles that were impasse factors when students were engaged in solving MEAs on the mathematical concepts of proportions and proportional reasoning. Two important factors are noteworthy in this study. Firstly, given the fact that students who participated in the study had no prior exposure to mathematical modelling, it is evident that there are definite gains to be made by permeating the curriculum with MEA-type activities. This, especially in relation to the sometimes elementary but key mathematical ideas—for instance, the variation in problem-solving approaches adopted by the students, for example, proportional reasoning, algebraic and pure arithmetic. Indeed, it is only fair to assume that further exposure to learning mathematics through mathematical modelling will go a long way in expanding the scope of approaches used. Secondly, the learning environment in South African schools is largely traditional in terms of the systematic approach to mathematics teaching and with a clear emphasis on high-stakes examinations. This makes it difficult for teachers to readily adopt modelling as a new way of teaching. Although teachers did not directly form part of the research team, the study provided useful and practical information to teachers on how MEAs can be used for teaching mathematics in schools.

The immediate implication for pedagogy and student struggles is that there appears to be a definite need to move students away from the (often limiting) understanding that mathematical solutions ('the answers') to mathematical problems are always unique, and either correct or incorrect—especially since real-life problems are generally open-ended. More importantly, the activity revealed some embedded conceptual misconceptions with respect to arithmetic operations involving fractions and translation from everyday language to mathematical operations or representations, as well as with the mathematical interpretation of real-life contexts (mathematisation).

Further research should focus on understanding how choice of context in the MEAs influences student engagement with it; in particular, how students deal with assumptions. It could also explore how mathematical modelling in the context of model-eliciting activities can be used as a tool to help students bridge the gap between spoken language and mathematical language and, to make connections between different representations.

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Chapter 34
Using the Modelling Activity Diagram Framework to Characterise Students' Activities: A Case for Geometrical Constructions

Duncan Mhakure and Arne Jakobsen

Abstract This chapter reports from a study investigating the use of a modelling task that was given to sixty-two Grade 11 students from two schools in low socio-economic areas in South Africa. The Modelling Activity Diagram (MAD) framework was used to characterise students' mathematical thinking style when working on a real-world problem on geometrical constructions. Although students were able to find solutions to the scaffolded questions, they had problems with identifying key mathematical concepts required during the mathematisation process and assumptions required to solve the modelling task. Only 16% of the students successfully completed the modelling task.

Keywords Mathematical modelling · Modelling activity diagrams · High-stakes examinations · Mathematical thinking styles · Mathematical objects · Modelling competencies

34.1 Introduction

Over the past two decades, mathematical modelling has increasingly been viewed as a central instructional strategy in mathematics education, from elementary school to higher education settings. The notion that all mathematical objects are abstract makes mathematics as a school subject difficult for students to learn. By using the phrase *mathematical object*, we refer to: sets, numbers, matrix, real numbers, function, differentiation, and squares, to name but a few. For instance, it is easy to see square-like shapes in everyday life situations; however, it is difficult to see the abstract *object* of the square—without imposing the mental image of a square on a real-world

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thing (Stacey 2015). Perhaps not surprisingly, mathematicians refer to mathematical objects as if they are physical objects. During the modelling process, “abstract tools developed in one context can be applied to many other physical phenomena and social constructs of the human experience and science” (Stacey 2015, p. 58). The latter aligns itself with the construct of mathematical modelling, in which a real-world problem is mathematised or converted to an intra-mathematical problem using a process that applies abstract mathematical objects, which have been discovered and derived in different areas of application. In mathematics education contexts, mathematical modelling as an instructional strategy is seen as a *modelling vehicle* (Julie 2002)—an approach for teaching mathematical concepts and enhancing students' abilities in solving real-world problems (Erbas et al. 2014). It also true that in the past two decades, many educational studies using mathematical modelling as a teaching strategy have been carried out, with a number of them detailing how mathematical modelling has been applied to school curricula (Erbas et al. 2014; Kaiser and Sriraman 2006).

Whilst this chapter is foregrounded in the South African mathematics education context, a more global perspective of the construct of mathematical modelling is adopted. The South African National Curriculum Statement (NCS) and the Curriculum and Assessment Policy Statement (CAPS) for Further Education and Training Phase Grades 10-12 states that: “Mathematical modelling is an important focal point of the curriculum. Real life problems should be incorporated into sections whenever appropriate. Examples should be realistic and not contrived” (Department of Basic Education (DBE) 2011, p. 8). Modelling contexts need to come from various everyday life situations—these include “issues relating to health, social, economic, cultural, scientific, political and environmental issues where possible” (DBE 2011, p. 8). A dominant feature of the South African schools' mathematics curriculum is the high-stakes examinations, which in some way determine the discourses around how mathematics should be taught and/or learned in schools. Given that mathematical modelling is not highly prioritised in high-stakes examinations in the South African schools' mathematics curriculum, it seems that, anecdotally, teachers spend less time pursuing the modelling goals of the curriculum.

Evidence from research shows that, whilst there is consensus on mathematical modelling as a cyclical process and its general phases, the challenge is that it does not explicitly show students' work as they engage in the process (Albarracin et al. 2019; Czocher 2016). This view is shared by Blum (2002) and Pelesko et al. (2013) who posit that cognitive aspects from individual students' work during the modelling cycle should be the focus of further research. Niss (2013) contends that the real-world problems being solved do influence the way in which the mathematics modelling cycle is applied, and by implication the instruction that is intended to support students' work on the problem being investigated. This chapter is a response to these two calls. We investigated students' individual mathematical thinking when working of a real-world problem. Hence, the research question is: *What are the characteristics of students' mathematical thinking styles when working on a real-world problem on geometrical constructions.* We use the MAD framework (see Sect. 34.2 on theoretical framework), to characterise students' mathematical thinking styles when solving the modelling task, we show how the mathematical modelling cycle relates to the MAD framework.

34.2 Theoretical Framework

In this section, key elements of the theoretical framework on mathematical modelling are presented. First, the cyclical characteristic of the mathematical modelling process is presented. Second, the MAD framework categories and their respective descriptions are analysed, together with their linkage to the mathematical modelling process.

34.2.1 Mathematical Modelling Cycle

The mathematical modelling process, as a construct, is cyclical and can be represented by different modelling cycles, some of them also include representations of individual students' cognition and interpretation of the problem in context (Czocher 2017; Maaß et al. 2018). In addition, Borromeo Ferri (2010) posits that during the modelling process, students are often engaged in loops as they navigate between the real-world model and the mathematical model. In Fig. 34.1, we present the mathematical modelling cycle (Kaiser and Stender 2013) which outlines the knowledge and skills necessary to build modelling competencies as illustrated in words in italics.

The expressions written in the rectangles in Fig. 34.1 give an end-product of each of the five phases of the modelling cycle. A real-life situation constitutes a real-life practical problem which needs to be solved by collecting important information and writing down the necessary assumptions, thus converting it into a real model. An appropriate mathematical approach)—could be a formula or construction)—is used

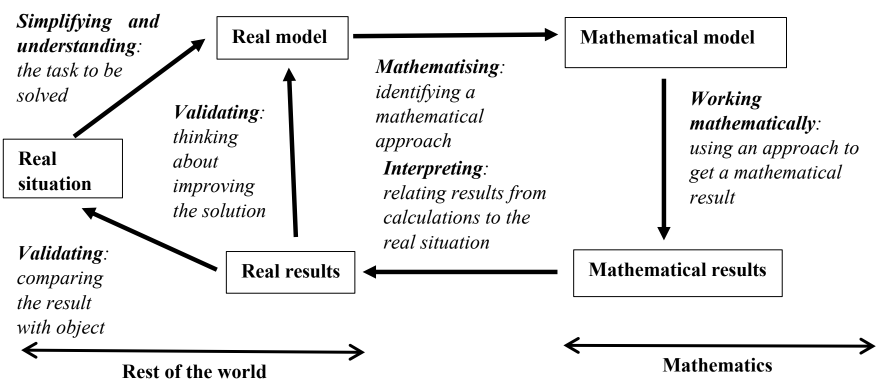


Fig. 34.1 Mathematical modelling cycle (Adapted from Kaiser & Stender, 2013, p. 279)

Table 34.1 Descriptions of MAD framework categories (Albarracín et al. 2019, p. 215)

Categories of MAD	Description
Reading	Unpacking the context and understanding it
Modelling	Transitioning from a real-world context to a mathematical interpretation of the task
Estimating	Making sense of the quantitative estimates in the problem in context
Calculating	Using simple mathematical concepts to calculate the missing information on the sketched diagrams or figures
Validating	Interpreting, verifying and validating the mathematical calculations from the model, and making sense and meaning of the calculations as they relate to the real-world problem
Writing	Giving a brief summary of the findings in a report, and how they relate to the original task or problem, and the processes leading to finding the solution to the problem

to translate a real model to a mathematical model. The chosen approach—(calculating and/or constructing)—are used to work out the mathematical results. Real results are obtained as students relate the mathematical results to the real situation through interpreting. In order to validate the real results, students can check if the real results fit the real situation, and check for the scope of improving the solution.

34.2.2 Modelling Activity Diagram Framework

While the mathematical modelling cycle contain the general phases of the modelling cycle, it does not sufficiently provide a framework for detailed analyses of the students' cognitive processes during modelling activities (Albarracín et al. 2019). Due to this limitation, we use the MAD framework (Ärlebäck and Bergsten 2013) to analyse the mathematical thinking styles of students engaged in a modelling activity (Borromeo Ferri 2010). Mathematical thinking style is a construct that was founded by Borromeo Ferri (2010), and it refers to how individual students use their mathematical abilities to solve the modelling task in ways which are unique to them. Table 34.1 below describes the categories of the MAD framework that we used to analyse the students' mathematical thinking styles on a modelling task on geometrical constructions. The six categorisations are presented in the order in which they are operationalised: reading, modelling, estimation, calculating, validating and writing.

34.3 Methodology

This study is part of a bigger project whose aim is to improve the teaching and learning of mathematics in secondary schools located in low socio-economic areas in South Africa, schools that are characterised by a lack of teaching and learning resources. Equally important is that the students from these schools have not been exposed to solving open-ended, unstructured and complex everyday real-world problems, which are the types of tasks envisaged in the mathematical modelling cycle. Teaching and learning in these schools are often textbook driven, with a limited use of everyday real-world contexts in the teaching of mathematics. This study uses the grounded theory as a qualitative research approach to unravel and obtain theoretically dense explanations, predictions, and interpretations of the students' mathematical thinking processes during the modelling process (Glaser and Strauss 2017). Using the MAD framework, the study characterised and analysed these mathematical thinking processes as students engaged and solved the modelling task.

A cohort of sixty-two Grade 11 students from two schools participated in this study, with thirty-one students from each school answering questions about the scaffolding modelling task. In addition, students were audio-recorded, and transcriptions were done to analyse students' utterances during the modelling activity, where they were asked to explain their solutions.

34.3.1 The Modelling Task and Data Collection

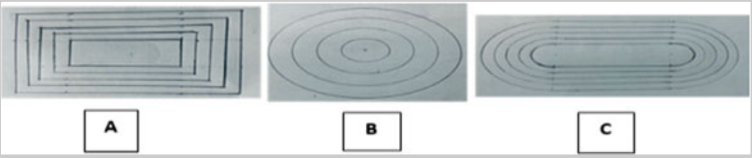
The modelling task: ***Design a four-lane 400-metre athletics track on a rectangular school ground with dimensions 180 m and 90 m***.

When the task was piloted at another school, we discovered that the Grade 11 students in that school experienced difficulties in understanding the task. Hence, we decided to scaffold the task by using two supporting questions. Question 1 sought to probe students' general knowledge about the shape of a 400-metre track, knowledge about South African athletes, why staggering in the lanes is important, and in which of the three proposed track designs the athletes can run fastest around the track. It was an attempt to have students scaffold around the modelling task by bringing the students into the context of the task. Question 1a) asked if students knew who Mr Wade van Niekerk was. He is a South Africa athlete and the current (2020) world record holder for the International Association of Athletics Federations (IAAF) athletics 400-metre track event. The second question required the students to calculate missing dimensions on the shape representing the inner track. This task is an example of a modelling-eliciting problem, where the key goal is the students' ability to find the required mathematical concepts to enable them to solve the problem task in context (Kang and Noh 2012; Ng 2013). In this task, students need to apply elementary mathematical concepts such as: perimeter of shapes—including circles and semi-circles, calculation of length of arcs given angles to name but a few. The task can be classified as a level-3 problem since it is open-ended, messy, unstructured, and complex (Kang and Noh 2012; Ng 2013).

Question 1

(a) Have you ever watched Mr Wade van Niekerk competing in a 400-metre track event? YES or NO (circle the correct answer).

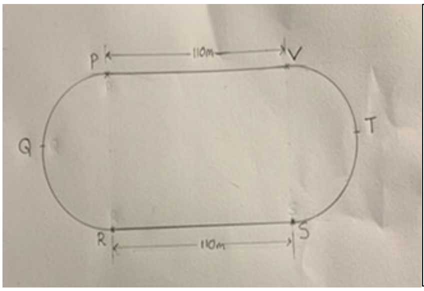
(b) Which of the following diagrams labelled A, B or C represent a shape of an athletic track? Explain in two lines.



(c) Explain in which of the three track diagrams above the athlete will be able to run fastest. Respond in two lines.

(d) Explain how four athletes running the 400 m in the first four lanes can be positioned at the start to ensure that each one of them runs 400 m when they reach the finish line.

Question 2 See Fig. 34.2.



(a) The length of the perimeter PQRSTV of a running track is 400 metres. The straight lengths PV and RS each measure 110 meters. PQR and STV are semicircles. Calculate the distance PR.

(b) Design a four-lane four hundred metres athletics track on a rectangular school ground with dimensions 180 metres and 90 metres.

Fig. 34.2 The scaffolded modelling task

34.4 Findings of the Study

In this section, the main findings of the study are presented. First, we give some descriptive analyses of how the students performed on the scaffolding activities of the task as shown in Fig. 34.2, including some examples of the students' narratives on their modelling process. Second, we discuss students' performance within the MAD categories.

34.4.1 Descriptive Analyses of Students' Work on the Modelling Tasks

The analyses below are based on three levels of achievements of the modelling task: students who "completed both questions 1 and 2"; students who "completed both questions 1 and 2(a)"; and students who "completed question 1 only".

Completed both questions 1 and 2: 42 students, of which only 14 students (36%) indicated that they had watched Mr Wade van Niekerk running 400 m. All 42 identified the correct figure C for the shape of the track. The notion of staggering athletes at the starting point was understood by 28 students (67%). 30 students (71%) correctly calculated the distance PR.

Completed questions 1 and 2(a): 13 students, of these five students (38%) acknowledged that they had watched Wade running, in addition to identifying alternative C as the correct shape of the tracking field. While 11 students (85%) acknowledged the importance of staggering athletes at the starting point, 10 (77%) had the correct calculation of the distance PR.

Completed question one only: 7 students, of which four (57%) students had watched Wade running; although 6 students (86%) acknowledge the importance of staggering athletes at the beginning of the race, they failed to calculate the distance PR, equating it with the length of the semi-circle PQR as 90 m.

Whilst the scaffolding of the task helped students to understand its context, only ten students of the sixty-two correctly sketched the four-lane track. The calculations of the exact staggering positions of athletes remained a challenge to all the students. One of the challenges students faced is that they were looking for a single approach to solve the task—instead of realising that there were multiple ways in which the tasks could be solved. This could have been due to the students' limited exposure to solving open-ended and unstructured problems. On communicating his/her solution, one student said, "After making my four lanes I will use a wheel metre to measure 400 m from the finish line clockwise in each lane until I get 400 m to get the starting positions of each of the athletes". This was a solution proposed by a student who realised that calculating the staggering positions was going to be difficult for him/her.

34.4.2 Using the MAD Framework to Characterise Students' Modelling Activities

The MAD framework is useful in characterising the students' mathematical thinking style within the phases of the modelling cycle. In Table 34.2, we characterise the

Table 34.2 Characterising students' mathematical thinking styles during the modelling proses using the MAD framework

Categories of MAD	Description of modelling activities of students from the problem in context
Reading	**Unpacking and understanding the task**: The scaffolding of the task led to the students thinking about the problem in context. Students were forced to think and focus on the problem context by answering the initial questions—on Mr van Niekerk, choosing the appropriate shape of the 400-metre athletics track, the notion of staggering the athletes at the start of the event, and which shape of the track field allows the athletes to run fastest
Modelling	**Transitioning from a real-world context to a mathematical interpretation of the task**: Students formulated a representation, in the form of a figure and/or diagram, of the problem in context by building a mathematical model that highlighted the shape of the track, how staggering can be achieved, and the width of the lanes of the 400-metre track. Few students managed to carry out this transition from real-world context to a mathematical model
Estimating	**Making sense of the quantitative estimates in the problem in context**: Students list the quantitative estimates required from the figure and/or diagram they sketched under the "modelling" category above. The students decide on which quantities are given and which ones need to be calculated—and with what accuracy. Using estimations, the students draw a rough sketch or representation of the 400-metre track. Students decide on the width of the track (standard line is 1.2 m)
Calculating	**Using simple mathematical concepts to calculate the missing information on the sketched diagrams or figures**: Students identify the concept of perimeter of a single lane of 400-m as a central concept. They calculate the perimeters of the semi-circles, ensuring that each lane length is 400 m from start to finish. Finally, they must calculate the exact starting positions in all four lanes – finding the final arc length and the corresponding angle leading to an accurate calculation of the actual staggering positions
Validating	**Interpreting, verifying and validating the mathematical calculations**: Students make sense of the mathematical results, including calculations, within the problem in context: *design a 400-metre four lane track*. Students are given opportunities to critique and compare their solutions with their peers, in addition to justifying their own solutions or designs
Writing	Students report on and communicate about their track designs, and thus have an opportunity to re-visit the mathematical modelling cycle activities

students' mathematical thinking styles visible during the modelling process within the MAD framework (Borromeo Ferri 2010).

From this characterisation of students' mathematical thinking styles, we observed that although students were able to find solutions to the scaffolded questions, they nonetheless experienced challenges with the formulation of the mathematical model. In other words, students had problems with identifying key mathematical concepts required during the mathematization process, and assumptions required to solve the modelling task, such as the width of the lane. For example, whilst students understood the notion of staggering athletes within lanes, often the students' mathematical styles did not lead to tangible solutions. As a result, only 16% of the students successfully completed the modelling task.

34.5 Conclusion

Reforming education through the introduction of new instructional approaches, such as, mathematical modelling, to classrooms environments dominated by the traditional teacher-centred and textbook driven practices can be challenging. In the context of this study, students were expected for the first time to solve open-ended, unstructured, and complex everyday real-world problems, which are the types of tasks envisaged in mathematical modelling. The findings of the study support the notion that the MAD framework was useful in characterising the students' mathematical thinking styles within the phases of the modelling cycle. As expected, students struggled to solve the open-ended task. In order to support teaching for understanding, the modelling task was scaffolded to bridge and narrow the gap between students' current approaches and the proposed modelling instructional approach. The scaffolding of the modelling task allowed students to familiarise and get a better understanding of the task. As a way forward on adopting the modelling as an instructional approach in the identified schools, further studies are required to focus on supporting teachers on how to implement mathematical modelling approaches in their teaching practices and enhancing the teachers' skills on designing open-ended tasks.

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Chapter 35
Sense-Making in Mathematics with Activities of Mathematical Modelling: The Case of Multiplication at Primary School

Simone Passarella

Abstract Mathematical modelling can be seen as simulations of problem-solving situations that, starting from realistic and rich contexts, favours understanding, reasoning and sense-making. In this study, we designed a model eliciting sequence with the aim of bringing out formal mathematical concepts from students, in order to help them give meaning to new mathematical knowledge and sense to their mathematical activity. The teaching case was conducted in a primary school class during regular mathematics lessons dealing with multiplication as iterated sum. The study supports the fact that the implementation of model eliciting activities can foster emergent modelling, i.e. the students' attitude to discover and (re-)create new mathematical concepts.

Keywords Model eliciting activity · Emergent modelling · Teaching case · Teaching practice at primary school · Artifacts · Realistic mathematics education

35.1 Introduction

Thinking mathematically can be seen as interpreting situations mathematically, in a close interaction between mathematical understanding and the understanding of the complexity and variety of the natural and social phenomena of contemporary world. In this direction, mathematical modelling represents a critical tool to understand the reality or society in general. Teaching students to interpret critically the communities they live in and to understand its codes and messages should be an important goal for education (Bonotto 2007), in order to give students not only mathematical competencies but also to prepare them to situations they will have to face in an increasingly complex world. However, students' reasoning and critical thinking are not favoured by the current school practice. Indeed, several studies have shown a discontinuity between mathematical competencies in and out of school. Mathematical problems

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turned in stereotyped exercises in the four basic operations solved through the application of mechanical procedures. Moreover, students seem to have established a set of rules of which include: (i) any problem is solvable and makes sense; (ii) there is a single, correct and precise (numerical) answer which must be obtained by performing one or more arithmetical operations with numbers given in the text; (iii) violations of personal knowledge about the everyday would may be ignored (Greer et al. 2007). The main consequences of this situation are an increasing gap between mathematics and real-world (Gravemeijer 1997) and a suspension of sense-making (Schoenfeld 1991) that do not favour mathematical modelling (Blum and Niss 1991).

On the contrary, according to the Realistic Mathematics Education (RME) perspective, a connection between mathematics and reality in order to improve students' critical thinking and reasoning should be fostered with activities based on realistic and rich contexts (Gravemeijer and Doorman 1999). The teaching of mathematics might be seen as a human activity of *guided reinvention* (Freudenthal 1991), in which students are active participants in the learning process, in a balance between students' freedom of invention and the power of teacher's guidance. Modelling is a powerful educational strategy to improve the teaching of mathematics in a guided reinvention approach, offering students opportunities to attach meaning to the mathematical constructs they develop while solving problems.

This study is part of a research project whose overall aim is to develop prototypes of practices available for teachers of primary and secondary schools. The project is divided in three main phases. The first phase consisted of a questionnaire that was administrated by the author to in-service primary and secondary mathematics teachers. The findings indicated that despite teachers implement regularly modelling activities, they ask for materials to deepen their preparation and practice. The second phase expected the implementation of several teaching experiments based on mathematical modelling. The third phase will consist in the development of professional teaching courses based on mathematical modelling. In this chapter, we will focus on the second part of the project. In particular, we will present a teaching case at primary school. The aim of the study is to help students give meaning to new mathematical knowledge and sense to their mathematical activity.

35.2 Theoretical Framework

Mathematical modelling is commonly identified with the process of structuring, generating real world facts and data, mathematizing, working mathematically, interpreting, validating and evaluating. Related to this definition of modelling, we considered the inter-connection between the *Model Eliciting Activity* (MEA) approach and the *emergent-modelling* one.

MEA consists in simulations of real-life problem-solving situations in which the central goal is to develop, test, revise and refine powerful, sharable and re-usable conceptual tools. It is evident that the process of modelling itself is at the core of MEA. In this perspective, in fact, students construct a model through repeated steps following the pattern: (i) expression of possible mathematical approaches; (ii) tests; (iii) revision based on their tests, discussions and interpretation of results (Leavitt and Ahn 2010). In order to implement model eliciting activities, Lesh et al. (2003) proposed a *model development sequence*, whose simplified version is made by three subsequent phases: *warm-up*; *model construction*; *presentation and discussion.* The warm-up phase involves activities whose aims are to introduce or test preliminary notions and to help students be confident with the context of the modeling activity. The central phase of the sequence is the model construction one, in which students, working in groups, create a model to solve a problem based on a real context. Each group of students makes a project in which the model they developed is explained and their findings are shared. Finally, in the presentation and discussion phase, students present in a whole class discussion their projects.

The starting point of a modelling activity is the choice of the real context to be modeled, but what do we mean with *real* context? One of the characterizing principles of RME is the *reality principle.* According to this principle, contexts for mathematical activities, and in particular for modelling ones, should be *realistic* and *rich.* A context is realistic if it is meaningful, sense-making for students. As a consequence, situations should come from the real world, but also from a fantasy world or from the mathematics itself, until they are experientially real for the student (Van den Heuvel-Panhuizen and Drijvers 2014). However, this realistic connotation is not sufficient to have a valuable mathematical problem. The context, indeed, must be also rich (Freudenthal 1991), i.e. poor in words and rich in mathematical concepts. An example is represented by cultural artifacts that thanks to their richness in mathematical meaning can stimulate students to connect mathematics and everyday contexts (Bonotto 2013).

A fundamental component of modelling is *mathematization*, in its double nature of *horizontal mathematization* and *vertical mathematization* (Treffers 1987; Freudenthal 1991). Horizontal mathematization refers to the movement from the real world and the world of symbols (mathematical objects, structures, methods), and viceversa. Students describe, translate a concrete situation in mathematical terms and use mathematical tools to solve real problems. Vertical mathematization, instead, refers to an internal movement in the world of mathematics. Here, students reflect on their own mathematical activity, recognize mathematical relations and work with them.

A nature of modelling that fosters vertical mathematization is *emergent modelling* (Gravemeijer 2007). Emergent modelling was introduced with the meaning of supporting the emergence of formal mathematical ways of knowing. It is a dynamic process from a *model of* students' situated informal mathematical strategies to a *model for* more formal mathematical reasoning. Students do not previously need at their disposal mathematical tools, instead the process of modelling becomes itself a way to develop new mathematical concepts and applications (Greer et al. 2007). As a consequence, the role of the model shifts during the learning process, from being situation-related to becoming more general. In conclusion, emergent modelling can be seen as a long-term process that favours understanding, reasoning and sense-making.

35.3 Research Question

The aims of this study are: (i) to foster the emergent nature of modelling; (ii) to enhance the understanding of some aspects of the multiplicative structure in a meaningful way; (iii) to provide teachers with activities on the educational strategy of modelling. In the specific, the research question we investigated was the following:

- How can emergent modelling be fostered to help students in understanding some aspects of the multiplicative structure?

In order to answer to our research question, we implemented a teaching experiment. In the specific, our hypothesis was that a modelling activity designed following a model eliciting sequence (Lesh et al., 2003) with the use of suitable artifacts could actually foster the emergent nature of modelling. In the following section, the phases of the model eliciting activity will be presented together with the materials developed for its implementation. As a consequence, the activity represents also a practical material that could be used and/or adapted by teachers in the future.

35.4 Teaching Case

The study was conducted in a second-grade class (age 7) composed by nineteen students during two weeks of regular mathematics lessons. The class had never been engaged in a modelling activity before the teaching experiment. At the time of the activity, students were working on multiplication in the set of natural numbers. In particular, multiplication as iterated sum was introduced by the official mathematics teacher one week before the teaching experiment. Students were able to perform basic multiplications between numbers with one digit. We decided to design a modelling sequence with the aim of enhancing the understanding of some aspects of the multiplicative structure. In the specific, the designed materials ought to foster students' re-invention of the distributivity property of multiplication over addition.

The research method for the data analysis was qualitative. The aim of the data analysis was to reconstruct the classroom progress, which resulted in an empirical grounded understanding of students' reasoning during the classroom activity. In order to be able to reconstruct the learning process and verify our hypothesis, different kinds of data were collected: transcriptions of classroom dialogs; observations of group working; students' final projects. In the next section, we will present some extracts that highlights students' reasoning during the activity and that permit to answer to our research question.

The modelling activity was implemented by the author with the cooperation of the regular mathematics teacher. The designed sequence involved the three model eliciting phases: warm up, model construction, presentation and discussion. At the time of the modelling activity, the school in which the teaching case took place was under building renovation. As a consequence, we decided to choose as real task the following *Tiling Problem*:

The Tiling Problem

The school director decided to renovate the school. Students can design a floor tiling of their own classroom. The floor of your classroom was divided in six equal strips. Each group of students should tile a strip, using all the available types of floor tiles.

The first part of the activity (two hours) was dedicated to the warm-up phase. The text of the task was given to each student together with: (i) the figure of the classroom divided in six stripes; (ii) the figure of each stripe to be tiled; (iii) a brochure with the shapes of the available tiles (triangular, square, rectangular) with the corresponding costs. Finally, the task was repeated in a clearer form (see Fig. 35.1). This brochure represented a cultural artifact that, thanks to its richness in (mathematical) meaning created a sort of hybrid space that connects mathematics and everyday contexts. During the warm-up phase, students, firstly individually and then in groups (of three or four), were asked to answer some questions dealing *The Tiling Problem*. Questions were about comprehension of the task and reasoning on the relations between different tiles and their cost (Fig. 35.2). The second phase (five hours) of the modelling activity consisted in the model construction. In this phase, each group created a poster in which they designed the floor tiling and explained the strategies followed to calculate its total cost. In the final phase of the activity (two hours), presentation and discussion, each group presented to the classroom their project explaining the steps followed to solve the task. Each member of the group had to take part to the presentation.

35.4.1 Results

In this section we report some results from the modelling activity. During the model construction phase, each group of students created a poster in which they designed the floor tiling and explained the strategies followed to calculate its total cost. In Fig. 35.3, there are some examples of students' group working and final posters. While solving the task, all the groups developed a similar strategy to obtain the total cost. The strategy consisted in two steps. The first one consisted in counting the number of all the tiles of the same type and multiply the number obtained with the relative cost. For example, one group counted fifty square tiles, twenty-six triangular tiles, and fifteen rectangular tiles. Then, the number of each type of tile was multiplied by its relative cost. In our example, students had to perform 50×6, 26×4, 15×10. This step highlights the notion of multiplication as iterated sum, already known by the students. While performing multiplications similar to the latter one, the groups

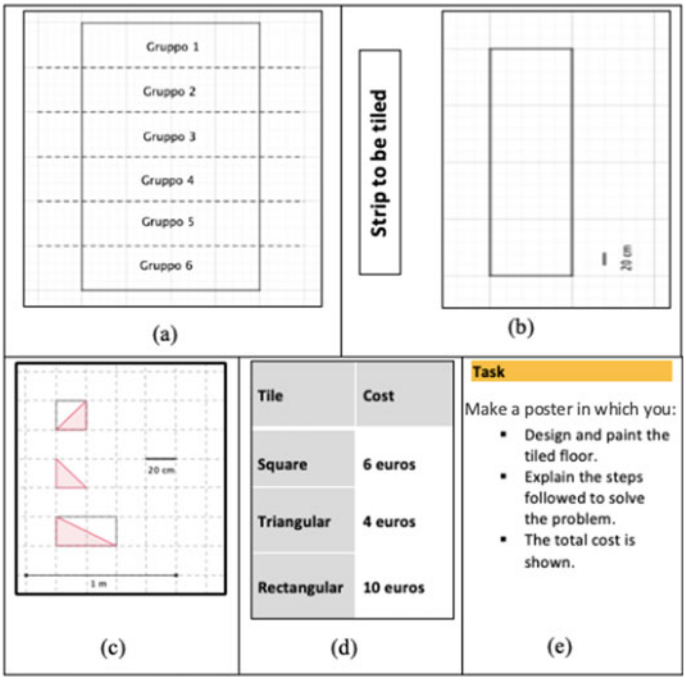
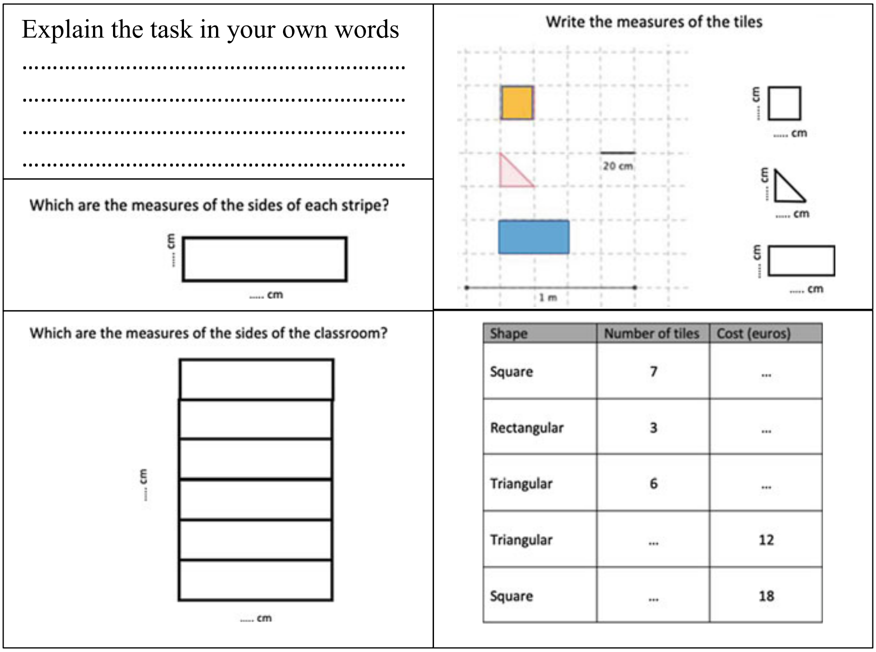


Fig. 35.1 **a** Classroom divided in six stripes **b** Stripe to be tiled **c** Available tiles **d** Cost of the tiles **e** Task given to students

encountered the problem of multiply a number with one digit and a number with two digits. Since in several groups, students were not able to find a way to solve this problem, the researcher decided to reason about it in a whole class discussion. Some students suggested the strategy reported in the following dialogue (R = researcher; S1 = first student; S2 = second student) to calculate 6×57:

S1: I write $6 \times 57 = 57 \times 6$.
Then I divide 57 as 50 and 7…
R: Divide?
S1: Write…?
R: Decompose.
S1: Yes, I decompose 57 as 50 plus 7!
Then I calculate 50×6.
S2: That is 300!
S1: Then 6×7
S2: 42
R: Excellent, and with these number? (pointing 300 and 42)
S1: I put them together!
R: How?
S1: I compose them…


Explain the task in your own words
……………………………………………………
……………………………………………………
……………………………………………………
……………………………………………………

Which are the measures of the sides of each stripe?
…… cm
…… cm

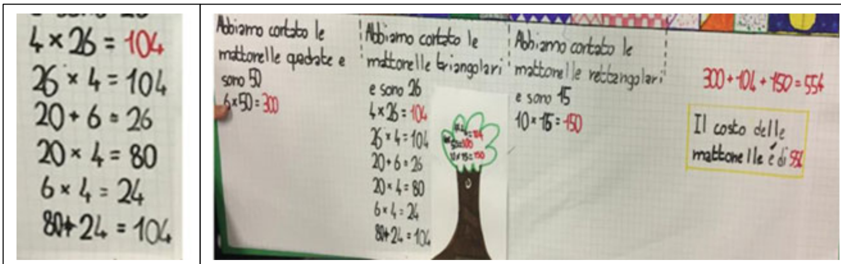
Which are the measures of the sides of the classroom?
…… cm
…… cm

Write the measures of the tiles
20 cm
1 m
…… cm
…… cm
…… cm
…… cm
…… cm
…… cm

Shape	Number of tiles	Cost (euros)
Square	7	…
Rectangular	3	…
Triangular	6	…
Triangular	…	12
Square	…	18

Fig. 35.2 Warm-up questions

Fig. 35.3 Students' group working and some final projects



(a) (b)

Fig. 35.4 **a** Students' strategy to calculate 26 × 4, **b** Calculation of the total cost

R: What does it mean?

S1: I make the sum!

After the discussion that included also other examples solved by students, each group applied the strategy suggested by their peers to perform their operations. In the group of our first example, students were able to calculate 26 × 4, as shown in Fig. 35.4a.

The second step to solve *The Tiling Problem* was to sum the costs of each shape of tiles. In our example, students having calculated 50 × 6 = 300, 26 × 4 = 104, 15 × 10 = 150, summed 300 + 104 + 150 = 504 that represented the total cost in euros of their tiling design (see Fig. 35.4b).

35.4.2 Discussion

In agreement with the process of emergent modelling, the assignment given to the students stimulated them to create and work with new mathematical concepts they did not know before. In the specific, the strategy developed by students to solve the task that consisted in grouping the tiles with the same shape and then multiply by the associated costs, showed that they were able to re-invent two fundamental mathematical concepts: (i) factorize by grouping; (ii) distributivity property of multiplication respect to addiction. The distributivity property is evident in the extract of the dialogue proposed in the previous section and in Fig. 35.4a, in which students, guided by the interaction with the teacher and peers, were able to reason and explain this property that would be at the base of their following strategies of calculus. In this way, properties of mathematical operations become meaningful for students, because no longer mechanical rules but rooted in their experience, directly constructed by students to solve a concrete problem in a meaningful context. This re-invention process was possible not only thanks to the designed sequence, but also to the use of a suitable artifact (brochure Fig. 3.1). Having given students the shapes of the tiles to be used and the constraint to use all of that shapes, guided them to face with the problem of performing multiplications between numbers with more than one digit, and consequently to the reformulation of the distributivity property of multiplication over addition. As a consequence, model eliciting activities together with suitable artifacts could foster the emergent nature of modelling that confirms our hypothesis. Moreover, the understanding of some aspects of the multiplicative structure in a meaningful way was enhanced. Therefore, answering to our research question, the integration of artifacts in a model eliciting sequence can actually foster emergent modelling.

The use of an artifact established a connection not only between mathematics and real world but also between various mathematical topics and other subjects (arithmetic, geometry, art). We remark that to achieve such results, the role of the teacher is fundamental. The teacher, indeed, has to encourage students to use their own methods; stimulate students to articulate and reflect on their personal beliefs, misconceptions and informal problem-solving and modelling strategies. Consequently, learning become a constructed understanding through a continuous interaction between teacher and students, that can be synthetized, using Freudenthal words, in teaching and learning as *guided reinvention*, reinforcing in this way mathematical reasoning and sense-making.

35.5 Conclusions

This chapter presented a teaching case designed following the phases of a model eliciting sequence, providing teachers with a practical material to be adapted in their classrooms.

The results show that complex modelling activities can be implemented also at the first grades of primary school. The study supports the fact that the implementation of model eliciting activities can foster emergent modelling, i.e. the students' attitude to discover and (re-)create new mathematical concepts and tools. This process was reinforced by the use of suitable cultural artifacts that represent realistic and rich contexts for modelling activities. Moreover, activities based on real contexts help students give meaning to new mathematical knowledge and give sense to their mathematical activity.

We believe that teachers professional development courses should be increased, in order to make teachers able to design and/or adapt valuable modelling activities in their daily school practice, enhancing students' reasoning and sense-making.

For the future, we will focus on the development of teachers professional development courses based on modelling. In the specific, we would work with teachers to make them able to recognize cultural artifacts starting from the needs and interests of the students present in their classrooms.

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Chapter 36
Modelling the Fundamental Theorem of Calculus Using Scientific Inquiry

Andrzej Sokolowski

Abstract This study provides an example of a mathematical modelling activity that utilizes scientific reasoning to support the learning of mathematics concepts. The method of merging mathematics concepts with scientific reasoning was developed using research on and recommendations about designing effective exploratory STEM modelling activities. A calculus topic, the Fundamental Theorem of Calculus (FTC), was converted into a modelling activity. The FTC was selected due to its significance in integral calculus and calls for designing activities that would develop students' covariational reasoning crucial to understand the theorem. A group of 21 high school students participated in this study. The students' responses showed positive effects of this activity in understanding the FTC. Suggestions for further studies conclude this paper.

Keywords Modeling · Inquiry · The first fundamental theorem of calculus · Accumulation · Change of function value · Covariational reasoning

36.1 Introduction

Mathematical modelling serves traditionally two main purposes: to solve a particular problem or to develop individuals' modelling skills (Stillman et al. 2013) and as such it generates positive learning effects (e.g. Sokolowski 2015). This study sought to expand the purposes of modelling and aimed to design an activity to merge mathematical modelling with scientific reasoning. Many researchers (e.g. Honey et al. 2014) are concerned that mathematics is underrepresented in the STEM paradigm. Hämäläinen et al. (2014) posited that mathematics concepts should be considered as processes of embracing mathematical structures to theoretical knowledge and empirical observations. Given that students' familiarities with mathematical modelling are one of the priority skills to succeed in STEM university programs (Deeken et al. 2019), it is seen that the role of mathematics in STEM can increase through

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mathematical modelling. Such intertwining can also serve as a way of improving students' mathematical reasoning in science courses (Sokolowski 2019b). Inquiry as a learning method in mathematics has been widely discussed (Honey et al. 2014). While learning via multifaceted explorations plays a vital role in developing students' skills in science and mathematics (Crouch and Haines 2004; Pollak 2015), verifying how scientific methods can support modelling in mathematics has been rarely undertaken. It is believed that merging mathematical modelling with scientific methods will support not only understanding of mathematics concepts but also mathematical reasoning whose importance is gaining momentum in contemporary science practices (Pospiech et al. 2019).

36.2 The FTC; Teaching Challenges and Research Recommendations

The FTC was developed independently by Newton and Leibniz in the sixteenth century (Boyer 1959) and has been seen as a hallmark of differential and integral calculus. Sobczyk (2013) suggested that the geometric form of the FTC provides a significant improvement and generalization over other types used in teaching this theorem. Tall and Bakar (1992) found out that conceptual understanding of calculus theorems is much more profound if students are given opportunities to explore its ideas. He advocated for designing activities to have students re-constract the calculus concepts. Connally et al. (2019) suggested three rules for effective calculus teaching: graphical, numerical and analytical. These rules served as platforms through which calculus students were supposed to explore the concept to gain understanding. Carson et al. (2003) highlighted understanding of the notation of the FTC as equally important as its conceptual counterpart. Blum (2011) pointed out that many students' concept image of integrals is related to area interpretation which, according to Blum diminishes the idea of integral function. Thompson (1994) introduced covariational reasoning as critical mental action to understand the virtue of the FTC. Covariational reasoning is about coordinating the buildup of accumulation under the graph of $f'(x)$ with a change of $f(x)$. It requires coordination of images of two varying quantities while attending to determine how they change in relation to each other (Carlson et al. 2002). Thompson (1992) found out that the development of images of the rate of change starts with learners' image of change in some quantity (e.g. object's displacement) and progresses to a loosely coordinated image of two quantities (e.g. change of velocity) which progresses to an image of the covariation of these quantities. The idea of motion and covariation of the area under the velocity graph and object's displacement will be a backbone of the activity. Among learning theorems, active learning providing discovery approach (Cummins 1960) has been suggested as the most effective in teaching calculus.

36.3 The Theoretical Framework of the Activity Design

Scientific inquiry is central to developing students' scientific reasoning skills (Prince and Felder 2006). Mathematical modelling provides a framework that allows for inducing consistent quantification methods of scientific investigations. Therefore, scientific inquiry methods merged with mathematical modelling deem to be a prominent learning method. The fundamental elements of scientific inquiry are hypotheses, nature of the inquiry, type of elicited model and the verification stage. How are scientific methods linked with mathematical modelling in the current research? The literature does not provide many examples of such intertwining. The hypothesis is defined as the investigator's proposed theory explaining why something happens based on the learner's prior knowledge (Felder and Brent 2004). Its cognitive purpose is to confirm or correct an investigator's content understanding. Dunbar (2019) claimed that the hypotheses translate into the mathematical structure that becomes the heart of the mathematical model. Lim et al. (2009) found out that hypotheses were often formulated qualitatively and focused on testing isolated mathematical concepts. Klymchuk et al. 2008 noted that students fail to validate formulated mathematical structures or have difficulty with a contextual interpretation of the derived model. Scientific methods can be supported by inductive or deductive inquiry (Prince and Felder 2006). The inductive inquiry that is about reasoning from specific observations to reaching a general conclusion has been proven to play a prominent role in concept learning and the development of mathematical expertise (Haverty et al. 2000). Because exploratory mathematical modelling activities are defined as pattern formulation and generalization (Lesh and Harel 2003), inductive inquiry seemed to be the right choice to guide such practices. Figure 36.1 depicts an outline of the laboratory design. It was inspired by an earlier study (Sokolowski 2018) and attempted to provide opportunities to improve conceptual understanding of mathematical concepts while attending to scientific contexts.

This framework was applied to develop the laboratory instructional support. Michelsen (2006) advocated replacing the current monodisciplinary approaches with an interdisciplinary one, where mathematics and science are woven continuously together. This modelling scheme can be considered a proposal for such an interdisciplinary approach.

36.4 Methods

This study can be classified as one-group quasi-experimental (Shadish et al. 2002). Quasi-experimental research shares a range of similarities with an experimental design. The following question guided the study: *Can an activity that merges mathematical modelling with scientific inquiry support conceptual understanding of the FTC*? A group of 21 high school calculus students (12 males and 9 females, age range 17–18 years) from a suburban high school participated in this study. The students

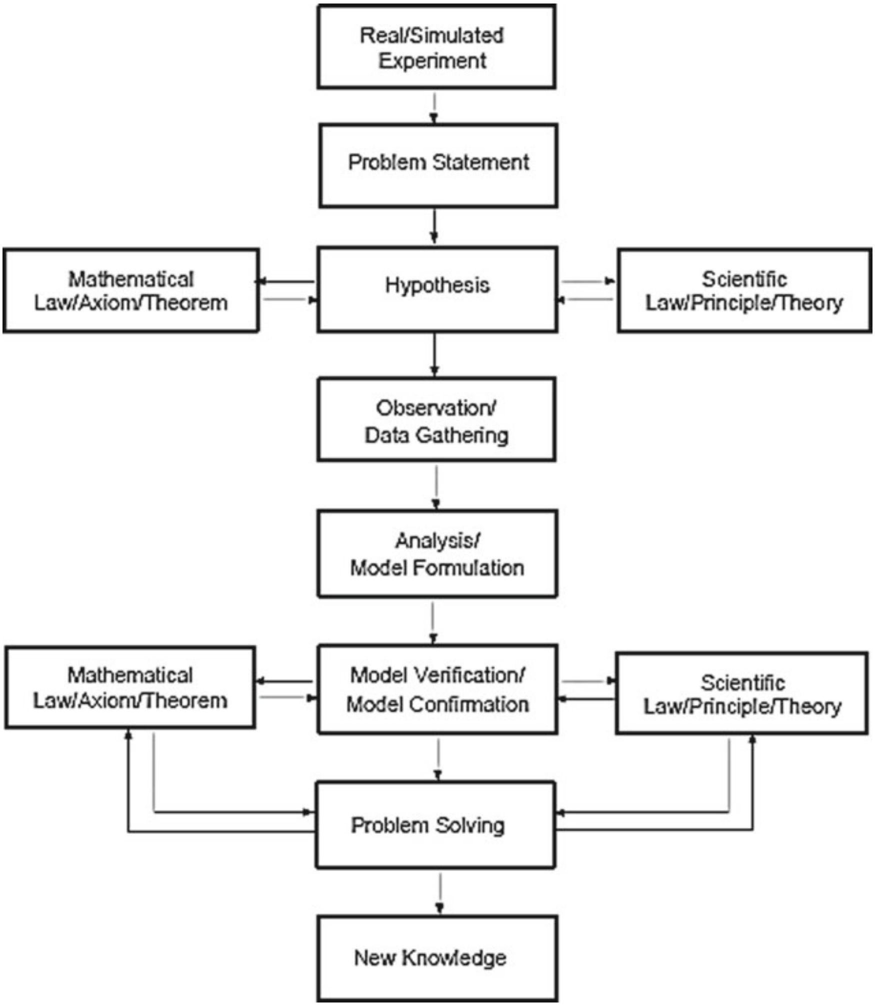

Fig. 36.1 Merging mathematical modelling with scientific inquiry

were introduced to the techniques of evaluation of integrals and the interpretation of accumulation. Yet, the formal definition of the FTC was not presented. The students took a pre-test, and after the laboratory conduct, they took the post-test. The quality of the verbal responses served as an indicator of their understanding.

36.5 Conduct of the Instructional Unit

This section contains a summary of the critical elements of the laboratory design and conduct. The scientific background was supplied by a physics simulation designed by Interactive Simulation Project PhET that is downloadable for free from http://phet.colorado.edu/en/simulation/moving-man.

36.5.1 Mathematical Knowledge and Scientific Context

The activity was designed to help students develop a conceptual understanding of the FTC of the form $\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$ where F is an antiderivative of f, that is, a function such that $F' = f$ (Stewart 2012). FTC has a vast range of applications in calculus and science. To focus students' attention on developing covariational reasoning, a cognitive load of computing the accumulation under the derivative graph was reduced to applying geometry formulas. The interactive sim called *Moving Man* was selected because of its high capability of displaying the relations between the movement of the man and the corresponding position $x(t)$, velocity $v(t)$, and acceleration $a(t)$ functions. This capability enhanced further the covariational reasoning and allowed for linking the sides of the FTC using two independent covariate methods—areas under $v(t)$ and a difference in the values of the antiderivative, $x(t)$. Reasoning about the Fundamental Theorem of Calculus involves mental actions of coordinating the accumulation of rate-of-change with the change antiderivative (Thompson 1994). By observing simultaneously the man's movement $x(t)$ and $v(t)$ graphs, students were to connect the scientific underpinnings of the phenomena with its mathematical representations. This linkage was to guide the students to discover that the accumulation under $f'(x)$ is equal to the change of values of $f(x)$. Before initiating the laboratory conduct, the students were supposed to state their hypotheses. Dual mathematical and scientific nature of the hypothesis that students were to formulate and the duality of the verification phase of the activity intended to support the intertwining. The culminating stage of the activity—discovering equity between the man's change of position using two different methods interrelated the left and right side of the algebraic forms that resulted in reconstructing the FTC. Further deployment of the discovered pattern in problem solving extended the theorem's applicability.

36.5.2 Details of the *Laboratory* Conduct

Students were provided with instructional support that contained snapshots of the simulation along with the problem statement and tasks to follow. The instructor opened the sim and entered motion parameters—precisely, -10 m for the initial

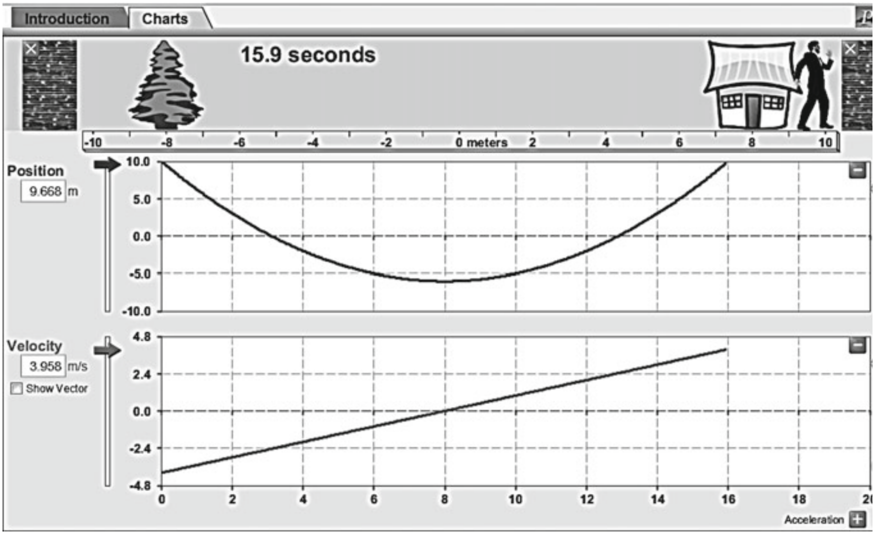

Fig. 36.2 Snapshot of the sim with position and velocity graphs. *Source* PhET Interactive Simulations (n.d.)

position, 4 m/s for the initial velocity and −0.5 m/s^2 for the acceleration—and let the man walk while simultaneously displaying his position and velocity-time graphs (see Fig. 36.2). The man's movement was restricted—by the simulation design—to his initial and final locations, which were −10 m and 10 m; thus, the motion, due to these restrictions was stopped after 16 s.

The students were to hypothesise the answer to the following question: Can the accumulation under the velocity function be used to compute the change of the man's position? Students were to support their answers using calculus and science knowledge. The sim displayed the primary function and its derivatives, but it did not display the area under the graphs explicitly. Thus, the students were to apply their prior knowledge about accumulation to formulate their hypotheses. They gathered the data and proved/disproved their hypotheses. More specifically, they computed the area under the velocity graph, and calculated change of the man's position using the position graph. Separate screenshots of enlarged graphs of the velocity and position functions allowed for taking precise data. Students were supposed to calculate:

$$\text{a)} \int_0^4 v(t)\mathrm{d}t = \_\_\_\_\_\_\_\_ \quad \text{b)} \int_4^8 v(t)\mathrm{d}t = \_\_\_\_\_\_\_\_ \quad \text{c)} \int_0^{16} v(t)\mathrm{d}t = \_\_\_\_\_\_\_\_$$

These computations constituted the left side of the FTC. The following differences represented the right side of the FTC.

$$a)x(4) - x(0) = \_\_\_\_\_\_ \quad b)x(8) - x(4) = \_\_\_ \quad c)x(16) - x(0) = \_\_\_\_\_$$

Students were to pay attention to the units of calculated quantities. This nuance was explicitly highlighted because the equity of the magnitudes, coupled with the correlation of the physical units were to support the virtue of the FTC. Once the data was gathered and calculations performed, students were guided to search for patterns. They concluded that the computations of areas and respective change of the antiderivatives were the same. Some were prompted to correct their computations, especially when the accumulations were negative. Finally, considering their findings, the students were to link the sides of the computed quantities (e.g. a with a, b with b and c with c) by selecting the correct sign either > , < , or = .

a) $\int_0^4 v(t)\mathrm{d}t \_\_\_ x(4) - x(0)$ b) $\int_4^8 v(t)\mathrm{d}t \_\_\_ x(8) - x(4)$ c) $\int_0^{16} v(t)\mathrm{d}t \_\_\_ x(16) - x(0)$

Students were to realize that not only the numerical values of the computed expressions were supposed to be equal but also that their physical units—metres—corresponded. Thus, both the mathematical theorem and scientific laws were to lead them to formulate the FTC. The experimental counterpart enhanced the meaning of the discovery and provided prompts for its validity. To extend the applicability of the discovery, the students gathered data and formulated similar inferences using velocity and acceleration graphs. The instructor summarized their findings and concluded that by establishing equities between both methods of computations: accumulation under the function derivative and change of function values, they discovered and proved the validity of the FTC. They further applied the ideas in other contexts related to science and engineering.

36.6 Data Analysis

The activity was well-received by the students. It seemed that the friendly and straightforward scientific environment enhanced the concept of understanding, making the abstract calculus structure more tangible. The following day, the students were supposed to explain the meaning of the FTC using their own words. Table 36.1 includes selected correct (marked with *) and incorrect pre-test and post-test students' responses. Response was marked correct if it contained the terms accumulation, derivative, antiderivative and their logical linkage that would reflect the interpretation of the FTC.

Majority of the students (76%, $N = 16$) correctly explained the meaning of the FTC on the post-test as opposed to (33%, $N = 7$) on the pre-test. The pre-test showed that many students associated the algebraic statement of the theorem, referring only to the accumulation of the derivative. For example "The left side gives you the accumulation, the right side explicitly solves for the accumulation". This statement indicated that the student just explained the procedure, not the concept. After the laboratory, the students have changed their perceptions. They more often referred to the right side of the theorem as the values of the antiderivative. It is seen that

Table 36.1 Selected students' pre-test–post-test descriptions of the FTC

Pre-test	Post-test
The fundamental theorem is utilized in calculus to find the derivative and its relations It helps you to find the rate of change and average values in relation to its derivative The left side gives you the accumulation. The right side explicitly solves for the accumulation The main idea is applying the derivative to solve the problems or taking the antiderivative *The integral of a function from point a to point b is equal to the rate of change of the function antiderivative *The accumulation of a function will be the same as $f'(b) - f'(a)$ It is an easier way to calculate rates of change It allows me to understand the idea of the rate of change and its relation to its derivative The left side is the formal accumulation which is equal to the process of accumulation There are two different ways of calculating the same number	The antiderivative of the derivative equals the original function Finding antiderivative of value will give you the displacement of the antiderivative *The accumulation of the derivative $f'(x)$ represents the value of the $f(x)$ *The change of $F(x)$ might be obtained by the integral of $f(x)$ The purpose is to access all of the info from $x(t)$, $v(t)$ and $a(t)$ while starting with data from a single function *The antiderivative from a to b of some function is equal to the integral of this function from a to b The relations between integrals of $v(t)$ and $a(t)$ *Accumulation of the derivative is equal to the function values of the antiderivative *The accumulation is equal to the change of the antiderivative The area comes out to equal the function solutions

observing the construction of two different graphs $x(t)$ and $v(t)$ helped students realize that both sides of the FTC referred to two different algebraic structures that produced the same answer. Most of the students who did not verbalize the theorem had difficulties conceptualizing the accumulation under the derivative graph and were confused about distinguishing between the conventional meanings of $F(x)$ vs its lower $f(x)$ notation. Some students explained the theorem strictly referring to the contexts of the lab. It was to note that the students did not use the term *change of antiderivative values* often when interpreting the right side of the theorem. This imprecision though did not invalidate the answers.

36.7 Discussion

While the results are encouraging, room for improvement exists. Further studies on how students would transfer the meaning of the theorem to solve traditional textbook problems or examination assessment items would shed more light on the learning effects of the activity. The activity served as an example of a lesson that attempted to link scientific methods with mathematical reasoning to enhance the understanding of FTC. The goal of this activity was also to have students realize that science and mathematics are not isolated disciplines and that the theorems of mathematics can support the laws of motion used in science. While science provided the context, mathematics provided concise ways of quantifications. Both views merged, proving that both disciplines are unified. Blum (2009) pointed out that many students' concept image of the integral function is weak and called for more attention to develop such a notion. While the goal of this activity was to develop a conceptual understanding of the FTC, it is seen that it could also address this concern. Such an extension would broaden the interpretation of the FTC and more meaningfully relate both its sides. This could be exercised after having students realize that $\int_{t_1}^{t_2} v(t)\mathrm{d}t = x(t_1) - x(t_2)$. Changing the limits of integration to $\int_0^t v(t)\mathrm{d}t = x(t) - x(0)$ and solving for $x(t) = x(0) + \int_0^t v(t)\mathrm{d}x$ generate an integral function that could be used to link the properties of derivative and antiderivative further. While this lab focused on conceptualizing an advanced calculus topic, the theoretical framework of the design can be extended to lower mathematics courses, for instance, to explore the slope conceptualization (Sokolowski 2019a) or to discover limitations of function transformations. Exploratory mathematical modelling activities have the potential to solidify their role in STEM education. They provide meaningful and tangible interpretations that enhance understandings of abstract mathematics theorems and laws. It is believed that such actions not only integrate mathematics conceptual and procedural knowledge but also enhance students' scientific skills.

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Chapter 37
Mathematical Modelling Activities Within a Context-based Approach in Thai Classrooms

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Abstract In this chapter, mathematical modelling activities were designed by integrating the relevant processes following a context-based approach. The activities emphasised authentic situations that were closely related to the real life of ninth grade students to encourage them to integrate mathematical knowledge, skills, and processes in the creation of mathematical models to understand and solve problems. During the activities, most of the students engaged in mathematical modelling processes with their friends. This allowed them to use and practically connect mathematics with real situations and problems encountered during their daily lives.

Keywords Mathematical modelling processes · Context-based approach · Mathematical modelling activities · Students' real-life problems · Connect mathematics · Secondary student

37.1 Introduction

Knowledge of mathematics allows students to apply their skills to solve extracurricular real-life problems, especially those closely related to real-life daily activities. About half of 15-year-old Thai students did not attain the international basic proficiency level at mathematics in PISA 2009 and PISA 2012. They also had little or no experience in applying their mathematical knowledge and skills to solving extracurricular problems (Klainin 2015). Therefore, mathematical activities that enhance students' mathematical knowledge and skills to solve class-based problems require enhanced development.

In several countries, mathematical modelling is an essential educational topic that enables students to deal with real-world problems. This supports mathematical

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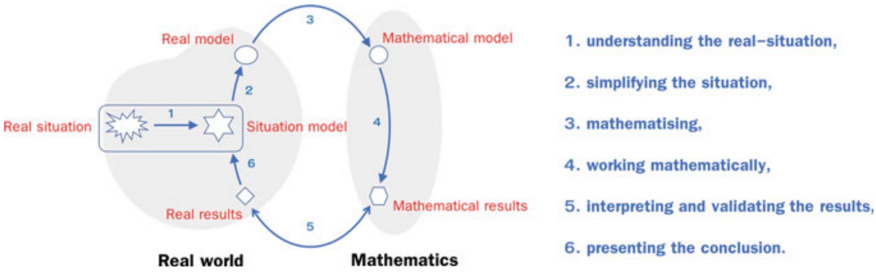

Fig. 37.1 Modelling process adapted from Blum and Borromeo Ferri (2009)

learning (Blum and Borromeo Ferri 2009). However, in Thailand, although the ability to deal with real-world problems has been recognised, both students and teachers have little experience in the context of mathematical modelling.

Hence, we were interested in designing activities to make mathematical modelling accessible to teachers and students in Thailand. Many situations that are relevant to daily life provide interesting issues that can be used to encourage students to connect classroom and extracurricular mathematics. Descriptions of relevant activities that can be subjected to mathematical modelling processes and students' responses are presented in this chapter.

37.2 Theoretical Framework

37.2.1 Mathematical Modelling Processes

We were particularly interested in the mathematical modelling cycle described by Blum and Borromeo Ferri (2009). They developed an ideal modelling process that includes seven phases or steps allowing cognitive activities to solve the modelling tasks. In this chapter, we adapted these phases into six design activity processes as shown in Fig. 37.1.

Figure 37.1 combines the interpretation and validation of the results that support students to relate everyday real-life situations with mathematical concepts. Moreover, students presented their results based on the assumptions of real-world situations and the situation model.

37.2.2 Context-based Approach

When designing the activities, we adapted the idea of using real-world situations as a context-based approach to provide opportunities to connect mathematics with daily life following Wijaya et al. (2014). This idea related to the development of modelling tasks by Borromeo Ferri (2018). Contexts or situations in each modelling activity must be closely related to students' experiences, as well as connect to mathematical knowledge that they have assimilated to deal with the problems and activity processes. Moreover, the context should encourage and engage students to integrate mathematical knowledge, skills, and processes to think outside of the box and create mathematical models to understand and solve real-life problems (Beswick 2011; Sullivan et al. 2003). We restricted real-world situations to interesting and accessible problems that were closely related to students' experiences.

37.3 Design of the Study

The purpose of this study was to describe the activities and record the students' responses. Two questions were posited as (1) the students' responses for each mathematical modelling process and (2) similarities and differences in the students' responses for each activity.

37.3.1 Participants

The study participants were divided into two groups. The first consisted of three mathematics teachers and one expert mathematics educator who had more than 15 years of experience teaching in Chiang Mai, while the second comprised 30 students in the ninth grade who had no previous experience of mathematical modelling in the classroom based on the Basic Education Core Curriculum in Thailand.

37.3.2 Designing the Activities

The goal was to encourage the students to think and create models to understand and solve real-life problems through mathematical modelling processes. We adapted the ideas of a context-based development of modelling tasks from Borromeo Ferri (2018) as follows: (1) The situations must closely relate to students' experiences as well as connect to mathematical knowledge that the students have learned to use within their learning processes; (2) Each activity should include further material (such as calculators) or require actual experiments to support the students to model the problem.

The activities were inspired by three real-life local problems that occur in Chiang Mai, Thailand and closely relate to students' experiences. We acquired information concerning students' experiences and previous knowledge relevant to the activities by conducting interviews with teachers and students to confirm that the local problems of each activity both encouraged and engaged students. After that, we developed the activities and verified their correctness, opportuneness and relevance with the assistance of three mathematics teachers and one expert mathematics educator. We gave them hands-on data as lesson plans and teaching materials to consider the three selected activities. Their opinions concerned time management, questioning and guiding the students during the modelling processes since the students had no previous experience of mathematical modelling techniques.

The first activity, *the street market problem*, considered the relationship between the sizes and numbers of booths that can reasonably be arranged in a street market context when considering the size of the street. This activity was closely related to students' lives because street markets are central locations where students often walk and shop with their parents or friends. Besides, schools regularly organise exhibitions as walking streets to display students' work, products, and handicrafts. Most of the students assist their teachers to set up and manage these exhibitions.

The Street Market Problem

In Chiang Mai, a large Sunday street market is located in the centre of the old walled city area and close to your school. The products on sale are mainly handicrafts that are made in and around Chiang Mai. You and your team members are given the relevant materials. Draw your own conclusions and answer the following questions:

1. What is the appropriate size of a standard booth used for selling and displaying products at the street market?
2. What is the relationship between the size and the number of booths that can be reasonably arranged in the street and what is the size of the street?

The second activity, *the dust particle problem*, concerned air pollution and was closely and related to the daily lives of students. The dust particle problem was a continuous occurrence in the province.

The Dust Particle Problem

In Thailand, Chiang Mai is one of many provinces struggling with the dust particle problem. Chiang Mai's PM2.5 annual average is well above the safety limit set by the World Health Organisation. You and your team members are given the relevant materials. Draw your own conclusions and present answers to the following questions:

1. What does "PM 2.5" mean?
2. What is the relationship between the size of the dust particle and visible real objects? (e.g., hair, pencil lead, table tennis ball, and beads).
3. When does the dust particle problem usually occur in Chiang Mai, Thailand? Why does this usually happen at the same time each year?

Fig. 37.2 **a** Stone table and four chairs and **b** Seating arrangements modified by students

The third activity, *the bench problem,* involved benches that students use for relaxing with their friends. Regular benches in the school were constructed from lightweight stone that two students could easily lift and move to another position.

The Bench Problem

In schools and some public parks, lightweight stone tables are often provided for relaxation and other activities. Generally, the stone tables have four chairs, as shown in Fig. 37.2a. Students arrange the stone tables depending on the number of friends who want to sit together, as depicted in Fig. 37.2b.

You and your team members are given square tiles (tables), bottle caps (chairs) and writing materials. Draw your own conclusions and answer the following questions:

1. What is the optimal appropriate style of arranging the garden stone tables and chairs to allow more people to sit together?
2. How many garden stone tables and chairs should you arrange for 12 people, 15 people and over 15 people to sit together?
3. What is the relationship between the number of tables and chairs in your arranged seating plans and unused chairs in the general set up?

37.3.3 Experimenting the Activities in the Classroom

A qualitative approach was adopted in the form of a case study. Results of the three experiments were analysed for students' responses to each step of the activities. Each experiment lasted for 90 min and was conducted in a classroom with 30 students who did not have previous experience of mathematical modelling. The students were organised into six groups of five students with wide-ranging abilities in mathematics and varied social backgrounds. The teacher and the researchers implemented the activities. The teacher's role was as a facilitator, encouraging and guiding a small group of students to deal with problems and activity processes. Responses were gathered through observations, written work, and interviews with the students.

37.4 Results of the Study

The responses of the students showed that most of them fully engaged in the mathematical modelling processes with their friends under the guidance and support of the teacher. Gradually, they learned how to use and connect mathematics with local problems. Details of students' responses that related to the mathematical modelling processes in each of the three activities were as follows.

37.4.1 Students' Responses to the Street Market Problem

Initially, the teacher showed pictures and discussed the student's experiences at the street market. All of the students participated in the discussion. They stated that they used to go there to walk and shop with their parents or friends. The teacher then posed questions and the students first considered and then tried to *understand the real-life situation* with their friends. Some groups of students discussed with their friends about the school's exhibition that was regularly organised along similar lines to the walking street to show students' work, products and handicrafts. Then, they were asked to make assumptions about the shape of the booths and the size of the street to *simplifying the situation.* The students were given paper, a ruler, a measuring tape, and writing materials to measure the length and width of the street and the approximate distance of the street from the school. Then, the teacher gave the students time to think about the appropriate mathematical knowledge and *mathematising* that related to solve the problem.

After that, the students *worked mathematically* based on the assumptions. To answer the first question, most of the students used a ruler and measuring tape to determine and discuss the appropriate size of a rectangular booth. Each group of students settled for a similar size of booth as 1.5×1.5 m^2 and 2×2 m^2 and then they considered that the appropriate size of the booth depended on the length of the street and related to the number of booths along the length of the street. To answer the next question, the teacher guided the students to use the appropriate size of a rectangular booth to find the number of booths that could be reasonably arranged in a 10 m length of the street, a 50 m length of the street and in general. The students concluded that the relationship between the length of the booth denoted as B, the number of booths denoted as nB, and the length of the street denoted as S consisted of two cases. (1) If the booths were arranged on both sides of the street, in general terms the relationship could be written as $nB = 2 \times (S/B)$ and (2) If the booths were arranged on only one side of the street, then in general terms the relationship could be written as $nB = (S/B)$.

The teacher then asked the students to *interpret and check* the mathematical results. The students reviewed the appropriate size of a booth. They used measuring tape and their experience to confirm the size of the booth. The size of a standard rectangular booth should be limited to $2 \times 2\ \text{m}^2$ by the management teams of the market. Moreover, they found that the relationship between the length of the booth, the number of booths, and the length of the street was not feasible in general conditions, particularly the width of the street. The space in the middle of the street was appropriate for arranging booths on both sides of the street. In this case, they added the necessary condition as the width of the street. This must be more than 1.5 m and double the width of the booths.

In the final step, the teacher allowed the students to prepare for *presenting and sharing their ideas with the whole class*. One group of students used the ratio of the length of the paper to the corresponding distance on the street (scale map) and size of the booths. This showed that the students not only applied mathematical knowledge to solve the problem but also to model their working idea. The students determined that the lengths of the street and the booths were related. For example, the length of the street should be divisible by the length of the booths and the length of the booths was a factor of the length of the street. These students tried to connect their mathematical knowledge using whole numbers to determine if the other answers were reasonable.

37.4.2 Students' Responses to the Dust Particle Problem

During the first step, students were shown a news clip explaining the dust particle problem in Chiang Mai. The teacher then discussed the meanings of some essential vocabulary in the video and the cause and effect of the dust particle problem. Questions were posed to the whole class to ensure that they *understood the real-world situation*. Next, the teacher presented model pictures of dust particles. The shape of the dust particles was regarded as spherical to *simplify the situation* and allow assumptions to be made when solving the problem (e.g. shape, size and the unit for measuring dust particles). Then, the students used the appropriate mathematical knowledge, *mathematising*, to answer the questions.

The students *worked mathematically*. For the first question, most students were able to explain the meaning of PM 2.5 and they recognised how to represent vary small numbers in exponential and scientific notation that referred to the size of the dust particles. However, all the students were confused about how to use equivalent ratios to compare the size of dust particles to the size of visible objects in the second question. In this case, the teacher guided the students to review how to compare two numbers as a ratio by discussing with friends. Then, the teacher presented additional examples to the whole class to recall the concepts of ratio, equivalent ratios and proportions. After reviewing each concept, the groups of students were able to compare and identify relationships between the size of dust particles and visible objects. They produced similar answers based on approximating the size of the objects. To answer the third question, the teacher presented real data concerning the air quality index and the amount of dust particles in the atmosphere from Thailand's Pollution Control Department. The data sheets contained real secondary data sets without cleaning. The students had to detect the irregular units of measurement presenting the number of dust particles. Then, they converted all the irregular units to the same unit before analysing and interpreting the data to predict the periods when dust particle problems occurred.

The teacher instructed the students to *interpret and check* all the results and then mathematically review some parts of the workings. Some students used the size of real objects to ensure that their results in the second question were reasonable. They also used their smartphones to search for information about dust particle problems on the Internet to support their possible results. In the final step, the teacher asked the students to *present* how they solved the problem and their conclusions. Reflecting on this activity, one group of students stated that they initially used only dots to represent the amount of dust particle data. Later, they revised their data as a line graph to represent changes over time. Moreover, the teacher imparted more details about how to use a line graph. One of the students surprisingly announced that on returning home he would tell his parents how small dust particles were by comparing their size to a pencil lead. This case showed that the student realised that the dust particle problem was hazardous and could explain the results using mathematical knowledge.

37.4.3 Students' Responses to the Bench Problem

Initially, the teacher showed pictures and conducted a discussion about the general arrangement of the stone tables (Fig. 37.2a). Then, the teacher posed the first question to the class. This guided the students to *understand the real-world situation*. The students discussed the problem with their friends and analysed the appropriate patterns of arranging the stone tables. They made assumptions about the types of chair (one-person chair) and style of arranging the stone tables (Boardroom style, Fig. 37.2b) to *simplify the situation*. Then, the students used mathematical knowledge, *mathematising*, to answer the questions.

The students *worked mathematically* based on the assumptions made in the previous step. First, the students tried to model table patterns by drawing diagrams. Next, the teacher gave the students some square tiles and bottle caps to make sample arrangements and observe and model the relationship among the number of tables, used chairs and unused chairs. Some groups of students used three or four square tiles and bottle caps to make samples of arrangements. By contrast, others thought that more people sitting together would require more square tiles and bottle caps. This showed that the students did not look for a pattern when arranging the tables. The teacher noticed that some students failed to observe all the data from the experiment of arranging the tables, especially the number of tables, used chairs and unused chairs. The teacher guided the students to take notes, collect the data and look for patterns in the table arrangements. After that, most of the students were able to answer the second and third questions. To achieve this, they used the square tiles and bottle caps to *check their results* of arranging tables as well as to *test the relationship* among the number of tables, used chairs, and unused chairs. Moreover, they *interpreted* the number of used chairs in a real-life context. They realised that the number of used chairs was equal to the number of people sitting together. Some students said that if they knew how many people wanted to sit together, then they would know the required number of tables and chairs. This case showed that the students could connect their experiences by arranging tables for many people seated together to solve this problem.

As the final task, the teacher asked the students to prepare, *present and share* their ideas with the whole class. Most of the students were uncertain about how to use a variable to represent a relationship, such as a variable number n. In this relationship, the variable n represents the number of tables and must therefore be a positive integer. One group of students said that their group could use the ratio to describe the relationship. Moreover, some students said that even if someone could move the garden stone tables and chairs into several patterns, they should not do that because the assets might be damaged.

37.4.4 Similarities and Differences Between Students' Responses to the Activities

Because of the observations made during the experiments, we compared students' responses by discussing each of the mathematical modelling processes.

First, for *understanding the real-world situation* and *simplifying the situation,* there were several similarities between the street market problem and the bench problem. All the students satisfactorily discussed the details of each situation with their friends. They explained what the problem entailed and they connected to their experiences and knowledge about the problem to consider a suitable solution. These major similarities indicated that using context-based or local problems helped the students to understand real-world situations.

However, there were minor differences between the dust particle problem and the two other activities. In the dust particle problem, the teacher explained the meaning of some essential vocabulary related to dust particles and presented model pictures of dust particles to the whole class because they had never seen the shape of dust particles. These major differences indicated that although the dust particle problem occurred annually in their province, most students were unaware of the pollution. This case showed that using modelling to resolve local problems encouraged the students to integrate their mathematical knowledge, skills, and processes with everyday real-life activities.

Second, students' responses in each activity were similar. They could use the appropriate mathematical knowledge to solve local problems, *mathematising,* after discussions with their friends and support from the teacher. Most students also *worked mathematically* in the group correctly because they were stimulated to recheck their work before interpreting the results. Moreover, *interpreting and validating the results* showed similar students' responses in each activity in terms of the tools used to validate the conclusions. The students used extra materials and conducted experiments to confirm their conclusions. For example, they used a measuring tape and their experiences to confirm the optimal size of the booth in the street market problem and they used a few square tiles and bottle caps to arrange, observe, and model the relationship between the number of tables, used chairs, and unused chairs in the bench problem. In the dust particle problem, students used the size of real objects to ensure the validity of their results about the size of dust particles and their smartphones to garner information. Lastly, when *presenting their conclusions,* all the students reflected on their knowledge gained from mathematical modelling processes to ensure accurate and reasonable inferences.

37.5 Conclusions and Recommendations

Based on the implication of activities in the classroom, most of the students readily engaged in mathematical modelling processes. They were able to use and connect mathematics with other situations as *mathematical connection ability*. Moreover, most of the students stated that these activities increased and enhanced their awareness of the utility of mathematics and encouraged them to continue learning. This confirmed that modelling activities generated a positive *attitude towards learning mathematics*. In future, to implement mathematical modelling activities with non-experienced students, the teacher should first analyse the students' knowledge and use scaffolding strategies to help solve problems. Furthermore, the teacher should design sub-activities or use manipulative aids that engage students to investigate by trial and error, collect data, observe and make conjectures to solve real-life problems by themselves. The teacher can assist with student progress using guided questions if necessary.

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Chapter 38
How to Integrate Mathematical Modelling into Calculus Teaching?

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Abstract Integrating mathematical modelling into a calculus course teaching is an effective way to cultivate students' innovative and practical abilities. Moreover, it is a significant direction of the reform of calculus course. In this chapter, from the perspective of teachers, we explore how to infiltrate the mathematical modelling in calculus teaching, such as including definition introduction, theorem application and practice training. In particular, three examples, that is, *circle cutting*, *table placing* and *investment cost* are presented in detail to illustrate the three aspects, respectively.

Keywords Mathematical modelling · Calculus · Teaching · Definition introduction · Theorem application · Practice training

38.1 Introduction

Modelling is considered a vehicle for supporting students' endeavours to create and develop their primitive mathematical knowledge and models (Erbas et al. 2014). More and more teachers from universities in China realize that the teaching of mathematical modelling is very important. Specifically, mathematical modelling may guide students how to think, how to use mathematical tools to solve various practical problems. Therefore, it is necessary to incorporate mathematical modelling in the courses of university mathematics in China.

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Many scholars have proposed suggestions to integrate mathematical modelling into the courses of university mathematics. Jiang et al. (2007) introduced a two-year project on incorporating ideas and methods of mathematical modelling into the teaching of main mathematical courses in Chinese universities and colleges. Ye (2012) proposed and designed several mathematical modelling modules and integrated them into calculus teaching. Silva et al. (2015) proposed using mathematical modelling in the curriculum and software to cultivate students' learning skills to model mathematically and offered different approaches to reinterpret the calculus point of view. Sokolowski et al. (2011) recommended an alternative to strengthen the process of teaching mathematical modelling—utilization of computer-based science simulations. Shi et al. (2019) put forward to use the classical model to analyze the calculus knowledge system.

Calculus is one of the key courses of university mathematics in China. In teaching calculus, teachers always argue about how to deal with the relationships between concrete and abstract, practice and cognition. In fact, a focus on mathematical modelling and applications would be a good breakthrough. It can cultivate students' application awareness, stimulate students' interest in active learning and help students understand the abstract definitions and theorems. In this chapter, how to integrate the idea of mathematical modelling into calculus courses will be discussed from the perspective of teachers.

38.2 Problem and Background

The China Undergraduate Mathematical Contest in Modelling (CUMCM) is the most popular competition in China. In 2019, more than 130,000 students from nearly 1500 universities participated in CUMCM. These students have learned mathematical modelling and shown competence in their subsequent courses, projects and later careers. Most of them have learned calculus for at least one or two semesters. Many students do not understand why they have to spend so much time in studying calculus, and why it is important for their future careers. As a result, their study lacks motivation and initiative. In order to solve the problem, in our university, the coaching team of mathematics modelling has insisted on integrating mathematical modelling ideas into calculus teaching and has exploited and utilized calculus in the guidance of mathematical modelling competition. The authors will argue how to integrate mathematical modelling and applications into calculus teaching on the basis of their experience of teaching calculus and mathematical modelling.

38.3 Permeating the Idea of Mathematical Modelling into Calculus

In this section, we will discuss how to integrate the idea of mathematical modelling into calculus course through three aspects, definition introduction, theorem application and practice training.

38.3.1 Definition Introduction

Generally speaking, the definitions of function, limit, derivative, integral and series are abstracted from some quantitative relation or spatial form of objective things (Jiang 1993). Accordingly, these definitions should be naturally elicited from their actual prototype and examples in students' daily life, so that students realize that the definitions are not rigid rules, but closely related to real life. Therefore, when introducing these definitions, teachers should try to combine them with practice, set up appropriate mathematical models, provide abundant and intuitive background materials in observation, experiment, operation, conjecture, induction and verification.

For example,the formal and precise definition of limit in calculus is difficult for beginners of calculus to understand. As a result, students often regard it as obscure mathematical symbols and memorize it without understanding. To help students understand the idea of limit more intuitively, we consider an interesting problem: the area of a regular polygon is used to approximate the area of a circle. As the number of polygon sides increases, the accuracy of the approximation will become higher. This problem can be considered as a model of *circle cutting* (as shown in Fig. 38.1).

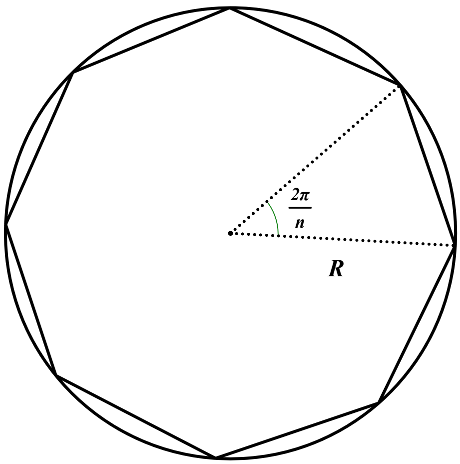

Fig. 38.1 *Circle cutting*: Approximating the area of a circle with the area of a regular polygon

In Fig. 38.1, we can easily find that the area of a circle can be regarded as an approximation of the area of its inscribed regular polygon. The area of the inscribed regular polygon can be expressed as

$$S_n = \frac{1}{2} n R^2 \sin \frac{2\pi}{n}, \tag{1}$$

where n denotes the number of sides of the inscribed regular polygon and R is the radius of the circle. Moreover, the area of the inscribed regular polygon of this circle is less than the area of the circle. As the number of sides of the inscribed regular polygon increases, its area will gradually increase. Accordingly, the area of the circle is equal to the limit of the area of the regular polygon, that is,

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1}{2} n R^2 \sin \frac{2\pi}{n} = \pi R^2 \cdot \lim_{n \to \infty} \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} = \pi R^2. \tag{2}$$

Based on the procedure of this modelling, students realize that the area of the circle goes from approximate to precise. Furthermore, exploring the thinking of problem-solving and generalizing of the law of problem-solving can help students understand the definition of limit. In addition, we may also use computer software to demonstrate the graphical and numerical changes. Such treatment would also be applied to the definition of integral by computing the area bounded by a curve with trapezia.

Many definitions in calculus have a good practical background. In definition introduction, these resources should be utilized to guide and inspire students to discover and create. The modelling of definitions is helpful. Students can experience the formation of mathematical definitions and improve their abilities of generalization and application.

38.3.2 Theorem Application

There are many theorems in calculus. In fact, some theorems are difficult for students to understand at the beginning. In order to help students grasp the ins and outs of the theorems, we can apply the theorems to practical problems. Whereupon, the conditions of the theorems are regarded as the assumptions of the models and the conclusions of the theorems can be used to solve the problems.

For example, we discuss the application of the theorem of root existence. The function $f(x)$ is continuous in the closed interval $[a, b]$, and $f(a)$ and $f(b)$ have different signs (that is, $f(a) f(b) < 0$). Then, there is at least point ξ in the open interval (a, b), so that $f(\xi) = 0$.

We consider a practical problem: on an uneven surface, can you find an appropriate place to locate a table so that all four legs of the table touch the surface at the same time?

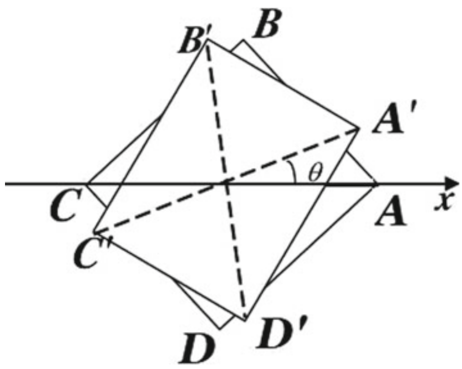

Fig. 38.2 Mathematical model of the rectangular table

In order to solve this problem by modelling the theorem of root existence, we make some assumptions. First, the four legs of the table constitute a strict rectangle on the plane. Second, the height of the ground is continuously changing, and there will be no discontinuities in any direction, that is, the ground can be regarded as a mathematically continuous smooth surface. Third, the four legs of the table are the same length. Fourth, the ground is relatively flat, so that at least three legs of the table touch the ground at the same time.

We consider the centre of the rectangular table as the coordinate origin. When the rectangular table rotates around this centre, the angle formed by the diagonal connecting vector *CA* and the *x*-axis is θ (see Fig. 38.2). Let the distances between the four legs and the ground be, respectively, $h_A(\theta)$, $h_B(\theta)$, $h_C(\theta)$ and $h_D(\theta)$. For any θ, three of $h_A(\theta)$, $h_B(\theta)$, $h_C(\theta)$ and $h_D(\theta)$ are always zero. In addition, $h_A(\theta)$, $h_B(\theta)$, $h_C(\theta)$ and $h_D(\theta)$ are all continuous functions with respect to θ. Thus, this problem can be transformed into a mathematical model. Specifically, $h_A(\theta)$, $h_B(\theta)$, $h_C(\theta)$ and $h_D(\theta)$ are all positive continuous functions with respect to θ, and three of $h_A(\theta)$, $h_B(\theta)$, $h_C(\theta)$ and $h_D(\theta)$ are always zero for any θ. Our goal is to find some $\theta \geq 0$, so that $h_A(\theta) = h_B(\theta) = h_C(\theta) = h_D(\theta) = 0$. Therefore, we can apply the theorem of root existence to solve this model.

The advantage of the above modelling is that a variable θ is used to describe the position of the table, and a function of θ is used to indicate the distance between the four legs of the table and the ground. Of course, we can also make other assumptions, such as the four legs of the table form a square or trapezoid. At this time, the method and result of this problem may be different.

38.3.3 Practice Training

Mathematical modelling has been considered a way of improving students' ability to solve problems in real life (Lesh and Lehrer 2003; Niss et al. 2007). In China, students consolidate knowledge mostly by using definitions, theorems and formulas to finish theoretical questions in calculus. This approach to learning makes it difficult for students to be creative. In order to make up for this shortcoming, mathematical modelling can also be incorporated into practice training.

> We can properly arrange some practical problems so that students have more space to think. In the traditional ways, students should do their homework independently. Therefore, we must encourage students to form discussion groups in order to strengthen their communication and cultivate their cooperation capability. They will experience mathematics, understand mathematics and master the thought and method of mathematical modelling.

We consider a problem of applying a series to investment costs. The government plans to build a new wooden bridge or cable-stayed steel bridge over a river. The cost of constructing a steel bridge is RMB 3,800,000. The steel bridge needs to be painted every 10 years, and the cost of each painting is RMB 400,000. The expected life of the steel bridge is 40 years. The cost of building a wooden bridge is RMB 2,000,000. The wooden bridge needs to be painted every 2 years, and the cost of each painting is RMB 200,000. The expected life of the wooden bridge is 15 years. If the annual interest rate is 10%, which bridge is more economical to build?

To solve this problem simply, students would make some assumptions. First, the prices of steel and wood are not affected by market, and their prices are relatively fixed during the construction of the bridge. Second, the interest rate remains unchanged.

Suppose the initial investment is p, the annual interest rate is r, and the investment is repeated once every t years. So, the cost of the first update is pe^{-rt}, and the cost of the second update is pe^{-2rt}. In this way, the investment cost D is the sum of the following proportional series:

$$D = p + pe^{-rt} + pe^{-2rt} + \cdots + pe^{-nrt} + \cdots = \frac{p}{1 - e^{-rt}}. \tag{3}$$

The cost of building a bridge includes two parts, i.e., building a bridge and painting. For the steel bridge, $p = 3{,}800{,}000$, $r = 0.1$, $t = 40$, and the cost of building the steel bridge can be expressed by

$$D_1 = \frac{3,800,000}{1 - e^{-4}} = 3,870,908. \tag{4}$$

when $p = 400{,}000$, $r = 0.1$, $t = 10$, the cost of painting the steel bridge can be expressed by

$$D_2 = \frac{400,000}{1 - e^{-1}} = 632,788. \tag{5}$$

Therefore, the total cost of building a steel bridge is $D = D_1 + D_2 = 4,503,696$.

For the wooden bridge, $p = 2{,}000{,}000$, $r = 0.1$, $t = 15$, and the cost of building the wooden bridge can be expressed by

$$D_1 = \frac{2,000,000}{1 - e^{-1.5}} = 2,574,400. \tag{6}$$

when $p = 200{,}000$, $r = 0.1$, $t = 2$, the cost of painting the wooden bridge can be expressed by

$$D_2 = \frac{200,000}{1 - e^{-0.2}} = 1,102,438. \tag{7}$$

Therefore, the total cost of building a wooden bridge is $D = D_1 + D_2 = 3,676,838$.

Thus, a wooden bridge is more economical.

Actually, the prices of steel and wood are usually affected by market. Under this assumption, the annual increase in the prices of steel and wood can be denoted as b. At this time, Eq. (3) can be rewritten as

$$D = p + pe^{-(r-b)t} + pe^{-2(r-b)t} + \cdots + pe^{-n(r-b)t} + \cdots = \frac{pe^{(r-b)t}}{e^{(r-b)t} - 1}. \tag{8}$$

Thus, different b will lead to different results. For example, when $b = 7\%$, a steel bridge is more economical.

Through the processes of mathematical modelling and applications, students realize that calculus is a useful tool to solve practical problems. We should try to conduct the principle of combining theory and practice in teaching, and improves the students' abilities to analyze and solve practical problems.

38.4 Conclusion

In our university, students enjoy these teaching practices. Students are very interested in mathematical modelling and applying calculus to solve practical problems. Many students had preliminary understanding of mathematical modelling during their freshman year, and later also participated in CUMCM and achieved good results. Meanwhile, their experiences also built a foundation for the subsequent courses.

Calculus is the wisdom crystallization of human mathematics development. Mathematical modelling is an important tool to solve practical problems. How to integrate mathematical modelling into calculus teaching is worth discussing. This chapter explains integrating mathematical modelling into calculus from three aspects, including definition introduction, theorem application and practice training. The ideas presented here go some way towards what Haines and Crouch (2007) described as helping "novices develop into experts to define concise descriptors of behaviours related to mathematical modelling and applications". In addition, some examples are provided in this chapter, just to set the ball rolling. We hope to offer some remarks for teachers engaged in calculus or mathematical modelling teaching.

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Part VII
Examples

Chapter 39
Economic Applications in the International Debates on Modelling and Applications Since the 1980s

Dirk De Bock, Johan Deprez, and Ann Laeremans

Abstract In many countries, secondary school students learn about mathematical applications and modelling through examples and contexts exclusively taken from physics or other natural sciences. However, there are good reasons to argue in favour of changing this situation and, in particular, to more intensively include applications from economics, business, or finance in secondary school mathematics. In order to identify the role of such applications in the mathematical–educational debates since the 1980s, we scrutinized all Proceedings of past ICTMA conferences as a representative body of research and development in this field. We came to the conclusion that economic applications were indeed not well represented in these debates. However, a positive trend was revealed since ICTMA12, the first ICTMA whose conference theme explicitly referred to economics.

Keyword Economic applications · Economic modelling · Historical overview · ICTMA · Literature review · Mathematical modelling

39.1 Introduction

In many countries, secondary school students learn about mathematical applications and modelling through examples and contexts exclusively taken from physics or other natural sciences. Although mathematical and statistical methods are increasingly used in the social sciences and humanities, applications from these scientific fields are often neglected in current mathematics education. There are several good

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reasons to argue in favour of changing this situation and, in particular, to more intensively include applications from economics, business, or finance in secondary school mathematics. First, applications from the various fields of economics often require less domain-specific knowledge than the traditional applications of mathematics in the natural sciences, and are therefore easier to understand for many students. Second, applications from these fields can improve students' economic and financial literacy, and are worth studying from a citizen's perspective on mathematics education. Third, extending the domains of application would enrich students' view on mathematics and its power to model a broad range of situations, both in real life and in a scientific domain other than the natural sciences. Fourth, it can ensure that also students in study streams with a strong mathematical component are introduced to some economic principles and laws, as well as to a corresponding quantitative approach. At present, this is not always the case and it could enable these students to consider economic-oriented studies at the tertiary level. Fifth, a sound understanding of mathematical concepts, for instance the integral and the differential quotient, often benefits from exemplifications and interpretations in a variety of disciplines in which these concepts are used and have a slightly different meaning (de Lange 1987; Freudenthal 1973).

39.2 Economic Applications in Mathematics Education

From a historical perspective, we find advocates of closer relations between mathematics education and economic applications as early as a century ago. Webster (1918) argued that "the pendulum has swung too far to the cultural side and that we must introduce the commercial and industrial factors" (p. 194). In his paper, he suggested two kinds of changes in the mathematics curriculum. On the one hand, he proposed to include financial mathematics, i.e. a topic with an explicit economic focus. On the other hand, he promoted the use of graphs, which is a more generally applicable skill, using arguments based in economics, i.e. because "all large corporations are using this mode of interpretation" (p. 195). Also Schaaf (1934) wanted to reform mathematics education in high school "so that as adults they [the pupils] will be able and willing to think about social-economic problems in quantitative terms" (p. 374). He argued in favour of what he calls "quantitative thinking", which encompasses statistics (e.g. the concept of frequency distribution), elementary probability theory (e.g. the concept of probability) and calculus (e.g. the concepts of functional relationship between two variables and of rate of change) and against conventional algebra and geometry, which are, in his words, "rather inadequate for an intelligent understanding of the vital problems that beset us today" (p. 374). Finally, we mention the contribution at the Royaumont Seminar of Tucker (1961), who refers to new types of problems, coming e.g. from economics, to suggest changes in the mathematics curriculum, like the study of matrices and linear programming. For a more detailed account of Tucker's contribution, we refer to De Bock and Zwaneveld (2020).

An exploration of literature yielded several descriptions of teaching projects featuring a diversity of economic applications in mathematics classes, both from recent and older times. Whereas older sources introduce financial mathematics in mathematics classes from the point of view of a future employee working in the financial sector, Mariner and Miller (2009) use high school algebra and pre-calculus to discuss financial mathematics in relation to the problems in the housing market during the financial crisis. Schaaf (1951) and Ruppel (1982) show how business arithmetic can profit from including slightly more evolved mathematics, notably elementary algebra and graphs. Olson and Sindt (1982) developed classroom activities around topics in consumer mathematics, i.e. how the rate of inflation determines the time to double price of an item and how quickly growth rate of consumption can deplete a finite resource. Nievergelt (1988) uses income tax as an economic context to study three representations of functions. Also, more theoretical concepts from economic science are included in a number of teaching projects: Van Dyke (1998) shows how cost, revenue and profit functions can be visualized. Kang, Letze and Letze (2017) use the concept of Production Possibilities Frontier in their project.

This abundance of practice-oriented papers is in sharp contrast with the virtual absence of papers reporting on empirical research concerning economic applications in mathematics education. Although a comprehensive review of the literature on economic applications in mathematics education is beyond the scope of this chapter, a systematic search of *Educational Studies in Mathematics* (ESM) and *Journal for Research in Mathematics Education* (JRME), two renowned journals with a very long tradition in our field, was undertaken. [1] This search, however, yielded no research papers on this topic, apart from studies linking primary mathematics education to money in daily life. Given our focus on secondary and, although to a lesser extent, on primary education, we did not search for evidence of economic applications in scientific journals that typically publish papers at university level or deal with advanced mathematics.

39.3 An Analysis of the ICTMA Proceedings

39.3.1 Research Questions

What role did economic applications play in the international debates on the role of applications of mathematics in education? Which applications were involved and can certain trends be observed?

[1] We used the following keywords, including a wildcard character in order to search efficiently for related words: economic* (referring to, e.g., economic, economics, economical, etc.). financ*, cost*, income*, profit*, market*, business*, consum*, money*, fiscal* and mortgage*.

39.3.2 Methodology

To review the role of economic applications in the mathematics education debate since the 1980s, we scrutinized all ICTMA Proceedings to date, from ICTMA1 to ICTMA17, that means a total of 18 edited books[2], as a representative body of research and development on mathematical modelling and applications. Chapters[3] in these Proceedings having economics as a main focus, or having at least a clear economic angle, were inventoried and analysed in more detail. This means that for each of these chapters, we identified the: (1) educational level, (2) methodology, (3) strength of the economic angle, (4) economic domain and (5) mathematical domain. For educational level, a distinction was made between primary, secondary and tertiary education, with the latter category being further subdivided into whether or not the chapter was related to teacher education. When more than one educational level was applicable, the most dominant was assigned. For research-oriented contributions, methodology referred to the applied research methodology (quantitative, qualitative or mixed methods, which means a combination of quantitative and qualitative methods). Chapters without a clear research focus, e.g. descriptions of teaching practices, programs or projects, were labelled as practice-oriented.[4] Finally, methodology could also relate to literature reviews. For strength of the economic angle, we distinguished the chapters with economics as the main focus and those with an auxiliary focus on economics. The latter meant that, although the main focus of the chapter was not economic, there was still an economic angle, or that the examples cited came mainly from economics. This inclusion criterion is quite broad, but we did not include chapters with only one, not further elaborated economic example, and chapters in which only references to an economic domain were made without further information or discussion.

With regard to economic domain, in which economics is generally defined as the social science that studies the production, distribution and consumption of goods and services, we distinguished between (a) macroeconomics, commonly defined as the "branch of economics dealing with the performance, structure, behaviour, and decision-making of an economy as a whole" ("Macroeconomics", n.d., n.p.)[5], (b) microeconomics, the "branch of economics that studies the behaviour of individuals and firms in making decisions regarding the allocation of scarce resources and the interactions among these individuals and firms" ("Microeconomics", n.d., n.p.), (c) business economics, the "field in applied economics which uses economic

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[2]The Proceedings of ICTMA2 were published in two separate volumes: Volume 1 dealt with "Mathematical modelling methodology, models and micros", Volume 2 was titled "Mathematical modelling courses". A list of the ICTMA Proceedings is available on the ICTMA website ("Literature", n.d.).

[3]These chapters are further elaborations of papers presented at the respective ICTMA conferences.

[4]In older ICTMA Proceedings (see, for instance, de Lange, Keitel, Huntley and Niss 1993) such contributions were commonly referred to as "case studies". Nowadays, however, this term has a research-methodological meaning. We have chosen the label "practice-oriented" in order to clearly distinguish these chapters from those with a research orientation.

[5]In addition to the classical concepts, such as inflation, price levels, rate of economic growth, national income or gross domestic product, also measures of social inequality, such as those based on income and wealth distributions, are included in the macroeconomics domain.

theory and quantitative methods to analyse business enterprises and the factors contributing to the diversity of organizational structures and the relationships of firms with labour, capital and product markets" ("Business economics", n.d., n.p.) and (d) finance, the "field that is concerned with the allocation (investment) of assets and liabilities over space and time, often under conditions of risk or uncertainty" ("Finance", n.d., n.p.). Applications related to operational research were typically classified in the domain of business economics.

With regard to mathematical domain, we basically relied on the content categories used in the PISA surveys for assessing mathematical literacy of 15-year-old students (OECD 2018): (a) change and relationships, (b) space and shape, (c) quantity and (d) uncertainty and data. Change and relationships deal with describing, modelling and interpreting change phenomena using algebra and functions, "including algebraic expressions, equations and inequalities, and tabular and graphical representations" (OECD 2018, p. 59). This category is most closely related to the common school subject algebra. Because ICTMA chapters often involve more advanced mathematical modelling by students who are older than the PISA target group, an extension proved necessary. Therefore, we divided change and relationships into two subcategories: (i) algebra and pre-calculus, including calculating with logarithms and the study of functions without the calculus apparatus (linear and quadratic functions, inverse proportional relationships, and basic exponential, logarithmic and trigonometric functions), and (ii) calculus, including derivatives, integrals and differential equations and their applications on more complex functional relationships. Space and shape encompasses plane and spatial "patterns, properties of objects, positions and orientations, representations of objects and shapes, decoding and encoding of visual information, navigation, …" (OECD 2018, p. 59). The school subject geometry serves as a foundation for space and shape, but the scope of this category is broader including also areas such as spatial visualization and measurement. The category quantity incorporates quantitative relationships and involves "measurements, counts, magnitudes, units, indicators, relative size, and numerical trends and patterns" (OECD 2018, p. 59). It is most closely related to the school subjects number and number operations. The last category, uncertainty and data, focusses on presenting and interpreting data, and evaluating conclusions drawn from data in situations where uncertainty is central. It also includes "recognising the place of variation in processes, having a sense of quantification of that variation, acknowledging uncertainty and error measurement, and knowing about chance" (OECD 2018, p. 60). The school subjects that are closest to this category are probability and statistics.

The analysis of all ICTMA Proceedings, according to a standardized analysis scheme based on the categorizations we discussed above, was conducted by two independent researchers. In case of inconsistencies, the criteria were refined until a full consensus was reached.

39.3.3 Results

Table 39.1 shows the distribution of the economic-related chapters in the respective ICTMA Proceedings. Each relevant chapter is given a tracking number. Table 39.2 provides an analysis of these chapters with the tracking number as a unique identifier.

Table 39.1 Distribution and identification of chapters related to economics in the different ICTMA Proceedings

ICTMA Proceedings: serial number/number of chapters/number of pages (from Chap. 1)	Number of economic-related chapters/corresponding total number of pages	Identification and assigned chapter's tracking number (between brackets) of economic-related chapters
1/38/483	3/40	pp. 11–25 (1), pp. 257–268 (2), pp. 269–281 (3)
2–Vol. 1/23/306	0/0	
2–Vol. 2/25/267	2/33	pp. 58–69 (4), pp. 188–208 (5)
3/70/449	6/37	pp. 159–165 (6), pp. 187–191 (7), pp. 195–200 (8), pp. 207–212 (9), pp. 280–285 (10), pp. 348–354 (11)
4/44/427	2/22	pp. 147–157 (12), pp. 249–259 (13)
5/37/392	3/23	pp. 235–243 (14), pp. 297–302 (15), pp. 385–392 (16)
6/24/334	0/0	
7/32/401	4/51	pp. 51–61 (17), pp. 183–202 (18), pp. 331–341 (19), pp. 385–393 (20)
8/34/344	2/17	pp. 51–61 (21), pp. 183–202 (22)
9/38/422	6/63	pp. 15–29 (23), pp. 62–71 (24), pp. 90–98 (25), pp. 119–129 (26), pp. 227–234 (27), pp. 381–390 (28)
10/26/330	3/41	pp. 16–29 (29), pp. 101–110 (30), pp. 267–283 (31)
11/23/267	1/15	pp. 3–17 (32)
12/49/492	3/34	pp. 25–42 (33), pp. 110–119 (34), pp. 288–293 (35)
13/53/648	6/66	pp. 111–118 (36), pp. 119–129 (37), pp. 173–188 (38), pp. 223–233 (39), pp. 255–264 (40), pp. 399–408 (41)
14/68/722	1/10	pp. 289–298 (42)

(continued)

Table 39.1 (continued)

ICTMA Proceedings: serial number/number of chapters/number of pages (from Chap. 1)	Number of economic-related chapters/corresponding total number of pages	Identification and assigned chapter's tracking number (between brackets) of economic-related chapters
15/52/617	5/53	pp. 67–76 (43), pp. 153–163 (44), pp. 241–251 (45), pp. 517–526 (46), pp. 607–617 (47)
16/50/603	10/107	pp. 67–79 (48), pp. 241–250 (49), pp. 251–261 (50), pp. 317–326 (51), pp. 327–337 (52), pp. 351–361 (53), pp. 407–416 (54), pp. 477–486 (55), pp. 557–566 (56), pp. 579–589 (57)
17/52/637	8/86	pp. 37–47 (58), pp. 107–116 (59), pp. 211–221 (60), pp. 337–347 (61), pp. 443–453 (62), pp. 491–501 (63), pp. 577–586 (64), pp. 615–625 (65)

Out of a total of 738 chapters, 65 (or 8.81%) were found with an economic angle. These chapters represented 698 (or 8.57%) of the 8141 pages in the 18 Proceedings that were analysed. The majority of these chapters referred to secondary (31) or tertiary education (29), the latter group including only three chapters about teacher education. The vast majority (39) were not research-oriented and therefore labelled as practice-oriented. A main economic angle was only identified in 15 chapters. Economic domains varied, although microeconomics and business economics were best represented (with each 22 chapters). With regard to the mathematical domain, Change and relationships were best represented with 25 chapters on algebra or pre-calculus and 23 chapters on calculus.

39.3.4 Discussion

In line with our expectations, we found that economic applications were not well represented in the debates at ICTMA since the 1980s. Although we used a very broad inclusion criterion, less than 10% of the contributions had a link to economics. Inspired by the historical overview of ICTMA ("The first twenty-five years" 2000-2019), we present a more fine-grained qualitative analysis of the results and try to identify some trends.

Economic-related chapters in the ICTMA Proceedings are mainly related to the tertiary and secondary levels. However, whereas in the 1980s, the pioneering years

Table 39.2 Analysis of economic-related chapters in the Proceedings of ICTMA 1 to 18

Category	Chapter's tracking numbers	Total number of chapters
Educational level		
Primary	32, 37, 41, 44, 62	5
Secondary	5, 6, 7, 8, 9, 12, 13, 17, 18, 21, 24, 26, 29, 30, 34, 35, 40, 43, 45, 49, 50, 51, 52, 53, 54, 55, 57, 58, 59, 60, 65	31
Tertiary–Teacher education	14, 42, 63	3
Tertiary–Other	1, 2, 3, 4, 10, 11, 15, 16, 19, 20, 22, 23, 25, 27, 28, 31, 33, 36, 38, 39, 46, 47, 48, 56, 61, 64	26
Methodology		
Quantitative	28, 34, 45	3
Qualitative	14, 35, 42, 43, 44, 46, 49, 51, 52, 54, 55, 56, 58, 60, 62, 63	16
Mixed methods	27, 29, 41, 50, 59	5
Practice-oriented	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 30, 31, 32, 33, 36, 37, 38, 39, 40, 47, 48, 53, 57, 64, 65	39
Literature review	15, 61	2
Strength of economic angle		
Main	2, 3, 4, 8, 11, 15, 20, 33, 36, 39, 45, 47, 50, 55, 61	15
Auxiliary	1, 5, 6, 7, 9, 10, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 35, 37, 38, 40, 41, 42, 43, 44, 46, 48, 49, 51, 52, 53, 54, 56, 57, 58, 59, 60, 62, 63, 64, 65	50
Economic domain		
Macroeconomics	2, 13, 18, 31, 33, 45, 48	7
Microeconomics	4, 5, 12, 16, 21, 22, 23, 24, 27, 32, 35, 43, 47, 49, 51, 53, 54, 56, 57, 60, 62, 64	22
Business economics	1, 3, 9, 14, 17, 19, 20, 28, 29, 30, 34, 36, 37, 38, 39, 41, 42, 44, 46, 58, 63, 65	22
Finance	6, 8, 10, 11, 25, 26, 50, 52, 55, 59	10
More than one domain	7, 15, 40, 61	4

(continued)

Table 39.2 (continued)

Category	Chapter's tracking numbers	Total number of chapters
Mathematical domain		
Change and relationships– Algebra and pre-calculus	2, 3, 4, 6, 7, 8, 9, 12, 14, 16, 17, 21, 24, 30, 36, 40, 43, 45, 46, 49, 50, 51, 53, 60, 64	25
Change and relationships– Calculus	1, 5, 11, 15, 18, 19, 20, 22, 23, 25, 26, 27, 28, 34, 35, 39, 47, 48, 52, 54, 55, 57, 61	23
Space and shape	62, 65	2
Quantity	29, 32, 37, 41, 44, 56, 59	7
Uncertainty and data	10, 13, 31, 33, 38, 42, 58, 63	8

of ICTMA, almost all chapters involved tertiary (engineering) education, this has changed since the end of the 1980s. From then on, secondary education receives more and even as much attention as tertiary education. For a first contribution with an economic angle to primary education, we must wait until ICTMA11 (English 2003), and also thereafter, such contributions remain rare. In the 1980s, and to a lesser extent also in the 1990s, chapters related to economics were practice-oriented. The emphasis was on teaching modelling and various mathematical models were discussed. From around 2000 onward, this focus has shifted slowly to research on mathematical modelling and this research focus has become predominant for the past ten years.

Although we observed a slightly positive trend over the years, the share of economic-related chapters remained small. This applies in particular to chapters with economics as the core focus. A slight turnaround appeared since ICTMA12 (2005), the first ICTMA with a conference theme explicitly referring to economics and welcoming a plenary lecture on "economic modelling" by Barker (2007), member of the Monetary Policy Committee of the Bank of England. Barker discussed the range of issues which arise from the use of economic models, generally defined as theoretical constructs representing economic processes by sets of variables and logical and/or quantitative relationships between them. In particular, she focussed on the importance of recognizing the context when selling particular models to the bank's clients.

39.4 Final Comments

Taking the ICTMA Proceedings as a representative body of research and development, we reviewed the situation of economic applications in mathematics education debates since the 1980s. This led us to a descriptive scheme of such applications within the chapters of these Proceedings. On that basis, we could conclude that the role of this genre of applications in debates at ICTMA conferences was, in particular up to ICTMA11, rather limited. However, we did not answer the question whether economics was under-represented in the ICTMA debates compared to other scientific disciplines. This would require similar analyses for various disciplines that make frequent use of mathematics, such as physics, chemistry, biology, linguistics or psychology.

Future research could also examine the role of economic applications in recent debates more thoroughly through the analysis of more wide-ranging sources, including a broad range of journals, curricula and textbooks. As far as journals are concerned, we systematically searched ESM and JRME, two flagship journals in the scientific discipline of mathematics education, but neither usually publish research papers at the university level or deal with advanced mathematics. It would be worth exploring for evidence of applications to economics in journals that typically treat more advanced topics in mathematics, such as *International Journal of Mathematical Education in Science and Technology*; *International Journal of Research in Undergraduate Mathematics Education*; *Journal of Mathematical Behaviour*; *Problems, Resources, and Issues in Mathematics Undergraduate Studies* (PRIMUS); or *Teaching Mathematics and its Applications*. The analysis of textbooks for secondary or primary schools is particularly interesting because it can provide valuable insights into the role of economic applications in mathematics lessons as they actually take place at these levels. Such analyses could also map out the situation from one country or region to another. To further improve current educational practices, it is critical to provide teachers with meaningful economic applications that they can integrate in their courses, applications that can empower their students with basic mathematical and economic ideas.

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Chapter 40
Inquiry-based Orbital Modelling to Build Coherence in Trigonometry

Celil Ekici and Cigdem Alagoz

Abstract Mathematics modelling education has potential to support pedagogical innovations by fostering coherence within mathematics as well as supporting interdisciplinary connections. Students often struggle to build coherence in trigonometry presented in multiple mathematical frames such as triangle, circle, vector and complex numbers. Design-based research experiments are conducted to extend the modelling of circular motion to the modelling of advanced periodic orbits from a series of trigonometric functions. Inquiry-based orbital modelling allows students experiment with modelling of periodic orbits with technology-rich tasks in interpreting the meaning and connections of periods and amplitudes of circular functions and the emergent patterns. The results show that learners experience coherence while interpreting, comparing and validating their orbital models in circular, functional and complex trigonometry with connections in between.

Keywords Coherence · Trigonometry · Mathematical modelling · Periodic functions · Fourier analysis · Integrated STEM learning

40.1 Introduction

The learning tasks for inquiry-based mathematics education come not only from applications of mathematics but also from mathematical objects themselves (Schoenfeld and Kilpatrick 2013). Trigonometry is a subject traditionally rich with modelling tasks involving triangles, circles and the periodic phenomena from secondary schools to college (Bressoud 2010). Yet, it is often problematic for students to build coherence with trigonometric functions due to the multiplicity of trigonometric frames

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involving triangle, circle, periodic functions, vectors and complex numbers (Altman and Kidron 2016; Ekici, 2010).

Inquiry-based mathematical modelling is a cyclic process including selecting mathematical framings, determining relationships, interpreting and validating (Anhalt and Cortez 2015). New loops can be initiated during mathematizing and validating. Inquiry-based modelling of periodic orbits serves as a driving theme across modelling tasks to connect trigonometric frameworks. Orbital modelling is the modelling of periodic orbits traced by the trajectory of a point on a circle rotated about another circle. For example, an orbital pattern with four petals is mathematized by an ordered pair (cos t + cos($-3t$), sin t + sin($-3t$) where t is a real number. We use mathematical modelling as a vehicle for coherent learning progressions across the trigonometric frameworks while mathematizing periodic phenomena.

40.2 Problem and Theoretical Framework

While there is a research in covering the gap from triangle to circle trigonometry (Moore and LaForest 2014; Weber 2005) largely due to students' conceptual difficulty with radians, there is a need for research on learning progressions for coherent learning of trigonometry connecting circle, functional and complex trigonometry (Ekici 2010). This study is conducted to address this gap with a sequence of mathematical modelling tasks to facilitate coherence of learning trigonometric functions in real and complex domain. The orbital modelling learning trajectory is designed to deepen student knowledge of college trigonometry for freshmen for a connected learning of circle, functional and complex trigonometry for applied mathematics education. The modelling task sequences are intended to provide a balanced perspective in modelling education to advance mathematical ideas and connections (Doerr 2016; Hjalmarson et al. 2008; Zbiek and Conner 2006). Orbital modelling allows us to use time as a variable to support learners overcome their conceptual difficulty with the angle measure with radians in circular motion.

Our underlying mathematical framework has three main stages of learning progressions from circle to complex trigonometry through orbital modelling. In the first stage, the circle trigonometry is initially used to model a trajectory of a point around a single circular orbit mathematized by an ordered pair of single sine and cosine functions with $(\mathbf{x}(t), \mathbf{y}(t)) = (r \cos(2\pi\ \omega\ t), \sin(2\pi\ \omega\ t))$ with t representing time, and ω representing the frequency. $\mathbf{x}(t)$ and $\mathbf{y}(t)$ trace the trajectory of the point along a circular orbit. In second stage, the circle trigonometry is extended to periodic functions using multiple circles to model periodic orbits $(\mathbf{x}(t), \mathbf{y}(t))$ as a sum of trigonometric sine or cosine functions in Cartesian plane. Third stage of progression is to model the orbital patterns by using complex trigonometry. This is conceptually based on the Euler formula to combine the ordered pair $(\mathbf{x}, \mathbf{y})$ as a single number $\mathbf{z}$ so that the sum of trigonometric functions are now, $\mathbf{z}(t) = r_1 e^{i\omega_1 t} + r_2 e^{i\omega_2 t} + \ldots$.

40.2.1 Learning Trajectories as Modelling Sequences Towards Building Coherence

Learning trajectories provide an integrated approach to develop mathematical knowledge for teaching mathematical modelling through analysis of tasks and their sequences, students' experience of learning, connecting, anticipating and attending to discourse and assessment (Sztajn et al. 2012). The instructional design theory of Realistic Mathematics Education (RME) (Gravemeijer 2004) is adopted with the guided reinvention and emergent modelling with instructional sequences designed to challenge learners to organize circle trigonometry at one level to produce new understanding at a higher level. In exploring the multiplicity of mathematizing phase, frame shifts and comparisons of our modelling process across mathematical frames are gradual and incremental across the semester to build a strong metacognition in modelling practice to build connections within discipline of trigonometry. This design-based research approach aligns with Streefland's (1993) orientation on shifting of perspectives across purposeful modelling task sequences as a context for deepening students' understanding of mathematics and mathematical connections. Building on Lesh and Doerr's models and modelling approach, we facilitate students' work on "real-life" modelling challenges such as modelling double Ferris Wheel rides, that are designed to require students to invent, refine and generalize powerful mathematical constructs (Lesh and Doerr 2003).

In this study, learning trajectories are connected with an orbital modelling task sequence on designing advanced double Ferris Wheel rides. These orbital patterns are investigated and mathematized by a pair of periodic functions as a combination of circular functions or a single function in Euler form with complex numbers. Combined circular functions essentially form a series of trigonometric functions to model any periodic function or pattern.

Underlying justification behind teaching and learning mathematical modelling of orbital paths is to facilitate the emergence of intuitive ideas for functional analysis and Fourier analysis ideas as a foundation for applied and engineering mathematics education. The intuition for Fourier Analysis lies in modelling a complex periodic pattern as a combination of sine waves with different frequencies and amplitudes. The goal for research-based design experiments is to build a coherent trigonometry practice with orbital modelling task sequences across trigonometric frames with an ultimate mathematical horizon on Fourier analysis. Learning trajectories are investigated across a series of orbital modelling tasks towards building a coherent trigonometry practice through modelling. Design-based research involves iterative series of teaching cycles conjecturing, enacting, assessing and revising modelling task sequences to build coherence along learning trajectories (Simon 1995).

40.2.2 Learning Trajectory in Trigonometry for Orbital Modelling

The trigonometry students are often given opportunities to study trigonometric functions as simple periodic functions that can be associated to a circle modelled by ($r \cos \omega t$, $r \sin \omega t$). In teaching trigonometry, circular functions are commonly introduced to model periodic phenomena such as the changes in the vertical and horizontal position of a rider on a Ferris Wheel with circular functions. Here, we expand the practice of modelling with circular functions to the modelling of periodic functions as a sum of trigonometric functions.

We experiment with learning trajectories in teaching practice of college trigonometry by modelling periodic orbits with a series of trigonometric functions. We aim to provide a transition from a [trigonometric] function as a “model of” to a sum of [trigonometric] functions as a “model for”. We refine our modelling sequences to facilitate coherent learning trajectories based on our classroom experimentation informed by our experience with students. We identify the patterns across trigonometric objects in orbital modelling observed through its variations within a trigonometry frame and across frames building on students’ experience during the teaching experiments. For example, the learning trajectory along model sequences is conjectured to elicit the relationships across periods in building a certain orbital pattern in a parametric pair of trigonometric functions ($x(t)$, $y(t)$). Once established, this learning trajectory offers a connection within a frame. Then model task sequences orients students’ inquiry into whether the observed connection as a pattern still holds when the modelling problem reframed in complex representation as ($x(t) + iy(t)$).

Local instructional practices around the learning trajectories are designed, assessed and revised to help to coordinate students’ and teachers’ understanding across trigonometric frames. Inquiry-oriented modelling sequences are designed to facilitate student discovery of trigonometric and mathematical ideas and connections behind orbital modelling. Using dynamic applets, students learn to treat trigonometric functions as objects that can be manipulated and combined to build and study advanced periodic orbital patterns. Informal approach to analysis with trigonometric functions allows students to develop an intuition and demand towards a deeper mathematics that can be revisited and expanded in later courses for college students.

Inquiry-based math modelling task sequences on orbital modelling theme is designed to support learning trajectories building a series of connected mathematical ideas across circle, functional and complex framings of trigonometry. This modelling approach is aimed to foster student understanding of trigonometric functions as building blocks to model any periodic behaviour by using alternative frames of mathematizing with circular functions, a series of circular functions in real and complex domain. This pluralist approach to mathematising phase is kept across mathematical modelling tasks sequences to revisit, experimenting with the orbital modelling through its alternative mathematical representations. This approach is designed to offer students opportunities to build connections and develop coherent meanings with the trigonometric objects in modelling orbital patterns.

40.3 Method and Data

Design-based teaching experiments were conducted in the context of college trigonometry class taught by the first author to develop and implement inquiry-based modelling task sequences with a theme on orbital modelling. The class met for two 75-minute sessions each week for 14 weeks, with a common learning objective of developing triangle, circle and complex trigonometry perspective. Participants were freshmen without prior background on trigonometry. Students did not have prior experience with GeoGebra prior to this class. Orbital modelling activities are done in a span of four weeks to connect circle trigonometry and complex trigonometry with modelling and technology integration. Students work samples and classroom artefacts including the GeoGebra investigations are collected and analysed as they worked on their modelling projects. The authors collaborated on design and analysis of learning trajectories during designing, assessing and revising modelling task sequences. Observation and reflection logs were kept. Data was analysed following an open and axial coding approach to identify emerging patterns in students' responses to tasks or connections between these responses (Corbin and Strauss 2008). Researchers worked on themes emerging from data regarding participants' experience with modelling sequences to support coherence within mathematics during the mathematization, interpretation and validation of alternative trigonometric models for periodic orbits.

40.3.1 Orbital Modelling Tasks in Trigonometry

Inquiry-based modelling activities supported students' experimentation in extending the study of trigonometric functions from modelling with single sinusoids to modelling periodic functions as a combination of circular functions. Inquiry-based modelling of periodic orbits is a driving theme across multiple modelling task sequences. The learning trajectories are designed to revamp trigonometry for mathematical scientists including computer scientists and engineers.

Orbital models are extensions of the unit circle representation depicting the horizontal and vertical position of a rider with a parametric pair $(x(t), y(t))$ as t changes. Dynamic GeoGebra applets are designed to help students examine orbital behaviour visually and algebraically to investigate the emerging patterns between periods and amplitudes with their implications to orbital behaviour.

$$x(t) = r_1 \cos(w_1 t) + r_2 \cos(w_2 t) + \ldots. + r_n \cos(w_n t)$$
$$y(t) = r_1 \sin(w_1 t) + r_2 \sin(w_2 t) + \ldots. + r_n \sin(w_n t)$$

Cross-cutting activity is here to design complex Ferris Wheel rides (such as Sky Wheel) modelled by combined circular functions. Students start with modelling of a *Ferris Wheel,* develop models of periodic orbits formed by the trajectory of a rider located on several circles on circles simultaneously rotating with different periods and radii. Starting in week 2, initial modelling sequence works on a single wheel with different rotational speeds, boarding positions using the specifications of famous models such as High Roller in Las Vegas. Using resources over Internet, students find videos and information about the design features of famous Ferris Wheels with their radii and periods. Students work on modelling horizontal and vertical position of the rider at time *t*. Students are expected to model the orbits by mathematizing first with cartesian and complex trigonometric models extending their understanding of basic circular orbits parametrized as $(r\cos(\omega t), r\sin(\omega t))$. Experimenting with GeoGebra[1], students examine the characteristics of modified circular functions generated by adding circular functions with different amplitudes and periods in modelling periodic orbits. Participants make observations on the emergent phenomena by building conjectures about the patterns among parameters in generating different orbit families. Students compare the models and build connections across frames in modelling the periodic orbits by trigonometric functions. The functional trigonometry perspective as an extension of circle trigonometry is advanced here to develop an intuitive understanding of Fourier analysis which is a fundamental idea for engineers and applied mathematicians.

40.4 Results

40.4.1 Representative Orbital Modelling Tasks

Students were introduced the following orbital modelling task with their follow-up sequences.

40.4.1.1 Design Orbital Path for the Sky Wheel

> ***Design Your Sky Wheel***
> You are commissioned to design a Double Ferris Wheel that should provide a thrill ride as seen in Fig. 40.1a. Develop a mathematical model for the rider's orbital path during the entire *Sky Wheel* ride. Investigate the impact of the periods and radii on the orbital patterns to come up with your thrilling ride.

During the first phase of the *Sky Wheel Modelling* task, students watch a video of a *Double Ferris Wheel* in action as depicted in Fig. 40.1a. The modelling challenge is to describe orbital pattern traced by the position of a *Sky Wheel* rider during the entire

[1]GeoGebra (Hohenwarter et al. 2018) is an interactive software for mathematics learning from primary to college.

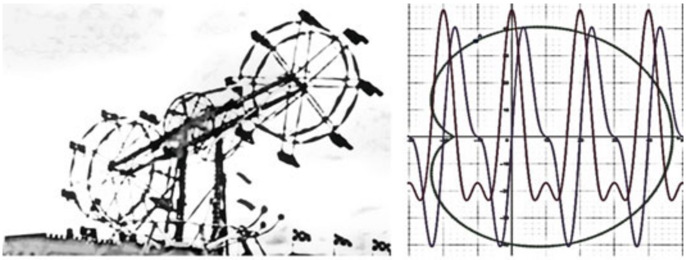


Fig. 40.1 *Sky Wheel* as Double Ferris Wheel https://www.youtube.com/watch?v=2LUNVPtRn0A

ride. The rotating arm connects centres of two wheels. From the video and Internet research, students make estimations for the radii and the periods of the rotating arm and two small Ferris Wheels. By using the command *curve* (*a* cos(*b t*), *a* sin(*b t*), *t*, 0, T) in GeoGebra, students discover the meanings of the parameters in generating an orbital pattern and observe that the constants *a* is linked to the *radius* and *b* is linked to the *period T* as 2π/T. The resulting orbital model is depicted in Fig. 40.1b.

During this orientation phase for inquiry-based learning, students start posing *what if* questions on the design of Sky Wheel examining the impact of different radii and periods on orbital patterns. Initial teaching experiments helped us observe that students preferred using period compared to the frequency to visualize what happens to the period of the combined periodic functions. This modification allowed them to use real time as an angle measured in radian.

$$x(t) = r_1 \cos(\frac{2\pi}{T_1}t) + r_2 \cos(\frac{2\pi}{T_2}t),$$
$$y(t) = r_1 \sin(\frac{2\pi}{T_1}t) + r_2 \sin(\frac{2\pi}{T_2}t).$$

Expanding the realistic model for Sky Wheel, a GeoGebra applet is created for a generalized double Ferris Wheel model describing the orbit of a rider on a multiple circle system by connected rotating circles (see https://www.geogebra.org/m/ezwnvma8). Based on the initial explorations in generating orbital patterns in GeoGebra, this applet is collectively developed in class with students for a dynamic experimentation. This applet facilitates the discovery of orbital patterns by a dynamic construction and comparison of graphical and algebraic representations. The parameters are changed by sliders for radii and periods of revolutions for the central arm and the Ferris Wheels. Sky Wheel provides a concrete generative model that can be extended to study and design more thrilling rides. This modelling challenge allows students to study orbital modelling by a series of circles rotating with different period and radii. Students individually or in groups generate and compare their orbital patterns by different radii and periods.

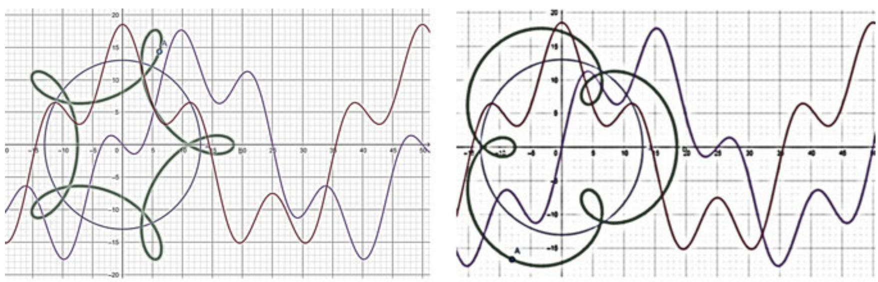
40.4.1.2 Challenge of Creating Loops and Turns

Students are given the challenge to examine the characteristics of an orbital pattern interpreting from the relationships between the periods, their ratios and the signs as positive or negative. Students design a thrill ride of their choice building alternative trigonometric models, examining orbital patterns, comparing their models, and justifying their arguments with peers.

> **The Rides with Loops Task**
>
> Discover how you design your Sky Wheel so that riders can experience any desired number of loops during their ride, say 1, or 5. You may use the applet to examine the relationships between the periods of the wheels and the rotating central arm for your generalized Sky Wheel. For a rider to experience any given number of loops, identify the relationship between periods for the rotating circles (wheels or arm) in your design.

During this *Rides with Loops Task* activity students experiment individually or in groups with the orbital patterns to determine the relationships between periods in generating loops. Students argue the meaning and the impact of the negative period if it is permissible in the problem context. Students conclude that the negative sign of a period should be interpreted to indicate the clockwise rotation of the wheel or the central arm. In response to this task, Henry's response is a common across students. He offers his favourite Double Ferris Wheel design with an orbital path with five loops as in Fig. 40.2a. The parametric trigonometric functions modelling the orbit for the rider is, contrasted with the positive periods 50 and 12.5 given in Fig. 40.2b. The equations are $y(t) = 13\ \sin(2\pi/50\ t) + 5.5\ \sin(2\pi/(-12.5)\ t)$ and $x(t) = 13\ \cos(2\pi/50\ t) + 5.5\ \cos(2\pi/(-12.5)\ t)$ where t is time.

Get to know Your Family of Your Sky Wheel Task is the next modelling activity exploring orbital paths with the same periodic ratios. It is designed to develop an



(a) (b)

Fig. 40.2 Henry's designs with 5 loops versus 3 loops with period ratios −4 and +4 as in (**a**) and (**b**) intuitive idea of functional analysis by examining orbital curve families. This inquiry-based task provides a learning progression along orbital modelling sequence to determine the impact and the constraints of the students' choices of radii in the orbital path designs. Holding ratios of periods constant, students observe the impact of radii on the orbits and resulting impact on the experience for the riders. The idea behind this modelling sequence is to examine the resemblance of orbital characteristics when radii is being manipulated for orbits with different periodic ratios. This manipulation is facilitated by GeoGebra applet. Using the applets, students manipulate their designs so that they can compare the orbital patterns based on their choices of radii and periods. Interpretation is a critical part of this analysis.

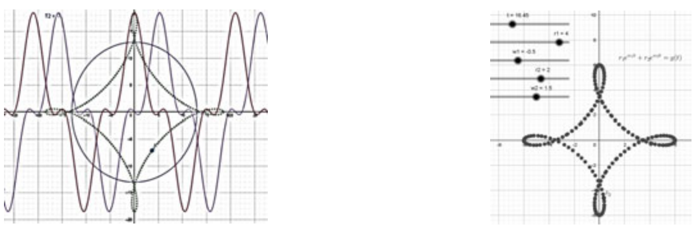
Rachel: When Ts are multiples it makes a loop for each multiple and subtracts one, number of loops are T_2/T_{1-1}. For negative multiples, it makes a loop for every multiple plus an extra loop $|T_2/T_1| + 1$.

Rachel's observation is another typical expected learning outcome observed across students connecting the orbital loop patterns to the periods of T_1 and T_2. Henry's designed pentagonal or five-pedal patterns exemplify how a family of five-pedal resemblance of orbital models are generated by period ratios of -5 or $-1/5$.

40.4.1.3 Modelling a Given Periodic Orbital Pattern by Different Frames for Trigonometry

Next task in modelling sequence asks students model a given periodic orbit with four loops as in Fig. 40.3. This activity is reintroduced after mathematizing a circular orbit using the complex framing of a circular orbit as $r\mathrm{e}^{i\omega t} = r(\cos(\omega t) + i\,\sin(\omega t))$. This framing allows students to combine $(x(t), \mathrm{y}(t))$ as an orbital model into one equation as $x(t) + i\,y(t) = r_1\mathrm{e}^{i\omega_1 t} + r_2\mathrm{e}^{i\omega_2 t}$.

Students' parametric models have the periods of 12, -4, and the radii 13 and 5.5 for wheel and the central arm as in Fig. 40.3a. Multiple solutions are generated for the same orbital pattern using $\{T_1, T_2, r_1, r_2\} = \{21, -7, 13, 5.5\}$, with $T_1/T_2 = -3$ or $T_1/T_2 = -3$. Using GeoGebra, students are able to compare and connect the parametric and complex representations of this pattern as in Fig. 40.3.



(a) Mario's parametric framing for orbital modelling (x(t), y(t))

(b) Jasmine's complex framing of orbital modelling of Sky Wheel z(t)=$r_1\,e^{i\,w_1 t} + r_2\,e^{i\,w_2 t}$

Fig. 40.3 Comparison of student modelling work using multiple framings of *Sky Wheel Modelling*

40.4.2 Assessment

We built the following framework for a collaborative formative assessment of mathematical modelling problems inspired by the work by Diefes-Dux et al. (2012). First dimension asks students examine with their peers what it is their mathematical model describing about the double Ferris Wheel ride. Students are expected to elaborate on the procedure they follow in developing their mathematical models. Students assess how well their mathematical model addresses the problem of modelling riders' path in a double Ferris Wheel. Students are expected to demonstrate fluency with alternative mathematical representations, whether algebraic, graphical, or verbal and switching in between clearly. Students revisit and reassess their models if it is sufficiently descriptive of the desired periodic orbits for their own Sky Wheels.

Second dimension asks students to assess how their modelling process works if reused in modelling new yet similar periodic orbits. This allows them to look back the steps in their modelling procedures across similar orbits with different periods and radii. This helps to reveal students' strategic competence with their modelling process at metacognitive level examining the characteristics of periodic patterns and radii in generating loops and turns for desired rides.

Third dimension asks students consider other problem contexts that can be modelled using the mathematical modelling procedure for periodic orbits. This dimension helps student to think of different modelling situations that can have similar structure with periodic orbits. Recontextualization and transfer of process are harder for students.

Articulation of the assumptions in context is challenging for students that require deeper understanding of physics and research into interdisciplinary context (Galbraith and Stillman 2001). We ask students for rationales for their steps during the modelling process reflecting their understanding of the relevant context. Freshmen students question the constraints and the physical implications in realistic content, but their conceptual knowledge of speed and acceleration is not well formed yet, not having taken a physics or a calculus course earlier. Students listed the defining attributes of the sine wave as period, amplitude and phase, as embodied by the attributes of a rotating object as revolution per minute, radius and initial angle. Students early on missed the phase as a defining attribute for sine waves in describing orbital patterns. This shortcoming is in part due to the experimental setup and constraints with the GeoGebra applets. Learning trajectory can be modified to facilitate stronger connection between the phase and initial angle before rotation for the circular motion.

Throughout the orbital modelling sequence, students build more complex orbital paths by manipulating simple circular functions. The culminating idea at the end is a conceptual orientation towards looking into a periodic behaviour as a composition of a circular functions. This is to provide readiness towards more formal approach towards the Fourier analysis as a process to find the set of cycle periods, amplitudes and phases to match any periodic function, $f(\mathrm{t})$. At the end students' work have generated simple examples for this reverse conceptual orientation, that is "How do we decompose a complex periodic function or a signal into a group of circular functions with their dominant amplitudes, fundamental frequencies and phases?".

40.5 Conclusions

The integration of modelling into teaching trigonometry and other school subjects works well when there is a "balancing act" during the instruction between developing students' modelling skills on the one hand and using modelling to help them learn mathematics and sciences on the other. Aligning with Zbiek and Conner (2006), Schoenfeld and Kilpatrick (2013), we use mathematical modelling as a context for deepening students' understanding of mathematics. The learning trajectories allow the shifting of perspectives across modelling task sequences, as inspired by Streefland (1993). Focusing on conceptual connections across trigonometry while designing and experimenting the modelling sequences along a learning trajectory builds more coherence to the content. Since the focus is on deepening mathematical connections through modelling, doing more mathematics with less contextual shifts is found to be working as pedagogically intendedd here in the learning trajectories for building foundations of Fourier analysis through rich modelling contexts. Contextual variation is kept at minimum across modelling tasks when we shift the focus from one modelling task to the next. The new trigonometric frames and representations during the mathematization phase are compared with the previous models to help students build more advanced understanding of trigonometry as a subject providing multiple frameworks for mathematizing orbital modelling problems. This learning trajectory offers a learning process with an iterative function: looking back and looking forward with the same modelling activity with different mathematical lenses (Kilpatrick 1985). Purposeful variations along modelling task sequences provides students and teachers opportunities for reflection and discussion. For future research, orbital modelling provides authentic interdisciplinary learning opportunities for students to investigate, build, and synthesize their knowledge of and experience with physics, engineering and technology.

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Chapter 41
Video-based Word Problems or Modelling Projects—Classifying ICT-based Modelling Tasks

Gilbert Greefrath and Pauline Vos

Abstract Mathematical modelling tasks increasingly feature the use of digital tools and media. In this chapter, we discuss the wide variety of these. Until now, classifications for modelling tasks did not consider the use of tools and media. Therefore, we developed a new classification for ICT-based modelling tasks. One class relates to mathematics; the others differentiate across (1) modelling aspects unrelated to tool and media, (2) the task context, (3) the digital tools and media (CAS, Wikipedia, type of feedback, etc.) and (4) students' anticipated activities guided by task regulations, such as group work or time restrictions. The classification was validated with three example tasks. A visual presentation based on the classification system enables the evaluation of qualities of a given ICT-based modelling task and can give insight into potential adaptations.

Keywords Mathematical modelling · Modelling tasks · Task analysis · Digital tools · Digital media · Classification

41.1 Introduction

All over the world, mathematical modelling is entering mainstream mathematics education, not just in classroom activities, but also in curricula and assessments (Frejd 2011; Vos 2013). Simultaneously, digital tools and media are embraced by education, and the combination of both has led to a wide variety of mathematical modelling tasks (Drijvers et al. 2016). On the one hand, there are open-ended modelling research projects within technology-rich environments, and on the other hand, there are tasks that are questionable to label as 'modelling tasks', yet these allow students to use digital tools (e.g. CAS, DGS). In this chapter, we first explore

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the wide variety between these extrema. Then, we review existing classifications of modelling tasks and classify aspects specific to ICT-based modelling tasks. The purpose of a classification is to obtain a plausible evaluation of the quality of ICT-based modelling tasks. We validate the new classification by applying it to three example tasks. A visualisation based on this classification allows us to describe strengths and weaknesses of a given ICT-based mathematical modelling task.

41.2 Examples of ICT-Based Modelling Tasks

To illustrate the variety, we present three examples of ICT-based modelling tasks. In the *Maypole Task*(see Fig. 41.1), the task designers offers a video to illustrate a traditional group dance in Bad Dinkelsdorf. Such media replace verbal descriptions and assist students in understanding the task context. Students are asked to process this situation with the help of mathematics. They are asked to write a number into an answer field, making this task potentially suitable for a digital test with automatic grading. Students can use a *clickable* calculator, which enables teachers/researchers to log students' calculation activities. This task is not based on a problem from the dancers or the choreographer, and the focus is on a number answer. This makes the *Maypole Task* effectively a word problem on the application of Pythagoras' theorem. In real life, the radius of the dancers' circle is adapted to the available space, whereby the dancers shorten the ribbons by winding these around their hands. Also, one may

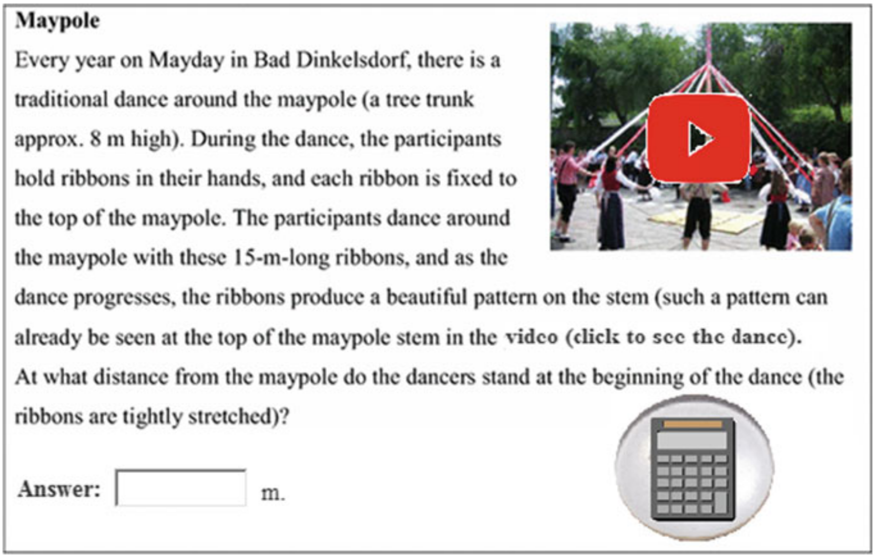

Fig. 41.1 Maypole task—using Pythagoras' theorem. Adapted from Rellensmann and Schukajlow (2017)

In the movie you will see a recording from the cockpit of a glider. Watch the movie carefully, especially the altimeter (left) and the variometer (right). Compare the two displays and describe how they interrelate.

Fig. 41.2 Glider task—introduction of the derivative function (www.net-mathebuch.de)

note that the town of Bad Dinkelsdorf does not really exist. So, the city and the problem for the dancers are inauthentic, but the video shows an authentic dance.

The *Glider Task* (see Fig. 41.2) includes a movie showing the take-off by a glider plane, together with the displays of the altimeter indicating the altitude and the variometer indicating the rate of climb or descent. Just like in the *Maypole Task*, this movie shows an authentic situation, as demonstrated by the details like the dirt on the windscreen and the tiny features in the horizon. The students are asked to compare the two displays. The video playback can be moved back and forth by students to explore the situation. The openness regarding the approach to the task and the openness regarding the final answer invite students' discussion. This makes the task potentially suitable for group work. It is anticipated that students will recognise a connection that will lead them to develop an intuitive and informal, yet meaningful understanding of the derivative function.

The *Algal Bloom Task* from Geiger and Redmond (2013) is a project-based task in a technology-rich environment. It starts from a large, authentic data set on the CO_2 concentration in the Darling River, together with explanations about algae blooming, sunlight deprivation and the potential death of all life in the river. The question asks whether the present data are a cause for concern. This open-ended task allows for various approaches, does not target a single correct answer and is covered in two lessons, in which students work in pairs.

The above three example tasks show the wide variety in how digital tools and media can interact with mathematical modelling tasks. There is no linear scale from less digital to more digital, when comparing between traditional paper-and-pencil mathematical modelling tasks that request students to use digital tools like CAS or DGS, and tasks, in which students are asked to explore situations through digital media, like in the *Glider Task*. Also, there is no linear scale from less modelling to more modelling, when comparing between tasks with embedded digital media illustrating the task context, or tasks with *clickable* links to the world beyond the classroom. The variety in ICT-based modelling tasks is multi-dimensional, and our study aims at finding ways to describe, compare and evaluate these.

In this chapter, we will not discuss how digital tools and media shape and change communication, organisation, cognitive levels and other aspects of modelling activities. For this, we refer to recent research (e.g. Molina-Toro et al. 2019; Monaghan 2016; Williams and Goos 2012). Also, we will not study whether or not the use of digital tools and media within tasks lead to new modelling activities, more realistic contexts, more intense group work and so forth. Rather, we will exclusively focus on mathematical modelling tasks, in which digital tools and media are integrated. Since we want to describe, compare and evaluate these, we need criteria for important aspects across the tasks. Our study was guided by the following questions: (1) *Which criteria are suitable to describe and compare ICT-based modelling tasks? (2) How can we classify and evaluate qualities of different ICT-based modelling tasks?* To answer these, we studied the literature. After several rounds of adapting and improving, we formulated a classification system. We used the three described tasks to validate the classification and evaluate the qualities of these tasks.

41.3 Classifying Tasks

There are a variety of ways to classify mathematical modelling tasks. The assessment framework from OECD (2013) distinguishes the following aspects: mathematical topics (e.g. geometry), mathematical concepts (e.g. angle, perimeter), mathematical competencies (e.g. reasoning, representing), modelling activities (mathematize; work mathematically; interpret/evaluate), task difficulty, task format (e.g. multiple choice), students' digital tools (calculator, spreadsheet) and whether a task is presented on paper or screen. This framework gives us a first basis. However, it was typically developed for large-scale testing; it does not include, among others, criteria on openness or group work. This framework includes 'task difficulty', which is important in testing and can be determined for large student groups. This aspect is framed by the testing regime; if students were given more time or free access to Internet, the 'difficulty' could be different. Therefore, we will not include task difficulty into our classification, but rather include regulations that frame a task, such as allowing students to have ample time, peer collaboration or access to resources.

A comprehensive classification tailored to modelling tasks was created by Maaß (2010). This classification caters for a wide variety of modelling tasks and gives us a base to build on (see below). However, it does not include the use of tools and media. For this, we will use a description of digital aspects within modelling tasks by Geiger and Redmond (2013), but these authors only considered open project-based tasks in rich digital environments and not the less open tasks. So, we started from the classification system by Maaß (2010), restructured it and obtained five main classes. The first class pertains to the mathematics needed to solve the task, such as the topic (e.g. geometry) and the concepts (e.g. angles, perimeter). This class is substantial to a modelling task. The other four classes are explained below. Since we aimed for a classification that would enable a comparison across tasks, and an evaluation of qualities, we focused on developing classes that could be rated for higher or lower quality. Only the first class, regarding mathematical topic and concepts, cannot be rated. A summary of the classes, subclasses and ratings will be presented at the end of this chapter.

41.3.1 Modelling Tasks Without Considering Digital Tools and Media

Starting from Maaß (2010) and OECD (2013), we found classes for mathematical tasks describing competences required to solve the task (e.g. reasoning, representing). In some classifications of competencies, mathematical modelling is a subclass among other mathematical activities (e.g. Blomhøj and Jensen 2007). However, many mathematical activities can alternatively be perceived as sub-activities within mathematical modelling. In this chicken-and-egg dilemma, we chose the latter perspective, namely to view any given *mathematical activity* as potentially being a subclass of mathematical modelling, in particular as part of 'working mathematically'. In our classification, we included this class, with the rating 0–4 for the number of competencies needed to solve the task. Another class from Maaß (2010) distinguishes between holistic modelling (students undertake the whole process) and atomistic modelling (students undertake a partial process, like only setting up the real model). We adapted this class by rating the number of *modelling activities*, in which the students were asked to engage in. Also, we included a class from Maaß (2010) regarding the *information given* in a task: superfluous (making for an overdetermined task), missing (underdetermined task), inconsistent (both over- and underdetermined) and matching.

In her classification, Maaß (2010) had three further classes, but these needed reconsideration when looking through the lens of a classification of ICT-based modelling tasks. One class was 'nature of the relationship to reality', which needed adaptation when considering virtual worlds, which have their own digital reality. The second class needing reconsideration was 'type of representation', which described texts and pictures, but not animations, video or other interactive representations. The third class pertained to openness (in solution methods), which we shall extend to openness to tools. We will return to these below.

41.3.2 Task Context in ICT-Based Modelling Tasks

In this class, we assert that a modelling task always contains a context with some problem that needs to be tackled mathematically. A first subclass here is the *reality* *reference* of the task context, which is the way the context is presented compared to the actual real world. For example, a task context can be designed as intentionally *artificial* (e.g. to simplify it to students). An artificial context can be perceived as a *digital reality*, like in games. If the task context is closer to the real world of humans, it can be *realistic* when it is experientially real and imaginable for students, even if not convincingly originating from real life. In the case where the presentation of the context contains evidence of its genuine existence, for example through a video, the task context, or parts of it, can be *authentic* (Vos 2018).

Modelling tasks always contain both a task context and a question or a request. So, we classify the relation between context and the problem posed, which is the *question reference* of the task context. A presented context can 'beg' for a question; one could imagine a task involving a video of citizens who present their context and ask for help to find a solution that has used value to them. When there is convincing evidence that people in the task context genuinely require an answer to their question, it is an *authentic* question. However, often the question is not presented with such urgency and authenticity; nevertheless, it can still be a *realistic* question. We assert that a task cannot have an artificial task context and an authentic question. However, a modelling task can have a meaningless, artificial question based on an authentic context (e.g. authentic data), which makes it a dressed-up word problem. In the case where both the task context and the question relate to the students' current or future lives, we speak of a *relevant* question, distinguishing between student relevance (relevance from the students' point of view) and relevance to life (relevance to the students' future situations) (Greefrath et al. 2017).

Regarding the task context, we also include its representation. These can be text, diagrams or picture, which are static. A video can be played back, thus offering some interactivity. We can also imagine interactive animations that offer the students the possibility to explore the situation, for example through sliders to manipulate variables.

41.3.3 Aspects of the Digital Tool or Medium Within ICT-Based Modelling Tasks

In this chapter, we use the term *digital tools and media* as shorthand for overlapping terms like *ICT,* digital *technology*, digital *environments*, digital *worlds*, digital *products* and so forth. There are some ambiguities in these terms. For example, a video is generally perceived as a medium, but it can also be a tool for a designer to explain a task context, or a product created by students to report on their modelling project. We shall distinguish between *digital tools for students* to solve the task, like pocket or graphical calculators, CAS, DGS, spreadsheets, Wikipedia and so forth. We can also look at how the use of tools and media is regulated (*openness of tool use*). A task designer can encourage or restrict the students' use of a certain digital tool or medium ("solve this task using CAS"). Digital tools and media are also available to designers, *teachers* and examiners, who can use tools for the presentation of a task, but also to administer students' activities (logging answers) or for evaluation purposes. When a task is offered within a digital environment, there can be different *types of feedback*: a short response (right/wrong), or more elaborate feedback providing 'an explanation about why a specific response was correct or not' (Shute 2008, p. 160). The tool or medium can also allow a designer to frame the timing of the feedback (immediate or delayed).

41.3.4 Students' Anticipated Activities and Task Regulations

Any task designer, teacher or examiner will anticipate certain *students' activities* that are expected to be triggered by a given task. However, many mathematical modelling tasks are open towards approaches, to the use of tools and media or to different interpretations of contexts or answers. This implies that designers, teachers and examiners cannot (and should not) be fully able to foresee what students will do. Nevertheless, we included a subclass for students' anticipated activities, in which the variation in students' activities is rated, and we acknowledge that rating this quantitatively will be somewhat subjective.

A different subclass pertains to whether or not the task designer creates an *open* task, offering solution openness and/or answer openness. Also, Maaß (2010) had included this subclass, but we elaborate it with digital learning environments in mind. *Open* tasks are those that allow, for example, multiple solutions (at different levels). Open tasks can be classified according to the clarity of the initial and final states and the clarity and ambiguity of the transformation. When a modelling task is offered with an answer field in a digital environment, like in the Maypole task (Fig. 41.1), the mere presentation already announces that the task has little answer openness; nonetheless, the problem is open with regard to solution strategies. Some tasks can be classified as *(un)clear tasks*, including both a subjective and an objective component. The subjective component means that the perceived clarity depends on the students' competencies and on the regulations that enable a student to gain further clarifications. The objective component refers to whether task-specific information can only be tapped with limited accuracy, even by experts with the best tools and media, like with some Fermi problems (see Ärlebäck & Bergsten 2010).

The subclass of *task regulations* pertains to rules set by designers, teachers or examiners. One such regulation is whether or not group work is allowed and whether students perform the work independently or may consult with others (including experts). Also, we can consider whether students have ample time to explore or be creative, or whether they are subject to a regime of time restrictions. When a task is used for a high stakes test, there will be pressure on students to find the answer that an authority will judge as 'correct'. A task can also be geared towards the application of a certain formal mathematical concept. Oftentimes, such a task asks for theorems or algorithms that were recently learned in lessons. The Maypole task is a typical task on applying Pythagoras' theorem, although one could conceivably estimate the length of the ribbon based on experience, based on a drawing, or based on a role play. In this subclass, we considered that the greater student's independence and ownership, the better the task aligns with the spirit of mathematical modelling.

41.4 Results

Using the criteria and descriptions presented in Sect. 41.3, the three task examples, *Maypole*, *Glider* and *Algal Bloom task,* were rated by the authors within the scope of a qualitative research process. Based on the literature, we had reached a classification with five classes, of which the first regarding mathematical topic and concepts cannot be rated. The other four classes had subclasses theoretically derived, and these were ordinal and thus made accessible for quantifying. At the end of the process, the example tasks were rated for each subclass. In the few cases, in which the raters disagreed, the raters discussed the issue until agreement was reached on a common, final rate. This resulted in values assigned to the example tasks for each of the above-mentioned criteria (see Table 41.1).

The *Maypole task* mainly focuses on a few modelling activities and the digital tools and media play a subordinate role, with little openness regarding approaches, tool use or possible answers. The *Glider task* shows a higher overall potential in

Table 41.1 Results of the classification (A *Maypole task*, B *Glider task*, C *Algal Bloom task*)

			A	B	C
3.0	Mathematics	Topic, concepts (not rated)			
3.1	Competencies	Number of competencies: argue mathematically, solve problems mathematically, communicate mathematically, model mathematically (0–4)	2	4	4
	Focus of modelling activity	Number of different sub-competences: understand, simplify, mathematize, work mathematically, interpret, validate, expose, holistic (0–8)	3	8	8
	Quality of the information given	Exact, overdetermined or underdetermined, both (0–2)	1	2	2
3.2	Reality reference of task context	Artificial, digital reality, realistic, authentic (0–3)	2	3	3
	Question reference of task context	Artificial, realistic, authentic (0–2)	0	1	2
	Representations	Verbal (text only), static (table, picture, diagram), semi-interactive (video), interactive (0–3)	1	2	1
3.3	Digital tools for students	Number of used tools: no tool, one, more than one (0–2)	1	2	2
	Openness of tool use	Prescribed tool use, some choice, free choice (0–2)	1	2	2
	Digital tools for teachers	Number of activities, e.g. presentation, evaluation (0–2)	1	1	1
	Types of feedback by the tool	No feedback, simple feedback, elaborate feedback (0–2)	0	0	2
3.4	Students' activities	Number of anticipated activities: exploring, computing, drawing, checking, presenting, etc. (0–6)	2	4	6
	Openness	Number of open states: initial state, final state (0–2)	1	1	2
	Task regulations	No group work, group work, group work and consult others (0–2); yes/no time restrictions (0–1); high/low evaluation norms (0–1); high/low formal mathematics demand (0–1)	1	3	5

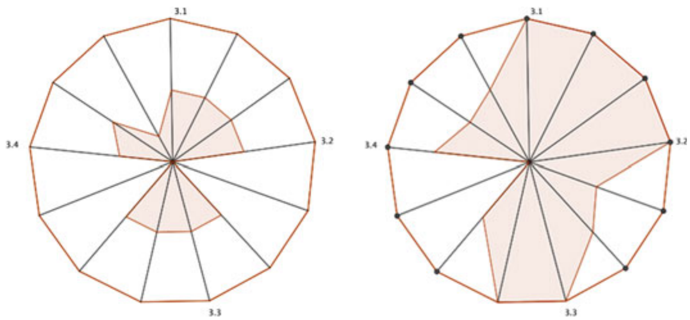

Fig. 41.3 Net diagram of the *Maypole task* and the *Glider task*

terms of modelling and digital tools and media. The modelling properties are particularly noticeable. And finally, the *Algal Bloom task*, in which many in the ICTMA community will consider the only 'real' mathematical modelling task of the three, scores highly overall.

A visual way to represent the classification of individual ICT-based modelling tasks is a net diagram. The lowest value of the scales is in the middle of the diagram. The potential of each example is directly apparent. A larger area indicates greater potential. The classes on the left generally express the use of digital media, and the classes on the right generally express the modelling potential (Fig. 41.3).

The diagrams show that the modelling potential of the *Glider task* is strong, whereas the *Maypole task* is limited in every class. Due to space limitations, we could not include the diagram of the *Algal Bloom task*, which was a nearly regular tridecagon.

41.5 Discussion, Conclusion, Recommendations

In applying the classification to the example tasks, we obtained a plausible evaluation of the quality of the tasks. We see strengths in the approach, but acknowledge its limitations. It would require further tasks to be evaluated by more raters to confirm the validity of the classification scheme and the reliability of the rates. The three selected examples already show that there can be no unambiguous weighting of the different criteria, since both the criteria for the modelling and the criteria for the digital tools and media describe different facets of the tasks, which cannot directly be compared to one another. One might observe that tasks with a high modelling potential seem to be visualised by a high degree of authenticity, relevance and a manifold of different modelling sub-competences. On the other hand, there are interactive multimedia tasks that perform more strongly on certain aspects relating to digital tools (e.g. CAS or DGS) and digital media (e.g. video). Weaknesses in modelling classes cannot be compensated by strengths in tool and media use and vice versa. The classification system shows that all these tasks aspects are difficult to compare, and that one needs a multi-dimensional view in describing, comparing and evaluating such tasks. Our classification system refines earlier classifications with respect to the use of digital tools and media. Also, the classification assists in analysing ICT-based mathematics tasks from the perspective of mathematical modelling education. A strength of this system is that it reveals possible ways to improve the quality of an ICT-based modelling task and how tasks can be improved by making suitable use of digital tools and media (like tasks embedded into virtual worlds). Our classification also shows that some tasks gain little from the use of tools and media, like the *Maypole task* (see Fig. 41.1). Other modelling tasks could not exist without digital tools and media. In the *Glider task* (see Fig. 41.2), the digital medium offers information that could not otherwise be given. Finally, we note that future developments in task development will undoubtedly require new criteria to extend the classification system.

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Chapter 42
South African and Norwegian Prospective Teachers' Critical Discussions About Mathematical Models Used in Society

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Abstract The chapter concerns prescriptive mathematical modelling and the development of critical competence. The potential of such modelling examples to promote critical discussions about the role of mathematical models in society is argued. A study is presented in which Norwegian and South African prospective teachers critically discuss a task dealing with a mathematical model, the Body Mass Index. The question posed in this chapter focuses on critical issues that come to the fore during the discussions. Four themes were identified, connected to several limitations of the model and to possible alternative models. The themes are discussed in relation to prior research on critical discussions of mathematical models in society and teacher education.

Keywords Critical discussions · Prescriptive modelling · Prospective teachers · Socio-critical perspective · Teacher education

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42.1 Introduction

The Norwegian and the South African mathematics curricula require citizens to critically assess information expressed in mathematical forms. Mathematical modelling is highlighted in the new Norwegian curriculum (Ministry of Education and Research 2020) as one of the six core elements aiming to generate insights into the use of mathematics in everyday life, working life, and society in general. The South African school mathematics curriculum states that "Mathematical modelling is an important focal point of the curriculum [and] Contextual problems should include issues relating to health, social, economic, cultural, scientific, political and environmental issues whenever possible" (Department of Basic Education 2011, p. 8). Prospective teachers (PTs) will thus have to deal with critical competence and mathematical modelling with their students. The crucial role of teachers and their competences in mathematical modelling processes are recognized by Blum (2015). Consequently, the role of teacher education in educating PTs who can teach modelling is emphasized.

Critical competence in mathematics consists of being able to use and reflect on mathematics in different contexts, as well as discuss its role in society (Skovsmose 2014). One of the purposes of working with modelling in schools, at different levels, is to understand the role of mathematics and its connections to the real world. Blum (2015) identified this with the socio-critical perspective on modelling. Skovsmose (2014) argued that *critical discussions* about how the use of mathematical models in different contexts can influence society and how models' limitations act out in the represented phenomena are important for developing the critical competence. Current and future mathematics teachers will have to support their students in participating in such critical discussions. They need to participate themselves in critical discussions and develop their critical competence *in* and *with* mathematics to be able to support their students. In our study, we created opportunities for PTs to engage in discussions about mathematical models that influence society. Research on mathematical modelling (e.g. Blum and Ließ 2007) shows that several evaluations are made while building mathematical models, such as which variables to take into account and which not. Starting from those evaluations and choices, a mathematical model is not a finally settled model, and different mathematical models can be developed depending on the purpose.

In our study, we use the Body Mass Index (BMI), a mathematical model expressed as m/h^2, with m as the mass in kilograms and h the height in metres. The statistician, Quetelet, developed the BMI formula in the nineteenth century using data obtained from groups of conscripts (see Hall and Barwell 2015). The BMI is generally used to classify individuals as of normal weight or overweight. Adults are deemed to be of normal weight if their BMI falls in the interval 18.5–24.9. The index has, however, several limitations (see, e.g. Kacerja et al. 2017) because it ignores relevant variables. However, the model is simple to use, and people can determine their BMI by inserting their mass and height in a relevant application on the internet. The limitations, as well as the simplicity of determining the BMI, have consequences for individuals. Our contention is therefore that by using BMI, the PTs have the potential to engage with critical discussions about mathematical models used in society. The potential is explored in this chapter, based on results from Norwegian and South African prospective teachers' discussions presented here. The underlying research question reported on is: What critical issues come to the fore when PTs discuss the mathematical model of BMI?

42.2 Theoretical Background

Mathematical modelling is the process that takes place when constructing some model, using mathematics, of a problem or phenomenon from an extra-mathematical domain. There are several versions of the modelling cycle that represent the iterative stages of a modelling process (e.g. Blum and Ließ 2007). Skovsmose (2014) viewed a model as not representing but re-presenting reality in a new way. Modelling thus shows one part of reality in a certain way, and not a one-to-one correspondence between reality and a mathematical model. Thus, the BMI can be viewed as but one model representing obesity (and underweight). There can be other models or indicators for the same phenomenon. Discussing existing models is a way to bring attention to, and investigate, the mathematics in action in society, Skovsmose argued. A mathematical model serves different purposes depending on the real problem it answers. This chapter focuses on the prescriptive purpose, which according to Niss (2015, p. 69) is "to design, prescribe, or structure certain aspects" of the world. Niss referred to this as prescriptive modelling and argued for including it in mathematics classrooms. The Body Mass Index, which has a prescriptive purpose as it defines obesity in populations, was under scrutiny. Hall and Barwell (2015) argued the BMI's potential to promote discussions about the formatting power of mathematics in society.

Barbosa (2006) presented modelling as critique and Kaiser (2007, p. 111) viewed this critique as a sub-competency "to challenge solutions". In the socio-critical perspective of modelling (Barbosa 2006; Blum 2015), the development of critical thinking is at the centre, and closely connected to understanding and criticizing the nature and the role of mathematical models in society. Students should therefore be given opportunities to develop this critical competency (Skovsmose 2014). Consequently, it is as important for PTs to engage in critical discussions of an existing mathematical model. Indeed, research in modelling has argued about the importance of training mathematics teachers in teaching mathematical modelling, and examples of proper modelling courses in teacher education have been presented (e.g. Blum 2015; Borromeo Ferri 2017). Doerr (2007) argued for the need for teacher education to work with both subject matter knowledge and pedagogical knowledge of PTs in modelling and provide them with possibilities to reflect on meta-levels about their own modelling processes. Borromeo Ferri (2017) added the relevance of prospective teachers' written reflections about their work with students and emphasized the need to foster their reflective competency (p. 132). In a socio-critical study by Villarreal et al. (2015), PTs were asked to pose mathematical modelling projects. In addition to carrying out modelling projects of social issues, the PTs were able to reflect about mathematics, its social role and the models created, as well as about the potential of modelling with school students.

Niss (2015) suggested criticizing models and meta-validating them as possible and desirable in prescriptive modelling. Amongst the issues to consider in such interrogations are, according to Niss: investigating hidden assumptions behind the model, interrogating which variables were and were not taken into account; the reasons for choice of variables; the consequences of the use of the model and contemplating possible alternative models through the consideration of changes of the variables used. Skovsmose (2014) recommended the studying of a mathematical model in its structural relationship to the object it represents and to the theory, the interests, and intentions for building and for using the model. The consequences for using the mathematical model are also important to be able to critically analyse it. There are therefore similarities between the recommendations of Niss and Skovsmose for critically engaging with mathematical models. These recommendations are used in this chapter to discuss the results from prospective teachers' discussions.

42.3 Methods

The study presented here is a collaboration between two institutions, one in Norway and one in South Africa. The aim was to jointly design a toolkit for teaching indexes as mathematical models to be implemented in prospective mathematics teacher education courses at both institutions. Kacerja et al. (2017) presented some reflections about in-service primary school teachers' discussions of the BMI and its use in society. Based on that study, the toolkit activity sheet was translated and further developed for the current study. The following three groups of questions were focused on: exploring the mathematics of the BMI; exploring and discussing the BMI and the uses of other indexes in society, and on the use of BMI and other indexesin schools as a basis for developing critical mathematical discussions. A picture of a muscular male sportsperson with a high BMI value was included to inspire the students. Also, the BMI formula and a respective graph with the categories were provided.

The question sheet was meant to ensure that the discussions of all three aspects were realized within the time frame. One set of data was collected in autumn 2017 from 11 female Norwegian PTs (presented as N1, N2, etc.) in the 1st year of their master's programme. The PTs were divided in three groups of 3–4 students. The discussions lasted around 60 min, after an introduction on critical mathematics education by the first author. A similar procedure was followed in spring 2018 with 14 South African PTs (3 females and 11 males, presented as S1, S2, etc.), in their 4th year of study, who discussed the question sheet in 3 groups (of 6, 5 and 3 students). All the groups were audiotaped, except one group that was videotaped, with their permission. Teacher educators were present during group discussions, mostly to clarify any questions and sometimes to ask questions to help the group go forward.

Fitness for purpose • Muscles vs fat • No gender differentiation • The body shape, the individual look, the body construction • Self-picture and knowledge to interpret • Alternative existing formulas • Reasons for using the BMI	*Social factors* • Not taken into account • Differences between countries • Accessibility of BMI (technology) • Problematic for vulnerable groups
Lifestyle • Eating habits • Physical activities • Metabolic age	*Engagement with mathematics* • Why is height squared? What if not? • Heart rate as variable • Measurement issues

Fig. 42.1 Themes from the data analysis

Thematic analysis was used to analyse the transcribed data, aiming to provide rich thematic descriptions (Braun and Clarke 2006). The authors had several face-to-face joint analysis sessions, reading and re-reading the transcriptions and listening to the audios and videos to gain familiarity with the data. First, the South African data was used to produce some codes. Afterwards, the Norwegian data was similarly explored to check if the South African codes applied and if any new codes would show up. The codes were sorted into four themes according to their topical similarity and are presented in the next section. As is the nature of discussions, the themes are not hermeneutically sealed and do overlap. Figure 42.1 presents the four identified themes and the respective codes. Because of the limited space in this chapter, only some of the codes will be discussed in the next section.

42.4 Findings and Discussion

Before rendering the four themes from the data analysis, it is important to note that neither the South African nor the Norwegian PTs rejected the BMI. Rather, the South African PTs regarded the BMI as relevant but inadequate, as one participant said: "… it can't be the only thing to determine whether someone is healthy or unhealthy". This is in line with Skovsmose's (2014) idea of a model showing only one part of reality. Similarly, the Norwegian PTs pointed out limitations by problematizing how the BMI is used for individuals and not "for larger groups to say something about what the average is". Ideas of meta-validation of the model (Niss 2015) include identifying assumptions behind the model and its uses, as the PTs did. Both groups were critical towards the use of mathematics of the BMI.

42.4.1 Fitness for Purpose of the BMI

When discussing the BMI, its formula and function, as well as its use in society, the PTs identified several variables not taken into account that make it an imperfect index. Often the identifications started with personal examples. The PTs' deliberations of the limitations of the formula about its purpose are discussed in this sub-section.

The PTs criticized the model's prescriptive use and meta-validated it, as Niss (2015) emphasized, by identifying variables that are not considered such as not differentiating between *muscles and fat.* This is a limitation identified by all six groups. In the beginning of the discussions, a South African PT referring to the picture of the muscular sportsperson said: "He is not unhealthy, he is pretty fit. He is a rugby player and obviously, he is fit, but his BMI is 35.9. Look at the category, he falls in the second last one. But he is not obese". The PTs used the contrast between the sportsperson who "is pretty fit" with "his BMI is 35.9", a high value which characterized him as obese. The obese categorization was made by using the supplied BMI graph categorizing persons with a BMI value between 35 and 39.9 kg/m^2 as obese class II. S2 disputed this by arguing that the player might have a high BMI, "but he is not obese". In this case, the BMI was not deemed adequate for use, and the PTs found its uncritical use as problematic. South African PTs often referred to extreme cases of sportspeople such as bodybuilders or boxers to argue that the "BMI is restricted in its use" and thus people in sports "should be in a different kind of scale".

Norwegian PTs discussed the lack of a variable that differentiated between muscles and fat:

N2: Yes. But I know that it is very [much] discussed in relation to, for example, people who have a lot of muscles, it is in a way not fair. Because then … being overweight …

N3: Muscles weigh more than fat.

The PTs referred to muscular people for whom the categorization by the BMI "is in a way not fair" because "muscles weigh more than fat". This meant that they are likely to have a higher BMI and be categorized as obese, even though they trained and had muscles, and were as such far from being obese.

In their discussions of the use of BMI in society, all the groups of PTs problematized the inappropriateness of some of its uses. They pointed to the need of having *knowledge to interpret* the BMI number, what is behind it, what it is good for and what it is not good for. A South African PT related this as:

S3: Look at the classic examples they are using nowadays, with the Barbie presenting like a certain body type in the … So if you don't fall then … and if you don't realise what variables they're taking into account, then you might get a negative view about yourself.

S3 started with reality, with Barbies being used as ideals. He used this as a parallel to BMI, and returned to the BMI again, "so if you don't fall then", referring to the

BMI values that do not fall into the "normal" category. He mentioned the "variables they're taking into account", expressing awareness about the construction of the BMI as a mathematical model showing one part of reality as Skovsmose (2014) discussed. S3 ended with "you might get a negative view about yourself" which he saw as the consequence of teenagers being placed in the categories of BMI without knowing what is behind the model. He drew attention to young people needing to be able to interpret the BMI carefully before drawing unwarranted conclusions based on using only the BMI. These considerations of the potential consequences the BMI can have on people and in society indicated that the PTs were interacting with the model from a socio-critical modelling perspective (cf. Barbosa 2006; Skovsmose 2014).

Also coming to the fore in this theme was the *reasons for using the formula.* As a Norwegian PT said: "I think they have made it so that it will be a simple tool", as Hall and Barwell (2015) noted that the BMI is simple to apply to large populations to keep track of obesity. Another PT confirmed this: "Different countries would operate with the same index; they would also be able to say something about those countries in relation to each other".

42.4.2 Social Factors

In their deliberations on the role of the BMI in society, PTs raised the issue of *social factors* as a missing variable when judgements related to obesity and health based on the BMI are made. S5 drew attention to the pervasiveness of social factors by stating "… where we are, it's very difficult to ignore the social factors". The "where we are" points to the low socio-economic areas where they were from and were likely to teach in.

In relation to the decontextualized use of the BMI, its universality was questioned since there might be *differences between countries*, as exemplified here:

S1: And for each country, I've heard that this obese thingie differs. They say like in America, you, even when you're thin you are classified as obese because of the unhealthy food they eat and I think in South Africa it will be totally [different]

By saying "this obese thingie differs" and mentioning the USA (America), S1 said that differences in the obesity phenomenon can be connected to diet differences or eating habits. This is further underpinned when S1 referred to "the unhealthy food they eat" in the USA. It is however not quite clear what she meant by "even when you're thin you are classified as obese", but it can be interpreted that she argued that the use of the BMI should be country specific rather than universal. This reference to other countries, both more and less developed than South Africa, with different social conditions was further alluded to in their discussions.

When discussing the role of the BMI in society, a link was made to the ease of *accessibility of BMI.* For the Norwegian PTs, this was problematic: "I think it is all too easily available when you know how crazy misleading it can be". They referred to the several internet versions of the BMI formula, where you easily get a BMI number and a category with prescriptions by just putting in the mass and height. The problem with such accessibility is in this way related to the consequences for using the model.

This category indicates that it is not only the model construction issues, such as the absence of certain perceived variables, that occupy the deliberations on the BMI, but also social factors which might not be easy to include in the operative formula. In addition to the meta-validation of the BMI model by discussing variables not taken into account as Niss (2015) recommended, the PTs interrogated the BMI in relation to the contexts in which it is used. They found it problematic as a universal measure due to the differing social conditions in different countries. They discussed hence the consequences of its use, engaging in what Skovsmose (2014) called the mathematics in action.

42.4.3 Lifestyle

PTs commented on *eating habits,* in USA and South Africa, affecting the BMI of an individual. With respect to the USA, S1 said: “Look at Americans. They’ve got so much stuff pumped in their foods. They flipping huge”. About South Africa, the same student said:

S1: [I] often … just see them just eating packets and packets of lambamba beefs [a kind of potato chip] … they not eating anything that’s good for them. So, their BMI might say they normal, but are they healthy?

Another aspect not accounted for in the BMI formula, a missing variable, related to *physical activities.* The South African PTs discussed different kinds of sports requiring different kinds of exertions as, for example, chess and rugby. They felt that the BMI formula did not consider a person’s *metabolic age*—a measure of how fast your body burns energy. To them, a very fit person would have a lower metabolic age than a person who leads a sedentary lifestyle. The PTs also commented that the BMI values and how they are interpreted related to the “ideal type”, what they called the “average person”. This suggests that BMI values and their interpretation are based on many observations and that a particular BMI value attached to a specific person cannot be seen in isolation to lifestyle factors. The research literature is replete with observations regarding lifestyle and BMI (e.g. Kushner and Choi 2010).

42.4.4 Engagement with Mathematics

The prospective teachers’ engagement with the mathematical constructs was one of the themes during their discussions of the BMI. The first set of questions on the task sheet referred to issues such as what is measured by the formula depicting the BMI. Mathematical engagement concerns the prospective teachers’ engagement with mathematical ideas, how they try to give meaning to the formula and how they use mathematical entities in their deliberations. These issues are connected to the meta-validation Niss (2015) proposed for prescriptive modelling.

The Norwegian PTs wondered about the reasons for using the square of the height in the formula. They do not come much further in those discussions, remaining at a hypothesis level. One of the groups was talking about an alternative formula, mass divided by height:

N2: Then you get 30.19, then …
N1: Yes, so you are in the borderland of obesity if you do not use it squared [the height] …

N2 used a value to calculate the ratio of mass to height and obtained 30.19 kg/m^2 as a result. This was done after they used the same values in the original formula. N1's assertion "you are in the borderland of obesity" referred to the BMI intervals where people with a BMI higher than 30 kg/m^2 are considered obese. The PTs were trying alternative versions of the formula, but keeping the same intervals. N3 entered the conversation at this point by saying: "But if, think if it was not squared, right? Then the kilos would in a way get divided in how many metres we are, right?" N3 was trying to consider the formula as mass divided by height through the utterance "think if it [the height] was not squared". With this hypothesis, N3 continued by saying, "the kilos would in a way get divided in how many metres we are". She sought confirmation by adding "right?" and tried to make sense of the ratio m/h, as a physical explanation for kilos divided by the height in metres. Teachers in Kacerja et al. (2017) had similar ideas in their discussions of the BMI formula. The PTs did not pursue this idea further.

The South African PTs suggested adding heart rate as a third variable that would affect the BMI intervals so that the sportsperson in the task sheet picture would not be considered obese. One student pointed thus to the need for changing the formula as well. S4 proposed:

S4: So, say you run for two minutes and then your heart rate is 74, say 80 to make it easy. Then they could give, depending on how much you want the scale to move, they could have the heart rate over 3 or 2, so it will be either plus or minus that heart rate. I mean, based on that, that value will be moved, the additional plus or minus where the point lies on this [the chart with BMI accompanying the task sheet]

S4 explained the role of the heart rate and a possible way it could be included as a variable "the heart rate over 3 or 2". This would be "either plus or minus" and thus added or subtracted, based on what "they" wanted the scale to do. S4 pointed to the chart and continued "that value will be moved, the additional plus or minus where the point lies on this". He suggested that the BMI for the sportsperson could be moving to the left or the right of the chart, being placed closer to the normal or the obese intervals. With the intervention of the lecturer, S4 came up with the formula "kg/m^2 + heart rate/2". The lecturer drew their attention to the issue of adding or subtracting quantities with different dimensions. That question remained unanswered.

42.5 Concluding Comments

The research presented here suggests that when PTs engage with mathematical models that are used in society, they can point out several critical issues that come with the use of models. Examples of prescriptive modelling, such as the BMI used in this study, have the potential to engage PTs in discussions about the mathematical aspects, such as relevant missing variables and meta-validation processes; they also have the potential to encourage critical discussions about the role of mathematical models in society, which is important for developing critical competence. Given the emphasized relevance of prospective teachers' engagement with mathematical modelling in teacher education, it is also crucial that teacher educators facilitate for critical discussions so that the PTs can realize the potential of mathematical models used in society. Several of the elements from prescriptive modelling (Niss 2015) and critical perspectives in mathematics education (Barbosa 2006; Hall and Barwell 2015; Skovsmose 2014) are identified in prospective teachers' discussions, showing the relevance of combining the two perspectives in modelling. Similar to Villarreal et al. (2015), our PTs were able to reflect on the role of mathematical models in society, even though they did not create the models themselves. Overall, the study reveals that it is possible to incorporate indexes as mathematical models in teacher education courses, so that the socio-critical perspective of modelling forms an integrated component of their mathematical modelling experiences.

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Chapter 43
Carbon Footprints Calculators and Climate Change

Lisa Steffensen and Suela Kacerja

Abstract From a socio-critical modelling perspective, we research how students reflect when using a Carbon Footprint Calculator (CFC) in their work with climate change in the mathematics classroom. The findings show that lower secondary school students reflect on issues such as: how the calculators work; how variables impact the output; how to make sense of the calculator by breaking down its components, exploring extreme values, and comparing their own results to others; the people behind the calculator and its limitations; individual, national and global emissions, and finally, the "take-home" and "bring forward" message. It is suggested, based on these findings, that CFCs have the potential to bring about critical reflections on mathematical models with the power to impact people's lives.

Keywords Socio-critical modelling · Critical mathematics education · Prescriptive modelling · Climate change · Carbon footprint calculator · Students' reflections

43.1 Introduction

Climate change is a big challenge in society, and greenhouse gas emissions contribute to climate change (IPCC 2014). There is a global focus on reducing greenhouse gas emissions, however, such emissions are challenging to quantify or to visualize. Different greenhouse gases have different warming potential for the atmosphere and a common unit, CO_2e, is used to make it easier to compare emissions from gases such as CO_2 and CH_4. CO_2e refers to the amount of greenhouse gas that, over a given period, causes the same integrated radiative forcing as an emitted amount of CO_2 (IPCC 2014). It is thus important to identify which emission that contributes

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the most. Wynes and Nicholas (2017) described that governments focused on lower-impact actions (e.g. recycling and changing lightbulbs), and recommended that more effective actions should be promoted (e.g. living car-free and eating plant-based food).

One approach to make individual emission quantifiable is to use mathematical models such as CFCs. Salo, Mattinen-Yuryev and Nissinen (2019) described CFCs as a soft-policy measure for raising public awareness of the individual carbon footprint, and as a tool to steer consumption. They emphasized that reflection and awareness are not equal to actions, but that the output of the CFCs (the amount of CO_2e) provide an initiator for discussions and potentially a change in behaviour. Similarly, Edstrand (2015) highlighted that CFCs are used as an educational tool for environmental awareness. She argued that CFCs could be a way of making the invisible visible by calculating the individual CO_2e-emissions. The global CO_2-emission has increased from about 5 billion tonnes in 1950, to over 36 billion tonnes in 2017 (Ritchie and Roser 2019). The CO_2-emission varies from countries, for instance, Americans and Australians emit about 15–17 metric tons of CO_2 per year, Norwegians about 8–9 metric tons of CO_2, while Samoans emit about one metric ton of CO_2 (Ritchie and Roser 2019). The CFC convert a person's consumption into a specific number, and thereby one gets an impression of one's own contribution. On the one hand, CFCs, as a way of quantifying emissions on an individual level, could bring the attention towards the relationship between individual human activities and the global emission; and potentially lead to greater individual responsibility for one's own actions. On the other hand, CFCs are not a precise tool, and different CFCs can produce quite different results for the same person. They are also quite complex, and there are many hidden assumptions and hidden mathematics behind the output of CFCs. Carbon emissions are highly relevant in public debates on climate change and considering how models such as CFCs and the mathematics behind them can be used to steer citizens, it is crucial that citizens possess critical competencies in order to make informed decisions.

In this chapter, we argue for the relevance of including CFCs—as mathematical models in the classroom—for several reasons. Firstly, to raise awareness of the use of CFCs as soft-policy measure; secondly, to make students aware of the hidden mathematics behind such models; thirdly, to make the invisible visible by engaging with critical reflections; and lastly, to let students become aware of, and to critically reflect on, their own carbon footprint. The empirical data is from lower secondary school. The focus of this research is *how students reflect when using a CFC when working with climate change in the mathematics classroom.* We use student groups' reflections while exploring the CFC, as a starting point to reflect on how CFCs could be used as educational tools in mathematics education.

43.2 Theoretical Foundation

This study takes a critical perspective in which the perception of mathematics as objective and representing “the truth” is questioned. Instead, a Critical Mathematics Education (CME) perspective aims to highlight that mathematics reflects the interests and values of the people involved, and mathematics is considered as neither objective, neutral nor value-free (Skovsmose 2014). In a society where mathematical models are used to describe phenomena from everyday life and to even prescribe rules and regulations such as the CO_2-tax that affects citizens’ lives, it becomes important for citizens to identify and critique how mathematics shapes society. This side of mathematics, which Skovsmose (2014) called the formatting power, gives agency to people who initiated the model, to people who built it, and to those who use it. Climate change is a challenge in our society where scientists rely heavily on mathematics to describe, predict and communicate around the issue (Barwell 2018). Barwell (2018) argued for the mathematical formatting of climate change and thus the importance for mathematics educators and researchers to be involved in discussing this challenge. These arguments point to the potential for students to be exposed to climate change issues in their mathematics classes as one way for them to be involved in discussions about mathematical models and the formatting power of mathematics.

Mathematical modelling is the process of finding mathematical solutions to problems from real life. What different models of the modelling process (e.g. Blum and Leiß 2007; Niss 2015) seem to have in common is the relationship reality-mathematics where modelling happens and the cyclic nature of the process. Reality is the starting point for modelling, used to decide the mathematical variables, and the ending point to refer to whenever a mathematical solution is found. Also typical is the translation between reality and mathematics all the time, as well as the several cycles until a satisfactory solution is found. Such a cycle could describe the work done by the designers of the CFC, with the aim to make visible emission quantities by categorizing people based on their input data. A mathematical model is a result of a mathematical modelling process.

Niss (2015) described prescriptive modelling as aiming to design, prescribe, organize or structure society. There, “the ultimate aim is to pave the way for taking action based on decisions resulting from a certain kind of mathematical considerations” (2015, p. 69). This kind of modelling exerts some power, in the sense that it is being used to model the way we look at certain phenomena from reality, and not just to describe something mathematically. The prevalence of CFCs might not be spread throughout the population; however, they could be considered as a way of regulating an individual’s consumptions and way of life. By assigning people into different categories, CFCs define and prescribe what is acceptable and not acceptable in terms of amounts of emission one releases. An often-expressed goal is that individuals should adjust their consumption after becoming aware of carbon emission. Based on the definition by Niss (2015), CFCs can thus be considered as examples of prescriptive modelling. While the modelling process cycle has proved fruitful as an analytical tool in different research examples (see Kaiser et al. 2006), such a cycle will not do the same in cases when students work with examples of prescriptive modelling (Niss 2015). In such cases, the students explore mathematical models which other people, the specialists, have developed. Niss (2015) suggested two possible focal points for engaging students in complex prescriptive modelling. One such point is to identify what assumptions and prerequisites are behind the models, and how they influence the model. In terms of CFCs, this would mean to identify variables taken into account, and the assumptions for choosing those variables. The other focal point is the meta-validation in which one model of CFC is compared to alternative models whenever possible, considering what the consequences of using the model are, and what changes in variables would mean for the model. Both focal points from Niss are in line with the ideas of CME, where looking at the mathematics, the ways and contexts it is being used, and the consequences for its use, define what mathematical competence should look like.

Barbosa (2006) introduced a socio-critical perspective in modelling which is based on ideas from CME such as the formatting power of mathematics. The aim here is to develop a critical understanding of society in a social and political context (Stillman et al. 2013). Therefore, understanding the function of mathematical modelling and the nature and role of mathematical models in society is the focus of this modelling perspective (Barbosa 2006). In Skovsmose's (2014) ideas of CME, it is important for students to critically reflect on how mathematics affects our lives. Similar to Skovsmose, Blomhøj (2009) highlighted reflection and critique as important and emphasized that the object of such critical reflection can be the modelling process, the actual applications of a mathematical model, or societal issues. Enabling students to critically reflect on societal issues is a way to empower them through mathematical modelling. Although critique and reflection on societal issues are imperative, a socio-critical perspective could also involve action. Mellin-Olsen (1987) emphasized that "a call for a critical awareness excluding action may be interpreted as just another discussion" (p. 204). So, mathematics education involving societal issues should, according to Mellin-Olsen, consider how students could take actions upon these challenges. The action could be a way to enable students coping with the sometimes-overwhelming hopelessness when facing problems like climate change (Freire 1992).

One problem with some mathematical models used in society is that the mathematics behind those models is hidden, and it is not clear for the user what the assumptions for that model are. The model serves thus as a black box in which with certain numerical data one gets a certain number or category as output. Therefore, it becomes important to find ways to explore the model's characteristics, hidden assumptions, and the model's appropriate use. Working with existing mathematical models has, therefore, the potential to offer students opportunities to experience and engage with the formatting power of mathematics.

In order to understand the relevance of CFCs as mathematical models, we present some features of one CFC (Norwegian Broadcasting Corporation 2014). A CFC requires some input data from the users and provides the amount of CO_2e an individual emits. While some CFCs require very detailed input and provide detailed output, the chosen CFC had a relatively low entry threshold for submitting answers (input) and provided fewer details (output). The designers of the CFC selected four main areas: housing, consumption, transport and food. Housing includes variables such as house type, number of persons, type of heating and electricity use. Consumption includes variables such as the monthly income, loan and general consumption, while transportation involves domestic and non-domestic flights and daily transportation. Food includes numbers of meat, fish and vegetarian dinners.

The mathematics behind the model is not openly observable. One has to explore the output values for different input values to understand more about the variables that have the bigger impact. Some description on which numbers the calculations are based on is provided online. For instance, the designers stipulate 3.2 kilo CO_2e for a meat dinner, 1.8 CO_2e for a fish dinner and 1.5 CO_2e for a vegetarian dinner. In addition, an average of 670 CO_2e is added each year for the other meals. Concerning daily transportation, the designers stipulate 2.9 kilo CO_2e per litre for gasoline and diesel, and 121 grams CO_2e per kilometre for electric vehicles. These estimates are based on numbers calculated by Norwegian research centres (Steen-Olsen, personal communication, April 27, 2020), and considerations such as user-friendliness were taken when designing the CFC, which influenced their modelling process. Different representations of the output are provided: a "speedometer" for the total CO_2e-emission, numbers displayed in four boxes (the four areas), a sector diagram, and two written responses. The first response compares one's CO_2e-emission with the average Norwegian and the average world citizen: "Each Norwegian emits 11.5 tons of CO_2e per year when we include consumption. A world citizen releases 6.5 tons of CO_2e per year". The second comment depends on the amount of CO_2 emitted and comes in three categories (with fictive numbers inserted):

1. My carbon footprint is 8.42 tons of CO_2. I am an environmental angel, at least compared to other Norwegians. But there is no reason to relax anyway!
2. My carbon footprint is 10.13 tons of CO_2. I am neither a climate pig nor an environmental angel, but an average environmental Norwegian. Maybe time to start walking to work?
3. My carbon footprint is 35.20 tons of CO_2. I am a climate pig. Now it's time to sharpen!

We use the theoretical ideas of critical mathematics education and formatting power of mathematics, focal points for working with prescriptive modelling, and reflection and critique of models and modelling, to analyse students' investigations of the CFCs.

43.3 Methods

The empirical data was collected through a one-year research partnership with three mathematics and natural science teachers and their 15-16-year-old students. The focus was on the facilitation of critical mathematical competencies in the context of climate change. The students had different kinds of activities during the year, such as excursion, discussions and debates. Here, we use data from one lesson where the students prepared a poster for an exhibition. The students were working in groups of 3–5 students. The teacher had made a task with the wording: “Choose a number that plays a central role in climate change. […] Show why this is an important number and what this tells us about global warming”.

One group of three boys came across a CFC while doing research on the Internet for their poster. The empirical data analysed here consists of video- and audio recordings from this group, and the students’ poster. The data was transcribed and coded in NVivo to get an impression of the different reflections. The initial coding resulted in seven main categories: practical-related issues, global temperatures, graphs/interpretation, prognoses/uncertainty, global/individual impact, CO_2 (e.g. sources, emission), and CFC. Some of the utterances overlapped and were coded at several categories. For instance, when the students explored the CFC, one utterance concerned both CO_2-emission and individual impact. In this chapter, we focus on one theme, the CFC. This theme was further categorized to find out how the students reflected while working with the CFC. This resulted in two main categories; “Investigating the CFC” and “Global, national, and individual emission”. Both themes had sub-categories, and an overview of these categories is presented in Fig. 43.1. Due to page-limitation, not all sub-categories are elaborated on, and the selected examples were chosen because they are relevant, interesting, or representative examples for the research question.

43.4 Findings, Analysis and Discussion

The focus of this research was on how students reflected using a CFC when working with climate change and mathematics. Figure 43.1 presents an overview of themes emerging from the analysis, and the two main themes, “Investigating the CFC”, and “Global, national and individual emission”, are discussed in the following subchapters.

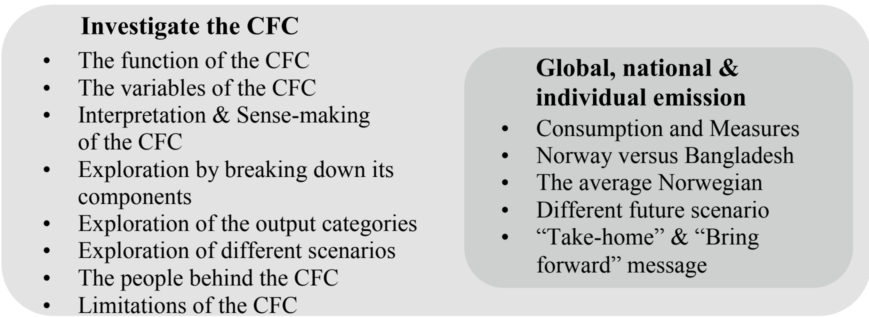

Fig. 43.1 Utterances concerning the CFC were categorized into these themes

43.5 Global, National and Individual Emissions

At the start of the lesson, the students clarified with the teacher that individual, national and global emission were all relevant. The theme *Consumption and Measures* captures the idea of individual emission, and it involves discussions about how consumptions can be measured. The first step in measuring is to define the object or the phenomenon to be measured and the attributes that better represent it. Some typical examples of the attributes mentioned were "how long you shower and how many products you buy [...] how much CO_2 you use in commuting to work". Given that some actions have a higher impact than others, it could be relevant to understand and to differentiate between high-impact and low-impact actions as highlighted by Wynes and Nicholas (2017). In terms of mathematical modelling, this means that students need to discuss between those variables that are more important for the phenomena being measured, critically reflect on the assumptions and prerequisites behind the model (c.f. Niss (2015) focal point), thus choosing between the given variables.

The theme of *Norway versus Bangladesh* captures the idea of national emission and involves one article they found on the Internet. Mark read out loud part of the information:

> Norwegians have 39 times higher emissions per capita than the inhabitants of Bangladesh. It only takes 9 days for the average Norwegian to emit as much CO_2 as a resident of Bangladesh does during a whole year. With its 149 million inhabitants, Bangladesh overall contributes less to climate change than 4.6 million Norwegians. (Tajet 2007, p. 1).

Afterwards, he said: "This is totally crazy [...] We Norwegians contribute with 39 times more emission than Bangladesh does. 39 times!" His choice of words, along with his tone of voice and facial expression, can be interpreted as the numbers were astonishing. When Swedish students used CFC, they compared themselves with Americans with a higher carbon footprint (Edstrand 2015). Contrary to the Swedish students, these three boys compared their emission with citizens with much lower impact. In Norwegian public debates, an often-used argument is that since Norway has only 5 million inhabitants, it would not make a difference what citizens do. When the students reflect on the quantification and contrasting between Norway and Bangladesh, it becomes possible for them to critically understand an abstract notion as Norway's footprint and their society, in line with the aims of socio-critical modelling as highlighted by Stillman et al. (2013). It is also in line with what Blomhøj's (2009) emphasis on the relevance of reflecting on issues such as different ways of living and differences in rich and poor countries/citizens. Comparative reasoning relates to measurement, and because of a lack of familiarity with emission quantities and their measurement, by comparing and contrasting emission quantities between these two different countries, the students found a way to create an idea of those quantities.

The theme *"take-home" and "bring forward" message* concerned the messages students included as important to bring forward to themselves and to others. An example of the "take-home" message was when Mark and Henrik manipulated the type of food inserted in the CFC. They explored what the change in the variables would mean for the model, the CFC, in line with Niss (2015) focal points. Mark said: "If you only eat two meat dinners and say five fish dinners, then you're suddenly down at 9.92 [tons of CO_2e per year]". The CFC stipulated 3.2 kilo CO_2e for a meat dinner, 1.8 CO_2e for a fish dinner and 1.5 CO_2e for a vegetarian dinner, and although the students did not question how these figures were estimated, they used them to lower their carbon impact. They continued to adjust other variables and after a while, Mark said: "3 meat meals, 2 fish meals, and a vegetarian dinner. That's realistic […] This is possible". His tone of voice indicated that he considered the combination of different meals, as something he could do. He had critically reflected and accepted the model and reflected on possible actions in his own life, like those Mellin-Olsen (1987) argued for should accompany a critical awareness. An example of the "bring forward" message was observed at the students' poster: "Help our planet. Pollute less, reuse, use public transport, and eat less meat. Doing these little changes will help save the planet". This statement can be argued as having a clear message towards acting more sustainable, by highlighting the message on a public poster on an exhibition. In addition, it can be argued that it brings hope that action is useful, what Freire (1992) highlighted as important. Although we cannot draw the conclusion that the CFC is the reason for this message, nor that the CFC has changed students towards a more environmentally friendly consumption, like Salo et al. (2019) highlighted, it did facilitate reflections on these issues.

43.6 Investigating the Carbon Footprint Calculator

Aspects of investigating how CFCs works are included in the theme *the function of the CFC*. Mark investigated how the income variable affected the carbon footprint by comparing two different income levels. He concluded: "So if you make more money, the carbon footprints grow". By changing only one area, Mark noticed a positive correlation between income and carbon footprint and thus identified an assumption in the model, compared with Niss (2015) focal points. Being challenged by the researcher about the correctness of this correlation, Mark replied: "So this is just how they envisage you spending money". Mark did not specify who "they" are, but it is reasonable to think that he referred to the designers of the CFC, or those initiating the design. Mark's answer about *people behind the CFC* can be connected to the formatting power of mathematics (Skovsmose 2014). This example can also be connected to the students' reflections about the modelling process that takes place. In this case, the designers decide how to calculate certain values by deciding, e.g. a priori that the more money one makes, the higher is the footprint and not taking into account other variables.

The theme *exploration of the output categories* concerned the three categories "climate pig", "environmental angel", and "average Norwegian". These comments from the CFC added a humoristic element to the output, as seen when Peter laughed out loud at several occasions. In one of these incidents, he uttered: "I've written an insane number of kilometres". He read aloud from the feedback: "Ohh! I am a climate pig, now it's time to sharpen". This incident resulted in that all three boys tried out extreme values of the variables of the CFC and explored what inputs gave the higher output (in CO_2e), as one way to get familiar with the hidden nature of the mathematics behind the CFC highlighted by Skovsmose (2014). However, the normative aspects of prescriptive modelling are not always uncomplicated, and . for example in the Scandinavian countries, a recent focus has been on shaming people who drive, fly, eat meat, et cetera. Based on Peter's quantitative input, the CFC categorized him as a climate pig, and one could argue that mathematics and technology, by measuring and categorizing using predefined criteria, is intended to format his behaviour in line with the formatting powers of mathematics (Skovsmose 2014).

A reflection by Mark illustrates the theme *limitations of the CFC*: "If you could choose to buy short-travelled food more often, or yes… then the average might decrease. But now they have only imagined that you buy regular meat products that may have travelled far to get to you". Mark started by pointing to a limitation, no option for short-travelled food. He critically reflected on how such an option could have decreased the average. He highlighted that "they" (the people behind the CFC) "only imagined" people buying regular products. Mark's reflections are related to the ones highlighted by Blomhøj (2009), criticizing specific mathematical modelling processes and the authentic applications of this in a real-life situation. It is also related to what Niss (2015) suggested, to identify or analyse the hidden assumptions underlying the modelling process. In building the CFC, as happens in modelling, the designers have to choose which variables to include. One such variable is the amount of food one consumes, which is then given a numerical value for measuring its impact. In this case, the designers have given one single value for food, not distinguishing between imported or local food which makes, in fact, a difference in the impact. Many other examples of prescriptive modelling include such decisions to use offset variables, which is a critical point one can discuss related to the impact that the model has when used for different purposes.

43.7 Concluding Comments

The focus of this research is on how students reflect when using a CFC when working with climate change in the mathematics classroom. We consider the CFC as an example of prescriptive modelling where the mathematics is hidden. Our findings show that students' reflections displayed a variety of issues, and they explored the model corresponding to some of the focal points recommended by Niss (2015) with prescriptive modelling. They reflected on how the CFC works, how the different variables impact the result; attempted to give meaning to the CFC by breaking down its components; explored extreme values and compared the individual average; reflected on the people behind the CFC, on its hidden assumptions, and its limitations; reflected on the global, national and individual emission; and reflected on the "take-home" and "bring forward" message. The students reflected about some sides of the modelling process where the designers include certain offset variables, such as the decision to give a certain value to food types independent of their length of travel.

It is important to highlight what the students did not focus on. For instance, they did not analyse, question or critique: whether the numbers were reliable, trustworthy or relevant; how the calculations were carried out, what figures the designers had chosen to base the calculation on; if the selected areas were representative, nor did they identify or question the purpose of the CFC. Investigating these types of questions is crucial in CME. It can contribute to more awareness of the way mathematics affects our lives and are relevant when working with prescriptive modelling and CFCs. The student group in this chapter accidentally came across the calculator. However, the students' explorations on their own show the potential that working with such examples has for their competence in exploring mathematical models, their initiating awareness about the humans behind the models, and their reflections and actions towards discussing and dealing with critical societal issues such as the climate change. Further research would be desirable to focus on situations that facilitate critical discussions on environmental issues with prescriptive models, such as comparing and contrasting different carbon footprint calculators in the mathematics classroom.

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Part VIII
Tertiary Level

Chapter 44
Understanding Links Between Mathematics and Engineering Through Mathematical Modelling—The Case of Training Civil Engineers in a Course of Structural Analysis

Saúl Ernesto Cosmes Aragón and Elizabeth Montoya Delgadillo

Abstract We studied mathematical modelling in the engineer training program by contrasting the problems presented by an engineering professor in a course that is part of the structural analysis professional education core and the modelling carried out in a project that was developed by students within the class workshop, which corresponds to the practical section of said course. The results provide evidence that the modelling competence is promoted and developed in engineering and that it is possible to consider it as a connector between various training cores, identifying mathematical models that will allow us to understand and establish relationships between such training cores.

Keywords Mathematics modelling · Structural analysis · Engineer training · Modelling cycle · Modelling competence · Mathematical work

44.1 Introduction

Education of engineering students calls for the development of competencies that will allow them to address situations both at the school and their future professional development levels. Various accreditation programs for engineering training consider modelling as a competence to be promoted. Despite the broad evidence on the importance of modelling in the educational system, research is still limited regarding how it can be incorporated into the classroom (Borromeo Ferri 2018; Blum 2015), and more specifically, engineering lacks research about how modelling, under a perspective of mathematics didactics, is addressed in engineering education (Alpers 2017;

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Romo Vázquez 2014). We studied the training core aspects and chose the professional training aspect to study the role of (mathematical) modelling in a program that declares that its engineers are trained in this competence.

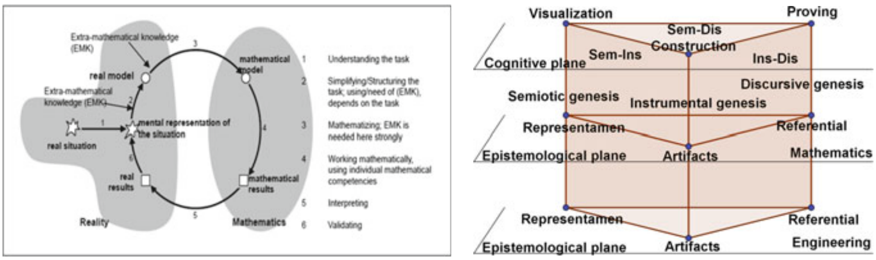
Specifically, in engineering, Alpers (2017) states that engineering students face situations in which they must solve problems through the construction and use of mathematical models, and therefore he considers that the competence of mathematical modelling is central to engineering education. In addition, Kaiser and Brand (2015) state that although the *mathematical modelling competence* construct has been addressed in research, the role of the notion of competence and how it can be characterized in different academic levels are still to be studied. It is worth mentioning that, for them, the modelling competence is more than an ability, but it is also the disposition of the individual towards facing a modelling situation.

Niss and Hojgaard (2019) within the framework of the mathematical learning and competences (KOM for its name in Danish) define eight mathematical competences, one of them being the *mathematical modelling competence.* In addition, they established that competences are interrelated. It is worth noting that in their definition of mathematical modelling competence, the authors consider that extra-mathematical knowledge must be explicit and have a genuine relationship with the mathematical world. In our case, this extra-mathematical domains are the engineering problems that are solved both in the context of the class as well as the project, where the extra-mathematical is related to the understanding of situation analysis of structures and what is related to mathematics are the models that are used and constructed from calculus notions. In this way, we address the following research question: How does modelling training happen in engineering?

44.2 Theoretical Framework

We considered the mathematical working spaces (MWS) (Kuzniak et al. 2016). This theoretical framework allows to understand the mathematical work developed by subjects that are facing problems within a school context, but it also allows the development of didactic sequences to be implemented in the educational system. MWS is formed by two dimensions, one is epistemological, related with mathematical knowledge, and the other is cognitive, related to the process of construction of mathematical knowledge on the user's part. The levels are articulated through genesis, a semiotic genesis that allows to make mathematical objects tangible through the transition between the poles of the representation (icons, index, symbols) and visualization (interpretation and decoding of the signs through semiotic processes). An instrumental genesis allows to make artefacts operational through the use of symbolic or technological artefacts, connecting the artefacts and construction poles and a discursive genesis that connects the referential and proving poles, where the proof must be limited to the use of theory more than experimental intuitions.

We consider the importance to incorporate a didactic theory to analyse mathematics that are present and not only understand the modelling process. The Borromeo



(a) (b)

Fig. 44.1 **a** Modelling cycle. (Borromeo Ferri 2006, p. 92) **b** Extended MWS for engineering. Adapted from Moutet (2016)

Ferri modelling cycle, from now on MC (see Fig. 44.1a) (Borromeo Ferri 2006, p. 92) has allowed us to organize and study how problems are addressed in the engineering class. The MC is based on a real-world component, from which the situations to be modelled are from and to which we ascribe the extra-mathematical contexts, and another component related to the mathematical world, which provides elements to vertically mathematize the situation that needs modelling, but at the same time provides stages that articulate both components through mathematization and interpretation processes that allow to confront mathematics with the real world.

Based on Moutet's (2016) study, who proposes an extended MWS to link physics and mathematics, we propose a connection between mathematics and engineering, where the epistemological level (see Fig. 44.1b) is composed by signs, artefacts and theoretical references pertaining to engineering.

The MC allows to identify subcompetences of modelling in particular and the modelling competency at a global level. It is in the subcompetence of working mathematically where the MWS intervenes to analyse student production, the class of a teacher or when this is used for designing modelling situations.

44.3 Methodological Elements

This research was developed through a qualitative methodology under the instrumental-type case study method, as in Stake (1995). In particular, the case studies the civil engineering program of a Chilean university. Out of the cores of basic sciences, engineering sciences and professional education, we chose the latter one in the area of training in structures.

We considered three sources. Source 1: an interview to a teacher of structural engineering who has professional experience in the area and informed us about the role of modelling not only in the professional training field, but also in the workplace. Source 2: problems worked in the structural analysis class throughout the academic semester. Source 3: a structural analysis project developed by students (4 teams of

3 members each), assigned during the workshop of the course. For this chapter, we present the analysis of the collected data in source 2, and from source 3 we show a first part that explains the types of problem students face.

The analysis of the problems was performed based on categories established with the theoretical references used. For the extended MWS, we considered the way in which the epistemological and cognitive levels were articulated through working with the genesis, and for the MC we considered the subcompetences of modelling in order to identify four moments in the class taught by the teacher, inspired in said subcompetencies, without being exactly equivalent. We have called them M1, M2, M3 and M4 (see Sect. 4.2).

44.4 Results

The following are the scientific academic situations addressed in the class, a problem solved in class (calculation of deformations) and a student project.

44.4.1 Scientific Academic Situations Addressed in the Course

The situations correspond to the treatment of structures through the use of models associated with structures such as beams, frames and trusses. Two other types of problems were addressed, which were related to the management of structures that are statically determined and those that are statically undetermined. The theoretical models were used to ensure their capacity to resist and suffer appropriate deformations, and this is a goal for the modelled problems.

We observed that throughout the course, the solutions to the problems in general were built on two hypotheses: the hypothesis of a linear elastic structure behaviour, this is, that the relation between the applied effort and its deformation is a linear relation, where proportionality is measured through an elasticity module (E). The second hypothesis we observed is geometrical linearity, this means that the structure can only endure small deformations, in such a way that these would be virtually invisible to the naked eye.

Therefore, from the exploration of problems developed in class, we proceeded to perform a rigorous analysis of six academic situations in the sense of Niss and Hojgaard (2019). We based our selection on the aforementioned themes and the present mathematical objects. In this way, the linear model (or linearity) is present during the complete development of this course and students learn how to solve and model structures under this mathematical principle that has a meaning in the structure analysis for engineering core.

For the theme of statically determined structures, the mathematical objects that emerged were those associated with basic notions of calculus (function, derivative, integral) and in the statically undetermined structures, notions related to linear algebra emerged (mainly work matrices and notions such as singular matrix, inverse matrix, transformation matrix, which facilitated what is called the assembly of local matrices to a global matrix of the system).

From the identification of objects and associated models, we decided to select those that were related to calculus notions, in order to go in depth into the analysis. In the following, we present a problem solved by the teacher in class and some preliminary results of a project developed by the students of the class, in the context of a workshop.

44.4.2 The Frame Problem by the Professor

The problem consists of determining the structure deformation of a frame-type model of a statically determined structure.

The problem consists of determining the rotation of the N node (angular deformation) of the structure (see Fig. 44.2), which has a modulus of elasticity $E = 2,45E + 06\,\text{N/cm}^2$. The structure that we want to approach is a *frame* type, which is formed by two elements (AB element and BC element), and we can observe that element BC must support a uniformly distributed linear load of 10000 N/m.

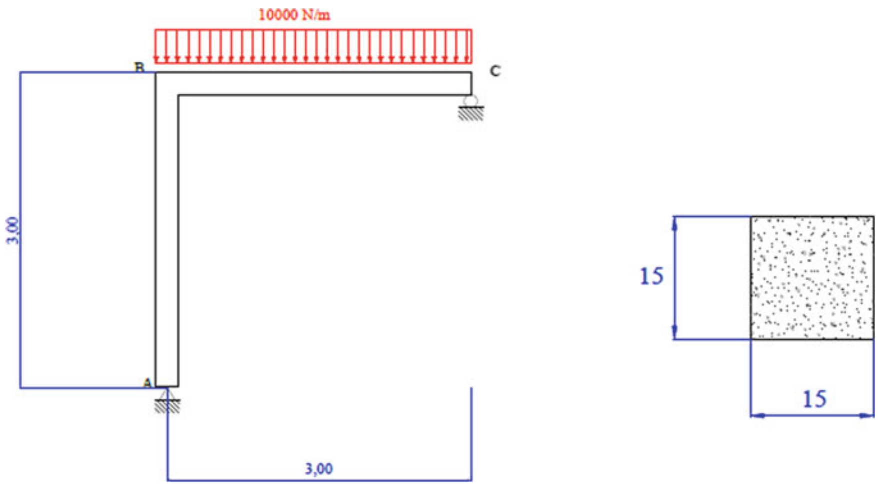


Fig. 44.2 Frame-type structure

Moment 1. Presentation of a problem and understanding the problem to be solved.

It can be observed that in the program a scheme of the structure is provided, with distances and given loads, so we can say that stage 3 of the MC is initiated, when working with a real, simplified and structured model. Regarding the extended MWS, we encounter the referential of the epistemological level of engineering, and so the discursive genesis is activated, because there are engineering concepts present in the understanding of the situation such as the frame-type structural model, conditions of material elasticity and types of bearing that support the structure (fixed in A and movable in C).

Moment 2. Considerations that arise from the extra-mathematical context presented, with the purpose of structuring and simplifying the situation.

Here, we observed work with stage 3 of the MC, because there is work with the real model presented, and a free body diagram (FBD) of the structure is elaborated (see Fig. 44.3). It can be observed that extra-mathematical knowledge of the situation strongly influences this stage, because there are support artefacts that originate the unknowns that are external to the structure (FBD). Regarding the extended MWS, we observed the activation of the semiotic genesis of the epistemological level of engineering, because there are signs associated with the supports, type of load and elements that compose the frame-type model and activate, in this way, visualization and the semiotic genesis.

In addition, we observed the activation of the instrumental genesis. Since the conditions of structure balance and the sum of forces are used, these provide symbolic artefacts coming from the epistemological level of engineering. The equilibrium situation is considered as part of the artefact of engineering and the calculations performed with the vectors that represent the force that composes the FBD are considered activators of the mathematical artefacts.

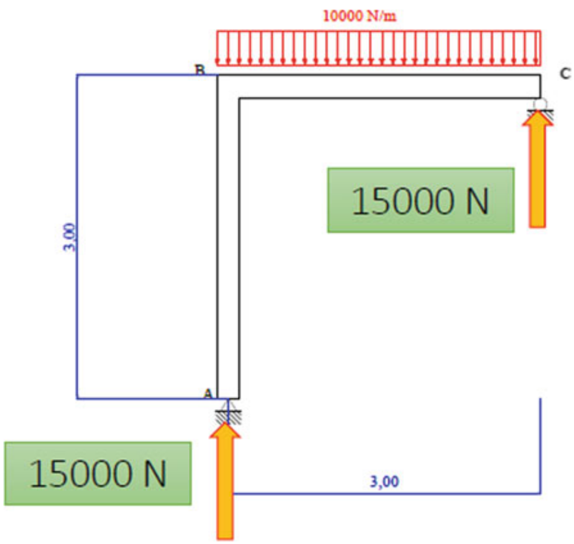


Fig. 44.3 Free body diagram of the structure

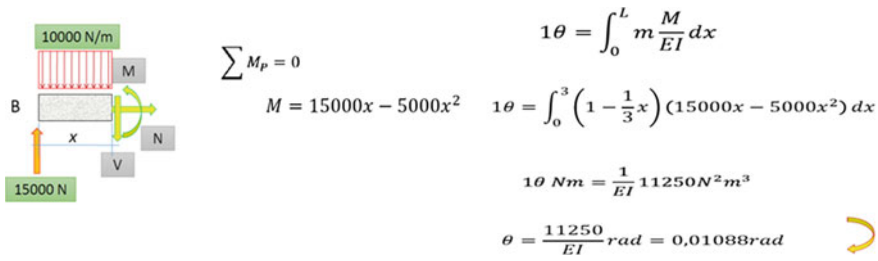


Fig. 44.4 Use of the moment function in the integral

Moment 3. Identification of the presence of a model (engineering) associated with the situation. To analyse this moment, we classified two sub-moments according to the types of engineering models, which we present next.

We identified the presence of a sub-moment 1 associated with models that stem from the conditions of balance for the calculation of external reactions and a sub-moment 2 associated with the work with models that come from the internal stresses. What follows is the development of the latter.

We identified stage 4 of the MC (work with the mathematical model) which uses the integral model $\theta = \int_0^L \frac{mM}{EI} \mathrm{d}x$, through the determination of the angular rotation of the B node with the virtual work[1] method. For this, moment functions built in sub-moment 1 are used (see Fig. 44.4), and one of them is the real moment (M), caused by a unitary virtual force applied to the point where you want to calculate the deformation (m) and the flexural stiffness of the structure (El) for vertical deformations.

Regarding the extended MWS, the semiotic genesis of engineering is activated when visualizing the signs present in the models, which will allow first the determination of the moment functions and second, the determination of the deformation by rotations in the B point of the structure. Next, the semiotic genesis of mathematics is activated when considering the equations that represent engineering models. Then, we observed the transition to the instrumental genesis of engineering, first when considering the equilibrium equations as artefacts that will allow the building of models (moment functions and shear functions, as well as the model for the determination of the deformation by rotation).

Moment 4. Presentation of the obtained results and analysis of them with the purpose of confirming the performed processes and validate them: We were able to observe that although results are presented, validation is restricted only to verification of the balance of the structure in relation to the FBD but its angular deformation results are not validated, therefore performing an internal validation as in Borromeo Ferri (2006).

In general, we observed that modelling was present, but the nature of such modelling corresponded to a structural modelling where mathematical modelling elements were present in an implicit manner. According to our analyses, the basis

[1]The virtual work method considers a virtual force that according to Hibbeler (2012) is defined as an imaginary force because it is not part of the actual load.

of the problems was a dialogue between structural models and the use, in the sense of Alpers (2017), of theoretical models pertaining to structural analysis that are influenced by the notions of disciplines such as physics and mathematics.

44.4.3 Working in a Project: Practical Work of Students

Students must model a structure using a software (SAP2000), which is used in engineering for the analysis and design of civil works such as bridges, dams and buildings. They receive plans with measurements to scale, and this is their first encounter with a problem since phase 1 of the MC in the course. We will present 3 of 4 stages of the project, since stage 4 is the armature modelling and it is virtually a process that repeats previous stages.

For stage 1, the geometrical model of a building related to social housing, offices, library and hotel (a type of building per team) was requested, whose dimensions are obtained from the plans. We observed that students presented difficulties of the instrumental kind, not only related to the use of software, but also related to working a (real) structure to scale with the plans. This was evidenced by Team 4 that worked with a hotel:

[Team 4]: The most difficult thing when modelling was to interpret the plan in AutoCAD, which was subdivided in sections of different lengths and we needed to be especially careful with measurements to then transcribe them in the SAP2000. In addition, to work for the first time with a plan at such scale resulted overwhelming at first sight (…).

For stage 2, they were asked to create and assign the sections and materials corresponding to each element of the building (structure), in addition, to generate the finite element mesh[2] for the roofs and to assign cases and load combinations and add live loads[3], and dead loads[4] to the modelling. In this stage, students worked in feeding the software with the physical and geometrical properties of various structural elements. In addition, they generated the finite element mesh with the goal of preparing the model for software analysis. We observed that students developed the modelling ability in relation to being aware of the kinds of building materials and its physical properties. This is evidenced by what is stated by Team 3:

[Team 3]: Regarding the assigned loads to our group (300kgf/cm^2) in reading areas and hallways, (400kgf/cm^2) in book areas, we consider it correct since by norm the reading area is assigned a lighter load.

For stage 3, they were asked to generate the analysis of the complete structure and assign seismic mass to the building. The assigning of loads data was provided in each project since this is a topic studied in a different course called structural design

[2]Finite element mesh: an operation performed with the goal of generating a model for the analysis of finite elements.

[3]Live loads (associated to the load of people, furniture).

[4]Dead loads (associated with the weight of the structure).

and seismic design of buildings. Students obtained the axial force, shear strength and bending moment values from the software (run analysis), for each one of the structural elements of the given structures. Students specifically stated that they had modelled a structure, which means that it is viable for building, as per the diagram with data provided by the software. They had to interpret and make decisions with calculations and during the development of the course they had to obtain by manual calculations.

44.4.4 Modelling Problems and the Role of Mathematical Models

From the analysis of the selected problems and the work with the modelling of a building by the students, we observed that the modelling competence is promoted in the training of a civil engineer. To inquire the mathematical objects that are present in modelling of structures is important to understand the relationship between mathematics and engineering.

When analysing problems that are taught for modelling, we identified calculation of mathematical objects that are present in the course that are models used (most of them in an implicit manner) for structural analysis, and we selected 6 (see Table 44.1).

To understand the nature of the problems allowed us to have evidence to identify modelling in engineering and mathematical modelling. To understand the role of diagrams, the use of professional software that not only makes mathematics invisible but that allows to model a real structure and at scale are aspects to be considered for mathematical modelling.

Table 44.1 Analysed problems in connection with calculation objects

Problem	Activity to determine	Mathematical Objects
Beam resistance analysis	Diagrams that represent the structure resistance. Shear and moment diagrams	Proportionality, function, first and second derivative of a function, definite integral, equation
Frame reaction calculations	External reactions of the structure	Function, equation solution
Internal forces in truss bars	Values of the internal forces in the bars that form the truss	Equation/system of equations and equation solution
Linear deformation on a beam	Value of a linear deformation	Function, equation, integral
Angular deformation on a frame	Value of an angular deformation	Function, equation, integral

44.5 Discussion and Conclusion

Through the analysed problems, we can see that modelling in engineering strengthens what was established by Niss and Hojgaard (2019) regarding the modelling competence, in which the extra-mathematical context results key. In theoretical terms, there was a rich interaction between the MC proposed by Borromeo Ferri (Borromeo Ferri 2018) and the articulation of the epistemological dimensions of engineering and mathematics with the cognitive dimension, validating in this way in the proposal of Moutet (2016) and that we have extended to engineering.

We observed that during the course, the subcompetence of building a real model (stages 1 and 2 of MC), in general, is not developed, because of the presence of problems with structures that start from the real model, with simplifications and previously performed structures. In the case of the project developed by students in the class workshop, the subcompetence is developed, but on this occasion, the subcompetence of working mathematically with a mathematical model would be subordinated to the use of software (SAP2000). The software invisibilizes the mathematical work process as it works as a black box in the sense of the mathematical operations that are performed for the analysis of the structural model. In this finding, we differ from Gainsburg (2013), who found that modelling was not clearly promoted within the instruction for engineering courses. To the contrary, our findings evidence an explicit teaching of modelling, although divided, as we have mentioned before.

It becomes key to understand the role of models and mathematical and engineering objects. For the resistance of beam-type structures, the function model emerged from calculus of external reactions and internal stresses of the beam, through the construction of shear functions ($V_{(x)}$) and moment functions ($M_{(x)}$). At the same time, for the calculation of deformations in structures of a frame type, the integral (area) takes a key role where a function model emerges as part of the integrand. Moreover, we observed diversity of representations in the solution of the problem, where the importance of the graphic representation stands out, in order to visualize the maximum values that would be critical when designing the structure.

The study allows us to talk about a mathematical engineering modelling as we make evident the existence of an indissolubility between mathematics and engineering, so this provides elements to legitimize the inclusion of modelling in the mathematical classes in the context of engineer training. In this sense, we agree with Gainsburg (2006), who studied a community of structural engineers at their workplace and conclude that mathematics alone do not allow for the engineer to give a solution to the problem, but rather it is a group of norms, judgements and decisions, where mathematics are implicit. Therefore, we do not talk of mathematical modelling, but of a mathematical engineering modelling.

We consider that mathematical modelling is an articulator that strengthens the integration of several training areas that compose the engineering curriculum (Basic Sciences, Engineering Sciences, and Professional Training), so it is an approach that can contribute not only to the transversality of mathematical knowledge but also to its interdisciplinarity.

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Chapter 45
How Does the Teaching Design Influence Engineering Students' Learning of Mathematical Modelling? A Case Study in a South African Context

Rina Durandt, Werner Blum, and Alfred Lindl

Abstract This chapter reports on empirical results about the influence of two different teaching designs on the development of engineering mathematics students' modelling competency. 144 first-year students were exposed to a modelling unit (entrance test, pre-test, five lessons with ten tasks, post-test) following two teaching designs, similar to the German DISUM study. One group of participants was offered an independence-oriented teaching style, aiming at a balance between students' independent work and teacher's guidance, while two other groups were taught according to a more traditional teacher-guided style. Linear mixed regression models were used to compare pre- and post-test results. The results show that all groups had significant learning progress and that the group taught according to the independence-oriented design had the biggest competency growth.

Keywords Engineering students · Modelling competency · Solution plan · Student independence · Teaching design · Teacher guidance

45.1 Quality Mathematics Teaching

The transition from school to university is a substantial hurdle in students' learning of mathematics, also because students often lack basic skills and abilities and have a constricted disposition towards mathematics (compare to Rach and Heinze 2017). The learning of mathematical modelling is particularly challenging. Nonetheless,

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problem-solving abilities for real life situations, in particular mathematical modelling competencies (for this construct see Kaiser and Brand 2015 Niss and Højgaard 2011, 2019), are crucial in the professional development of engineering students. Reports over a number of years show that South African students at school level have severe shortcomings in basic mathematics skills. The teaching of mathematics seems to follow still mostly a traditional, strongly teacher-directive style with little room for students' engagement in mentally challenging activities (compare the TIMSS findings from Reddy et al. 2016).

What do we know empirically and theoretically about the effective teaching of mathematics and in particular of mathematical modelling? The general (e.g. Hattie 2009) and the subject specific literature (e.g. Kunter et al. 2013; Schlesinger et al. 2018) reveals that certain necessary conditions have to be fulfilled if teaching ought to have visible effects on students' knowledge, skills and abilities. We briefly summarise these conditions by the following five *criteria for quality teaching* (see Blum 2015):

1. *Effective classroom management* (comprising subject-independent aspects such as structuring lessons clearly, using time effectively, separating learning and assessment recognisably or varying methods and media flexibly).
2. *Student orientation* (progressing adaptively, linking new content with students' pre-knowledge, using language sensibly, giving diagnose, feedback and support individually, using students' mistakes constructively, and encouraging individual solutions).
3. *Cognitive activation of students* (stimulating students' mental activities by maintaining a permanent balance between students' independence and teacher's guidance, avoiding wrong dichotomies such as teacher guided instruction versus students working alone, "direct teaching" versus "discovery learning", or teacher "explains" versus teacher "moderates", and instead intertwining those elements).
4. *Meta-cognitive activation of students* (stimulating accompanying and retrospective reflections, and advancing strategies).
5. *Demanding orchestration of topics* (creating permanent opportunities for students to practice the aspired competencies by means of substantial tasks, fostering learning with understanding and intelligent practicing/repeating, emphasising justifications, and linking between topics as well as between the subject and the real world).

45.2 The DISUM Project

In the German interdisciplinary research project DISUM[1], "quality teaching" formed the conceptual frame (for more information about DISUM, see Schukajlow et al. 2012; Blum and Schukajlow 2018). The main aim of the project was to find out

[1]**D**idaktische **I**nterventionsformen für einen **s**elbständigkeitsorientierten aufgabengesteuerten **U**nterricht am Beispiel **M**athematik—in English: Didactical intervention modes for mathematics teaching oriented towards students' self-regulation and guided by tasks. The project was carried out 2002–2013 and directed by W. Blum, R. Messner and R. Pekrun.

empirically how students' mathematical modelling competency can be advanced effectively in everyday teaching practice, with a focus on the lower secondary school level (11-16-year-olds). In the DISUM main study, a more *independence-oriented* teaching style was compared with a more traditional *teacher-guided* style. The key principles of the independence-oriented style, called *operative-strategic* design, were:

- Teacher's guidance aiming at students' active and independent work on tasks, maintaining a balance between teacher's guidance and students' independence according to Aebli's "Principle of minimal support" (Aebli 1985), and encouraging individual solutions.
- Realising that balance by adaptive teacher interventions (which allow students to continue their work without losing their independence; see Blum 2011; Stender and Kaiser 2016), in particular strategic interventions (such as "Read the text carefully!", "Imagine the situation clearly!", "Make a sketch!", "What do you aim at?", "What is missing?", "Does this result make sense for the real situation?").
- Changing between independent work in groups and whole-class activities (for students' presentations and retrospective reflections).
- Teacher's diagnose and support based on a four-step modelling cycle, the "Solution Plan" (which was not in students' hands; see Schukajlow et al. 2015).

The key principles of the teacher-guided style, called *directive* design (which is the common style in everyday German classrooms), were:

- Development of common solution patterns guided by the teacher, oriented towards the average ability of the class.
- Changing between whole-class teaching (oriented towards the "average student") and students' individual work in exercises.

Both the "operative-strategic" and the "directive" teaching were implemented as *optimised* teaching styles, oriented towards aspects of quality teaching (classroom management, cognitive demand). The teachers were particularly trained on two days.

The DISUM main study followed a classical design—mathematics ability test, pre-test, treatment and questionnaires, post-test, and follow-up-test. The teaching unit consisted of 10 lessons (with 45' each) with altogether 7 introductory tasks and 14 modelling tasks (all tasks identical in both designs), and it was implemented in altogether 24 grade 9 classes at the medium track ("Realschule") of the German school system. The results show that both teaching designs had significant and very similar effects on students' technical mathematical skills, but only the "operative-strategic" design had significant effects on students' modelling competency. However, from a normative point of view these learning gains were still unsatisfactory, and compared with the above-mentioned criteria of "quality teaching" there were obvious shortcomings of the operative-strategic design, in particular some promising directive elements were missing such as individual practising and a teacher demonstration in the beginning of how modelling tasks may be solved (the teacher as a "model" according to cognitive apprenticeship, see Brown et al. 1989). In addition, the "Solution Plan" was not in students' hands (while several studies have shown how favourable meta-cognitive support can be for modelling activities; for an overview see Niss and Blum 2020, Chap. 6). This led to the construction of a new "*method-integrative*" teaching design as a blend of the operative-strategic design with those directive elements and with the "Solution Plan" for students. A case study in two classes with the same research design showed significantly higher learning gains compared to the operative-strategic design.

An obvious question is: Will similar effects be visible in other environments (such as other topic areas and other educational levels)? At the tertiary level, in particular, where mathematics is often taught in a rather teacher-centred way as a service subject for other disciplines, it would be interesting to see whether a more independence-oriented teaching style may lead to similar results or how the results may differ due to the different motivations and expectations of the students. More generally, knowing that mathematics learning and in particular learning mathematical modelling is dependent on the specific learning conditions and contexts, we need more studies that repeat certain elements and change others in order to better understand which influence certain variables have and to what extent empirical results may be generalised beyond the specific circumstances (compare to Schukajlow et al. 2018, as well as Niss and Blum 2020, Chap. 8). Thus, in 2018 the idea arose to conceive and carry out a similar research study, following the same conceptual framework, with engineering students in a tertiary environment in a South African context.

45.3 Research Design

The purpose of this study, carried out in February and March 2019 in a South African University with 144 first year engineering students in an extended curriculum programme, is to enhance the teaching and learning of mathematical modelling at the tertiary level, guided by applying principles of quality teaching. The students were exposed to a mathematical modelling unit following two different teaching styles, analogous to the method-integrative and the directive style described in Sect. 45.2. The broad *research question* is: *How do the modelling competency and the mathematical competency of the students develop through the modelling unit, depending on the teaching styles*? Here, *modelling competency* refers to the students' ability to solve modelling tasks at the pre-calculus level.

For organisational and administrative reasons, the sample was randomly assigned to three distinct class groups (called "Pink", "Blue 1" and "Blue 2"). The modelling unit was embedded in the students' first semester mathematics course. The intervention started with a diagnostic entrance test, informed by the South African mathematics school curriculum (CAPS) and by prior knowledge components for calculus (the content of the course), then a pre-test, followed by a modelling unit with 5 lessons including 10 different tasks, and at the end a post-test. One group (Pink) was offered a *method-integrative* teaching design with the four-step "Solution Plan" as a meta-cognitive aid. In this particular design, analogous to the DISUM project, the lecturer aimed at guidance by adaptive interventions. Students were to develop individual solutions, various solutions were later discussed and compared in the whole class, and the lecturer demonstrated in the second lesson how to solve modelling tasks by using the "Solution Plan". The other two groups (Blue 1 and Blue 2) were taught according to a more traditional *teacher-directive* design that is very common and well known at the tertiary level in the South African context. In an optimised form, again analogous to the DISUM project, the whole-class teaching was oriented towards the "average" student, the lecturer developed common solutions for the tasks and students followed. The lecturer was the same for both the Pink and Blue 1 groups and was experienced in teaching mathematics and mathematical modelling, while the lecturer for the Blue 2 group was only experienced in teaching mathematics. Irrespective of the teaching design, all groups received the same modelling tasks in the same order during implementation. Examples of tasks treated in the unit include the problem of how much air is in a balloon shown on a picture with a base jumper on top, the problem of what the optimal speed is for dense traffic on a one lane road, and the problem to approximate how tall a giant would be in order to fit the world's biggest shoes (2.37 m by 5.29 m).

Apart from minor everyday logistical problems (e.g. student transport problems, interruptions in electrical power supply), the implementation of the modelling unit was according to plan. Each lesson was planned in detail by the first two authors of this chapter. Additional steps were taken as control measures during implementation, such as discussions between the lecturers before and after every lesson regarding the particular teaching design for the group, possible task solutions and appropriate teacher actions. The lecturers also compiled in-class notes for record keeping on every lesson. Furthermore, all standard ethical matters were adhered to.

The *hypotheses* related to the research question were: the researchers expected equal and substantial progress in mathematics for all three groups (similar to the DISUM results), and equal but only slight progress in modelling for all three groups (unlike to the DISUM results, but an advantage of the Pink group in modelling seemed unreasonable because of the South African students' unfamiliarity with self-regulated teaching methods).

45.4 Test Instruments and Methodology

Data were collected from the entrance test, and both the pre- and the post-test. The *entrance test*, as a diagnosis of basic competencies from school, consists of 25 tasks (32 items with 38 marks) from the content areas algebra, geometry, functions, trigonometry, calculus, and modelling. The intention was to provide an overview of the mathematical entrance qualifications of the sample and to use this information as a covariate in further analyses. The *pre*- and the *post-tests* were specifically designed for this project and aligned with the modelling unit (see Table 45.1). They contained open modelling tasks (such as estimating the volume of a container shown on a photo), intra-mathematical tasks of varying complexity with topics relevant for the

Table 45.1 Pre- and post-test design, and alignment with the modelling unit

Section	Number of tasks/items/marks	Pre-test and post-test (versions 1 and 2)	Alignment with modelling unit
Section A	2 /2 /6	*Modelling tasks* with pictures (Beer Container or Straw Roll) resp. with given data	Lessons 1 and 5 resp. lesson 2
Section B	3 /6 /7	*Mathematical tasks* including proportional, linear and rational functions	Lessons 2, 3 and 4
Section C	5 /5 /10	*Multiple-choice tasks* selected in parallel for each version	Lesson 2

modelling unit, and multiple-choice modelling tasks selected from the well-known test of Haines et al. (2001). Both tests were administered in two versions with parallel items, following a rotation design, randomly and equally distributed to each group, which allowed also for comparing pre- and post-test results. By mistake, in group Blue 1 only one pre-test version and hence only one post-test version were distributed. The researchers did not expect this inaccuracy to be actually problematic because of the parallel items in each test version, but the results indicated some effect (see Sect. 45.5).

All analyses were conducted using the statistical software R (R Core Team 2019), and the raw values of the single items were combined to sum scores for each section as well as to total sum scores for each test. The internal consistencies of the scales were estimated with McDonald's Omega, whose values are comparable to Cronbach's Alpha with less stringent assumptions about the factorial test structure (McDonald 1999). To examine the respective effects of the teaching intervention, four linear mixed regression models with dummy-coded predictor variables were created that take into account the longitudinal data structure nested by participants (for intraclass correlations see Fig. 45.1a–d), that can handle missing values at different time points without excluding entire cases, and that have even for small sample sizes superior statistical power as well as less stringent requirements compared to conventional methods (for details see Hilbert et al. 2019). Therefore, the time variable (pre-test versus post-test), the indicator variable for the test version (A versus B) and the grouping variable (Blue 1, Blue 2, Pink) were dummy-coded (0/1) with Blue 2 as reference group for design-based reasons.

45.5 Results

As can be seen from Table 45.2, the values of McDonald's Omega are satisfactory for all three tests as a whole ($\omega \geq 0.59$), but relatively low for single sections, as expected, because they cover heterogeneous topics of the modelling unit with only a few items (see Table 45.1). For psychometric reasons (i.e., constant values,

Table 45.2 Internal consistencies (McDonald's Omega) per test (section), means and standard deviations per group and test (section) and one-way analyses of variance between groups per test (section)

Test (topic)	Number of items	Internal consistency	Group (N_{entr}/N_{pre}/N_{post})			ANOVA	
			Blue 1 (48/44/46)	Blue 2 (44/49/48)	Pink (41/46/47)		
		Omega (ω)	*M (SD)*	*M (SD)*	*M (SD)*	$F(2, N_i–3)$	*p*
Entrance test	30	0.79	12.65 (5.59)	10.23 (5.48)	12.98 (5.28)	3.30	0.04
Pre-test (total)	11	0.59	7.09 (2.86)	6.59 (1.84)	7.22 (2.65)	0.86	0.43
Section A	2	0.17	1.09 (1.25)	0.45 (.71)	1.07 (1.31)	5.01	< 0.01
Section B	6	0.48	3.41 (1.54)	3.61 (1.26)	3.67 (1.52)	0.42	0.66
Section C vers. 1	3	0.43	2.59 (1.53)	2.28 (1.28)	2.09 (1.31)	1.05	0.36
Section C vers. 2	3	0.41	–	2.79 (1.35)	2.87 (1.60)	0.03	0.86
Post-test (total)	11	0.60	10.11 (2.68)	9.08 (2.73)	10.89 (2.69)	5.38	< 0.01
Section A	2	0.19	2.65 (1.77)	1.10 (1.29)	2.64 (1.63)	15.18	< 0.01
Section B	6	0.49	4.35 (1.34)	4.98 (1.26)	5.17 (1.42)	4.78	< 0.01
Section C vers. 1	3	0.38	–	2.62 (1.61)	2.71 (1.16)	0.04	0.84
Section C vers. 2	3	0.40	3.11 (1.23)	3.38 (1.01)	3.48 (1.31)	0.85	0.43

Note. ω = McDonald's Omega; M = mean; SD = standard deviation; ANOVA = (one-way) analysis of variance; F = F-value; p = probability of committing a Type I error, N_i: $N_{entrance} = 143$, $N_{pre} = 139$, $N_{post} = 141$

negative correlations with the respective scales), two items had to be removed from the entrance test and two items from section C (in pre- and post-test). So, contrary to our expectation, some multiple-choice items in section C (Haines et al. 2001) were not equivalent. After removing these items, the two test versions became better comparable, but version 1 is still at all times slightly more difficult than version 2 (see Table 45.2). This favours group Blue 1 which by mistake (see Sect. 45.4) received only one test version, the easier one in the post-test.

Due to significant group mean differences in the entrance test (see Table 45.2), its total score was used as a covariate and controlled for in the subsequent linear mixed models. Regarding test section A (modelling tasks) as dependent variable, Fig. 45.1a does not only show that Pink and Blue 1 significantly differ from Blue 2 at the pre-test. Rather, it is evident that despite a significant gain in mathematical modelling competency between pre- and post-test by Blue 2 ($\beta = 0.69$, p = 0.01; β: unstandardised regression weight) Pink and Blue 1 have significant additional increases ($\beta = 0.80$, p = 0.04; $\beta = 0.98$, $p = 0.01$). Thus, referring to test section A, the results indicate that the competency of groups Pink and Blue 1 grows under the respective learning conditions (concerning Pink this is similar to the results of the DISUM study) more than twice as much as that of group Blue 2.

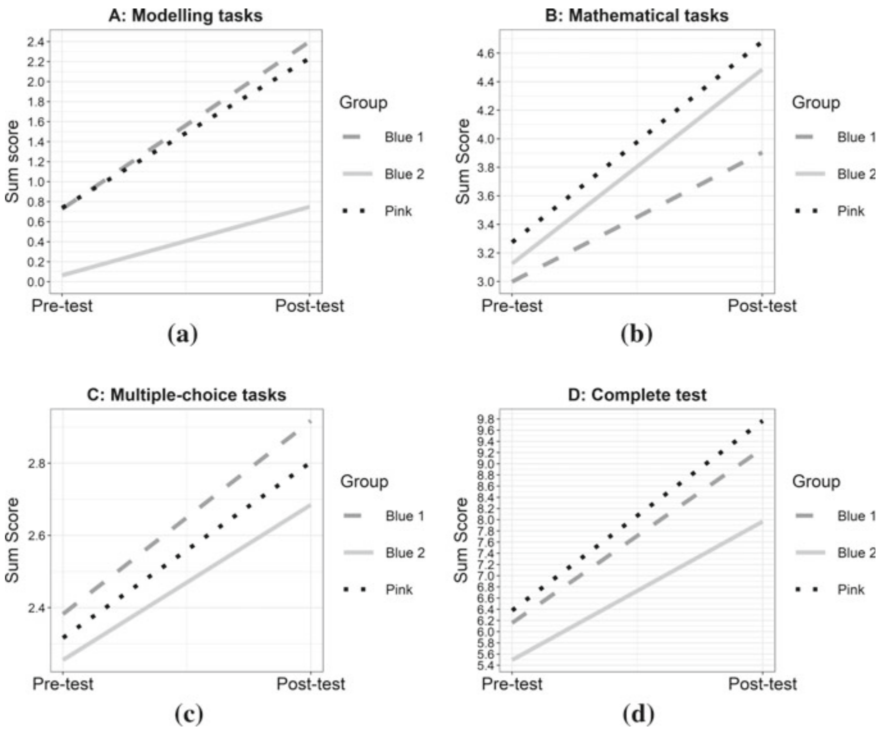

Fig. 45.1 **a** Line graph for modelling tasks (intraclass correlation: 0.08). **b** Line graph for mathematical tasks (intraclass correlation: 0.14). **c** Line graph for multiple-choice tasks (intraclass correlation: < 0.01). **d** Line graph for the complete test (intraclass correlation: 0.16)

In a second linear mixed model with regard to section B (mathematical tasks), the competency gain of Blue 2 is significant ($\beta = 1.36, p \leq 0.01$), and although the increase of Pink ($\beta = 0.04, p = 0.92$) and Blue 1 ($\beta = -0.43, p = 0.22$) deviates only slightly from that (also similar to the results of DISUM; see Fig. 45.1b), the average mathematical competency of Blue 1 is significantly smaller in the post-test (see Table 45.2).

A third mixed model for test section C (multiple-choice modelling questions), which additionally takes into account the different test versions, shows no significant result and only descriptively a low, nearly equivalent competency increase for all groups (Fig. 45.1c). This could be due to the fact that after shortening the scale of section C the number of remaining items was too small, their closed task format perhaps unsuitable or their content not sufficiently adapted to the modelling unit.

Finally, the results of the complete test are visualised in Fig. 45.1d for each group and analysed in a fourth mixed model (again considering test versions). In comparison with the competency growth of the reference group Blue 2 ($\beta = 1.05, p = 0.16$), Pink shows significant additional increases in both test versions (A: $\beta = 2.10$, B: $\beta = 2.71$, $p \leq 0.05$), while the gain of Blue 1 (just version B at post-test: $\beta = 2.82, p = 0.02$) is mixed with a test version effect, as further analyses suggest. So, Pink does not only outperform Blue 2 in the post-test significantly (see Table 45.2), but also Blue 1, and shows a significant improvement of competency that is mainly attributable to the learning intervention (similar to the results of DISUM).

45.6 Discussion and Perspectives

The purpose of this study, carried out in 2019, was to enhance the teaching and learning of mathematical modelling at the tertiary level, guided by principles of quality teaching. First-year engineering mathematics students were exposed to a modelling unit following two different teaching styles, analogous to the method-integrative and the directive style in the DISUM project. The results of this study indicate that quality teaching is promising also for the tertiary level and thus complement the results of the DISUM study. Both teaching designs, implemented in an optimised form by specifically trained lecturers, had effects, but for the development of mathematical modelling competency the *method-integrative* teaching style seems more favourable, analogous to DISUM but contrary to our cautious hypothesis. This may have consequences not only for engineering but also for teacher education in South Africa (see Jacobs and Durandt 2017). An interesting result is that the two Blue groups, both instructed according to the directive teaching style but with different lecturers, had distinctly different learning gains. Therefore, corresponding with other results (e.g. Kunter et al. 2013), the teacher variable seems important, also at the tertiary level and also for teaching modelling. This aspect should be further investigated, for instance, by taking into account the lecturers' professional competencies as a covariate.

From a normative point of view, both the pre- and the post-test results appear rather weak (see Table 45.2), so there is certainly a big potential for further improvement. To a considerable extent, these results are certainly due to the unsatisfactory university entrance qualifications of the students. However, there will also be room for a further improvement of the teaching design. One possibility is to extent the duration of the teaching unit and to include more phases for individual practising, with and eventually without teacher support. Another possibility is to link the modelling examples more closely to the engineering subjects and to those South African students' life contexts. This will be realised when the study is repeated in 2021; corresponding tasks have been developed.

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Chapter 46
Reflective Engagements by Practising South African Teachers on a Constructed Model for University Funding and Rankings

Cyril Julie

Abstract A model for funding and ranking of universities was constructed by a group of practising teachers during an introductory immersion course on mathematical modelling and the applications of mathematics. The group presented their model to other members of the participating cohort of teachers. Interactions between the group who constructed the model and their peers during this presentation were analysed. The overall analysis was anchored around the notions of internal and external reflections occurring during the deliberations of the two groups. These reflections were thematically analysed to ascertain the arguments used to critique and defend the presented model. The analysis rendered four themes of which two were distinctly aligned to internal reflections. The other two were an intertwinement between the external and external reflections.

Keywords Ranking models · Ratings · Internal reflections · External reflections · Practicing teachers · Index

46.1 Introduction

Rankings of all sorts are mathematical models—an extra-mathematical domain is taken and mathematised. They are widely reported in the public media. This makes them the most pervasive mathematical models the general public has contact with and experience of. Some of them are, at times widely (and passionately), debated in the public domain because of agreement or not with the outcomes of their application. At least, two kinds of ranking systems exist. One is a performance-based ranking system which is based on a system for the award of points such as league tables in sporting competitions and the rankings resulting from the Trends in Mathematics and Science Study (TIMSS). The other kind is based on data on certain issues gathered from institutions or persons of interest to construct some measurement model for

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ranking. The Human Development Index (HDI), which ranks countries according to their development status using certain indicators, is a ranking model of this nature.

In many instances, rankings result from ratings where an issue of import is placed on a pre-ordered scale by (knowledgeable) adjudicators. The World Economic Forum, for example, uses such an approach to rank countries according to the inclusion of creative and critical thinking skills in the teaching in the country. The respondents, executives in business, are requested to provide an opinion on the question "In your country, how do you assess the style of teaching?" The respondents have to assign a value of 1–7 [1 = frontal, teacher based, and focused on memorizing; 7 = encourages creative and critical individual thinking]" (Schwab 2019, p. 620) to indicate the response to the question. The values selected by the respondents from a country are used in various formulae to obtain a score for the particular country. For the issue, style of teaching, the computed values for each country is used to rank the countries on this issue. Although many rankings result from ratings, ratings are not without controversies as illustrated by Minton (1992) for the college football competition in the USA.

Rankings of universities are widely reported in the public media. The three most popular ranking systems for universities are the Academic Ranking of World Universities (ARWU), The Times Higher Education (THE) world university rankings and Quacquarelli Symonds (QS) system (Degenar 2014). All three ranking systems use research, although differently weighted, in conjunction with other indicators as an element of their system. In South Africa a major source of research income for universities from the government is based on research outputs include articles in journals in an approved list of journals, books approved by appointed expert committees and creative pieces such as music compositions. Frequent reports are produced of these research outputs. In these reports South African universities are ranked (see, e.g. Mouton 2019). This chapter reports on the reflections of practising teachers during the initial report-back of one group's preliminary model dealing with funding for research output for three hypothetical universities in South Africa.

The research question being pursued was: What kind of reflections do practising teachers engage with when they collectively deliberate on an alternative funding and ranking model constructed by a peer group for the ranking of South African universities?

46.2 Theoretical Considerations

As is in the postulated research question, the focus is on reflections. Cobb et al. (1997) define collective reflection as "the joint or communal activity of making what was previously done in action an object of reflection" (p. 258). They distinguish collective reflection from Piaget's reflective abstraction, a psychological activity. Blomhøj and Kjeldsen (2011) use a similar definition of reflection by referring to it as a "deliberate act of thinking about some actual or potential action aiming at understanding the action and improving it" (p. 386). They refine this notion of collective reflection by distinguishing between internal and external reflections students engage in during the construction and accompanying deliberations of a mathematical model. Internal reflections are linked to the cyclical mathematical modelling process and become visible or can be extracted from deliberations, if there is access to them, or recalls by the model constructors of the model construction process. Modelling competencies such as selection of variables, declaration of assumptions, mathematical work and validations are under discussion during internal reflections. External reflections are deliberations associated with the context within which a mathematical model is applied.

The research reported in this chapter focuses on the internal and external reflections practising teachers engaged in when they constructed an alternate model for funding and ranking South African universities.

46.3 Data Collection and Context

The data used in the study was collected during an elementary short introductory immersion course, as part of a continuing professional development programme for mathematics teachers, on mathematical modelling and the applications of mathematics to a group of practising teachers with no prior experience of the areas. In the South African curriculum mathematical modelling is an important focal point of the curriculum. It stated that problems related to health, economics, et cetera should be the real-life issues to be included (Department of Basic Education, DBE 2011). The participating teachers had limited, if any, experience of mathematical modelling and the applications of mathematics since mathematics courses they encountered in their pre-service teacher education were generally anchored around pure mathematics. Their experience is primarily with the use of word problems at the end of a mathematical topic. Another experience they have is the use of ready-made models which must be particularised and then calculate some results using given information (Julie 2015). Thus teachers had limited, if any, experience of the construction of a mathematical model.

The duration of the course was 16 hours over a weekend (4 hours the Friday and Sunday and 8 hours the Saturday) at a venue where participants stayed for the entire weekend. Ten high school teachers, one primary school teacher and a mathematics curriculum advisor participated in the course. I was the facilitator and there were five females and seven males.

The teaching procedure started with the participants being given the 2016 Summer Olympic rankings of the participating countries. The teachers had to develop an alternate ranking system. This was followed by a lecture on the modelling cycle using the depiction of the mathematical modelling process by Stillman (1998). Some components of the mathematical modelling cycle were exemplified by appropriate instances which arose during the teachers' construction of the alternate ranking system for the Summer Olympics. Since the anticipation was that teachers would attempt to construct a ranking model comprising multiple disparate components, the Human

Development Index and the normalisation process to obtain non-dimensional entities for calculations were also discussed.

Upon completion of the 2016 Summer Olympic rankings activity, it was discussed and linkages to the modelling process were drawn out. Participants were then provided with a set of situations—a parking lot design, control of an elephant herd in a national park in South Africa, a ranking system for South African universities based on publications output and the placement of emergency service to mathematically model. They had to work in voluntary selected groups and each group had to select one situation to deal with. Three groups were formed and one group chose the ranking system for South African universities based on publications output. The focus the research reported in this chapter is on the work of this group. This ranking system is, as explained above, one where the institutions annually submits their research outputs to the relevant government authority and an expert committee assesses and audits which research outputs qualify for research funding from the government. Of particular interest are the reflections that took place when the group reported the first draft of their model to the rest of the cohort of participants. The draft can be characterised as an "unrealistic model" (Spooner 2019). It was, in a sense a provisional model which could hopefully lead to an informed revision of the model based on the reflections.

The report-back sessions were video-recorded and this was the data source that was used for analysis. Although the data are restricted to the video record, it is reasonable to accept that there were deliberations amongst participants in spaces, which I was not privy to, other than the work venue. The newsprint recording, Fig. 46.1 below, that unfolded during the group's report was also part of the data.

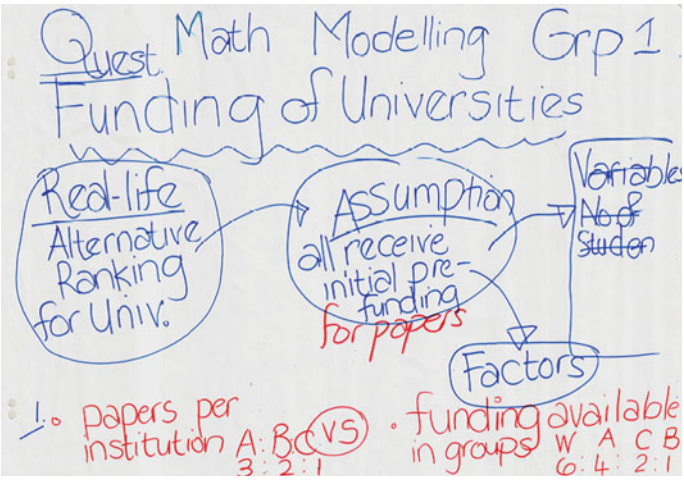

Fig. 46.1 Newsprint recording of ongoing work

46.4 Methodological Approach

The 45 min of video-recorded interactions were transcribed and the transcription with the recordings of ongoing work was analysed. The analysis was conducted using an adapted version of thematic analysis Terry et al. (2017). Thematic analysis is a search for themes inherent in the data. Six phases of analysis followed: familiarisation with the data, initial code generation, creating themes, reviewing themes, definition of themes and doing a report of the analysis. The process is non-linear and iterative. Normally only the communicative acts of the research participants are used. I, however, was a participant and not divorced from the ongoing deliberations. Hence the reference to adapted version of thematic analysis.

The coding was primarily inductive but the constructs of internal and external reflection were kept in mind. Hence, there were elements of deductive coding present during the entire process. An indifferent stance was taken with respect to the “correctness” of the model. Twenty codes resulted and these were converted into themes. Four overlapping themes resulted and these are presented in the next section.

46.5 Results and Discussion

In the transcripts below PG(number) is used for a member of the group who presented the model and RG(number) a member of the group who responded. Briefly the model that the group presented was one where universities will receive initial funding indicated as “initial pre-funding for papers”. This model entails the redistribution of the invested funds (28 million Rand) in the ratio 3:2:1 for universities C, B and A, respectively. The ratios were based on consideration of the distribution of articles per institution and distribution of income per demographic group. This model of redistribution was the dominant aspect for the deliberations.

46.5.1 Clarification of Issues Within the Modelling Cycle

During the deliberations, there were explicit references to aspects related to elements of the mathematical modelling cycle. A member of the responding group requested clarification on the origin of the ratio “6:4:2:1” by asking “Explain to me that ratio” [RG1] (refers to 6:4:2:1 in figure). A member of the presenting group [PG1] responded “According the average household income”. PG1 was supported by another member of the presenting group by stating “Yes we’ve done it. We’ve got all the calculations…” [PG2].

Referring to the calculations (and having done it) is part of the mathematical work done during the model construction process. This mathematical work is not strictly during the construction of the final model but rather during the stage of “formulating the mathematical problem” as given by Stillman (1998).

The discussion proceeded by RG2 stating, “You guys obviously looked at a relationship between funding and papers published. … Is that what you guys are assuming?” Attention was thus drawn to the underlying assumptions of the presented model. The response was that it was about funding to which RG2 responded “… you guys are looking at it from only one angle”. The presenting group was adamant that this was not the case because “We looked at a lot of things” [PG1].

The member of the responding group is thus drawing attention to possible flaw in the presented model. Because the deliberations focused primarily on aspects of the model, the reflections at stake here are internal reflections.

46.5.2 Suggestions for an Alternative Model

After about 25 min, attention is drawn to the task as set out on the activity sheet. Reference is made that the reporting group should not have developed a distribution model but one that had its focus on the articles produced by the university academics. This is viewed as a problem by the responding group as revealed in the transcription excerpt below.

> RG2: This problem that we have there. The funding that we have there is per lecturer, per academic staff. Did you guys work out how many of those academics do not produce papers? Then you’ll find out that there are more lecturers at university A that is not producing publications.
>
> RG6: If I look at the first statement there (refers to task sheet) then the first row is termed the academic staff members, then the next row is average annual publications. And then when I look at, when I look at university C they are having two hundred and sixty-one staff members. And if that figure…I divide by one thirty-eight then I get an answer of about one point eight which is the higher number than [for] university B and university A. Which means that C is the one that is producing more publications than the other universities.

At this point, I intervened by suggesting that RG6 was calculating the number of lecturers per publication and if this calculation is inverted it will be publications per lecturer.

Despite this interchange the reporting group, however, stuck to the model of redistribution of the invested funds. As with the first theme above, the deliberations are also of an internal reflection nature. However, in terms of Blomhøj and Kjeldsen’s (2011) definition of reflections also triggering improvements, there were no indications of the reporting group’s willingness to construct a revised model based on the internal reflection.

46.5.3 Addressing Discrimination of the Past

Discrimination of the past and possible corrective actions are widely discussed and debated in the public media in South Africa. RG2's comment, "Because we linked to apartheid and because we linked to previous disadvantage. That's a normal thing to do.", shifted the discussion in the direction of racial demography being a major driver of the presented model. PG2 responded affirmatively. RG2 continued to bring this actual issue into sharper focus by stating "If this was a room full of White people they might have handled this completely differently and make it even and spread it out and let every university get the same amount of funding. It's weird that you didn't go that way?" The presenting group defended their position by emphasising the factors they preferred captured in the utterance "No, no, no. We went this way because of the factors that we were favour(ing)" (PG2).

The responding group also raised concerns about penalising university A and not recognising their efforts as having some benefit for the future. RG4 articulated this position as "Why crush a good thing because … university A having produced more papers…that's good…Why deny that progress [and]…concentrate on the future. Then instead of promoting those who are producing less [you] might have handled this completely differently…".

The presenting group responded to these political issues by referring to fairness which is the next theme being discussed. This theme leans more towards external reflections since the discussions were not explicitly concerned about the internal mechanisms of the model construction process but on political-ideological concerns.

46.5.4 Equity as Purpose of the Constructed Model

Early in the discussions the presenting group made it clear that their model was about "something that should be done to make sure that there should be fair funding for the three universities." [PG]. Doubt was expressed whether the model is actually complying with the criterion of fairness with RG4 saying "you not saying the same thing [about] A. You decreasing A [and] I don't think that is fair allocation of resources". Attention is here drawn to decreasing the allocation to university A as unfair. Further elaborations of the fairness issue are delved into when attention was drawn to an apparent discrepancy between the quest for fairness and the redistribution of the available funds to increase the production of research articles. This discussion proceeded as follows:

RG1: I have to disagree, the problem is on the funding model not on promoting producing more.

PG1: [But] the funding is depending on the publications …

RG1: Is the system we use fair? That's the thing. So we're working on this as the only problem. So we're saying let's promote this organisation because they are at the bottom. It's like we're saying is the model we are [presented with] fair? [It is] Not saying let's increase this one.

The reporting group, however, maintained their stance on fairness since they considered the demographic make-up of the three institutions and the distribution of household income as the way to bring about equity. As with the last-mentioned theme this one also leans towards external reflection since the deliberations engaged were not strictly about model construction process.

46.6 Concluding Discussion and Comments

Two of the themes, discussed, clarification of issues within the modelling cycle and suggestions for an alternative model, were internal reflections forthcoming from the communicative acts. The other two, addressing discrimination of the past and equity as purpose of the constructed model, are not strictly external ones. Rather, they are intertwined with the external ones being dominant but not completely separate from the internal ones.

This study raises various issues for further investigation. One is linked to task design for the development of ranking models which need to include composite dimensions. The participants in this study focused on one issue, redistribution of existing resources based on invested funds. The other information did not configure in the preliminary model but was more used as justifications despite the respondents offering seeds for reconsidering a different model which can include more dimensions. This points in the direction that for practising teachers with limited experience of mathematical modelling and the applications courses should also focus on the development of non-dimensional indexes that are constructed using multiple dimensions and using a process of normalisation to make numerical calculations so as to ranking from the indexes, as is the case for the Human Development Index.

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APPENDIX: Activity Sheet

Funding of Universities

One factor taken into account for the funding and ranking of universities is the number academic publications per year. This factor is expressed as the number of publications per academic staff member. The situation for three hypothetical South African universities is given below:

University	A	B	C
Academic staff members	782	428	261
Average annual publications	477	248	138
Average annual income (Million Rand)	385	200	130
Invested funds (Million Rand)	20	5	3
Number of students	18 119	12 041	6 519
Institutional Distribution students according to Apartheid-designated Groups (%)			
Institution	A	B	C
African	20	45	91
Coloured	16	47	5
Asian	12	6	3
White	52	2	1
Average Household Income (Statistics South Africa, 1995)			
Apartheid-designated Group	Average Household Income (Rand)	Average Household Income (Dollars)	Adjusted Average Household Income (Dollars) according to HDI adjustment formula
African	23 000	2 191	2 191
Coloured	32 000	3 048	3 048
Asian	71 000	6 762	6 046
White	103 000	9 810	6 114

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Chapter 47
Mathematical Modelling Problems in a Mathematics Course for Engineers: A Commognitive Perspective

Svitlana Rogovchenko

Abstract Mathematical knowledge is very important for engineers; both quantitative and qualitative analysis of physical systems and manufacturing processes require solid scientific principles and appropriate mathematical tools. Teaching of mathematical modelling at the university contributes to the development of mathematical competencies, motivates the interest to mathematics and plays a significant didactical role in promoting advanced mathematical thinking. These views are supported by the analysis of students' work on a mathematical modelling assignment in a university mathematics course for future engineers with the help of the commognitive framework. Students' narratives in written solutions and oral discussions reveal different components of students' precedent-search-space and confirm the progressive development of exploratory routines.

Keywords Mathematical discourse · Commognition · Mathematical modelling · CAS · University engineering education

47.1 Introduction

Mathematics plays an important role in sciences, engineering and technology. Engineering students are usually taught several mathematical courses designed to provide them with a solid mathematical knowledge necessary for future job and motivating further interest in mathematical subjects. However, this purpose is not always achieved; mathematics is often seen by the students as a discipline that teaches mostly procedures completely irrelevant to their future careers whereas "for them the focus on practical elements of the mathematical parts of the course is particularly important" (Sazhin 1998, p. 146).

Studies show that after a course in differential equations engineering students are still unable to use their knowledge in advanced engineering courses (Czocher and Baker 2010). In many universities, mathematics as service course is taught for

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students coming from a very wide range of engineering areas (civil, electrical, electronic, computer engineering, etc.) and selection of modelling tasks that are interesting and important for all specialisations is a big educational challenge. Lack of mathematics knowledge leads to the situation when engineering graduates need to take additional courses in "selected topics in mathematics" required by engineering departments. It is often realised at this point that many students have little or no experience of solving applied mathematical problems and the task of bridging the gap between students' mathematical and engineering knowledge becomes quite demanding.

Nowadays, the progress in the design of university mathematics curricula is observed: mathematical modelling courses, project-based courses, problem-solving courses are offered to engineering students at various universities (Wedelin and Adawi 2015). At the same time, several mathematics courses, for example, differential equations, are taught with the focus on modelling perspective and applications that use modelling techniques to enhance teaching and learning quality (Czocher 2017). Using mathematical modelling as a teaching approach, various goals can be attained including building the relationship between elements of real-world systems and their mathematical representations, testing and adjustment of models, developing students' mathematical competencies, cognitive and decision-making skills in different realistic situations, improving retention of mathematical knowledge, etc.

47.2 Mathematical Modelling and Technology in Engineering Curriculum

Blum and Niss (1991) suggested several ways of including mathematical modelling tasks in mathematics instruction: the separation approach, the two-compartment approach, the islands approach, the mixing approach, the mathematics curriculum integrated approach and the interdisciplinary integrated approach. They argued that "in 'mathematics as a service subject' programmes, all approaches can be encountered, but probably the two-compartment, the islands, and the mixing approaches are the ones most widely used" (p. 62). One of the popular teaching approaches in engineering curricula, also employed by the author, is the mixing approach where "elements of applications and modelling are invoked to assist the introduction of mathematical concepts" and "newly developed mathematical concepts, methods and results are activated towards applicational and modelling situations whenever possible" (Blum and Niss 1991, p. 61). Assignments with real-word problems are classified as higher-level cognitive demand tasks because they focus on "comprehension, interpretation, flexible application of knowledge and skills, and assembly of information from several different sources to accomplish the work" (Doyle 1988, p. 171). From the learning perspective, such problems help students to interrelate different areas of mathematical knowledge and connect it to other forms of knowledge in science and engineering, motivating thus particular ways of interdisciplinary thinking and acting.

Nowadays, mathematics lecturers become more aware of the importance of technology in university teaching. Undergraduate engineering curricula include different programming courses; programming skills acquired by students can be successfully used in mathematics classes. Kramarski and Hirsch (2003) argue that computer algebra systems (CASs) positively contribute to the strengthening of students' mathematical thinking offering numerous advantages for the teaching and learning of mathematical modelling; "in particular, implications for collaborative group activity in modelling and applications work are profound" (Galbraith et al. 2003, p. 122). Computers can be used at different steps of the modelling task; application of digital tools has been successfully integrated into a seven-step modelling cycle producing an extended modelling cycle (Greefrath 2011, p. 302). Theoretical and empirical studies confirm that "while many of the uses of the computer are essentially computational, enabling students to investigate problems involving 'messy' real world data, others are meant to facilitate both procedural and conceptual learning of mathematical topics" (Selden 2005, p. 133). Use of technologies in a differential equations class promotes the development of mathematical modelling competencies in engineering students (Gallegos and Rivera 2015, p. 443). Integration of Maple, MATLAB, Mathematica into applied mathematics projects for engineers showed promising results in enhancing "the capabilities of engineering students to use mathematics for solving problems in larger projects as well as to communicate and present mathematical content" (Alpers 2004).

47.3 Theoretical Framework: Mathematical Discourse and Commognition

From a commognitive perspective (Sfard 2008; Lavie et al. 2018), learning is viewed as routinisation. Routine is defined as a task-procedure pair, "the routine performed in a given task situation is the task, as seen by the performer, together with the procedure she executed to perform the task" (Lavie et al. 2018, p. 9). Commognition theory considers the following elements of mathematical discourse: *words and their use* (words that are specific to the discourse); *visual mediators* (graphs, diagrams and specific symbols); *routines* (repetitive patterns characteristic of a discourse); *narratives* (sequences of utterances about the mathematical objects of the discourse or the processes on the objects). The theory distinguishes practical and discursive routines which in turn are classified as process-oriented (rituals) and product-oriented (deeds or explorations). Narratives are endorsed, modified, or rejected by the community using substantiation, construction and recall routines.

It may happen in a mathematical task situation that the interpretation of the task by the student differs from what was meant by the teacher. The learner acts relying on a precedent-search-space (*PSS*) consisting of situations relevant to the current task. For a mathematical modelling problem, the student's PSS can be represented by the collection of mathematical methods that can be used for solution; familiar problems, relevant to the given one; student's previous life experience with respect to the given problem; mathematical skills, etc. The members of PSS applicable in a task situation are found with the help of precedent identifiers. According to commognitive approach, the learner can participate in a new discourse only in a ritualised way and ritualised routines are expected to gradually transform to *explorations* (de-ritualisation) (Heyd-Metzuyanim and Graven 2019). Routines characterised by *flexibility* (multiple solutions); *bondedness* (absence of redundant steps); *applicability* (potential use); performer's *agentivity* (the number of autonomous decisions, ability to carry out more complex tasks without help); *objectification* (ability to think in terms of mathematical objects); *substantiability* (the performer has the criteria to assess his/her own performance) are classified as explorative routines. Commognitive approach can be used for analysing various mathematical discourses, both verbal and written, everyday (spontaneous), presentations for school students, scholar publications, etc. (Sfard 2008, p. 132).

Mathematical modelling (MM) problems contribute to the development of advanced mathematical thinking by engineering students which encompasses the use of nonalgorithmic, complex ways of thinking that often lead to multiple solutions and involve interpretation and use of several different criteria. Analysis of the meaning and the structure of problems containing uncertainty often requires self-regulation and considerable mental effort (Resnick 1987). Commognition theory has been recently used to analyse the learning and teaching at the university level (Viirman and Nardi 2018; Treffert-Thomas et al. 2018) and "it seems a promising and rewarding task to formulate a commognitive perspective of mathematical models and modelling" (Ärlebäck and Frejd 2013, p. 48). Commognitive framework was also employed to identify modelling routines in MM activities by analysing verbal mathematical discourse of prospective teachers engaged in a summer camp activity (Shahbari and Tabach 2017). The authors of this study were able to trace "a change from using a non-systematic routine to using a systematic routine and from routines focusing on choosing specific cases to routines focusing on eliciting criteria for making choices" (p. 185).

The focus in this chapter is on mathematical routines and the use of mathematical language in students' discussions and written reports. Commognitive theory was chosen as an appropriate theoretical framework to analyse students' mathematical communication, both verbal and written, and to draw conclusions about students' use of available mathematical arsenal and the ways they build mathematical (and modelling) routines. Certainly, interpretations of the word use, visual mediators, narratives and routines are subjective; the teacher acts as a representative of the mathematical (professional) community that assesses the level of the mathematical maturity and "literacy" of students' discourse.

47.4 Methodology and Data Collection

The study took place in the engineering department of the medium-sized Norwegian university. Modelling tasks were introduced in a differential equations course for the fourth-year engineering students who by that time took several courses in engineering, chemistry and physics. Students in this class neither had modelling courses nor previous modelling experience in undergraduate studies which included the sequence of courses Mathematics 1, 2 and 3 covering mostly topics in calculus and linear algebra. Mathematics courses were not much coordinated with engineering and physics courses taught at the same time. The intention to modify the differential equations course was motivated by the idea to use the mixing approach connecting the knowledge students gained in different areas including physics and mathematics. Since there was no dedicated course in MM offered at the university, the syllabus for a differential equations course was modified to introduce several MM assignments in the form of course projects. The introduction of the MM assignment had several pedagogic purposes: to enrich students' mathematical narratives about the nature of differential equations, promote students' advanced mathematical thinking and use of mathematical language, contribute to the development of general modelling routines, explain how known mathematical routines can be used and combined to produce new mathematical routines, motivate transformation of ritualised mathematical routines into explorative routines in the process of MM as a particular problem solving strategy. In addition, students' work in small groups introduces important elements of collaborative learning in mathematics courses and enhances students' social skills, although this was not the focus of the study.

Forty students (38 males and 2 females, all in their twenties) were given different sets of modelling problems. Small groups of two to three students worked for one week on the assignment, discussed their solutions to problems and produced individual written reports. MM tasks were mostly linked to the subject area of engineering studies (mechatronics); the analysis of the mathematical model with respect to the real-world problem was required. The level of complexity of the problems was different, ranging from closed to open-ended problems. Students were asked to employ mathematical methods for finding solutions and use CAS (Maple or MATLAB) to support their work. They were expected to use previous programming experience and computational skills (PSS); some programming tutoring was also provided in the course. Students audio-recorded group discussions in the absence of the lecturer and provided their recordings at her request for research purposes. Group solutions were presented by each group in a whole-class session. Students' individual written reports were evaluated as a part of the course work; the mark counted towards the final grade.

The research questions addressed in this study are: *How does the engagement with MM problems motivate the development of mathematical discourse and mathematical routines? To what extent does the use of CASs contribute to such development?* The analysis focuses primarily on students' work on several important steps of modelling cycle related to mathematisation, choice of solution method, model validation and interpretation of results.

47.5 Data Analysis

The study topic is Existence and Uniqueness Theorems for the initial-value problems for differential equations. Contrary to traditional teaching practices, students engaged with modelling problems to work with the theorem. One of such problems is presented below.

Problem. Consider a cylindrical bucket of constant cross-sectional area A with a hole of cross-sectional area a. The small hole is plugged, and the bucket is filled to depth h_0. A clock is started as the plug is removed and the water begins to leak out of the hole. Construct a DE model to determine the height $h(t)$ (m) with respect to time $t(s)$. Take $g = 10$ m/s^2. Choose your values for A and a so that the ratio $\frac{A}{a} = \sqrt{5}$.

(a) Explain all your steps while setting the model. (b) Take $t_0 = 0, \quad h_0 = 4$, set the IVP, explain the meaning. (c) Solve the problem and observe that the solution if defined for all t but after some time it stops being a realistic description of the height. What physical event occurs at this moment? (d) Build a realistic continuous solution to this problem and show that the solution is valid for all t. Is this solution continuously differentiable? (e) Do these results contradict the Existence and Uniqueness Theorem? Explain your reasoning in detail. (f) Plot the solutions found in part (c) and part (d) and analyse the graphs.

Students suggested several physical descriptions of the problem (illustrated in Excerpt 1) and discussed corresponding mathematical models which demonstrates the *flexibility* of the modelling routine. Many students used diagrams (*visual mediators*) as a tool for translating verbal description to mathematical mode. They relied on *PSS* using combination of several familiar routines in the process of solution: setting up an initial-value problem—it was not a trivial task requiring several steps: solving a differential equation, identifying the general solution and applying the initial conditions to find a particular solution. The mathematical discourse presented in students' written solutions suggests that they have developed the ability "to express things in the language of mathematics" (Schoenfeld 1992, p. 337) known as *objectification*, successfully selected an appropriate problem-solving strategy available in their *PSS*, and completed the solution. In the Excerpt 1 from the discussion in Group 1 one clearly observes the "repetitiveness, and thus patterns which is the source of communicational effectiveness" (Sfard 2008, p. 195). It was important for this group to agree on the common solution method and "indorse" the narrative, but not all groups came to an agreement; in these cases, students presented their individual versions of solutions.

S11: I used Bernoulli equation to set the differential equation.
S12: I did something similar, but I started from the Conservation of Energy Law to find the velocity out…
S13: I also used the energy law, and worked with some constants and found a nice equation…
S11: Yes, we can use different values for constants, but I chose to have the simplest…

Students in Group 2 used physical laws to derive the following initial-value problem:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -2\sqrt{h}, \quad h(0) = 4.$$

Formal integration yields the exact solution $h(t) = (2 - \mathrm{t})^2$. which is valid on the entire real axis but should be considered only on the interval [0, 2] until the bucket empties. From the instant $t = 2$, the bucket is empty and the second "piece" of the solution to the problem on the half-axis $[2, \infty)$, $h(t) = 0$, can be obtained by the reasoning in context. The conditions of the Existence and Uniqueness Theorem are not satisfied for $h = 0$; this occurs at $t = 2$ and leads to the multiplicity of solutions to the given initial-value problem. Talking about the "realistic" solution, students also discussed the existence of a "mathematical" solution when time is "negative" (reversing the time), and solution's meaning in terms of physical phenomena ("water level will go to infinity"). This investigation was facilitated by the use of CAS; the students' discussion illustrated in Excerpt 2 below relates to *substantiability* and demonstrates the explorative character of the mathematical discourse.

S21: The solution we got is a parabola. … After $t = 2$ the solution is no longer realistic. What physical event occurs at this moment? What occurs is that the tank become empty and then some sort of filling start to happen at the tank, which would not obviously happen at the real tank… It would be very practical for my car (laughter), but unfortunately this is not the case.

S22: We agree that the solution is not realistic after this point, like the tank starts to magically fill again (laughter).

S21: The way I started to solve this is to make the solution the piece-wise function and say that it follows the original solution up to the point when the tank is empty and the second part of the piece-wise function is zero for all values of t after 2.

S23: Yes, we can use different values for constants, but I chose to have the simplest… I also tried to fit the exponential function, like saying it is linearly independent, but it did not fit very well so I ended up splitting the function.

The attempt of student S23 to fit exponential function can be interpreted as explorative routine in the use of CAS. Other students in this group solved the differential equation and plotted graphs manually; none of them opted for solving the differential equation analytically with CAS although they did this when asked explicitly. In Fig. 47.1, the student plotted the formal solution to the problem and a piece-wise defined function corresponding to the "realistic" solution. The decision not to use the formal solution described geometrically by the parabola and construction of the realistic solution demonstrates performer's *agentivity* and *applicability,* two characteristics of explorative routines. For Fig. 47.1a, the student reasoned: "As our solution is a parabola, the reasonable thing to suspect is that after the level has decreased to its bottom value, it will start increasing again. As we plot the graph, we can see that at $t = 2$ the container is empty, and mathematically it starts filling again. So, the

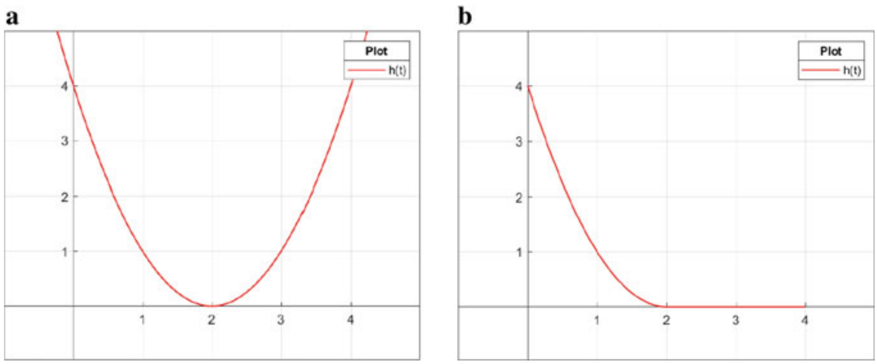

Fig. 47.1 **a** Student's graph of formal solution. **b** Student's graph of "realistic" solution

solution stops being a realistic description of the height after the container becomes empty". For the "realistic" solution in Fig. 47.1b, the student defined a function $h(t)$ by two expressions, $(2-t)^2$ for $0 \leq t \leq 2$ and 0 for $t \geq 2$.

In the development of the theoretical mathematical discourse, engineering students were less confident with the use of mathematical concepts of continuity and differentiability and relied on geometric arguments to support their argumentation in written reports, see Fig. 47.2.

The use of mathematical language in this fragment can be described as immature, the explanation is given on a rather intuitive level, but at the same time there is an evidence that the visual image from the computer simulation helped to develop the mathematical discourse. It is confirmed by the fact that the similar sets of graphs are present in the student's report twice: in the explanation of the solution as shown in Fig. 47.1 and in the answer to part (f) of the problem where the explanations to the graphs were explicitly required.

From out first part of the piece-wise function:

$$\frac{dh(t)}{dt} = -2 * (2-t), \qquad t \in [0, 2\rangle$$

$$\lim_{t \to 2} \frac{dh(t)}{dt} = 0.$$

This gives a line that starts at (0, -4) and ends in (2, 0). Differentiating our second definition gives us 0 for all t and is also continuous. The differentiated functions meet at the point (2, 0). We can therefore say that the function is continuously differentiable and of class C^1. Which is enough for our problem.

Fig. 47.2 Student's reasoning regarding continuity and differentiability of the solution

47.6 Concluding Remarks

The analysis of written reports shows that students relied on different representations (realisations) of the modelling task: mathematical description with the help of a diagram (visual), mathematisation using an appropriate differential equation (symbolic), graph plotting (visual), solution of the differential equation with the help of the CAS (symbolic). Mathematical routines employed by students possess characteristics pointing towards their explorative nature whereas routines associated with the use of CAS were more ritualised. Students' written reports document striking differences in their ability to use CASs and demonstrate that technology was mainly used as a computational and verification tool and, to some extent, as a visualising tool, but it did not become a transformational or data collection and analysis tool; this agrees with the findings of Doerr and Zangor (2000).

This study confirms that MM tasks motivate the development of new explorative mathematical routines thus improving students' mathematical and programming skills. Engagement in MM furnishes students with more confidence in mathematical knowledge and confirms their ability to apply it in practice. If MM tasks explicitly requires the use of CASs, special attention should be paid to the establishment of solid connections between mathematical and CAS routines; students should be encouraged to use programming not only for informal mathematical explorations but also as a powerful learning tool. In view of students' poor skills in the analysis of symbolic and numeric solutions for continuity and differentiability, teaching the use of CAS as a helpful tool for rigorous mathematical analysis could improve students' ability to explore important properties of mathematical objects.

Ärlebäck and Frejd (2013) argued that the "attempt to use the commognitive perspective on the mathematical models and modelling is challenging since we are all 'newcomers' to this particular research discourse" but "commognition naturally facilitates conceiving mathematical modelling as an interdisciplinary subject including a number of different disciplines" (p. 55). Our study promotes the use of commognitive approach as a useful tool for analysing students' mathematical discourse. Commognitive framework provides important insights into learning routines, helps to spot difficulties encountered by the students during the modelling process and prompts how these difficulties can be addressed.

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Chapter 48
Mathematical Modelling with Biology Undergraduates: Balancing Task Difficulty and Level of Support

Yuriy Rogovchenko

Abstract We report on extra-curricular activities with biology undergraduates focusing our attention on the selection of mathematical modelling tasks with different levels of cognitive demand and the level of teacher's guidance during students' collaborative work on the tasks.

Keywords Mathematical modelling · University education · Biology undergraduates · Teacher's guidance · Cognitive demand

48.1 Introduction

Mathematics is playing an increasingly important role in the life sciences. In the last decades, the interaction between mathematics and biology has become more and more intense with the two sciences positively influencing the advancement of each other as Cohen notes:

> In the coming century, biology will stimulate the creation of entirely new realms of mathematics. In this sense, biology is mathematics' next physics, only better. Biology will stimulate fundamentally new mathematics because living nature is qualitatively more heterogeneous than non-living nature…. Coping with the hyper-diversity of life at every scale of spatial and temporal organization will require fundamental conceptual advances in mathematics. (Cohen 2004, p. 2017)

A new relationship between mathematics and biology places higher demands on the education of biologists which is "burdened by habits from a past where biology was seen as a safe harbour for math-averse science students" (Steen 2005, p. 14). On the other hand, "the need for basic mathematical and computer science (CS) literacy among biologists has never been greater" (Gross et al. 2004, p. 85); "concepts from biology should be integrated within the quantitative courses that life science students take, and quantitative concepts should be emphasized throughout the life

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science curriculum" (p. 86). Benefits of mathematical modelling (MM) in biology are clearly identified by Odenbaugh (2005, p. 236):

> (1) Models are used to explore possibilities. (2) Models give scientists simplified means by which they can investigate more complex systems. (3) Models provide scientists with conceptual frameworks. (4) Models can be used to generate accurate predictions. (5) Models can be used to generate explanations.

Our research on how biology could be brought to the mathematics classroom is situated within an emerging area of tertiary mathematics education dealing with the teaching of mathematics to non-mathematicians. This category of interdisciplinary integration contextualises mathematical ideas for other subjects and "can loosely be termed thematic integration, where mathematics and other subjects come together around a particular topic or theme, while each retains their disciplinary nature" (Williams et al. 2016, p. 19). Several recent papers address the implementation of mathematics and MM in biology education. Integration of more mathematics and statistics in biology courses "did not have a negative effect on the performance of first-year students and can help more advanced students gain a better understanding of underlying biological principles and concepts" (Madlung et al. 2011, p. 52). The use of MM tasks in an introductory biology course led to "(i) improved equation literacy, (ii) greater conceptual and descriptive precision, (iii) formation of conceptual connections within and among disciplines, and (iv) more mature scientific judgment" (Weisstein 2011, p. 208). A mismatch between instructors' expectations and students' mathematical skills can be "immensely frustrating", but the situation is improved "by making quantitative reasoning an explicit objective of our course design" (Hester et al. 2014, p. 62).

Regular calculus courses can be designed to make them relevant to biology and pre-medical students; and it has worked very well for the majority of students (Rheinlander and Wallace 2011). However, "for a few students the uncertainties of not having a concrete answer and working with a big messy problem, even if fully acceptable in science, are not comfortable for them in math. For these students mathematics is about learning more math content and not how to apply the math they know in a creative, integrated and precise way" (p. 15). Another challenge is that "the natural tendency for life sciences students to understand how categories of life forms differ from one another is turned upside down in mathematics, where we wish to illustrate how seemingly disparate phenomena observed in unrelated applications are driven by identical mathematical descriptions" (Usher et al. 2010, p. 182).

Chiel et al. (2010) describe a course teaching MM to biology students and biology to students with strong backgrounds in mathematics, physics, and engineering. In addition to curriculum-related institutional constraints, they emphasise the cultural gap between biologists and quantitatively oriented sciences: the training process for biologists "tends to attract students who are good at memorization" and "repels students who are most interested in abstract principles" (p. 250). Although the majority of students did well in the course, "it is somewhat disappointing that biology students showed no significant improvement in their attitudes toward and their sense of competence in mathematics" (p. 262) and a one-semester course may not be sufficient for changing students' attitudes towards mathematics.

48.2 Theoretical Framework

Following Niss et al. (2007), we define *mathematical model* as "a triple (D, M, f) consisting of a domain D of the real world, a subset M of the mathematical world and a mapping from D to M" (p. 77) and interpret *modelling competency* as "the ability to construct and to use or apply mathematical models by carrying out appropriate steps as well as to analyse or to compare given models" (pp. 77–78). MM is known to be difficult to teach and learn as Blum explains:

> Mathematical modelling is a cognitively demanding activity since several competencies involved, also non-mathematical ones, extra-mathematical knowledge is required, mathematical knowledge and, in particular for translations, conceptual ideas … are necessary …, and appropriate beliefs and attitude are required, especially for more complex modelling activities. These cognitive demands are responsible for *empirical difficulty*.(Blum 2015, p. 78, emphasis in original)

By *cognitive demands*, we mean "the kind and level of thinking required of students in order to successfully engage with and solve the task" (Stein et al. 2009, p. 11). Recent empirical and theoretical research suggests that modelling competency amounts to the ability to successfully perform all steps in a modelling cycle. Models of a modelling cycle are helpful for the cognitive analysis of modelling tasks and serve as a tool for scaffolding, diagnosis, and intervention during students' work on modelling tasks. For research and teaching purposes, seven-step models of a modelling cycle (e.g. Blum and Leiß 2007; Stillman et al. 2007) are used, whereas a simpler four-step schema—understanding task → establishing model → using mathematics → explaining result—is more appropriate for students' work (Blum and Borromeo Ferri 2009). "All these steps are potential cognitive barriers for students as well as essential stages in actual modelling processes, though generally not in a linear order" (Blum amd Borromeo Ferri, p. 47).

For analysing students' cognitive difficulties, we rely on a research tool developed for "(a) identifying and classifying critical aspects of modelling activity within transitions between stages in the modelling cycle, and (b) identifying pedagogical insights for implementation through task design and organisation of learning" (Stillman 2015, p. 799). The first two transitions, Messy real-world situation → Real-world problem statement and Real-world problem statement → Mathematical model, are particularly useful.

The best students' learning is achieved on the basis of "quality mathematics teaching" which combines "a demanding orchestration of teaching the mathematical subject matter", "permanent cognitive activation of the learners" and "an effective and learner-oriented classroom management" (Blum and Borromeo Ferri 2009, p. 52). Strategic support "plays a prominnet role as the students are only supported to find a way to go on, but the solution itself must still be developed by the students themselves" (Stender et al. 2017, p. 469).

> For quality teaching, it is crucial that a permanent balance between (minimal) teacher's guidance and (maximal) students' independence is maintained. … In particular, when students are dealing with modelling tasks, this balance is best achieved by adaptive, independence-preserving teacher interventions. In this context, often strategic interventions are most adequate, that means interventions which give hints to students on a meta-level. (Blum and Borromeo Ferri 2009, p. 52)

One of the main difficulties is that learning depends on the specific learning context and MM has to be learnt specifically (Blum and Borromeo Ferri 2009). This sets particular demands on a teacher's ability to balance the level of students' support in the development of meta-knowledge needed for successful modelling and the level of students' independent learning.

> According to our observations, mathematics teacher's spontaneous interventions in modelling contexts were mostly not independence-preserving, they were mostly content-related or organisational, and next to never strategic. … A common feature of many of our observations was that the teacher's own favourite solution of a given task was often imposed on the students through his interventions, mostly without even noticing it. (p. 53)

The author of this chapter also found the following guidelines very useful for the organisation of teaching:

> Teaching aiming at students' active and independent constructions and individual solutions (realising permanently the aspired balance between students' independence and teacher‘s guidance); Systematic change between independent work in groups (coached by the teacher) and whole-class activities (especially for comparison of different solutions and retrospective reflections); Teacher‘s coaching based on the modelling cycle and on individual diagnoses. (Blum and Borromeo Ferri 2009, p. 55)

Presenting theoretical material and examples during whole-class analyses of group solutions and individual solutions to home assessments, the author emphasised important cognitive and metacognitive strategies and suggested several ways for overcoming blockages in the modelling process. A modelling diagram (Blum and Leiß 2007; Stillman et al. 2007) served well the purpose. Since "students normally do not have *strategies* available for solving real world problems" (Blum 2015, p. 80, emphasis in original), we organised the teaching and learning based on the knowledge that students' success with MM tasks requires:

> a well-developed repertoire of cognitive and metacognitive strategies as well as a rich store of mathematical concepts, facts, procedures, and experiences; vicarious general encyclopaedic knowledge of the world and word meanings; and truly experiential knowledge from personal experiences outside school or in more practical school subjects. (Stillman 2015, p. 796)

48.3 Design of the Study

Motivated by the idea that MM can serve as a "didactical vehicle both for developing modelling competency and for enhancing students' conceptual learning of mathematics" (Blomhøj and Kjeldsen 2013, p. 151) and knowing that case studies "that show, for example, student's views, motivations or performance, while learning in interdisciplinary lessons, are very helpful" (Williams et al. 2016, p. 22), the author designed extra-curricular MM activities for the first-year undergraduate biology students at a reputable research-intense Norwegian university. The twofold aim was (a) to demonstrate how even simple mathematics tools can be employed for solving biologically meaningful tasks, including unstructured problems, and (b) to increase students' motivation for learning mathematics.

The activities were organised in the form of five three-hour sessions. In addition to the author, a professor of mathematics, the project team included three mathematics educators with different levels of teaching and research experience: a first year PhD student, a postdoc with a recently earned PhD degree and a knowledgeable professor. The author coached a group of 12 volunteer students (9 female and 3 male) enrolled at the same time in a regular first-semester compulsory mathematics course. This general course emphasised practical use of mathematics and was designed to refresh and consolidate students' knowledge of school mathematics preparing them for further university studies. By the first meeting, students studied properties of periodic, power, exponential, logarithmic functions, limits, continuity, and derivatives. MM was briefly introduced in the course with exponential growth; differential equations were covered before we started using them in the activities.

The first four sessions were structured similarly: the author briefly presented relevant theory illustrating it with complete solutions to selected problems; then students worked collaboratively in three small groups on modelling tasks of various difficulty. Collaborative learning means that "participants are making a coordinated, continuing attempt to solve a problem or in some other way construct common knowledge" (Mercer and Howe 2012, p. 11). After each problem-solving block, group solutions were discussed along with "expert" solutions provided by the mathematician. Then a new portion of material was introduced, followed by students' work on the tasks. In each session, take-home assignments were handed out to motivate students' individual explorations between the sessions; solutions to these tasks were discussed in the beginning of the next session. In the last, fifth session, three team members joined small students' groups working on a MM task based on Newton's cooling law and presented as a "criminal story". Video recordings of all sessions were collected and transcribed. The dataset includes students' written work and answers on two self-administered questionnaires using a 5-point Likert scale exploring students' previous experience with mathematics, their perception of the subject and its relevance for biology.

48.4 Modelling Tasks

The author's expectations of students' capability to engage productively with modelling tasks were strongly influenced by his pedagogical practice, personal beliefs about teaching and learning, and passion for mathematics. The choice of tasks was motivated by the principles of effective teaching formulated by Swan and

Burkhardt (2014, p. 16), namely: “the tasks we use should be accessible, extendable, encourage decision-making, promote discussion, encourage creativity, encourage ‘what if’ and ‘what if not?’ questions”. To reflect all four groups of justifications for the inclusion of MM in teaching (Blum 2015), we selected open-ended tasks for “pragmatic” justification to illustrate the mathematical analysis of real-world situations (Task A below) and well-structured tasks accentuating modelling steps for “formative” justification (Task C). We have chosen mostly authentic tasks illustrating the value of mathematics to secure “cultural” justification and added entertaining tasks aimed at raising students’ interest in the subject (Task B) for “psychological” justification.

With respect to problems’ complexity, we employed two well-known frameworks which classify tasks by the degree of cognitive effort (low-level versus high-level, Stein et al. 2009) and by the type of reasoning, imitative (memorised and algorithmic) or creative (local and global creative mathematically founded) (Lithner 2008). In particular, creative mathematically founded reasoning (CMR) should fulfil all of the following criteria:

> Novelty. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created. Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible. Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning. (Lithner 2008, p. 266)

Both frameworks highlight the importance of cognitively demanding tasks for the development of students’ reasoning and problem-solving skills. Since MM is a cognitively demanding activity, we expected that students’ engagement with high-level tasks will have a positive impact on their learning and an overall satisfaction with the activities.

> The importance of starting with high-level, cognitively complex tasks if the ultimate goal is to have students develop the capacity to think, reason and problem solve. … Selecting and setting up a high-level task well does not guarantee students’ engagement at a high level. Starting with a good task does, however, appear to be a necessary condition, since low-level tasks almost never result in high-level engagement. (Smith and Stein 1998, p. 344)

Recent research indicates that task complexity is determined and fixed when the task is selected, whereas its difficulty varies from student to student; it is also known that poor performance with modelling tasks is more likely to be associated with no to low engagement, although high engagement with the task context is not necessarily associated with modelling success and appears to be task specific (Stillman 2015). Our ambition was to suggest to students several problems with high-level cognitive demands, also termed “doing mathematics” which (i) require complex and non-algorithmic thinking; (ii) require students to explore and understand the nature of mathematical concepts, processes, or relationships; (iii) demand self-monitoring or self-regulation of one’s own cognitive processes; (iv) require students to access relevant knowledge and experiences and make appropriate use of them; (v) require students to analyse the task and actively examine the task constraints that may limit possible solution strategies and solutions; and (vi) require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required (Stein et al. 2009). Research shows that both the setting of the tasks and the modelling of the process by teachers are important in the development of students' "meta-knowledge associated with the successful modelling of situations in their environment, students need tasks that require them to engage with the context in order to solve them successfully" (Stillman 2015, p. 795).

To illustrate the project in this chapter, we selected three tasks. Task A, a challenging open-ended assignment, *Rabbits on the Road*, is from the text by Harte (1988, pp. 211–213).

> Driving across Nevada, you count 97 dead but still easily recognizable jackrabbits on a 200-km stretch of Highway 50. Along the same stretch of highway, 28 vehicles passed you going the opposite way. What is the approximate density of the rabbit population to which the killed ones belonged?

Task B deals with the uncontrolled geometric growth of the bacterium *Escherichia coli* (*E. coli*). This problem was selected from the book by Enns (2011, pp. 9–10) where it is introduced to challenge the claim made by Michael Crichton, the author of the best-selling science fiction thriller *The Andromeda Strain*, that one cell of *E. coli* could produce in just one day a super-colony equal in size and weight to the planet Earth.

> If a single cell of *E. coli* divides every 20 min, how many *E. coli* would there be at the end of 24 h? The mass of *E. coli* bacterium is 1.7×10^{-12} g, while the mass of the Earth is 6.0×10^{27}. Is Crichton's claim accurate? How many hours should he have allowed for his statement to be correct?

Finally, Task C, *Growth of a yeast culture revisited* is adapted from a textbook by Giordano et al. (2014, pp. 11-14). The students were given two tables with the observed biomass p_n and the change in biomass Δp_n measured hourly for 7 h and then for further 11 h.

> The data in the table describe the growth of a yeast culture versus time in hours and come from the paper by R. Pearl, The growth of population, *Quart. Rev. Biol.* 2 (1927), 532–548. (a) Analyse first numerical data in the table. (b) Plot the data and analyse the graph. (c) Suggest a simple model based on a difference equation of the form $\Delta p_n = k_1 p_n$, where p_n is the size of the yeast biomass after n hours, $\Delta p_n = p_{n+1} - p_n$ is the change of biomass between two measurements, and k_1 is a positive constant. (d) What would be your expectations regarding

the predictive power of the model you constructed? Please explain. Analysing the data in the second part of the table, we note the change in population per hour becomes smaller as the resources become limited. (e) Plot the population against time and explore the shape of the graph. (f) What would you expect in the long run? (g) Based on the graph, we observe that the population appears to be approaching a limiting value, known in biology as the carrying capacity. What would be your expected value for the carrying capacity in this case? Please explain your answer.

Three sample tasks fall into different categories. An open-ended Task A requires careful mathematisation. Task B is solved by applying a growth model or by employing a geometric progression. Task C comes with the "embedded" solution plan; it develops model building skills and contributes to students' conceptual understanding of how the modelling cycle is implemented. Students' work in Task A is classified as "doing mathematics"; it also requires CMR. Complex computation or methods are not needed, but the solution calls for a novel reasoning sequence leading to the design of a strategy which must relate scarce information to additional assumptions needed for the solution. Being rather thoroughly scaffolded, formative Task C does not classify as "doing mathematics" and it does not require CMR; yet it is a higher cognitive level "procedure with connection tasks", "although general procedures may be followed, they cannot be followed mindlessly" (Smith and Stein 1998, p. 348).

Stewart et al. (2010) differentiate problem-solving reasoning for sciences into four categories: (1) *model-less* when the primary method is based on the use of algorithms; (2) *model-using* when the model is used to solve problems it can tackle; (3) *model-elaborating* when "the primary motivation for solving problems is to come to new understandings"; and (4) *model-revising* when "the real problem is to revise an existing model to fit anomalous data" (pp. 86-87). In this classification, Task A is model-elaborating since an "interesting reasoning occurs, and new insights may emerge" (p. 86). Task B falls into one of the first two categories, model-less if students use the reasoning based on the geometric progression or model-using if they exploit a growth model. Task C is designed as a model-using task since the students are directly prompted to use and develop further a familiar model.

48.5 Balancing Teacher's Guidance and Students' Independent Work

Zech suggested a five-level classification for the organisation of students' learning support:

1. Students are motivated only in a general way. 2. Positive feedback is given based on successful intermediate results. 3. Strategic support is given which takes the form of hints

> that refer on how to proceed without addressing content-related issues. 4. Content-related strategic support is offered; these are interventions, which also relate to the procedure, but content-related issues are involved. 5. A content-related intervention is completely related to the content of the task and contains the core of the solution. (as cited in Stender et al. 2017, p. 469)

The author organised activities in the project to minimise the teachers' guidance and maximise students 'independence, as suggested by Blum and Borromeo Ferri (2009). To avoid common mistakes in a MM classroom when teachers dramatically reduce students 'independence by premature content-related interventions or by imposing teacher-favoured solutions, we shifted strategic support to teaching blocks where metacognitive strategies based on a modelling cycle scheme were explained and promoted. During students 'work on MM tasks, all four team members were mostly engaged in non-verbal interventions, "tutors looked for the kind of work the students were doing without speaking to them, but noticed by the student" (Stender et al. 2017, p. 473), and only occasionally gave hints at meta-level ("read the task again", "what is the next step?", "what assumption might be useful to proceed?").

48.6 Discussion and Conclusions

At the beginning and at the end of the project, students rated their attitude to mathematics, its importance in biology, relevance of their mathematics course and knowledge, and evaluated their experience of activities (5 represents the strongest positive response). The averages in the pre-questionnaire were: mathematics is … interesting (2.75); enjoyable (2.25); important in biology (3.33); existing knowledge of mathematics (3.5); relevance of mathematics course to biology (3.33). The feedback in the post-questionnaire was very positive and encouraging: the activities were … interesting (4.17); enjoyable (3.92); challenging (4.67); contributed to the understanding of mathematics (3.33); biology (3.67); applications of mathematics to biology (4.17); usefulness of MM in regular mathematics classes (3.83). Despite the challenges and occasional frustrating moments, students warmly acknowledged the positive impact of the activities. Given that we selected tasks with different levels of complexity, each reflecting at least one of the four justifications for the inclusion of applications in modelling (Blum 2015), we conclude that both aims of the study were achieved. Students' responses on the questionnaire, to our satisfaction, do not support the hypotheses that "significant effects are less likely … (ii) for short term than sustained interventions; and (iii) for perceptions of students than teachers" (Williams et al. 2016, p. 19) and encourage the use of short term interventions as an efficient tool for stimulating students' interest in mathematics and its interdisciplinary applications. It goes without saying that "for quality teaching of applications and modelling, the teacher needs a lot of different competencies" (Blum 2015, p. 88).

What other lessons did we learn? Unexpectedly, a meticulously structured, restricting students' agency Task C confused students even more than the open-ended Task A (for instance, an unusual request to plot the change in biomass versus time). This suggests that even a very detailed solution plan embedded in the task may not necessarily lead to students' success if teacher's guidance is limited or unavailable. Surprisingly, students denied authenticity of some tasks, including, for instance, Task A, but did not question it for Task C. This issue, however, did not noticeably influence students' engagement with the tasks.

Students had difficulties with the mathematisation. Our study confirmed once again that "learners are afraid of making assumptions" (Blum 2015, p. 79). In MM tasks, students may need three different classes of assumptions associated with (i) model formulation; (ii) mathematical processes; and (iii) strategic choices in the solution process (Stillman 2015, p. 800). In Task A, one group of students made reasonable assumptions regarding the traffic intensity during the day and night and the time it takes to drive 200 km on a highway (i.e. car's speed) abolishing them later in favour of guessing the percentage of rabbits hit by the cars (difficulty with (i)). In Task C, students experienced difficulties with the assumptions in (ii) when they failed to replace the assumption of unlimited exponential growth of bacteria for a different one, associated with the "flattening" of the graph due to the nutrient and space restrictions. We can say that the "assumptions theme" was one of the leading learnings for the whole activity. In fact, answering the question "What was the best thing about the activity?" in the post-session questionnaire, many students mentioned the assumption building: "The way we got explained how we can make assumptions and overlook and add variables"; "It made me think in a different way than usual and it was exciting. Making assumptions was new to me"; "The assumptions. To understand what assumption is important and which is not".

Our experience in the project was very rewarding both for students and for the project team and in many aspects more positive compared to what was reported by the biologists who attempted incorporation of mathematics and MM into biology courses (Chiel et al. 2010; Hester et al. 2014; Madlung et al. 2011; Rheinlander and Wallace 2011; Usher et al. 2010).

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Chapter 49
Promoting Conditions for Student Learning in a Mathematical Modelling Course

Kerri Spooner

Abstract A case study looking into tertiary student learning experiences for mathematical modelling was carried out. The focus of the study was the student experience resulting from one lecturer's learning goals to address the research question: What is the student experience with mathematical modelling created by one lecturer of a first-year mathematical modelling course? Data were collected from student interviews. Student data were then inductively analysed, using reflective thematic analysis, to identify themes relating to their collective learning experiences. Results show that through guidance during lectures, students were able to have an independent modelling experience. To further enhance independence, it is recommended that lecturers work through problems unfamiliar to themselves during lectures.

Keywords Learning goals · Independent modelling · Reflective thematic analysis · Self discovery · Teaching of mathematical modelling · Tertiary teaching

49.1 Introduction

The purpose of this study was to explore the experience of first-year tertiary students with an open-ended modelling activity within a mathematical modelling course designed for engineering students. This study addresses the research question: What is the student experience with mathematical modelling created by one lecturer of a first-year mathematical modelling course? The lecturer's learning goals for students of the course, including the learning goals for the open-ended activity, were established in an earlier study (see Spooner 2020). A particular focus of this study was to gain insight into the student experience produced from the lecturer's learning goals. The modelling course consisted of lectures, tutorials, and a time-restricted, open-ended modelling project. This chapter considers the connection between the lecturer's learning goals and the resulting student experience. Literature on learning goals for mathematical modelling, current teaching practices concerning the use of

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open modelling projects at tertiary level and teaching support recommended for modelling projects will be presented. The learning goals of the lecturer used to inform the interpretation of the student data are referred to, followed by evidence of the student experiences these produced. This will then be discussed with literature support.

49.2 Literature Review

Consciously or subconsciously, lecturers have learning goals for students. There is a known link between learning goals and student educational experiences (Blomhøj and Kjeldsen 2006; Stronge 2018). An established global learning goal for secondary and tertiary mathematical modelling teaching is to develop modelling competency (Blomhøj and Kjeldsen 2011; Blum 2011). At the tertiary level, Caron and Bélair (2007) present and define communication skills, intervention skills, and evaluation skills as sub-competencies of modelling competency. Blomhøj and Kjeldsen (2011) promote reflecting internally on the modelling process and fostering the ability to reflect on the use and application of models as part of developing mathematical modelling competency. Interviews with mathematical modelling lecturers revealed modelling competency also includes connecting mathematics to reality, looking for patterns, exploring relationships, and being comfortable that the first attempt at producing a model will not likely produce a realistic model (Drakes 2012). In regard to developing generic modelling competency, a general discussion concerning a lecturer's subsidiary learning goals that underpin this general goal is lacking (Spooner 2020). A lecturer's subsidiary goals for developing mathematical modelling competency for his first-year modelling were presented in an earlier study (Spooner 2020). These goals, along with student outcomes created from these goals, form the basis of this study.

A generally agreed characteristic for developing modelling competency is to provide students with opportunities to actively participate in the modelling process (Blomhøj and Kjeldsen 2011, 2013; Caron and Bélair 2007). "Modelling is not a spectator sport and can only be learnt by doing" (Kaiser and Stender 2013, p. 280) with many studies suggesting that modelling is best learnt with authentic open-ended problems (Caron 2019). It is thought to be vital that students work with, and have their own experiences of creating, mathematical models of realistic situations (Blomhøj 2009; Kaiser and Sriraman 2006). Alongside active participation, evidence shows that mathematical modelling competencies can be developed using a combination of group work, classroom discussion, and individual work (Maaß 2006). Open-ended modelling activities, where the model to be developed is not immediately obvious, has similar characteristics.

At tertiary level, Duan et al. (2020), Blomhøj and Kjeldsen (2013), and Caron and Bélair (2007) studies, centre around the use of various open-ended modelling activities. To provide guidance for students, support is given, either internally or externally with open-ended mathematical activities. Support ranges from externally supplied heuristic strategies (Caron and Bélair 2007), that is, strategies that enable a person to discover or learn something themselves (Radford et al. 2015), to consultations between lecturers and students (Blomhøj and Kjeldsen 2013; Duan et al. 2020). Specific examples of heuristic strategies include: simplify as much as possible, use a simulation, draw and use diagrams, explain and discuss your thinking with someone else (Kaiser and Stender 2013; Stender 2019), and asking, have you seen a similar problem before? The support provides a scaffold for the activities, that is a structure or framework for students to refer to (Durandt 2018) as they progress through the modelling process.

Scaffolding approaches have developed out of constructivist teaching and learning methods (Kaiser and Stender 2013). One such approach is discovery learning. Alfieri et al. (2011) carried out two meta-analyses that concluded assisted discovery learning had better outcomes for students compared to unassisted discovery learning or direct instruction. Optimal approaches for assisted discovery learning included guided tasks with internal scaffolding, "tasks requiring learners to explain their ideas" (Alfieri et al. 2011, p. 12), and worked examples (Sweller et al. 2007). Klaher (2009) recommends using direct instruction initially to facilitate later discovery as an alternative approach for assisted discovery.

Developing mathematical modelling competency is central to modelling education. It is generally agreed that modelling competency is best developed through providing active opportunities to participate in modelling. The use of open-ended modelling activities is one way of providing this. Guidance can be provided internally or externally to the activity and is generally in the form of a scaffold designed to provide either directed or independent guidance.

49.3 Context of Study

To provide insight into the experience tertiary students have with an open-ended mathematical modelling activity and how this relates to their lecturer's student learning goals, an exploratory case study (see Yin 1981 for method) was conducted. To achieve this, the research question established was: What is the student experience with mathematical modelling created by one lecturer for a first year mathematical modelling course? The context of the study involved a first year summer school university mathematical modelling course in New Zealand designed for engineering students, its lecturer, and eight student participants. The five-week course consisted of two-hour classes (a combination of lecture and tutorial), four times a week.

Following an assessed pre-modelling project tutorial, students in groups of four undertook an assessed, open-ended modelling activity that formed 5% of their grade. Students were allocated seven hours to work on the one-day activity. A question applied on the day was: How high can you jump from a building without causing injury? The lecturer selected this question as it was ill-defined and open, enabling students ample opportunity to engage in, and independently develop, a group-directed modelling process. After working through the modelling process with the question, students submitted a written report outlining their process and showing the model produced. All student groups produced unique models in the form of a function or set of functions.

The lecturer is considered an expert lecturer, having received five teaching excellence awards. He is focused on developing modelling competencies within a holistic experience of modelling, that is, one that includes the full process of modelling (Haines et al. 2003). The lecturer's approach to provide a mathematical modelling experience for students included the following learning goals: Develop techniques for modelling; Acquire knowledge of the process of mathematical modelling; Engage in independent and critical thinking; Become familiar with skills for effective group/teamwork; Understand the criteria for the assessment of modelling project; Relate mathematics to genuine contexts; Participate in the modelling process as a group, independent of the lecturer; Communicate thought processes. See Spooner (2020) for additional details. These previously established learning goals were used in this study to inform the interpretation of the student experience data.

As part of the development of techniques for modelling, a *generic modelling cycle* was introduced during lectures and tutorials. This consisted of: Clarify the problem, list and classify the factors, make assumptions, formulate the problem statement, formulate the model, solve the model, interpret the model, and compare with reality. The lecturer utilised this, along with a "keep it simple" principle when role modelling the modelling problems and in assessment solutions.

Students enrolled in the course were invited to participate. Three male and five female students, out of eighty students, took up the invitation. Student participants were from five of the sixteen groups who participated in the open-ended activity. For all students, this course is their first university experience of mathematical modelling.

49.4 Data Collection and Analysis

The student participant data was collected through video recordings and field notes of individual post-course interviews conducted within two weeks of course completion. Student interview questions were designed to provide insight into the learning experience created for the students during the course, with a particular focus on a student's participation in the one-day modelling project. A large part of the questions was directed at establishing the approach undertaken by the student group during the modelling project and any associated challenges.

To gain insight into the students' learning experience of the course, and in particular, the one-day modelling project, reflective thematic analysis was used (Terry et al. 2017). This is an iterative method with an interpretative orientation where the objective is to make sense of what is going on within the data rather than classifying data into categories. The method involved first becoming familiar with the data set by reading and rereading, followed by inductive coding, producing both descriptive and analytical codes. Analytical codes encapsulate deeper meaning, and descriptive codes are codes that capture the primary topic of the data extract (Gibbs 2007). "I went over to another group at some point and they were doing something else" was coded as *More than one way*, and is an example of descriptive coding. For the data extract, "With not having the pressure of having a right or wrong answer is nice", a deeper meaning is implied; consequently, this was coded within the analytical code *Freedom to explore*. Once initial coding was carried out, the data were then re-examined using the learning goals of the lecturer as deductive codes to enable codes relevant to the lecturer's learning goals to be examined. Draft mind maps were then created to explore possible clusters and relationships between codes. A table of candidate themes, including potential codes within themes, was created to help define the themes, including their boundaries and how each theme might fit with others to "tell an overall story of the data" (Terry et al. 2017, p. 28). Each candidate theme was then checked to see if it truly captured the meaning of the coded data segments and the theme was adjusted accordingly. This was repeated across the dataset until there was coherency across the themes, associated data extracts and the story the themes told. The reviewing of the themes was central to the credibility of the analysis (Terry et al. 2017).

49.5 Findings

The findings are presented with the researcher's voice interweaved with the participants' voice to directly capture the student experience relating to the themes (Corden and Sainsbury 2006). Students' quotes have been selected to be representative of all participants and pseudonyms are used. Themes, sub-themes, and codes are shown in italics.

The lecturer used lectures, tutorials, and the one-day modelling project to provide opportunities for students' outcomes in response to his learning goals. The one-day modelling project enabled students to *participate in the modelling process as a group, independent of lecturer*, to use *techniques for modelling*, apply *modelling processes acquired* to develop a model for the problem situation. *Independent and critical thinking* combined with *communicating thoughts effectively* within *a group* were characteristics of group work undertaken during the one day modelling project. Lectures were used to teach students mathematical techniques, methods, and tools and to give exposure to the process of mathematical modelling, including providing, and frequently alluding to a generic modelling scaffold.

Results of data analysis from the student experience of the one-day modelling project revealed three main themes (Table 49.1): *giving students responsibility, students need for reassurance* and *change of course culture*. These all relate to a fourth theme, *students' independent discovery of the modelling process*. Table 49.1 includes both themes and their related codes. These will now be elaborated.

Table 49.1 Student experience themes and related codes

Themes	Giving students responsibility	Students need for reassurance	Change of course culture	Independent discovery of the modelling process
Codes	Freedom to explore Student ownership of the problem Adult-like responsibility Discover their own process	Unsure if it's right Unrealistic results Querying learning Correct information Receive knowledge from others	Exploration Unrealistic first time around No model answer No wrong or right way of approaching Everyone having their own approach Past culture of getting things right	Responsibility Change of course culture Student's need for reassurance Key modelling behaviours Turning points

49.5.1 Giving Students Responsibility

Through the use of the one-day modelling project, the acquiring of knowledge and techniques for modelling occurred by students being given the opportunity to be *responsible* for their own modelling process. *Responsibility* gave students *freedom to explore, ownership of the problem* and *adult-like responsibility*. Having the project as a low stakes assessment "minimised competition" amongst students. This enabled students to have "*freedom to explore* and not think too much about the end product" allowing them to *discover their own process* with "not having the pressure of having to get it right". Being able *to explore* meant students "learnt to do it by themselves" with all students saying this was how they learnt to formulate the model. "Doing the question is more than we could learn from just watching the lecturer do it" (Sophie). All students expressed that being given *responsibility*, through participation in the open project, "developed confidence in their ability to model". *Student ownership* was apparent with groups "defining the problem", "making [their] own assumptions" and "making the question [their] own". "We spent the first hour or so just really defining or really thinking about how we were going to answer it" (Shelly). By being given responsibility to work independently and "*discover their own process*", the lecturer was supportive of students "not wanting to be treated like a child", as firmly expressed by David.

The generic modelling cycle that was explicitly taught, used by the lecturer when role modelling problems and in past standard assessment items, was utilised by the students as a scaffold. "Because Peter had taught us beforehand, some of the techniques, we knew what to do when things went wrong and we knew where to start. So you need to have pre-learning, and another thing you've got to have is freedom" (Drew). Steph commented that there had been exposure to the modelling process during lectures such that it had become internalised "so it's really engrained after awhile". "Keep it simple", drawing diagrams, and planning were strategies used by students. "'Cause you can access everything and like, it's up to you to bring it together in a way that makes sense, but that's where the mathematic modelling process is helpful, have some order, order all your thoughts, like a real structured way" (Don).

49.5.2 Students' Need for Reassurance

Results show that while students are being given *responsibility* for *discovering their own process* they still expressed a need for *reassurance. Unsure if it's right, no model answer, unrealistic results, unrealistic first time around* and *querying learning correct information* come under this theme. The need for reassurance was demonstrated, both with reference to coursework and within the project. For coursework, one student (David) expressed needing the assurance that the information he was receiving was the truth. He liked to "*receive knowledge from others*" and did not like having to explore answers himself saying when asked "No, because you're not sure – you kinda wanna be sure." Interesting to note, this student's preferred learning style was from a textbook and he expressed wanting answers to assignments to "learn" while he does the problem. "Give me the answer and I will figure out how you got it". For the project, even though students were demonstrating modelling behaviour within their groups, there was doubt expressed that they were actually modelling. "I don't know if that's what mathematical modelling is like really" was a representative student comment. All students expressed there were times throughout the day where the way forward was not always clear.

49.5.3 Change of Course Culture

Exploration, unrealistic model first time around, appreciation *for not being right first attempt, no model answer, no wrong or right way of approaching*, *everyone having their own approach* and "for something not to necessarily be a perfect fit" are all part of modelling culture. This culture is different to that previously experienced by New Zealand school and university students for mathematics. Students are used to a mathematics *culture of getting things right*. "Changing that culture is hard 'cause it starts back at high school, I remember it's all about getting the answers right" (Don).

49.5.4 Independent Discovery of the Modelling Process

This *change of culture*, where students take *responsibility* utilising materials provided by the lecturer outside of the modelling project, allowed students to *discover their own process*. "We did some practice problems, but it's not like that. The four or five steps helped us bring it together" (Sophie). Specific *turning points* for groups in discovering their process were: discovering to "keep it simple" by "making assumptions to focus in on the most simplest way to get things up and running". "We got stuck….so we had to assume a lot. It was a case of simplifying things down" (Steph). For another group, learning how to recognise when a model foundation would be useful was key to being able to develop a useful model. They discovered that work that did not necessarily produce a viable model was not wasted. "I think though it [our first attempt] didn't give us the right answer … it gave us more details about the question and so you could understand it better" (Sophie). This was something that was not apparent to students before participating in the project. All students experienced and learnt to expect an *unrealistic model first time around* and observed that there was *no one ideal model* for the situation. "I went over to another group at some point and they were doing something else. But I didn't feel bad that we were doing our thing, 'cause you never feel like, ooa that's the right way to do it" (Drew).

The analysis revealed *key modelling behaviours* include being *mathematically creative*, having behaviours to cope with *new unfamiliar territory*, being *open to change*, being able to draw on *internal and external resources* and utilising *strategies to go forward*.

49.6 Discussion and Conclusion

An independent student self-discovery learning experience for developing mathematical modelling competencies was provided by the lecturer through the use of the open modelling project day. The modelling project used an authentic open-ended problem (Caron 2019) and provided opportunities for students to work independently of the lecturer in small groups (Maaß 2006), both conditions recommended for the development of modelling competencies (Caron 2019; Maaß 2006). Students working in groups explaining, sharing, and discussing their ideas with each other, assisted each other in their discovery of the modelling process (Alfieri et al. 2011; Kaiser and Stender 2013; Maaß 2006).

The results show that the support given externally during lectures and tutorials was taken up by students and utilised during the modelling activity. Support utilised was worked examples, a generic modelling process, and heuristic strategies. External support is in contrast to some of the recommendations for secondary school modelling experiences (Kaiser and Stender 2013), though supportive of Klaher's (2009) recommendation for self-discovery learning where direct instruction firstly takes place to aid future discovery. The students used past worked problems as guidance and they used the generic modelling cycle provided during lectures for assisting self discovery as recommended by Sweller et al. (2007) and as scaffolding for the process (Blomhøj and Kjeldsen 2013). These activities provided external scaffolding (Klaher 2009) as an alternative to the internal scaffolding for self-discovery learning, as suggested by Alfieri et al. (2011). Students utilised heuristic strategies recommended for modelling by Stender (2019) that included the saying "keep it simple", and drawing and using diagrams to communicate ideas and planning. The results show that a combination of the support given externally in class was taken up by students and utilised during the modelling activity. This demonstrates that the lecturer's learning objective to provide an independent student learning experience for modelling, with support external to the activity, was achieved.

Results show that the students discovered their own process, how to overcome challenges, and had an adult-like experience, in which they were responsible for their process and therefore learning from the experience. They discovered that their first attempt of a model will not be their last, a known modelling behaviour of practising modellers (Drakes 2012). This change in working culture experience left some students displaying modelling behaviour, though unsure if what they were doing was modelling. This suggests that even though the modelling cycle was explicitly taught and frequently alluded to during the course (as recommended by Caron and Bélair 2007), there is further need for the lecturer to expose the modelling behaviours of uncertainty and unfamiliarity, and provide additional strategies for dealing with these in lectures and tutorials. It is recommended that lecturers work through problems unfamiliar to themselves during lectures in order to role-model these behaviours that are integral to a modelling experience. This would involve a change of culture not only for students but also for lecturers, that is moving from a culture of the lecturer presenting pre-meditated solutions to one of role-modelling strategies for dealing with uncertainty. Strategies could include exploration, heuristic strategies, and not producing a perfect model to fit the situation.

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Part IX
Other Subjects

Chapter 50
The Red Book Activity—A Model Eliciting Activity to Introduce and Initiate a Section on Statistics Focusing on Variability and Sampling

Jonas Bergman Ärlebäck and Peter Frejd

Abstract This chapter analyses and discusses nine groups of upper secondary students' work on the question "How many red books are there in the library?". The students devised and implemented a plan, collected data, calculated estimates and reflected on what aspects and factors might have influenced their results caused by their adopted strategy. The analysis of the students' work focused on reconstructing and categorizing the models the students devised and implemented, as well as the sources and types of variability that the activity elicited. The results show how the central statistical idea of variability is manifested in the models developed and implemented by the students, and how these can be further explored and applied as central and bearing ideas for organizing a whole section of statistics at the upper secondary level.

Keywords Models and modelling perspective · Model eliciting activities · Sampling · Statistics · Teaching · Secondary level · Variability

50.1 Introduction

Statistics and mathematical modelling have lately been promoted as increasingly important for students to learn in order to be successful in society as well as in their professional and everyday lives (Franklin et al. 2007; Gal 2002; Niss et al. 2007; OECD 2013). Given the recent emergence of extremely large and complex data sets and the growing interest in data science due to increased technological abilities to collect and analyse data (Manyika et al. 2011), a critical understanding about the use and role of statistics in our world is now arguably even more important. Indeed,

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Franklin et al. (2007) write that "[o]ur lives are governed by numbers. Every high-school graduate should be able to use sound statistical reasoning to intelligently cope with the requirements of citizenship, employment, and family and to be prepared for a healthy, happy, and productive life" (p. 1). Numbers—or more specifically data as "numbers with context"—in themselves do not provide any information or knowledge, but rather require being organized and looked at using models in order to result in useful insights and knowledge (Manyika et al. 2011).

Research has shown that students often learn statistics from a procedural point of view with no or little understanding of what they do, why it works, or when to apply a certain method or way of reasoning (Batanero et al. 2011; Shaughnessy 2007). Horvath and Lehrer (1998) put forward that the essence of statistical thinking can be thought of as centred around developing, testing, interpreting, and revising models in order to understand our world and its diverse phenomena. Drawing on this view, Ärlebäck et al. (2015) argued that there are many unexploited parallels between this conceptualization of statistics and the on-going discussion on the use and role of mathematical modelling in the teaching and learning of mathematics. Building on the research suggesting that sequences of well-designed and structured modelling tasks can provide students with rich opportunities to learn more productively than in traditional settings (Doerr and English 2003; Lesh et al. 2003), we investigated the potential in adapting a modelling approach to the teaching and learning of statistics. Focusing on different types of variability in data (Franklin et al. 2007), we draw on the models and modelling perspective on teaching and learning and analyse groups of students' work on a model eliciting activity, called *The Red Book Problem*, designed to elicit students' initial models of variability (Lesh and Doerr 2003; Lesh et al. 2000). The two research questions addressed in the chapter are: *What models do students develop and implement when solving The Red Book Problem?* and *What sources and types of variability do students' work on the task elicit?*

50.2 Theoretical Background

50.2.1 Statistics and Statistical Inquiry

Statistics as a discipline can be conceptualised and understood in various ways (e.g. Wild et al. 2018). Many frameworks for statistical investigation build on the PPDAC framework (Problem, Plan, Data, Analysis, Conclusions) by MacKay and Oldford (1994). For example, Wild and Pfannkuch's (1999) four-dimensional framework of statistical thinking in statistical enquiry is expressed in terms of *the investigated cycle*, *types of thinking*, *the interrogative cycle* and *dispositions*. The data modelling perspective on statistics (e.g. Lehrer and English 2018) also draws on the PPDAC model. In both of these perspectives, variability is situated as a centrepiece of statistical thinking.

50.2.1.1 Variability

In Wild and Pfannkuch (1999), variability is portrayed as one of five fundamental aspects in statistical thinking and is composed of four components: *noticing and acknowledging*; *measuring and modelling*; *explaining and handling*; and *developing enquiry strategies*. Adding to this list, Reading and Shaughnessy (2004) argued that also *describing* and *representing variability* are needed to capture all critical and important aspects of variability. In this chapter, we use the framework by Franklin et al. (2007) who build on the aforementioned work and describe the nature of variability in terms of *measurement*-, *natural*-, *induced*-, and *sampling variability*. *Measurement variability* refers to the fact that whenever repeated measurements are collected, the result will entail variability and depend on a number of factors such as the precision of measuring devices used or changes in the system being measured. *Natural variability* captures the idea that variability is inherent in the physical world and nature. Individual and intrinsic differences in the objects being measured will induce variability in collected data. *Induced variability* in contrast to natural variability aims at describing variability that are due to external factors influencing the attributes being studied (such as an introduced treatment in a controlled experiment, for example). Lastly, *sample variability* focuses on the variability in a sample statistic arising as a consequence of the use of different sample sizes and sample strategies.

50.2.2 The Models and Modelling Perspective (MMP)

We situate our work with respect to the models and modelling perspective (MMP) on teaching and learning (Lesh and Doerr 2003). In this perspective, a model is defined as a general system consisting of elements, relationships, rules and operations that can be used to make sense of, predict, describe or explain some other system. In particular, a mathematical model focuses specifically on the structural characteristics of the system in question. The MMP is grounded in the ideas of Vygotsky, Piaget and Dienes as well as the American pragmatists' tradition represented by Mead, Peirce and Dewey, and Kaiser and Sriraman (2006) discuss the MMP as an example of a contextual perspective.

The MMP provides a framework to facilitate students' development of models towards a given learning goal through three different types of structurally related tasks organized in so-called model development sequences (MDSs). These sequences are purposefully designed to (i) elicit the ideas the students bring to the activity (MEAs—model eliciting activities); (ii) focus on the underlying mathematical structure of the models elicited by students (MXAs—model exploration activities); and (iii) students applying their model in similar or new contexts (MAAs—model application activities). In all of these tasks, students iteratively engage in exploring, expressing, testing, revising and developing their models to make sense of different situations and contexts (Lesh et al. 2003; Lesh and Doerr 2003).

The motivation for the work presented in this chapter is our goal to design and develop MDSs focusing on multiple key concepts in statistics. To this end, we sought to design and use a type of MEA that we are calling a *grounding and anchoring MEAs*, that provides students with an experience that elicits an ensemble of different central ideas within a given mathematical area (rather than a more focused learning objective as in the traditional use of MEAs). The goal of a grounding and anchoring MEA is that students' ideas and experiences elicited by the activity become both the basis for learning the mathematical content at hand, and the basis for connecting back to and tying the area of study together. In this chapter, we investigate the potential of a grounding and anchoring MEA in the context of a model development sequence focusing on statistics centred around sampling and variability.

50.2.2.1 Model Eliciting Activities (MEAs)

MEAs are tasks in which students are presented with a context where they need to develop a model that can be used to describe, explain or predict the behaviour of some situation or phenomenon (Lesh et al. 2000). MEAs have been used in a range of disciplines and contexts to support and investigate students' learning. The research by Lesh et al. (2000) have resulted in the following six design principles for MEAs: (a) *the reality principle*—the context should be meaningful to the students and connect to the students' previous experiences; (b) *the model construction principle*—that there is an inherent need for the students to develop a model when engaging in the activity; (c) *the self-evaluation principle*—that the task permits the students to assess their models and work; (d) *the model documentation principle*—that the situation in which the students are working requires the students to externally express their models; (e) *the model generalization and sharable principle*—that the model elicited in the MEA not is too narrow but rather is sharable, generalizable and applicable to similar situations; and (f) *the simplicity principle*—the situation in, and the formulation of, the MEA is as simple as possible facilitating that a focused model (learning goal) is targeted (Lesh et al. 2000; Lesh and Doerr 2003).

50.3 Setting and Method

The study was carried out in collaboration with an upper secondary mathematics teacher in her class of 28 11th-grade students (17–18-year-olds) taking their second mathematics course in a social study programme. As part of the mathematics courses, Swedish students in grades 10–12 learn about randomness, probability, descriptive statistics, measures of spread, correlation, causality, regression, and the normal distribution (Skolverket 2011). The teacher participated in a research project aiming to develop learning activities in statistics using a modelling approach, and the possibility to use an MEA was suggested by the researchers in the project. Before the intervention, the learning environment was typical for Swedish conditions, meaning that classes typically start with the teacher lecturing and presenting material followed by the students working in their textbooks on assigned problems.

50.3.1 The Red Book Activity

In designing the learning activity, the goal was to formulate an open and rich task that could function as an anchoring and grounding MEA for a 6-week section of statistics. We drew on a selection of the design principles of MEAs, and in particularly wanted the activity to facilitate the students to explore various approaches (*the model construction principle*) as well as have the students to express as many statistical ideas as possible explicitly (*the model documentation principle*). In addition, we wanted the problem situation to be accessible to the students and to have an objective that was easy to understand (*the simplicity principle*). We choose the context of the school library which facilitated a hands-on-experience for the students by going to the library to collect real data (*the reality principle*):

The Red Book Activity

Find out how many red books are there in the library.

1. Devise and write down a strategy and plan for how to answer this question.
2. Carry out your plan and document your work and result.
3. Reflect on the process and write down factors that might influence the answer.

Although this context does not present a realistic real-world problem situation per se, it mimics and captures many central aspects of real-world modelling problems, such as an ecologist's problem to estimate wildlife population sizes. For example, the task naturally forces the students to consider different types and sources of variability such as how to define what should count as a red book, and in addition also naturally elicits students' ideas with respect to sampling, since it is not feasible to count and examine all the individual books in the library.

The model generalization and sharable principle and the self-evaluation principle were considered in designing the follow-up lesson where the students' work on the activity was discussed and built upon. This will however not be presented and discussed in this chapter.

50.3.2 Data Collection and Analysis

For this study, our data consisted of students' written work carried out by nine small groups. The work on the task by one of the groups was video- and audio-recorded.

We also video- and audio-recorded the teacher in the classroom; and the whole class discussion following up the students' work in the second lesson. In this chapter, we limit the analysis to focus on the written work the students produced and the key statistical concept of variability in terms of (i) using the models and modelling perspective (Lesh and Doerr 2003) to reconstruct and categorize the models the students devised and implemented; and (ii) identifying the sources of variability that surfaced in the work and reflections of the students using open coding (Strauss and Corbin 1998), and mapping these codes to the types of variability (measurement-, natural-, induced- and sampling variability) from Franklin et al. (2007).

50.4 Results

The analysis of the students' work shows that three principally different models were used to determine an estimate of the number of red books in the library. We will now elaborate on these in more detail.

50.4.1 Students' Models to Determine the Number of Red Books in the Library

The three different models the students used to answer to *The Red Book Problem* were to (I) explicitly focus on the actual number of red books on each shelf; (II) focus on the proportion of red books on one or more shelves; or (III) use an area–density measure of red books (# red books per m^2). Two of the groups developed and implemented a model focusing on the actual number of red books in each shelf (I), six of the groups made use of a proportional approach (II), and two of the groups used an area–density measure of red books (III). Group 7 solved the problem twice using first model I and then model II. All three models developed entailed a sampling method, and four of the groups further applied some type of stratification (here referring to an explicit choice to gather information by considering different subsections of the library) in their sampling process to achieve the numerical values they needed to come up with an estimate for the sought-after quantity (see Table 50.1 for details).

The two groups who explicitly focused on the actual number of red books on each shelf (model I) both used stratified sampling and calculated the average number of red books per shelf. The groups used different sample size: one bookcase (group 7) and three bookcases (group 4). The groups then multiplied their average number of red books per shelf with the total number of shelves in the library to find the total number of red books (see Fig. 50.1). The total number of shelves in the library was either directly counted (group 7) or calculated for the three different types (sizes) of bookcases found, by multiplying the (counted) number of shelves in each type of bookcase with the (counted) number of bookcases of that type (group 4). Since the

Table 50.1 Models, samples and answer the groups used/derived

Group	Model	Sample		Estimated # red books
		Samples selected/used	Stratification[a]	
1	II—proportion	1 shelf	No	1296
2	II—proportion	8 shelves	Yes[c]	1231
3	III—area–density	4 areas of 1 × 1 m^2	No	604
4	I—number on shelf	3 bookcases	Yes[b+c]	600
5	II—proportion	541 books	No	1109
6	II—proportion	12 shelves	Yes[b+c]	1652
7	Both I/II	1 bookcase/5 shelves	Yes[c]	512.6 or 513.56
8	III—area–density	Area of 1 bookcase	No	582
9	II—proportion	15 shelves	No	431

[a]*Stratification* here refers to an explicit choice to gather information by considering a subsection of the library
[b]The sample of red books stratified
[c]The sample of total numbers of books stratified
[b+c]Both the sample of red books and of the total numbers of books stratified

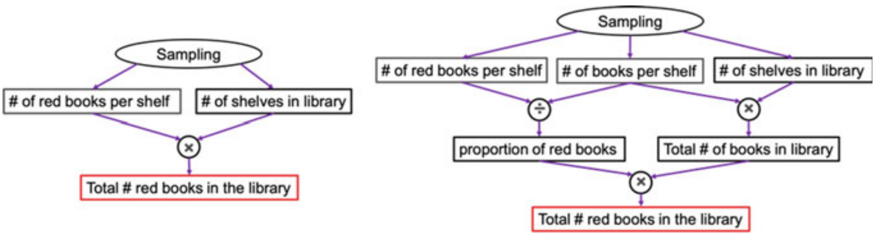

Fig. 50.1 Models based on the strategy of focusing on: (left) the number of red books per shelf (model I); and (right) the proportion of red books per shelf (model II)

library had three different types of bookcases, and the model used by these groups involve an estimate of the number of red books per shelf and the number of shelves in the library, the model naturally induced stratification in the sampling process regarding both quantities (number of red books per shelf and number of shelves in the library respectively) needed to come up with an estimate. However, only group 4 realised this potential and divided the library into three separate sections/rooms. Hence, this latter group sampled one whole bookcase, as well as counted the total number of shelves, in each section.

Six of the groups built their models around finding the proportion of red books on one or more shelves (model II). Five of these groups sampled between one and 15 shelves in the library, and counted both the total number of books on each shelf and the number of red books on each shelf. From their sample, and after having counted the total number of shelves in the library, they calculated the (mean-)proportion of red books and the total number of books in the library, before multiplying these

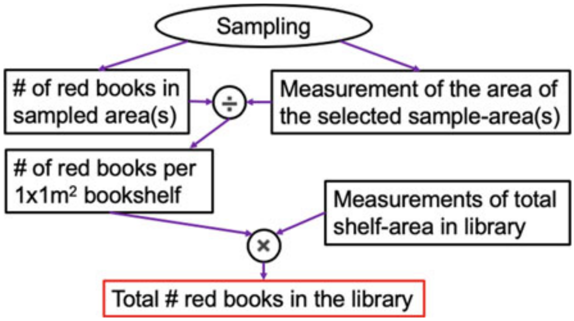

Fig. 50.2 Model based on using an area–density measure of red books (model III)

quantities together to get an estimate of the total number of red books (see Fig. 50.1). Three of the six groups adapting this type of model used some kind of stratification in their sampling process by dividing the library into different section and to sample and calculate the total number of (red) books in these individual sections. Two of the groups based their stratification based on the different rooms in the library. One group based their stratification on a division between fiction and fact/science literature, arguing that there might be different "colour-trends" within the two genres, which in addition prompted them to also sample the number of red books in the stratified sample. Group 5 used the same model, but the actual number of shelves they sampled is not explicitly specified in their written solution; only that they in total sampled 541 books (see Table 50.1).

Two groups created area–density models (model III) of the number of red books in the library. One group sampled four different 1×1 m^2 areas of bookcases in the library and created a new measure: the mean area–density of red books per square meter of bookcase by counting and averaging the number of red books in the four 1×1 m^2 areas. The other group sampled one bookcase, counted all the red books in the selected bookcase, took measurements of the bookcase, and then calculated the area–density of red books per square meter. To find the total bookcase-area of the whole library, both groups made measurements of all the bookcases, added these, and then multiplied with their area–density of red books to find an estimate of the total number of red books in the library (see Fig. 50.2).

50.4.2 Types and Sources of Variability Elicited by the Students' Work

The groups' written answers to the question about what factors might influence their answer were generally short such as "What one considers as 'red'" or "There are not the same number of books on each shelf". In all, 23 sources of variability could be identified in the students' answers, and six themes were identified and connected to the four different types of variability by Franklin et al. (2007); see Table 50.2.

Table 50.2 Sources of variability identified in the students' reflections

Themes (*nature of variability*[a])	Student identified sources of variability (*# of groups expressing this source*)	[b]Total (*# groups*)
Basic definitions lacking (*M*)	– What counts as a red book not well-defined (*5*) – What counts as a book more generally not well-defined (*1*)	6 (*5*)
Physical variability (*N*)	– Books come in different sizes (*1*) – Bookshelves come in different dimensions (*1*)	2 (*1*)
Population size unknown (*I*)	– The exact number of books in the library is not known (*1*) – Don't know how many books are checked out (*2*)	3 (*3*)
Distribution of (red) books (*I, N*)	– Different number of (red) books on shelves (*8 (= (4) + 4)*) – Different trends/patterns different in sections of the library (*1*)	8 (*5*)
Sampling (*S*)	– Larger sample gives a greater/better estimate answer (*2*) – Selection bias: "You sort of pre-scanned for red books…" (*1*)	3 (*3*)
Calculating (*M*)	– Rounding when doing calculations (*1*)	1 (*1*)

[a]Here *M* Measurement variability; *N* Natural variability; *I* Induced variability; and *S* Sampling variability (c.f. Franklin et al. 2007)
[b]Some of the groups expressed more than one source as variability

In Table 50.2, the theme *Population size unknown* captures variability that mirrors the uncertainty of books present in the library due to induced variability (*I*) from people reading books on site and having checked out books. The students' realization that books and bookshelves come in different sizes in the theme *Physical variability* focuses on the natural variability (*N*) within the population of books. Sampling variability (*S*) is expressed in the theme *Sampling*, and with respect to measurement variability (*M*) the students mentioned factors related to *Calculating* and the fact that *Basic definitions [were] lacking*. The theme *Distribution of (red) books* can be connected to both induced and natural variability (*I, N*) depending on what aspect is emphasised.

50.5 Conclusions and Discussion

Our analysis of the students' written work on *The Red Book Activit*y shows that (i) they based their models on three different sampling strategies (actual number of red books on shelf, proportion of red books, and an area–density measure) and created novel mathematical constructs such as various counts, proportions and area–density to find the number of red books in the library; and (ii) elicited all four types of variability (measurement-, natural-, induced-, and sampling variability) discussed by Franklin et al. (2007). Hence, we conclude that the key statistical idea of variability is manifested in the models and sampling strategies developed and implemented by the students, suggesting that these can be further explored and developed in a model development sequence focusing on variability using The Red Book Activity as a MEA. Indeed, many of the ideas and considerations (such as applying some kind of randomized sampling) that influenced the students' choices and work surfaced and were further discussed in more detail in the whole class discussion that followed the activity in the subsequent lesson.

The chapter also briefly introduces the notion of *anchoring and grounding MEAs*, and besides showing that *The Red Book Activity* is promising for developing models of variability, it in addition potentially also elicits other key statistical ideas: questions and issues related to (the role of random and stratified) sampling; the students' various estimated answers (see Table 50.1) might be used to develop models of distributions; the collected data to introduce various graphical data representations; and, the whole activity as such to discuss statistical problem solving or to start developing the ability of making (informal) statistical inferences. This, however, needs to be studied in future research.

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Chapter 51
Connections and Synergies Between the Teaching and Learning of Statistics and Modelling—A Pilot Study

Peter Frejd and Jonas Bergman Ärlebäck

Abstract We present a pilot study aiming at characterising the potential connections and synergies between the teaching and learning of statistics and modelling. Using grounded theory and the software NVivo, we analysed the 17 ICTMA books published to date and the books from ICME-6 and the 14th ICMI study. The results present identified themes based on the contexts in which the notions of statistic* are used in the books: teach and learn statistics/modelling; ICT; curriculum and theory. The analysed literature provides suggestions for how to teach statistics using a modelling approach, but seldom discusses the theoretical aspects of the relationship between mathematical and statistical modelling. The results also describe the potential of teaching and learning modelling using statistics as content and digital technology. Limitations of the methodological approach and suggestions for how to overcome these to develop a more robust methodology are also discussed.

Keywords Mathematics · Mathematical modelling · Statistics · Literature review · Grounded theory · Statistical modelling

51.1 Introduction, Research Goal and Research Question

The teaching and learning of statistics and its applications are the focus of the research field of statistics education, and substantial progress has been made in this area in the last 20 years (Ben-Zvi et al. 2018). However, as pointed out and argued by Ärlebäck et al. (2015), there are many unexploited and potentially productive parallels between the conceptualisations of statistics in statistics education and the ongoing discussion on the use and role of mathematical modelling in the teaching and learning of mathematics. Crossovers between the fields exist, such as the work by Doerr et al. (2017),

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who used a modelling approach to develop students' informal inferential reasoning. However, in this chapter, we want to investigate what evidence can be found of these alleged connections and synergies between the teaching and learning of statistics and mathematical modelling in research on mathematical modelling. We do this by developing and piloting a methodological literature review approach centred around a computer-aided keyword search, and apply this to the books from the past ICTMA conferences with the aim of characterising the themes that come to the fore in this selection of modelling literature, alluding to the (potential) connections and synergies between the teaching and learning of statistics and mathematical modelling. The research question we explore is: *In what contexts, and how, is the notion of "statistics" elaborated on in ICTMA books?*

In addition, we also discuss some limitations of the methodological approach applied, and how this can further be developed in order to provide a more informative, adequate and accurate picture of connections and synergies in the teaching and learning of statistics and mathematical modelling.

51.2 Statistical Models and Modelling from a Statistical Perspective

In statistics education, the teaching and learning of statistics are often framed and discussed in terms of *statistical thinking*, *statistical literacy* or *statistical reasoning* (e.g. Gal 2004; Garfield 2002; Pfannkuch and Wild 2004). However, these notions are not mutually exclusive. *Statistical thinking* is, according to Wild and Pfannkuch (1999), "concerned with learning and decision making under uncertainty. Much of that uncertainty stems from omnipresent variation. Statistical thinking emphasises the importance of variation for the purpose of explanation, prediction and control" (p. 227). *Statistical literacy* is related to the ability to interpret and validate quantitative information found in different contexts, and the ability to communicate and use statistical information to underpin opinions as well as personal and professional decisions (Gal 2004). *Statistical reasoning* is an ability to interpret statistical information, use statistical ideas in reasoning, explain statistical processes, connect statistical concepts to each other and make sense of statistical results (Garfield 2002).

The research fields of modelling and statistics education both have strong connections to other disciplines and contexts outside mathematics (Niss et al. 2007; Shaughnessy 1992). Indeed, statistics as a discipline arose as a tool to be used in other disciplines to cope with and handle statistical data (Cobb and Moore 1997), and the notions of *statistical models* and *modelling* are central within statistical education to make sense of our world (Shaughnessy 1992). Hence, statistical models foremost consider aspects such as variability, uncertainty and context (Lehrer and English 2018), often visualised and communicated using different representations such as regression lines, plots, diagrams, and algebraic and numerical expressions. Statistical modelling is a working process that uses statistical ideas and concepts to develop or use statistical models. Lehrer and English (2018) describe statistical modelling as a cyclic process that involves *posing* (researchable) *questions* that can be answered by *designing and conducting investigations*. This entails *generating and selecting attributes* to investigate, as well as deciding how to *measure the attributes*, which in turn gives rise to a *sample* representing the phenomena studied, but causing variability. In order to *model the variability*, it is important to *organise and structure data* as well as *measuring and representing data*. After getting a grip on the variability, it is possible to *make inferences* and express uncertainty about the inferences. So-called modelling cycles are also frequently used within the research literature of mathematical modelling, referring to the iterative process of structuring an extra mathematical problem situation, deciding on the use of suitable mathematics and working mathematically within the problem, and then interpreting and evaluating the results with regard to the extra mathematical situation (Niss et al. 2007).

51.3 Methodology

The method we developed to characterise the potential connections and synergies between the teaching and learning of statistics and mathematical modelling essentially follows the three basic steps for conducting a traditional qualitative literature review, namely to (a) identify adequate research literature related to the research question; (b) organise and analyse the identified research literature and (c) report the outcomes of the analysis (Bryman 2004).

In piloting our methodology, we elected to analyse the past ICTMA books and the official published outcomes from the 14th ICMI study and ICME-6. We acknowledge that this is a non-exhaustive sample and potentially excludes much relevant research and many authors, but for the purpose of testing our methodological approach, it is considered adequate, since this relatively small selection is guaranteed to provide a rich and varied selection of research on modelling from a multitude of perspectives, understandings and foci.

To organise and analyse the chapters in the selected books, we developed a methodological approach using the computer software NVivo. The idea was to exploit a word frequency query tool within NVivo, which automatically identifies and highlights all words entered into the NVivo search string. To use this functionality in NVivo, we defined a *set of search words* comprised of all notions and concepts related to different definitions of statistics. This set of search words included words like *mean*, *statistics*, *statistical literacy*, *variability* and *t-test*. The search words were then used to identify sections of text in the chapters of the books in our selection that use and discuss these statistical notions and concepts.

However, as single and isolated words by themselves do not provide any information on use and intended meaning, we chose the segments of texts around the words highlighted by NVivo as units of analysis to capture the contexts in which the words were used. These segments of texts will be called *context units* in this chapter, and each context unit typically contained one to three sentences or a small paragraph.

To categorise and classify our identified context units, a two-step coding strategy of *open coding* and *axial coding* (Strauss and Corbin 1998) inspired by grounded theory was applied. First, the context units were iteratively classified and organised into themes or open categories using the guiding question: *What are the contexts in which the words in the search set are used in the identified context units?* Then, the open codes were subjected to axial coding, which is the process of identifying relationships and conditions to link and group the open coded categories together into new and broader categories.

In this chapter, we present a pilot study of this method by focusing on a narrow subset of the set of search words, namely the search words *statistic** (where the asterisk captures all words starting with "statistic", such as statistic*s* and statistic*al*). Given that the set of search words is much larger (containing words and concepts such as *linear regression*, *distributions* and *t-tests*), we know that this only will provide a first indication and a partial insight with respect to our overall goal of characterising the potential connections and synergies between the teaching and learning of statistics and modelling. However, since the method of using NVivo together with context unit analysis needs to be evaluated and validated, we wanted to systematically pilot the method before undertaking the full analysis.

51.4 Results

The word frequency query in NVivo identified a total of 1300 words beginning with "statistic-". Figure 51.1 and Table 51.1 show the frequency of the words associated with "statistic*" found in the analysed literature. The frequency varies from three identified words in ICTMA12 to 218 words in ICTMA05. In all, 88 open codes emerged during the analysis. However, due to space limitation, we mainly focus and report on the results of the axial coding. Seven axial codes were identified:

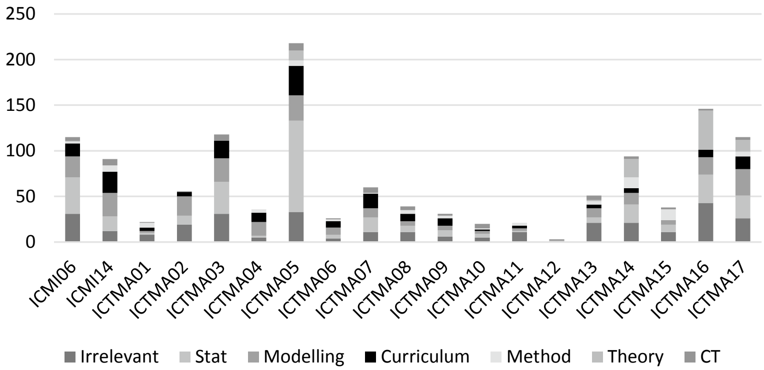

Fig. 51.1 Frequency of the words "statistic*"

Table 51.1 Number of context units coded in relation to axial codes and books

	Irre.	Stat.	Model	Curr	Metho	Theory	CT	Total
ICMI06	31	40	23	14	2	1	4	115
ICMI14	12	16	26	23	7	0	7	91
ICTMA01	8	2	2	4	5	0	1	22
ICTMA02	19	10	21	5	0	1	0	56
ICTMA03	31	35	26	19	0	0	7	118
ICTMA04	5	2	15	10	4	0	0	36
ICTMA05	33	100	28	32	6	11	8	218
ICTMA06	4	4	8	7	2	0	1	26
ICTMA07	11	16	10	16	1	0	6	60
ICTMA08	11	7	5	8	4	0	4	39
ICTMA09	6	7	5	8	3	0	2	31
ICTMA10	5	4	3	2	1	0	5	20
ICTMA11	11	1	3	3	3	0	0	21
ICTMA12	0	1	2	0	0	0	0	3
ICTMA13	21	6	10	4	4	1	5	51
ICTMA14	21	20	13	5	12	20	3	94
ICTMA15	11	8	5	0	12	0	2	38
ICTMA16	43	31	19	8	0	43	2	146
ICTMA17	26	25	29	14	5	13	3	115
Total	309	335	253	182	71	90	60	1300

Irrelevant (Irre), ***Methodology (Metho)***, ***Teach and learn statistic (Stat)***, ***Teach and learn modelling (Model)***, ***Theory***, ***Computer Technology (CT)*** and ***Curriculum (Curr)***, see Fig. 51.1 and Table 51.1.

Twenty-four per cent of the context units were coded as **Irrelevant** *codes*, which include open codes where the search word appeared as part of References, Contents, Name of course, Affiliations, Book titles, etc. Five per cent of the context units is categorised as **Methodology**, encompassing aspects focusing on research methodology that captures context units discussing how statistical methods have been used to validate the research results in the analysed literature. However, context units categorised as **Irrelevant** and **Methodology** do not contribute towards characterising the potential connections and synergies between the teaching and learning of statistics and modelling.

The axial code **Teach and learn modelling** (19% of the context units) captures context units where examples are given of how statistics is used as a content or context within the teaching and learning of modelling, with an emphasis on the modelling process. It entails context units that discuss the potential to introduce modelling using the topic of statistics, as illustrated by Dyke (1987): "*A good starting topic is statistics, which all students realise is useful, leading up to simulation and the first* *modelling …*" (p. 43). As suggested in this quotation, several context units show that the modelling problems students worked on in modelling courses and project work are of a statistical nature, and students often find statistics relevant for engaging in mathematical modelling connected to real life situations. Other context units discuss statistics as a tool in the modelling process. Some authors argue from a general point of view that statistics as a tool is important in project work, claiming for example that "*Statistics is clearly a useful and necessary tool required by large numbers of pupils for their project work in other subject areas*" (Rouncefield 1989, p. 154). Other researchers focus on specific parts of the modelling process where statistics might be a useful tool. According to Jablonka (1997):

> In surveys people possibly lie, are influenced by the way the question is posed or answer what seems desirable… so that validity is not easily guaranteed, though the measurement may be reliable. For assessment knowledge from statistics… as well as knowledge about the process of collection and recording of the data may be needed… (p. 44)

Here, Jablonka discusses the process of validating mathematical models and highlights a connection that knowledge in statistics is one of the key components in that process.

The axial code **Teach and learn statistics** (26% of the context units), in contrast to **Teach and learn modelling**, collects context units focusing on the teaching and learning of statistics rather than on the learning of modelling. Not surprisingly, within the analysed literature, suggestions for teaching statistics are frequently described using modelling as a vehicle. In their chapter *In-depth use of modeling in engineering coursework to enhance problem solving*, Clark et al. (2010) discuss a modelling activity with an aim to develop students' understanding of statistics, and write that statistics is useful to

> search for measurements or data… They could use this data to formulate and test statistical hypothesis developed on their own. The benefit would be increased understanding of the origins of data and the implications of using it… The students would be encouraged to make full use of library resources, the internet, practitioners, and other experts, thereby making the exercise more of a real-world, interactive, and multimedia experience. (Clark et al. 2010, p. 185)

The students in the quotation participated in a designed modelling activity which brings up aspects that are commonly addressed in statistical courses, such as exploring data, formulating hypotheses and applying statistical tests. In addition, the quotation connects to workplace mathematics, where statistics may be a useful tool in situations such as estimating the efficiency of industrial processes (Bungartz 1991). Evidence is found in the literature that teaching and learning statistics using a modelling approach develops students' statistical competence, as displayed in the following examples: "*the students' mathematical and statistical knowledge develops over this period, they are able to draw upon a continuously widening range of material and techniques necessary to back up the model building activities*" (Hamson 1987, p. 85); and "*It shows that the data-oriented modelling approach in the applied mathematics course improved statistical thinking skills*" (Engel and Kuntze 2011, p. 404). These two quotations indicate that teaching statistics through modelling increases students' understanding of statistics, pointing to synergetic outcome effects between engaging in modelling for learning statistics and engaging in statistics for learning modelling.

The axial code **CT** (Computer Technology, 5% of the context units) captures context units involving the role of computers. Many context units in **CT** are found in chapters explicitly focusing on ICT such as *Teaching and Learning Statistics by Computer* (Kemme 1993), and often emphasise the use of computers for analysing statistical data. In Hodgson's (1997) study, "*the students entered the data into a spreadsheet, graphed the data, and identified best-fitting models using the statistics capabilities of the spreadsheet*" (p. 214), illustrating how students developed mathematical models using regression. Different types of digital statistical toolkits are identified in multiple empirical studies, like Microtab, calculators, Excel, SPSS, Fathom, MATLAB, etc. The axial code **CT** also captures ideas of how students may simulate professional practice using both technology and statistical data, with a reference to so-called data-generated modelling, found in workplaces that include the handling of statistics and statistical toolkits (Frejd 2017).

Curriculum (14% of the context units) is an axial code that collects context units about statistical content or processes to be taught (as stated in curricula), standards for assessment and issues about statistical skills. An example of a context unit explicitly concerning such learning goals is found in Hamson (2001): "*The aims of the module can be summarised as: (i) to formulate and develop mathematical models of environmental processes (ii) to learn more advanced topics in the statistics of data analysis*" (p. 261). The two learning goals of the aforementioned module have the explicit aims of developing students' statistical and modelling abilities. Some of the literature focused on standards for assessment and developing criteria for assessing modelling in statistical enquiries (Izard 1997), whereas other literature focused on statistical skills. As previously argued, statistical knowledge may be improved by modelling activities (Engel and Kuntze 2011; Hamson 1987). However, concerns about the lack of statistical abilities are also found in the analysed literature, together with explicit suggestions that curriculums should focus more on these issues:

> The information society requires its members to be prepared to be able to interpret information… However, these audiences are in general not well prepared to handle these information carriers. Descriptive statistics, especially its use and more importantly its misuse, should have a central part in the mathematics curricula. (de Lange 1993, p. 6)

As shown in this quotation, de Lange (1993) raises concerns about citizens' statistical abilities and argues that more learning activities about descriptive statistics should be included in the classroom.

The axial code **Theory** (7% of the context units) collects context units focusing on theoretical aspects of statistical modelling and its relationship to mathematical modelling. The terms *statistical models* and *statistical modelling* are frequently used in the reviewed literature, but are usually not explicitly clarified in any greater detail. A few context units, however, explicitly discuss theoretical issues. Graham (1993) sets out to "*draw out three frameworks which underlie certain key features of statistics, and [will attempt to] use these to make suggestions as to how the subject might* *be taught more effectively*" (p. 177), and, for example, discusses a cyclic statistical working process describing how to work with statistical enquiries. The notions of statistical literacy and its relationship to mathematical modelling are discussed by Engel and Kuntze (2011). They "*look at the relationship between modelling competencies and statistical literacy and provide empirical evidence that proficiency in these areas can be jointly improved*" (p. 396), which presents some parallels between statistics in statistics education (statistical literacy) and mathematical modelling (modelling competence) in the teaching and learning of mathematics. Engel and Kuntze (2011) draw on the signal–noise metaphor to explain the core of statistical modelling and statistical thinking. Campos et al. (2015) include a theoretical section dealing with statistical reasoning, statistical thinking and statistical literacy. These authors also introduce a theoretical approach described as *critical statistics education*, aiming to foster a productive disposition towards important social and political issues, using realistic statistical enquiries together with ICT.

51.5 Conclusion and Discussion

In this chapter, we have developed and piloted a methodological approach to conduct a literature review aimed at characterising the potential connections and synergies between the teaching and learning of statistics and mathematical modelling in research on mathematical modelling. By focusing on contexts in which the words *statistic** are discussed in the selected literature, the application of the methodology highlighted and revealed connections and synergies in terms of *the teaching and learning of statistics using modelling as a vehicle* and *teaching and learning modelling using the content of statistics*. In addition, *the role of computers and other information technology*, *aspects related to curriculum policy* and *assessment* and *theoretical discussions* were also topics identified in the analysis.

Regarding the methodological approach, the NVivo software proved to be helpful in implementing the grounded theory-inspired approach, especially when finding context units mentioning the word *statistic**. However, implementing the two-step analysis was complex and time consuming, since the context units could potentially refer to different things. In particular, the axial coding was challenging when coordinating different meanings within the same context units, and this uncertainty needs to be considered when interpreting the relative frequency of context units within a particular theme. However, the coding was discussed between the two authors to arrive at a consensus on their meaning in order to increase reliability. An alternative method might have been to explore and categorise the identified chapters where the word *statistic** appeared most frequently, rather than coding all context units containing the search words. Due to technological difficulties with the ICTMA12 proceedings, not all chapters were fully readable by NVivo, which affected our results but probably not to a significant extent.

The variation in frequency of the words statistic* within the analysed literature as depicted in Fig. 51.1 and Table 51.1 is partly explained by some of the books (ICTMA5, 16 and 17) having chapters that focus explicitly on the teaching and learning of statistics. The earlier proceedings tend to include more examples connected to courses in statistics, whereas more recent literature concerns other aspects such as statistical literacy. Many references and quotations in this chapter within the axial codes of the teaching and learning of statistics using modelling as a vehicle, teaching and learning modelling using the content of statistics, and aspects related to curriculum policy are from older ICTMA books highlighting the large number codes connected to these books, as shown in Table 51.1. The axial code theory includes references from more recent literature in this chapter, and the role of computers is found to be discussed with a low frequency in most of the books, which can also be seen in Table 51.1. However, our results also show that issues related to deciding on attributes, sampling, variability and making inferences (e.g. Lehrer and English 2018) are rather spares in the analysed sample, and it could be fruitful to explore these areas further from a mathematical modelling perspective in future research, based on the preliminary results from this pilot study. The next phase in our research is to work on the limitations of our methodological approach and to conduct a complete analysis with all the words in the full set of search words, and to include more literature from the statistics education research community, to see how this might alter our findings and to point out further research areas.

In summary, the results from this study show that the teaching and learning of statistics and modelling have many things in common. For example, modelling can be a useful tool to develop statistical competences (e.g. Engel and Kuntze 2011; Hamson 1987), working with statistical enquiries can develop modelling ability (e.g. Clark et al. 2010), and connecting modelling and statistics is part of a professional modelling practice (Frejd 2017). We agree with Engel and Kuntze's (2011) argument that coordinating and stimulating these connections and synergies through a closer collaboration between the research communities of modelling and statistics education would benefit the development of both research fields.

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Chapter 52
System Dynamics: Adding a String to the Modelling Bow

Peter Galbraith and Diana M. Fisher

Abstract This chapter illustrates system dynamics modelling as a means of real-world problem-solving. Distinctive aspects are related to the familiar modelling cycle. Differential equations are used comparatively, with simple models, to introduce properties of software that enable solution by simulation; essential when equations are simultaneous and non-linear. Examples of common structures (archetypes) are used to demonstrate application to problems made tractable by software, newly free and online. Increasingly national curricula emphasise the importance of students being enabled to apply mathematics in the workplace, as citizens, and for private purposes. We illustrate how system dynamics provides (now accessible) methods, that are directly relevant to these purposes at secondary level and beyond.

Keywords Archetype · Causality · Delays · Feedback · Simulation · System dynamics

52.1 Introduction

National curricula, including those of our respective home countries, emphasise the importance of enabling students to use their mathematical knowledge in the workplace, in their private lives, and as responsible citizens. In addition to contributing to emerging demands to be expected in the first of these, a knowledge of system dynamics (SD) adds skills that are directly relevant to the latter two purposes. System dynamics exemplifies the role of mathematical modelling as real-world problem-solving. Its recently extended accessibility adds a further dimension to the types of modelling problems that can be addressed at secondary school level and beyond.

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System dynamics provides a methodology for analysing how actions and reactions cause and influence each other, and how and why elements and processes in a system change. As expressed, some time ago by Jay Forrester, the founder of the field:

> Complex systems differ from simple ones in being 'counter intuitive', i.e. not behaving as one might expect them to…They stubbornly resist policy changes. They contain influential pressure points, often in unexpected places, which can alter system-steady states dramatically… They often react to a policy change in the long run, in a way opposite to their reaction in the short run. Intuition and judgment generated by a lifetime of experience with the simple systems that surround one's every action create a network of expectations and perceptions that could hardly be better designed to mislead the unwary when he moves into the realm of complex systems. (Forrester in Miller 1972, p. 50)

The behaviour of complex systems (and many simple ones) is non-linear, which means that model equations require simulation for solution. While relevant icon-based software has been available for a number of years, costs have restricted opportunities to organisations, individuals and institutions with sufficient interest and resources to purchase the software. Caron (2019) has commented usefully on some earlier endeavours restricted in this way. Recently, free online versions of Stella (ISEE Systems 2020), and Powersim Express (2020) have enabled the building of models, which while small, allow a selection of genuine modelling contexts to be explored. This availability means that a new family of modelling problems becomes accessible to students, teachers, and others at secondary and tertiary levels. Of course, it is necessary to invest some effort into understanding the software, so as to use opportunities to advantage—in the same way that spreadsheet documentation needs to be mastered before their potential can be realised. Anyone familiar with first-order differential equations is equipped to fully understand and use the design properties of the software, for which documentation is excellent. But additionally, with secondary students in mind, essential skills can be developed directly to a productive level, independently of an initial knowledge of calculus (Fisher 2011).

The focus of this chapter is methodological, geared to an audience familiar with modelling. Distinctive components of systems dynamics are introduced and related to familiar modelling processes. We illustrate how system dynamics modelling software operates, beginning with simple models, solvable alternatively and familiarly by differential equations. We indicate how these models are equivalently solved through simulation, and then move to non-linear problems where simulation is the only way to proceed. Some examples of commonly occurring model structures (archetypes) illustrate how systems dynamics modelling is used to provide access to widely occurring real-world problems that persistently resist solution. The approach includes an orientation for readers to characteristics of the freely available software that is referenced in the chapter. These are introduced in Sect. 52.2 and further illustrated by the examples in Sect. 52.4.

52.2 Features of System Dynamics Models

52.2.1 Types of Problems Addressed by System Dynamics

There is indeed no call to cry over spilt milk as the spill is easily mopped up without further implications; but a "spill" of pollutants is different—some major problems being faced today are the result of toxic emissions by industrial processes of past decades. Typified by this case, and fundamental to problems suited to system dynamics analysis, is the identification of feedback systems (circular chains of cause and effect) often containing substantial delays. An information feedback system exists, whenever assessment of the environment leads to a decision resulting in actions which in turn affect that environment, and thereby influence future decisions and outcomes.

52.2.2 Model Structure

Noting that system dynamics addresses problems involving feedback, direct influences between variables are displayed by means of causal links (arrows). A '+' sign on an arrow means that a change in the value of the variable at the *tail* of the arrow causes the variable at the *head* of the arrow to move (or tend to move) in the SAME direction. A '−' sign means that SAME is replaced by OPPOSITE. When a succession of arrowed links forms a closed chain one of two types of feedback loop is formed.

52.2.3 Reinforcing (Positive) Feedback

Reinforcing (or positive) feedback is typified by structures that embody exponential growth. Figure 52.1a contains a *causal loop diagram* (CLD) for the familiar situation of a population growing at a constant net fraction per annum (say 5%). An increase

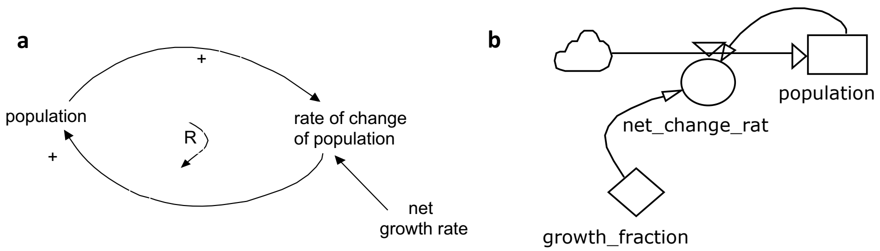


Fig. 52.1 **a** Causal loop diagram (+ve feedback), **b** Model diagram (+ve feedback)

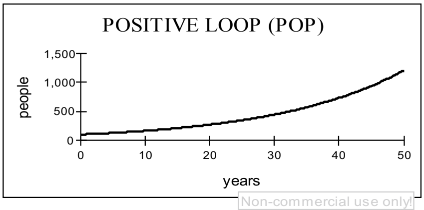
in population will lead to an increase in its rate of change, and this increase will lead to a further increase in population and so on. This closed loop comprises just two links, and the label 'R' indicates its reinforcing nature. A CLD is useful for displaying and explaining the essence of a modelling system but is insufficient for developing the mathematical model itself. For this purpose, model structure needs to be built using components of the software being used: shown in Fig. 52.1b for Powersim (e.g. Powersim Express 2020). Different software products use closely similar imagery—rather like different spreadsheet versions in comparison with each other.

52.2.4 Symbolism

The tank symbol (rectangle) in Fig. 52.1b is used for any quantity (*stock*) that accumulates, rising and falling over time in response to inputs and outputs (flows). In consequence of its measurable level ("height") at any time, a stock is designated as a *level* variable. Here the *level* variable is population, but in other situations, may represent goods, buildings, dollars, food, sick people, timber reserves, and so on. *Flows* are inrates and outrates that augment or deplete levels—symbolised by valves as in Fig. 52.1b for net_change_rate. While we can see they are happening, we cannot put a number on the instantaneous value of flow magnitudes through observation, as we can in reading the value of a level. They are defined in terms of the known values of other model components—here the net_change_rate is given by the product of the level (population) and the constant annual growth_fraction. *Constants* (*parameter values*) are fixed values of a model entity that apply throughout a given simulation. They are characteristic of the system being modelled—symbolised by a diamond. Here there is just one parameter: growth_fraction (5% per year).[1] Other *auxiliary* variables are usually present. Shown as circles, they are used simply to make life easier, by enabling complex formulae to be entered as component parts of rate specifications. Arrows are used to *link* variables that are directionally related in the model, as they are in the real world.

Model output for the above is displayed in Fig. 52.2 below, depicting exponential growth—obtained alternatively using the familiar differential equation format shown beside Fig. 52.2. This serves to demonstrate the equivalence of the two representations. The way the software works is shown in the box following.

[1]The growth_fraction as employed here is the result of combining increases due to births plus immigration and decreases due to deaths plus emigration.



Differential equation (traditional) format

Consider a population (initially 100) growing at 5% per year.

$dP/dt = 0.05P$, $P(0) = 100$

Gives $P = 100e^{0.05t}$

Fig. 52.2 Graphical output for positive loop structure

Sample computation of model output (*generated using numerical integration and graphed by the software)*
Beginning with the specified starting values, variable values are calculated iteratively at each timestep (defined by the chosen value of *dt*): Here

population (0) = 100 (people)

growth-fraction = 0.05 (1/year)

time step *dt* = 0.25 (year)

net_change_rate (0.25) = 0.05*100 = 5 (people/year)

population (0.25) = 100+5*0.25 = 101.25 (people)

net_change_rate (0.5) = 0.05*101.25 = 5.06 (people/year)

population (0.5) = 101.25 + 5.0625* 0.25 = 102.52 (people) And so on.

52.2.5 Balancing (Negative) Feedback

Balancing feedback is typified by processes with inbuilt compensating mechanisms, such as an air conditioning system set to adjust room temperature to a goal value.

The completed loop in Fig. 52.3a, which here consists of two positive links and one negative link, is a *balancing (or negative) loop*, as indicated by the 'B' symbol inside the loop. Such loops are "balancing" or "stabilising" since an increase (decrease) in a loop variable leads via the closed path of causality to a decrease (increase) in the same variable. (*Balancing loops* contain an *odd* number of *negative links*.) Figure 52.3b contains the corresponding stock and flow model diagram.

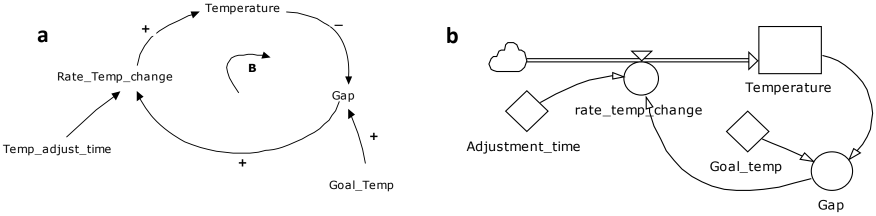


Fig. 52.3 **a** Causal loop diagram (−ve feedback), **b** Model diagram (−ve feedback)

Software computations (heating)

Goal_temp = 25 (deg)
Temp_adjust_time = 3 (min)
Temperature (0) = 15 (deg)
Gap (0) = 10 (deg)
dt = 0.25 (min)
Rate_temp_change (0) = 10/3 = 3.33 (deg/min)
Temperature (0.25) = 15 + 3.33*0.25 = 15.83 (deg) and so on.

Differential equation (neg feedback)

$dT/dt = (25 - T)/3,\ T(0) = 15$
$\int dT/(25 - T) = \int dt/3$
$T = 25 - 10e^{-t/3}$
'T' asymptotes to 25

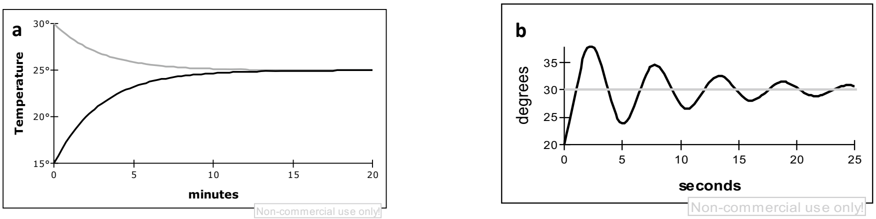


Fig. 52.4 **a** Temp control (−ve feedback), **b** Temp control (−ve feedback with delay)

All negative loops have a common feature (*a goal*); and a gap between the goal and the current state that is the source of action. The action aims to close the gap over an adjustment time. Here the gap is the temperature difference between the setting on the air conditioner (goal) and the current temperature, and the adjustment time is determined by the power of the device. Illustrations of software computations and the corresponding differential equation representation are shown below.

Figure 52.4a shows the asymptotic behaviour generated by the single negative feedback loop controlling temperature—consistent with the differential equation solution. Both heating and cooling situations are depicted.

52.2.6 Delays

Delays are the third principal entity responsible for influencing the behaviour of systems containing feedback. Figure 52.4b illustrates the effect of inserting a delay between the identification of a gap, and subsequent action in a heating system (think adjusting the hot tap in a shower). The delay results in corrective activity that successively overshoots and undershoots the desired target.

52.3 The System Dynamics Modelling Process

The process of mathematical modelling as real-word problem-solving is familiarly represented as a cyclic process, summarised, for example, as in the box below. The arrows indicate the logical order in which a problem solution develops and is reported.

It is not a picture of an individual's thinking processes, which typically move back and forth between stages as a solution is developed and refined. In system dynamics the same overall stages apply, but there are some distinctive features—for example:

Specify the mathematical modelling question→ formulate a mathematical model → solve the mathematics → interpret outcomes → validate/evaluate the outcomes in terms of the real context → revisit the solution process and/or document and report outcomes.

- In SD, it is common practice to sketch representations of problematic behaviours to act as evaluation criteria for model structure and output. Where such data do not yet exist, as in modelling within an emerging system, there needs to be agreement on the model representation of processes being set in train.
- System boundary is a specific SD concept. It needs to be drawn wide enough to include all feedback processes judged important for addressing the problem.
- Feedback processes that contribute to model structure need to be specified fully—this involves choosing variables, parameters and delays. In SD models, every variable and parameter has a precise counterpart in the real world, and is labelled accordingly.
- Model development features iterative rounds of formulation, solution (simulation), interpretation and evaluation. Formulation is more structured (by feedback processes) than is typical for modelling in general.
- Sensitivity testing features widely. In conventional modelling, it is typically used to test the robustness of a solution in terms of its real-world usefulness. In SD, it serves an essential purpose within formulation, by testing properties of an emerging simulation model. For example, extreme parameter values are chosen to check that even under extreme conditions, quantities that are never negative in the real world do not so behave in simulations.
- Model output is obtained in terms of behaviour modes exhibited by variables of interest, not in terms of point predictions. For example, persistent, wave-like behaviour tells a useful story when the accurate pinpointing of peaks and troughs is not possible due to imprecision of available data.

Software documentation provides illustrations and explanations of all features.

52.4 System Dynamics Modelling in Action

So, what kind of problems lend themselves to SD modelling? Typical problems involve situations where outcomes for interacting variables are compromised by feedback compounded by delays. Investigating the future carrying capacity of planet earth (IMMC 2019) is a problem-inviting SD analysis, with a past extensive modelling enterprise using this approach documented in Meadows et al. (1992).

A second family of problems involves analysing archetypical behaviour of simple systems. System archetypes (Senge 1990) are recurring structures that appear

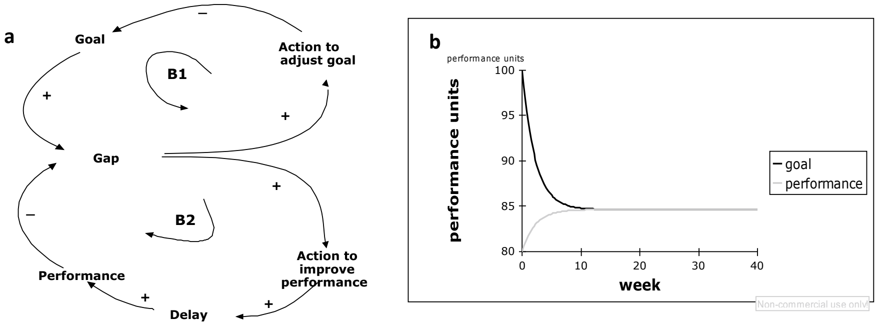


Fig. 52.5 **a** CLD (eroding goals) **b** Simulation output (eroding goals)

wearing different clothes in various real contexts. Recognition of a common model structure, together with its behaviour, provides a basis for anticipating, or improving or avoiding, consequences of potentially damaging decisions. It is risky to predict behaviour from CLD structure alone—simulation is needed to test intuitions and examine claims analytically (see Galbraith 2004). Once verified, there is a basis for action to improve outcomes, in whatever domain the given structure is identified. Three examples of archetypical behaviour are developed below. See Fisher (Chap. 3 this volume) for an illustration of system dynamics applied to an extended problem.

52.4.1 Eroding Goals (When Under Pressure Hold the Line)

When a gap exists between a goal and current performance, there are two sets of pressures—to raise performance (bottom loop in Fig. 52.5a), or to lower the goal and hence degrade future performance (top loop in Fig. 52.5a). Letting goals slide is easy, particularly when we evaluate achievement in terms of previous performance (well we did 90 per cent as well as last year!) instead of some external standard that we hold to.

Differential Equations for eroding goals based on Fig. 52.5a.

If x_1 denotes performance, and x_2 denotes goal then:

(1) $\mathrm{d}x_1/\mathrm{d}t = (x_2 - x_1)/T_1$, where $x_1(0) = X_1$ and T_1 is adjustment time for *performance*

(2) $\mathrm{d}x_2/\mathrm{d}t = -(x_2 - x_1)/T_2$, where $x_2(0) = X_2$ and T_2 is adjust time for *goal*: $T_2 < T_1$.

Although simple in form, these simultaneous non-linear differential equations can only be solved by simulation.[2] This is achieved by translating the structure into model equations—as illustrated earlier for positive and negative feedback. Typical output is

[2]DEs are included for comparative methodological purposes—they are not needed for model building. Model equations are input directly into a software-generated diagram.

shown in Fig. 52.5(b), where reduced performance attains a degraded goal. Examples include: governments meeting employment targets by weakening definitions of “full employment”; universities graduating more students by shortening degree times and pretending nothing has been lost; pass rates enhanced by lowering performance standards; and companies maintaining sales by hiding cost increases (e.g. shaving material from popular items such as chocolate bars).

52.4.2 *Escalation (Anything You Can Do I Can Do Better)*

American President: “There is a serious imbalance. The Soviets have the capacity to destroy the world three times over—we only have the capacity to destroy it twice” (Cold war cartoon).

Two parties in competition each see their respective well-being in terms of the relative advantage of one over the other. If one gets ahead, the other feeling threatened acts aggressively to re-establish its position. This in turn threatens the first, promoting further aggressive activity, and so on. The resulting build up is damaging to both.

Differential Equations for Escalation based on Fig. 52.6.

For the left loop (1) and right loop (2) in Fig. 52.6(a), we have respectively:

(1) $dx_1/dt = (r_1 - x_1/x_2)x_1/T_1$; $x_1(0) = X_1$, $r_1 = (x_1/x_2)$ desired by A, $T_1 =$ time to close gap by A

(2) $dx_2/dt = (x_1/x_2 - r_2)x_2/T_2$; $x_2(0) = X_2$, $r_2 = (x_1/x_2)$ desired by B, $T_2 =$ time to close gap by B

Output from a software-generated model is shown in Fig. 52.6b, confirming the escalating behaviour. Examples include the Arms Race; universities spending increasingly on advertising efforts to outdo each other; price cutting by businesses in competition; keeping up with the Jones’s; and wanting the last word in arguments.

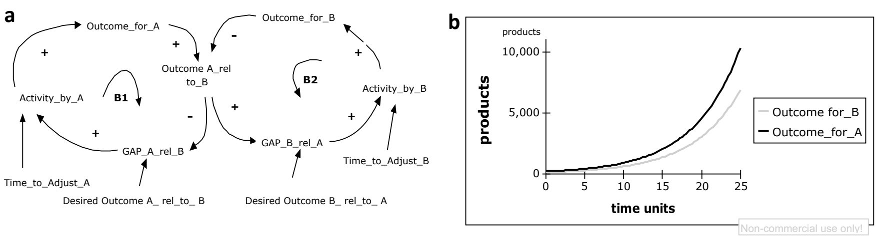


Fig. 52.6 **a** CLD (Escalation) **b** Simulation output (Escalation)

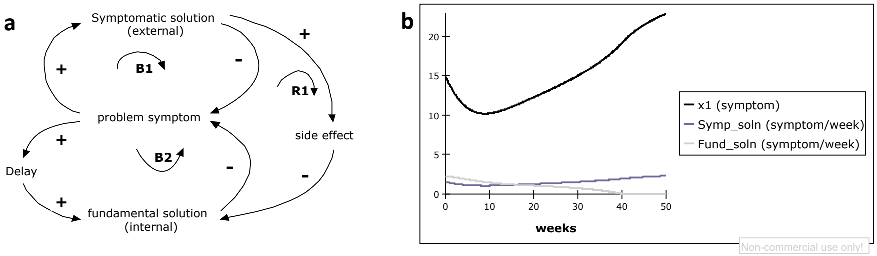


Fig. 52.7 **a** CLD (Shifting the Burden) **b** Simulation output (Shifting the Burden)

52.4.3 Shifting the Burden (Scratching a Persistent Itch)

An underlying problem produces symptoms but the problem is far deeper than the symptoms, requiring fundamental treatment to fix. As the fundamental treatment involves more resources, the burden of "treatment" is shifted to the symptoms.

A side effect of the repeated use of these "quick fixes" is to weaken the capacity of the fundamental solution (e.g. by starving it of resources or de-skilling workers). By repeatedly applying a superficial remedy that alleviates symptoms, but leaves the underlying problem unaddressed, the problem ultimately becomes worse, and more seriously, the capacity to deal with it is impaired. Figure 52.7a contains a CLD for *shifting the burden*, with output from the corresponding model shown in Fig. 52.7b.

Differential Equations for Shifting the Burden based on Fig. 52.7a.

If x_1 denotes "symptoms", and x_2 denotes "side effect":

(1) $dx_1/dt = g - [s + (f - cx_2)]x_1$ if $x_2 \leq f/c$; $dx_1/dt = g - sx_1$ if $x_2 > f/c$, where $x_1(0) = X_1$: g = generation rate of symptoms; s and f are the respective fractional rates at which symptoms are normally eliminated by the symptomatic and fundamental solutions ($s < f$), c is a multiplier through which the side effect impairs the normal effectiveness of the fundamental solution.
(2) $dx_2/dt = bsx_1$ where $x_2(0) = 0$ and b is a multiplier translating repeated use of the symptomatic solution to a growing side effect.

The graph in Fig. 52.7b illustrates how repeated use of an external (symptomatic) solution results in short-term improvement. Ultimately, the capacity of the fundamental solution is degraded through the generation of an unwanted side effect. As a result, symptoms increase again as the major controlling mechanism has been damaged. Examples of shifting the burden are: repeated substance abuse that masks the need to change a lifestyle damaged by other pressures; use of new credit cards to pay debts on "maxed out" predecessors, generating a debt burden only effectively addressed by budget discipline; and university faculties that address debt by sacking staff, then wonder why debt returns, not comprehending the impact on earnings of closure of specialist areas.

52.5 Summary

System dynamics facilitates non-linear modelling problems involving feedback and delays, where simulation is essential to solution processes. Specialised software enables the formulation and solution of such models, bypassing otherwise intractable differential equations. Many problem situations contain social content, which connects with common experience of students at all levels (e.g. archetypes). National curricula, including Australia and the USA, emphasise the importance of students being able to use their mathematics to address problems in their living environment. System dynamics provides skills directly relevant to this purpose. There is renewed opportunity to develop and apply these modelling skills through the agency of fully documented software, free versions of which are now available online (e.g. Powersim Express 2020; Stella: ISEE Systems 2020).

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Chapter 53
The Extended Theoretical Framework of Mathematical Working Space (Extended MWS): Potentialities in Chemistry

Laurent Moutet

Abstract The aim is to show how the extended mathematical working space (extended MWS) theoretical framework can be used to analyse the tasks implemented during a few stages of a modelling cycle in a chemical problem. This chapter studies a teaching sequence, including an experimental session in chemistry and graph construction for students in the last year of secondary school (grade 12) in France. The extended MWS theoretical framework makes it possible to study the multidisciplinary aspect of the different tasks that students must perform when working on problem solving.

Keywords Modelling · Interdisciplinarity · Chemistry · Extended MWS · GeoGebra · Geometry

53.1 Presentation of the Task Given to the Students

The work given to the students is centred on the notion of titration commonly taught in the last year of high school (grade 12). It is the same teacher who teaches physics and chemistry in France. A pre-test document is given to collect students' conceptions of equivalence and stoichiometry. The study concerns 16 grade 12 students of a public high school situated in the north of France in Abbeville. One hour was spent in the chemistry laboratory to perform the titration experiment. Two additional hours are required in the computer room to work with GeoGebra, a dynamic geometry software. The work is then to be finalised at home before being returned. The notions of equivalence or stoichiometry seen the previous year are not remobilised (only by 2 students out of 14). This is only the last part of the teaching sequence that will be presented with work using GeoGebra to model the situation.

An aqueous solution containing diiodine is placed in an Erlenmeyer flask, and an aqueous solution containing sodium thiosulfate is placed in a graduated burette (see

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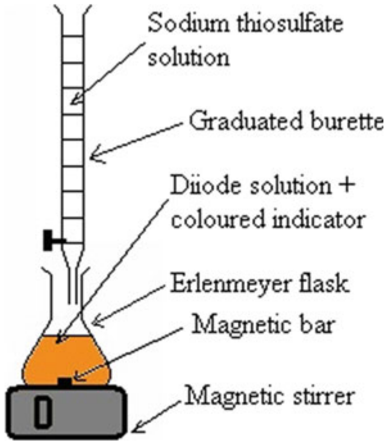

Fig. 53.1 Titration of a diiodine solution

Fig. 53.1). A chemical transformation takes place between the diiodine molecule I_2 and the thiosulfate ion $S_2O_3{}^{2-}$. Equivalence corresponds to the moment when both species have completely reacted. It is shown experimentally by changing the colour of the solution containing a coloured indicator, previously introduced into the Erlenmeyer flask containing the reaction mixture, when equivalence is reached. Knowing the quantity of sodium thiosulphate introduced at equivalence, it is possible to deduce from this, by using an algebraic relationship, the concentration of diiodine initially present. That's the purpose of this titration.

It is assumed that an expert visualises the equivalence of a titration by closely associating it with the notion of stoichiometry. It could therefore be relevant to use in this case a graphical construction using the reaction progress $x(t)$ with GeoGebra. The notion of stoichiometric relationship corresponds to the disciplinary work targeted here. The equation for the reaction of the diiodine titration and the algebraic calculations necessary to find the C_{I_2} diiodine concentration are presented below (see Fig. 53.2). $C_{S_2O_3^{2-}}$ corresponds to the thiosulfate ion concentration that is initially known, V_{I_2} corresponds to the volume of diiodine solution initially introduced, $V_{S_2O_3^{2-}}$ corresponds to the equivalent volume found during the experiment, n_{I_2} correspond to the quantity of diiodine matter and $n_{S_2O_3^{2-}}$ correspond to the quantity of thiosulfate ion matter. The only unknown is C_{I_2}.

Using the modelling cycle proposed by Blum and Leiss (2005), the *real situation* would correspond to the experimental titration with the observed colour change. The *model situation* would be associated with understanding the existence of a chemical

$$I_{2(aq)} + 2\ S_2O_3{}^{2-}{}_{(aq)} \rightarrow 2\ I^-{}_{(aq)} + S_4O_6{}^{2-}{}_{(aq)}$$

$$n_{I_2} = \frac{n_{S_2O_3^{2-}}}{2}\ ;\ n_{I_2} = C_{I_2} \times V_{I_2}\ ;\ n_{S_2O_3^{2-}} = C_{S_2O_3^{2-}} \times V_{S_2O_3^{2-}}\ ;\ C_{I_2} = \frac{C_{S_2O_3^{2-}} \times V_{S_2O_3^{2-}}}{2 \times V_{I_2}}$$

Fig. 53.2 Algebraic calculations associated with the dosage of diiodine

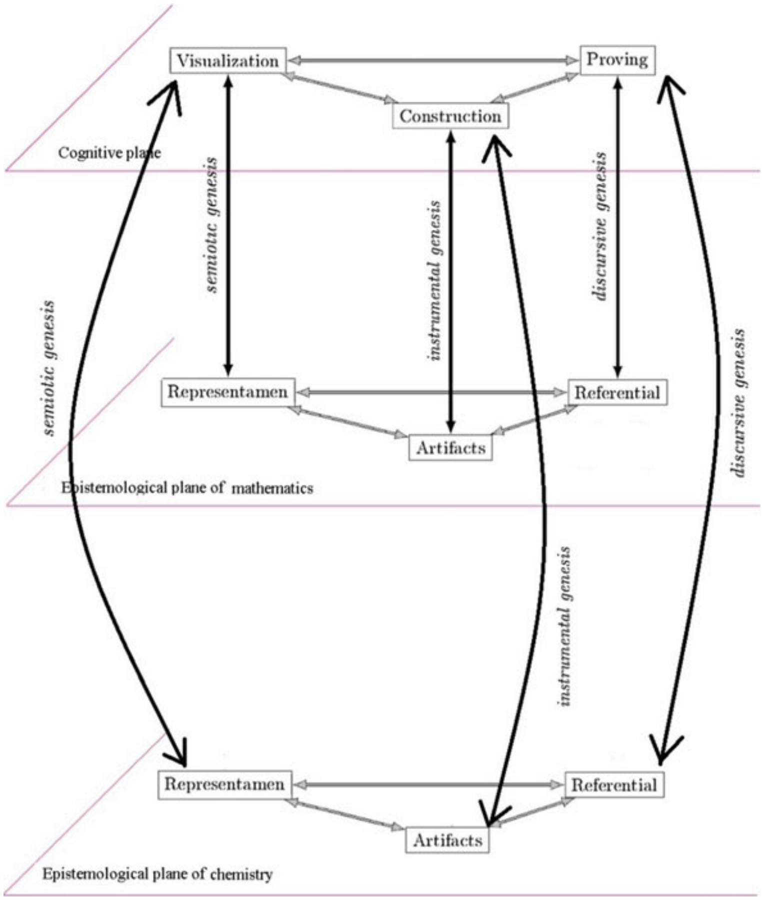

Fig. 53.3 Extended MWS model

reaction during this titration. The *real model* would be associated with understanding the chemical equation of the titration, the chemical species that react and the relevant parameters to be considered. This is the mathematical step usually involving an algebraic register that will be modified with a partially geometric register. It is suggested that the task to be performed by students to find the unknown concentration could be facilitated at the beginning of the learning process using an instrumental genesis with a dynamic geometry software such as GeoGebra.

Students should check the mass percentage of iodinated polyvidone in a newly manufactured bottle of betadine, a common antiseptic. They must perform a determination with a sodium thiosulfate solution and use GeoGebra to automate mass titter calculation when the equivalent volume of sodium thiosulphate solution added fluctuates. An aqueous solution containing a diluted solution of betadine is placed in an Erlenmeyer flask, and a solution containing sodium thiosulfate is placed in a graduated burette. Calculations are performed in relation with the experiment to determine the mass percentage of iodinated polyvidone in the betadine bottle using the measurement of the equivalent volume of thiosulfate added and considering the dilution factor. The task given to the students is described as follows:

You are part of a control laboratory in a pharmaceutical manufacturing plant and you must check the mass percentage of iodinated polyvidone in a newly manufactured betadine bottle. To do this, you must make a titration with a sodium thiosulphate solution 2 Na^+, $S_2O_3{}^{2-}$. You must use software to automate the determination of this mass titter.

1. Represent the quantities of matter of the reactants that have reacted at any time as a function of a reaction quantity. Use GeoGebra to represent the corresponding graphics.
2. Using GeoGebra's "cursor" tool, locate the quantity of sodium thiosulfate solution matter introduced at equivalence and then display on the software the molar concentration in diiodine of the diluted betadine extract.
3. Display on the software the mass titter in iodinated polyvidone of the analysed betadine, conclude.

Two additional documents are given. They provide information on betadine, iodinated polyvidone and reminders on titrations and equivalence. The chemical reaction equation of the titration is also provided. There is a simplification of the parameters relevant for understanding the phenomenon. The equation $y = 2.x$ is associated with the quantity of thiosulfate ion matter $S_2O_3{}^{2-}$ and the equation $y = x$ is associated with the quantity of diiodine matter I_2. Cursors are used to identify on the graph using a horizontal line, the quantity of thiosulfate ion matter added to the equivalence and then to infer the unknown concentration of diiodine. The construction of graphs allows to work on the notion of stoichiometry. The link to download the GeoGebra file is in the Appendix to this chapter (see Link 1).

53.2 Methodological and Theoretical Frameworks Used

A class sequence was designed using the methodological principles of didactic engineering described by Artigue and Perrin-Glorian (1991). The preliminary analyses consist in performing an epistemological study of the different concepts that will be addressed in the teaching sequence and reviewing the difficulties of students that may be listed in the literature. A priori analysis and conception consist in developing a teaching sequence (with eventually one or more pilot sessions) and analysing the different tasks that students will have to perform using an appropriate theoretical framework. The experimentation phase makes it possible to describe the conditions for data collection (audio recording, videos, interviews or analysis of paper and pencil activities, etc.) as well as the context of the study (number of students, grade level, type of school, etc.). Here the teacher is also the researcher involved in the study. The researcher-practitioner approach with practice-based evidence is applied in this study (Fichtman Dana 2016). Research using classroom experiments generally uses a comparative approach based on a statistical comparison of the results of experimental and reference groups. Didactic engineering, on the other hand, is at the level of case studies using a comparison between a priori and a posteriori analysis. The tasks performed by the students to solve the given problem are analysed during a posteriori analysis and validation stages. Comparison with the a priori analysis helps validate or invalidate the hypotheses developed when the research questions were established.

The extended MWS theoretical frameworks (Moutet 2018, 2019a, b) were used to carry out the a priori analyses. It makes it possible to specifically analyse interactions by considering the cognitive aspect and epistemological aspects in chemistry or mathematics. A problem in chemistry is studied (Moutet 2019a; Gauchon 2008). The data collections consist of written questionnaires or GeoGebra files. The modelling cycle proposed by Blum and Leiss (2005) is used to position the teaching sequence. The *real model* can be considered here as an *idealised model*. The sequence dealing with the chemistry of the solutions can be described by the steps: *real situation* → *real results*.

Two research questions guided this work: (1) How does the extended MWS framework allow the analysis of the sets of rationality frameworks between mathematics and chemistry, during a sequence with students in the final year of secondary school via a geometric approach? (2) To what extent does the analysis of the use of dynamic geometry software by the extended MWS framework show that it promotes a conceptualisation in students?

53.3 Presentation of the Extended MWS Theoretical Framework

The mathematical working space (MWS) was first developed by Kuzniak et al. (2016) to analyse mathematical work involved in teaching sequences. The MWS diagram was transformed by Moutet (2018, 2019a, b) by adding an epistemological plane corresponding to the rationality framework of physics or chemistry (Moutet 2019a) and this theoretical framework was compared with the Anthropological Theory of the Didactics (Moutet 2018). The extended MWS used in this chapter has three levels: one of a cognitive nature in relation to the student and two others of an epistemological nature in relation to the mathematical content studied and that involving chemistry (see Fig. 53.3). The cognitive plane contains a visualisation process (representation of space), a construction process (function of the tools used) and a discursive process (justification or reasoning). The epistemological plane contains a set of representations (signs used), a set of artefacts (instruments or software) and a theoretical reference set (definitions and properties). The placement of the three planes is not important here. Only the interactions between each epistemological level and the cognitive level are examined. Interactions within a plane as well as interactions between two epistemological planes are not used.

The separation between the epistemological plane of mathematics and that of chemistry depends on the task studied and the level of knowledge associated with it. At elementary levels, a single epistemological plan involving concepts in chemistry or mathematics will be enough to describe a school task. At more elaborate levels, one epistemological plane of chemistry and another of mathematics may be pertinent when the representamen, artefacts or theoretical referential are significantly different. The student problem described in this chapter can be analysed here by an extended MWS with three planes because the tasks performed lead to two sufficiently different epistemological planes. The theoretical referential associated with the epistemological plane of mathematics studied concerns Euclidean geometry or algebra. The tasks to be accomplished are associated with the construction of lines, segments, projections passing through a point and parallel to an axis, manipulation of quantities and use of simple relations. The theoretical referential associated with the epistemological plane of chemistry is associated with the chemistry of solutions, the notion of stoichiometry, the quantity of matter as well as molar and mass concentration.

It was chosen to keep only one cognitive plane because mathematical or chemical work are analysed by describing the articulations between the cognitive plane and the two epistemological planes. There are thus specific geneses between each epistemological plane and the cognitive plane. They are represented by double vertical arrows on the extended MWS model (see Fig. 53.3). Three geneses can be described: an instrumental genesis (operationalisation of artefacts), a semiotic genesis (based on the register of semiotic representations) and a discursive genesis (presentation of mathematical or chemical reasoning). It is possible to associate several geneses by following the work of Kuzniak et al. (2016). The different phases of the mathematical or chemical works associated with a task can be highlighted using vertical planes on the extended MWS diagram. The semiotic-instrumental interactions lead to a process of discovery and exploration of a given academic problem. Those of the instrumental-discursive type lead to reasoning based on experimental evidence. Finally, those of the semiotic-discursive type are characteristic of more elaborate reasoning.

53.4 A Priori and a Posteriori Analysis of the Tasks

The epistemological plane of chemistry and the cognitive plane are mobilised during the beginning of the resolution of the first question with semiotic-discursive interactions because students should find that they need to use the reaction progress $x(t)$. The students should then construct with GeoGebra the lines $x(t)$ and $2.x(t)$ corresponding to the quantities of matter of I_2 and $S_2O_3{}^{2-}$. The construction of these two lines requires semiotic-instrumental interactions between the epistemological plane of mathematics and the cognitive plane. The meaning of these two lines requires semiotic-discursive interactions between the epistemological plane of chemistry and

Table 53.1 A priori analysis of the different questions

Question-task	Plane[a]	Genesis[b]
1–1	Chem-Cog	Sem-Dis
1–2	Chem-Cog	Sem-Dis
1–2	Math-Cog	Sem-Inst
2–1	Chem-Cog	Sem-Dis
2–1	Math-Cog	Sem-Inst
2–2	Chem-Cog	Sem-Dis
2–2	Math-Cog	Sem-Inst-Dis
3	Chem-Cog	Sem-Dis

Note [a]Chem = Chemistry; Math = Mathematics; Cog = Cognitive; [b]Sem = Semiotic; Inst = Instrumental; Dis = Discursive

the cognitive plane. The mobilisation of the different epistemological and cognitive planes as well as the different genesis between the planes are summarised each time (see Table 53.1, 1–1 and 1–2).

In the beginning of second question, students should construct a cursor to locate the amount of $S_2O_3^{2-}$ matter introduced using a horizontal segment cutting the line $y = 2.x(t)$ at a point A for example. Cursor and segment constructions are analysed by semiotic-instrumental interactions between the epistemological plane of mathematics and the cognitive plane. The need to construct segments intersecting the line $y = 2.x(t)$ and the determination of the quantities of $S_2O_3^{2-}$ matter are analysed by semiotic-discursive interactions between the epistemological plane of chemistry and the cognitive plane. At the end of the second question, the students should construct a vertical segment starting from point A and crossing the line $y = x(t)$ at a point B, for example. A last horizontal segment starting from point B allows to find graphically the quantities of matter in I_2. These different constructions can be analysed by semiotic-instrumental and discursive interactions between the epistemological plane of mathematics and the cognitive plane. A reasoning is necessary to realise the geometrical constructions. Finally, the concentration of the diluted diiodine solution is calculated from the graphical determination of the quantities of matter found. Semiotic-discursive type interactions are mobilised between the epistemological plane of chemistry and the cognitive plane because students should find the concentration from the graphical method used (see Table 53.1, 2–1 and 2–2).

In the final question, students should find the mass of polyvidone-iodine in a 100 mL bottle of betadine to find and display the mass percentage of polyvidone-iodine. The epistemological plane of chemistry and the cognitive plane are mobilised with interactions of a semiotic-discursive type (see Table 53.1, 3).

A posteriori analysis of a group of two students was carried out using the extended MWS theoretical framework (see Table 53.2). The GeoGebra file rendered shows that the cursor is available and that the graphic construction is correct. This is analysed using the extended MWS theoretical framework with tasks involving the epistemological plane of mathematics and the cognitive plane that are correctly

Table 53.2 A posteriori analysis of the tasks performed by a group of students

Question-task	Plane[a]	Genesis[b]	Performed task
1–1	Chem-Cog	Sem-Dis	Correct
1–2	Chem-Cog	Sem-Dis	Correct
1–2	Math-Cog	Sem-Inst	Correct
2–1	Chem-Cog	Sem-Dis	Partially correct
2–1	Math-Cog	Sem-Inst	Correct
2–2	Chem-Cog	Sem-Dis	Incorrect
2–2	Math-Cog	Sem-Inst-Dis	Correct
3	Chem-Cog	Sem-Dis	Incorrect

Note [a]Chem = Chemistry; Math = Mathematics; Cog = Cognitive; [b]Sem = Semiotic; Inst = Instrumental; Dis = Discursive

performed. They are essentially of a semiotic-instrumental type. The whole geometrical construction requires a slightly more elaborate reasoning characterised by an interaction of the discursive type. Students use the reaction progress $x(t)$ and they construct with GeoGebra the lines $x(t)$ and $2.x(t)$ corresponding to the quantities of I_2 and $S_2O_3^{2-}$ matters. This shows that semiotic-discursive interactions are mobilised between the epistemological plane of chemistry and the cognitive plane. The quantities of $S_2O_3^{2-}$ matter are incorrect because there is a conversion error in the volume (mL–L) but it is well represented graphically. Errors are present in the determination of the diiodine concentration of the dilute solution because they find incorrect quantities of I_2 matter and they use a wrong volume for concentration determination. Errors are also present in the determination of the iodinated polyvidone mass titter because they did not consider the dilution of the betadine solution or the presence of iodinated polyvidone in betadine. This is the reason why the semiotic-discursive interactions mobilised between the epistemological plane of chemistry and the cognitive plane are either partially or incorrectly realised. The link to download the GeoGebra file proposed by the two students is available in the Appendix to this chapter (see Link 2).

The problem given to the students allows the stoichiometric relationship between the chemical reagents to be treated geometrically, but the overall resolution of the chemical problem here is relatively disappointing.

53.5 Conclusions

The theoretical framework of the extended MWS allowed for a more detailed analysis and evaluation of the types of tasks associated with certain stages of the modelling cycle for a problem involving the chemistry of aqueous solutions. It considers the mobilisation of the epistemological planes of mathematics and/or chemistry for each of the required tasks. It also shows that GeoGebra develops specific genesis in relation to a paper–pencil activity. An additional semiotic genesis is highlighted by the dynamic character of the software. An additional instrumental genesis corresponds to the manipulation of the dynamic geometry software with the cursor functionality allowing to simply change the experimental conditions. Finally, an additional discursive genesis allows to conclude here on the relation between the quantities of the reagents matter under stoichiometric conditions. The theoretical framework of the extended MWS permits the study of the multidisciplinary aspect of the different tasks that students must perform when working on a problem-solving exercise. Preliminary results tend to show that the genesis and epistemological planes of mathematics and chemistry are not mobilised in the same way according to the stage of the modelling cycle. This type of analysis will be used in future studies to develop assessments or problem-solving training for beginning teachers.

Appendix

The GeoGebra files described in this chapter can be downloaded using the following links:

1. https://drive.google.com/file/d/1DmeAGKjXWUHeWFJ3Gn17mS5QmSu2Ry0F/view?usp=sharing
2. https://drive.google.com/open?id=1I8CQux45Af9brSRBhn_cWH6uUauyY9hB

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Chapter 54
An Analytical Scheme to Characterise the Mathematical Discourse of Biology Tasks

Floridona Tetaj

Abstract The chapter describes an analytical scheme designed for investigating the mathematical discourse of biology tasks. The scheme was developed in the context of analysing tasks that are part of a fisheries management graduate-level course at a Norwegian university. Grounded in the commognitive perspective, the scheme focuses on the following aspects of the tasks: the mathematical content, its relation to biology discourse and students' expected engagement with both discourses. To illustrate the potential of analysis, I present and justify the choice of the categories included in the scheme, exemplify its use on one specific task and discuss some of the limitations of this approach to task analysis.

Keywords Task analysis · Mathematical models · Commognitive perspective · Mathematical discourse

54.1 Introduction

Mathematical models (MMs) play an important role in the field of natural resources' management. Particularly, starting from the twentieth century, modern fisheries management has been heavily dependent on mathematical methods which help to understand how fish populations respond to exploitation and regulate the way fishermen harvest fish population (Allen 1975). Despite the controversies and scepticism sometimes expressed among fisheries biologists concerning the accuracy with which the existing MMs describe and predict the real fisheries world (Kolding and van Zwieten 2011), these models are taught in various fisheries management university courses. This chapter which is part of a larger PhD project aimed at investigating how biology students engage with MMs centres around a Norwegian graduate-level course called Ecosystems and Fisheries Assessment Models (EFAM) that introduces various MMs which describe the dynamics of fish population. Specifically, I focus

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on developing a scheme for characterising the mathematical discourse of biology tasks which make use of MMs.

According to Lesh and Lehrer (2003), a MM is understood as the mathematical description of situations of particular system of practice, which has a purpose, an underlying conceptual system and a medium in which the model is expressed. Stillman (2019) separates MMs into descriptive models, describing the reality as it is required for the problem; and normative models, defining a part of the reality. Meanwhile, Smith et al. (1997) categorise MMs in the science of biology. First, they distinguish between theoretical models, developing the biological understanding of the underlying processes, and descriptive models, describing the phenomena as accurately as possible. A second distinction is between ad hoc models, created to fit the data, and first principles models, fitting known mathematical relationships to a biological setting.

MMs introduced in university non-mathematics graduate courses are often complex and sophisticated. These models make use of mathematical concepts that often are not dealt with in the mathematics courses that non-mathematics students take as part of their university studies. Therefore, it could be expected that these MMs might be introduced briefly, without professors going very deeply into their justification and mathematical details. The focus might be on the applications and the limitations of the models rather than on explaining how they were developed. In particular, this is the case in EFAM course. Although some insights into the assumptions of the MMs are provided by the professor, little attention is given to the mathematisation of the models. Students are mostly assessed on how well they can apply these MMs and analyse the obtained results using statistical methods.

According to Vos (2013), when students are assessed on tasks that make use of an already constructed MM and do not engage with the modelling cycle (Blum and Leiss 2006), they skip important elements of the modelling process such as the structuring, simplifying and mathematising. She argues that in such a case “students are not owners of the model and they cannot demonstrate competencies such as simplifying or structuring” (p. 485). On the other hand, mathematics education researchers widely acknowledge the idea that tasks which students engage with can shape their learning opportunities and their experience with the mathematics (Johnson et al. 2017). According to Rezat and Strässer (2012), a mathematical task can also act as a tool in facilitating students’ learning of mathematics. Therefore, analysing the aims of the tasks prior to analysing students’ work is of great importance particularly when the researcher did not participate in their design.

Additionally, one of the principles of the commognitive perspective in which this study is grounded claims that the discourse is situated in a particular context and prior to trying to make sense of the discourse that students engage with, the researcher needs to understand the context of the discourse, that is, where, when and how the discourse is initiated. Various frameworks have been introduced with the purpose of classifying and analysing mathematical tasks (Palm et al. 2011). However, little research has emerged on how to address the nature and the role that mathematics plays in tasks designed for graduate biology (or indeed other non-mathematics) students. Hitherto, the commognitive perspective has not been widely used for analysing tasks.

Morgan and Sfard (2016) developed a discursive framework for tracing changes over time in high-stakes mathematics examinations, building on social semiotics (Halliday 1978) and commognition (Sfard 2008). This framework analyses the mathematical discourse of tasks in order to examine their mathematical content, level of difficulty, guidance and support, the complexity of language, the use of diagrams and the non-mathematical context in the tasks.

Following Morgan and Sfard's example, Alshwaikh and Morgan (2013, 2018) developed an analytical framework for examining the nature of mathematics and mathematical activity in Palestinian and English textbooks. In their work, Alshwaikh and Morgan focus on two main questions: "What is mathematics and what kind of activity are construed as mathematical in school mathematics?" and "To what extent is specialised mathematical language used?" (ibid. 2018, p. 1044). These two questions are then operationalised according to the properties of the mathematical discourse by analysing the nature of verbal (written or spoken) text, particularly addressing the multimodal nature of the textbooks.

Consistent with these two studies, and particularly with the work of Morgan and Sfard (2016), here, a similar analytical scheme is presented with the purpose of exploring the mathematical discourse of biology tasks. The scheme is built by making use of the properties of discourse in order to characterise the mathematical content of the tasks, investigate how MMs interact with biology discourse, and how students are expected to engage with both discourses. In what follows, I present some aspects of the commognitive perspective (Sfard 2008), elaborate the setting where the tasks were used, introduce the analytical framework and lastly, discuss some initial results and the limitations of this approach.

54.2 Theoretical Framework

In the commognitive perspective (Sfard 2008), the (mathematical) knowledge is conceived through a community's established modes of communication, called discourses. Every discourse is distinguished by four characteristic features: words and their use, visual mediators, narratives and routines. A discourse has specific vocabulary or "common" words that are used in a specific way in this discourse, i.e. catch, fishing mortality, recruitment. Visual mediators are visible objects and artefacts that one operates upon while engaging in the discourse, for example, a graph representing the relation between two variables such as catch (C) and effort (f). The term narrative refers to descriptions of objects, relations between objects or processes with objects that are subject to endorsement or rejection by a discourse community. For instance, an endorsed narrative in fisheries management discourse is the following: "fishing mortality" refers to the removal of fish from the stock due to fishing activities. The term object/s refers to "special discursive constructs created by means of metaphorical projection from discourses on physical reality" (Morgan and Sfard 2016. p. 101).

Lastly, routines are defined using the notions of task situation, task and procedure (Lavie et al. 2018). First, a task situation is understood as a setting in which "a person considers herself bound to act—to do something" (ibid., p. 7). Second, a task is "the set of all the characteristics of the precedent events that she (the person) considers as requiring replication" (ibid., p. 9). The task refers to a person's interpretation of a task situation, and by precedent event is meant all that happened in the precedent task situation. Third, a procedure is "the prescription for action that fits both the present performance and those on which it was modelled" (ibid., p. 9). Building on this, a routine is the task, as seen by the performer, together with the procedure the person executes to perform the task in response to a given task situation. For example, a routine for finding the value of fish survived in the next year (N_{i+1}) when the number of fish in the current year (N_i) and total mortality (Z) are known can be applying the procedure $N_{i+1} = N_i \cdot e^{(F_i+M_i)}$. Depending on their aims, mathematical routines can be separated into explorations and rituals. If the performance of a routine is oriented towards the outcome then it is an exploration; if it is a process-oriented performance, then the routine is a ritual. While an explorative routine aims at producing a new "historical" fact about mathematical objects, a ritual is appreciated for its performance and not for its product. According to Sfard (2008), exploration routines are divided into three categories: construction (a performance resulting in a new endorsable narrative), substantiation (deciding whether to endorse previously constructed narratives) and recall (the process of citing a narrative that was endorsed in the past).

In addition to stories about mathematical objects, Sfard (2008) also calls attention to stories about people as an important factor that help the researcher to characterise the nature of the discourse. She claims that the mathematical discourse involves stories about the mathematical objects, which she refers to as the mathematising of the discourse—doing mathematics, and stories about humans and their actions, which she terms subjectifying—the performer of the mathematics. The aim of the analytical scheme presented in this chapter can now be more precisely stated: to analyse the mathematisation of the discourse at the level of vocabulary, visual mediators, narratives and their relation to fisheries narratives and routine use, and to investigate some aspects of subjectification such as the autonomy of students in engaging with the mathematical and fisheries narratives.

54.3 Setting and Methods

The EFAM course introduces the basic principles of modelling of natural fish populations to provide answers to management-related questions. Characteristics such as age, growth, natural and fishing mortality, maturation and recruitment of fish populations are discussed together with the MMs describing these processes. Models for estimating yield and abundance and their underlying assumptions are explained as well. The course was designed for first year graduate students, and it is a compulsory 10 ECTS credits study unit for students who want to pursue career in marine or fisheries biology. EFAM is a lecture-based course with twice-weekly meetings over one semester. During the lectures, new concepts are introduced and then, after each chapter's material is covered, students are asked to work individually or in groups on home assignments. They are allowed seven to ten days (depending on the volume of the assignment) to submit the solutions, which are then discussed by the professor in the classroom during the next lecture session.

For this intended analysis, I use a case study approach (Yin 2013) and the main source of data are the home assignments[1] (seven in total, and each composed of several tasks) with which the students enrolled in EFAM engaged during the spring semester 2019. Secondary sources of data for the task analysis are an interview with the professor of the course, who designed the assignments, together with video recordings of the lectures where he discusses their solutions. These data, together with observations and video recordings of two groups of students (six students in total) in which they engaged collaboratively in solving the assignments, are part of the larger data set collected for the PhD project. The assignments differ in several respects, for instance, the number of tasks posed, the particular content and the structure. The use of Excel or FiSAT (a software package developed to conduct various fisheries analyses) is an integrated part of the assignments, and all models and methods are described in such a way that they can be implemented in a spreadsheet.

Specifically, the assignments cover methods or procedures that predict and describe certain fisheries parameters, i.e. estimate the growth of fish population by using length frequency distributions; or estimate the mortality by using linearised length converted catch curves. Students are asked to engage with models such as: the so-called yield-per-recruits, Beverton–Holt model, Richer model, VPA method or biomass dynamic (also known as surplus production models). In order to engage with these MMs, first, students are asked to organise the data (to put them in columns' sheet and organise them according to age or length of fish in Excel or FiSAT). Then, they are asked to calculate various parameters by using MMs, mathematical methods such as linear regression, or by applying a given formula. After, students are asked to find the fitness of the models. In the cases, when students use two equivalent models, they are asked to compare which method or model fits the data best. Lastly, after students have analysed the data and found the fitness of MMs, they are asked to discuss the implications of their results and make certain fishery recommendations.

In order to illustrate the relevance of the analytical scheme and to help the reader better understand the character of the assignments, the scheme is applied on an assignment called the "VPA exercise". This assignment was chosen since it compresses various aspects of discourse that students are expected to engage with. VPA stands for *virtual population analysis* and is a method for estimating the population of fish (N) of a certain age (i). The idea behind this method is to analyse parameters that can be measured (the catch C) and estimated through the biological properties of the fish (the mortality M) in order to calculate the population that should be available in

[1]The "assignment" refers to the whole document provided by the professor to the students. An assignment may include various questions and a question may contain more than one task, in the sense of Lavie et al. (2018).

the water to produce the measured catch. Due to space limitation, the details of the model cannot be discussed here, but after some calculations (e.g. Sparre and Venema 1998), the catch equation, also known as the Baranov equation, is written as follows:

$$C_i = \left[1 - \frac{M_i}{\ln(N_i) - \ln(N_{i+1})}\right](N_i - N_{i+1}).$$

This is a nonlinear difference equation that does not have an exact solution for the variables N_i or N_{i+1}, thus iterative numerical methods (i.e. Newton–Raphson) are used to find approximations of N_i or N_{i+1}, when $C_i = f(N_i)$. The VPA exercise is distributed to students as a seven-page PDF document and it provides explanations on how Baranov equation can be converted and applied into a macro in Visual Basic. Students should mimic the procedure described in the document. After the number of fish for each group-age has been calculated, students are asked to calculate other parameters such as the total mortality ($Z_i = \ln(N_i) - \ln(N_{i+1})$) and fishing mortality ($F_i = Z_i - M_i$), and draw graphs representing their relationship. Then, students are asked to use the so-called Pope's approximation of the catch equation, which is a linear simplification of the Baranov equation, and find all the parameters again. Lastly, students are expected to compare the results of the two methods.

The intended data analysis is to be conducted in two stages. First, an overview of possible solution paths of the tasks is made, building on the video recordings of the lectures, the suggested literature and the (PowerPoint) slides used during lectures as well as the transcript of the interview with the professor of the course. Then, using the scheme elaborated below, the characteristics of the task discourse are categorised, with the aim of exposing relationships between MM and their use in fisheries biology, and how students are expected to engage with this relationship.

54.4 The Analytical Scheme

As mentioned earlier, the scheme introduced in this chapter is the result of an adaptation of the discursive framework of Morgan and Sfard (2016). In their framework, Morgan and Sfard specifically focus on two aspects of the discourse: mathematisation and subjectification. For each aspect, they develop guiding questions and answer them by finding corresponding textual indicators. The same strategy will be used here. However, there are two important differences between the context of this study and that of Morgan and Sfard. First, in their case, the tasks are already given in mathematical discourse, and students are expected to show that they are able to execute certain mathematical procedures. The tasks in this context, however, are strongly related to the fisheries discourse. Students are expected to put much effort into elaborating the non-mathematical descriptions of processes in mathematical discourse. Second, the aim of Morgan and Sfard's analysis was to compare and trace changes in the tasks over time; meanwhile, although the evolution of the students' expected engagement with the task during the semester is one of the aims of this analysis, it is not the main focus. The analysis in my case is of a descriptive nature and is intended to be used for further effort to analyse and assess students' performance on the tasks. Thus, considering these two differences, some of Morgan and Sfard's categories are excluded and some others are extended as will be visible in the rest of this section.

In order to investigate the mathematisation of the tasks, as suggested by Morgan and Sfard, the focus is on three characteristics of the discourse: word use, visual mediators and mathematical routines. Since the task discourse in the EFAM assignments is understood as a synthesis of mathematical and fisheries discourse, the scheme aims to understand the nature of the relation of these two discourses. To this end, I draw on the work of Paxson (1996), discussing the modes of interaction between two disciplines. He asserts that when two disciplines interact with each other by exchanging ideas, techniques or perspectives, the nature and the depth of this interaction influences the outcome in terms of scientific progress. In commognitive terms, an interaction between two disciplines happens when there is an exchange or combination of specific characteristics of these discourses, for example, when one borrows the vocabulary or routines of one of the discourses and applies them in another discourse. Paxson classifies four levels of interaction ranging from low-level to high-level impact between two or more disciplines. In this work, his classification is included in order to help us characterise the types of narratives that the tasks contain. When the task discourse only "takes notice" of the mathematical discourse but there is no engagement with it, then this interaction is categorised as first level. Second-level interaction denotes situations where the endorsed mathematical narratives are used to impact or change the narratives of the task discourse. The last two categories involve more advanced interactions and they occur when a phenomenon is analysed using the endorsed narratives of each discourse and this leads to a growth in both discourses, respectively, or when fundamental endorsed narratives of one discourse are used to change the endorsed narratives of the other discourse. These two interactions usually involve the communities of experts of each discourse; thus, they should not be expected to be present in the kind of tasks that the scheme aims to analyse. Thus, while Morgan and Sfard's framework sheds light on what kind of mathematics emerges in the fisheries discourse, Paxson's classification helps to investigate how the mathematical and fisheries discourses are interacting. In what follows, the categories of the mathematisation and subjectification of the discourse, respectively, are presented and discussed.

The main aim of investigating the mathematisation of the tasks is to understand what kind of stories about mathematical objects are present in the task discourse, and how these stories are brought into the fisheries discourse. The question guiding the analysis is: What kind of mathematical language is being used in the task? This question is answered by finding two types of textual indicators. First, lexical items that are used in accordance with mathematical definitions considered at the level of vocabulary, sentence or text unit are found. Second, other lexical items that may invite mathematical actions or refer to mathematical narratives are observed. In VPA exercise, items such as: "analytical solutions", "average", "function" or "graph" are examples of words of the first type. On the other hand, items such as: "fishing pattern"—which refers to the graph representing the relation between the age and the number of fish, or “the survivors and the natural deaths in number”—which mean the numerical values of these two variables, are examples of wordings that refer to mathematical concepts.

Concerning visual mediators, the guiding question is: What kind of mathematical mediators does the task make use of? Here, a distinction is made between mediators such as tables, diagrams or algebraic notations and the spreadsheet as a mediator in its own right. In the VPA exercise, the presence of tables or graphs demonstrating the relation between variables seems to be essential. They are the initiators of the engagement with the fisheries data collected in the field. It can be also noted that the assignment makes use of different modes, i.e. from an algebraic version of Baranov catch equation to a macro, or from a table to a graph in a spreadsheet. Thus, the second question included in this category is: What transitions need to be made between different mediators? This is answered by looking at the presence of or demand for two or more equivalent mediators. For example, the change in the number of fish calculated with the help of a table in the spreadsheet is expected to be presented in a graph. Making evident the vocabulary and the transitions between different mediators will be helpful for the future analysis of students’ work, particularly when trying to understand how students engage with mathematical concepts and cope with transitions from one mode to another.

Mathematical routines are the other characteristic included in the scheme. Since every routine is a pair of a task and procedure, the focus of the analysis is on identifying the task and then finding an associated procedure that may be applied for solving the task. For this purpose, I make use of video recordings of the professor (who is the task setter) where he discusses his expected solutions to the assignments. Thus, the question with respect to the nature of the task is: What is the task and what are the characteristics of the procedures? To answer this question, a differentiation between algorithmic procedures and heuristic procedures is made. In the first six pages of the VPA exercise, all the tasks are given as step-by-step procedures which suggest that they have an algorithmic nature. Meanwhile, the last part of the assignment requires students to “compare the two methods” which might be interpreted as an invitation to compare the results that one obtains using Newton–Raphson approximation and Pope’s approximation graphically or to explore the mathematical differences; thus, leaving the interpreter to believe in the heuristic nature of the task. It is worth mentioning that an initial analysis of the assignments suggests that the tasks differ quite a lot from each other with respect to the procedures required. The assignments towards the start of the semester are more algorithmic in nature and the expected involvement with mathematics is more ritualised, while the rest of the assignments are more explorative. Thus, analysing these differences and how the assignments evolve during the semester will help understand how students are expected to engage and learn MMs.

After having identified the routines that students are expected to engage with, the concept of endorsed narratives is used to investigate the interaction between the mathematical and the fisheries discourses considering MMs as contributors that help to make sense of the fisheries discourse. The question is posed: “how does mathematical discourse interact with fisheries discourse?” which is answered and operationalised using the classification of Paxson (1996). In the VPA exercise, there is a dynamic interaction between discourses. For example, the engagement with the Baranov equation could be classified as first level interaction. In the assignment, it is explained that the Baranov equation is a MM of the catch and it is "transcendental"—meaning that it does not have analytical solutions. However, there is no explicit request for engagement with any of the mathematics behind the model. Meanwhile, students use mathematical narratives such as the formula of fishing mortality ($F_i = Z_i - M_i$), and depending on their numerical values, students are asked to discuss if there is overfishing or not.

Lastly, considering that the nature of the assignments was quite often very algorithmic and other times would be more heuristically oriented, in the scheme a subjectifying aspect of the tasks, such as, students' autonomy is included. The following question is asked: "what decisions are students expected to make, and what decisions are already present in the task with respect to interpreting the task and to design the path to follow?" To answer it, the complexity of mathematical procedures and narratives that students are expected to produce and whether the tasks are indicated explicitly are considered. In the VPA assignment, most of the decisions are already given in the tasks and students are only asked to copy certain procedures. However, in latter assignments, the task interpretations are more complex. This will help to understand the decisions that students are expected to make with respect to their engagement with MMs.

54.5 Final Remarks

The scheme presented in this chapter was developed from a need for an analytical tool that would allow the researcher to better understand the nature of the engagement that biology tasks that make use of MMs demand from graduate students. The scheme makes use of the characteristics of discourse and focuses on two main aspects. First, it elaborates the nature of the mathematical content and its interaction with the biology discourse. Second, it explores the ways that students are expected to engage with the assignments. This choice was made having in mind the hypothesis that with respect to the task discourse, these two concerns would be the main challenges that biology students would face while engaging with the tasks. This said, I do not exclude the possibility that during students' work, other factors that influence students' engagement with the assignments may emerge. A pitfall of this kind of analysis is that it can be time consuming since it requires a deep engagement with the content of the tasks. It should also be recognised that this approach to task analysis is interpretative and dependent on the researcher's understanding of the context, yet I claim that when the researcher is not part of the design of the task and the task setter is the professor, such detailed analyses can only be helpful towards understanding students' engagement with the tasks.

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Chapter 55
Mandatory Mathematical Modelling in High School: How to Implement Within and Outside the Classroom to Meet Students' Diverse Needs

Xiaming Chen

Abstract This chapter focuses on studying the diverse demands of mathematical modelling in secondary schools and strategies to meet those demands under the new curriculum standard. This study analyses students' learning needs and the horizontal distribution of multidimensional modelling in compulsory conditions. Then, it proposes a corresponding 5-direction modelling courses. It offers a curriculum structure in mathematical modelling suitable for different levels of need, featuring a core 'three-person study community' and an 'echelon system' for teaching mathematical modelling at higher levels. A learning method of 'mathematical modelling by item' has been developed, and satisfactory results have been achieved. A system to evaluate mathematical modelling has been established and applied to daily teaching and the Shanghai Joint Secondary School Mathematical Modelling Activity.

Keywords Diverse needs · Mathematical modelling · Echelon system · Operating mechanism · Core trio · Curriculum evaluation system

55.1 Introduction

The National Standards of Mathematics Curriculum for Regular High School (2017 Edition) (Ministry of Education, 2018) in China incorporates mathematical modelling as a compulsory element in the high school mathematics curriculum. This requirement influences the entire mathematical modelling curriculum since the target group for instruction has changed from a small group to all students, from students who show keen interest and better performance in mathematics to students at all levels.

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Research on teaching mathematical modelling has focused mainly on developing processes for mathematical modelling (Blum & Leiss, 2007), designing and conducting empirical research on cases for teaching mathematical modelling (Blum, 2011; Blum & Ferri, 2009; Blum & Leiss, 2007), designing ways to assess mathematical modelling (Frejd, 2013) and collecting specific cases and evaluation strategies (Garfunkel & Montgomery, 2016). Ferri (2017) pointed out that an educator needs abilities in mathematical modelling in four dimensions: theory, task, pedagogy and diagnosis. Ferri then used case studies to explain these dimensions further. That work established a theoretical basis for developing a system of mathematical modelling education for students of all levels, allowing us to focus on more prominent problems.

Thus, I want to focus on three problems:

(1) How to construct a mathematical modelling curriculum within the mandatory high school requirements.
(2) How to construct an effective framework for mathematical modelling course implementation.
(3) How to build a diversified mathematical modelling course that crosses campus boundaries.

55.2 Theoretical Framework

55.2.1 Classification of Different Mathematical Modelling Needs

Mathematical modelling is a graduation requirement enforced by the Chinese Regular High School Mathematics Curriculum Standard (Ministry of Education, 2018). It is also embodied in the heart of various subjects, including physics, chemistry, biology and computer science. Since students have different career plans, we believe that different students reveal different needs in their mathematical modelling education. Those who plan to pursue a career in history or social studies need a less-detailed understanding of the application of models compared to their peers. Their need is to form basic cognitive competencies in mathematical modelling (dimension 1). Those who plan to study business-related subjects must master mathematical modelling principles and simple applications (dimension 2), since business and management fields require the applications of different models to real-world problems. Those who plan to take on a STEM career must understand the core ideas of mathematical modelling (dimension 3), since STEM courses have stronger prerequisites on the ability to derive and apply models. Those who plan to conduct actual scientific research and/or mathematical modelling competitions would need to be able to apply mathematical models to solve complex problems (dimension 4) because of the nature of such activities.

This classification of student needs is certainly not absolute. In many circumstances, because of intrinsic interests, students may want to challenge themselves to take courses in higher levels of mathematical modelling. Maaβ (2006) pointed out that mathematical modelling ability is positively correlated with mathematics ability. Since it is clear that there are different levels of mathematical ability among students, it is not appropriate to require students to exceed their own capabilities and zone of proximal development when laying down course requirements for mathematical modelling. Thus, it is feasible to use the four dimensions of student needs above as a basis for developing course requirements.

55.2.2 *Using the Division of Labour to Teach Mathematical Modelling*

According to Frejd (2016), mathematical modelling is a professional activity which requires an appropriate division of labour. Earlier, Blum (2007) established a process for mathematical modelling, dividing the activity into the following segments: understanding or constructing, simplifying or structuring, mathematising, working mathematically, interpreting and validating. This gives rise to a possible division labour among students. CCSSM (2010) introduced a segment of computer programming into Blum's process, which gave rise to the independent role of the programmer. In fact, because of the high complexity and big data-driven nature of real-life problems, programming has come to be an increasingly important role in solving actual problems. In the meantime, since CCSSM (2010) also defined an 'interpretation and presentation' segment in the process of mathematical modelling and since the primary way to present the results of mathematical modelling is by means of an academic paper, the role of a paper writer is inevitable. That was also illustrated in Blum's (2007) definition of the professional process. Not only is ensuring an appropriate division of labour in a team required for comprehensive modelling, but it also coincides with the opinion that atomic modelling can improve the effectiveness of solving mathematical modelling problems (Kaiser & Brand, 2015).

55.3 Methods

55.3.1 *Practical Frameworks and Operating Mechanism*

Given the diversity of students, we divide the course into five elements:

1. In-class mathematical modelling activities to familiarise students with the basic principles and processes of mathematical modelling by completing simple modelling tasks under the guidance of teachers.
2. Advanced mathematical modelling courses that enable students to grasp the full mathematical modelling skill set and accomplishing simple tasks independently through self-study and group discussion.

Table 55.1 Relationship of student needs to elements of a mathematical modelling curriculum

Need	Element				
	Element 1	Element 2	Element 3	Element 4	Element 5
Theory	☆				
Task	☆	☆	☆		
Pedagogy	☆	☆	☆	☆	
Diagnosis	☆	☆	☆		☆

3. Extracurricular clubs or societies for mathematical modelling, which mainly involves independent study and group discussion.
4. Research-oriented mathematical modelling studies, with practice in mathematical modelling competitions.
5. Projects commissioned by the school or social agencies.

The relationships between needs and elements are shown in Table 55.1.

A mathematical modelling education system for all students must integrate **basic mathematical modelling** (basic modelling mindsets, basic modelling methods for in-class applied problems and extended problems), **advanced mathematical modelling** (advanced modelling mindsets, comprehensive modelling methods and various modelling competitions with extended questions), **student independent development** (specialised unique activities with individualised results facilitate the course improvement in the school) and **cultivation of social responsibility** (realistic needs of the school and problems of society addressed with a variety of mathematical modelling approaches).

55.3.2 *The Basic Mathematical Modelling Phase*

In the basic mathematical modelling phase, taking regular applied questions as a starting point and the questions that the students can study in the textbook as the examples, we raise the standards of algorithm design and programming, so that students can fully understand how to produce an academic paper. Students are encouraged to experience the charm of mathematical modelling, to apprehend the ideas and methods of mathematical modelling and to improve the understanding and answering of applied questions in the class. This completes the first academic 'baptism'. The elements of the basic mathematical modelling phase are: a mathematical essay (accurately illustrate a problem, show reasonable analysis and process so that the model and its conclusions are meaningful), applied questions in the mathematics class (e.g. individual income tax problems, city taxi problems), junior high school mathematical modelling foundation courses (analysis of basic cases, trying to develop a program), formal report from a modelling club (modelling analysis of some phenomenon, e.g. programming simulation of dance moves) and a mathematics class assignment (complete the modelling of a small problem).

Cases for teaching mathematical modelling are designed using different assumptions and simplifications of practical problems. Four levels of cases can be generated, which are suitable for regular applied questions, small research topics, mathematical short papers and advanced mathematical modelling problems, respectively. Preparing advanced mathematical modelling cases can enhance teachers' understanding of basic mathematics modelling teaching cases and improve the teaching effectiveness of in-class applied questions.

55.3.3 The Advanced Mathematical Modelling Phase

In the advanced mathematical modelling phase, the *core trio study community* is constructed with the general courses and special projects as a setting and case teaching as the principal teaching method. This community includes '1 modeller, 1 programmer and 1 paper writer'.

(Based on experience, as the number of hours dedicated to studying and practising the skill set accumulates, every group member could undertake 2 or 3 roles at the same time.) Using the 'core trio', we developed an academic growth and self-management mechanism supported by a tutor system with accountable team leaders. This has been successful in developing student teams. The structure of the advanced mathematical modelling phase includes **comprehensive and specialised training** (case analysis and discussion activities, self-learning or learning with a senior member), **competitions and oral presentations** (paper writing, creating presentations and basic requirements of presentation), **topics and school missions** (e.g. analyse traffic problems or conduct a feasibility study of the school's strategy) and the **study community and tutor system** (with the 'core trio' as the study community, build a student tutor system and a leader responsibility system). This approach demands substantial student mathematical modelling expertise and cooperation between teammates.

55.3.4 Objectives of the Curriculum and Implementation Plans

The whole mathematical modelling education system has been reorganised to reflect the practical needs of teaching. It is divided into the following four parts: regular mathematical application questions with extensions, extended courses in mathematical modelling, mathematical modelling learning for students with special needs including relevant practices and mathematical modelling clubs.

Regular mathematical application questions with extensions. By selecting appropriate word problems in the textbook to adapt and expanding the problem according to the actual situation, students can understand the problems in relation to modelling and establishing assumptions. Students can connect applied questions and practical problems, and they can learn to use the modelling mindset to think about and solve some small problems around them and to complete a small paper.

Extended courses in mathematical modelling. Students learn how to formulate reasonable questions through case studies. They can gain a holistic understanding of the problems that must be solved. They can make reasonable assumptions about the problems, and they can adjust and correct them during the process of problem-solving. Students try to analyse real-world situations around them mathematically and report their analysis in a mathematical modelling paper.

Mathematical modelling learning for students of special needs including relevant practices. Through mathematical modelling, students solve practical problems such as the arrangement of a schedule to accommodate both the new college entrance examination and the school examination arrangement under the 3 + 3 mode. By participating in the International Mathematical Modelling Challenge (IMMC) finalist presentation competitions and other activities, they gain a deeper understanding of all aspects of the problem in preparing for the presentation, and they gain new insights into the ways and methods of problem solving.

Mathematical modelling clubs. By developing these clubs, activities can spread the seeds of modelling across the entire school. More junior high school students can be introduced to mathematical modelling and to learn what it is all about. Through reports from the club community, it is possible to use the power of the entire club to disseminate the knowledge of modelling beyond the school walls. The all-round growth of modellers is also better promoted with college students interacting with high school students and the seniors talking with the juniors.

55.4 Results

55.4.1 Construction and Implementation of Evaluation Systems

These systems have the following elements:

Paper presenting and reporting after modelling activities. After each modelling event, a paper-sharing session is routinely held. Each group of students has 20 min to introduce their thesis, analyse the major difficulties and ideas and give a framework solution and a detailed explanation of the paper. During the process, members of the audience can ask questions. This role facilitates the timely discussion and sharing of ideas of modelling and the directions and methods of analysing problems. In this way, the abilities of students to solve practical problems are improved in a substantial way.

Self-reflection rules after modelling activities. Reflections mainly involve discussing aspects of time management, basic discipline knowledge, teamwork, literature review and reading ability, communication and expression skills, basic modelling and specialised abilities. Students develop targeted reflections based on their own development and the problems exposed in the modelling process. Routine reflection features not only the combination of personal performance and team problems, which centres on personal faults, but also that of professional skills and cooperation. This is very effective for the improvement of the individual's academic level and the development of the team.

Reciprocal rating of team leaders and team members. The purpose of setting up the team leader's rating form and the participant's rating form is to enable the students in the group to and appreciate their teammates after the modelling activity. When team leaders evaluate team members, the strength of team members can be recognised, the problems and aspects that need improving can be pointed out and the creative contributions of team members can be awarded. When team members evaluate team leaders, they get an objective analysis of their performance, covering both the specialised work and the leadership skills of the team leader.

A 'cascade' system of specialised ability and evaluation. Developing students' mathematical modelling has an unusual feature—a 'cascade' of expertise from higher to lower grades. The core trio study community (see Fig. 55.1) is the basis for teaching modelling, and the leader responsibility system and student tutor system are established. The training of the new team members is handed over to an older, more experienced core trio. Each member is responsible for guiding new students. They

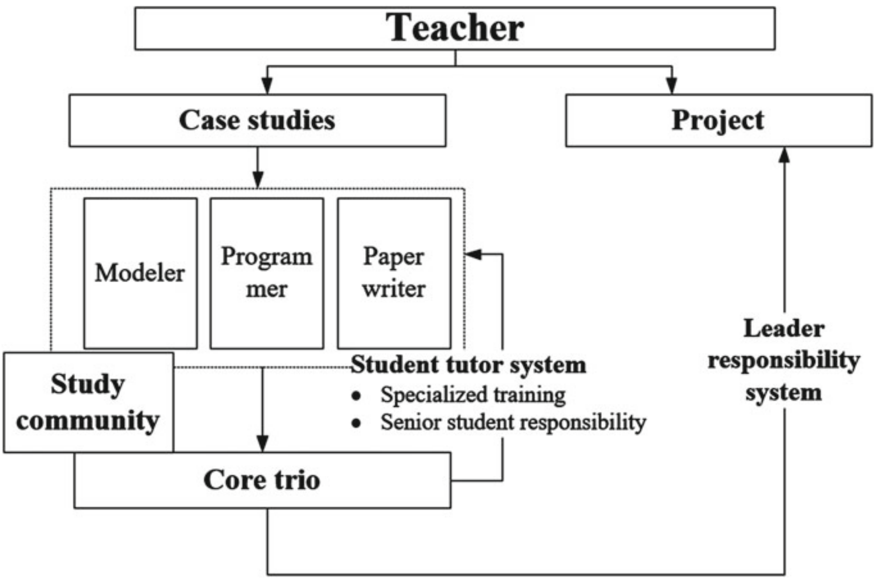

Fig. 55.1 Structure of mathematical modelling cultivation

are responsible for day-to-day assignments, progress checks, procedural assessments and interim reports. By establishing a cross-age study community and gradually cultivating the modelling expertise of lower-grade students, the leadership, cooperation and communication skills and academic guidance of senior students can be improved.

In Fig. 55.1, we introduce two protective systems. The first is the group leader system in the ‘core trio study community’. This system enables student leaders to be in full control of various aspects of their team, including time management, task allocation, construction of work environment and monitoring the implementation of projects. It also enables teachers to switch their role from being a director to that of an advisor and pivot their job from controlling students to supporting them. The second is the upperclassmen responsibility system for new member growth. The personal improvement process of mathematical modelling members is special. Since we base our mathematical modelling education on the ‘core trio study community’, experienced upperclassman communities undertake the responsibility for training new members. Each member of the trio is responsible for advising underclassmen in the same role, including assigning daily tasks, monitoring progress, evaluating procedures and preparing interim reports. The upperclassman assumes primary responsibility for whether their advisee(s) could be trained to become an acceptable member(s).

55.4.2 Evaluation of Mathematical Modelling Competence

After the success of a small-scale pilot project, we launched the Shanghai Joint Secondary School Mathematical Modelling Activity (SJMMA) based on IMMC. The aim was to build a more general Evaluation System of Regular High School Students’ Mathematical Modelling competence and thus drive the development of mathematical modelling activities in Shanghai. In this, we used a scoring table of five ranks; every paper was judged by at least two judges on a 5-point scale. The total rank distribution is shown in Fig. 55.2 for each aspect of performance.

We found that students scored low in ‘assumptions’ and ‘programming and model evaluation’. Using surveys, we conducted further analysis of the underlying reasons

Total Score Distribution							
score(High to low)	5	4	3	2	1	Mini	Average
Summary	21	33	46	43	37		2.77
Problem Analysis	21	37	43	51	28		2.84
Assumptions	5	36	49	46	44		2.51
Modeling Process	12	38	47	56	27		2.73
Programming & Model Evaluation	9	34	25	50	54		2.28
Academic Norms	17	25	39	66	36		2.61
Mini							

Fig. 55.2 Distribution of SJMMA2019 paper score

for the poor performance on those two indicators, and we discovered a critical factor that contributed to this phenomenon.

For 'assumptions', the main factor was that in regular math classes, the description of problems is definite; the lack of open-ended questions traps students in a closed system when solving problems. When students encounter problems that require independent analysis and making assumptions, they are less prepared. As a result, students lack understanding about the principles of making assumptions, so they designate some problem variables as constants, making the model less credible. For 'programming and model evaluations', the principal reasons were: (1) students rarely had opportunities to do hands-on programming projects outside of mathematical modelling, resulting in a lack of practice; (2) even when some computer science courses involved some level of programming, the problems were often artificially simplified so they were very different from real-life problems; (3) because of the complex nature of the models themselves, a high level of understanding and visualisation of models was required, and this level can be hard to reach given the students' current abilities.

55.5 Discussion

By analysing the resulting data of SJMMA, we saw that the mathematical modelling ability of most students was low, especially in making assumptions and computer programming. We made three recommendations for teachers. First, consider splitting students into groups of core trio communities during initial training, such that at least one member of each team can accomplish each task. This would ensure the successful completion of a modelling project. Second, increase the weight of problem analysis in teaching, such that each student can grasp the critical abilities required to make appropriate assumptions and analyse problems independently. This would help them translate real-life problems into mathematical problems, paving their way to success. Third, emphasise computer programming within the mathematical modelling activity and increase the proportion of computer program realisation in solving mathematical modelling problems. This would improve the students' abilities to design algorithms and implement programs.

55.6 Summary and Conclusion

By analysing the diverse needs for mathematical modelling under the new curriculum standard, the four-dimensional model of student needs for mathematical modelling is refined, and the five elements of the curriculum system of mathematical modelling are established. To reduce the barriers between normal mathematics curriculum and mathematical modelling activities, a teaching system for mathematical modelling covering all students is established. This links basic mathematical modelling and advanced mathematical modelling. By establishing the core trio study community and modelling cascade system for across-grade peer-to-peer tutoring, a mathematical modelling training programme is developed to improve students' mathematical modelling competence in an all-round way, which highlights a combination of general knowledge and specialised training.

By analysing data collected in school mathematical modelling teaching and SJMMA, along with index analysis, the real difficulties of students in mathematical modelling activities are identified. In the analysis, the factors underlying student difficulties and teaching strategies are clarified. In future research, we will track the relevant teaching further and use the information gained in actual situations to make timely adjustments.

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