One-Dimensional Axisymmetric Model of the Stress State of the Adhesive Joint

Kostiantyn Barakhov(✉), Daria Dvoretska, and Oleksandr Poliakov

National Aerospace University "Kharkiv Aviation Institute", 17 Chkalova Str., Kharkiv 61070, Ukraine
kpbarakhov@gmail.com, o.poliakov@khai.edu

Abstract. Local damage repair of modern aircraft structures can be done by creating patchs that are glued to the main structure. The patch takes on part of the load, unloading the damaged area. This method of repair provides tightness and aerodynamic efficiency of the structure. The stress state of such glued structures is calculated, as a rule, using the finite element method. Classical models of the lap joint stress state are one-dimensional. I.e. the change in the stress state is considered only along one coordinate. In this case, the joint is considered rectangular. The aim of this work is to create a mathematical model of the stress state of circular axisymmetric adhesive joints, and to build an appropriate analytical solution to the problem. It is assumed that there is no bending and the deformations of the plates are uniform in thickness. The adhesive layer only works on shear. Both plates – the main plate and the patch are assumed to be isotropic. The solution is built in the polar coordinate system, in which the stress state of the joint depends only on the radial coordinate, i.e. is one-dimensional. The solution is obtained in an analytical form. This mathematical model is a generalization of the Wolkersen classical adhesive joint model for a circular or annular domain and is considered for the first time. The model problem is solved. The calculation results are compared with the calculations performed using the finite element method.

Keywords: Adhesive joint · Axisymmetric model · Analytical solution · Circular plate

1 Introduction

Considerable attention is paid to the local repair problem of composite and metal structures in aerospace engineering [1–4]. In this case, as a rule, two types of crippling of the main panel or plate are considered – in the form of a circular hole or in the form of a crack. In the overwhelming number of works devoted to the study of the stress state of the plates and patchs joints, the calculations are performed using the finite element method [5–8]. There are analytical methods for the stress state calculation of patched plates with holes [9–11], where it is assumed that the patch is fixed with the main plate along the line, while in the real design the patch is often fixed over the lining area. A plate with a glued circular patch in the adhesive area can be considered as a

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three-layer plate with a soft joining layer. To build a solution, it is necessary to use models of three-layer plates, which have been used in the study of the adhesive joint stress state [12]. Well-known analytical solutions to the problem of the reinforcing patch and lap joints assume the rectangular geometry of the patch and plates to be joined, as well as a uniform distribution of stresses by the joint width [13–15]. There are also several different approximate two-dimensional models and corresponding methods for solving the problems of the adhesive joint stress state [16–22], where also the shape of the patch is assumed to be rectangular. Therefore, these approaches do not allow to obtain an analytical solution to the problem for a plate with a circular hole and a circular patch. It is obviously, that the axial symmetry of the structure under consideration requires the use of a polar coordinate system. This allows us to reduce the problem to a one-dimensional problem, as far as there is no dependence of stresses on the angular coordinate.

One-dimensional models of lap joints are used not only for glued beams calculation, but also for the axisymmetric coaxial tubes joint stress state calculation [23–25]. In this case, axial symmetry actually provides a uniform distribution of stresses along the circumferential coordinate. However, the stress state problem for the axisymmetric joint of a circular patch and plate is solved for the first time.

The aim of this work is to build an analytical solution and study the stress state of the lapped adhesive joint of a circular plate which has a circular hole with a coaxial circular patch. In the future, the proposed model can be developed for dynamic problems that arise, for example, during impacts on aerospace multilayer structures, including glazing [26] or attachment knots of honeycomb panels [27].

2 Formulation of the Problem

Consider the adhesive joint of two circular plates of the same thickness, shown in Fig. 1. The main plate is loaded with symmetrical biaxial tension. The radius of the hole in the main plate is R_1 patch radius is R_2. The main plate has a thickness δ_1 the patch has a thickness δ_2. The base and patch are made of isotropic materials, the elastic modulus of which, respectively E_1 and E_2, Poisson's ratio μ_1 and μ_2. The plates are joined using a joint layer, the thickness of which is δ_0, and the shear modulus is G_0.

Note that in practice, most often repairs are made using the same sheet-material the main structure is made, i.e. the thickness and properties of the patch coincide with the thickness and properties of the base. In addition, to reduce the bending of the structure, repairs can be performed using two identical patches glued to the base from two sides. In this case, the thickness of the patch is two times less than the thickness of the base, and due to the symmetry, the problem can be reduced to the one solved in this paper.

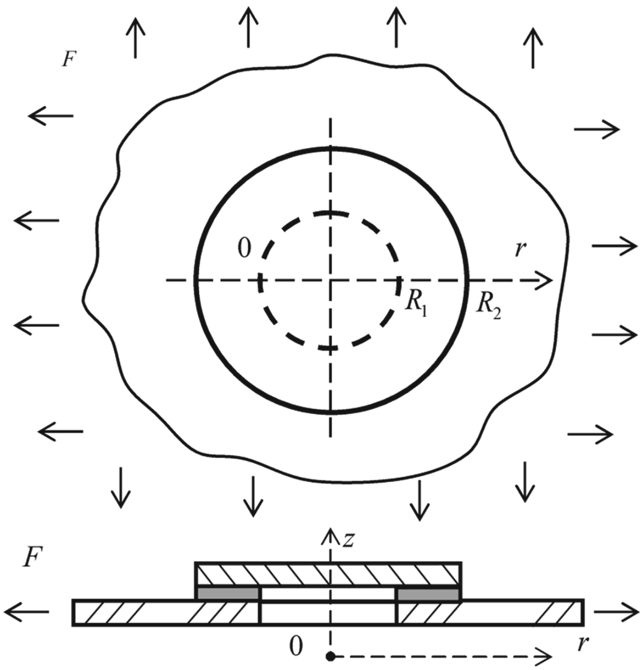

Fig. 1. Diagram of the adhesive joint of the plate with the patch.

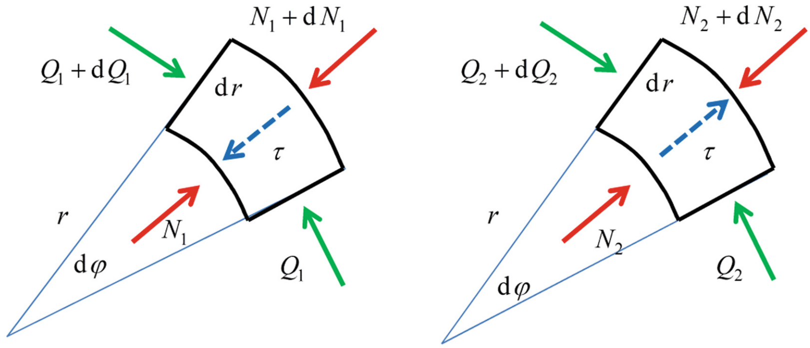

Fig. 2. Differential elements of the joint.

Due to axial symmetry, the tangential forces in the base layers Q_1 and Q_2 are independent on the angular coordinate, tangential forces in the base layers are absent. The subscript "1" corresponds to the main plate, and the subscript "2" corresponds to a circular plate within the gluing area $x \in [R_1;\ R_2]$.

Setting the direction of tangential stresses in the adhesive layer, we can write down the equilibrium equations of the base layers elements in the following form

$$\frac{N_1 - Q_1}{r} + \frac{dN_1}{dr} - \tau = 0,\ \frac{N_2 - Q_2}{r} + \frac{dN_2}{dr} + \tau = 0, \tag{1}$$

where N_k, Q_k are radial and tangential forces in the base layer k, $k = 1, 2$; τ is the tangential stresses in the adhesive layer in the radial direction.

We assume that the stresses in the adhesive are proportional to the difference of the shifts sided to the adhesive layer of the both plates.

$$\tau = P(U_1 - U_2), \tag{2}$$

where P is a shear rigidity of the adhesive layer, $P = G_0/\delta_0$; U_k is a radial shifts of the layers, $k = 1, 2$.

The physical law equations for the plates have the form:

$$N_k = B_k\left(\varepsilon_{k,r} + \mu_k \varepsilon_{k,\varphi}\right),\ Q_k = B_k\left(\varepsilon_{k,\varphi} + \mu_k \varepsilon_{k,r}\right), \tag{3}$$

where $B_k = \delta_k E_k \big/ (1 - \mu_k^2)$ is a membrane rigidity of plates; $\varepsilon_{k,r}$ and $\varepsilon_{k,\varphi}$ are radial and tangential deformation of the layer k.

Kinematic relations of the elasticity theory are

$$\varepsilon_{k,r} = \frac{dU_k}{dr},\ \ \varepsilon_{k,\varphi} = \frac{U_k}{r}, \tag{4}$$

3 Constructing the Solution

Equations (1), using (4) and (2), can be represented as:

$$\frac{\tau}{B_1} + \frac{d^2 U_1}{dr^2} + \frac{1}{r}\frac{dU_1}{dr} - \frac{U_1}{r^2} = 0,\ -\frac{\tau}{B_2} + \frac{d^2 U_2}{dr^2} + \frac{1}{r}\frac{dU_2}{dr} - \frac{U_2}{r^2} = 0. \tag{5}$$

Differentiating (2) and using the equations obtained above, we can obtain:

$$\frac{d^2\tau}{dr^2} + \frac{1}{r}\frac{d\tau}{dr} - \left(\frac{P}{B_1} + \frac{P}{B_2} + \frac{1}{r^2}\right)\tau = 0. \tag{6}$$

This equation has an analytical solution.

$$\tau = C_1 I_1(\lambda r) + C_2 K_1(\lambda r). \tag{7}$$

where $\lambda = \sqrt{\frac{P}{B_1} + \frac{P}{B_2}}$; I_1, K_1 are modified Bessel functions; C_1, C_2 an arbitrary constants.

It can be noted that in the adhesive joint stress state problem of rectangular plates, the tangential stresses in the adhesive are described by a linear combination of exponential functions [12, 15, 17]. In the simplest formulation of the problem, the so-called Wolkersen models [12], tangential stresses in the adhesive can be represented as a superposition of the hyperbolic sine and hyperbolic cosine. In the axisymmetric circular patch problem which is under consideration, the unlimited and non-periodic modified Bessel functions act as an analog of these hyperbolic and exponential functions.

Substituting tangential stresses (7) into one of Eqs. (5), then solving the nonhomogeneous Euler differential equations that was early obtained, and using relation (2), we get

$$\begin{aligned} U_1 &= -\frac{C_1}{\lambda^2 B_1} I_1(\lambda r) - \frac{C_2}{\lambda^2 B_1} K(\lambda r) + C_3 r + \frac{C_4}{r}, \\ U_2 &= \frac{C_1}{\lambda^2 B_2} I_1(\lambda r) + \frac{C_2}{\lambda^2 B_2} K(\lambda r) + C_3 r + \frac{C_4}{r}. \end{aligned} \quad (8)$$

4 Shifts Outside the Adhesive Area and Boundary Conditions

Movements in the inner area ($r < R_1$) and in the external area ($r > R_2$), i.e. outside of adhesive domain, are described by the well-known deformation equations for round plates

$$\frac{d^2 U}{dr^2} + \frac{1}{r}\frac{dU}{dr} - \frac{U}{r^2} = 0.$$

We denote the radial shifts of the patch in the inner part of the joint as U_3 and shifts of the main plate outside the joint as U_4. The equations which was given above have solutions

$$U_3 = c_1 r + \frac{c_2}{r}, \; U_4 = c_3 r + \frac{c_4}{r}. \quad (9)$$

Shifts (8) and (9), as well as relations (4) and (3), make it possible to find forces in the main plate and patch both in the adhesive area and beyond.

Constants $C_1, \; C_2, C_3, C_4$, and $c_1, \; \ldots, c_4$ we find from the boundary conditions and the shift conjugation conditions with boundary forces of the domains.

Suppose that the main plate has a radius R_3. Boundary conditions at the outer boundary of the main plate are:

$$N_4(R_3) = F;$$

Conditions at the adhesive area border for the main plate are:

$$U_1(R_2) - U_4(R_2) = 0; \;\; N_1(R_2) - N_4(R_2) = 0; \;\; N_2(R_2) = 0.$$

Conditions at the adhesive area border for the hole are:

$$U_2(R_1) - U_3(R_1) = 0;\;\; N_2(R_1) - N_3(R_1) = 0;\;\; N_1(R_1) = 0.$$

And we find two other constants from the zero longitudinal shift conditions for the patch and the finite value of its transverse shifts at the origin point ($r = 0$):

$$c_2 = 0.$$

Thus, we obtain a system of eight linear equations with respect to eight unknown constants.

5 Model Problem

When defining the geometry of the domain, we assume that the main plate has a very large radius R_3.

Model parameters are: $R_1 = 30$ mm, $R_2 = 50$ mm, $\delta_1 = \delta_2 = 3$ mm, $\delta_0 = 0,1$ mm, $E = 70$ GPa (aluminum alloy), $\mu = 0.28$, $E_0 = 0.8$ GPa, $G_0 = 0.3125$ GPa. Linear tensile forces F applied around the perimeter of the main plate, the outer radius of which is R_3 we will consider infinitely large ($R_3 = \infty$).

In Fig. 3 it is shown the graphs of tangential stresses in the adhesive layer, calculated according to the proposed model (a), and also calculated using the finite element method (b). The stresses in the presented graphs are shown in dimensionless form.

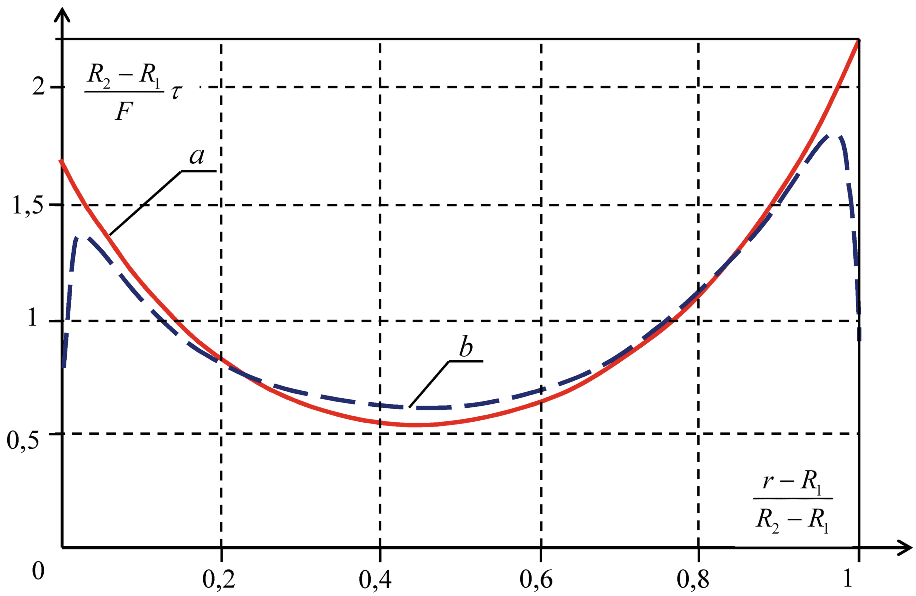

Fig. 3. Stresses in adhesive.

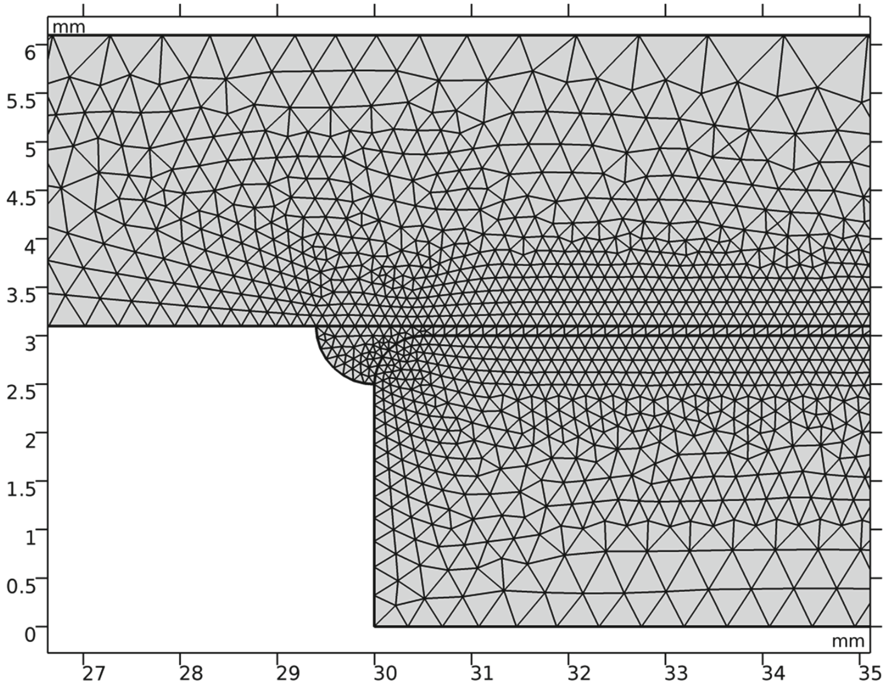

Fig. 4. Finite element model fragment.

The presented calculations were verified using finite element modeling in COMSOL Multiphysics 5.3a. For comparison, stresses were taken in the middle plane of the adhesive layer. Chamfers and squeezed glue excess are added to the finite element joint model. The neighborhood of the adhesive line edge and the grid of the finite elements are shown in Fig. 4.

Calculations showed that stresses calculated using the proposed model and using finite element modeling coincide in almost the entire adhesive area. Differences are observed only in small areas at the edges of the glue line, the length of this area being on the order of the thickness of the adhesive layer. The differences found do not exceed a few percent, and the graphs almost coincide in the results. Moreover, the approach proposed in this paper gives somewhat overrated results, which is acceptable for design problems. The described small differences between the results can be explained by the fact that the outer edge of the adhesive joint has a load-free border, as a result of which the tangential stresses at the edge of the adhesive layer should be zero, this fact cannot be effected by the proposed model. This feature of modeling the adhesive stress state by (4) is well known [12] and can be overcome using more accurate approaches to the description of the adhesive layer stress state [28].

To illustrate how the patch unloads the hole, we consider the graphs of the radial and tangential force relations in the main plate. In Fig. 5, radial and tangential forces are shown in dimensionless form in a certain neighborhood of the hole. It is obviously, that at infinity we have a uniform stress state $N_4 = Q_4 = F$. The ratio of radial forces to F is denoted as “a” on the graph, and the ratio of tangential forces to F as “b”.

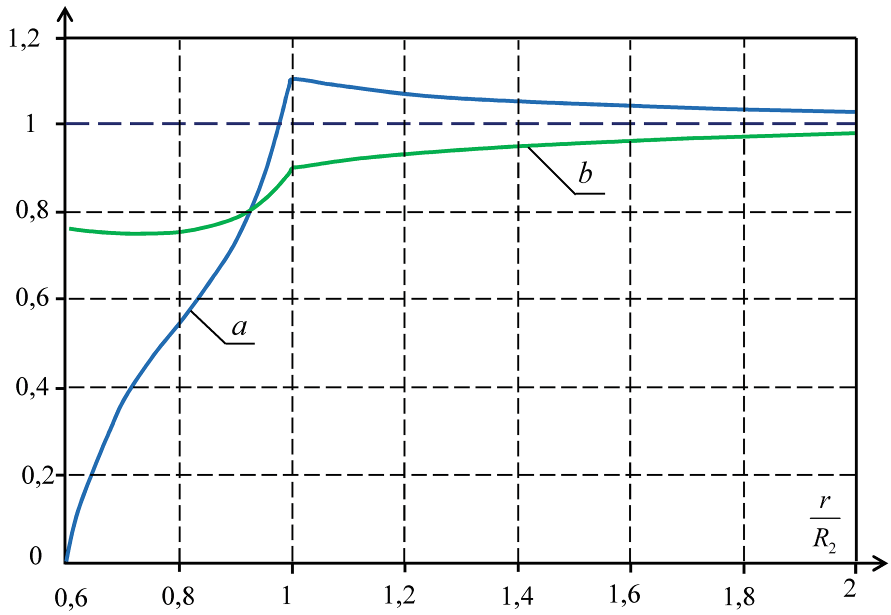

Fig. 5. Stresses in the main plate in the neighborhood of the hole: (a) – N/F; (b) – Q/F

The sharp bend in the graphs corresponds to the patch border.

In the absence of a patch at the hole edge, normal forces would be equal to zero, and tangential forces would equal $2F$. In this case, we have only a small increase in radial forces to 1.1 F. That is, as expected, the patch significantly unloads the hole. Calculations show that in the patch over the hole $Q_3 = N_3 < 0.9F$.

6 Conclusion

A stress state mathematical model for the axisymmetric plate with a circular hole, which is overlapped by an adherent coaxial circular patch, is proposed. The problem is reduced to a linear differential equation with respect to tangential stresses in the adhesive layer. This equation has an analytical solution within Bessel functions.

A feature of the problem is that, in contrast to beam joints under unidirectional loading, not all forces from the main plate are transferred to the patch by the adhesive layer. The patch a little bit unloads the hole, reducing the tangential stresses in the main plate in the neighborhood of the hole.

Possible directions for the development of the proposed model are

– taking into account the plates bending;
– inclusion into the model the temperature deformations;
– inclusion into the model the inertia forces and study the dynamic stresses;
– generalization to the case of uniaxial stress state of the base at infinity.

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