# Multidimensional Central Limit Theorem of the Multiclass M/M/1/1 Retrial Queue

Anatoly Nazarov[1], Tuan Phung-Duc[2], and Yana Izmailova[1(✉)]

[1] Institute of Applied Mathematics and Computer Science, National Research Tomsk State University, 36 Lenina ave., 634050 Tomsk, Russian Federation
nazarov.tsu@gmail.com, evgenevna.92@mail.ru

[2] Faculty of Engineering Information and Systems, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8573, Japan

**Abstract.** In this paper, we consider the multiclass M/M/1/1 retrial queueing system. Customers of each class arrive from outside the system according to a Poisson process. The service times of customers are assumed to be exponentially distributed with the parameter corresponding to the type of the customer. If the server is busy incoming customers join the orbit according to their type and make a delay for an exponentially distributed time. Equations for the characteristic function of the multi-dimensional probability distribution of the numbers of customers in the orbits are obtained. These equations are investigated by method of asymptotic analysis under the long delay condition of customers in the orbits. It is shown that the probability distribution can be approximated by a multi-dimensional Gaussian distribution. Equations are obtained for finding the parameters of this probability distribution.

**Keywords:** Retrial queueing system · A multiclass system · Asymptotic analysis

## 1 Introduction

Retrial queues have become popular in the queueing research due to the challenging in the analysis as well as the needs of modelling retrial phenomenon in real world systems, e.g., telecommunication systems [10,15], call center and other service systems [6]. Retrial queues reflect the situations that customers who arrive at a service system when the system is fully occupied, do not wait but retry to access the system in a later time. For example, customers of a call center may make a phone call again if all the operators are busy [6]. Retrial queues with single class of customers have been extensively studied in the literature [1,2]. For a survey on advances of retrial queues, we refer to [14]. The main difficulty in the analysis of retrial queues arises from the fact that customers retry independently leading to inhomogeneous transition structures of the underlying Markov

The reported study was funded by RFBR according to the research project No. 18-01-00277.


V. M. Vishnevskiy et al. (Eds.): DCCN 2020, LNCS 12563, pp. 298–310, 2020.
https://doi.org/10.1007/978-3-030-66471-8_23

chains. As a result, even for the pure Markovian model (i.e. Poisson arrivals and exponential service times), explicit results are found in only some special cases only. Single server model with pure Markovian assumptions is explicitly analyzed [2]. For the case of more than one server, generating functions for the number of customers in the orbit is represented in terms of hypergeometric functions for the case of two servers [2,4], while the distribution of the number of customers in the orbit is expressed in terms of continued fraction for the case of three and four servers [11,12] and matrix continued fraction for the case of arbitrary number of servers [5,13]. The main difficulty of the analysis is that the generating functions of the joint queue length distribution are solutions of a system of differential equations whose solution cannot be explicitly obtained in general.

For multiclass retrial queues, the analysis is even more difficult and analytical solution for the joint stationary distribution has not been obtained even for single server case. To the best of our knowledge, only the stability conditions [7,9] and moments of the number of customers in the orbit have been obtained [2,8]. The difficulty is the fact that the joint generating functions of the numbers of customers in the orbits are the solution of a system of partial differential equations. In this paper, we consider the system under an asymptotic regime of slow retrials. First, we consider obtain the first order asymptotic result that the scaled numbers of customers in the orbits converge to the constants having clear physical meaning. Next, we obtain the second order asymptotic result which states that the joint distribution of the centered numbers of customers in the orbits converges to a Gaussian distribution with explicit mean and covariance matrix.

The rest of our paper is organized as follows. Section 2 presents the model and problem formulation. Section 3 show the detailed analysis of the first and second order asymptotics. Section 4 presents some numerical examples while concluding remarks are presented in Sect. 5.

## 2 Model Description and Problem Statement

We consider a multiclass retrial queueing system. Let $N$ be the number of classes of incoming customers. Customers of each class arrive from outside the system according to a Poisson process with a rates $\lambda_n, n = \overline{1, N}$. If an arriving customer finds the server free, the customer occupies the server and gets a service. The service times for each class of customers are assumed to be exponentially distributed with service rates $\mu_n, n = \overline{1, N}$ depending on the class. If the server is busy incoming customers join the orbit according to their type and make a delay for an exponentially distributed time with rate $\sigma_n, n = \overline{1, N}$ then repeat their request for service.

Let $i_n(t), n = \overline{1, N}$ be the random processes of the numbers of customers in the orbits. We denote in vector notation as $\mathbf{i}(t) = [i_1(t) \dots i_N(t)]$. The aim of the current research is to derive the stationary probability distribution of this vector process. Let $k(t)$ be the random process that defines the server states: 0

if the server is free, $n$ if the server is busy serving an incoming call of $n$-th type, $n = \overline{1, N}$.

The process $\mathbf{i}(t)$ is not Markovian, therefore we consider the $(N+1)$-dimensional continuous time Markov chain $\{k(t), \mathbf{i}(t)\}$.

Denoting $P_k(\mathbf{i}, t) = P\{k(t) = k, i_1(t) = i_1, \dots, i_N(t) = i_N\}, k = \overline{0, N}$ it is possible to write down the following equalities

$$P_0(\mathbf{i}, t + \Delta t) = P_0(\mathbf{i}, t) \prod_{m=1}^{N} (1 - \lambda_m \Delta t)(1 - i_m \sigma_m \Delta t) + \sum_{m=1}^{N} P_m(\mathbf{i}, t) \mu_m \Delta t + o(\Delta t),$$

$$P_n(\mathbf{i}, t + \Delta t) = P_n(\mathbf{i}, t)(1 - \mu_n \Delta t) \prod_{m=1}^{N} (1 - \lambda_m \Delta t) + P_0(\mathbf{i}, t) \lambda_n \Delta t$$

$$+ P_0(\mathbf{i} + \mathbf{e}_n, t)(i_n + 1)\sigma_n \Delta t + \sum_{\nu=1}^{N} P_n(\mathbf{i} - \mathbf{e}_\nu, t) \lambda_\nu \Delta t + o(\Delta t), n = \overline{1, N}.$$

Here $\mathbf{e}_n$ is the vector whose $n$-th component is equal to unity, and the rest are zero.

We will consider the system in a steady state regime under the stability condition [9]:

$$\sum_{m=1}^{N} \frac{\lambda_m}{\mu_m} < 1.$$

We denote $P_k(\mathbf{i}) = \lim_{t \to \infty} P_k(\mathbf{i}, t)$ the stationary probability distribution of the system states $\{k(t), \mathbf{i}(t)\}$.

Let us write the system of equations for the probability distribution

$$\{P_0(\mathbf{i}), P_1(\mathbf{i}), \dots, P_N(\mathbf{i})\}, \mathbf{i} \geq 0,$$

using equalities the above:

$$P_0(\mathbf{i}) \sum_{m=1}^{N} (-\lambda_m - i_m \sigma_m) + \sum_{m=1}^{N} P_m(\mathbf{i}) \mu_m = 0,$$

$$- P_n(\mathbf{i}) \left( \mu_n + \sum_{m=1}^{N} \lambda_m \right) + P_0(\mathbf{i}) \lambda_n + P_0(\mathbf{i} + \mathbf{e}_n)(i_n + 1)\sigma_n \tag{1}$$

$$+ \sum_{\nu=1}^{N} P_n(\mathbf{i} - \mathbf{e}_\nu) \lambda_\nu = 0, n = \overline{1, N}.$$

Here it is assumed that $P_k(\mathbf{i}) = 0, k = \overline{0, N}$, if at least one component of the vector $\mathbf{i}$ is negative.

Let us introduce the multidimensional partial characteristic functions

$$H_k(\mathbf{u}) = \sum_{i_1=0}^{\infty} \dots \sum_{i_N=0}^{\infty} P_k(i_1, \dots, i_N) \exp\left\{ j \sum_{m=1}^{N} u_m i_m \right\}$$

$$= \sum_{\mathbf{i}=0}^{\infty} e^{j\mathbf{u}^T \mathbf{i}} P_k(\mathbf{i}), k = \overline{0, N}, \tag{2}$$

where $j = \sqrt{-1}$ is an imaginary unit, $\mathbf{u}$ – vector with components $u_n, n = \overline{1, N}$.

Substituting functions (2) into (1), the following system of equations is obtained.

$$\begin{aligned} &-H_0(\mathbf{u}) \sum_{m=1}^{N} \lambda_m + j \sum_{m=1}^{N} \frac{\partial H_0(\mathbf{u})}{\partial u_m} \sigma_m + \sum_{m=1}^{N} H_m(\mathbf{u}) \mu_m = 0, \\ &-H_n(\mathbf{u}) \left( \mu_n + \sum_{m=1}^{N} \lambda_m \right) + H_0(\mathbf{u}) \lambda_n - j \sigma_n e^{-j u_n} \frac{\partial H_0(\mathbf{u})}{\partial u_n} \\ &\quad + \sum_{m=1}^{N} H_n(\mathbf{u}) \lambda_m e^{j u_m} = 0, n = \overline{1, N}. \end{aligned} \tag{3}$$

## 3 Asymptotic Analysis Under the Long Delay Condition

Denote

$$\sigma_n = \sigma \gamma_n, n = \overline{1, N}.$$

The main idea of this paper is to find the solution of system (3) by using an asymptotic analysis method under the limit condition of the long delay customers in the orbits, i.e., when $\sigma \to 0$.

### 3.1 Asymptotic of the First-Order

We make the following substitutions in the system (3):

$$\sigma = \epsilon, \mathbf{u} = \epsilon \mathbf{w}, H_k(\mathbf{u}) = F_k(\mathbf{w}, \epsilon), k = \overline{0, N}.$$

As the result, we get the following equations:

$$\begin{aligned} &-F_0(\mathbf{w}, \epsilon) \sum_{m=1}^{N} \lambda_m + j \sum_{m=1}^{N} \frac{\partial F_0(\mathbf{w}, \epsilon)}{\partial w_m} \gamma_m + \sum_{m=1}^{N} F_m(\mathbf{w}, \epsilon) \mu_m = 0, \\ &- F_n(\mathbf{w}, \epsilon) \left( \mu_n + \sum_{m=1}^{N} \lambda_m \right) + F_0(\mathbf{w}, \epsilon) \lambda_n - j \gamma_n e^{-j \epsilon w_n} \frac{\partial F_0(\mathbf{w}, \epsilon)}{\partial w_n} \\ &\quad + \sum_{m=1}^{N} F_n(\mathbf{w}, \epsilon) \lambda_m e^{j \epsilon w_m} = 0, n = \overline{1, N}. \end{aligned} \tag{4}$$

Denoting the asymptotic solution of the system of Eqs. (4) in the form $F_k(\mathbf{w}) = \lim_{\epsilon \to 0} F_k(\mathbf{w}, \epsilon), k = \overline{0, N}$, we obtain solution named as "first-order asymptotic". We prove the following theorem.

**Theorem 1.** *The first-order asymptotic characteristic function of the probability distribution of the numbers of customers in the orbits has the form:*

$$F_k(\mathbf{w}) = R_k \exp \left\{ \sum_{m=1}^{N} j w_m x_m \right\}, k = \overline{0, N},$$

*where parameter*

$$R_n = \frac{\lambda_n}{\mu_n}, n = \overline{1, N}, R_0 = 1 - \sum_{m=1}^{N} \frac{\lambda_m}{\mu_m} \tag{5}$$

*is the stationary probability distribution of the state server (*$\mathbf{R} = \{R_k\}, k = \overline{0, N}$ *in matrix form),*

$$x_n = \frac{\lambda_n}{\gamma_n} \frac{1 - R_0}{R_0}, n = \overline{1, N}. \tag{6}$$

*Proof.* In system (4), we take the limit as $\epsilon \to 0$. Then, we get the system of equations:

$$\begin{gathered} -F_0(\mathbf{w}) \sum_{m=1}^{N} \lambda_m + j \sum_{m=1}^{N} \frac{\partial F_0(\mathbf{w})}{\partial w_m} \gamma_m + \sum_{m=1}^{N} F_m(\mathbf{w}) \mu_m = 0, \\ - F_n(\mathbf{w}) \mu_n + F_0(\mathbf{w}) \lambda_n - j \gamma_n \frac{\partial F_0(\mathbf{w})}{\partial w_n} = 0, n = \overline{1, N}. \end{gathered} \tag{7}$$

We will look for a solution the above system of equations in the following form

$$F_k(\mathbf{w}) = R_k \Phi(\mathbf{w}), k = \overline{0, N}. \tag{8}$$

Substituting (8) into (7) and multiplying the equations of the system by $\frac{1}{\Phi(\mathbf{w})}$, we derive equations:

$$\begin{gathered} -R_0 \sum_{m=1}^{N} \lambda_m + j R_0 \sum_{m=1}^{N} \frac{\partial \Phi(\mathbf{w}) / \partial w_m}{\Phi(\mathbf{w})} \gamma_m + \sum_{m=1}^{N} R_m \mu_m = 0, \\ - R_n \mu_n + R_0 \lambda_n - j \gamma_n R_0 \frac{\partial \Phi(\mathbf{w}) / \partial w_n}{\Phi(\mathbf{w})} = 0, n = \overline{1, N}. \end{gathered} \tag{9}$$

The solution of Eq. (9) is as follows:

$$\Phi(\mathbf{w}) = \exp\left\{ \sum_{m=1}^{N} j w_m x_m \right\}. \tag{10}$$

Substituting this expression into the system (9) yields

$$\begin{gathered} -R_0 \sum_{m=1}^{N} \lambda_m - R_0 \sum_{m=1}^{N} \gamma_m x_m + \sum_{m=1}^{N} R_m \mu_m = 0, \\ - R_n \mu_n + R_0 \lambda_n + \gamma_n x_n R_0 = 0, n = \overline{1, N}. \end{gathered}$$

We express $R_n$ from the second equation of system and get relation

$$R_n = \frac{\lambda_n + \gamma_n x_n}{\mu_n} R_0. \tag{11}$$

We sum the equations of the system (4) in order to get the equation

$$\sum_{m=1}^{N}\lambda_m(e^{j\epsilon w_m}-1)\sum_{m=1}^{N}F_m(\mathbf{w},\epsilon)+j\sum_{m=1}^{N}\frac{\partial F_0(\mathbf{w},\epsilon)}{\partial w_m}(1-e^{-j\epsilon w_m})\gamma_m=0. \quad (12)$$

Let us use expansion

$$e^{j\epsilon w_m}=1+j\epsilon w_m+o(\epsilon)$$

where $o(\epsilon)$ is an infinitesimal of the order greater than $\epsilon$, in Eq. (12). Dividing these equations by $j\epsilon$ and making the transition $\epsilon\to 0$, one obtains the following equation:

$$\sum_{m=1}^{N}\lambda_m w_m\sum_{m=1}^{N}F_m(\mathbf{w})+j\sum_{m=1}^{N}\frac{\partial F_0(\mathbf{w})}{\partial w_m}w_m\gamma_m=0. \quad (13)$$

Substituting (8) and (10) to (13), we have

$$\sum_{m=1}^{N}\lambda_m w_m\sum_{m=1}^{N}R_m-R_0\sum_{m=1}^{N}w_m\gamma_m x_m=0.$$

Using condition of standardization $\sum\limits_{m=0}^{N}R_m=1$, we obtain

$$\sum_{m=1}^{N}\lambda_m w_m-R_0\sum_{m=1}^{N}(\lambda_m+\gamma_m x_m)w_m=0. \quad (14)$$

After some transformations, one obtains the following equation:

$$\lambda_m-R_0(\lambda_m+\gamma_m x_m)=0.$$

We obtain an expressions for $x_n, n=\overline{1,N}$, which coincide with (6).

Using this expression, we can write

$$R_0=\frac{\lambda_m}{\lambda_m+\gamma_m x_m}.$$

Substituting this expression in formula (11), one obtains expression (5).

The values $x_n$ represent the average values of the numbers of customers in the orbits normalized by the value $\sigma$.

### 3.2 Asymptotic of the Second-Order

In the system (3) let us denote

$$H_k(\mathbf{u})=H_k^{(2)}(\mathbf{u})\exp\left\{\sum_{m=1}^{N}j\frac{u_m}{\sigma_m}\gamma_m x_m\right\},k=\overline{0,N}. \quad (15)$$

The functions $H_k^{(2)}(\mathbf{u})$ are the partial characteristic functions of the values of centered random processes $i_m(t) - \dfrac{x_m}{\sqrt{\sigma}}$. Substituting

$$\sigma_n = \sigma\gamma_n, \sigma = \epsilon^2, \mathbf{u} = \epsilon\mathbf{w}, H_k^{(2)}(\mathbf{u}) = F_k^{(2)}(\mathbf{w}, \epsilon), k = \overline{0, N}. \tag{16}$$

and expression (15) into the system (3) we get:

$$\begin{gathered}
-F_0^{(2)}(\mathbf{w},\epsilon)\sum_{m=1}^{N}(\lambda_m + \gamma_m x_m) + j\epsilon\sum_{m=1}^{N}\frac{\partial F_0^{(2)}(\mathbf{w},\epsilon)}{\partial w_m}\gamma_m + \sum_{m=1}^{N}F_m^{(2)}(\mathbf{w},\epsilon)\mu_m = 0,\\
-F_n^{(2)}(\mathbf{w},\epsilon)\left(\mu_n + \sum_{m=1}^{N}\lambda_m(1 - e^{j\epsilon w_m})\right) + F_0^{(2)}(\mathbf{w},\epsilon)(\lambda_n + \gamma_n x_n e^{-j\epsilon w_n})\\
-j\epsilon\gamma_n e^{-j\epsilon w_n}\frac{\partial F_0^{(2)}(\mathbf{w},\epsilon)}{\partial w_n} = 0, n = \overline{1, N}.
\end{gathered} \tag{17}$$

Denoting the asymptotic solution of the system of Eqs. (17) in the form $F_k^{(2)}(\mathbf{w}) = \lim\limits_{\epsilon \to 0} F_k^{(2)}(\mathbf{w}, \epsilon), k = \overline{0, N}$, we obtain this solution, named as "second-order asymptotic". We prove the following theorem.

**Theorem 2.** *The second-order asymptotic characteristic function of the probability distribution of the number of customers in the orbits has the form:*

$$F_k^{(2)}(\mathbf{w}) = R_k \exp\left\{-\frac{1}{2}\sum_{\nu=1}^{N}\sum_{m=1}^{N} w_\nu K_{\nu m} w_m\right\}, k = \overline{0, N}, \tag{18}$$

*where parameters $K_{\nu m}$ are the solution of the following system:*

$$\begin{gathered}
\gamma_m R_0 K_{mm} - \lambda_m R_0 \sum_{l=1}^{N}\frac{\gamma_l}{\mu_l}K_{lm} = \lambda_m(1 - R_0)(1 - R_m) + \lambda_m^2\sum_{l=1}^{N}\frac{R_l}{\mu_l}, \nu = m,\\
\gamma_m R_0 K_{m\nu} + \gamma_\nu R_0 K_{\nu m} - \lambda_m R_0\sum_{l=1}^{N}\frac{\gamma_l}{\mu_l}K_{l\nu} - \lambda_\nu R_0\sum_{l=1}^{N}\frac{\gamma_l}{\mu_l}K_{lm}\\
= 2\lambda_m\lambda_\nu\sum_{l=1}^{N}\frac{R_l}{\mu_l} - (R_m\lambda_\nu + R_\nu\lambda_m)(1 - R_0), \nu \neq m.
\end{gathered} \tag{19}$$

*Proof.* We will look for a solution of (17) in the following form:

$$F_k^{(2)}(\mathbf{w}, \epsilon) = \Phi_2(\mathbf{w})\left(R_k + \sum_{m=1}^{N} j\epsilon w_m f_{km} + o(\epsilon)\right). \tag{20}$$

Substituting (20) and the expansion $e^{j\epsilon w_m} = 1 + j\epsilon w_m + o(\epsilon)$ into (17), we obtain

$$-\Phi_2(\mathbf{w})\left(R_0 + \sum_{m=1}^{N} j\epsilon w_m f_{0m}\right)\sum_{m=1}^{N}(\lambda_m + \gamma_m x_m) + \Phi_2(\mathbf{w})\sum_{m=1}^{N}\mu_m \sum_{\nu=1}^{N} j\epsilon w_\nu f_{m\nu}$$
$$+\Phi_2(\mathbf{w})\sum_{m=1}^{N}\mu_m R_m + j\epsilon\sum_{m=1}^{N}\gamma_m(R_0 + j\epsilon\sum_{\nu=1}^{N} w_\nu f_{0\nu})\frac{\partial\Phi_2(\mathbf{w})}{\partial w_m} + o(\epsilon) = 0,$$
$$\Phi_2(\mathbf{w})\left(R_n + \sum_{\nu=1}^{N} j\epsilon w_\nu f_{n\nu}\right)\left(j\epsilon\sum_{m=1}^{N}\lambda_m w_m - \mu_n\right) + \Phi_2(\mathbf{w})R_0(\lambda_n + \gamma_n x_n)$$
$$-j\epsilon\Phi_2(\mathbf{w})R_0 w_n \gamma_n x_n + \Phi_2(\mathbf{w})(\lambda_n + \gamma_n x_n)\sum_{\nu=1}^{N} w_\nu f_{0\nu}$$
$$-j\epsilon\gamma_n R_0\frac{\partial\Phi_2(\mathbf{w})}{\partial w_n} + o(\epsilon) = 0, n = \overline{1,N}.$$

Using equality (11) and multiplying the above equation by $\dfrac{1}{j\epsilon\Phi_2(\mathbf{w})}$, we obtain

$$-\sum_{m=1}^{N}(\lambda_m + \gamma_m x_m)\sum_{\nu=1}^{N} w_\nu f_{0\nu} + \sum_{m=1}^{N}\mu_m\sum_{\nu=1}^{N} w_\nu f_{m\nu} + \sum_{m=1}^{N}\gamma_m R_0\frac{\partial\Phi_2(\mathbf{w})/\partial w_n}{\Phi_2(\mathbf{w})} = 0,$$
$$R_n\sum_{\nu=1}^{N}\lambda_\nu w_\nu - \mu_n\sum_{\nu=1}^{N} w_\nu f_{n\nu} - R_0\gamma_n x_n w_n \tag{21}$$
$$+ (\lambda_n + \gamma_n x_n)\sum_{\nu=1}^{N} w_\nu f_{0\nu} - \gamma_n R_0\frac{\partial\Phi_2(\mathbf{w})/\partial w_n}{\Phi_2(\mathbf{w})} = 0, n = \overline{1,N}.$$

The solution of Eq. (21) is as follows:

$$\Phi_2(\mathbf{w}) = \exp\left\{-\frac{1}{2}\sum_{m=1}^{N}\sum_{\nu=1}^{N} w_m K_{m\nu} w_\nu\right\}, \tag{22}$$

where the quantities $K_{m\nu}$ are elements of the covariance matrix $\mathbf{K} = \{K_{m\nu}\}$.

Substituting (22) into (21), we rewrite (21) as follows:

$$\sum_{m=1}^{N}(\lambda_m + \gamma_m x_m)\sum_{\nu=1}^{N} w_\nu f_{0\nu} - \sum_{m=1}^{N}\mu_m\sum_{\nu=1}^{N} w_\nu f_{m\nu} + \sum_{m=1}^{N}\gamma_m R_0\sum_{\nu=1}^{N} w_\nu K_{m\nu} = 0,$$
$$R_n\sum_{\nu=1}^{N}\lambda_\nu w_\nu - \mu_n\sum_{\nu=1}^{N} w_\nu f_{n\nu} - R_0\gamma_n x_n w_n + (\lambda_n + \gamma_n x_n)\sum_{\nu=1}^{N} w_\nu f_{0\nu} \tag{23}$$
$$+ \gamma_n R_0\sum_{\nu=1}^{N} w_\nu K_{n\nu} = 0, n = \overline{1,N}.$$

From the system (23) it follows that we can write the following equalities:

$$\begin{aligned} R_n\lambda_n - R_0\gamma_n x_n + R_0\gamma_n K_{nn} - \mu_n f_{nn} + (\lambda_n + \gamma_n x_n) f_{0n} = 0, \nu = n, \\ R_n\lambda_\nu + R_0\gamma_n K_{n\nu} - \mu_n f_{n\nu} + (\lambda_n + \gamma_n x_n) f_{0\nu} = 0, \nu \neq n. \end{aligned} \tag{24}$$

We then rewrite (24) as:

$$\begin{aligned} f_{nn} = R_n^2 - R_n(1 - R_0) + \frac{\gamma_n R_0}{\mu_n} K_{nn} + \frac{R_n}{R_0} f_{0n}, \nu = n, \\ f_{n\nu} = R_n \frac{\lambda_\nu}{\mu_n} + \frac{\gamma_n R_0}{\mu_n} K_{n\nu} + \frac{R_n}{R_0} f_{0\nu}, \nu \neq n. \end{aligned} \tag{25}$$

Summing equations of the system (17) and using expansion $e^{j\epsilon w_m} = 1 + j\epsilon w_m + \frac{(j\epsilon w_m)^2}{2} + o(\epsilon^2)$, we get the relation

$$j\epsilon \sum_{m=1}^{N} \lambda_m \left( w_m + \frac{j\epsilon w_m^2}{2} \right) \sum_{m=1}^{N} F_m(\mathbf{w}, \epsilon) + (j\epsilon)^2 \sum_{m=1}^{N} \gamma_m w_m \frac{\partial F_0(\mathbf{w}, \epsilon)}{\partial w_m}$$
$$- j\epsilon \sum_{m=1}^{N} \gamma_m x_m \left( w_m - \frac{j\epsilon w_m^2}{2} \right) F_0(\mathbf{w}, \epsilon) + o(\epsilon^2) = 0.$$

Substituting in the above equation expansion (20) and dividing each part of this equation by $j\epsilon$, we obtain the following equation for the function $\Phi_2(\mathbf{w})$

$$\sum_{m=1}^{N} \lambda_m \left( w_m + \frac{j\epsilon w_m^2}{2} \right) \sum_{m=1}^{N} \left( R_m + j\epsilon \sum_{\nu=1}^{N} w_\nu f_{m\nu} \right) - \sum_{m=1}^{N} \gamma_m x_m \left( w_m - \frac{j\epsilon w_m^2}{2} \right) R_0$$
$$- \sum_{m=1}^{N} \gamma_m x_m \left( w_m - \frac{j\epsilon w_m^2}{2} \right) j\epsilon \sum_{\nu=1}^{N} w_\nu f_{0\nu}$$
$$+ j\epsilon \sum_{m=1}^{N} \gamma_m w_m R_0 \frac{\partial \Phi_2(\mathbf{w}) / \partial w_m}{\Phi_2(\mathbf{w})} + o(\epsilon) = 0.$$

Using Eq. (14), we divide both sides of it by $j\epsilon$ and take the limit $\epsilon \to 0$. We then have:

$$\sum_{m=1}^{N} \lambda_m (1 - R_0) w_m^2 + \sum_{m=1}^{N} \gamma_m R_0 w_m \frac{\partial \Phi_2(\mathbf{w}) / \partial w_m}{\Phi_2(\mathbf{w})}$$
$$= \sum_{m=1}^{N} \gamma_m x_m w_m \sum_{\nu=1}^{N} w_\nu f_{0\nu} - \sum_{m=1}^{N} \lambda_m w_m \sum_{l=1}^{N} \sum_{\nu=1}^{N} w_\nu f_{l\nu}.$$

Substituting (22), we have

$$\sum_{m=1}^{N} \lambda_m (1-R_0) w_m^2 - \sum_{m=1}^{N} \gamma_m R_0 w_m \sum_{\nu=1}^{N} K_{m\nu} w_\nu$$
$$= \sum_{m=1}^{N} \gamma_m x_m w_m \sum_{\nu=1}^{N} w_\nu f_{0\nu} - \sum_{m=1}^{N} \lambda_m w_m \sum_{l=1}^{N} \sum_{\nu=1}^{N} w_\nu f_{l\nu}.$$

We get the system of equations for $K_{m\nu}, m = \overline{1,N}, \nu = \overline{1,N}$. Considering the above equation

$$\lambda_m (1-R_0) - \gamma_m R_0 K_{mm} = \gamma_m x_m f_{0m} - \lambda_m \sum_{l=1}^{N} f_{lm}, \nu = m,$$

$$-R_0(\gamma_m K_{m\nu} + \gamma_\nu K_{\nu m}) = -\lambda_m \sum_{l=1}^{N} f_{l\nu} - \lambda_\nu \sum_{l=1}^{N} f_{lm} + \gamma_m x_m f_{0\nu} + \gamma_\nu x_\nu f_{0m}, \nu \neq m$$

and then, substituting (25), it can be rewritten as

$$\gamma_m R_0 K_{mm} - \lambda_m R_0 \sum_{l=1}^{N} \frac{\gamma_l}{\mu_l} K_{lm} = \lambda_m (1-R_0)(1-R_m) + \lambda_m^2 \sum_{l=1}^{N} \frac{R_l}{\mu_l}, \nu = m,$$

$$\gamma_m R_0 K_{m\nu} + \gamma_\nu R_0 K_{\nu m} - \lambda_m R_0 \sum_{l=1}^{N} \frac{\gamma_l}{\mu_l} K_{l\nu} - \lambda_\nu R_0 \sum_{l=1}^{N} \frac{\gamma_l}{\mu_l} K_{lm}$$
$$= 2\lambda_m \lambda_\nu \sum_{l=1}^{N} \frac{R_l}{\mu_l} - (R_m \lambda_\nu + R_\nu \lambda_m)(1-R_0), \nu \neq m.$$

Thus, the proof is completed.

Replacing (18) to (16) and (15), we can write expression for approximation the partial characteristic function at small values of $\sigma$:

$$H_k(\mathbf{u}) \approx R_k \exp\left\{ j \sum_{m=1}^{N} \frac{u_m}{\sigma} x_m - \frac{1}{2} \sum_{m=1}^{N} \sum_{\nu=1}^{N} \frac{u_m}{\sqrt{\sigma}} K_{m\nu} \frac{u_\nu}{\sqrt{\sigma}} \right\}, k = \overline{0,N}.$$

Summing up all values $k = \overline{0,N}$, we obtain an approximation of the characteristic function of probability distribution of number customers in the orbits

$$H(\mathbf{u}) \approx \exp\left\{ j \sum_{m=1}^{N} \frac{u_m}{\sigma} x_m - \frac{1}{2} \sum_{m=1}^{N} \sum_{\nu=1}^{N} \frac{u_m}{\sqrt{\sigma}} K_{m\nu} \frac{u_\nu}{\sqrt{\sigma}} \right\}.$$

Thus, the number of customers in the orbits in the multiclass retrial queueing system is asymptotically Gaussian.

**Table 1.** The model of parameters

| The rate of arrival flow,$\lambda_n$ | The service rate, $\mu_n$ | The rate of a delay in the orbit, $\sigma_n = \sigma\gamma_n$ |
|---|---|---|
| $\lambda_1 = 0.7$ | $\mu_1 = 2$ | $\sigma_1 = 0.01$ |
| $\lambda_2 = 0.6$ | $\mu_2 = 3$ | $\sigma_2 = 0.02$ |
| $\lambda_3 = 0.5$ | $\mu_3 = 4$ | $\sigma_3 = 0.03$ |

## 4 Example

We consider particular: $n = 3$ (Table 1).

Given these values row-vector

$$\mathbf{x} = \begin{bmatrix} x_1 \; x_2 \; x_3 \end{bmatrix}$$

and row-vector

$$\mathbf{R} = \begin{bmatrix} R_0 \; R_1 \; R_2 \; R_3 \end{bmatrix}$$

are defined as:

$$\mathbf{x} = \begin{bmatrix} 1.454 \; 0.623 \; 0.346 \end{bmatrix}, \mathbf{R} = \begin{bmatrix} 0.325 \; 0.35 \; 0.2 \; 0.125 \end{bmatrix}.$$

Finally, we specify the matrix covariance

$$\mathbf{K} = \begin{bmatrix} 2.963 \; 0.739 \; 0.428 \\ 0.739 \; 1.018 \; 0.241 \\ 0.428 \; 0.241 \; 0.497 \end{bmatrix}.$$

Using these parameters, we find mean $\kappa_1$ and variance $\kappa_2$ of total number of customers in the orbits:

$$\kappa_1 = \frac{x_1 + x_2 + x_3}{\sigma}, \kappa_2 = \frac{K_{11} + K_{22} + K_{33} + 2K_{12} + 2K_{13} + 2K_{23}}{\sigma}$$

for $\sigma = 0.01$. Let us denote normal distribution function with moments $\kappa_1$ and $\kappa_2$ by $F(x)$, $P(i)$ be discrete distribution of nonnegative quantity which is defined by

$$P(i) = (F(i + 0.5) - F(i - 0.5))(1 - F(-0.5))^{-1}, i \geq 0.$$

The graph of asymptotic probability distribution $P(i)$ of total number of customers in the orbits is given in Fig. 1.

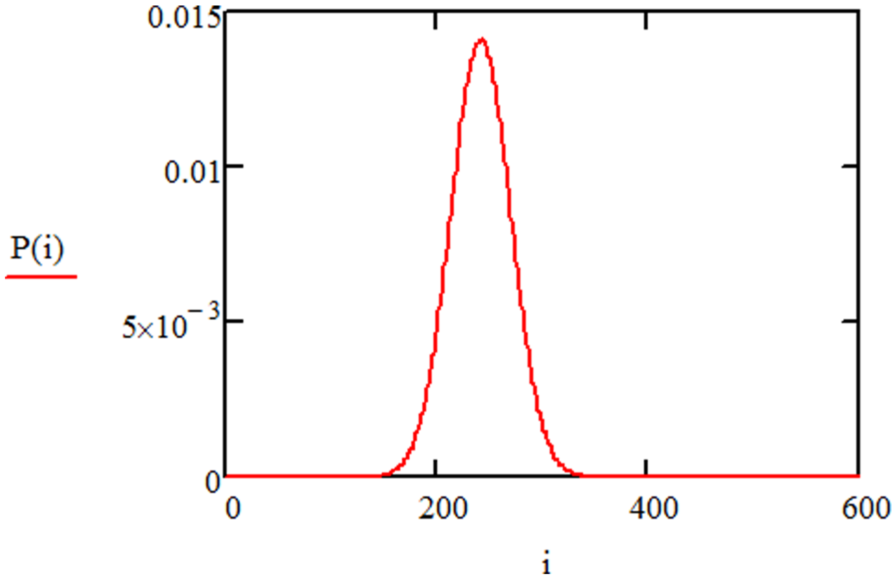


**Fig. 1.** Graph of distribution $P(i)$

## 5 Conclusion

In this paper, we considered a multiclass retrial queueing system. Equations for characteristic functions of the multi-dimensional probability distribution of the numbers of customers in the orbits are obtained. We then used the method of asymptotic analysis under condition of a long delay customers in the orbits to find the limiting probability distribution of the number of the customers in the orbits. This probability distribution turned out to be Gaussian. We as well derived the expressions for mean and stationary probability distribution of state server. Equations are obtained for finding the elements of the covariance matrix. In particular, we considered case $n = 3$. We obtained the values of mean and variance of total number of customers in the orbits. Graph for the probability distribution of the total number of customers in the orbits is given.

In the future it is planned to research a multiclass retrial queueing system in which service times of customers in a class follow an arbitrary distribution.

## References

1. Artalejo, J.R., Gómez-Corral, A.: Retrial Queueing Systems. Springer, Heidelberg (2008). https://doi.org/10.1007/978-3-540-78725-9
2. Falin, G., Templeton, J.G.: Retrial Queues. CRC Press, Boca Raton (1997)

3. Fiems, D., Phung-Duc, T.: Light-traffic analysis of random access systems without collisions. Ann. Oper. Res. **277**(2), 311–327 (2017). https://doi.org/10.1007/s10479-017-2636-7
4. Hanschke, T.: Explicit formulas for the characteristics of the M/M/2/2 queue with repeated attempts. J. Appl. Probab. **24**, 486–494 (1987)
5. Hanschke, T.: A matrix continued fraction algorithm for the multiserver repeated order queue. Math. Comput. Model. **30**, 159–170 (1999)
6. Hu, K., Allon, G., Bassamboo, A.: Understanding customers retrial in call centers: preferences for service quality and service speed. Available at SSRN 2838998D (2020)
7. Kim, B., Kim, J.: Stability of a multi-class multi-server retrial queueing system with service times depending on classes and servers. Queueing Syst. **94**(1), 129–146 (2019). https://doi.org/10.1007/s11134-019-09634-x
8. Kulkarni, V.G.: On queueing systems by retrials. J. Appl. Probab. **20**(2), 380–389 (1983)
9. Morozov, E., Phung-Duc, T.: Stability analysis of a multiclass retrial system with classical retrial policy. Perform. Eval. **112**, 15–26 (2017)
10. Phung-Duc, T., Masuyama, H., Kasahara, S., Takahashi, Y.: Performance analysis of optical burst switched networks with limited-range wavelength conversion, retransmission and burst segmentation. J. Oper. Res. Soc. Jpn. **52**(1), 58–74 (2009)
11. Phung-Duc, T., Masuyama, H., Kasahara, S., Takahashi, Y.: M/M/3/3 and M/M/4/4 retrial queues. J. Ind. Manag. Optim. **5**(3), 431–451 (2009)
12. Phung-Duc, T., Masuyama, H., Kasahara, S., Takahashi, Y.: State-dependent M/M/c/c+ r retrial queues with Bernoulli abandonment. J. Ind. Manage. Optim. **6**(3), 517–540 (2010)
13. Phung-Duc, T., Masuyama, H., Kasahara, S., Takahashi, Y.: A matrix continued fraction approach to multiserver retrial queues. Ann. Oper. Res. **202**(1), 161–183 (2013)
14. Phung-Duc, T.: Retrial Queueing Models: A Survey on Theory and Applications, to appear in by Dohi, T., Ano, K., Kasahara, S. (eds.) Stochastic Operations Research in Business and Industry. World Scientific Publisher, Singapore (2017)
15. Tran-Gia, P., Mandjes, M.: Modeling of customer retrial phenomenon in cellular mobile networks. IEEE J. Sel. Areas Commun. **15**(8), 1406–1414 (1997)