Chapter 7
On the Stark Units of Drinfeld Modules

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Abstract We present the notion of Stark units and various techniques involving it. The Stark units constitute a useful tool to study the unit and class modules of a Drinfeld module as defined by Taelman. We review some recent results on Drinfeld $\mathbb{F}_q[\theta]$-modules which make use of this notion. In particular, we present the "discrete Greenberg conjectures" which explain the structure of the class module of the canonical multi-variable deformations of the Carlitz module, and a result on the non vanishing modulo a given prime of a class of Bernoulli-Carlitz numbers.

7.1 Introduction

This text aims to constitute an introduction, largely accessible to non specialist readers, to the notion of *Stark units* of Drinfeld modules. The germs of the concept of Stark units can be found in [APTR16, APTR18]. The notion has been conceptualized in [ATR17] for Drinfeld modules over $\mathbb{F}_q[\theta]$ and then further developed in the general context of Drinfeld modules in [ANDTR17] and in [ANDTR20a] for t-modules.

Let $\mathbb{F}_q$ be a finite field with q elements, θ be an indeterminate over $\mathbb{F}_q$, $A = \mathbb{F}_q[\theta]$, and B be a finite integral extension of A, and denote by τ the map $x \mapsto x^q$. A Drinfeld A-module defined over B is a ring homomorphism $\phi : A \to B[\tau], a \mapsto \phi_a$ where $\phi_a \equiv a \pmod{\tau}$. We first define the z-deformation of ϕ which consists in twisting the Frobenius τ by a new variable z which commutes with τ. This can be obtained simply by the formula $\widetilde{\phi} : A \to B[z][\tau], a \mapsto \widetilde{\phi}_a$ where, if $\phi_a = \sum_{i=0}^r a_i \tau^i$, then $\widetilde{\phi}_a = \sum_{i=0}^r a_i z^i \tau^i$.

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This naive construction reveals its interest when one computes the unit module $U(\phi)$ of the Drinfeld module ϕ. This unit module, introduced by L. Taelman along with the class module (see [Tae12]), is, roughly speaking, the A-module of the elements which map to integral elements via the exponential map associated to ϕ. One can obtain a submodule of finite index of $U(\phi)$ by computing $U(\widetilde{\phi})$ and then evaluating at $z = 1$. This is the module of Stark units of ϕ.

The terminology of Stark units comes from the remark of Anderson from [And96] that the elements that he constructed play a role similar to the circular units, which generalize in the classical case to Stark units. The idea of considering Stark units indeed arose from investigations on log-algebraicity. A log-algebraicity result consists in the construction of a specific unit from the L-series of a Drinfeld module. The concept of log-algebraicity is due to D. Thakur and has been notably developed by G. Anderson in [And94, And96]. It has become a very lively topic in the current research. In a log-algebraicity statement, one in fact builds an element in $U(\widetilde{\phi})$, its evaluation at $z = 1$ is then always a Stark unit.

We can track this analogy in particular in Theorem 7.4.6 which states that the Fitting ideal of the quotient of $U(\phi)$ by the module of Stark units is equal to the Fitting ideal of the class module of ϕ.

The chapter is organized as follows. We start defining the basic notions involved in the theory of Drinfeld A-modules and introduce the tools which are necessary to state Taelman's class formula. The first three sections are meant to be self contained and present the general machinery of Stark units in the case of Drinfeld $\mathbb{F}_q[\theta]$-modules. This machinery has been generalized for Anderson A-modules with general A without difficulty. We invite the interested reader to [ANDTR17, ANDTR20a] for more details.

We present in Sect. 7.5 several class formulas and explain how Stark units appear in these formulas or can be computed from them.

We then turn to a slightly more general kind of objects, which are deformations of Drinfeld modules, in particular the multi-variable "canonical" deformations of the Carlitz module, which is canonical in the sense that the Carlitz module is deformed by its own shtuka function. This is a key object for arithmetic applications that we then review. First we show that the class module of the canonical deformation of the Carlitz module is, depending on the case, pseudo cyclic or pseudo null, which reminds of the classical Greenberg conjectures. Then we prove that, given a prime P, almost all Bernoulli-Carlitz numbers of a certain form do not vanish modulo P.

We finish with some words on Stark units in more general settings.

Some new proofs are given when possible and references are provided along the way. For the general references on Drinfeld modules, we refer the reader to [Gos96, Ros02, Tha04]. There are also obvious links between this survey and F. Pellarin's contribution [Pel20] to this volume, although the settings and notation might sometimes differ.

7.2 Background

After some notation, we present in this section the notions of Fitting ideals and ratios of covolumes which will be needed later, in particular in Sect. 7.5 to state class formulas.

7.2.1 Notation

We will use the following notation:

- $\mathbb{F}_q$: a finite field with q elements, of characteristic p,
- θ: an indeterminate over $\mathbb{F}_q$,
- $A = \mathbb{F}_q[\theta]$, $K = \mathbb{F}_q(\theta)$, $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$,
- v_∞: the valuation at the place ∞ such that $v_\infty(\theta) = -1$,
- $\mathbb{C}_\infty$: the completion of a fixed algebraic closure of K_∞,
- $\tau : \mathbb{C}_\infty \to \mathbb{C}_\infty, x \mapsto x^q$ the Frobenius endomorphism.

Note that K_∞ is the completion of K with respect to v_∞.

If k is a field containing $\mathbb{F}_q$, we set $(kK)_\infty = k\widehat{\otimes}_{\mathbb{F}_q} K_\infty = k((\frac{1}{\theta}))$. This is a field. If $x \in (kK)_\infty^\times$, we can write x uniquely as $x = \sum_{n\geq N} x_n \frac{1}{\theta^n}$, $x_n \in k$ with $x_N \neq 0$. Then we call $x_N \in k^\times$ the sign of x and write $\operatorname{sgn}(x) = x_N$. We say that such an $x \in (kK)_\infty$ is *monic* if $\operatorname{sgn}(x) = 1$. The valuation v_∞ extends naturally to $(kK)_\infty$ which is complete with respect to this valuation.

If L is a finite extension of K we denote by O_L the integral closure of A in L. We write $L_\infty = L \otimes_K K_\infty$ and if k is a field containing $\mathbb{F}_q$, $(kL)_\infty = L \otimes_K (kK)_\infty$. Note that $(kL)_\infty \simeq L_\infty$ when $k = \mathbb{F}_q$. As a finite dimensional $(kK)_\infty$-vector space, $(kL)_\infty$ is endowed with a natural topology. Moreover, O_{kL} or kO_L will denote the sub-k-vector space of $(kL)_\infty$ spanned by O_L. This is isomorphic to $k \otimes_{\mathbb{F}_q} O_L$.

The Frobenius homomorphism τ extends uniquely to a continuous homomorphism on $(kL)_\infty$ by putting $\tau(x) = x$ for all $x \in k$. We then have $\tau(O_{kL}) \subset O_{kL}$.

A case of particular interest in this text will be $k = \mathbb{F}_q(z)$ where z is a new indeterminate over $\mathbb{F}_q$. In this case, we will consider the Tate algebra

$$\mathbb{T}_z(L_\infty) := \left\{ \sum_{n\geq 0} a_n z^n \ ; \ a_n \in L_\infty, \lim_{n\to\infty} a_n = 0 \right\} \subset (\mathbb{F}_q(z)L)_\infty.$$

We have also the description $\mathbb{T}_z(K_\infty) \simeq \mathbb{F}_q[z][[\frac{1}{\theta}]]$ and more generally

$$\mathbb{T}_z(L_\infty) \simeq \mathbb{F}_q[z][[\frac{1}{\theta}]] \otimes_K L.$$

Remark that $\tau(\mathbb{T}_z(L_\infty)) \subset \mathbb{T}_z(L_\infty)$, and $\mathbb{T}_z(L_\infty) \cap O_{\mathbb{F}_q(z)L} = O_L[z]$.

It will be useful to also use the notation $(\mathbb{F}_q[z]L)_\infty = (L[z])_\infty = \mathbb{T}_z(L_\infty)$, and $O_{\mathbb{F}_q[z]L} = O_{L[z]} = O_L[z]$ so that if k denotes either $\mathbb{F}_q$, $\mathbb{F}_q(z)$ or $\mathbb{F}_q[z]$, then $(kL)_\infty$ stands respectively for L_∞, $(\mathbb{F}_q(z)L)_\infty$ or $\mathbb{T}_z(L_\infty)$, and O_{kL} for O_L, $O_{\mathbb{F}_q(z)L}$ or $O_L[z]$.

7.2.2 Fitting Ideals

In this section, we review basic facts on the theory of Fitting ideals. The standard references are the appendix to [MW84] and [Nor76, Eis95, Lan02].

We fix a commutative ring $\mathcal{R}$ and consider a finitely presented $\mathcal{R}$-module M. If for $a, b \in \mathbb{N}$,

$$\mathcal{R}^a \longrightarrow \mathcal{R}^b \longrightarrow M \longrightarrow 0$$

is a presentation of M, and if X is the matrix of the map $\mathcal{R}^a \to \mathcal{R}^b$ then one defines $\mathrm{Fitt}_M(\mathcal{R})$ to be the ideal of $\mathcal{R}$ generated by all the $b \times b$ minors of X if $b \leq a$, and $\mathrm{Fitt}_\mathcal{R}(M) = 0$ if $b > a$. This is independent from the presentation chosen for M. Note that if M is torsion, one has $b \leq a$.

In the case where $\mathcal{R}$ is a principal ideal domain (or more generally a Dedekind domain), the structure theorem asserts that if M is a torsion $\mathcal{R}$-module, then there exist ideals $I_1, \ldots, I_n$ of $\mathcal{R}$ such that M is isomorphic to the product $\mathcal{R}/I_1 \times \cdots \times \mathcal{R}/I_n$. This implies that $\mathrm{Fitt}_\mathcal{R}(M) = \prod_{i=1}^n I_i$. Fitting ideals are also multiplicative in exact sequences. That is, if $0 \to M_1 \to M \to M_2 \to 0$ is exact, then

$$\mathrm{Fitt}_\mathcal{R}(M_1) \cdot \mathrm{Fitt}_\mathcal{R}(M_2) = \mathrm{Fitt}_\mathcal{R}(M). \tag{7.1}$$

This can be deduced, for instance, from [Bou65, VII. §4 n.5 Proposition 10].

In the case where k is a field and $\mathcal{R} = k[\theta]$ we will denote by $[M]_{k[\theta]}$ the monic generator of $\mathrm{Fitt}_{k[\theta]}(M)$. Remark that in this case, there is a simple way to compute this quantity:

$$[M]_{k[\theta]} = \det_{k[Z]}(Z - \theta \mid M)\,|_{Z=\theta}. \tag{7.2}$$

We fix a field $k \supset \mathbb{F}_q$ such that $\mathbb{F}_q$ is algebraically closed in k. As an example, one can choose $k = \mathbb{F}_q(z)$. Let us write $\mathcal{R} = k[\theta]$. Let G be a finite abelian group whose order is prime to p. Let us denote by $\widehat{G} = \mathrm{Hom}(G, \overline{\mathbb{F}_q}^\times)$ the set of characters on G. For $\chi \in \widehat{G}$, we denote by $\mathbb{F}_q(\chi)$ the (finite, Galois) extension of $\mathbb{F}_q$ generated by the values of χ:

$$\mathbb{F}_q(\chi) := \mathbb{F}_q[\chi(g), g \in G].$$

And similarly,

$$k(\chi) := k[\chi(g), g \in G]$$

is the compositum of k and $\mathbb{F}_q[\chi]$ and is just isomorphic to $k \otimes_{\mathbb{F}_q} \mathbb{F}_q[\chi]$.

For $\chi \in \widehat{G}$, we define the idempotent

$$e_\chi := \frac{1}{|G|} \sum_{g \in G} \chi(g) g^{-1} \in \mathbb{F}_q(\chi)[G].$$

If $\chi \in \widehat{G}$, we also define:

$$[\chi] := \{\sigma \circ \chi \ , \ \sigma \in \mathrm{Gal}(\mathbb{F}_q(\chi)/\mathbb{F}_q)\} \subset \widehat{G}$$

and the corresponding idempotent:

$$e_{[\chi]} = \sum_{\psi \in [\chi]} e_\psi \in \mathbb{F}_q[G].$$

We define a map $e_{[\chi]}\mathbb{F}_q[G] \to \mathbb{F}_q(\chi)$ by associating, for $x \in \mathbb{F}_q[G]$, to $e_{[\chi]}x$ the unique $\lambda \in \mathbb{F}_q(\chi)$ such that $e_\chi x = \lambda e_\chi$ in $\mathbb{F}_q(\chi)[G]$. It is not hard to check that this is a well defined isomorphism, and thus it induces isomorphisms $e_{[\chi]}k[G] \to k(\chi)$ and $e_{[\chi]}\mathcal{R}[G] \to k(\chi)[\theta] = \mathcal{R}(\chi)$. Remark that the notion of a *monic element in* $e_{[\chi]}\mathcal{R}[G]$ is then well defined and does not depend on the choice of the representative χ of $[\chi]$.

Then, $\mathcal{R}[G]$ is the direct sum of its $[\chi]$-components $e_{[\chi]}\mathcal{R}[G]$. It is thus a principal ideal ring, and the notion of monic elements on each component leads to a natural notion of monic elements on $\mathcal{R}[G]$. Thus, if M is an $\mathcal{R}[G]$-module which is finite dimensional over k, then we can define $[e_{[\chi]}M]_{e_{[\chi]}\mathcal{R}[G]}$ for all character χ, and

$$[M]_{\mathcal{R}[G]} = \sum_\chi [e_{[\chi]}M]_{e_{[\chi]}\mathcal{R}[G]} \in e_{[\chi]}\mathcal{R}[G].$$

If M is now an $\mathcal{R}(\chi)[G]$ module which is finite dimensional over k, then we can define in a similar way $[e_\chi M]_{e_\chi \mathcal{R}(\chi)[G]} \in e_\chi \mathcal{R}(\chi)[G] = \mathcal{R}(\chi)e_\chi$. So if M is an $\mathcal{R}[G]$-module which is finite dimensional over k, we can set $M(\chi) := M \otimes_{\mathcal{R}} \mathcal{R}(\chi)$ and then we remark that:

$$\left[e_{[\chi]}M\right]_{e_{[\chi]}\mathcal{R}[G]} = \sum_{\psi \in [\chi]} \left[e_\psi M(\psi)\right]_{e_\psi \mathcal{R}[G]} \in \mathcal{R}[G].$$

If now M is a free $\mathcal{R}[G]$-module, then we also have the equality:

$$[M]_{\mathcal{R}[G]} = \det_{k[G][Z]} (Z - \theta \mid M) \mid_{Z=\theta}.$$

7.2.3 Ratio of Covolumes

We define here $k[\theta]$-lattices and the notion of ratio of covolumes which will be used to compare two lattices.

We fix k a field containing $\mathbb{F}_q$ and recall that $(kK)_\infty = k\widehat{\otimes}_{\mathbb{F}_q} K_\infty = k((\frac{1}{\theta}))$. In what follows, we fix V to be a finite dimensional $(kK)_\infty$-vector space endowed with the natural topology coming from $(kK)_\infty$.

Definition 7.2.1 A sub-$k[\theta]$-module M of V is a $k[\theta]$-*lattice* in V if M is discrete in V and if M generates V over $(kK)_\infty$.

Lemma 7.2.2 *Let M be a sub-$k[\theta]$-module of V. If M is discrete in V, then M is finitely generated over $k[\theta]$ and its rank is lower or equal to the dimension of V over $(kK)_\infty$. Equality holds if, and only if, M is a $k[\theta]$-lattice in V.*

Proof We choose a norm of $(kK)_\infty$-vector space on V. Let $e_1 \in M$ be an element of minimal norm among the non zero elements of M. Let d be the dimension of the $(kK)_\infty$-vector space generated by M. We build by induction a family $(e_1, \dots, e_d)$ of elements of M such that for $1 \leq i \leq d$, e_i is an element of minimal norm among the non zero elements of $M \setminus ((kK)_\infty e_1 \oplus \cdots \oplus (kK)_\infty e_{i-1})$. If $x \in M$, then there are $\lambda_1, \dots, \lambda_d \in (kK)_\infty$ such that $x = \sum_{i=1}^d \lambda_i e_i$. For $1 \leq i \leq d$, write $\lambda_i = \lambda_{i,0} + \lambda_{i,1}$ with $\lambda_{i,0} \in k[\theta]$ and $\lambda_{i,1} \in \frac{1}{\theta}k[\frac{1}{\theta}]$. Then

$$x - \sum_{i=1}^{d} \lambda_{i,0} e_i = \sum_{i=1}^{d} \lambda_{i,1} e_i \in M. \tag{7.3}$$

Let j be the maximal index, if it exists, for which $\lambda_{j,1} \neq 0$. Then (7.3) contradicts the minimality of e_j. We therefore must have $\lambda_{i,1} = 0$ for all i, and thus, $M = \bigoplus_{i=1}^d k[\theta]e_i$. We get the desired inequality.

This also proves that the dimension of the $(kK)_\infty$-vector space generated by M is the rank of M, whence the case of equality. □

As an immediate consequence, we can state:

Proposition 7.2.3 *Let M be a sub-$k[\theta]$-module of V. The following are equivalent:*

(i) M is a $k[\theta]$-lattice in V,
(ii) There exists a $(kK)_\infty$-basis $(e_1, \dots, e_n)$ of V such that M is the free $k[\theta]$-module of basis $(e_1, \dots, e_n)$,
(iii) M is discrete in V and its $k[\theta]$-rank is equal to the dimension of V over $(kK)_\infty$.

We can now proceed with the definition of ratio of co-volumes of lattices.

Let M and M' be two $k[\theta]$-lattices in V. Let $\mathcal{B}$ and $\mathcal{B}'$ be $k[\theta]$-bases of M and M', respectively. The ratio of co-volumes of M in M' is then defined as

$$\left[M' : M\right]_{k[\theta]} = \frac{\det_{\mathcal{B}'} \mathcal{B}}{\operatorname{sgn}(\det_{\mathcal{B}'} \mathcal{B})} \in (kK)_\infty.$$

Note that this is independent of the choices of $\mathcal{B}$ and $\mathcal{B}'$.

Remark 7.2.4

- The definition immediately implies that if M_0, M_1 and M_2 are lattices in V, then

$$[M_0 : M_1]_{k[\theta]} \, [M_1 : M_2]_{k[\theta]} = [M_0 : M_2]_{k[\theta]} \, .$$

- We also see that for two lattices M, M' in V, $\left[M' : M\right]_{k[\theta]} = \left[M : M'\right]_{k[\theta]}^{-1}$.

The two following results are also immediate:

Proposition 7.2.5 *Let M be a $k[\theta]$-lattice of V and u be a $(kK)_\infty$-automorphism of V. Then $u(M)$ is a lattice of V and*

$$[M : u(M)]_{k[\theta]} = \frac{\det u}{\operatorname{sgn}(\det u)}.$$

Proposition 7.2.6 *If M and M' are two $k[\theta]$-lattices of V and $M' \subset M$, then M/M' is a torsion $k[\theta]$-module and*

$$\left[M : M'\right]_{k[\theta]} = \left[M/M'\right]_{k[\theta]} .$$

Now let G be a finite abelian group whose order is prime to p. We suppose further that V is a free $(kK)_\infty[G]$-module. Write $\mathcal{R} = k[\theta]$. An $\mathcal{R}[G]$-lattice M in V is an $\mathcal{R}$-lattice in the $(kK)_\infty$-vector space V which is an $\mathcal{R}[G]$-submodule of V.

Let us fix a character $\chi \in \widehat{G}$. Then $e_{[\chi]}M$ is an $e_{[\chi]}\mathcal{R}[G]$ lattice in $e_{[\chi]}V$. Thus it makes sense to define for two $\mathcal{R}[G]$-lattices M and M' in V the ratio $\left[e_{[\chi]}M : e_{[\chi]}M'\right]_{e_{[\chi]}\mathcal{R}[G]}$. We then set

$$\left[M : M'\right]_{\mathcal{R}[G]} = \sum \left[e_{[\chi]}M : e_{[\chi]}M'\right]_{e_{[\chi]}\mathcal{R}[G]}$$

where the sum runs over the classes of characters $[\chi]$.

7.3 Drinfeld Modules

We review in this section the definition of Drinfeld modules and of the two fundamental associated maps: the exponential and the logarithm maps. We finish with the simplest example of a Drinfeld module, the Carlitz module, which allows some explicit computations. We also refer the reader to [Pel20, §3] where Drinfeld modules are presented for a general ring A.

7.3.1 Drinfeld Modules

In what follows, we fix $k = \mathbb{F}_q$ or $k = \mathbb{F}_q(z)$ and $R = kA$, that is, $R = A = \mathbb{F}_q[\theta]$ or $R = \mathbb{F}_q(z)[\theta]$. Let L be a finite extension of K. We write $S = O_{kL}$, that is $S = O_L$ if $R = A$ and $S = O_{\mathbb{F}_q(z)L}$ otherwise. We recall that S is endowed with the Frobenius homomorphism τ.

Definition 7.3.1 A *Drinfeld R-module* defined over S is a k-algebra homomorphism $\phi : R \to S[\tau]; a \mapsto \phi_a$ such that $\phi_a \equiv a \pmod{S[\tau]\tau}$ for all $a \in A$.

We remark that the data of ϕ_θ is sufficient to define the Drinfeld module ϕ. In particular, a Drinfeld A-module over O_L extends naturally to a Drinfeld $\mathbb{F}_q(z)[\theta]$-module over $O_{\mathbb{F}_q(z)L}$.

The degree $\deg_\tau \phi_\theta$ is called the *rank* of ϕ.

Example 7.3.2 We do not exclude the rank 0 case. In this case the Drinfeld module is the trivial map $\phi : a \mapsto \phi_a = a$.

Example 7.3.3 The *Carlitz module* is the Drinfeld A-module C over A defined by $C_\theta = \theta + \tau$. It is of rank 1. See Sect. 7.3.3 below.

Definition 7.3.4 Let ϕ be a Drinfeld A-module over O_L given by $\phi_\theta = \sum_{i=0}^n a_i \tau^i$ with $a_i \in O_L$. The *z-twist* of ϕ is the Drinfeld $\mathbb{F}_q(z)[\theta]$-module $\widetilde{\phi}$ over $O_{\mathbb{F}_q(z)L}$ given by $\widetilde{\phi}_\theta = \sum_{i=0}^n a_i z^i \tau^i$ and extended by $\mathbb{F}_q(z)$-linearity for any $a \in \mathbb{F}_q(z)[\theta]$.

If M is an $S[\tau]$-module and ϕ is a Drinfeld R-module over S, then ϕ induces a structure of R-module on M via $(a, m) \in R \times M \mapsto \phi_a(m)$. We then write $\phi(M)$ for the R-module M considered with this structure of R-module.

7.3.2 Exponential and Logarithm

We keep the notation of the previous section. Let ϕ be a Drinfeld kA-module over O_{kL}.

Let M be a finitely generated and free $(kL)_\infty$-module equipped with a semi-linear map τ, that is:

$$\forall a \in (kL)_\infty,\ \forall m \in M,\ \tau(a.m) = \tau(a).\tau(m).$$

We call such a module a *τ-module over* $(kL)_\infty$. It is in particular a finite dimensional $(kK)_\infty$-vector space, and all norms of $(kK)_\infty$-vector space on M are equivalent.

Proposition 7.3.5 *There exists a unique series* $\exp_\phi = \sum_{i\geq 0} e_i\tau^i \in kL[[\tau]]$ *such that:*

(i) $e_0 = 1$,
(ii) $\exp_\phi a = \phi_a \exp_\phi$ *holds in* $kL[[\tau]]$ *for all* $a \in A$.

Moreover, if $\|\cdot\|$ *is a norm of* $(kK)_\infty$*-vector spaces over* $(kL)_\infty$*, then*

$$\lim_{n\to\infty} \|e_n\|^{q^{-n}} = 0.$$

As a consequence, if M *is a* τ*-module over* $(kL)_\infty$*, then* $\exp_\phi$ *defines a function which converges everywhere on* M.

Proof We refer the reader to [And86, Proposition 2.1.4] for a proof of this classical result. Since this will be useful later on, we give a short proof of the last assertion: $\exp_\phi$ converges on the whole M.

We fix a norm $\|\cdot\|$ of $(kK)_\infty$-vector spaces on M. From the identification $(kK)_\infty \simeq k((\frac{1}{\theta}))$, we see that for all $x \in (kK)_\infty$, we have $|\tau(x)| \leq |x|^q$. Thus, since M is finite dimensional over $(kK)_\infty$, there exists some constant $\alpha \geq 1$ such that for all $x \in M$, $\|\tau(x)\| \leq \alpha\|x\|^q$. Thus for all $x \in M$ and all $n \geq 1$, we have: $\|\tau^n(x)\| \leq \alpha^{\frac{q^n-1}{q-1}}\|x\|^{q^n} \leq (\alpha\|x\|)^{q^n}$. Thus for all n,

$$\|e_n\tau^n(x)\| \leq \left(\|e_n\|^{q^{-n}}\alpha\|x\|\right)^{q^n} \tag{7.4}$$

which concludes the proof. □

We call $\exp_\phi$ the *exponential map* associated to the Drinfeld module ϕ.

Corollary 7.3.6 *If* M *is a* τ*-module over* $(kL)_\infty$*, then the exponential map* $\exp_\phi : M \to M$ *is locally an isometry.*

Proof We use the same notation as in the previous proof. Let us write $m = \max_n \|e_n\|^{q^{-n}}$. From Inequality (7.4), we get that for all $n \geq 1$, and for all $x \in M$ such that $\|x\| \leq (m\alpha)^{-1}$,

$$\|e_n\tau^n(x)\| \leq (m\alpha\|x\|)^q .$$

Thus, if $\|x\| < \min\left((m\alpha)^{-1}, (m\alpha)^{\frac{q}{1-q}}\right)$, and for all $n \geq 1$, $\|e_n\tau^n(x)\| < \|x\|$. It implies that $\|\exp_\phi(x)\| = \|x\|$. The proof is finished. □

Proposition 7.3.7 *There exists a unique series* $\log_\phi = \sum_{i\geq 0} l_i\tau^i \in kL[[\tau]]$ *such that:*

(i) $l_0 = 1$,
(ii) $\log_\phi \phi_a = a\log_\phi$ *holds in* $kL[[\tau]]$ *for all* $a \in A$.

Moreover, we have $\exp_\phi \log_\phi = \log_\phi \exp_\phi = 1$ *in* $kL[[\tau]]$ *and if* $\|\cdot\|$ *is a norm of* $(kK)_\infty$*-modules over* $(kL)_\infty$*, then* $\|l_n\|^{q^{-n}}$ *is bounded. As a consequence, if* M *is a* τ*-module over* $(kL)_\infty$*, then* $\log_\phi$ *converges on a neighborhood of* 0 *in* M*.*

Proof The construction of $\log_\phi$ is standard: if it exists, then we must have $\exp_\phi \log_\phi = \log_\phi \exp_\phi = 1$. So it can be obtained as the inverse series (in τ) of the exponential map, and this gives both (i) and (ii). Note that it can also be constructed directly by solving the equation $\log_\phi \phi_\theta = \theta \log_\phi$.

Let $m = \max(1, \max_n \|e_n\|)$. We prove by induction that for all n, $\|l_n\| \leq m^{q^n}$. The case $n = 0$ is trivial. The inequality

$$\|l_n\| = \| - \sum_{i=0}^{n-1} l_i e_{n-i}^{q^i}\| \leq \max_i \|l_i e_{n-i}^{q^i}\| \leq \max_{i \leq n-1} m^{2q^i} \leq m^{q^n}$$

concludes the proof. □

We call $\log_\phi$ the *logarithm map* associated to the Drinfeld module ϕ.

Corollary 7.3.8 *The logarithm map* $\log_\phi$ *is an isometry on a neighborhood of* 0*.*

Proof The proof can be done along the same lines as that of Corollary 7.3.6. It is also a consequence of the fact that the logarithm map is formally an inverse map of the exponential map, that it converges on a neighborhood of 0 and that the exponential map is locally an isometry. □

If ϕ is a Drinfeld A-module over O_L, and $\widetilde{\phi}$ denotes its z-twist, then we have $\exp_{\widetilde{\phi}}(\mathbb{T}_z(L_\infty)) \subset \mathbb{T}_z(L_\infty)$, and if $x \in \mathbb{T}_z(L_\infty)$ and $\log_{\widetilde{\phi}}(x)$ converges in $(\mathbb{F}_q(z)L)_\infty$, then it converges in $\mathbb{T}_z(L_\infty)$. Thus Corollary 7.3.6 and 7.3.8 remain true on $\mathbb{T}_z(L_\infty)$.

7.3.3 The Carlitz Module

The Carlitz module is often considered as the first case of a Drinfeld module, and we can make a lot of the constructions completely explicit here. We give a short overview of these explicit constructions and refer the reader to [Pel20, §4] or, for instance, to [Gos96, §3] for more details.

Let us recall that the Carlitz module is the Drinfeld A-module C over A defined by $C_\theta = \theta + \tau$. We define $D_0 = 1$, and for $i \geq 1$, $D_{i+1} = D_i^q(\theta^{q^{i+1}} - \theta)$, so that $v_\infty(D_i) = -iq^i$. Then the exponential map associated to C is $\exp_C = \sum_{i \geq 0} \frac{1}{D_i}\tau^i$. Similarly, if $l_0 = 1$, and for $i \geq 1$, $l_{i+1} = l_i(\theta - \theta^{q^i})$, then $\log_C = \sum_{i \geq 0} \frac{1}{l_i}\tau^i$. The kernel of $\exp_C : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a rank one A-module. One can give an explicit description of a generator of this kernel as

$$\widetilde{\pi} := (-\theta)^{\frac{1}{q-1}} \theta \prod_{j \geq 1} \frac{1}{1 - \theta^{1-q^j}} \tag{7.5}$$

where $(-\theta)^{\frac{1}{q-1}}$ is a fixed $q-1$-st root of $-\theta$ in $\mathbb{C}_\infty$. We call $\widetilde{\pi}$ "the" period of the Carlitz module (uniquely determined up to $\mathbb{F}_q^\times$).

7.4 Stark Units

We come to the definition of the Stark units. We first review Taelman's class and unit modules. Then we will be able to define the module of Stark units which is a submodule of the unit module. The section ends with some words on Anderson's [And94] which inspired the notion of Stark units.

7.4.1 Taelman Modules

We define here the class module and the unit module of a Drinfeld module as introduced by L. Taelman in [Tae12].

Let L/K be a finite extension and let ϕ denote a Drinfeld A-module over O_L. We define the *unit module* of ϕ to be

$$U(\phi; O_L) = \left\{ x \in L_\infty, \exp_\phi(x) \in O_L \right\}$$

and the *class module* of ϕ to be

$$H(\phi; O_L) = \frac{\phi(L_\infty)}{\phi(O_L) + \exp_\phi(L_\infty)}.$$

Since $\exp_\phi$ is a homomorphism of A-modules, those are naturally A-modules.

We also write $\widetilde{\phi}$ for the z-twist of ϕ and define the corresponding Taelman modules:

$$U(\widetilde{\phi}; O_{\mathbb{F}_q(z)L}) = \left\{ x \in (\mathbb{F}_q(z)L)_\infty, \exp_{\widetilde{\phi}}(x) \in O_{\mathbb{F}_q(z)L} \right\}$$

and

$$H(\widetilde{\phi}; O_{\mathbb{F}_q(z)L}) = \frac{\widetilde{\phi}((\mathbb{F}_q(z)L)_\infty)}{\widetilde{\phi}(O_{\mathbb{F}_q(z)L}) + \exp_{\widetilde{\phi}}((\mathbb{F}_q(z)L)_\infty)}.$$

And finally, at the "integral" level, we define:

$$U(\widetilde{\phi};\, O_L[z]) = \left\{x \in \mathbb{T}_z(L_\infty),\, \exp_{\widetilde{\phi}}(x) \in O_L[z]\right\}$$

and

$$H(\widetilde{\phi};\, O_L[z]) = \frac{\widetilde{\phi}(\mathbb{T}_z(L_\infty))}{\widetilde{\phi}(O_L[z]) + \exp_{\widetilde{\phi}}(\mathbb{T}_z(L_\infty))}.$$

We fix from now on a Drinfeld A-module ϕ over O_L and write $k = \mathbb{F}_q, \mathbb{F}_q(z)$ or $\mathbb{F}_q[z]$ and $\varphi = \phi$ in the first case and $\varphi = \widetilde{\phi}$ otherwise.

Proposition 7.4.1

1. *The class module $H(\varphi;\, O_{kL})$ is finitely generated over k, thus a finitely generated and torsion kA-module.*
2. *Suppose that $k = \mathbb{F}_q$ or $k = \mathbb{F}_q(z)$. The unit module $U(\varphi;\, O_{kL})$ is a kA-lattice in $(kL)_\infty$.*

Proof We use the proof of [Dem14, Proposition 2.6].

For Part 1, since $\exp_\varphi$ is locally an isometry on $(kL)_\infty$, we can find a neighborhood V of 0 such that $\exp_\varphi$ is an isometry on V, $\exp_\varphi(V) = V$ and $V \cap O_{kL} = \{0\}$. We remark that $\frac{(kL)_\infty}{O_{kL}+V}$ is finitely generated over k. But we have a surjection $\frac{(kL)_\infty}{O_{kL}+V} \twoheadrightarrow H(\varphi;\, O_{kL})$ so that $H(\varphi;\, O_{kL})$ is also finitely generated.

For Part 2, since $\exp_\varphi$ is locally an isometry, we get that $U(\varphi;\, O_{kL})$ is discrete in $(kL)_\infty$. The exponential map induces a short exact sequence of kA-modules:

$$0 \longrightarrow \frac{(kL)_\infty}{U(\varphi;\, O_{kL}) + V} \longrightarrow \frac{\varphi((kL)_\infty)}{\varphi(O_{kL}) + V} \longrightarrow H(\varphi;\, O_{kL}) \longrightarrow 0.$$

Since the vector space in the middle is finite dimensional over k, then so is the first one. If $U(\varphi;\, O_{kL})$ did not generate $(kL)_\infty$ over $(kK)_\infty$, we could find $x \in (kL)_\infty$ such that $(kK)_\infty U(\varphi;\, O_{kL}) \cap (kK)_\infty x = \{0\}$. But, there is an injection $O_{kL} \hookrightarrow \frac{(}{k}L)_\infty V$, and $\frac{(kL)_\infty}{U(\varphi;O_{kL})+V}$ is the cokernel of the natural map $U(\varphi;\, O_{kL}) \to \frac{(}{k}L)_\infty V$. We deduce that the kA-ranks of O_{kL} and $U(\varphi;\, O_{kL})$ must coincide. Thus $U(\varphi;\, O_{kL})$ is a lattice in $(kL)_\infty$. □

Proposition 7.4.2 *We have:*

1. $U(\widetilde{\phi};\, O_{\mathbb{F}_q(z)L}) = \mathbb{F}_q(z)U(\widetilde{\phi};\, O_L[z]) \subset (\mathbb{F}_q(z)L)_\infty$,
2. $H(\widetilde{\phi};\, O_{\mathbb{F}_q(z)L}) \simeq \mathbb{F}_q(z) \otimes_{\mathbb{F}_q[z]} H(\widetilde{\phi};\, O_L[z])$.

Proof For Part 1, we mimic the proof of [APTR16, Proposition 5.4].

The inclusion $\mathbb{F}_q(z)U(\widetilde{\phi};\, O_L[z]) \subset U(\widetilde{\phi};\, O_{\mathbb{F}_q(z)L})$ is clear.

We have that $\mathbb{F}_q(z)\mathbb{T}_z(L_\infty)$ is dense in $(\mathbb{F}_q(z)L)_\infty$. We fix a neighborhood V of 0 in $\mathbb{T}_z(L_\infty)$ such that $\exp_{\widetilde{\phi}}(V) = V$. We write V' for the closure of

$\mathbb{F}_q(z)V$ in $(\mathbb{F}_q(z)L)_\infty$. We still have $\exp_{\widetilde{\phi}}(V') = V'$. We then have $(\mathbb{F}_q(z)L)_\infty = \mathbb{F}_q(z)\mathbb{T}_z(L_\infty) + V'$. Let $f \in U(\widetilde{\phi}; O_{\mathbb{F}_q(z)L})$. We can write $f = g + h$ with $g \in \mathbb{F}_q(z)\mathbb{T}_z(L_\infty)$ and $h \in V'$. We get:

$$\exp_{\widetilde{\phi}}(h) = \exp_{\widetilde{\phi}}(f) - \exp_{\widetilde{\phi}}(g) \in \left(O_{\mathbb{F}_q(z)L} + \mathbb{F}_q(z)\mathbb{T}_z(L_\infty)\right) \cap V'.$$

But

$$\left(O_{\mathbb{F}_q(z)L} + \mathbb{F}_q(z)\mathbb{T}_z(L_\infty)\right) \cap V' = \mathbb{F}_q(z)\mathbb{T}_z(L_\infty) \cap V' = \mathbb{F}_q(z)V.$$

Thus, $h \in \mathbb{F}_q(z)V$ and $f \in \mathbb{F}_q(z)\mathbb{T}_z(L_\infty)$. This proves Part 1.

Part 2 is a consequence of the fact that $\mathbb{F}_q(z)\mathbb{T}_z(L_\infty)$ is dense in $(\mathbb{F}_q(z)L)_\infty$ and $\exp_{\widetilde{\phi}}$ is locally an isometry. □

Proposition 7.4.3 *The $A[z]$-module $H(\widetilde{\phi}; O_L[z])$ is a finitely generated and torsion $\mathbb{F}_q[z]$-module, with no z-torsion.*

Proof We copy the proof of [ATR17, Proposition 2].

By Proposition 7.4.1, $H(\phi; O_L[z])$ is finitely generated over $\mathbb{F}_q[z]$. Since $\exp_{\widetilde{\phi}} \equiv 1 \pmod{L[z][[\tau]]z\tau}$, we get:

$$\mathbb{T}_z(L_\infty) = z\mathbb{T}_z(L_\infty) + \exp_{\widetilde{\phi}}(\mathbb{T}_z(L_\infty)).$$

We deduce that the multiplication by z is surjective on $H(\widetilde{\phi}; O_L[z])$. Thus, if we denote by $H(\widetilde{\phi}; O_L[z])[z]$ the z-torsion of $H(\widetilde{\phi}; O_L[z])$, the multiplication by z induces an exact sequence of finitely generated $\mathbb{F}_q[z]$-modules:

$$0 \longrightarrow H(\widetilde{\phi}; O_L[z])[z] \longrightarrow H(\widetilde{\phi}; O_L[z]) \longrightarrow H(\widetilde{\phi}; O_L[z]) \longrightarrow 0.$$

By the structure theorem for finitely generated modules over $\mathbb{F}_q[z]$, this implies that $H(\widetilde{\phi}; O_L[z])[z] = 0$ and that $H(\widetilde{\phi}; O_L[z])$ is a torsion $\mathbb{F}_q[z]$-module. □

Corollary 7.4.4 *The class module $H(\widetilde{\phi}; O_{\mathbb{F}_q(z)L})$ vanishes.*

Proof This is a consequence of the previous proposition and Proposition 7.4.2. □

7.4.2 The Module of Stark Units

We define here the module of Stark units, and compute its covolume in the unit module.

We keep the notation of Sect. 7.4.1. The evaluation $z \mapsto 1$ induces a map $\mathrm{ev} : \mathbb{T}_z(L_\infty) \to L_\infty$.

Definition 7.4.5 The *module of Stark units* is defined as:

$$U_{\mathrm{St}}(\phi; O_L) = \mathrm{ev}\left(U(\widetilde{\phi}, O_L[z])\right).$$

We observe that $U_{\mathrm{St}}(\phi; O_L) \subset U(\phi; O_L)$. We will now prove the following theorem by using the proof of [ATR17, Theorem 1] or [ANDTR17, Proposition 2.7].

Theorem 7.4.6 *The A-module $U_{\mathrm{St}}(\phi; O_L)$ is an A-lattice in L_∞ and*

$$\left[\frac{U(\phi; O_L)}{U_{\mathrm{St}}(\phi; O_L)}\right]_A = [H(\phi; O_L)]_A.$$

We introduce a map on L_∞:

$$\alpha : \begin{cases} L_\infty \to & \mathbb{T}_z(L_\infty) \\ x \mapsto & \frac{\exp_{\widetilde{\phi}}(x) - \exp_\phi(x)}{z-1}. \end{cases}$$

The map is well defined since $\mathrm{ev}(\exp_{\widetilde{\phi}}(x)) = \exp_\phi(x)$ so that $z - 1$ divides $\exp_{\widetilde{\phi}}(x) - \exp_\phi(x)$ in $\mathbb{T}_z(L_\infty)$.

Proposition 7.4.7 *The map α induces an isomorphism of A-modules:*

$$\overline{\alpha} : \frac{U(\phi, O_L)}{U_{\mathrm{St}}(\phi; O_L)} \simeq H(\widetilde{\phi}; O_L[z])[z-1]$$

where $H(\widetilde{\phi}; O_L[z])[z-1]$ is the $(z-1)$-torsion of $H(\widetilde{\phi}; O_L[z])$.

Proof Let us first show that $\alpha : U(\phi, O_L) \to H(\phi; \mathbb{T}_z(L_\infty))$ is a homomorphism of A-modules. Let $x \in U(\phi, O_L)$ and $a \in A$. Write $\phi_a = \sum_{i=0}^n a_i \tau^i$ with $a_i \in O_L$. Thus,

$$\begin{aligned} \alpha(ax) &= \frac{\exp_{\widetilde{\phi}}(ax) - \exp_\phi(ax)}{z-1} \\ &= \frac{\widetilde{\phi}_a(\exp_{\widetilde{\phi}}(x)) - \phi_a(\exp_\phi(x))}{z-1} \\ &= \widetilde{\phi}_a(\alpha(x)) + \sum_{i=0}^n a_i \frac{z^i - 1}{z-1} \tau^i(\exp_\phi(x)) \end{aligned}$$

and this equals $\widetilde{\phi}_a(\alpha(x))$ in $H(\widetilde{\phi}; O_L[z])$ since $\exp_\phi(x) \in O_L$.

We now prove that the image of $U(\phi, O_L)$ in $H(\widetilde{\phi}; O_L[z])$ through α lies in $H(\widetilde{\phi}; O_L[z])[z-1]$.

Let $x \in U(\phi; O_L)$. We then have

$$(z-1)\alpha(x) = \exp_{\widetilde{\phi}}(x) - \exp_\phi(x) \in \exp_\phi(\mathbb{T}_z(L_\infty)) + O_L[z]$$

so that it vanishes in $H(\widetilde{\phi}; O_L[z])$.

We now show that α is surjective on $H(\widetilde{\phi}; O_L[z])[z-1]$.

Let $x \in \mathbb{T}_z(L_\infty)$ be such that its image in $H(\widetilde{\phi}; O_L[z])$ lies in $H(\widetilde{\phi}; O_L[z])[z-1]$. Thus, $(z-1)x \in \exp_{\widetilde{\phi}}(\mathbb{T}_z(L_\infty)) + O_L[z]$. Write $(z-1)x = \exp_{\widetilde{\phi}}(u) + v$ with $u \in \mathbb{T}_z(L_\infty)$ and $v \in O_L[z]$. Write $u = u_1 + (z-1)u_2$ with $u_1 \in L_\infty$ and $u_2 \in \mathbb{T}_z(L_\infty)$ and $v = v_1 + (z-1)v_2$ with $v_1 \in O_L$ and $v_2 \in O_L[z]$. Then we have

$$(z-1)x = \exp_{\widetilde{\phi}}(u_1) + v_1 + (z-1)(\exp_{\widetilde{\phi}}(u_2) + v_2)$$

so that, by evaluating at $z = 1$, we get $\exp_\phi(u_1) + v_1 = 0$. Thus $u_1 \in U(\phi; O_L)$. Moreover, we get:

$$\begin{aligned} \alpha(u_1) &= \frac{\exp_{\widetilde{\phi}}(u_1) - \exp_\phi(u_1)}{z-1} \\ &= \frac{\exp_{\widetilde{\phi}}(u_1) + v_1}{z-1} \\ &= x - \exp_{\widetilde{\phi}}(u_2) + v_2 \end{aligned}$$

so that the images of $\alpha(u_1)$ and x in $H(\widetilde{\phi}; O_L[z])$ coincide.

We claim that the kernel κ of $\alpha : U(\phi; O_L) \to H(\widetilde{\phi}; O_L[z])$ equals $U_{\mathrm{St}}(\phi; O_L)$. We start with the inclusion $U_{\mathrm{St}}(\phi; O_L) \subset \kappa$.

Let $x \in U_{\mathrm{St}}(\phi; O_L)$, it is the evaluation at $z = 1$ of some $u \in U(\widetilde{\phi}; O_L[z])$, so there exists $v \in \mathbb{T}_z(L_\infty)$ such that $x = u + (z-1)v$. Thus

$$\alpha(x) = \frac{\exp_{\widetilde{\phi}}(u) - \exp_\phi(x)}{z-1} + \exp_{\widetilde{\phi}}(v)$$

but $\exp_\phi(x)$ is the evaluation at $z = 1$ of $\exp_{\widetilde{\phi}}(u) \in O_L[z]$. Thus $\alpha(x) \in O_L[z] + \exp_\phi(\mathbb{T}_z(L_\infty))$.

Lastly, we show the other inclusion: $\kappa \subset U_{\mathrm{St}}(\phi; O_L)$. Let $x \in U(\phi; O_L)$ be such that $\alpha(x)$ vanishes in $H(\widetilde{\phi}; O_L[z])$, that is, $\alpha(x) \in O_L[z] + \exp_{\widetilde{\phi}}(\mathbb{T}_z(L_\infty))$. Thus $(z-1)\alpha(x) = \exp_{\widetilde{\phi}}(x) - \exp_\phi(x) = (z-1)u + \exp_{\widetilde{\phi}}((z-1)v)$ for some $u \in O_L[z]$ and $v \in \mathbb{T}_z(L_\infty)$. Thus $x - (z-1)v \in U(\widetilde{\phi}; O_L[z])$ and its evaluation at $z = 1$ is x, that is, $x \in U_{\mathrm{St}}(\phi; O_L)$. □

Proposition 7.4.8 *We have:*

$$\left[H(\widetilde{\phi}; O_L[z])[z-1]\right]_A = [H(\phi; O_L)]_A \,.$$

Proof The evaluation map ev induces an exact sequence of A-modules:

$$0 \longrightarrow (z-1)H(\widetilde{\phi};\, O_L[z]) \longrightarrow H(\widetilde{\phi};\, O_L[z]) \longrightarrow H(\phi;\, O_L) \longrightarrow 0$$

from which we get the exact sequence of finitely generated k-vector spaces

$$0 \to H(\widetilde{\phi};\, O_L[z])[z-1] \to H(\widetilde{\phi};\, O_L[z]) \xrightarrow{z-1} H(\widetilde{\phi};\, O_L[z]) \to H(\phi;\, O_L) \to 0.$$

By (7.1), the multiplicativity of the Fitting ideal in exact sequences, we obtain

$$\left[H(\widetilde{\phi};\, O_L[z])[z-1]\right]_A = [H(\phi;\, O_L)]_A\,.$$

□

Proof of Theorem 7.4.6 It only remains to show that $U_{\mathrm{St}}(\phi;\, O_L)$ is an A-lattice. It is a direct consequence of the fact that $\frac{U(\phi;\, O_L)}{U_{\mathrm{St}}(\phi;\, O_L)}$ is a finite dimensional $\mathbb{F}_q$-vector space. □

Let now E/L be a finite abelian extension of degree prime to p and let $G = \mathrm{Gal}(E/L)$. Then $U(\phi;\, O_E)$ and $U_{\mathrm{St}}(\phi;\, O_E)$ are both $A[G]$-lattices in $E_\infty = E \otimes_K K_\infty$ and $H(\phi;\, O_E)$ is naturally an $A[G]$-module. We remark that the map $\overline{\alpha}$ of Proposition 7.4.7 is G-equivariant, so that the equivalent of Theorem 7.4.6 remains true in the equivariant setting:

Proposition 7.4.9 *We have*

$$\left[\frac{U(\phi;\, O_E)}{U_{\mathrm{St}}(\phi;\, O_E)}\right]_{A[G]} = [H(\phi;\, O_E)]_{A[G]}\,.$$

An example will be given in Theorem 7.5.10 below in the context of the equivariant class formula.

7.4.3 Link with Anderson's Special Points

Let us finish this section with a few words on the origin of the notion of Stark Units. This notion grew up from attempts to understand the fundamental work [And94] of Anderson. Following Thakur, Anderson considers the formal power series for integers $m \geq 0$:

$$l_m(X, Z) := \sum_{a \in A \text{ monic}} \frac{C_a(X)^m}{a} Z^{q^{\deg a}} \in K[X][[Z]]$$

where τ acts on X and Z via $\tau(X) = X^q$ and $\tau(Z) = Z^q$. He shows [And94, Theorem 3] the following log-*algebraicity* result:

$$S_m(X, Z) := \exp_C(l_m(X, Z)) \in A[X, Z].$$

Let us fix now a monic irreducible polynomial $P \in A$ of degree d and define $\lambda := \exp_C(\frac{\widetilde{\pi}}{P})$. Then $L = K(\lambda)$ is the "cyclotomic" extension associated with P. We refer the reader to [Ros02, Chapter 12] for more details on this extension. Anderson considers the A-submodule $\mathcal{S}$ of $C(O_L)$ generated by $S_m(\lambda, 1)$ for all $m \geq 0$. He (see [And94, §4.5]) calls $\mathcal{S}$ the *module of special points* and remarks that the special points play a role analogue to the circular units in the classical setting of cyclotomic fields.

It turns out that those special elements are just the images under the exponential map of what we called Stark units. More precisely (see [AT15, §7, in particular Theorem 7.5]):

$$\mathcal{S} = \exp_C(U_{\mathrm{St}}(C; O_L)).$$

Stark units are therefore a generalization of the analogue of circular units for the Carlitz module, which explains their name.

7.5 Class Formulas

This section is devoted to class formulas: the original Taelman class formula from [Tae12] and some generalizations, in particular in the equivariant setting. We also give some explicit examples.

In what follows, we keep considering a finite extension L of K and a Drinfeld $\mathbb{F}_q[\theta]$-module ϕ defined over O_L.

7.5.1 Taelman's Class Formula

We present Taelman's class formula and how it can be expressed in terms of the regulator of Stark units.

Let I be a non-zero ideal of O_L. Then O_L/IO_L is a finite dimensional $\mathbb{F}_q$-vector space. Since $\tau(I) \subset I$, it makes sense to define both $[O_L/IO_L]_A$ and $[\phi(O_L/IO_L)]_A$.

Remark that the first one is easy to compute:

Lemma 7.5.1 *Let I be a non-zero ideal of O_L and denote by $N_{L/K}$ the norm map from the ideals of O_L to the ones of A. Then $[O_L/IO_L]_A$ is the monic generator of $N_{L/K}(I)$.*

Proof The equality $\mathrm{Fitt}_A(O_L/IO_L) = N_{L/K}(I)$ is immediate from the definitions. □

If $\mathfrak{P}$ is a prime ideal of O_L, the Euler factor at $\mathfrak{P}$ is then the quotient $[O_L/\mathfrak{P}O_L]_A \,/\, [\phi(O_L/\mathfrak{P}O_L)]_A$. By putting together all these local factors, we obtain the L-series:

$$L(\phi/O_L) := \prod_{\mathfrak{P}} \frac{[O_L/\mathfrak{P}O_L]_A}{[\phi(O_L/\mathfrak{P}O_L)]_A} \tag{7.6}$$

where the product runs over all the non-zero prime ideals of O_L.

Lemma 7.5.2 *Let I be a non-zero ideal of O_L. Let $n \geq 1$. Then:*

$$\begin{aligned}\left[O_L/I^n O_L\right]_A \cdot [\phi(O_L/IO_L)]_A \\ = \left[\phi(O_L/I^n O_L)\right]_A \cdot [O_L/IO_L]_A .\end{aligned}$$

Proof We prove this equality by induction on n. The case $n = 1$ is clear. The short exact sequence

$$0 \to I^n O_L/I^{n+1} O_L \to O_L/I^{n+1} O_L \to O_L/I^n O_L \to 0$$

gives

$$\left[O_L/I^{n+1} O_L\right]_A = \left[O_L/I^n O_L\right]_A \cdot \left[I^n O_L/I^{n+1} O_L\right]_A .$$

Similarly, we have the short exact sequence

$$0 \to \phi(I^n O_L/I^{n+1} O_L) \to \phi(O_L/I^{n+1} O_L) \to \phi(O_L/I^n O_L) \to 0$$

but for any $x \in I^n O_L$, $a \in A$, $\phi_a(x) \equiv ax \pmod{I)^{q^n} O_L}$, thus

$$\phi(I^n O_L/I^{n+1} O_L) \simeq I^n O_L/I^{n+1} O_L,$$

so that

$$\left[\phi(O_L/I^{n+1} O_L)\right]_A = \left[\phi(O_L/I^n O_L)\right]_A \cdot \left[I^n O_L/I^{n+1} O_L\right]_A .$$

Putting altogether we get the desired result. □

The previous lemma, together with the Chinese Remainder Theorem allows to write the L-series as:

$$L(\phi/O_L) := \prod_{P} \frac{[O_L/PO_L]_A}{[\phi(O_L/PO_L)]_A} \tag{7.7}$$

where the product runs over all the monic irreducible polynomials P of A. In this form, the numerator is also very easy to compute:

$$[O_L/PO_L]_A = P^{[L:K]}.$$

The main result of [Tae12] is the following class formula:

Theorem 7.5.3 (Taelman) *The product defining $L(\phi/O_L)$ converges in K_∞, and the following equality holds:*

$$L(\phi/O_L) = [O_L : U(\phi; O_L)]_A \, [H(\phi; O_L)]_A \, .$$

Corollary 7.5.4 *We have:*

$$L(\phi/O_L) = [O_L : U_{\mathrm{St}}(\phi; O_L)]_A \, .$$

Proof This is immediate from Taelman's class formula and Theorem 7.4.6. □

The co-volume of the Taelman units or the Stark units in O_L is very similar to the classical notion of a regulator, so that the previous corollary can nicely translate as: the L-value attached to ϕ is the regulator of its module of Stark units.

Remark that, as in (7.6), we can also define the z-twisted version of the L-series:

$$L(\widetilde{\phi}/O_{\mathbb{F}_q(z)L}) := \prod_{\mathfrak{P}} \frac{\left[O_{\mathbb{F}_q(z)L}/\mathfrak{P}O_{\mathbb{F}_q(z)L}\right]_{\mathbb{F}_q(z)A}}{\left[\widetilde{\phi}(O_{\mathbb{F}_q(z)L}/\mathfrak{P}O_{\mathbb{F}_q(z)L})\right]_{\mathbb{F}_q(z)A}}$$

where the product runs over all the non-zero prime ideals of O_L. Here again, the numerator of the local factor at $\mathfrak{P}$ is

$$\left[O_{\mathbb{F}_q(z)L}/\mathfrak{P}O_{\mathbb{F}_q(z)L}\right]_{\mathbb{F}_q(z)A} = N_{L/K}(\mathfrak{P}).$$

And, similarly to (7.7), we have the alternative expression:

$$L(\widetilde{\phi}/O_{\mathbb{F}_q(z)L}) := \prod_{P} \frac{\left[O_{\mathbb{F}_q(z)L}/PO_{\mathbb{F}_q(z)L}\right]_{\mathbb{F}_q(z)A}}{\left[\widetilde{\phi}(O_{\mathbb{F}_q(z)L}/PO_{\mathbb{F}_q(z)L})\right]_{\mathbb{F}_q(z)A}}$$

where the product runs over all the monic irreducible polynomials P of A. And again:

$$\left[O_{\mathbb{F}_q(z)L}/PO_{\mathbb{F}_q(z)L}\right]_{\mathbb{F}_q(z)A} = P^{[L:K]}.$$

By Demeslay's adaptation of the work of Taelman, [Dem14, Theorem 2.7], we also have the class formula:

Theorem 7.5.5 (Demeslay) *The product defining* $L(\widetilde{\phi}/O_{\mathbb{F}_q(z)L})$ *converges in* $(\mathbb{F}_q(z)K)_\infty$*, and the following equality holds:*

$$L(\widetilde{\phi}/O_{\mathbb{F}_q(z)L}) = \left[O_{\mathbb{F}_q(z)L} : U(\widetilde{\phi};\, O_{\mathbb{F}_q(z)L})\right]_{\mathbb{F}_q(z)A} \left[H(\widetilde{\phi};\, O_{\mathbb{F}_q(z)L})\right]_{\mathbb{F}_q(z)A}.$$

Remark that, because of Corollary 7.4.4, this result can simply be stated as

$$L(\widetilde{\phi}/O_{\mathbb{F}_q(z)L}) = \left[O_{\mathbb{F}_q(z)L} : U(\widetilde{\phi};\, O_{\mathbb{F}_q(z)L})\right]_{\mathbb{F}_q(z)A}.$$

Corollary 7.5.6 *The* L*-series* $L(\widetilde{\phi}/O_{\mathbb{F}_q(z)L})$ *converges in* $\mathbb{T}_z(K_\infty)$*.*

Proof For any monic irreducible polynomial $P \in A$, we have:

$$\left[\widetilde{\phi}(O_L/PO_L)\right]_{\mathbb{F}_q(z)A} = \det_{\mathbb{F}_q(z)[Z]}\left(Z - \theta \mid \widetilde{\phi}(O_L/PO_L)\right)|_{Z=\theta}$$

which is a polynomial in z which evaluates to $P^{[L:K]}$ at $z = 0$. But

$$\deg_\theta\left(\left[\widetilde{\phi}(O_L/PO_L)\right]_{\mathbb{F}_q(z)A}\right) = \dim_{\mathbb{F}_q} O_L/PO_L = \deg_\theta P^{[L:K]}.$$

We deduce that the local factor at P belongs to $\mathbb{T}_z(K_\infty)$. The convergence of $L(\widetilde{\phi}/O_{\mathbb{F}_q(z)L})$ in $(\mathbb{F}_q(z)K)_\infty$ then implies its convergence in $\mathbb{T}_z(K_\infty)$. □

7.5.2 The Equivariant Class Formula

We present now the class formula in the equivariant setting.

We consider as previously a Drinfeld A-module ϕ defined over O_L, and E/L a finite abelian extension of degree prime to p and we let $G = \mathrm{Gal}(E/L)$.

In this context, we can define an equivariant L-series via:

$$L(\phi/(O_E/O_L), G) := \prod_{\mathfrak{P}} \frac{[O_E/\mathfrak{P}O_E]_{A[G]}}{[\phi(O_E/\mathfrak{P}O_E)]_{A[G]}}$$

where the product runs over the non-zero prime ideals of O_E. As in (7.7), it is equivalent to taking the product over the non-zero prime ideals of O_L or of A. And we have the z-twisted version:

$$L(\widetilde{\phi}/(O_{\mathbb{F}_q(z)E}/O_{\mathbb{F}_q(z)L}), G) := \prod_{\mathfrak{P}} \frac{\left[O_{\mathbb{F}_q(z)E}/\mathfrak{P}O_{\mathbb{F}_q(z)E}\right]_{\mathbb{F}_q(z)A[G]}}{\left[\widetilde{\phi}(O_{\mathbb{F}_q(z)E}/\mathfrak{P}O_{\mathbb{F}_q(z)E})\right]_{\mathbb{F}_q(z)A[G]}}.$$

The convergence of the L-series $L(\phi/(O_E/O_L), G)$, and an equivariant class formula involving it was proved, in an even more general setting, by Fang in [Fan18, Theorem 1.12]:

Theorem 7.5.7 (Fang) *We have:*

$$L(\phi/(O_E/O_L), G) = [O_E : U(\phi; O_E)]_{A[G]} [H(\phi; O_E)]_{A[G]} .$$

The equivariant class formula has its origin in [AT15, Theorem A] for the Carlitz module. We also signal to the reader the recent work [FGHP20] of Ferrara, Green, Higgins and Popescu where an equivariant class formula is proved without the restrictions that G is abelian and of order prime to p.

Following the proof of [AT15, Theorem A] (the details can be found in [ATR17, Proposition 4]), one can show the z-twisted version:

Theorem 7.5.8 *The L-series $L(\widetilde{\phi}/(O_{\mathbb{F}_q(z)E}/O_{\mathbb{F}_q(z)L}), G)$ converges in $(\mathbb{F}_q(z)K)_\infty[G]$ and we have:*

$$L(\widetilde{\phi}/(O_{\mathbb{F}_q(z)E}/O_{\mathbb{F}_q(z)L}), G) = \left[O_{\mathbb{F}_q(z)E} : U(\widetilde{\phi}; O_{\mathbb{F}_q(z)E})\right]_{\mathbb{F}_q(z)A[G]} .$$

As for $L(\widetilde{\phi}/O_{\mathbb{F}_q(z)L})$, the convergence of $L(\widetilde{\phi}/(O_{\mathbb{F}_q(z)E}/O_{\mathbb{F}_q(z)L}), G)$ in $(\mathbb{F}_q(z)K)_\infty[G]$ implies that it actually converges in $\mathbb{T}_z(K_\infty)[G]$. We can then evaluate it at $z = 1$, and we see that the result is just $L(\phi/(O_E/O_L), G)$.

Combining Theorem 7.5.7 with Proposition 7.4.9, we also get:

Theorem 7.5.9

$$L(\phi/(O_E/O_L), G) = [O_E : U_{\mathrm{St}}(\phi; O_E)]_{A[G]} .$$

In the case where $L = K$, we have a simple description of the Stark units in terms of the equivariant L-series (see [ATR17, Theorem 2]):

Theorem 7.5.10 *Let ϕ be a Drinfeld A-module defined over A and E/K be a finite abelian extension of degree prime to p, and $G = \mathrm{Gal}(E/K)$. We have:*

$$U(\widetilde{\phi}; O_E[z]) = L(\widetilde{\phi}/(O_{\mathbb{F}_q(z)E}/\mathbb{F}_q(z)A), G)O_E[z]$$

and

$$U_{\mathrm{St}}(\phi; O_E) = L(\phi/(O_E/A), G)O_E.$$

Proof Since $A[G]$ and $\mathbb{F}_q(z)A[G]$ are principal ideal rings, we see that O_E is a rank 1 free $A[G]$-module, and that $O_{\mathbb{F}_q(z)E}$ and $U(\widetilde{\phi}; O_{\mathbb{F}_q(z)E})$ are free $\mathbb{F}_q(z)A[G]$-modules of rank 1. By Theorem 7.5.8, we then have:

$$L(\widetilde{\phi}/(O_{\mathbb{F}_q(z)E}/\mathbb{F}_q(z)A), G)O_{\mathbb{F}_q(z)E} = U(\widetilde{\phi}; O_{\mathbb{F}_q(z)E}).$$

And since $L(\widetilde{\phi}/(O_{\mathbb{F}_q(z)E}/\mathbb{F}_q(z)A), G)$ converges in $\mathbb{T}_z(K_\infty)[G]$, we get:

$$L(\widetilde{\phi}/(O_{\mathbb{F}_q(z)E}/\mathbb{F}_q(z)A), G)O_E[z] \subset U(\widetilde{\phi}; O_E[z]).$$

If conversely $x \in U(\widetilde{\phi}; O_E[z]) \subset \mathbb{T}_z(E_\infty)$, there is $y \in O_{\mathbb{F}_q(z)E}$ such that

$$x = L(\widetilde{\phi}/(O_{\mathbb{F}_q(z)E}/\mathbb{F}_q(z)A), G)y.$$

Since $L(\widetilde{\phi}/(O_{\mathbb{F}_q(z)E}/\mathbb{F}_q(z)A), G)$ has sign 1, this implies that $y \in O_E[z]$. Thus

$$U(\widetilde{\phi}; O_E[z]) = L(\widetilde{\phi}/(O_{\mathbb{F}_q(z)E}/\mathbb{F}_q(z)A), G)O_E[z].$$

The second assertion comes now from the evaluation at $z = 1$. □

7.5.3 Examples

Let us now work out some examples of the class formula. We first treat the Carlitz module C with $L = K$. We refer to Sect. 7.3.3 for the basic facts and notation on the Carlitz module. The L-series associated to C is easily computed. Let $P \in A$ be monic and irreducible. Then obviously $[A/PA]_A = P$. Moreover, as $C_P \equiv \tau^{\deg P}$ (mod $PA[\tau]$), we get $C(A/PA) \simeq A/(P-1)A$ so that the local factor at P is just $(1 - \frac{1}{P})^{-1}$ and

$$L(C/A) = \prod_P (1 - \frac{1}{P})^{-1} = \sum_{a \in A_+} \frac{1}{a}$$

where A_+ stands for the subset of monic polynomials in A. This is also the zeta value at 1 as defined by Carlitz. The other values are, if $n \geq 0$:

$$\zeta_A(n) = \sum_{a \in A_+} \frac{1}{a^n}.$$

Note that at a negative integer, the zeta value is also defined as the (finite!) sum, for $n \geq 0$:

$$\zeta_A(-n) = \sum_{d \geq 0} \sum_{a \in A_+, \deg a = d} a^n.$$

Let us define

$$\mathcal{N} = \{x \in K_\infty, v_\infty(x) > -1\}.$$

Because $v_\infty(D_i) = -iq^i$, we can make Corollary 7.3.6 explicit: $\exp_C$ is isometric on $\mathcal{N}$, so that $\exp_C(\mathcal{N}) = \mathcal{N}$. Consequently, $\exp_C(K_\infty) + A = K_\infty$ so that $H(C; A) = \{0\}$. Hence, by Theorem 7.4.6, $U(C; A) = U_{\mathrm{St}}(C; A)$. This is a rank one A-module, and since $1 \in \mathcal{N}$, we see that $U(C; A) = A\log_C(1)$. The class formula for C can then be written as:

$$\zeta_A(1) = L(C/A) = [A : U(C; A)]_A = \log_C(1).$$

We thus recover this well-known equality which is a consequence of a result of Carlitz [Gos96, Theorem 3.1.5].

Let us now fix an integer $d \geq 0$ and consider the Drinfeld A-module ϕ over A defined by $\phi_\theta = \theta + (-\theta)^d\tau$. We see that if $a \in A$ and $C_a = \sum_{i=0}^k a_i\tau^i$ then $\phi_a = \sum_{i=0}^k a_i(-\theta)^{d\frac{q^i-1}{q-1}}\tau^i$. Let $P \in A$ be monic and irreducible. We thus get that $\phi_P \equiv (-\theta)^{d\frac{q^{\deg P}-1}{q-1}}\tau^{\deg P} \pmod{PA[\tau]}$. But

$$\theta^{\frac{q^{\deg P}-1}{q-1}} = \theta^{1+q+\cdots+q^{\deg P-1}} \equiv (-1)^{\deg P}P(0) \mod P.$$

We deduce that $\phi_{P-P(0)^d}$ is identically zero on A/PA and since for any $Q \in A$, ϕ_Q is a polynomial of $A[\tau]$ of degree $\deg Q$ in τ, $P(X) - P(0)^d$ is the minimal polynomial of ϕ_θ, that is $\phi(A/PA) \simeq A/(P - P(0)^d)A$. Thus $[\phi(A/PA)]_A = P - P(0)^d$. We get:

$$L(\phi/A) = \prod_P \left(1 - \frac{P(0)^d}{P}\right)^{-1} = \sum_{a\in A_+} \frac{a(0)^d}{a}.$$

These computations are also consequences of Sect. 7.6.2 below. See in particular Eq. (7.9). Let us now describe the units and Stark units of ϕ. For that purpose, we use results that will be proved later on. We have by Proposition 7.6.5:

$$U_{\mathrm{St}}(\phi; A) = L(\phi/A)A.$$

There are now two different cases, whether $n \equiv 1 \pmod{q-1}$ or not. This difference is linked to the fact that the kernel of $\exp_\phi : K_\infty \to K_\infty$ is non trivial if and only if $n \equiv 1 \pmod{q-1}$.

In the case $n \not\equiv 1 \pmod{q-1}$, by the proof of Theorem 7.7.1 we have $H(\phi; A) = \{0\}$ and thus

$$U(\phi; A) = U_{\mathrm{St}}(\phi; A) = L(\phi/A)A.$$

In the case $n \equiv 1 \pmod{q-1}$, the unit module is the kernel of $\exp_\phi$ if $n \neq 1$ and more generally the inverse image of the A-torsion submodule of $\phi(K_\infty)$ if $n = 1$.

More explicitly, if $n = 1$:

$$U(\phi; A) = \frac{\widetilde{\pi}}{(-\theta)^{\frac{1}{q-1}}\theta} A$$

and if $n > 1$:

$$U(\phi; A) = \frac{\widetilde{\pi}}{(-\theta)^{\frac{1}{q-1}}\theta^{\frac{n-1}{q-1}}} A$$

where $(-\theta)^{\frac{1}{q-1}}$ is the fixed $(q-1)$-st root of $-\theta$ (see Eq. (7.5)). Thus, if $n > 1$, there is $B_n \in A$ of degree $\frac{n-q}{q-1}$ such that

$$(-\theta)^{\frac{1}{q-1}}\theta^{\frac{n-1}{q-1}} L(\phi/A) = \widetilde{\pi} B_n.$$

Taelman's class formula (Theorem 7.5.3) tells us that $[H(\phi; O_L)]_A = B_n$. Moreover, $[H(\phi; O_L)]_A$ just vanishes when $n = 1$.

7.6 The Multi-Variable Deformation of a Drinfeld A-Module

7.6.1 The Multi-Variable Setting

We have presented in the previous section the z-deformation of a Drinfeld module ϕ, which, roughly speaking, "evaluates" at $z = 1$ to ϕ. It turns out that there are other natural ways to twist a Drinfeld module using multiple variables. The idea here is still to twist the Frobenius τ by a polynomial in the new variables. It is also of interest to combine those two deformations and define Stark units for the multiple variable deformation of our Drinfeld module. Let us now give more precise statements:

Let $t_1, \ldots, t_n$ be new variables, with $n \geq 1$. We will denote by $\mathbf{t}$ the set of variables $t_1, \ldots, t_n$. We fix some additional notation:

- $k = \mathbb{F}_q(\mathbf{t}) = \mathbb{F}_q(t_1, \ldots, t_n)$,
- $\mathbb{A} = k[\theta]$, $\mathbb{K} = k(\theta)$, $\mathbb{K}_\infty = k((\frac{1}{\theta}))$,
- v_∞ the valuation at the place ∞ such that $v_\infty(\theta) = -1$, extending the valuation on K_∞.

We fix a complete algebraically closed extension of $\mathbb{K}$ and we identify $\mathbb{C}_\infty$ with the completion of the algebraic closure of K in this extension. For L a fixed finite extension of K, $\mathbb{L}$ will denote the compositum of L and $\mathbb{K}$, and $O_{\mathbb{L}}$ the integral closure of $\mathbb{A}$ in $\mathbb{L}$. We set $\mathbb{L}_\infty = \mathbb{L} \otimes_{\mathbb{K}} \mathbb{K}_\infty$. We extend τ to $\mathbb{L}$ by k-linearity and thus to $\mathbb{L}_\infty$.

Then, the theory developed in the previous sections remain valid by replacing $\mathbb{F}_q$ by k. We leave to the reader as an exercice to check that the arguments carry over. We will then be interested in Drinfeld $\mathbb{A}$-modules ϕ defined over $O_{\mathbb{L}}$ with an obvious definition. The existence of the exponential and logarithmic maps and their properties described in Sect. 7.3.2 remain valid and we can define the $\mathbb{A}$-modules $U(\phi; O_{\mathbb{L}})$ and $H(\phi; O_{\mathbb{L}})$. By Demeslay's work [Dem14], we have in particular:

Theorem 7.6.1 (Demeslay) *Let ϕ be a Drinfeld $\mathbb{A}$-module defined over $O_{\mathbb{L}}$. Then:*

1. *the unit module*

$$U(\phi; O_{\mathbb{L}}) = \left\{x \in \mathbb{L}_\infty, \exp_\phi(x) \in O_{\mathbb{L}}\right\}$$

is an $\mathbb{A}$-lattice in $\mathbb{K}_\infty$,

2. *the class module*

$$H(\phi; O_{\mathbb{L}}) = \frac{\phi(\mathbb{L}_\infty)}{\phi(O_{\mathbb{L}}) + \exp_\phi(\mathbb{L}_\infty)}$$

is a finite dimensional k-vector space and an $\mathbb{A}$-module via ϕ,

3. *the infinite product*

$$L(\phi/O_{\mathbb{L}}) := \prod_P \frac{[O_{\mathbb{L}}/PO_{\mathbb{L}}]_{\mathbb{A}}}{[\phi(O_{\mathbb{L}}/PO_{\mathbb{L}})]_{\mathbb{A}}},$$

where the product runs over the monic irreducible polynomials $P \in A$, converges in $\mathbb{L}_\infty^\times$ and we have the class formula:

$$L(\phi/O_{\mathbb{L}}) = [O_{\mathbb{L}} : U(\phi; O_{\mathbb{L}})]_{\mathbb{A}}[H(\phi; O_{\mathbb{L}})]_{\mathbb{A}}.$$

Proof Part 1 and Part 2 follow from [Dem14, Proposition 2.6] and Part 3 from [Dem14, Theorem 2.7] □

As previously, we can define the z-twist $\widetilde{\phi}$ of a Drinfeld $\mathbb{A}$-module ϕ defined over $O_{\mathbb{L}}$ by twisting the frobenius τ by z. It is thus a Drinfeld $k(z)\mathbb{A}$-module over $O_{k(z)\mathbb{L}}$. Demeslay's work also applies to this case and we have similarly:

Theorem 7.6.2 (Demeslay) *Let ϕ be a Drinfeld $\mathbb{A}$-module defined over $O_{\mathbb{L}}$ and $\widetilde{\phi}$ be its z-twist. Then:*

1. *the unit module*

$$U(\widetilde{\phi}; O_{k(z)\mathbb{L}}) = \left\{x \in (k(z)\mathbb{L})_\infty, \exp_{\widetilde{\phi}}(x) \in O_{k(z)\mathbb{L}}\right\}$$

is a $k(z)\mathbb{A}$-lattice in $(k(z)\mathbb{K})_\infty$,

2. *the class module*

$$H(\widetilde{\phi}; O_{k(z)\mathbb{L}}) = \frac{\widetilde{\phi}((k(z)\mathbb{L})_\infty)}{\widetilde{\phi}(O_{k(z)\mathbb{L}}) + \exp_{\widetilde{\phi}}((k(z)\mathbb{L})_\infty)}$$

is a finite dimensional $k(z)$-vector space and a $k(z)\mathbb{A}$-module via $\widetilde{\phi}$,

3. *the infinite product*

$$L(\widetilde{\phi}/O_{k(z)\mathbb{L}}) := \prod_P \frac{\left[O_{k(z)\mathbb{L}}/PO_{k(z)\mathbb{L}}\right]_{k(z)\mathbb{A}}}{\left[\widetilde{\phi}(O_{k(z)\mathbb{L}}/PO_{k(z)\mathbb{L}})\right]_{k(z)\mathbb{A}}},$$

where the product runs over the monic irreducible polynomials $P \in A$, converges in $(k(z)\mathbb{L})_\infty^\times$ and we have the class formula:

$$L(\widetilde{\phi}/O_{k(z)\mathbb{L}}) = [O_{k(z)\mathbb{L}} : U(\widetilde{\phi}; O_{k(z)\mathbb{L}})]_{k(z)\mathbb{A}}[H(\widetilde{\phi}; O_{k(z)\mathbb{L}})]_{k(z)\mathbb{A}}.$$

Remark 7.6.3 As in Proposition 7.4.3, $H(\widetilde{\phi}; O_{\mathbb{L}}[z])$ is a finitely generated torsion $k[z]$-module, so that the class module $H(\widetilde{\phi}; O_{k(z)\mathbb{L}})$ vanishes, which simplifies the class formula.

We now want to work at the integral level in $\mathbb{A}$ or $\mathbb{K}_\infty$. We then suppose that $\phi_\theta \in O_L[\mathbf{t}][\tau]$. We can thus consider ϕ either as a Drinfeld $\mathbb{A}$-module defined over $\mathbb{L}$ or as a Drinfeld $A[\mathbf{t}]$-module defined over $O_L[\mathbf{t}]$. We denote by $\mathbb{T}_n(L_\infty)$ the Tate algebra in variables $t_1, \ldots, t_n$ and coefficients in L_∞ and we define the Taelman modules:

$$U(\phi; O_L[\mathbf{t}]) = \left\{x \in \mathbb{T}_n(L_\infty), \exp_\phi(x) \in O_L[\mathbf{t}]\right\} \subset U(\phi; O_{\mathbb{L}})$$

and

$$H(\phi; O_L[\mathbf{t}]) = \frac{\phi(\mathbb{T}_n(L_\infty))}{\phi(O_L[\mathbf{t}]) + \exp_\phi(\mathbb{T}_n(L_\infty))}.$$

Since ϕ is defined over $O_L[\mathbf{t}]$, by using the functional equation $\phi_\theta \exp_\phi = \exp_\phi \theta$, one shows that $\exp_\phi$ has coefficients in $L[\mathbf{t}]$, so that $\exp_\phi(\mathbb{T}_n(L_\infty)) \subset \mathbb{T}_n(L_\infty)$. We deduce that:

$$U(\phi; O_L[\mathbf{t}]) = U(\phi; O_{\mathbb{L}}) \cap \mathbb{T}_n(L_\infty).$$

By the same argument as in Proposition 7.4.2, we also have

$$U(\phi; O_{\mathbb{L}}) = kU(\phi; O_L[\mathbf{t}])$$

and

$$H(\phi;\, O_L[\mathbf{t}]) \otimes_{\mathbb{F}_q[\mathbf{t}]} k \simeq H(\phi;\, O_{\mathbb{L}}).$$

By evaluation at $z = 1$ of the unit module, we have a well defined notion of the module of Stark units $U_{\mathrm{St}}(\phi;\, O_{\mathbb{L}})$. Let us be more explicit for the construction at the integral level. We denote by $\mathbb{T}_{n,z}(L_\infty)$ the Tate algebra in variables $t_1, \dots, t_n, z$ and coefficients in L_∞. Then we define

$$U(\widetilde{\phi};\, O_L[\mathbf{t}, z]) = \left\{x \in \mathbb{T}_{n,z}(L_\infty), \exp_{\widetilde{\phi}}(x) \in O_L[\mathbf{t}, z]\right\}$$

and

$$H(\widetilde{\phi};\, O_L[\mathbf{t}, z]) = \frac{\widetilde{\phi}(\mathbb{T}_{n,z}(L_\infty))}{\widetilde{\phi}(O_L[\mathbf{t}, z]) + \exp_{\widetilde{\phi}}(\mathbb{T}_{n,z}(L_\infty))}.$$

The evaluation at $z = 1$ of $U(\widetilde{\phi};\, O_L[\mathbf{t}, z])$ is our module of Stark units $U_{\mathrm{St}}(\phi;\, O_L[\mathbf{t}]) \subset U(\phi;\, O_L[\mathbf{t}])$.

Theorem 7.4.6 remains true here, in particular we have the following version (see [ATR17, Proposition 6]):

Proposition 7.6.4 *The map*

$$\alpha : \begin{cases} \mathbb{T}_n(L_\infty) \to & \mathbb{T}_{n,z}(L_\infty) \\ \quad x \quad\;\; \mapsto & \frac{\exp_{\widetilde{\phi}}(x) - \exp_\phi(x)}{z-1} \end{cases}$$

induces an isomorphism of $A[\mathbf{t}]$*-modules:*

$$\frac{U(\phi;\, O_L[\mathbf{t}])}{U_{\mathrm{St}}(\phi;\, O_L[\mathbf{t}])} \simeq H(\widetilde{\phi};\, O_L[\mathbf{t}, z])[z-1].$$

7.6.2 The Canonical Deformation of the Carlitz Module

We focus here on a natural multi-variable deformation of the Carlitz module built by means of its shtuka function.

Let ϕ be a Drinfeld A-module defined over O_L and $f(\mathbf{t}) = f(t_1, \dots, t_n) \in O_L[\mathbf{t}]$. Then we can use f to twist ϕ: if $a \in A$ and $\phi_a = \sum_{i=0}^m a_i \tau^i$, then

$$\widehat{\phi}_a = \sum_{i=0}^m a_i (f(\mathbf{t})\tau)^i = \sum_{i=0}^m a_i \left(\prod_{j=0}^{i} \tau^j(f)(\mathbf{t})\right) \tau^i.$$

Remark that, as for the z-twist, we in fact twist here the action of the Frobenius τ by $f(\mathbf{t})$, which induces the deformation of ϕ. We get a Drinfeld $A[\mathbf{t}]$-module $\widehat{\phi}$ defined over $O_L[\mathbf{t}]$.

From now on, we will be only interested in the case of the Carlitz module C. Let us recall (see Sect. 7.3.3) that C is the Drinfeld A-module defined over A by $C_\theta = \theta + \tau$. To such a Drinfeld module one can associate a so-called *shtuka function* (see e.g. [Gos96, §7.11], or [Tha93]), from which one recovers the Drinfeld module, and which encodes its arithmetic properties. In the case of the Carlitz module, the shtuka function is simply $t - \theta$. There is therefore a natural n variable twist of the Carlitz module, which we call the *canonical deformation of the Carlitz module*, given by

$$f(\mathbf{t}) = \prod_{i=1}^{n}(t_i - \theta).$$

We thus consider the Drinfeld $A[\mathbf{t}]$-module $\varphi = \widehat{C}$ defined over $A[\mathbf{t}]$ by

$$\varphi_\theta = \theta + f(\mathbf{t})\tau = \theta + \prod_{i=1}^{n}(t_i - \theta)\tau.$$

We will denote for $k \geq 0$, by $f_k(\mathbf{t})$ the polynomial appearing in the formula $(f(\mathbf{t})\tau)^k = f_k(\mathbf{t})\tau^k$, that is:

$$f_k(\mathbf{t}) = \prod_{i=1}^{n}\prod_{j=0}^{k}(t_i - \theta^{q^j}).$$

We get the exponential map $\exp_\varphi = \sum_{i\geq 0} \frac{1}{D_i} f_i(\mathbf{t})\tau^i$ and the logarithm map $\log_\varphi = \sum_{i\geq 0} \frac{1}{l_i} f_i(\mathbf{t})\tau^i$.

We also introduce the Anderson-Thakur ω function:

$$\omega(t) := (-\theta)^{\frac{1}{q-1}} \prod_{j\geq 0}\left(1 - \frac{t}{\theta^{q^j}}\right)^{-1} \in \mathbb{T}_1(K_\infty)^\times.$$

We see from (7.5) that $-\widetilde{\pi}$ is the residue of ω at $t = \theta$ and that ω enjoys the functional equation:

$$\tau(\omega(t)) = (t - \theta)\omega(t).$$

Thus, we get

$$\exp_\varphi = \left(\prod_{i=1}^{n}\omega(t_i)\right)^{-1} \exp_C\left(\prod_{i=1}^{n}\omega(t_i)\right).$$

In particular, we obtain:

$$\ker(\exp_\varphi : \mathbb{T}_n(\mathbb{C}_\infty) \to \mathbb{T}_n(\mathbb{C}_\infty)) = \frac{\widetilde{\pi}}{\prod_{i=1}^n \omega(t_i)} A[\mathbf{t}]. \tag{7.8}$$

And we remark that this kernel is included in $\mathbb{T}_n(K_\infty)$ if, and only if, $n \equiv 1 \pmod{q-1}$.

The L-series associated to φ can be computed similarly to the one of C (see Sect. 7.5.3). We have

$$\varphi_P \equiv f_{\deg P}(\mathbf{t})\tau^{\deg P} \pmod{PA[\mathbf{t}][\tau]}$$

but

$$(t-\theta)(t-\theta^q)\cdots(t-\theta^{q^{\deg P-1}}) \equiv P(t) \pmod{PA[t]}$$

so that

$$\varphi_P \equiv P(t_1)\cdots P(t_n)\tau^{\deg P} \pmod{PA[\mathbf{t}][\tau]}.$$

We deduce that $P(X) - P(t_1)\cdots P(t_n)$ is an annihilating polynomial of ϕ_θ acting on $A/PA(\mathbf{t})$. Since it is also a monic irreducible polynomial in $\mathbb{F}_q(\mathbf{t})[X]$, of degree $\deg P$, it is its characteristic polynomial and we get by (7.2):

$$\left[\frac{A}{PA}(\mathbf{t})\right]_{\mathbb{A}} = P - P(t_1)\cdots P(t_n).$$

Putting all the local factors together, we obtain

$$L(\varphi/\mathbb{A}) = \prod_P \left(1 - \frac{P(t_1)\cdots P(t_n)}{P}\right)^{-1} = \sum_{a\in A_+} \frac{a(t_1)\cdots a(t_n)}{a} \in \mathbb{T}_n(K_\infty)^\times. \tag{7.9}$$

Similar calculations for the z-twist of φ lead to:

$$\begin{aligned} L(\widetilde{\varphi}/k(z)\mathbb{A}) &= \prod_P \left(1 - \frac{z^{\deg P} P(t_1)\cdots P(t_n)}{P}\right)^{-1} \\ &= \sum_{a\in A_+} z^{\deg P} \frac{a(t_1)\cdots a(t_n)}{a} \in \mathbb{T}_{n,z}(K_\infty)^\times. \end{aligned}$$

Let us compute the units:

Proposition 7.6.5 *We have*

$$U(\widetilde{\varphi}; A[\mathbf{t}, z]) = L(\widetilde{\varphi}/k(z)\mathbb{A})A[\mathbf{t}, z]$$

so that

$$U_{\mathrm{St}}(\varphi; A[\mathbf{t}]) = L(\varphi/\mathbb{A})A[\mathbf{t}].$$

Moreover, $[H(\varphi; \mathbb{A})]_{\mathbb{A}} \in A[\mathbf{t}] \cap \mathbb{T}_n(K_\infty)^\times$ *and*

$$[H(\varphi; \mathbb{A})]_{\mathbb{A}}\, U(\varphi; A[\mathbf{t}]) = L(\varphi/\mathbb{A})A[\mathbf{t}].$$

Proof We give the proof for $U(\varphi; A[\mathbf{t}])$. The other assertion can be proved in a similar way, since, by Remark 7.6.3, $H(\widetilde{\varphi}; k(z)\mathbb{A})$ vanishes.

First, since φ has coefficients in $A[\mathbf{t}]$ and because we can compute $[H(\varphi; \mathbb{A})]_{\mathbb{A}}$ as a determinant by Eq. (7.2), we see that $[H(\varphi; \mathbb{A})]_{\mathbb{A}} \in A[\mathbf{t}]$.

Since the unit module has rank 1, by the class formula (Theorem 7.5.5), we get $[H(\varphi; \mathbb{A})]_{\mathbb{A}}\, U(\varphi; \mathbb{A}) = L(\varphi/\mathbb{A})\mathbb{A}$. Since $U(\varphi; \mathbb{A}) = kU(\varphi; A[\mathbf{t}])$, we can find $\eta \in U(\varphi; A[\mathbf{t}])$ such that $U(\varphi; \mathbb{A}) = \mathbb{A}\eta$. We can, and will, also assume that η is primitive in $\mathbb{T}_n(K_\infty)$, that is, not divisible by a non constant polynomial $\delta \in \mathbb{F}_q[\mathbf{t}]$. We get $[H(\varphi; \mathbb{A})]_{\mathbb{A}}\, \eta\mathbb{A} = L(\varphi/\mathbb{A})\mathbb{A}$, so that

$$L(\varphi/\mathbb{A}) = \lambda\, [H(\varphi; \mathbb{A})]_{\mathbb{A}}\, \eta$$

for some $\lambda \in \mathbb{F}_q^\times$. In particular, $[H(\varphi; \mathbb{A})]_{\mathbb{A}} \in \mathbb{T}_n(K_\infty)^\times$. We get:

$$\begin{aligned} U(\varphi; A[\mathbf{t}]) = U(\varphi; \mathbb{A}) \cap \mathbb{T}_n(K_\infty) &= ([H(\varphi; \mathbb{A})]_{\mathbb{A}}^{-1}\, L(\varphi/\mathbb{A})\mathbb{A}) \cap \mathbb{T}_n(K_\infty) \\ &= [H(\varphi; \mathbb{A})]_{\mathbb{A}}^{-1}\, L(\varphi/\mathbb{A})A[\mathbf{t}] \end{aligned}$$

whence the result. □

We set

$$\mathcal{N} = \left\{ x \in \mathbb{T}_n(K_\infty),\, v_\infty(x) \geq \frac{n}{q-1} - 1 \right\}.$$

Lemma 7.6.6 *If* $x \in \mathcal{N}$, $v_\infty(\exp_\varphi(x) - x) > v_\infty(x)$ *and* $v_\infty(\log_\varphi(x) - x) > v_\infty(x)$. *In particular, both* $\exp_\varphi$ *and* $\log_\varphi$ *define isometries* $\mathcal{N} \to \mathcal{N}$.

Proof For $k \geq 0$, we compute: $v_\infty(D_k) = -kq^k$, $v_\infty(l_k) = -q\frac{q^k-1}{q-1}$ and $v_\infty(f_k(\mathbf{t})) = -n\frac{q^k-1}{q-1}$. Thus, if $x \in \mathcal{N}$, and $k > 0$,

$$v_\infty\left(\frac{f_k(\mathbf{t})}{D_k}\tau^k(x)\right) = v_\infty(x) + (q^k - 1)\left(v_\infty(x) + k - \frac{n}{q-1}\right) + k > v_\infty(x)$$

and

$$v_\infty\left(\frac{f_k(\mathbf{t})}{l_k}\tau^k(x)\right) = v_\infty(x) + (q^k - 1)\left(\frac{q-n}{q-1} + v_\infty(x)\right) > v_\infty(x)$$

whence the result. □

Remark 7.6.7 If $n \leq 2q-2$, we have $\mathbb{T}_n(K_\infty) = \mathcal{N} + A[\mathbf{t}] \subset \exp_\varphi(\mathbb{T}_n(K_\infty)) + A[\mathbf{t}]$ so that $H(\varphi; A[\mathbf{t}]) = \{0\}$ and

$$U(\varphi; A[\mathbf{t}]) = U_{\mathrm{St}}(\varphi; A[\mathbf{t}]) = L(\varphi/\mathbb{A})A[\mathbf{t}].$$

7.7 Applications

7.7.1 Discrete Greenberg Conjectures

As a first application of the notion of Stark Units, we present a pseudo-nullity and a pseudo-cyclicity result from [ATR17] for the class module of the canonical deformation of the Carlitz module. These theorems are reminiscent of the Greenberg conjectures, in particular after evaluation at characters.

We keep the notation of all the previous sections. In particular, we recall that:

$$\mathcal{N} = \left\{x \in \mathbb{T}_n(K_\infty), v_\infty(x) \geq \frac{n}{q-1} - 1\right\}.$$

We denote now

$$\mathbb{B}_n(\mathbf{t}) = [H(\varphi; \mathbb{A})]_{\mathbb{A}} \in A[\mathbf{t}] \cap \mathbb{T}_n(K_\infty)^\times.$$

By Remark 7.6.7, we have $\mathbb{B}_n(\mathbf{t}) = 1$ if $1 \leq n \leq 2q - 2$. We also introduce the special elements:

$$u_n(\mathbf{t}, z) = \exp_{\widetilde{\varphi}}(L(\widetilde{\varphi}/k(z)\mathbb{A})) \in A[\mathbf{t}, z]$$

and

$$u_n(\mathbf{t}) = u_n(\mathbf{t}, 1) = \exp_\varphi(L(\varphi/\mathbb{A})) \in A[\mathbf{t}].$$

By Proposition 7.6.5, those elements generate the $A[\mathbf{t}, z]$-module (via $\widetilde{\varphi}$) $U(\widetilde{\varphi}; A[\mathbf{t}, z])$ and the $A[\mathbf{t}]$-module (via φ) of Stark units $U_{\mathrm{St}}(\varphi; A[\mathbf{t}])$.

If $1 \leq n \leq q-1$, $L(\varphi/\mathbb{A}) \in \mathcal{N}$; by Lemma 7.6.6, we have $u_n(\mathbf{t}) \in \mathcal{N} \cap A[\mathbf{t}] = \mathbb{F}_q$ and u_n has the same sign as $L(\varphi/\mathbb{A})$. Thus in this case, $u_n(\mathbf{t}) = 1$.

As we have seen in (7.8), $\exp_\varphi$ is injective on $\mathbb{T}_n(K_\infty)$ if and only if $n \not\equiv 1 \pmod{q-1}$. This leads us to distinguish the two cases, where different phenomena occur.

7.7.1.1 Case $n \not\equiv 1 \pmod{q-1}$

We prove in this case the following pseudo-nullity result (see [ATR17, Theorem 3]):

Theorem 7.7.1 *We have* $\mathbb{B}_n(\mathbf{t}) = 1$*, that is,* $H(\varphi; A[\mathbf{t}])$ *is a finitely generated and torsion* $\mathbb{F}_q[\mathbf{t}]$*-module.*

Proof Let $r \in \{2, \dots, q-1\}$ be such that $n \equiv r \pmod{q-1}$. We denote by ψ the r-variable twist of the Carlitz module:

$$\psi_\theta = (t_1 - \theta) \cdots (t_r - \theta)\tau + \theta.$$

We set:

$$\Xi := \frac{L(\psi/\mathbb{F}_q(t_1, \dots, t_r))}{\omega(t_{r+1}) \cdots \omega(t_n)} \in \mathbb{T}_n(K_\infty)^\times.$$

We get for $a \in A[\mathbf{t}]$:

$$\begin{aligned}\exp_\varphi(a\Xi) &= \frac{\exp_\psi(aL(\psi/\mathbb{F}_q(t_1, \dots, t_r)))}{\omega(t_{r+1}) \cdots \omega(t_n)} \\ &= \frac{\psi_a(u_r(t_1, \dots, t_r))}{\omega(t_{r+1}) \cdots \omega(t_n)} = \frac{\psi_a(1)}{\omega(t_{r+1}) \cdots \omega(t_n)}.\end{aligned}$$

Remark now that $\mathcal{N} = \left\{x \in \mathbb{T}_n(K_\infty), v_\infty(x) \geq \frac{n-r}{q-1}\right\}$ so that

$$\mathbb{T}_n(K_\infty) = A[\mathbf{t}] \oplus \mathcal{N} \oplus \bigoplus_{k=1}^{\frac{n-r}{q-1}-1} \theta^{k-\frac{n-r}{q-1}} \mathbb{F}_q[\mathbf{t}].$$

We then define for $1 \leq i, j \leq \frac{n-r}{q-1} - 1$, $\beta_{ij} \in \mathbb{F}_q[\mathbf{t}]$ by the formula:

$$\exp_\varphi\left(\theta^i \Xi\right) - \sum_{j=1}^{\frac{n-r}{q-1}-1} \theta^{j-\frac{n-r}{q-1}} \beta_{ij} \in A[\mathbf{t}] \oplus \mathcal{N}.$$

Our theorem is now equivalent to $\det(\beta_{ij}) \neq 0$. Since $\det(\beta_{ij}) \in \mathbb{F}_q[\mathbf{t}]$, it will be enough to show that its evaluation at $t_1 = \cdots t_n = 0$ does not vanish. Let us denote by $\mathrm{ev}_0 : \mathbb{T}_n(K_\infty) \to K_\infty$ this evaluation. We have:

$$\mathrm{ev}_0(\exp_\varphi\left(\theta^i\,\Xi\right)) = \frac{\psi'_{\theta^i}(1)}{(-\theta)^{\frac{n-r}{q-1}}} \in \sum_{j=1}^{\frac{n-r}{q-1}-1} \theta^{j-\frac{n-r}{q-1}}\,\mathrm{ev}_0(\beta_{ij}) + A + \mathrm{ev}_0(\mathcal{N})$$

where $\psi'_\theta = (-\theta)^r\tau + \theta$. An immediate induction now shows that for $i \geq 1$,

$$\psi'_{\theta^i}(1) - \theta^i \in \theta^{i+1}A.$$

Thus $\mathrm{ev}_0(\det(\beta_{ij})) \neq 0$ and $\det(\beta_{ij}) \neq 0$. □

7.7.1.2 Case $n \equiv 1 \pmod{q-1}$

Let us first describe the unit module in this case:

Proposition 7.7.2 *If $n = 1$ then*

$$U(\varphi; A[\mathbf{t}]) = \frac{\widetilde{\pi}}{(t_1-\theta)\omega(t_1)}A[\mathbf{t}].$$

and if $n > 1$, then

$$U(\varphi; A[\mathbf{t}]) = \frac{\widetilde{\pi}}{\prod_{i=1}^n \omega(t_i)}A[\mathbf{t}].$$

Proof Since $\frac{\widetilde{\pi}}{\prod_{i=1}^n \omega(t_i)}A[\mathbf{t}] = \ker \exp_\varphi$, it is clearly included in $U(\varphi; A[\mathbf{t}])$. As the unit module has rank 1, we deduce that if $x \in U(\varphi; A[\mathbf{t}])$, then $y = \exp_\varphi(x)$ is a torsion point for φ, that is there is $a \in A[\mathbf{t}]$ such that $\varphi_a(y) = 0$. But, if $v_\infty(x) \leq 0$, we see that

$$v_\infty((t_1-\theta)\cdots(t_n-\theta)(\tau(x))) = qv_\infty(x) - n \ \text{ and } \ v_\infty(\theta x) = v_\infty(x) - 1.$$

If $n > 1$, the first quantity is strictly lower than the second, this easily implies that no non trivial torsion point can exist: if $a \in A[\mathbf{t}]$, $\varphi_a(x)$ has the same (negative, and even explicitly computable) valuation as $\varphi_{\theta^{\deg_\theta(a)}}(x)$. With the same argument in the case $n = 1$ we see that if x is a torsion point, it must have valuation 0, so $x \in \mathbb{F}_q(t)$. Conversely, for $x \in \mathbb{F}_q(t)$, we have $\varphi_{(\theta-t)}(x) = 0$. □

Remark that in both cases we have the decomposition of $\mathbb{F}_q[t]$-modules: $\mathbb{T}_n(K_\infty) = \mathcal{N} \oplus U(\varphi; A[\mathbf{t}])$. In particular, if $n > 1$:

$$\exp_\varphi(\mathbb{T}_n(K_\infty)) = \mathcal{N}. \tag{7.10}$$

In the case $n = 1$, we know that $\mathbb{B}_n(\mathbf{t}) = 1$, so that, units and Stark units coincide, we deduce that $L(\varphi/\mathbb{A})$ equals, up to the sign, $\frac{\widetilde{\pi}}{(\theta-t_1)\omega(t_1)}$. But both have sign 1, so that we recover Pellarin's formula (see [Pel12]):

$$L(\varphi/\mathbb{A}) = \frac{\widetilde{\pi}}{(\theta - t_1)\omega(t_1)}.$$

If $n > 1$, we obtain another description of $\mathbb{B}_n(\mathbf{t})$:

$$\mathbb{B}_n(\mathbf{t}) = [H(\varphi; \mathbb{A})]_{\mathbb{A}} = (-1)^{\frac{n-1}{q-1}} L(\varphi/\mathbb{A}) \frac{\prod_{i=1}^n \omega(t_i)}{\widetilde{\pi}}.$$

We deduce in particular that $\mathbb{B}_n(\mathbf{t})$ has degree in θ equal to $\frac{n-q}{q-1}$. In particular, when $n = q$, we recover that $\mathbb{B}_q(\mathbf{t}) = 1$ so that

$$L(\varphi/\mathbb{A}) = \frac{\widetilde{\pi}}{\prod_{i=1}^q \omega(t_i)}.$$

More generally, if one can explicitly compute $\mathbb{B}_n(\mathbf{t})$, this gives us an explicit formula for $L(\varphi/\mathbb{A})$. We also stress that $L(\varphi/\mathbb{A})\frac{\prod_{i=1}^n \omega(t_i)}{\widetilde{\pi}} \in A[\mathbf{t}]$ is one of the main results of [AP15] where it is obtained without using the class formula.

Recall from Proposition 7.4.7 that we can build a map $\overline{\alpha} : \frac{U(\varphi; A[\mathbf{t}])}{U_{\mathrm{St}}(\varphi; A[\mathbf{t}])} \to H(\widetilde{\varphi}; A[\mathbf{t}, z])[z-1]$. We can compose it with the evaluation at $z = 1$ and obtain a map $\beta : \frac{U(\varphi; A[\mathbf{t}])}{U_{\mathrm{St}}(\varphi; A[\mathbf{t}])} \to H(\varphi; A[\mathbf{t}])$. Let us remark that β is induced by:

$$\exp_{\varphi}^{(1)} \begin{cases} U(\varphi; A[\mathbf{t}]) \to & \mathbb{T}_n(K_\infty) \\ \quad x & \mapsto \sum_{k \geq 1} k \frac{f_k(\mathbf{t})}{D_k} \tau^k(x) \end{cases}$$

since we essentially differentiate $\exp_{\widetilde{\varphi}}$ at 1 with respect to z.

Let us denote by $H^{(1)}(\varphi; A[\mathbf{t}]) \subset H(\varphi; A[\mathbf{t}])$ the image of β.

We devote the rest of this section to the proof of the following pseudo-cyclicity result (see [ATR17, Theorem 4]):

Theorem 7.7.3 *Let $n \geq q$. There is an isomorphism of $A[\mathbf{t}]$-modules:*

$$H^{(1)}(\varphi; A[\mathbf{t}]) \simeq \frac{A[\mathbf{t}]}{\mathbb{B}_n(\mathbf{t})A[\mathbf{t}]}$$

and the quotient $\frac{H(\varphi; A[\mathbf{t}])}{H^{(1)}(\varphi; A[\mathbf{t}])}$ is a finitely generated and torsion $\mathbb{F}_q[\mathbf{t}]$-module.

Proof Since $\frac{U(\varphi; A[\mathbf{t}])}{U_{\mathrm{St}}(\varphi; A[\mathbf{t}])}$ is an $A[\mathbf{t}]$-module isomorphic to $\frac{A[\mathbf{t}]}{\mathbb{B}_n(\mathbf{t})A[\mathbf{t}]}$ generated by the image of $\frac{\widetilde{\pi}}{\prod_{i=1}^n \omega(t_i)}$, we are led to compute $\exp_{\varphi}^{(1)}(\frac{\widetilde{\pi}}{\prod_{i=1}^n \omega(t_i)})$. But we have once again:

$$\exp_{\varphi}^{(1)}\left(\frac{\widetilde{\pi}}{\prod_{i=1}^{n}\omega(t_i)}\right) = \frac{1}{\prod_{i=1}^{n}\omega(t_i)}\exp_C^{(1)}(\widetilde{\pi})$$

where $\exp_C^{(1)} = \sum_{k\geq 1} k\frac{1}{D_k}\tau^k$. The proof that can be found in [ATR17, Theorem 3] relies on computations involving $\exp_C^{(1)}(\widetilde{\pi})$. We will give here a slightly different proof, more similar to that of Theorem 7.7.1 above.

We denote by ψ the q-variable twist of the Carlitz module:

$$\psi_\theta = (t_1 - \theta)\cdots(t_q - \theta)\tau + \theta.$$

We first compute $u' = \exp_{\psi}^{(1)}\left(\frac{\widetilde{\pi}}{\prod_{i=1}^{q}\omega(t_i)}\right)$. Since $\mathbb{B}_q(\mathbf{t}) = 1$, we have $u' \in A[t_1, \ldots, t_q] \oplus \mathcal{N}_q$ where $\mathcal{N}_q = \left\{x \in \mathbb{T}_q(K_\infty), v_\infty(x) \geq 1\right\}$. But $v_\infty(\frac{\widetilde{\pi}}{\prod_{i=1}^{q}\omega(t_i)}) = 0$ and for $k \geq 1$, $v_\infty(D_k) = -kq^k$, $v_\infty(l_k) = -q\frac{q^k-1}{q-1}$ and $v_\infty(f_k(\mathbf{t})) = -n\frac{q^k-1}{q-1}$.

$$v_\infty(\frac{f_k(t_1, \cdots, t_q)}{D_k}) = kq^k - q\frac{q^k - 1}{q-1} = q^k(k - \frac{q}{q-1}) + \frac{q}{q-1}$$

which is positive if $k > 1$ and equals 0 if $k = 1$. Thus $\frac{(t_1-\theta)\cdots(t_q-\theta)}{\theta^q-\theta}\frac{\widetilde{\pi}}{\prod_{i=1}^{q}\omega(t_i)}$ has sign 1, we obtain that $u' \in 1 + \mathcal{N}_q$.

We get for $a \in A[\mathbf{t}]$:

$$\begin{aligned}\exp_{\varphi}^{(1)}\left(a\frac{\widetilde{\pi}}{\prod_{i=1}^{n}\omega(t_i)}\right) &= \frac{\exp_{\psi}^{(1)}(a\frac{\widetilde{\pi}}{\prod_{i=1}^{q}\omega(t_i)})}{\omega(t_{q+1})\cdots\omega(t_n)} \\ &\equiv \frac{\psi_a(u')}{\omega(t_{q+1})\cdots\omega(t_n)} \pmod{\mathcal{N} + A[\mathbf{t}]} \\ &\equiv \frac{\psi_a(1)}{\omega(t_{q+1})\cdots\omega(t_n)} \pmod{\mathcal{N} + A[\mathbf{t}]}.\end{aligned}$$

Remark now that

$$\mathbb{T}_n(K_\infty) = A[\mathbf{t}] \oplus \mathcal{N} \oplus \bigoplus_{k=1}^{\frac{n-q}{q-1}} \theta^{k-\frac{n-1}{q-1}}\mathbb{F}_q[\mathbf{t}].$$

We then define for $1 \leq i, j \leq \frac{n-q}{q-1}$, $\beta_{ij} \in \mathbb{F}_q[\mathbf{t}]$ by the formula:

$$\exp_{\varphi}^{(1)}\left(\theta^i\frac{\widetilde{\pi}}{\prod_{i=1}^{n}\omega(t_i)}\right) - \sum_{j=1}^{\frac{n-q}{q-1}}\theta^{j-\frac{n-1}{q-1}}\beta_{ij} \in A[\mathbf{t}] \oplus \mathcal{N}.$$

The injectivity of β is now equivalent to $\det(\beta_{ij}) \neq 0$. It is again enough to show that its evaluation at $t_1 = \cdots t_n = 0$ does not vanish. Let us denote by $\mathrm{ev}_0 : \mathbb{T}_n(K_\infty) \to K_\infty$ this evaluation. We have:

$$\mathrm{ev}_0(\exp_\varphi^{(1)}\left(\theta^i \frac{\widetilde{\pi}}{\prod_{i=1}^n \omega(t_i)}\right)) = \frac{\psi'_{\theta^i}(1)}{(-\theta)^{\frac{n-q}{q-1}}} \in \sum_{j=1}^{\frac{n-q}{q-1}} \theta^{j-\frac{n-1}{q-1}} \mathrm{ev}_0(\beta_{ij}) + A + \mathrm{ev}_0(\mathcal{N})$$

where $\psi'_\theta = (-\theta)^q \tau + \theta$. But again, for $i \geq 1$,

$$\psi'_{\theta^i}(1) - \theta^i \in \theta^{i+1} A.$$

Thus $\mathrm{ev}_0(\det(\beta_{ij})) \neq 0$ and $\det(\beta_{ij}) \neq 0$.

Finally, $H^{(1)}(\varphi; A[\mathbf{t}])$ is a sub-$\mathbb{F}_q[\mathbf{t}]$-module of $H(\varphi; A[\mathbf{t}])$ with same rank, which gives the last assertion. □

7.7.1.3 Evaluation at Characters

Let us now very briefly explain some consequences of Theorems 7.7.1 and 7.7.3 above. We refer the reader for instance to [APTR16, §9] for more details. Let a be a non constant and square free element in A and $\chi : A/aA \to \overline{\mathbb{F}}_q$ be a Dirichlet character mod a. Let us denote by k_a the extension of $\mathbb{F}_q$ generated by the roots of a. Then one can find $\zeta_1, \ldots, \zeta_n \in k_a$ (in fact all of the ζ_i's are roots of a) such that for all $b \in A$, $\chi(b) = b(\zeta_1) \cdots b(\zeta_n)$. We then have a natural homomorphism of $\mathbb{F}_q$-vector spaces $\mathrm{ev}_\chi : \mathbb{T}_n(K_\infty) \to (k_a K)_\infty$ which evaluates t_i to ζ_i for all $1 \leq i \leq n$.

We get for instance:

$$\mathrm{ev}_\chi(L(\varphi/\mathbb{A})) = L(C/A, \chi) := \sum_{b \in A+} \frac{\chi(b)}{b}.$$

In order to define the class module associated to χ, we define $\tau_a : K_\infty \otimes_{\mathbb{F}_q} k_a K_\infty \otimes_{\mathbb{F}_q} k_a$ by $\tau_a = \tau \otimes \mathrm{id}$. We use it to define the Drinfeld A-module C' over $A \otimes_{\mathbb{F}_q} k_a$ by $C'_\theta = \theta + \prod_{i=1}^n (1 \otimes \zeta_i - \theta \otimes 1)\tau_a$. Then:

$$H_\chi := \frac{C'(K_\infty \otimes_{\mathbb{F}_q} k_a)}{\exp_{C'}(K_\infty \otimes_{\mathbb{F}_q} k_a) + C'(A \otimes_{\mathbb{F}_q} k_a)}.$$

In fact, ev_χ also induces a surjection $H(\varphi; A[\mathbf{t}]) \to H_\chi$. Moreover, although the number n of variables involved in this construction is not unique, it is unique modulo $q - 1$. The minimal number n that can be used is called the *type* of χ. There is a well defined notion of "almost all characters of type n" which is, roughly speaking, all but a Zariski closed non trivial subset.

Then, Theorems 7.7.1 and 7.7.3 imply:

Theorem 7.7.4

1. *If* $n \not\equiv 1 \pmod{q-1}$*, then for almost all Dirichlet character* χ *of type* n*, we have* $H_\chi = \{0\}$.
2. *If* $n \equiv 1 \pmod{q-1}$*, then for almost all Dirichlet character* χ *of type* n*,* H_χ *is a cyclic* $A \otimes k_a$*-module.*

These two results remind of the celebrated Greenberg conjectures. For details on the analogy between the two contexts we refer the reader to [ATR17, Introduction].

7.7.2 On the Bernoulli-Carlitz Numbers

As a second application, we show the non vanishing of families of Bernoulli-Carlitz numbers modulo monic irreducible polynomials P for almost all P. This is a striking result as it is a stronger function field version of an open conjecture on Bernoulli numbers.

The classical Bernoulli numbers have been discovered and studied by Jacob Bernoulli during the late seventeenth century. They can be defined as the coefficients $B_m, m \geq 0$ which appear in the power series equality

$$\frac{t}{e^t - 1} = \sum_{m \geq 0} B_m \frac{t^m}{m!}. \tag{7.11}$$

Euler computed the zeta values $\zeta(n) = \sum_{k \geq 1} k^{-n}$ for even positive integers n with the help of the Bernoulli numbers: if $n > 0$ is even then

$$\zeta(n) = \frac{-1}{2} \frac{(2i\pi)^n}{n!} B_n. \tag{7.12}$$

For more background on Bernoulli numbers, we refer the reader for instance to [IR90, Chapter 15 §1].

In 1935, Carlitz introduced analogues of the Bernoulli numbers. Those *Bernoulli-Carlitz* numbers are linked with the polynomials $\mathbb{B}_n(\mathbf{t})$. We prove in this section a quite surprising result on the Bernoulli-Carlitz numbers with the help of $\mathbb{B}_n(\mathbf{t})$. Let $N > 1$ be an integer and $N = \sum_{i=0}^r n_i q^i$ be its q-expansion. Then we define the Carlitz factorial as:

$$\Pi(N) = \prod_{i=0}^{r} D_i^{n_i} \in A$$

where we recall (see Sect. 7.3.3) that $D_0 = 1$, and for $i \geq 1$, $D_{i+1} = D_i^q(\theta^{q^{i+1}} - \theta)$. The Bernoulli-Carlitz numbers are defined as the coefficients BC_N, $N \geq 0$ which appear in the power series equality (similar to (7.11)):

$$\frac{t}{\exp_C(t)} = \sum_{m \geq 0} BC_N \frac{t^N}{\Pi(N)}.$$

We also recall that for $N \geq 1$, we have the Carlitz zeta value:

$$\zeta_A(N) = \sum_{a \in A_+} \frac{1}{a^N}.$$

Then the N-th Bernoulli-Carlitz number is $BC_N = 0$ if $N \not\equiv 0 \pmod{q-1}$ and, if $N \equiv 0 \pmod{q-1}$,

$$\zeta_A(N) = \frac{\widetilde{\pi}^N}{\Pi(N)} BC_N$$

reminding of Euler's formula (7.12). (Remark that the role of 2 is played here by $q - 1$.)

If we have the q-expansion $N = \sum_{i=0}^r n_i q^i$, then we denote $\ell_q(N) = \sum_{i=0}^r n_i$ and define the evaluation map $\mathrm{ev}_N : \mathbb{T}_{\ell_q(N)}(K_\infty) \to K_\infty$ by $\mathrm{ev}_N(t_j) = \theta^{q^k}$ if $\sum_{i=0}^{k-1} n_i < j \leq \sum_{i=0}^k n_i$, so that

$$\mathrm{ev}_N(a(t_1) \cdots a(t_{\ell_q(N)})) = a^N.$$

We recall the link between Bernoulli-Carlitz numbers and the polynomials $\mathbb{B}_n(\mathbf{t})$:

Proposition 7.7.5 *Let $N \geq 2$, $N \equiv 1 \pmod{q-1}$. Let $P \in A$ be a monic irreducible polynomial of degree $d > 1$, such that $q^d > N$. Then $BC_{q^d-N} \equiv 0 \pmod{P}$ if and only if $\mathrm{ev}_N(\mathbb{B}_{\ell_q(N)}(\mathbf{t})) \equiv 0 \pmod{P}$.*

We do not give the proof, which can be found in [ANDTR19, Proposition 4.3] or in [AP14, Theorem 2]. Let us just sketch the main ideas: starting with the identity in $\ell_q(N)$ variables:

$$(-1)^{\frac{\ell_q(N)-1}{q-1}} \frac{L(\varphi/\mathbb{A})}{\widetilde{\pi}} \prod_{i=1}^{\ell_q(N)} \omega(t_i) = \mathbb{B}_{\ell_q(N)}(\mathbf{t}).$$

We then apply τ^d and evaluate with ev_N so that, up to some terms, the left hand side becomes $\frac{\zeta_A(q^d-N)}{\widetilde{\pi}^{q^d-N}} = \frac{BC_{q^d-N}}{\Pi_{q^d-N}}$, and the right hand side is congruent to $\mathrm{ev}_N(\mathbb{B}_{\ell_q(N)}(\mathbf{t})) \bmod P$ since for all $a \in A$, $a^{q^d} \equiv a \pmod{P}$.

As a consequence of Proposition 7.7.5, we see that if $\mathrm{ev}_N(\mathbb{B}_{\ell_q(N)}(\mathbf{t})) \neq 0$, then for all P not dividing $\mathrm{ev}_N(\mathbb{B}_{\ell_q(N)}(\mathbf{t}))$ and such that $q^{\deg P} > N$, that is, for almost all P, we have $BC_{q^d-N} \equiv 0 \pmod{P}$. In fact, we have the more precise result:

Theorem 7.7.6 *Let $N \geq 2$, $N \equiv 1 \pmod{q-1}$. Let $P \in A$ be a monic irreducible polynomial of degree $d > 1$, such that $q^d > N$. If $d \geq \frac{\ell_q(N)-1}{q-1}N$, then $BC_{q^d-N} \not\equiv 0 \pmod{P}$.*

This is a strong version of the following open conjecture on classical Bernoulli numbers:

Conjecture 7.7.7 *Let $N \geq 3$ be an odd integer, then there exist infinitely many prime numbers p such that $B_{p-N} \not\equiv 0 \pmod{p}$.*

It seems however reasonable to expect that the equivalent of Theorem 7.7.6 does not hold for Bernoulli numbers. Namely, if $N \geq 3$ is an odd integer, then there should exist infinitely many prime numbers p such that $B_{p-N} \equiv 0 \pmod{p}$. This would be an example where number fields and function fields lead to different results.

Theorem 7.7.6 is the main theorem of [ANDTR19]. The key result is that $\mathrm{ev}_N(\mathbb{B}_{\ell_q(N)}(\mathbf{t}))$ is not zero. We actually prove more generally:

Theorem 7.7.8 *Let $n \geq 2$, $n \equiv 1 \pmod{q-1}$. Then for any evaluation homomorphism $\mathrm{ev} : A[\mathbf{t}] \to A$ such that $\mathrm{ev}(t_i)$ is non constant for all i, we have*

$$\mathrm{ev}(\mathbb{B}_n(\mathbf{t})) \neq 0.$$

Proof We give a proof different from the one of [ANDTR19]. Recall:

$$H(\varphi; \mathbb{A}) = \frac{\varphi(\mathbb{K})}{\exp_\varphi(\mathbb{K}) + \varphi(\mathbb{A})}.$$

And $\mathbb{B}_n(\mathbf{t}) = [H(\varphi; \mathbb{A})]_{\mathbb{A}}$, in particular:

$$\mathbb{B}_n(\mathbf{t}) = \det(Z - \varphi_\theta \mid H(\varphi; \mathbb{A}))_{|Z=\theta}.$$

We set $r = \frac{n-q}{q-1}$. As for (7.10), we have

$$\exp_\varphi(\mathbb{K}) = \left\{ x \in \mathbb{K}, v_\infty(x) \geq \frac{n}{q-1} - 1 \right\}.$$

Since $\frac{n}{q-1} - 1 = r + \frac{1}{q-1}$, a basis of $H(\varphi; \mathbb{A})$ is given by $\frac{1}{\theta^r}, \cdots, \frac{1}{\theta}$. We compute the matrix of φ_θ in this basis. It is the sum of a matrix M_n that we must determine and of a nilpotent matrix $N_n = (\delta_{i,j+1})_{1 \leq i,j \leq r}$ where $\delta_{i,j}$ is the Kronecker symbol. That is, the coefficients of N_n immediately above the diagonal are 1, and 0 elsewhere. Note that M_n is the matrix of $(t_1 - \theta) \cdots (t_n - \theta)\tau$. Since $q(r-k) = r + n - q(k+1)$, we get in $H(\varphi; \mathbb{A})$:

$$(t_1 - \theta) \cdots (t_n - \theta)\tau(\frac{1}{\theta^{r-k}}) = \sum_{j=0}^{r-1} \frac{\sigma(q(k+1) - j)}{\theta^{r-j}}$$

where

$$\sigma(j) = (-1)^{j-1} \sum_{i_1 < i_2 < \cdots < i_j} t_{i_1} \cdots t_{i_j}$$

if $0 \leq j \leq n$, and $\sigma(j) = 0$ otherwise. (Note that $\sigma(0) = -1$.) Thus,

$$M_n = (\sigma(jq - (i-1)))_{1 \leq i, j \leq r} .$$

We will replace the polynomials $\sigma(j)$ by symbols independent of the number of variables in order to proceed by induction on n. We define on $\mathbb{F}_q$ variables Σ_j, $j > 0$ and a valuation val on $\mathbb{F}_q[\Sigma_j, j > 0]$ such that $val(\Sigma_j) = j$ by stating that if

$$f = \sum_{k_1, \ldots, k_n \geq 0} \alpha_{k_1, \ldots, k_n} \prod_{j=1}^{n} \Sigma_j^{k_j}$$

then $val(f) = -\infty$ if $f = 0$ and $val(f) = \inf\{\sum_{j=1}^{n} jk_j ; \alpha_{k_1, \ldots, k_n} \neq 0\}$ otherwise.

We moreover set $\Sigma_0 = -1$ and $\Sigma_j = 0$ if $j < 0$. Let

$$\mathbb{M}_n = \left(\Sigma_{jq-(i-1)}\right)_{1 \leq i, j \leq r} .$$

We have the evaluation map $\mathrm{ev}_n : \mathbb{F}_q[\Sigma_j, j > 0] \to \mathbb{F}_q[\mathbf{t}]$ defined by $\mathrm{ev}_n(\Sigma_j) = \sigma(j)$ (recall that $\sigma(j) = 0$ if $n < j$). Then $val(f)$ equals the valuation of $\mathrm{ev}_n(f)$ with respect to the ideal $(t_1, \ldots, t_n)$, and

$$M_n = \mathrm{ev}_n(\mathbb{M}_n).$$

Developing now $\det(ZI_r - \mathbb{M}_n - N_n)$ with respect to the last column, we find

$$\det(ZI_r - \mathbb{M}_n - N_n) = Z \det(ZI_{r-1} - \mathbb{M}_{n-(q-1)} - N_{n-(q-1)}) + \epsilon$$

where ϵ is a sum of terms which are multiples of elements in the last column of $\mathbb{M}_n$, that is, $\Sigma_{rq-(i-1)}$, $0 \leq i \leq r$ all of them of valuation at least $rq - (r-1) = r(q-1) + 1$.

Thus, by induction, $\det(ZI_r - \mathbb{M}_n - N_n) = Z^r + \sum_{i=1}^{r} \beta_i Z^{r-i}$ with $val(\beta_i) \geq i(q-1) + 1$, and thus

$$\mathbb{B}_n = \theta^r + \sum_{i=1}^{r} B_i(\mathbf{t})\theta^{r-i}$$

where the valuation of $B_i(\mathbf{t}) \in \mathbb{F}_q[\mathbf{t}]$ with respect to $(t_1, \ldots, t_n)$ is at least $i(q-1)+1$. Thus for every evaluation homomorphism ev, $\mathrm{ev}(\mathbb{B}_n(\mathbf{t}))$ has valuation r at the place θ of A. □

7.8 Stark Units in More General Settings

In this final short section, we want to stress out that the machinery of Stark units carries over to more general settings than Drinfeld $\mathbb{F}_q[\theta]$-modules. The results presented in Sect. 7.4 have indeed been developed in [ANDTR17] for Drinfeld modules over a general A. More precisely, we replace K with a function field in which $\mathbb{F}_q$ is algebraically closed, fix a place ∞ of K and write A for the ring of functions regular outside ∞ (see [Pel20, §2.2]). If L/K is a finite extension, a Drinfeld A-module over O_L is an $\mathbb{F}_q$-algebra homomorphism

$$\phi : \begin{cases} A \to O_L[\tau] \\ a \mapsto \phi_a \end{cases}$$

such that $\phi_a \equiv a \pmod{\tau}$ for all $a \in A$. We refer the reader to [Pel20, §3] for a presentation of the Drinfeld modules in this general setting. We can define units in this setting, and follow the constructions presented in this text, that is, twist the Frobenius by a new variable z, define z-units and evaluate them at $z = 1$ to obtain Stark units.

Let K_∞ denote the completion of K at ∞ and $\mathbb{F}_\infty$ its residue field. We choose a *sign function* $\mathrm{sgn} : K_\infty^\times \to \mathbb{F}_\infty^\times$, that is, a group homomorphism which is the identity on $\mathbb{F}_\infty^\times$. A rank one Drinfeld module ϕ is *sign-normalized* if there is an $i \in \mathbb{N}$ such that

$$\forall a \in A \backslash \{0\}, \quad \phi_a = a + a_1\tau + \cdots + \mathrm{sgn}(a)^{q^i} \tau^{\deg a}.$$

Stark units are used in [ANDTR17] to obtain various results for sign normalized rank one Drinfeld modules: explicitly computing the Taelman units, obtaining a class formula and some log-algebraicity results, that is, constructing explicit units by the mean of the L-series. As in Sect. 7.6.2, canonical deformations of these Drinfeld modules are also introduced by means of their shtuka functions.

In [ANDTR20a], Stark units have been extended to Anderson t-modules (for $A = \mathbb{F}_q[\theta]$) which are defined as $\mathbb{F}_q$-algebra homomorphisms

$$E : \begin{cases} \mathbb{F}_q[\theta] \to M_n(O_L)[\tau] \\ a \mapsto E_a = E_{a,0} + E_{a,1}\tau + \cdots + E_{a,r\deg a}\tau^{r\deg a} \end{cases}$$

such that $(E_a - E_{a,0})^n = 0$ for all $a \in \mathbb{F}_q[\theta]$. For instance, the n-th tensor power of the Carlitz module is the Anderson t-module defined by

$$E_\theta := \begin{pmatrix} \theta & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \theta \end{pmatrix} + \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix} \tau.$$

We refer the reader to [AT90] for more details about these Anderson t-modules.

Once again, Stark units play a key role in [ANDTR20a] to determine the Taelman's units of t-modules which allows to prove that a large class of t-modules satisfy a conjecture of Taelman stated in [Tae09]. They are also used to establish log-algebraicity identities for the tensor powers of the Carlitz module.

One can finally extend the definition of t-module to a general A and define Stark units in this context where the machinery of Sect. 7.4 still works.

We also signal to the reader two very recent works involving Stark units: in [GND20] Green and Ngo Dac use Stark units to obtain log-algebraic identities for Anderson t-modules. They derive from it some logarithmic identities on multiple zeta values. In [ANDTR20b], the authors prove a class formula generalizing Theorem 7.5.3 to a large class of Anderson modules over a general A, which includes in particular all Drinfeld modules.

We will end this survey with a remark on the level of generality to which one can extend the notion of Stark units. At the beginning of this work, we had an exponential map, that is a power series in the Frobenius τ which satisfies a certain functional identity involving τ, and we wanted to study the Taelman units, that is the inverse image of the integral elements through the exponential map. We then introduced the Stark units by twisting the Frobenius τ with a new variable z and proceeded to the study of the z-units before evaluating at 1 to get a natural submodule of the Taelman units. If we now consider a difference field (K, τ) (see [DV20, §2]), then the above construction should carry over if we have a suitable exponential map. It would be interesting to work out Stark units in this general setting (which involves a definition of a *suitable* exponential map). Due to the formal nature of the construction, one would expect applications mainly in the case of non archimedean fields. L. Di Vizio's contribution [DV20] to this volume gives many examples of difference fields for which one could try to see what comes out from a construction of Stark units.

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