Chapter 6
Berkovich Curves and Schottky Uniformization II: Analytic Uniformization of Mumford Curves

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Abstract This is the second part of a survey on the theory of non-Archimedean curves and Schottky uniformization from the point of view of Berkovich geometry. It is more advanced than the first part and covers the theory of Mumford curves and Schottky uniformization. We start by briefly reviewing the theory of Berkovich curves, then introduce Mumford curves in a purely analytic way (without using formal geometry). We define Schottky groups acting on the Berkovich projective line, highlighting how geometry and group theory come together to prove that the quotient by the action of a Schottky group is an analytic Mumford curve. Finally, we present an analytic proof of Schottky uniformization, showing that any analytic Mumford curve can be described as a quotient of this kind. The guiding principle of our exposition is to stress notions and fully prove results in the theory of non-Archimedean curves that, to our knowledge, are not fully treated in other texts.

6.1 Introduction

In the first part [PT20] of this survey, we provided a concrete description of the Berkovich affine line over a non-Archimedean complete valued field $(k, |\cdot|)$ and investigated its main properties. It is a remarkable fact that, combining topology, algebra, and combinatorics, one can still get a very satisfactory description of more general analytic curves over k, in the sense of Berkovich theory.

If k is algebraically closed, for instance, one can show that a smooth compact Berkovich curve X can always be decomposed into a finite graph and an infinite number of open discs. If the genus of X is positive, there exists a smallest graph

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B. Anglès, T. Ngo Dac (eds.), *Arithmetic and Geometry over Local Fields*,
Lecture Notes in Mathematics 2275, https://doi.org/10.1007/978-3-030-66249-3_6

satisfying this property. It is classically called the *skeleton* of X, an invariant that encodes a surprising number of properties of X. As an example, if the Betti number of the skeleton of X is equal to the genus of X and is at least 2, then the curve X can be described analytically as a quotient $\Gamma \backslash O$, where O is an open dense subset of the projective analytic line $\mathbb{P}^{1,\mathrm{an}}_k$ and Γ a suitable subgroup of $\mathrm{PGL}_2(k)$. This phenomenon is known as *Schottky uniformization*, and it is the consequence of a celebrated theorem of D. Mumford, which is the main result of [Mum72a].

Obviously, D. Mumford's proof did not make use of Berkovich spaces, as they were not yet introduced at that time, but rather of formal geometry and the theory of Bruhat-Tits trees. A few years later, L. Gerritzen and M. van der Put recasted the theory purely in the language of rigid analytic geometry (using in a systematic way the notion of reduction of a rigid analytic curve). We refer the reader to the reference manuscript [GvdP80] for a detailed account of the theory and related topics, enriched with examples and applications.

In this text, we develop the whole theory of *Schottky groups* and *Mumford curves* from scratch, in a purely analytic manner, relying in a crucial way on the nice topological properties of Berkovich spaces, and the tools that they enable us to use: the theory of proper actions of groups on topological spaces, of fundamental groups, etc. We are convinced that those features, and Berkovich's point of view in general, will help improve our understanding of Schottky uniformization.

In this second part of the survey, we have allowed ourselves to be sometimes more sketchy than in the first part [PT20], but this should not cause any trouble to anyone familiar enough with the theory of Berkovich curves. We begin by reviewing standard material. In Sect. 6.2, we define the Berkovich projective line $\mathbb{P}^{1,\mathrm{an}}_k$ over k, consider its group of k-linear automorphisms $\mathrm{PGL}_2(k)$ and introduce the Koebe coordinates for the loxodromic transformations. In Sect. 6.3, we give an introduction to the theory of Berkovich analytic curves, starting with those that locally look like the affine line. For the more general curves, we review the theory without proofs, but provide some references. We conclude this section by an original purely analytic definition of Mumford curves. In Sect. 6.4, we propose two definitions of Schottky groups, first using the usual description of their fundamental domains and second, via their group theoretical properties, using their action of $\mathbb{P}^{1,\mathrm{an}}_k$. We show that they coincide by relying on the nice topological properties of Berkovich spaces. In Sect. 6.5, we prove that every Mumford curve may be uniformized by a dense open subset of $\mathbb{P}^{1,\mathrm{an}}_k$ with a group of deck transformations that is a Schottky group. Once again, our proof is purely analytic, relying ultimately on arguments from potential theory. To the best of our knowledge, this is the first complete proof of this result. We conclude the section by investigating automorphisms of Mumford curves and giving explicit examples.

We put a great effort in providing a self-contained presentation of the results above and including details that are often omitted in the literature. Both the theories of Berkovich curves and Schottky uniformization have a great amount of ramifications and interactions with other branches of mathematics. For the interested readers, we provide an appendix with a series of references that will hopefully help them to navigate through this jungle of wonderful mathematical objects.

The idea of writing down these notes came to the first author when he was taking part to the VIASM School on Number Theory in June 2018 in Hanoi. Just as the school was, the material presented here is primarily aimed at graduate students, although we also cover some of the most advanced developments in the field. Moreover, we have included questions that we believe could be interesting topics for young researchers (see Remarks 6.4.20 and 6.5.7). The appendix provides additional material leading to active subjects of research and open problems.

The different chapters in this volume are united by the use of analytic techniques in the study of arithmetic geometry. While they treat different topics, we encourage the reader to try to understand how they are related and may shed light on each other. In particular, the lecture notes of F. Pellarin [Pel20], about Drinfeld modular forms, mention several topics related to ours, although phrased in the language of rigid analytic spaces, such as Schottky groups (Section 5) or quotient spaces (Section 6). It would be interesting to investigate to what extent the viewpoint of Berkovich geometry presented here could provide a useful addition to this theory.

We retain notation as in [PT20]. In particular, $(k, |\cdot|)$ is a non-Archimedean complete valued field, k^a is a fixed algebraic closure of it, and $\widehat{k^a}$ is the completion of the latter.

6.2 The Berkovich Projective Line and Möbius Transformations

6.2.1 Affine Berkovich Spaces

We generalize the constructions of [PT20], replacing $k[T]$ by an arbitrary k-algebra of finite type. Our reference here is [Ber90, Section 1.5].

Definition 6.2.1 Let A be k-algebra of finite type. The *Berkovich spectrum* $\mathrm{Spec}^{\mathrm{an}}(A)$ of A is the set of multiplicative seminorms on A that induce the given absolute value $|\cdot|$ on k.

As in [PT20, Definition 5.3.1], we can associate a completed residue field $\mathscr{H}(x)$ to each point x of $\mathrm{Spec}^{\mathrm{an}}(A)$. As in [PT20, Section 5.4], we endow $\mathrm{Spec}^{\mathrm{an}}(A)$ with the coarsest topology that makes continuous the maps of the form

$$x \in \mathrm{Spec}^{\mathrm{an}}(A) \longmapsto |f(x)| \in \mathbb{R}$$

for $f \in A$. Properties similar to that of the Berkovich affine line still hold in this setting: the space $\mathrm{Spec}^{\mathrm{an}}(A)$ is countable at infinity, locally compact and locally path-connected.

We could also define a sheaf of function on $\mathrm{Spec}^{\mathrm{an}}(A)$ as in [PT20, Definition 5.5.1][1] with properties similar to that of the usual complex analytic spaces.

Lemma 6.2.2 *Each morphism of k-algebras $\varphi\colon A \to B$ induces a continuous map of Berkovich spectra*

$$\begin{array}{rccc} \mathrm{Spec}^{\mathrm{an}}(\varphi)\colon & \mathrm{Spec}^{\mathrm{an}}(B) & \longrightarrow & \mathrm{Spec}^{\mathrm{an}}(A) \\ & |\cdot|_x & \longmapsto & |\varphi(\cdot)|_x \end{array}.$$

Let us do the example of a localisation morphism.

Notation 6.2.3 Let A be a k-algebra of finite type and let $f \in A$. We set

$$D(f) := \{x \in \mathrm{Spec}^{\mathrm{an}}(A) : f(x) \neq 0\}.$$

It is an open subset of $\mathrm{Spec}^{\mathrm{an}}(A)$.

Lemma 6.2.4 *Let A be a k-algebra of finite type and let $f \in A$. The map* $\mathrm{Spec}^{\mathrm{an}}(A[1/f]) \to \mathrm{Spec}^{\mathrm{an}}(A)$ *induced by the localisation morphism $A \to A[1/f]$ induces a homeomorphism onto its image $D(f)$.* □

6.2.2 The Berkovich Projective Line

In this section, we explain how to construct the Berkovich projective line over k. It can be done, as usual, by gluing upside-down two copies of the affine line $\mathbb{A}_k^{1,\mathrm{an}}$ along $\mathbb{A}_k^{1,\mathrm{an}} - \{0\}$. We refer to [BR10, Section 2.2] for a definition in one step reminiscent of the "Proj" construction from algebraic geometry.

To carry out the construction of the Berkovich projective line more precisely, let us introduce some notation. We consider, as before, the Berkovich affine line $X := \mathbb{A}_k^{1,\mathrm{an}}$ with coordinate T, i.e. $\mathrm{Spec}^{\mathrm{an}}(k[T])$. By Lemma 6.2.4, its subset $U := \mathbb{A}_k^{1,\mathrm{an}} - \{0\} = D(T)$ may be identified with $\mathrm{Spec}^{\mathrm{an}}(k[T, 1/T])$.

We also consider another Berkovich affine line X' with coordinate T' and identify its subset $U' := X' - \{0\}$ with $\mathrm{Spec}^{\mathrm{an}}(k[T', 1/T'])$.

By Lemma 6.2.2, the isomorphism $k[T', 1/T'] \xrightarrow{\sim} k[T, 1/T]$ sending T' to $1/T$ induces an isomorphism $\iota\colon U \xrightarrow{\sim} U'$.

[1] Note however that the ring of global sections is always reduced, so that we only get the right notion when A is reduced. The proper construction involves defining first the space $\mathbb{A}_k^{n,\mathrm{an}} := \mathrm{Spec}^{\mathrm{an}}(k[T_1, \dots, T_n])$, then open subsets of it, and then closed analytic subsets of the latter, as we usually proceed for analytifications in the complex setting.

Definition 6.2.5 The Berkovich projective line $\mathbb{P}_k^{1,\mathrm{an}}$ is the space obtained by gluing the Berkovich affine lines X and X' along their open subsets U and U' via the isomorphim ι.

We denote by ∞ the image in $\mathbb{P}_k^{1,\mathrm{an}}$ of the point 0 in X'.

The basic topological properties of $\mathbb{P}_k^{1,\mathrm{an}}$ follow from that of $\mathbb{A}_k^{1,\mathrm{an}}$.

Proposition 6.2.6 *We have* $\mathbb{P}_k^{1,\mathrm{an}} = \mathbb{A}_k^{1,\mathrm{an}} \cup \{\infty\}$.

The space $\mathbb{P}_k^{1,\mathrm{an}}$ *is Hausdorff, compact, uniquely path-connected and locally path-connected.* □

For $x, y \in \mathbb{P}_k^{1,\mathrm{an}}$, we denote by $[x, y]$ the unique injective path between x and y.

6.2.3 Möbius Transformations

Let us recall that, in the complex setting, the group $\mathrm{PGL}_2(\mathbb{C})$ acts on $\mathbb{P}^1(\mathbb{C})$ via Möbius transformations. More precisely, to an invertible matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, one associates the automorphism

$$\gamma_A : z \in \mathbb{P}^1(\mathbb{C}) \longmapsto \frac{az+b}{cz+d} \in \mathbb{P}^1(\mathbb{C})$$

with the usual convention that, if $c \neq 0$, then $\gamma_A(\infty) = a/c$ and $\gamma_A(-d/c) = \infty$, and, if $c = 0$, then $\gamma_A(\infty) = \infty$.

We would like to define an action of $\mathrm{PGL}_2(k)$ on $\mathbb{P}_k^{1,\mathrm{an}}$ similar to the complex one. Let $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k)$.

First note that we can use the same formula as above to associate to A an automorphism γ_A of the set of rational points $\mathbb{P}_k^{1,\mathrm{an}}(k)$.

It is actually possible to deal with all the points this way. Indeed, let x be a point of $\mathbb{P}_k^{1,\mathrm{an}} - \mathbb{P}_k^{1,\mathrm{an}}(k)$. In [PT20, Section 5.3], we have associated to x a character $\chi_x : k[T] \to \mathscr{H}(x)$. Since x is not a rational point, $\chi_x(T)$ does not belong to k, hence the quotient $(a\chi_x(T)+b)/(c\chi_x(T)+d)$ makes sense. We can then define $\gamma_A(x)$ as the element of $\mathbb{A}_k^{1,\mathrm{an}}$ associated to the character

$$P(T) \in k[T] \mapsto P\Big(\frac{a\chi_x(T)+b}{c\chi_x(T)+d}\Big) \in \mathscr{H}(x).$$

This construction can also be made in a more algebraic way. By Lemmas 6.2.2 and 6.2.4, the morphism of k-algebras

$$P(T) \in k[T] \mapsto P\Big(\frac{aT+b}{cT+d}\Big) \in k\Big[T, \frac{1}{cT+d}\Big]$$

induces a map $\gamma_{A,1}\colon \mathbb{A}_k^{1,\mathrm{an}} - \{-\frac{d}{c}\} \to \mathbb{A}_k^{1,\mathrm{an}} \subseteq \mathbb{P}_k^{1,\mathrm{an}}$ (with the convention that $-d/c = \infty$ if $c = 0$).

Similarly, the morphism of k-algebras

$$Q(U) \in k[T'] \mapsto Q\Big(\frac{c + dT'}{a + bT'}\Big) \in k\Big[T', \frac{1}{a + bT'}\Big]$$

induces a map $\gamma_{A,2}\colon \mathbb{P}_k^{1,\mathrm{an}} - \{0, -\frac{b}{a}\} \to \mathbb{P}_k^{1,\mathrm{an}}$ (with the convention that $-b/a = \infty$ if $a = 0$).

A simple computation shows that the maps $\gamma_{A,1}$ and $\gamma_{A,2}$ are compatible with the isomorphism ι from Sect. 6.2.2. Note that we always have $-\frac{d}{c} \neq -\frac{b}{a}$. If $ad \neq 0$, it follows that we have $\big(\mathbb{A}_k^{1,\mathrm{an}} - \{-\frac{d}{c}\}\big) \cup \big(\mathbb{P}_k^{1,\mathrm{an}} - \{0, -\frac{b}{a}\}\big) = \mathbb{P}_k^{1,\mathrm{an}}$, so the two maps glue to give a global map

$$\gamma_A\colon \mathbb{P}_k^{1,\mathrm{an}} \to \mathbb{P}_k^{1,\mathrm{an}}.$$

We let the reader handle the remaining cases by using appropriate changes of variables.

Notation 6.2.7 For $a, b, c, d \in k$ with $ad - bc \neq 0$, we denote by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the image in $\mathrm{PGL}_2(k)$ of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

From now on, we will identify each element A of $\mathrm{PGL}_2(k)$ with the associated automorphism γ_A of $\mathbb{P}_k^{1,\mathrm{an}}$.

Lemma 6.2.8 *The image of a closed (resp. open) disc of* $\mathbb{P}_k^{1,\mathrm{an}}$ *by a Möbius transformation is a closed (resp. open) disc.*

Proof Let $A \in \mathrm{GL}_2(k)$. We may extend the scalars, hence assume that k is algebraically closed. In this case, A is similar to an upper triangular matrix. In other words, up to changing coordinates of $\mathbb{P}_k^{1,\mathrm{an}}$, we may assume that A is upper triangular. The transformation γ_A is then of the form

$$\gamma_A\colon z \in \mathbb{P}_k^{1,\mathrm{an}} \mapsto \alpha z \in \mathbb{P}_k^{1,\mathrm{an}}$$

or

$$\gamma_A\colon z \in \mathbb{P}_k^{1,\mathrm{an}} \mapsto z + \alpha \in \mathbb{P}_k^{1,\mathrm{an}}$$

for some $\alpha \in k$. In both cases, the result is clear. □

6.2.4 Loxodromic Transformations and Koebe Coordinates

Definition 6.2.9 A matrix in $\mathrm{GL}_2(k)$ is said to be *loxodromic* if its eigenvalues in k^a have distinct absolute values.

A Möbius transformation is said to be *loxodromic* if some (or equivalently every) representative is.

Lemma 6.2.10 *Let $a, b, c, d \in k$ with $ad - bc \neq 0$ and set $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k)$. Then A is loxodromic if, and only if, we have $|ad - bc| < |a + d|^2$.*

Proof Let λ and λ' be the eigenvalues of A in k^a. We may assume that $|\lambda| \leqslant |\lambda'|$.

On the one hand, if $|\lambda| = |\lambda'|$, then we have

$$|a + d|^2 = |\lambda + \lambda'|^2 \leqslant |\lambda'|^2 = |\lambda|\,|\lambda'| = |ad - bc|.$$

On the other hand, if $|\lambda| < |\lambda'|$, then we have

$$|a + d|^2 = |\lambda + \lambda'|^2 = |\lambda'|^2 > |\lambda|\,|\lambda'| = |ad - bc|.$$

□

Let $A \in \mathrm{PGL}_2(k)$ be a loxodromic Möbius transformation.

Fix some representative B of A in $\mathrm{GL}_2(k)$. Denote by λ and λ' its eigenvalues in k^a. We may assume that $|\lambda| < |\lambda'|$. The characteristic polynomial χ_B of B cannot be irreducible over k, since otherwise its roots in k^a would have the same absolute values. It follows that λ and λ' belong to k. Set $\beta := \lambda/\lambda' \in k^{\circ\circ}$.

The eigenspace of B associated to the eigenvalue λ (resp. λ') is a line in k^2. Denote by α (resp. α') the corresponding element in $\mathbb{P}^1(k)$.

Definition 6.2.11 The elements $\alpha, \alpha' \in \mathbb{P}^1(k)$ and $\beta \in k^{\circ\circ}$ depend only on A and not on the chosen representative. They are called the *Koebe coordinates* of A.

There exists a Möbius transformation $\varepsilon \in \mathrm{PGL}_2(k)$ such that $\varepsilon(0) = \alpha$ and $\varepsilon(\infty) = \alpha'$. The Möbius transformation $\varepsilon^{-1}A\varepsilon$ now has eigenspaces corresponding to 0 and ∞ in $\mathbb{P}^1(k)$ and the associated automorphism of $\mathbb{P}^{1,\mathrm{an}}_k$ is

$$\gamma_{\varepsilon^{-1}A\varepsilon} \colon z \in \mathbb{P}^{1,\mathrm{an}}_k \mapsto \beta z \in \mathbb{P}^{1,\mathrm{an}}_k.$$

We deduce that 0 and ∞ are respectively the attracting and repelling fixed points of $\gamma_{\varepsilon^{-1}A\varepsilon}$ in $\mathbb{P}^{1,\mathrm{an}}_k$. It follows that α and α' are respectively the attracting and repelling fixed points of γ_A in $\mathbb{P}^{1,\mathrm{an}}_k$.

The same argument shows that the Koebe coordinates determine uniquely the Möbius transformation A. In fact, given $\alpha, \alpha', \beta \in k$ with $\alpha \neq \alpha'$ and $0 < |\beta| < 1$, the Möbius transformation that has these elements as Koebe coordinates is given explicitly by

$$M(\alpha, \alpha', \beta) = \begin{bmatrix} \alpha - \beta\alpha' & (\beta - 1)\alpha\alpha' \\ 1 - \beta & \beta\alpha - \alpha' \end{bmatrix}, \tag{6.2.1}$$

In an analogous way, whenever $\infty \in \mathbb{P}^{1,\mathrm{an}}_k$ is an attracting or repelling point of a loxodromic Möbius transformation, we can recover the latter as:

$$M(\alpha, \infty, \beta) = \begin{bmatrix} \beta & (1 - \beta)\alpha \\ 0 & 1 \end{bmatrix} \text{ or } M(\infty, \alpha', \beta) = \begin{bmatrix} 1 & (\beta - 1)\alpha' \\ 0 & \beta \end{bmatrix}. \tag{6.2.2}$$

Remark 6.2.12 Let $A \in \mathrm{PGL}_2(k)$ be a Möbius transformation that is not loxodromic. Then, extending the scalars to $\widehat{k^a}$ and possibly changing the coordinates, the associated automorphism of $\mathbb{P}^{1,\mathrm{an}}_{\widehat{k^a}}$ is a homothety

$$z \in \mathbb{P}^{1,\mathrm{an}}_{\widehat{k^a}} \mapsto \beta z \in \mathbb{P}^{1,\mathrm{an}}_{\widehat{k^a}} \text{ with } |\beta| = 1$$

or a translation

$$z \in \mathbb{P}^{1,\mathrm{an}}_{\widehat{k^a}} \mapsto z + b \in \mathbb{P}^{1,\mathrm{an}}_{\widehat{k^a}}.$$

Note that those automorphisms have several fixed points in $\mathbb{P}^{1,\mathrm{an}}_{\widehat{k^a}}$ (η_r with $r \geqslant 0$ in the first case and $r \geqslant |b|$ in the second). It follows that A itself also has infinitely many fixed points in $\mathbb{P}^{1,\mathrm{an}}_k$.

6.3 Berkovich *k*-Analytic Curves

6.3.1 Berkovich $\mathbb{A}^1$-like Curves

In this section we go one step beyond the study of affine and projective lines, by introducing a class of curves that "locally look like the affine line", and see that there are interesting examples of curves belonging to this class.

A much more general theory of k-analytic curves exists but it will be discussed only briefly in this text in Sect. 6.3.2, in the case of smooth curves. For more on this topic, the standard reference is [Ber90, Chapter 4]. The most comprehensive account to-date can be found in A. Ducros' book project [Duc], while deeper discussions of specific aspects are contained in the references in the Appendix A.1 of the present text.

Definition 6.3.1 A *k-analytic $\mathbb{A}^1$-like curve* is a locally ringed space in which every point admits an open neighborhood isomorphic to an open subset of $\mathbb{A}_k^{1,\mathrm{an}}$.

It follows from the explicit description of bases of neighborhoods of points of $\mathbb{A}_k^{1,\mathrm{an}}$ (see [PT20, Proposition 5.4.11]) that each k-analytic $\mathbb{A}^1$-like curve admits a covering by virtual open Swiss cheeses. By local compactness, such a covering can always be found locally finite. It can be refined into a partition (no longer locally finite) consisting of simpler pieces.

Theorem 6.3.2 *Let X be a k-analytic $\mathbb{A}^1$-like curve. Then, there exist*

(i) a locally finite set S of type 2 points of X;
(ii) a locally finite set $\mathcal{A}$ of virtual open annuli of X;
(iii) a set $\mathcal{D}$ of virtual open discs of X

such that $S \cup \mathcal{A} \cup \mathcal{D}$ is a partition of X.

Proof Each virtual open Swiss cheese may be written as a union of finitely many points of type 2, finitely many virtual open annuli and some virtual open discs (as in Example 6.3.5 below). By a combinatorial argument that is not difficult but quite lengthy, the covering so obtained can be turned into a partition. □

Definition 6.3.3 Let X be a k-analytic $\mathbb{A}^1$-like curve. A partition $\mathcal{T} = (S, \mathcal{A}, \mathcal{D})$ of X satisfying the properties (i), (ii), (iii) of Theorem 6.3.2 is called a *triangulation* of X. The locally finite graph naturally arising from the set

$$\Sigma_{\mathcal{T}} := S \cup \bigcup_{A \in \mathcal{A}} \Sigma_A$$

is called the *skeleton* of $\mathcal{T}$. It is such that $X - \Sigma_{\mathcal{T}}$ is a disjoint union of virtual open discs.

A triangulation $\mathcal{T}$ is said to be *finite* if the associated set S is finite. If this is the case, then $\Sigma_{\mathcal{T}}$ is a finite graph. By the results of [PT20, Section 5.9], for each triangulation $\mathcal{T}$, $\Sigma_{\mathcal{T}}$ may be naturally endowed with a metric structure.

Remark 6.3.4 It is more usual to define a triangulation as the datum of the set S only. Note that S determines uniquely $\mathcal{A}$ and $\mathcal{D}$ since their elements are exactly the connected components of $X - S$, so our change of convention is harmless.

Example 6.3.5 Consider the curve

$$X := D^-(0, 1) - (D^+(a, r) \cup D^+(b, r))$$

for $r \in (0, 1)$ and $a, b \in k$ with $|a|, |b| < 1$, $|a - b| > r$. Set

$$S := \{\eta_{a,|a-b|}\},$$

$$\mathcal{A} := \{A^-(a, |a - b|, 1), A^-(a, r, |a - b|), A^-(b, r, |a - b|)\}$$

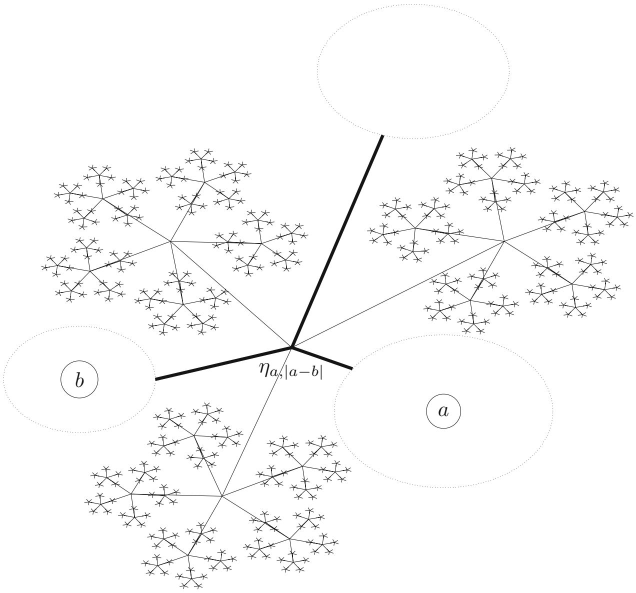

Fig. 6.1 The Swiss cheese X described in Example 6.3.5. Its skeleton Σ_X is the union of the three edges in evidence

and

$$\mathcal{D} := \{D^-(u, |a-b|), u \in k, |u-a| = |u-b| = |a-b|\}.$$

Then, the triple $\mathcal{T} := (S, \mathcal{A}, \mathcal{D})$ is a triangulation of X. The associated skeleton is a finite tree with three (half open) edges (Fig. 6.1).

Proposition 6.3.6 *Let X be a connected $\mathbb{A}^1$-like curve. Let $\mathcal{T} = (S, \mathcal{A}, \mathcal{D})$ be a triangulation of X such that $S \neq \emptyset$ or $\mathcal{A} \neq \emptyset$.*

There exists a canonical deformation retraction $\tau_{\mathcal{T}} \colon X \to \Sigma_{\mathcal{T}}$. Its restriction to any virtual open annulus $A \in \mathcal{A}$ induces the map τ_A from [PT20, Proposition 5.8.11] and its restriction to any connected component D of $A - \Sigma_A$ (which is a virtual open disc) induces the map τ_D from [PT20, Proposition 5.8.10].

In particular, for each $\eta \in \Sigma_A$, the set $\tau_{\mathcal{T}}^{-1}(\eta)$ is a virtual flat closed annulus. □

Definition 6.3.7 Let X be a k-analytic $\mathbb{A}^1$-like curve. The *skeleton* of X is the complement of all the virtual open discs contained in X. We denote it by Σ_X.

Remark 6.3.8 Let X be a k-analytic $\mathbb{A}^1$-like curve. It is not difficult to check that we have

$$\Sigma_X = \bigcap_{\mathcal{T}} \Sigma_{\mathcal{T}},$$

for $\mathcal{T}$ ranging over all triangulations of X. In particular, Σ_X is a locally finite metric graph (possibly empty).

Assume that X is connected and that Σ_X is non-empty. Then there exists a triangulation $\mathcal{T}_0$ of X such that $\Sigma_X = \Sigma_{\mathcal{T}_0}$. In particular, there is a canonical deformation retraction $\tau_X : X \to \Sigma_X$.

6.3.2 Arbitrary Smooth Curves

It goes beyond the scope of this survey to develop the full theory of Berkovich analytic curves. We only state a few definitions and general facts, to which we would like to refer later.

Definition 6.3.9 A *smooth k-analytic curve* is a locally ringed space X that is locally isomorphic to an open subset of a Berkovich spectrum of the form $\mathrm{Spec}^{\mathrm{an}}(A)$, where A is the ring of functions on a smooth affine algebraic curve over k.

For each smooth k-analytic curve X and each complete valued extension K of k, one may define the *base-change* X_K of X to K, by replacing each $\mathrm{Spec}^{\mathrm{an}}(A)$ by $\mathrm{Spec}^{\mathrm{an}}(A \otimes_k K)$ in its definition. It is a smooth K-analytic curve and there is a canonical projection morphism $\pi_{K/k} : X_K \to X$. The analogues of [PT20, Proposition 5.6.5] and [PT20, Corollary 5.6.6] hold in this more general setting.

Example 6.3.10 For each complete valued extension K of k, the base-change of $\mathbb{A}_k^{1,\mathrm{an}}$ to K is $\mathbb{A}_K^{1,\mathrm{an}}$.

If one starts with a smooth algebraic curve $\mathcal{X}$ over k, one may cover it by curves of the form $\mathrm{Spec}(A)$, with A as in Definition 6.3.9 above, and then glue the corresponding analytic spaces $\mathrm{Spec}^{\mathrm{an}}(A)$ to get a smooth k-analytic curve, called the *analytification* of $\mathcal{X}$, and denoted by $\mathcal{X}^{\mathrm{an}}$.

Example 6.3.11 The analytification of $\mathbb{A}_k^1$ is $\mathbb{A}_k^{1,\mathrm{an}}$.

As in the complex case, smooth compact k-analytic curves are automatically algebraic.

Theorem 6.3.12 *Let X be a smooth compact k-analytic curve. Then, there exists a projective smooth algebraic curve over k such that $X = \mathcal{X}^{\mathrm{an}}$.*

The invariants we have defined so far for the Berkovich affine line $\mathbb{A}_k^{1,\mathrm{an}}$ have natural counterparts for smooth k-analytic curves. Let X be a smooth k-analytic curve. For each point $x \in X$, the completed residue field $\mathscr{H}(x)$ is the completion of a finitely generated extension of k of transcendence degree less than or equal to 1. We may then define integers $s(x)$ and $t(x)$ such that $s(x) + t(x) \leqslant 1$ and the type of x, as we did in the case of $\mathbb{A}_k^{1,\mathrm{an}}$ (see [PT20, Definition 5.3.9]).

If x is of type 2, then, by the equality case in Abhyankar's inequality (see [PT20, Theorem 5.3.8]), the group $|\mathscr{H}(x)^\times|/|k^\times|$ is finitely generated, hence finite, and the field extension $\widetilde{\mathscr{H}(x)}/\tilde{k}$ is finitely generated.

Let us fix the definition of genus of an algebraic curve.

Definition 6.3.13 Let F be a field and let C be a projective curve over F, i.e. a connected normal projective scheme of finite type over F of dimension 1.

If F is algebraically closed, then C is smooth, and we define the *geometric genus of* C to be

$$g(C) := \dim_F H^0(C, \Omega_C).$$

In general, let $\bar{F}$ be an algebraic closure of F. Let C' be the normalization of a connected component of $C \times_F \bar{F}$. It is a projective curve over $\bar{F}$ and we define the *geometric genus of* C to be

$$g(C) := g(C').$$

It does not depend on the choice of C'.

Definition 6.3.14 Let X be a smooth k-analytic curve and let $x \in X$ be a point of type 2.

The *residue curve* at x is the unique (up to isomorphism) projective curve $\mathscr{C}_x$ over $\tilde{k}$ with function field $\widetilde{\mathscr{H}(x)}$. The *genus of* x is the geometric genus of $\mathscr{C}_x$. We denote it by $g(x)$.

The *stable genus of* x, is the genus of any point x' over x in $X_{\widehat{k^a}}$. We denote it by $g_{\mathrm{st}}(x)$. It does not depend on the choice of x'.

Example 6.3.15 Let $\alpha \in k$ and $r \in |k^\times|^{\mathbb{Q}}$. By [PT20, Example 5.3.10], the residue curve at the point $\eta_{\alpha,r}$ in $\mathbb{A}_k^{1,\mathrm{an}}$ is the projective line $\mathbb{P}^1_{\tilde{k}}$ over $\tilde{k}$. In particular, we have $g(\eta_{\alpha,r}) = 0$.

By [PT20, Lemma 5.3.11], any point of type 2 in $\mathbb{A}_k^{1,\mathrm{an}}$ (hence in any k-analytic $\mathbb{A}^1$-like curve) has stable genus 0.

The fact that the stable genus does not need to coincide with the genus is what motivates our definition. Let us give an example of this phenomenon.

Remark 6.3.16 Let $p \geqslant 5$ be a prime number. Consider the affine analytic plane $\mathbb{A}^{2,\mathrm{an}}_{\mathbb{Q}_p}$ with coordinates x, y. Let X be the smooth $\mathbb{Q}_p$-analytic curve inside $\mathbb{A}^{2,\mathrm{an}}_{\mathbb{Q}_p}$ given by the equation $y^2 = x^3 + p$ and let $\pi : X \to \mathbb{A}^{1,\mathrm{an}}_{\mathbb{Q}_p}$ be the projection onto the first coordinate x.

The fiber $\pi^{-1}(\eta_{0,|p|^{-1/3}})$ contains a unique point, that we will denote by a. One may check that $\widetilde{\mathscr{H}(a)}$ is a purely transcendental extension of $\mathbb{F}_p$ generated by the class u of px^3 (which coincides with the class of py^2):

$$\widetilde{\mathscr{H}(a)} \simeq \mathbb{F}_p(u).$$

In particular, we have $\mathscr{C}_a = \mathbb{P}^1_{\mathbb{F}_p}$ and $g(a) = 0$.

Let us now extend the scalars to the field $\mathbb{C}_p$, whose residue field is an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. Let b be the unique point of $X_{\mathbb{C}_p}$ over a. The field $\widetilde{\mathscr{H}(b)}$ is now generated by the class v of $p^{-1/3}x$ and the class w of $p^{-1/2}y$:

$$\widetilde{\mathscr{H}(b)} \simeq \overline{\mathbb{F}}_p(v)[w]/(v^3 - w^2 + 1).$$

In particular, $\mathscr{C}_b$ is an elliptic curve over $\overline{\mathbb{F}_p}$, and we have $g_{\mathrm{st}}(a) = g(b) = 1$.

We always have an inequality between genus and stable genus.

Lemma 6.3.17 *Let X be a smooth k-analytic curve and let $x \in X$ be a point of type 2. Then, we have $g(x) \leqslant g_{\mathrm{st}}(x)$.*

Proof Let x' be a point of $X_{\widehat{k^a}}$ over x. By definition, the residue curve $\mathscr{C}_x$ at x is defined over $\tilde{k}$ and the residue curve $\mathscr{C}_{x'}$ at x' is defined over an algebraic closure $\bar{\tilde{k}}$ of $\tilde{k}$.

The projection morphism $\pi_{\widehat{k^a}/k}\colon X_{\widehat{k^a}} \to X$ induces an isometric embedding $\mathscr{H}(x) \to \mathscr{H}(x')$, hence an embedding $\widetilde{\mathscr{H}(x)} \to \widetilde{\mathscr{H}(x')}$. It follows that we have a morphism $\mathscr{C}_{x'} \to \mathscr{C}_x$, hence a morphism $\varphi\colon \mathscr{C}_{x'} \to \mathscr{C}_x \times_{\tilde{k}} \bar{\tilde{k}}$. Its image is a connected component C of $\mathscr{C}_x \times_{\tilde{k}} \bar{\tilde{k}}$. The morphism φ factors through C, and even through the normalization $\widetilde{C}$ of C. By definition, we have $g(\widetilde{C}) = g(x)$ and $g(\mathscr{C}_{x'}) = g_{\mathrm{st}}(x)$. The result now follows from the Riemann–Hurwitz formula. □

Proposition 6.3.18 *Let X be a smooth k-analytic curve and let $x \in X$ be a point of type 2. There is a natural bijection between the closed points of the residue curve $\mathscr{C}_x$ at x and the set of directions emanating from x in X.* □

Example 6.3.19 Assume that k is algebraically closed. For $X = \mathbb{A}^{1,\mathrm{an}}_k$ and $x = \eta_1$, the result of Proposition 6.3.18 follows from [PT20, Lemma 5.4.9].

The structure of smooth k-analytic curves is well understood.

Theorem 6.3.20 *Every smooth k-analytic curve admits a triangulation in the sense of Theorem 6.3.2.*

The result of Proposition 6.3.6 also extends. If $\mathcal{T}$ is a non-empty triangulation of a smooth connected k-analytic curve X, then there is a canonical deformation retraction of X onto the skeleton $\Sigma_{\mathcal{T}}$ of $\mathcal{T}$, which is a locally finite metric graph.

We may also define the skeleton of X as in Definition 6.3.7, and it satisfies the properties of Remark 6.3.8.

Remark 6.3.21 With this purely analytic formulation, Theorem 6.3.20 is due to A. Ducros, who provided a purely analytic proof in [Duc]. It is very closely related to the semi-stable reduction theorem of S. Bosch and W. Lütkebohmert (see [BL85]): for each smooth k-analytic curve X, there exists a finite extension ℓ/k such that X_ℓ admits a model over ℓ° whose special fiber is a semi-stable curve over $\tilde{\ell}$, that is, it is reduced and its singularities are at worst double nodes.

If a smooth k-analytic curve X admits a semi-stable model over k°, then we may associate to it a triangulation of X. The points of S, $\mathcal{A}$ and $\mathcal{D}$ then correspond respectively to the irreducible components, the singular points and the smooth points of the special fiber of the model. Moreover, the genus of a point of S (which, in this case, coincides with its stable genus) is equal to the genus of the corresponding component. We refer to [Ber90, Theorem 4.3.1] for more details.

In the other direction, it is always possible to associate a model over k° to a triangulation of X, but it may fail to be semi-stable in general. The reader may consult [Duc, Sections 6.3 and 6.4] for general results.

Definition 6.3.22 Assume that k is algebraically closed. Let X be a smooth connected k-analytic curve. We define the *genus of* X to be

$$g(X) := b_1(X) + \sum_{x \in X^{(2)}} g(x),$$

where $b_1(X)$ is the first Betti number of X and $X^{(2)}$ the set of type 2 points of X.

If k is arbitrary, we define the genus of a smooth geometrically connected k-analytic curve X to be the genus of $X_{\widehat{k^a}}$.

This notion of genus is compatible with the one defined in the algebraic setting.

Theorem 6.3.23 *For each smooth geometrically connected projective algebraic curve $\mathcal{X}$ over k, we have*

$$g(\mathcal{X}) = g(\mathcal{X}^{\mathrm{an}}).$$

Let us finally comment that, among the results that are presented here, Theorem 6.3.20 is deep and difficult, but we will not need to use it since an easier direct proof is available for k-analytic $\mathbb{A}^1$-like curves (see Theorem 6.3.2). The others are rather standard applications of the general theory of curves.

6.3.3 Mumford Curves

Let us now return to $\mathbb{A}^1$-like curves over k. A special kind of such curves is obtained by asking for the existence of an open covering made of actual open Swiss cheeses over k rather than virtual ones. Recall that open Swiss cheeses over k are defined as the complement of closed discs in an open disc over k.

Definition 6.3.24 A connected, compact k-analytic ($\mathbb{A}^1$-like) curve X is called a *k-analytic Mumford curve* if every point $x \in X$ has a neighborhood that is isomorphic to an open Swiss cheese over k.

Remark 6.3.25 Such a curve is automatically projective algebraic by Theorem 6.3.12.

The following proposition relates the definition of a k-analytic Mumford curve with the existence of a triangulation of a certain type, and therefore with the original algebraic definition given by Mumford in [Mum72a]. Its proof uses some technical notions that were not fully presented in the first sections of this text, but we believe that the result of the proposition is important enough to deserve to be fully included for completeness.

Proposition 6.3.26 *Let X be a compact k-analytic curve.*

If $g(X) = 0$, then X is a k-analytic Mumford curve if and only if X is isomorphic to $\mathbb{P}^{1,\mathrm{an}}_k$.

If $g(X) \geqslant 1$, then X is a k-analytic Mumford curve if and only if there exists a triangulation $(S, \mathcal{A}, \mathcal{D})$ of X such that the points of S are of stable genus 0 and the elements of $\mathcal{A}$ are open annuli.

Proof

- Assume that $g(X) = 0$. If X is isomorphic to $\mathbb{P}^{1,\mathrm{an}}_k$, then it is obviously a Mumford curve.

  Conversely, assume that X is a k-analytic Mumford curve. By Theorems 6.3.12 and 6.3.23, it is isomorphic to the analytification of a projective smooth algebraic curve over k. Therefore, to prove that it is isomorphic to $\mathbb{P}^{1,\mathrm{an}}_k$, it is enough to prove that it has a k-rational point.

  By assumption, X contains an open Swiss cheese over k. In particular, it contains an open annulus A over k. Let x be a boundary point of the skeleton of A. By assumption, x has a neighborhood that is isomorphic to an open Swiss cheese over k. It follows that A is contained in a strictly bigger annulus A' whose skeleton strictly contains that of A. Arguing this way (possibly considering the union of all the annuli and applying the argument again), we show that X contains an open annulus over k of infinite modulus. At least one of its boundary points is a k-rational point, and the result follows.
- Assume that $g(X) \geqslant 1$. If X is a k-analytic Mumford curve, then it may be covered by finitely many Swiss cheeses over k. The result then follows from the

fact that every Swiss cheese over k admits a triangulation $(S, \mathcal{A}, \mathcal{D})$ such that the points of S are of stable genus 0 and the elements of $\mathcal{A}$ are annuli.

Conversely, assume that there exists a triangulation $(S, \mathcal{A}, \mathcal{D})$ of X satisfying the properties of the statement. Since $g(X) \geqslant 1$, we have $\mathcal{A} \neq \emptyset$. Up to adding a point of S in the skeleton of each element of $\mathcal{A}$, we may assume that all the elements of $\mathcal{A}$ have two distinct endpoints in X.

Let $x \in S$. Denote by $\mathcal{D}_x$ (resp. $\mathcal{A}_x$) the set of elements of $\mathcal{D}$ (resp. $\mathcal{A}$) that have x as an endpoint and set

$$U_x := \{x\} \cup \bigcup_{D \in \mathcal{D}_x} D \cup \bigcup_{A \in \mathcal{A}_x} A.$$

It is an open neighborhood of x in X. Let us now enlarge U_x in the following way: for each $A \in \mathcal{A}_x$, we paste a closed disc at the extremity of A that is different from x. The resulting curve V_x is compact, hence the analytification of a projective smooth algebraic curve over k, by Theorem 6.3.12. Since x is of stable genus 0, the genus of the base-change $(V_x)_{\widehat{k^a}}$ of V_x to $\widehat{k^a}$ is 0. By Theorem 6.3.23, we deduce that $(V_x)_{\widehat{k^a}}$ is isomorphic to $\mathbb{P}^{1,\mathrm{an}}_{\widehat{k^a}}$. Since V_x contains k-rational points (inside the pasted discs, for instance), V_x itself is isomorphic to $\mathbb{P}^{1,\mathrm{an}}_k$. We deduce that U_x is a Swiss cheese over k.

Since any point of X has a neighborhood that is of the form U_x for some $x \in S$, it follows that X is a Mumford curve.

□

Remark 6.3.27 If X is a compact k-analytic curve and k is algebraically closed, then Proposition 6.3.26 shows that the following properties are equivalent:

(i) X is a Mumford curve;
(ii) X is an $\mathbb{A}^1$-like curve;
(iii) the points of type 2 of X are all of genus 0.

Remark 6.3.28 Using the correspondence between triangulations and semi-stable models (see Remark 6.3.21), the result of Proposition 6.3.26 says that k-analytic Mumford curves are exactly those for which there exists a semi-stable model over k° whose special fiber consists of projective lines over $\tilde{k}$, intersecting transversally in $\tilde{k}$-rational points. This is indeed how algebraic Mumford curves are defined in Mumford's paper [Mum72a].

Corollary 6.3.29 *Let X be a k-analytic Mumford curve and $\mathcal{T}$ be a triangulation of X. Then the following quantities are equal:*

(i) the genus of X;
(ii) the cyclomatic number of the skeleton $\Sigma_{\mathcal{T}}$;
(iii) the first Betti number of X.

Proof We may assume that $\mathcal{T} = (S, \mathcal{A}, \mathcal{D})$ satisfies the conclusions of Proposition 6.3.26. We will assume that $\mathcal{A} \neq \emptyset$, the other case being dealt with similarly.

Consider the base-change morphism $\pi_{\widehat{k^a}/k} \colon X_{\widehat{k^a}} \to X$. By assumption, every element A of $\mathcal{A}$ is an annulus over k, hence its preimage $\pi^{-1}_{\widehat{k^a}/k}(A)$ is an annulus over $\widehat{k^a}$. In particular, $\pi_{\widehat{k^a}/k}$ induces a homeomorphism between the skeleton of $\pi^{-1}_{\widehat{k^a}/k}(A)$ and that of A. Since each point of S lies at the boundary of the skeleton of an element of $\mathcal{A}$, we deduce that each point of S has exactly one preimage by $\pi_{\widehat{k^a}}$.

It follows that the set $\mathcal{T}' = (S', \mathcal{A}', \mathcal{D}')$ of $X_{\widehat{k^a}}$, where

- S' is the set of preimages of the elements of S by $\pi_{\widehat{k^a}/k}$;
- $\mathcal{A}'$ is the set of preimages of the elements of $\mathcal{A}$ by $\pi_{\widehat{k^a}/k}$;
- $\mathcal{D}'$ is the set of connected components of the preimages of the elements of $\mathcal{D}$ by $\pi_{\widehat{k^a}/k}$

is a triangulation of $X_{\widehat{k^a}}$ and, moreover, that $\pi_{\widehat{k^a}/k}$ induces a homeomorphism between the skeleta $\Sigma_{\mathcal{T}'}$ and $\Sigma_{\mathcal{T}}$. In particular, their cyclomatic numbers are equal.

Since X is a Mumford curve, all the points of type 2 of the curve $X_{\widehat{k^a}}$ are of genus 0, hence the genus of $X_{\widehat{k^a}}$ coincides with its first Betti number, hence with the cyclomatic number of $\Sigma_{\mathcal{T}'}$, by Proposition 6.3.6. The equality between (i) and (ii) follows.

The equality between (ii) and (iii) follows from Proposition 6.3.6 again. □

6.4 Schottky Groups

Let $(k, |\cdot|)$ be a complete valued field. Some of the material of this section is adapted from Mumford [Mum72a], Gerritzen and van der Put [GvdP80] and Berkovich [Ber90, Section 4.4].

6.4.1 Schottky Figures

Let $g \in \mathbb{N}_{\geqslant 1}$.

Definition 6.4.1 Let $\gamma_1, \ldots, \gamma_g \in \mathrm{PGL}_2(k)$. Let $\mathcal{B} = (D^+(\gamma_i^\varepsilon), 1 \leqslant i \leqslant g, \varepsilon \in \{\pm 1\})$ be a family of pairwise disjoint closed discs in $\mathbb{P}^{1,\mathrm{an}}_k$. For each $i \in \{1, \ldots, g\}$ and $\varepsilon \in \{-1, 1\}$, set

$$D^-(\gamma_i^\varepsilon) := \gamma_i^\varepsilon(\mathbb{P}^1_k - D^+(\gamma_i^{-\varepsilon})).$$

We say that $\mathcal{B}$ is a *Schottky figure* adapted to $(\gamma_1, \ldots, \gamma_g)$ if, for each $i \in \{1, \ldots, g\}$ and $\varepsilon \in \{-1, 1\}$, $D^-(\gamma_i^\varepsilon)$ is a maximal open disc inside $D^+(\gamma_i^\varepsilon)$. (See Fig. 6.2 for an illustration.)

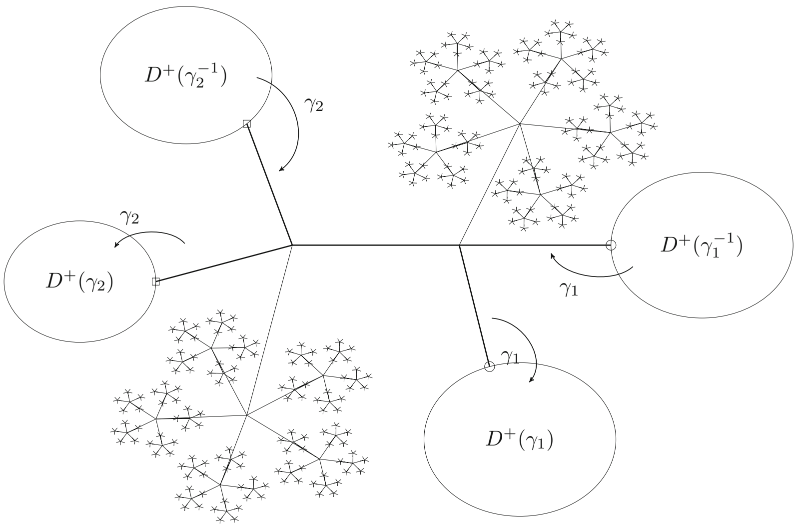

Fig. 6.2 A Schottky figure adapted to a pair (γ_1, γ_2)

Remark 6.4.2 Let $i \in \{1, \dots, g\}$. It follows from Remark 6.2.12 that γ_i is loxodromic. Moreover, denoting by α_i and α'_i the attracting and repelling fixed points of γ_i respectively, we have

$$\alpha'_i \in D^-(\gamma_i^{-1}) \text{ and } \alpha_i \in D^-(\gamma_i).$$

The result is easily proven for $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}$ and one may reduce to this case by choosing a suitable coordinate on $\mathbb{P}^{1,\mathrm{an}}_k$.

For the rest of the section, we fix $\gamma_1, \dots, \gamma_g \in \mathrm{PGL}_2(k)$ and a Schottky figure adapted to $(\gamma_1, \dots, \gamma_g)$, with the notation of Definition 6.4.1.

Notation 6.4.3 For $\sigma \in \{-, +\}$, we set

$$F^\sigma := \mathbb{P}^1_k - \bigcup_{\substack{1 \leqslant i \leqslant g \\ \varepsilon = \pm 1}} D^{-\sigma}(\gamma_i^\varepsilon).$$

Note that, for $\gamma \in \{\gamma_1^{\pm 1}, \dots, \gamma_g^{\pm 1}\}$, $D^+(\gamma)$ is the unique disc that contains $\gamma(F^+)$ among those defining the Schottky figure.

Remark 6.4.4 The sets F^- and F^+ are open and closed Swiss cheeses respectively.

Denote by ∂F^+ the boundary of F^+ in $\mathbb{P}^{1,\mathrm{an}}_k$. It is equal to the set of boundary points of the $D^+(\gamma_i^{\pm 1})$'s, for $i \in \{1, \dots, g\}$. The skeleton Σ_{F^+} of F^+ is the convex envelope of ∂F^+, that is to say the minimal connected graph containing ∂F^+, or

$$\Sigma_{F^+} = \bigcup_{x,y \in \partial F^+} [x, y].$$

The skeleton Σ_{F^-} of F^- satisfies

$$\Sigma_{F^-} = \Sigma_{F^+} \cap F^- = \Sigma_{F^+} - \partial F^+.$$

Set $\Delta := \{\gamma_1, \dots, \gamma_g\}$. Denote by F_g the abstract free group with set of generators Δ and by Γ the subgroup of $\mathrm{PGL}_2(k)$ generated by Δ. The existence of a Schottky figure for the g-tuple $(\gamma_1, \dots, \gamma_g)$ determines important properties of the group Γ. In fact, we have a natural morphism $\varphi \colon F_g \to \Gamma$ inducing an action of F_g on $\mathbb{P}^{1,\mathrm{an}}_k$. We now define a disc in $\mathbb{P}^1_k$ associated with each element of F_g. As usual, we will identify these elements with the words over the alphabet $\Delta^\pm := \{\gamma_1^{\pm 1}, \dots, \gamma_g^{\pm 1}\}$.

Notation 6.4.5 For a non-empty reduced word $w = w'\gamma$ over Δ and $\sigma \in \{-, +\}$, we set

$$D^\sigma(w) := w'\, D^\sigma(\gamma).$$

Lemma 6.4.6 *Let u be a non-empty reduced word over $\Delta^\pm$. Then we have $uF^+ \subseteq D^+(u)$.*

Let v be a non-empty reduced word over $\Delta^\pm$. If there exists a word w over $\Delta^\pm$ such that $u = vw$, then we have $uF^+ \subseteq D^+(u) \subseteq D^+(v)$. If, moreover, $u \neq v$, then we have $D^+(u) \subseteq D^-(v)$.

Conversely, if we have $D^+(u) \subseteq D^+(v)$, then there exists a word w over $\Delta^\pm$ such that $u = vw$.

Proof Write in a reduced form $u = u'\gamma$ with $\gamma \in \Delta^\pm$. We have $\gamma F^+ \subseteq D^+(\gamma)$, by definition. Applying u', it follows that $uF^+ \subseteq D^+(u)$.

Assume that there exists a word w such that $u = vw$ and let us prove that $D^+(u) \subseteq D^+(v)$. We first assume that v is a single letter. We will argue by induction on the length $|u|$ of u. If $|u| = 1$, then $u = v$ and the result is trivial. If $|u| \geqslant 2$, denote by δ the first letter of w. By induction, we have $D^+(w) \subseteq D^+(\delta)$. Since $\delta \neq v^{-1}$, we also have $D^+(\delta) \subseteq \mathbb{P}^1_k - D^+(v^{-1})$. The result follows by applying v.

Let us now handle the general case. Write in a reduced form $v = v'\gamma$ with $\gamma \in \Delta^\pm$. By the former case, we have $D^+(\gamma w) \subseteq D^+(\gamma)$ and $D^+(\gamma w) \subseteq D^-(\gamma)$ if w is non-empty. The result follows by applying v'.

Assume that we have $D^+(u) \subseteq D^+(v)$. We will prove that there exists a word w such that $u = vw$ by induction on $|v|$. Write in reduced forms $u = \gamma u'$ and $v = \delta v'$. By the previous result, we have $D^+(u) \subseteq D^+(\gamma)$ and $D^+(v) \subseteq D^+(\delta)$, hence $\gamma = \delta$. If $|v| = 1$, this proves the result. If $|v| \geqslant 2$, then we deduce that we have

$D^+(u') \subseteq D^+(v')$, hence, by induction, there exists a word w such that $u' = v'w$. It follows that $u = vw$. □

Proposition 6.4.7 *The morphism φ is an isomorphism and the group Γ is free on the generators $\gamma_1, \dots, \gamma_g$.*

Proof If w is a non-empty word, then the previous lemma ensures that $wF^+ \neq F^+$. The result follows. □

As a consequence, we now identify Γ with F_g and express the elements of Γ as words over the alphabet $\Delta^\pm$. In particular, we allow us to speak of the length of an element γ of Γ, that we denote by $|\gamma|$. Set

$$O_n := \bigcup_{|\gamma| \leqslant n} \gamma F^+.$$

Since the complement of F^+ is the disjoint union of the open disks $D^-(\gamma)$ with $\gamma \in \Delta^\pm$, it follows from the description of the action that, for each $n \geqslant 0$, we have

$$\mathbb{P}^{1,\mathrm{an}}_k - O_n = \bigsqcup_{|w|=n+1} D^-(w).$$

It follows from Lemma 6.4.6 that, for each $n \geqslant 0$, O_n is contained in the interior of O_{n+1}. We set

$$O := \bigcup_{n \geqslant 0} O_n = \bigcup_{\gamma \in \Gamma} \gamma F^+.$$

We now compute the orbits of discs under Möbius transformations $\mathbb{P}^{1,\mathrm{an}}_k$. Set $\iota := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{PGL}_2(k)$. It corresponds to the map $z \mapsto 1/z$ on $\mathbb{P}^{1,\mathrm{an}}_k$. The first result follows from an explicit computation.

Lemma 6.4.8 *Let $\alpha \in k^\times$ and $\rho \in [0, |\alpha|)$. Then, we have $\iota\big(D^+(\alpha, \rho)\big) = D^+\left(\frac{1}{\alpha}, \frac{\rho}{|\alpha|^2}\right)$.* □

Lemma 6.4.9 *Let $r > 0$ and let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathrm{PGL}_2(k)$ such that $\gamma\big(D^+(0, r)\big) \subseteq \mathbb{A}^{1,\mathrm{an}}_k$. Then, we have $|d| > r|c|$ and $\gamma\big(D^+(0, r)\big) = D^+\left(\frac{b}{d}, \frac{|ad-bc|\,r}{|d|^2}\right)$.*

Proof Let us first assume that $c = 0$. Then, we have $d \neq 0$, so the inequality $|d| > r|c|$ holds, and γ is affine with ratio a/d. The result follows.

Let us now assume that $c \neq 0$. In this case, we have $\gamma^{-1}(\infty) = -\frac{d}{c}$, which does not belong to $D(0, r)$ if, and only if, $|d| > r|c|$. Note that we have the following equality in $k(T)$:

$$\frac{aT + b}{cT + d} = \frac{a}{c} - \frac{ad - bc}{c^2} \frac{1}{T + \frac{d}{c}}.$$

By Lemma 6.4.8, there exist $\beta \in k$ and $\sigma > 0$ such that $\iota\big(D^+(\frac{d}{c}, r)\big) = D^+(\beta, \sigma)$. Then, we have $\gamma\big(D^+(0, r)\big) = D^+(\frac{a}{c} - \frac{ad-bc}{c^2}\beta, \big|\frac{ad-bc}{c^2}\big|\sigma)$ and the result follows from an explicit computation. □

Lemma 6.4.10 *Let $D' \subseteq D$ be closed discs in $\mathbb{A}_k^{1,\mathrm{an}}$. Let $\gamma \in \mathrm{PGL}_2(k)$ such that $\gamma D' \subseteq \gamma D \subseteq \mathbb{A}_k^{1,\mathrm{an}}$. Then, we have*

$$\frac{\text{radius of } \gamma D'}{\text{radius of } \gamma D} = \frac{\text{radius of } D'}{\text{radius of } D}.$$

Proof Let p be a k-rational point in D' and let τ be the translation sending p to 0. Up to changing D into τD, D' into $\tau D'$, γ into $\gamma\tau^{-1}$ and γ' into $\gamma'\tau^{-1}$, we may assume that D and D' are centered at 0. The result then follows from Lemma 6.4.9. □

Proposition 6.4.11 *Assume that $\infty \in F^-$. Then, there exist $R > 0$ and $c \in (0, 1)$ such that, for each $\gamma \in \Gamma - \{\mathrm{id}\}$, $D^+(\gamma)$ is a closed disc of radius at most $R\,c^{|\gamma|}$.*

Proof Let $\delta, \delta' \in \Delta^\pm$ such that $\delta' \neq \delta^{-1}$. By Lemma 6.4.6, we have $D^+(\delta'\delta) \subset D^-(\delta') \subseteq D^+(\delta')$. Set

$$c_{\delta,\delta'} := \frac{\text{radius of } D^+(\delta'\delta)}{\text{radius of } D^+(\delta')} \in (0, 1).$$

For each $\gamma \in \Gamma$ such that $\gamma\delta'$ is a reduced word, by Lemma 6.4.10, we have

$$\frac{\text{radius of } D^+(\gamma\delta'\delta)}{\text{radius of } D^+(\gamma\delta')} = \frac{\text{radius of } \gamma D^+(\delta'\delta)}{\text{radius of } \gamma D^+(\delta')} = c_{\delta,\delta'}.$$

Set

$$R := \max(\{\text{radius of } D^+(\delta) \mid \delta \in \Delta^\pm\})$$

and

$$c := \max(\{c_{\delta,\delta'} \mid \delta, \delta' \in \Delta^\pm, \delta' \neq \delta^{-1}\}).$$

By induction, for each $\gamma \in \Gamma - \{\mathrm{id}\}$, we have

$$\text{radius of } D^+(\gamma) \leqslant R\, c^{|\gamma|}.$$

□

Corollary 6.4.12 *Every element of* $\Gamma - \{\mathrm{id}\}$ *is loxodromic.*

Proof In order to prove the result, we may extend the scalars. As a result, we may assume that $F^- \cap \mathbb{P}^{1,\mathrm{an}}_k(k) \neq \emptyset$, hence up to changing coordinates, that $\infty \notin F^-$. Let $\gamma \in \Gamma - \{\mathrm{id}\}$. By Proposition 6.4.11 the radii of the discs $\gamma^n(D^+(\gamma))$ tend to 0 when n tends to ∞, which forces γ to be loxodromic, by Remark 6.2.12. □

Corollary 6.4.13 *Let* $w = (w_n)_{\neq 0}$ *be a sequence of reduced words over* $\Delta^\pm$ *such that the associated sequence of discs* $(D^+(w_n))_{n\geqslant 0}$ *is strictly decreasing. Then, the intersection* $\bigcap_{n\geqslant 0} D^+(w_n)$ *is a single k-rational point* p_w*. Moreover, the discs* $D^+(w_n)$ *form a basis of neighborhoods of* p_w *in* $\mathbb{P}^{1,\mathrm{an}}_k$.

Proof Let k_0 be a finite extension of k such that $F^- \cap \mathbb{P}^1(k_0) \neq \emptyset$. Consider the projection morphism $\pi_0 \colon \mathbb{P}^{1,\mathrm{an}}_{k_0} \to \mathbb{P}^{1,\mathrm{an}}_k$. For each $i \in \{1, \dots, g\}$, γ_i may be identified with an element $\gamma_{i,0}$ in $\mathrm{PGL}_2(k_0)$. The family $(\pi_0^{-1}(D^-(\gamma_i^{\pm 1}), 1 \leqslant i \leqslant g, \varepsilon = \pm 1)$ is a Schottky figure adapted to $(\gamma_{1,0}, \dots, \gamma_{g,0})$. We will denote with a subscript 0 the associated sets: F_0^-, $D_0^+(w)$, etc. Note that these sets are all equal to the preimages of the corresponding sets by π_0.

Up to changing coordinates on $\mathbb{P}^{1,\mathrm{an}}_{k_0}$, we may assume that $\infty \in F_0^-$. The sequence of discs $(D_0^+(w_n))_{n\geqslant 0}$ is strictly decreasing, so by Lemma 6.4.6, the length of w_n tends to ∞ when n goes to ∞ and, by Proposition 6.4.11, the radius of $D_0^+(w_n)$ tends to 0 when n goes to ∞. It follows that $\bigcap_{n\geqslant 0} D_0^+(w_n)$ is a single point $p_{w,0}$ of type 1 and that the discs $D_0^+(w_n)$ form a basis of neighborhood of $p_{w,0}$ in $\mathbb{P}^{1,\mathrm{an}}_{k_0}$.

Set $p_w := \pi_0(p_{w,0})$. It follows from the results over k_0 that $\bigcap_{n\geqslant 0} D^+(w_n) = \{p_w\}$ and that the discs $D^+(w_n)$ form a basis of neighborhoods of p_w in $\mathbb{P}^{1,\mathrm{an}}_k$.

It remains to show that p_w is k-rational. Note that p_w belongs to the closure of $\mathbb{P}^1(k)$, since it is the limit of the centers of the $D^+(w_n)$'s. Since k is complete, $\mathbb{P}^1(k)$ is closed in $\mathbb{P}^1(\widehat{k^a})$ and the result follows. □

Corollary 6.4.14 *The set* O *is dense in* $\mathbb{P}^{1,\mathrm{an}}_k$ *and its complement is contained in* $\mathbb{P}^1(k)$. □

Definition 6.4.15 We say that a point $x \in \mathbb{P}^{1,\mathrm{an}}_k$ is a *limit point* if there exist a point $x_0 \in \mathbb{P}^{1,\mathrm{an}}_k$ and a sequence $(\gamma_n)_{n\geqslant 0}$ of distinct elements of Γ such that $\lim_{n\to\infty} \gamma_n(x_0) = x$.

The *limit set* L of Γ is the set of limit points of Γ.

Let us add a short reminder on proper group actions.

Definition 6.4.16 ([Bou71, III, §4, Définition 1]) We say that the action of a topological group G on a topological space E is *proper* if the map

$$\begin{aligned} \Gamma \times E &\to E \times E \\ (\gamma, x) &\mapsto (x, \gamma \cdot x) \end{aligned}$$

is proper.

Proposition 6.4.17 ([Bou71, III, §4, Propositions 3 and 7]) *Let G be a locally compact topological group and E be a Hausdorff topological space. Then, the action of G on E is proper if, and only if, for every $x, y \in E$, there exist neighborhoods U_x and U_y of x and y respectively such that the set $\{\gamma \in \Gamma \mid \gamma U_x \cap U_y \neq \emptyset\}$ is relatively compact (that is to say finite, if G is discrete).*

In this case, the quotient space $\Gamma\backslash E$ is Hausdorff. □

We denote by C the set of points $x \in \mathbb{P}_k^{1,\mathrm{an}}$ that admit a neighborhood U_x satisfying $\{\gamma \in \Gamma : \gamma U_x \cap U_x \neq \emptyset\} = \{\mathrm{id}\}$. The set C is an open subset of $\mathbb{P}_k^{1,\mathrm{an}}$ and the quotient map $C \to \Gamma\backslash C$ is a local homeomorphism. In particular, the topological space $\Gamma\backslash C$ is naturally endowed with a structure of analytic space via this map.

Theorem 6.4.18 *We have $O = C = \mathbb{P}_k^{1,\mathrm{an}} - L$. Moreover, the action of Γ on O is free and proper and the quotient $\Gamma\backslash O$ is a Mumford curve of genus g.*

Set $X := \Gamma\backslash O$ and denote by $p\colon O \to X$ the quotient map. Let Σ_O, Σ_{F^+} and Σ_X denote the skeleta of O, F^+ and X respectively. Then, Σ_O is the trace on O of the convex envelope of L:

$$\Sigma_O = O \cap \bigcup_{x,y \in L} [x, y]$$

and we have

$$p^{-1}(\Sigma_X) = \Sigma_O \text{ and } p(\Sigma_O) = p(\Sigma_{F^+}) = \Sigma_X.$$

(See Fig. 6.3 for an illustration.)

Proof Let $x \in L$. By definition, there exist $x_0 \in \mathbb{P}_k^{1,\mathrm{an}}$ and a sequence $(\gamma_n)_{n\geqslant 0}$ of distinct elements of Γ such that $\lim_{n\to\infty} \gamma_n(x_0) = x$. Assume that $x \in F^+$. Since F^+ is contained in the interior of O_1, there exists $N \geqslant 0$ such that $\gamma_N(x_0) \in O_1$, hence we may assume that $x_0 \in O_1$. Lemma 6.4.6 then leads to a contradiction. It follows that L does not meet F^+, hence, by Γ-invariance, L is contained in $\mathbb{P}_k^{1,\mathrm{an}} - O$.

Let $y \in \mathbb{P}_k^{1,\mathrm{an}} - O$. By definition, there exists a sequence $(w_n)_{n\geqslant 0}$ of reduced words over $\Delta^\pm$ such that, for each $n \geqslant 0$, $|w_n| \geqslant n$ and $y \in D^-(w_n)$. Let $y_0 \in F^-$. By Lemma 6.4.6, for each $n \geqslant 0$, we have $w_n(y_0) \in D^-(w_n)$ and the sequence of

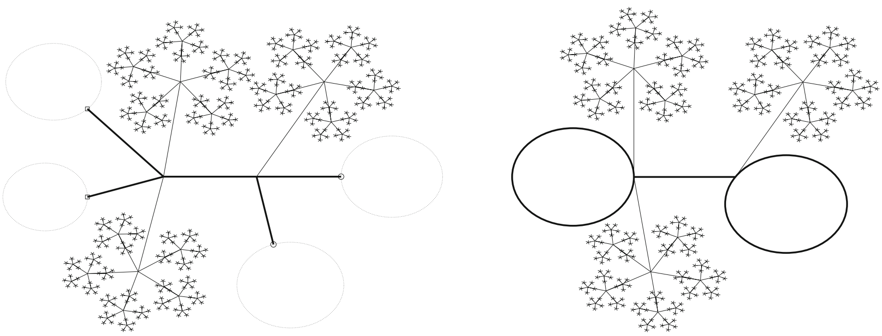

Fig. 6.3 The closed fundamental domain F^+ (on the left) of the Schottky group Γ is a Swiss cheese. The group Γ identifies the ends of the skeleton Σ_{F^+}, so that the corresponding Mumford curve (on the right) contains the finite graph Σ_X

discs $(D^+(w_n))_{n\geqslant 0}$ is strictly decreasing. By Corollary 6.4.13, $(w_n(y_0))_{n\geqslant 0}$ tends to y, hence $y \in L$. It follows that $\mathbb{P}^{1,\mathrm{an}}_k - O = L$.

Set

$$U := F^+ \cup \bigcup_{\gamma\in\Delta^\pm} \gamma F^- = \mathbb{P}^{1,\mathrm{an}}_k - \bigsqcup_{|\gamma|=2} D^+(\gamma).$$

It is an open subset of $\mathbb{P}^{1,\mathrm{an}}_k$ and it follows from the properties of the action (see Lemma 6.4.6) that we have $\{\gamma \in \Gamma \mid \gamma U \cap U \neq \emptyset\} = \{\mathrm{id}\} \cup \Delta^\pm$. Using the fact that the stabilizers of the points of U are trivial, we deduce that $U \subseteq C$. Letting Γ act, it follows that $O \subseteq C$. Since no limit point may belong to C, we deduce that this is actually an equality.

We have already seen that the action is free on O. Let us prove that it is proper. Let $x, y \in O$. There exists $n \geqslant 0$ such that x and y belong to the interior of O_n. By Lemma 6.4.6, the set $\{\gamma \in \Gamma : \gamma O_n \cap O_n \neq \emptyset\}$ is made of elements of length at most $2n + 1$. In particular, it is finite. We deduce that the action of Γ on O is proper.

The compact subset F^+ of $\mathbb{P}^{1,\mathrm{an}}_k$ contains a point of every orbit of every element of O. It follows that $\Gamma\backslash O$ is compact. The set F^- is an open k-Swiss cheese and the map p is injective on it, which implies that $p_{|F^-}$ induces an isomorphism onto its image. In addition, one may check that each subset of the form $D^+(\gamma) - D^-(\gamma)$ for $\gamma \in \{\gamma_1^{\pm 1}, \dots, \gamma_g^{\pm 1}\}$ is contained in an open k-annulus on which p is injective. It follows that any element of $\Gamma\backslash O$ has a neighborhood isomorphic to a k-Swiss cheese, hence $\Gamma\backslash O$ is a Mumford curve.

Set $\Sigma := O \cap \bigcup_{x,y\in L}[x, y]$. It is clear that no point of Σ is contained in a virtual open disc inside O, hence $\Sigma \subseteq \Sigma_O$. It follows from [PT20, Proposition 5.7.10] that $\mathbb{P}^{1,\mathrm{an}}_k - \Sigma$ is a union of virtual open discs, hence $\Sigma_O \cap (\mathbb{P}^{1,\mathrm{an}}_k - \Sigma) = \emptyset$. We deduce that $\Sigma_O = \Sigma$. Note that it follows that $\Sigma_{F^+} = \Sigma_O \cap F^+$.

Let $x \in O - \Sigma_O$. Then x is contained in a virtual open disc inside O. Assume that there exists $\gamma \in \Gamma$ such that $x \in \gamma F^-$. Then, the said virtual open disc is contained in γF^-. Since $p_{|\gamma F^-}$ induces an isomorphism onto its image, $p(x)$ is contained in a virtual open disc in X, hence $p(x) \notin \Sigma_X$. As above, the argument may be adapted to handle all the points of $O - \Sigma_O$. It follows that $p^{-1}(\Sigma_X) \subseteq \Sigma_O$.

Let $x \in \Sigma_O$. In order to show that $p(x) \in \Sigma_X$, we may replace x by $\gamma(x)$ for any $\gamma \in \Gamma$, hence assume that $x \in F^+ \cap \Sigma_O = \Sigma_{F^+}$. From the explicit description of the action of Γ on F^+, we may describe precisely the behaviour of p on $\Sigma_{F^+} = \Sigma_{F^-} \cup \partial F^+$: it is injective on Σ_{F^-} and identifies pairs of points in ∂F^+. It follows that $p(x)$ belongs to a injective loop inside X and Remark 6.3.8 then ensures that $p(x) \in \Sigma_X$. The results about the skeleta follow directly.

It remains to prove that the genus of $X = \Gamma \backslash O$ is equal to g. The arguments above show that $\Sigma_X \simeq \Gamma \backslash \Sigma_{F^+}$ is a graph with cyclomatic number g. The result now follows from Corollary 6.3.29. □

Example 6.4.19 (Tate Curves) If $g = 1$ in the theory above, one starts with the data of an element $\gamma \in \mathrm{PGL}_2(k)$ and of two disjoint closed discs $D^+(\gamma)$ and $D^+(\gamma^{-1})$ in such a way that $\gamma(\mathbb{P}_k^{1,\mathrm{an}} - D^+(\gamma^{-1}))$ is a maximal open disc inside $D^+(\gamma)$. Since γ is loxodromic, up to conjugation, it is represented by a matrix of the form $\begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix}$ for some $q \in k$ satisfying $0 < |q| < 1$. In other words, up to a change of coordinate in $\mathbb{P}_k^{1,\mathrm{an}}$, the transformation γ is the multiplication by q and hence the limit set L consists only of the two points 0 and ∞. The quotient curve obtained from applying Theorem 6.4.18 is an elliptic curve, whose set of k-points is isomorphic to the multiplicative group $k^\times/q^{\mathbb{Z}}$.

Remark 6.4.20 It follows from Theorem 6.4.18 and Corollary 6.4.13 that each point in the limit set may be described as the intersection of a nested sequence of discs of the form $\bigcap_{n \geqslant 0} D^+(w_n)$, for a sequence of words w_n whose lengths tend to infinity. This is a rather concrete description, that could easily be implemented to any precision on a computer. The complex version of this idea gave rise to beautiful pictures in [MSW15].

Actually, we highly recommend the whole book [MSW15] to the reader. It starts with the example of a complex Schottky group with two generators in a very accessible way and then carefully presents a large amount of advanced material, with an original and colorful terminology, enriched with many pictures. Among the subjects covered are the Hausdorff dimension of the limit set ("fractal dust"), the degeneration of the notion of Schottky groups when the discs in the Schottky figures become tangent ("kissing Schottky groups"), etc. We believe that it is worth investigating those questions in the non-Archimedean setting too. In particular, finding a way to draw meaningful non-Archimedean pictures would certainly be very rewarding.

6.4.2 Group-Theoretic Version

We now give the general definition of Schottky group over k and explain how it relates to the geometric situation considered in the previous sections. As regards proper actions, recall Definition 6.4.16 and Proposition 6.4.17.

Definition 6.4.21 A subgroup Γ of $\mathrm{PGL}_2(k)$ is said to be a *Schottky group over k* if

(i) it is free and finitely generated;
(ii) all its non-trivial elements are loxodromic;
(iii) there exists a non-empty Γ-invariant connected open subset of $\mathbb{P}^{1,\mathrm{an}}_k$ on which the action of Γ is free and proper.

Remark 6.4.22 Schottky groups are discrete subgroups of $\mathrm{PGL}_2(k)$. Indeed any element of $\mathrm{PGL}_2(k)$ that is close enough to the identity has both eigenvalues of absolute value 1, hence cannot be loxodromic.

Remark 6.4.23 There are other definitions of Schottky groups in the literature. L. Gerritzen and M. van der Put use a slightly different version of condition (iii) (see [GvdP80, I (1.6)]). This is due to the fact that they work in the setting of rigid geometry, where the space consists only of our rigid points. We chose to formulate our definition this way in order to take advantage of the nice topological properties of Berkovich spaces and make it look closer to the definition used in complex geometry.

D. Mumford considered a more general setting where k is the fraction field of a complete integrally closed noetherian local ring and he requires only properties (i) and (ii) in his definition of Schottky group (see [Mum72a, Definition 1.3]). The intersection with our setting consists of the complete discretely valued fields k.

When k is a local field, all the definitions coincide (see [GvdP80, I (1.6.4)] and Sect. 6.4.4).

Schottky groups arise naturally when we have Schottky figures as in Sect. 6.4.1. Indeed, the following result follows from Proposition 6.4.7, Corollary 6.4.12 and Theorem 6.4.18.

Proposition 6.4.24 *Let Γ be a subgroup of* $\mathrm{PGL}_2(k)$ *generated by finitely many elements $\gamma_1, \dots, \gamma_g$. If there exists a Schottky figure adapted to $(\gamma_1, \dots, \gamma_g)$, then Γ is a Schottky group.* □

We now turn to the proof of the converse statement.

Lemma 6.4.25 *Let γ be a loxodromic Möbius transformation. Let A and A' be disjoint virtual flat closed annuli. Denote by I the open interval equal to the interior of the path joining their boundary points. Assume that $\gamma A_1 = A_2$ and $\gamma I \cap I = \emptyset$. For $\varepsilon \in \{\emptyset, '\}$, denote by D^ε the connected component of $\mathbb{P}^{1,\mathrm{an}}_k - A^\varepsilon$ that does* *not meet I. Then, for $\varepsilon \in \{\emptyset, '\}$, A^ε is a flat closed annulus, D^ε is an open disc, $E^\varepsilon := D^\varepsilon \cup A^\varepsilon$ is a closed disc and we have*

$$\gamma D = \mathbb{P}^{1,\mathrm{an}}_k - E' \text{ and } \gamma E = \mathbb{P}^{1,\mathrm{an}}_k - D'.$$

Proof For each $\varepsilon \in \{\emptyset, '\}$, D^ε and E^ε are respectively a virtual open disc and a virtual closed disc. Note that the set $\mathbb{P}^{1,\mathrm{an}}_k - A^\varepsilon$ has two connected components, namely D^ε and $\mathbb{P}^{1,\mathrm{an}}_k - E^\varepsilon$, and that the latter contains I.

Since γ is an automorphism, it sends the connected component $\mathbb{P}^{1,\mathrm{an}}_k - E$ of $\mathbb{P}^{1,\mathrm{an}}_k - A$ to a connected component C of $\mathbb{P}^{1,\mathrm{an}}_k - \gamma A = \mathbb{P}^{1,\mathrm{an}}_k - A'$. Denote by η and η' the boundary points of A and A'. Let $z \in \mathbb{P}^{1,\mathrm{an}}_k - E$. The unique path $[\eta, z]$ between η and z then meets I. Its image is the unique path $[\eta', \gamma(z)]$ between $\gamma(\eta) = \eta'$ and $\gamma(z)$. If $\gamma(z) \notin E'$, then this path meets I, contradicting the assumption $\gamma I \cap I = \emptyset$. We deduce that $\gamma(z) \in E'$, hence that $C = D'$. It follows that we have

$$\gamma D = \mathbb{P}^{1,\mathrm{an}}_k - E' \text{ and } \gamma E = \mathbb{P}^{1,\mathrm{an}}_k - D',$$

as wanted.

In particular, D and D' contain respectively the attracting and repelling fixed point of γ. Since those points are k-rational, we deduce that D and D' are discs. The rest of the result follows. □

Theorem 6.4.26 *Let Γ be a Schottky group over k. Then, there exists a basis β of Γ and a Schottky figure $\mathcal{B}$ that is adapted to β.*

Proof By assumption, there exists a non-empty Γ-invariant connected open subset U of $\mathbb{P}^{1,\mathrm{an}}_k$ on which the action of Γ is free and proper. The quotient $X := \Gamma\backslash U$ is then an $\mathbb{A}^{1,\mathrm{an}}_k$-like curve in the sense of Sect. 6.3.1. Since U is a connected subset of $\mathbb{P}^{1,\mathrm{an}}_k$, it is simply connected, hence the fundamental group $\pi_1(X)$ of X is isomorphic to Γ. Since X is finitely generated, the topological genus g of X is finite.

Fix a skeleton Σ of X and consider the associated retraction $\tau\colon X \to \Sigma$. Fix g elements $\gamma_1, \ldots, \gamma_g$ of Γ corresponding to disjoint simple loops in Σ. Note that $\gamma_1, \ldots, \gamma_g$ is a basis of Γ.

For each $i \in \{1, \ldots, g\}$, pick a point $x_i \in \alpha_i$ that is not a branch point of Σ. Its preimage by the retraction $A_i := \tau^{-1}(x_i)$ is then a virtual flat closed annulus.

Let Y' be an open subset of U such that the morphism $Y' \to X$ induced by the quotient is an isomorphism onto $X - \bigcup_{1\leqslant i \leqslant A_i}$. We extend it to a compact lift Y of X in U by adding, for each $i \in \{1, \ldots, g\}$, two virtual flat annuli B_i and B'_i that are isomorphic preimages of A_i. Up to switching the names, we may assume that $\gamma_i B_i = B'_i$.

Let $i \in \{1, \ldots, g\}$. The complement of B_i (resp. B'_i) has two connected components. Let us denote by $D^-(\gamma_i)$ (resp. $D^-(\gamma_i^{-1})$) the one that does not meet Y. It is a virtual open disc. We set $D^+(\gamma_i^{-1}) = D^-(\gamma_i^{-1}) \cup B_i$ and $D^+(\gamma_i) = D^-(\gamma_i) \cup B_i'$.

By construction of Y', for each $\gamma \in \Gamma - \{\mathrm{id}\}$, we have $\gamma Y' \cap Y' = \emptyset$. It now follows from Lemma 6.4.25 that the family $(D^+(\gamma_i^\sigma), 1 \leqslant i \leqslant g, \sigma = \pm)$ is a Schottky figure adapted to $(\gamma_1, \ldots, \gamma_g)$.

□

Remark 6.4.27 The fact that Γ is free is actually not used in the proof of Theorem 6.4.26. As a result, Proposition 6.4.24 shows that it is a consequence of the other properties appearing in the definition of a Schottky group. It could also be deduced from the fact that the fundamental group of a Berkovich curve (which is the same as that of its skeleton) is free.

6.4.3 Twisted Ford Discs

We can actually be more precise about the form of the discs in the Schottky figure from Theorem 6.4.26. To do so, we introduce some terminology.

Definition 6.4.28 Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(k)$, with $c \neq 0$, be a loxodromic Möbius transformation and let $\lambda \in \mathbb{R}_{>0}$. We call open and closed *twisted Ford discs* associated to (γ, λ) the sets

$$D^-_{(\gamma,\lambda)} := \left\{ z \in k \;:\; \lambda|\gamma'(z)| = \lambda \frac{|ad-bc|}{|cz+d|^2} > 1 \right\}$$

and

$$D^+_{(\gamma,\lambda)} := \left\{ z \in k \;:\; \lambda|\gamma'(z)| = \lambda \frac{|ad-bc|}{|cz+d|^2} \geqslant 1 \right\}.$$

Lemma 6.4.29 *Let $\alpha, \alpha', \beta \in k$ with $\alpha \neq \alpha'$ and $|\beta| < 1$ and let $\lambda \in \mathbb{R}_{>0}$. Set $\gamma := M(\alpha, \alpha', \beta) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The twisted Ford discs $D^-_{(\gamma,\lambda)}$ and $D^+_{(\gamma,\lambda)}$ have center*

$$\frac{\alpha' - \beta\alpha}{1-\beta} = -\frac{d}{c}$$

and radius

$$\rho = \frac{(\lambda|\beta|)^{1/2}|\alpha-\alpha'|}{|1-\beta|} = \frac{(\lambda\,|ad-bc|)^{1/2}}{|c|}.$$

In particular, $\alpha' \in D^-_{(\gamma,\lambda)}$ if, and only if, $|\beta| < \lambda$.

The twisted Ford discs $D^-_{(\gamma^{-1},\lambda^{-1})}$ *and* $D^+_{(\gamma^{-1},\lambda^{-1})}$ *have center*

$$\frac{\alpha - \beta\alpha'}{1-\beta} = \frac{a}{c}$$

and radius $\rho' = \rho/\lambda$.

In particular, $\alpha \in D^-_{(\gamma^{-1},\lambda^{-1})}$ *if, and only if,* $|\beta| < \lambda^{-1}$. □

Lemma 6.4.30 *Let* $\gamma \in \mathrm{PGL}_2(k)$ *be a loxodromic Möbius transformation that does not fix* ∞ *and let* $\lambda \in \mathbb{R}_{>0}$. *Then, we have* $\gamma(D^+_{(\gamma,\lambda)}) = \mathbb{P}^{1,\mathrm{an}}_k - D^-_{(\gamma^{-1},\lambda^{-1})}$.

Proof Let us write $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since γ does not fix ∞, we have $c \neq 0$. Let K be a complete valued extension of k and let $z \in K$. We have $|-c\gamma(z)+a|\,|cz+d| = |ad-bc|$, hence

$$z \in D_{(\gamma,\lambda)} \iff \lambda\frac{|ad-bc|}{|cz+d|^2} \geq 1 \iff \lambda^{-1}\frac{|ad-bc|}{|-c\gamma(z)+a|^2} \leq 1.$$

Since we have $\gamma^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, the latter condition describes precisely the complement of $D^-_{(\gamma^{-1},\lambda^{-1})}$. □

Lemma 6.4.31 *Let* $\gamma \in \mathrm{PGL}_2(k)$ *be a loxodromic Möbius transformation. Let* $D^+(\gamma)$ *and* $D^+(\gamma^{-1})$ *be disjoint closed discs in* $\mathbb{P}^{1,\mathrm{an}}_k$. *Set*

$$D^-(\gamma) := \gamma(\mathbb{P}^{1,\mathrm{an}}_k - D^+(\gamma^{-1})) \text{ and } D^-(\gamma^{-1}) := \gamma^{-1}(\mathbb{P}^{1,\mathrm{an}}_k - D^+(\gamma)).$$

Assume that $D^-(\gamma)$ *and* $D^-(\gamma^{-1})$ *are maximal open discs inside* $D^+(\gamma)$ *and* $D^+(\gamma^{-1})$ *respectively and that they are contained in* $\mathbb{A}^{1,\mathrm{an}}_k$.

Then, there exists $\lambda \in \mathbb{R}_{>0}$ *such that, for each* $\sigma \in \{-,+\}$, *we have*

$$D^\sigma(\gamma) = D^\sigma_{\gamma,\lambda} \text{ and } D^\sigma(\gamma^{-1}) = D^\sigma_{\gamma^{-1},\lambda^{-1}}.$$

Proof Denote by α and α' the attracting and repelling fixed points of γ respectively. By the same argument as in Remark 6.4.2, we have $\alpha \in D^-(\gamma^{-1})$ and $\alpha' \in D^-(\gamma)$. Let $r, r' > 0$ such that $D^-(\gamma) = D^-(\alpha', r')$ and $D^-(\gamma^{-1}) = D^-(\alpha, r)$.

Write $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a, b, c, d \in k$. Since $\alpha, \alpha' \in \mathbb{A}^{1,\mathrm{an}}_k$, we have $c \neq 0$. By assumption, $\infty \in \gamma(D^-(\gamma^{-1}))$, hence $-d/c \in D^-(\gamma^{-1})$ and $D^-(\gamma^{-1}) = D^-(-d/c, r)$. Similarly, we have $D^-(\gamma) = D^-(a/c, r')$.

Writing

$$\frac{aT+b}{cT+d} = \frac{a}{c} - \frac{ad-bc}{c^2}\,\frac{1}{T+\frac{d}{c}},$$

it is not difficult to compute $\gamma(D^-(\gamma^{-1}))$ and prove that we have

$$r = \frac{|ad-bc|}{|c|^2\, r'} = \frac{|\beta|\,|\alpha-\alpha'|^2}{r'}.$$

Set

$$\lambda := \frac{r^2}{|\beta|\,|\alpha-\alpha'|^2} = \frac{r}{r'} = \frac{|\beta|\,|\alpha-\alpha'|^2}{(r')^2}.$$

Since $D^+(\gamma)$ and $D^+(\gamma^{-1})$ are disjoint, we have $\max(r, r') < |\alpha-\alpha'|$, hence $|\beta| < \min(\lambda, \lambda^{-1})$. It follows that $D^-_{\gamma,\lambda}$ and $D^-_{\gamma^{-1},\lambda^{-1}}$ contains respectively α' and α, hence

$$D^-_{(\gamma,\lambda)} = D^-(\alpha', r') = D^-(\gamma) \text{ and } D^-_{(\gamma^{-1},\lambda^{-1})} = D^-(\alpha, r) = D^-(\gamma^{-1}).$$

□

Corollary 6.4.32 *Let Γ be a Schottky group over k whose limit set does not contain ∞. Then, there exists a basis $(\gamma_1, \dots, \gamma_g)$ of Γ and $\lambda_1, \dots, \lambda_g \in \mathbb{R}_{>0}$ such that the family of discs $\big(D^+_{(\gamma_i^\varepsilon, \lambda_i^\varepsilon)}, 1 \leqslant i \leqslant g, \varepsilon \in \{\pm 1\}\big)$ is a Schottky figure that is adapted to $(\gamma_1, \dots, \gamma_g)$.*

Proof By Theorem 6.4.26, there exists a basis $\beta = (\gamma_1, \dots, \gamma_g)$ of Γ and a Schottky figure $\mathcal{B} = (D^+(\gamma_i^\varepsilon), 1 \leqslant i \leqslant g, \varepsilon \in \{\pm 1\})$ that is adapted to β. As in Sect. 6.4.1, define the open discs $D^-(\gamma_i^{\pm 1})$ and set

$$F^+ := \mathbb{P}^1_k - \bigcup_{\substack{1 \leqslant i \leqslant g\\ \varepsilon = \pm 1}} D^-(\gamma_i^\varepsilon).$$

By Theorem 6.4.18, since ∞ is not a limit point of Γ, there exists $\gamma \in \Gamma$ such that $\infty \in \gamma F^+$.

Set $\beta' := (\gamma\gamma_1\gamma^{-1}, \dots, \gamma\gamma_g\gamma^{-1})$. It is a basis of Γ and the family of discs $\mathcal{B}' := (\gamma D^+(\gamma_i^\varepsilon), 1 \leqslant i \leqslant g, \varepsilon \in \{\pm 1\})$ is a Schottky figure that is adapted to it. Since all the discs $\gamma D^+(\gamma_i^{\pm 1})$ are contained in $\mathbb{A}^{1,\mathrm{an}}_k$, we may now apply Lemma 6.4.31 to conclude. □

6.4.4 Local Fields

When k is a local field, the definition of a Schottky group can be greatly simplified. Our treatment here borrows from [GvdP80, I (1.6)] (see also [Mar07, Lemma 2.1.1] in the complex setting).

Lemma 6.4.33 *Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of loxodromic Möbius transformations such that*

(i) $(\gamma_n)_{n\in\mathbb{N}}$ has no convergent subsequence in $\mathrm{PGL}_2(k)$;
(ii) the sequence of Koebe coordinates $((\alpha_n, \alpha'_n, \beta_n))_{n\in\mathbb{N}}$ converges to some $(\alpha, \alpha', \beta) \in (\mathbb{P}^1(k))^3$.

Then, $(\gamma_n)_{n\in\mathbb{N}}$ converges to the constant function α uniformly on compact subsets of $\mathbb{P}^{1,\mathrm{an}}_k - \{\alpha'\}$.

Proof By definition, for each $n \in \mathbb{N}$, we have $|\beta_n| < 1$, which implies that $|\beta| < 1$.

Up to changing coordinates, we may assume that $\alpha, \alpha' \in k$. Up to modifying finitely many terms of the sequences, we may assume that, for each $n \in \mathbb{N}$, we have $\alpha_n, \alpha'_n \in k$. In this case, for each $n \in \mathbb{N}$, we have

$$\gamma_n =: \begin{bmatrix} \alpha_n - \beta_n\alpha'_n & (\beta_n - 1)\alpha_n\alpha'_n \\ 1 - \beta_n & \beta_n\alpha_n - \alpha'_n \end{bmatrix} \text{ in } \mathrm{PGL}_2(k).$$

The determinant of the above matrix is $\beta_n(\alpha_n - \alpha'_n)^2$. Since $(\gamma_n)_{n\in\mathbb{N}}$ has no convergent subsequence in $\mathrm{PGL}_2(k)$, we deduce that $\beta(\alpha - \alpha')^2 = 0$. In each of the two cases $\beta = 0$ and $\alpha = \alpha'$, it is not difficult to check that the claimed result holds.

□

The result below shows that the definition of Schottky group may be simplified when k is a local field. Note that, in this case, $\mathbb{P}^1(k)$ is compact, hence closed in $\mathbb{P}^{1,\mathrm{an}}_k$.

Corollary 6.4.34 *Assume that k is a local field. Let Γ be a subgroup of $\mathrm{PGL}_2(k)$ all of whose non-trivial elements are loxodromic.*

Let Λ be the set of fixed points of the elements of $\Gamma - \{\mathrm{id}\}$ and let $\bar{\Lambda}$ be its closure in $\mathbb{P}^{1,\mathrm{an}}_k$. Then, $\bar{\Lambda}$ is a compact subset of $\mathbb{P}^{1,\mathrm{an}}_k$ that is contained in $\mathbb{P}^1(k)$ and the action of Γ on $\mathbb{P}^{1,\mathrm{an}}_k - \bar{\Lambda}$ is free and proper.

Proof Since k is locally compact for the topology given by the absolute value, $\mathbb{P}^1(k)$ is compact. By [PT20, Remark 5.4.1], the topology on k given by the absolute value coincides with that induced by the topology on $\mathbb{A}^{1,\mathrm{an}}_k$. We deduce that $\mathbb{P}^1(k)$ is a compact subset of $\mathbb{P}^{1,\mathrm{an}}_k$. It follows that $\bar{\Lambda}$ is contained in $\mathbb{P}^1(k)$ and that it is compact, as it is closed.

The action of Γ is obviously free on $\mathbb{P}_k^{1,\mathrm{an}} - \bar{\Lambda}$. Assume, by contradiction, that it is not proper. Then, there exist $x, y \notin \bar{\Lambda}$ such that, for every neighborhoods U and V of x and y respectively, the set $\{\gamma \in \Gamma : \gamma U \cap V \neq \emptyset\}$ is infinite.

Since k is a local field, [PT20, Corollary 5.4.6] ensures that the space $\mathbb{A}_k^{1,\mathrm{an}}$ is metrizable. In particular, we may find countable bases of neighborhoods $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ of x and y respectively. By assumption, there exist a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of distinct elements of Γ and a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{P}_k^{1,\mathrm{an}} - \bar{\Lambda}$ such that, for each $n \in \mathbb{N}$, we have $x_n \in U_n$ and $\gamma_n(x_n) \in V_n$. In particular, $(x_n)_{n\in\mathbb{N}}$ converges to x and $(\gamma_n(x_n))_{n\in\mathbb{N}}$ converges to y.

Since all the non-trivial elements of Γ are loxodromic, by the same argument as in Remark 6.4.22, the group Γ is discrete. As a result, up to passing to a subsequence, we may assume that the assumptions of Lemma 6.4.33 are satisfied. Define α and α' as in this Lemma. Since x does not belong to $\bar{\Lambda}$, it cannot be equal to α'. It follows that the sequences $(\gamma_n(x_n))_{n\in\mathbb{N}}$ and $(\gamma_n(x))_{n\in\mathbb{N}}$ converge to the same limit $y = \alpha$, and we get a contradiction since $\alpha \in \bar{\Lambda}$. □

Corollary 6.4.35 *Assume that k is a local field. Then, a subgroup Γ of $\mathrm{PGL}_2(k)$ is a Schottky group if, and only if, it is finitely generated and all its non-trivial elements are loxodromic.* □

6.5 Uniformization of Mumford Curves

The main result of this section, Theorem 6.5.3, states that the procedure described in Sect. 6.4.1 can be reversed: any Mumford curve may be uniformized by an open subset of the Berkovich projective line $\mathbb{P}_k^{1,\mathrm{an}}$ with a Schottky group as group of deck transformations. The consequences of this result are many and far-reaching. Some of them are discussed in Appendix A.3.

This was first proved by D. Mumford in his influential paper [Mum72a], where he introduces this as a non-Archimedean analogue of the uniformization of handlebodies by means of Schottky groups in the complex setting. His arguments make a heavy use of formal models of the curves. Here, we argue directly on the curves themselves, following the strategy of [GvdP80, Chapter IV] and [Lüt16, Proposition 4.6.6]. Note, however that the proof in the first reference is flawed (since it relies on the wrong claim that every k-analytic curve of genus 0 embeds into $\mathbb{P}_k^{1,\mathrm{an}}$, see Remark 6.5.7) and that the second reference assumes that the curve contains at least three rational points.

As an application, we discuss how Theorem 6.5.3 can be used to study the automorphism groups of Mumford curves. This is far from being the sole purpose of uniformization. Other important consequences are mentioned in Appendix A.3.

6.5.1 The Uniformization Theorem

In this section, we prove that any analytic Mumford curve as defined in 6.3.24 can be obtained as the quotient of an open dense subspace of $\mathbb{P}^{1,\mathrm{an}}_k$ by the action of a Schottky group, leading to a purely analytic proof of Mumford's theorem. We begin with a few preparatory results.

Lemma 6.5.1 *Let L be a compact subset of $\mathbb{P}^1(k)$. Set $O := \mathbb{P}^{1,\mathrm{an}}_k - L$.*

(i) Every bounded analytic function on O is constant.
(ii) Every automorphism of O is induced by an element of $\mathrm{PGL}_2(k)$.

Proof

(i) Let $F \in \mathcal{O}(O)$. The function F is constant if, and only if, its pullback to $O_{\widehat{k^a}}$ is, hence we may assume that k is algebraically closed.

Assume, by contradiction, that F is not constant. Then, there exists $x \in O$ and a branch b at x such that $F(x) \neq 0$ and $|F|$ is monomial at x along b with a positive integer exponent. We may assume that x is of type 2 or 3. Then, there exists $y \in O - \{x\}$ and $N \in \mathbb{N}_{\geqslant 1}$ such that, for each $z \in [x, y]$, we have $|F(z)| = |F(x)|\, \ell([x, z])^N$.

Let us now consider a path $[x, y]$, with $y \in \mathbb{P}^{1,\mathrm{an}}_k$, with the following property: for each $z \in (x, y)$, $|F|$ is monomial at z with positive integer slope along the branch in (x, y) going away from x. By Zorn's lemma, we may find a maximal path $[x, y]$ among those.

We claim that y is of type 1. If y is of type 4, then, by [PT20, Theorem 5.10.10], $|F|$ is constant in the neighborhood of y in (x, y), and we get a contradiction. Assume that b is of type 2 or 3. Then, the exponent of $|F|$ at y along the branch corresponding to $[y, x]$ is negative. By [PT20, Corollary 5.10.12], there exists a branch b at y such that $|F|$ is monomial with positive exponent at y along b, which contradicts the maximality. Finally, y is of type 1.

By assumption, $|F|$ has a positive integer exponent everywhere on (x, y). It follows that, for each $z \in (x, y)$, we have $|F(z)| \geqslant |F(x)|\, \ell([x, z])$. Since y is of type 1, by [PT20, Lemma 5.9.12], we have $\ell([x, y]) = \infty$, hence F is unbounded. This is a contradiction.

(ii) Let σ be an automorphism of O.

Let us first assume that O contains at least 2 k-rational points. Up to changing coordinates, we may assume that $0, \infty \in O$. Let us choose an automorphism $\tau \in \mathrm{PGL}_2(k)$ that agrees with σ on 0 and ∞. Then $\tau^{-1} \circ \sigma$ is an automorphism of O that fixes 0 and ∞. In particular, it corresponds to an analytic function with a zero of order 1 at 0 and a pole of order 1 at ∞. Let us consider the quotient analytic function $\varphi := (\tau^{-1} \circ \sigma)/\mathrm{id}$. There exist a neighborhood U of 0 and a neighborhood V of ∞ on which φ is bounded. Since $\tau^{-1} \circ \sigma$ is an automorphism, it sends V to a neighborhood of ∞, hence it is bounded on $O - V$. It follows that φ is bounded on $O - (U \cup V)$, hence on O. By (i), we deduce that φ is constant, and the result follows.

Let us now handle the case where $O \cap \mathbb{P}^1(k) = \emptyset$. There exists a finite extension K of k such that O_K contains a K-rational point. Applying the previous argument after extending the scalars to K, we deduce that σ belongs to $\mathrm{PGL}_2(K)$. Since it preserves $\mathbb{P}^1(k)$, it actually belongs to $\mathrm{PGL}_2(k)$.

□

Lemma 6.5.2 *Let Y be a connected k-analytic $\mathbb{A}^1$-like curve of genus 0. Let $\mathcal{T} = (S, \mathcal{A}, \mathcal{D})$ be a triangulation of Y. Assume that $\mathcal{A}$ is non-empty and consists of annuli. Let U be an open relatively compact subset of $\Sigma_{\mathcal{T}}$. Then, there exists an embedding of $\tau_{\mathcal{T}}^{-1}(U)$ into $\mathbb{P}_k^{1,\mathrm{an}}$ such that the complement of $\tau_{\mathcal{T}}^{-1}(U)$ is a disjoint union of finitely many closed discs.*

Proof Recall that $\Sigma_{\mathcal{T}}$ is a locally finite graph (see Theorem 6.3.2). As a consequence, the boundary ∂U of U in $\Sigma_{\mathcal{T}}$ is finite. For each $z \in \partial U$, let I_z be an open interval in $\Sigma_{\mathcal{T}}$ having z as an end-point. Up to shrinking the I_z's, we may assume that they are disjoint.

Let $z \in \partial U$. Set $A_z := \tau_{\mathcal{T}}^{-1}(I_z)$. Since every element of $\mathcal{A}$ is an annulus, up to shrinking I_z (so that it contains no points of S), we may assume that A_z is an annulus. The open annulus A_z may be embedded into an open disc D_z such that the complement is a closed disc.

Let us construct a curve Y' by starting from $\tau_{\mathcal{T}}^{-1}(U)$ an gluing D_z along A_z for each $z \in \partial U$. By construction, the curve Y' is compact and of genus 0. Moreover, it contains rational points, as the D_z do. It follows from Theorems 6.3.12 and 6.3.23 that Y' is isomorphic to $\mathbb{P}_k^{1,\mathrm{an}}$. By construction,

$$\mathbb{P}_k^{1,\mathrm{an}} - \tau_{\mathcal{T}}^{-1}(U) = \bigcup_{z \in \partial U} D_z - A_z$$

is a disjoint union of finitely many closed discs. □

We now state and prove the uniformization theorem.

Theorem 6.5.3 *Let X be a k-analytic Mumford curve. Then the fundamental group Γ of X is a Schottky group. If we denote by L the limit set of Γ, then $O := \mathbb{P}_k^{1,\mathrm{an}} - L$ is a universal cover of X. In particular, we have $X \simeq \Gamma \backslash O$.*

Proof Assume that the genus of X is bigger than or equal to 2.

Let $p \colon Y \to X$ be the topological universal cover of X. Since p is a local homeomorphism, we may use it to endow Y with a k-analytic structure. The set Y then becomes an $\mathbb{A}^1$-like curve and the map p becomes a local isomorphism of locally ringed spaces. Note that the curve Y has genus 0.

We claim that it is enough to prove that Y is isomorphic to an open subset of $\mathbb{P}_k^{1,\mathrm{an}}$ whose complement lies in $\mathbb{P}^1(k)$. Indeed, in this case, Y is simply connected, hence the fundamental group Γ of X may be identified with the group of deck transformations of p. By Lemma 6.5.1, it embeds into $\mathrm{PGL}_2(k)$. It now follows from the properties of the universal cover and the fundamental group that Γ is a Schottky group (see Remark 6.2.12 for the fact that the non-trivial elements of Γ are loxodromic). Moreover, by Theorem 6.4.18, we have $Y \subseteq \mathbb{P}^{1,\mathrm{an}}_k - L$, where L is the limit set of Γ, hence $X = \Gamma\backslash Y \subset \Gamma\backslash(\mathbb{P}^{1,\mathrm{an}}_k - L)$. Since $\Gamma\backslash Y$ and $\Gamma\backslash(\mathbb{P}^{1,\mathrm{an}}_k - L)$ are both connected proper curves, they have to be equal, hence $Y = \mathbb{P}^{1,\mathrm{an}}_k - L$.

In the rest of the proof, we show that Y embeds into $\mathbb{P}^{1,\mathrm{an}}_k$ with a complement in $\mathbb{P}^1(k)$. Since X is a k-analytic Mumford curve of genus at least 2, it has a minimal skeleton Σ_X and the connected components of Σ_X deprived of its branch points are skeleta of open annuli over k. Its preimage $p^{-1}(\Sigma_X)$ coincides with the minimal skeleton Σ_Y of Y. Similarly, the connected components of Σ_Y deprived of its branch points are skeleta of open annuli over k. We denote by $\tau_Y : Y \to \Sigma_Y$ the canonical retraction.

Let $x_0 \in X$ and $y_0 \in p^{-1}(x_0)$. Let ℓ_X be a loop in Σ_X based at x_0 that is not homotopic to 0. It lifts to a path in Σ_Y between y_0 and a point y_1 of $p^{-1}(x_0)$. We may then lift again ℓ_X to a path in Σ_Y between y_1 and a point y_2 of $p^{-1}(x_0)$. Repeating the procedure, we obtain a non-relatively compact path $\lambda(\ell_X)$ in Σ_Y starting at y_0. Note that the length of $\lambda(\ell_X)$ is infinite since it contains infinitely many copies of ℓ_X.

More generally, all the maximal paths starting from y_0 in Σ_Y are of infinite length, since they contain infinitely many lifts of loops from Σ_X.

Since X is of genus at least 2, we may find two loops $\ell_{X,0}$ and $\ell_{X,1}$ based at x_0 in Γ_X that are not homotopic to 0 and not homotopic one to the other. Set $\ell_0 := \lambda(\ell_{X,0})$, $\ell_\infty := \lambda(\ell_{X,0}^{-1})$ and $\ell_1 := \lambda(\ell_{X,1})$. Away from some compact set of Y, the three paths $\ell_0, \ell_\infty, \ell_1$ are disjoint. Up to moving x_0 and y_0, we may assume that

$$\ell_0 \cap \ell_1 = \ell_\infty \cap \ell_1 = \ell_0 \cap \ell_\infty = \{y_0\}.$$

For $i \in \{0, 1, \infty\}$ and $r \in \mathbb{R}_{\geqslant 1}$, we denote by $\xi_{i,r}$ the unique point of ℓ_i such that $\ell([y_0, \xi_{i,r}]) = r$.

Let $n \in \mathbb{N}_{\geqslant 1}$. Set

$$U_n := \{z \in \Sigma_Y : \ell([y_0, z]) < 2^n\} \text{ and } Y_n := \tau_Y^{-1}(U_n).$$

We already saw that all the maximal paths starting from y_0 in Σ_Y are of infinite length, hence U_n is relatively compact in Σ_Y. Denote by ∂U_n the boundary of U_n in Σ_Y. For each $z \in \partial U_n$, we have $\ell([y_0, z]) = 2^n$.

By Lemma 6.5.2, there exists an open subset O_n of $\mathbb{P}^{1,\mathrm{an}}_k$ and an isomorphism $\varphi_n : Y_n \xrightarrow{\sim} O_n$ such that $\mathbb{P}^{1,\mathrm{an}}_k - O_n$ is a disjoint union of closed discs. For each $z \in \partial U_n$, we denote by p_z the end-point of $\varphi_n([y_0, z))$ in $\mathbb{P}^{1,\mathrm{an}}_k - O_n$ and by D_z the connected component of $\mathbb{P}^{1,\mathrm{an}}_k - O_n$ whose boundary point is p_z. To ease the notation, for $i \in \{0, 1, \infty\}$, we set $D_{i,n} := D_{\xi_{i,2^n}}$.

Let us fix a point at infinity on $\mathbb{P}^{1,\mathrm{an}}_k$ and a coordinate T on $\mathbb{A}^{1,\mathrm{an}}_k \subset \mathbb{P}^{1,\mathrm{an}}_k$. We may assume that, for each $i \in \{0, 1, \infty\}$, we have $i \in D_{i,n}$. By pulling back the analytic function T on O_n by φ_n, we get a analytic function on Y_n. We denote it by ψ_n. Recall that it is actually equivalent to give oneself φ_n or ψ_n, see [PT20, Lemma 5.5.11]. □

Lemma 6.5.4 *We have $\varphi_n(y_0) = \eta_1$. For each $r \in [1, 2^n)$, we have*

$$\varphi_n(\xi_{0,r}) = \eta_{1/r},\ \varphi_n(\xi_{\infty,r}) = \eta_r \text{ and } \varphi_n(\xi_{1,r}) = \eta_{1,1/r}.$$

Let C be a connected component C of $Y - (\ell_0 \cup \ell_\infty)$. For each $y \in C \cap Y_n$, we have

$$|\psi_n(y)| = \begin{cases} 1/r & \text{if the boundary point of } C \text{ is } \xi_{0,r};\\ r & \text{if the boundary point of } C \text{ is } \xi_{\infty,r}.\end{cases}$$

Let $N \in [\![1, n]\!]$. The image $\varphi_n(Y_N)$ is an open Swiss cheese. More precisely, there exist $d \in \mathbb{N}_{\geqslant 2}$, $\alpha_2, \ldots, \alpha_d \in k^*$ and, for each $j \in [\![2, d]\!]$, $r_j \in [2^{-N}, |\alpha_j|)$ such that $\varphi_n(Y_N)$ is the subset of $\mathbb{A}_k^{1,\mathrm{an}}$ defined by the following conditions:

$$\begin{cases} 2^{-N} < |T| < 2^N;\\ |T-1| > 2^{-N};\\ \forall j \in [\![2, d]\!],\ |T - \alpha_j| > r_j.\end{cases}$$

Proof It follows from the construction that, for each $i \in \{0, 1, \infty\}$, $\varphi_n([y_0, \xi_{i,2^n})$ is an injective path joining $\varphi_n(y_0)$ to the boundary point of a disc centered at i. Since those paths only meet at $\varphi_n(y_0)$, the only possibility is that $\varphi_n(y_0) = \{\eta_1\}$.

Let $r \in [1, 2^n)$. Since lengths are preserved by automorphism (see [PT20, Proposition 5.5.14]), for each $i \in \{0, 1, \infty\}$, we have $\ell([\eta_1, \varphi_n(\xi_{i,r})]) = r$. Since $\varphi_n(\xi_{\infty,r})$ belongs to $[\eta_1, \infty]$, it follows that $\varphi_n(\xi_{\infty,r}) = \eta_r$. By a similar argument, we have $\varphi_n(\xi_{0,r}) = \eta_{1/r}$ and $\varphi_n(\xi_{1,r}) = \eta_{1,1/r}$.

Recall that we have $I_0 = \{\eta_r : r \in \mathbb{R}_{\geqslant 0}\} \subset \mathbb{A}_k^{1,\mathrm{an}}$. Let C be a connected component of $\mathbb{A}_k^{1,\mathrm{an}} - I_0$ and let η_r be its boundary point. Then, for each $z \in C$, we have $|T(z)| = r$.

We have $\varphi_n^{-1}(I_0 \cap O_n) = (\ell_0 \cup \ell_\infty) \cap Y_n$. By definition of ψ_n, for each $y \in Y_n$, we have $|\psi_n(y)| = |T(\varphi_n(y))|$. It follows that, for each connected component C of $Y - (\ell_0 \cup \ell_\infty)$ and each $y \in C \cap Y_n$, we have

$$|\psi_n(y)| = \begin{cases} 1/r & \text{if the boundary point of } C \text{ is } \xi_{0,r};\\ r & \text{if the boundary point of } C \text{ is } \xi_{\infty,r}.\end{cases}$$

The set O_n is an open Swiss cheese. The set $\varphi_n(U_N)$ is a connected open subset of its skeleton and $\varphi_n(Y_N)$ is the preimage of it by the retraction. It follows that $\varphi_n(Y_N)$ is an open Swiss cheese too, hence the complement in $\mathbb{P}_k^{1,\mathrm{an}}$ of finitely many closed discs $E_\infty, E_0, \ldots, E_d$. Let $z_\infty, z_0, \ldots, z_d$ denote the corresponding boundary points. The set $\varphi_n(Y_N)$ contains $\varphi_n(y_0) = \eta_1$ and, by construction of Y_N, for each $i \in \{\infty\} \cup [\![0, d]\!]$, we have $\ell([\eta_1, z_i]) = 2^N$.

Since 0, 1 and ∞ do not belong to O_n, some of those discs E_i contain those points. Since $\varphi_n(Y_N)$ contains η_1, those discs are disjoint. We may assume that, for each $i \in \{0, 1, \infty\}$, we have $i \in E_i$. The length property then implies that we have $z_\infty = \eta_{2^N}$, $z_0 = \eta_{2^{-N}}$ and $z_0 = \eta_{1,2^{-N}}$. In other words,

$$\mathbb{P}_k^{1,\mathrm{an}} - (E_\infty \cup E_0 \cup E_1) = \{x \in \mathbb{A}_k^{1,\mathrm{an}} : 2^{-N} < |T(x)| < 2^N,\ |T - 1| > 2^{-N}\}.$$

For $j \in [\![2, d]\!]$, let α_j be a k-rational point of E_j. The boundary point z_j of E_j is then of the form η_{α_j, r_j} for some $r_j \in \mathbb{R}_{\geqslant 0}$. Since E_j does not contain 0, we have $r_j < |\alpha_j|$. Moreover, the condition $\ell([\eta_1, \eta_{\alpha_j, r_j}]) = 2^N$ implies that $r_j \geqslant 2^{-N}$ (see [PT20, Example 5.9.11]). The result follows. □

Let $N, n, m \in \mathbb{N}_{\geqslant 1}$ with $n \geqslant m > N$. The analytic function ψ_m has no zeros on Y_m, hence the quotient $\psi_{n|Y_m}/(\psi_m)$ defines an analytic function on Y_m. Set

$$h_{n,m} := \frac{\psi_{n|Y_m}}{\psi_m} - 1 \in \mathcal{O}(Y_m).$$

Lemma 6.5.5 *For $N, n, m \in \mathbb{N}_{\geqslant 1}$ with $n \geqslant m > N$, we have $\|h_{n,m}\|_{Y_N} \leqslant \max(2^{N-m}, 2^{-m/2})$.*

Proof By Lemma 6.5.4, for each $y \in Y_m$, we have $|\psi_n(y)| = |\psi_m(y)|$. It follows that $\|h_{m,n}\|_{Y_m} \leqslant 1$. We now distinguish two cases.

- Assume that $|h_{n,m}|$ is not constant on Y_N.

By [PT20, Corollary 5.10.16], there exists $y \in \partial Y_N$ such that $\|h_{n,m}\|_{Y_N} = |h_{n,m}(y)|$ and $|h_{n,m}|$ has a negative exponent at y along the branch entering Y_N. By harmonicity (see [PT20, Theorem 5.10.14]), there exist a branch b at y not belonging to Y_N such that the exponent of $|h_{n,m}|$ along b is positive. Repeating the procedure, we construct a path joining y to a boundary point y' of Y_m such that $|h_{n,m}|$ has a positive exponent at each point of $[y, y')$ along the branch pointing towards y'. It follows that we have

$$\|h_{n,m}\|_{Y_n} \geqslant |h_{n,m}(y)|\, \ell([y, y']) \geqslant \|h_{n,m}\|_{Y_N}\, 2^{m-N},$$

hence

$$\|h_{n,m}\|_{Y_N} \leqslant 2^{N-m}.$$

- Assume that $|h_{n,m}|$ is constant on Y_N.

Let N' be the maximum integer smaller than or equal to m such that $|h_{n,m}|$ is constant on $Y_{N'}$. Then, for every $r \in [1, 2^{N'})$, we have $|h_{n,m}(\xi_{1,r})| = \|h_{n,m}\|_{Y_{N'}}$. We also have

$$\begin{aligned}|h_{n,m}(\xi_{1,r})| &= \frac{|(\psi_n - \psi_m)(\xi_{1,r})|}{|\psi_m(\xi_{1,r})|}\\ &= \frac{|(\psi_n - \psi_m)(\xi_{1,r})|}{|T(\eta_{1,1/r})|}\\ &\leqslant \max(|(\psi_n - 1)(\xi_{1,r})|, |(\psi_m - 1)(\xi_{1,r})|)\\ &\leqslant |(T-1)(\eta_{1,1/r})|\\ &\leqslant \frac{1}{r}.\end{aligned}$$

We deduce that $\|h_{n,m}\|_{Y_{N'}} \leqslant 2^{-N'}$.

If $N' < m$, it follows from the previous case that we have $\|h_{n,m}\|_{Y_{N'}} \leqslant 2^{N'-m}$.

In any case, we have

$$\|h_{n,m}\|_{Y_N} \leqslant 2^{-m/2}.$$

□

It follows from Lemma 6.5.5 that the sequence $(\psi_n)_{n>N}$ converges uniformly on Y_N. Let $\psi^{(N)}$ be its limit. It is an analytic function on Y_N.

The functions $\psi^{(N)}$ are compatible, by uniqueness of the limit, which gives rise to an analytic function $\psi \in \mathcal{O}(Y)$. By [PT20, Lemma 5.5.11], there exists a unique analytic morphism $\varphi\colon Y \to \mathbb{A}^{1,\mathrm{an}}_k$ such that the pull-back of T by φ is ψ.

Let $N \in \mathbb{N}_{\geqslant 1}$. By Lemma 6.5.5, there exists $m > N$ such that, for each $n \geqslant m$, we have $\|h_{n,m}\|_{Y_N} \leqslant 2^{-2N}$. (For instance, one could choose $m = 4N$.) By Lemma 6.5.4, we have $\|\psi_m\|_{Y_N} = \|T\|_{\varphi_m(Y_N)} = 2^N$. It follows that $\|\psi_n - \psi_m\|_{Y_N} \leqslant \|\psi_m\|_{Y_N} \|h_{n,m}\|_{Y_N} \leqslant 2^{-N}$. By passing to the limit over n, we deduce that

$$\|\psi - \psi_m\|_{Y_N} \leqslant 2^{-N}.$$

Lemma 6.5.6 *We have $\varphi(Y_N) = \varphi_m(Y_N)$ and $\varphi_{|Y_N}$ is an isomorphism onto its image.*

Proof By Lemma 6.5.4, there exist $d \in \mathbb{N}_{\geqslant 2}, \alpha_2, \dots, \alpha_d \in k^*$ and, for each $j \in [\![2, d]\!]$, $r_j \in [2^{-N}, |\alpha_j|)$ such that $\varphi_m(Y_N)$ is the subset of $\mathbb{A}^{1,\mathrm{an}}_k$ defined by

$$\begin{cases} 2^{-N} < |T| < 2^N;\\ |T-1| > 2^{-N};\\ \forall j \in [\![2, d]\!],\ |T - \alpha_j| > r_j. \end{cases}$$

For $t \in (1, 2^N)$, let W_t be the subset of $\mathbb{A}_k^{1,\mathrm{an}}$ defined by

$$\begin{cases} 2^{-N} t \leqslant |T| \leqslant 2^N t^{-1}; \\ |T-1| \geqslant 2^{-N} t; \\ \forall j \in [\![2, d]\!],\ |T - \alpha_j| \geqslant r_j\, t. \end{cases}$$

Each W_t is compact and the family $(W_t)_{t \in (1, 2^N)}$ is an exhaustion of $\varphi_m(Y_N)$.

Let $n \geqslant m$. For $t \in (1, 2^N)$, the set $\varphi^{-1}(W_t) \cap Y_N$ is the subset of points $y \in Y_N$ such that

$$\begin{cases} 2^{-N} t \leqslant |\psi(y)| \leqslant 2^N t^{-1}; \\ |\psi(y)-1| \geqslant 2^{-N} t; \\ \forall j \in [\![2, d]\!],\ |\psi(y) - \alpha_j| \geqslant r_j\, t. \end{cases}$$

From the inequality $\|\psi - \psi_m\|_{Y_N} \leqslant 2^{-N}$, we deduce that $\varphi^{-1}(W_t) \cap Y_N = \varphi_m^{-1}(W_t) \cap Y_N$.

It follows that $\varphi(Y_N) = \varphi_m(Y_N)$ and that the morphism $\varphi_{|Y_N} : Y_N \to \varphi(Y_N)$ is proper. Since Y_N is a smooth curve and $\varphi_{|Y_N}$ is not constant, it is actually finite.

To prove that $\varphi_{|Y_N}$ is an isomorphism, it is enough to show that it is of degree 1. We will prove that, for each $r \in [1, 2^N)$, we have $\varphi_{|Y_N}^{-1}(\xi_{\infty,r}) = \{\eta_r\}$. This implies the result, by [PT20, Theorem 5.10.17].

Let $r \in [1, 2^N)$. Let $y \in Y_N$ such that $\varphi(y) = \eta_r$. To prove that $y = \xi_{\infty,r}$, we may extend the scalars to $\widehat{k^a}$. The point η_r of $\mathbb{A}_k^{1,\mathrm{an}}$ is characterized by the following equalities:

$$\begin{cases} |T(\eta_r)| = r; \\ \forall \alpha \in \widehat{k^a} \text{ with } |\alpha| = r,\ |(T-\alpha)(\eta_r)| = r. \end{cases}$$

Since $\varphi(y) = \eta_r$, we have

$$\begin{cases} |\psi(y)| = r; \\ \forall \alpha \in \widehat{k^a} \text{ with } |\alpha| = r,\ |\psi(y) - \alpha| = r. \end{cases}$$

Since $\|\psi - \psi_m\|_{Y_N} \leqslant 2^{-N} < r$, the same equalities hold with ψ_m instead of ψ. It follows that $\psi_m(y) = \eta_r$, hence $y = \xi_{\infty,r}$ since ψ_m is injective. □

It follows from Lemmas 6.5.4 and 6.5.6 that, for each $N \in \mathbb{N}_{\geqslant 1}$, $\mathbb{P}_k^{1,\mathrm{an}} - \varphi(Y_N)$ is a disjoint union of closed discs with radii smaller than or equal to 2^{-N}. It follows that

$$\mathbb{P}_k^{1,\mathrm{an}} - \varphi(Y) = \bigcap_{N \geqslant 1} \mathbb{P}_k^{1,\mathrm{an}} - \varphi(Y_N)$$

is a compact subset of $\mathbb{P}^1(k)$ (see the proof of Corollary 6.4.13 for details on k-rationality). By Lemma 6.5.6 again, φ induces an isomorphism onto its image.

We briefly sketch how the proof needs to be modified to handle the case of genus 0 and 1. One may use similar arguments but the paths $\ell_0, \ell_\infty, \ell_1$ have to be constructed in a different way. In genus 0, one first proves that X has rational points and consider paths joining y_0 to them. (In this case, one may also argue more directly to prove that X is isomorphic to $\mathbb{P}^{1,\mathrm{an}}_k$ by Theorems 6.3.12 and 6.3.23.) In genus 1, the skeleton provides two paths and we can use a rational point to construct the third one. Such a point has to exist, since any annulus over k whose skeleton is of large enough length contains some.

Remark 6.5.7 The most difficult part of the proof of Theorem 6.5.3 consists in proving that the k-analytic curve Y, which is known to be of genus 0, may be embedded into $\mathbb{P}^{1,\mathrm{an}}_k$. Contrary to what happens over the field of complex numbers, this is not automatic. This problem was studied extensively by Q. Liu under the assumption that k is algebraically closed. He proved that the answer depends crucially on the maximal completeness of k. If it holds, then any smooth connected k-analytic curve of finite genus may be embedded into the analytification of an algebraic curve of the same genus (hence into $\mathbb{P}^{1,\mathrm{an}}_k$ in the genus 0 case), see [Liu87b, Théorème 3] or [Liu87a, Théorème 3.2]. Otherwise, there exists a smooth connected k-analytic curve of genus 0 with no embedding into $\mathbb{P}^{1,\mathrm{an}}_k$, see [Liu87b, Proposition 5.5]. Q. Liu also prove several other positive results that hold over any algebraically closed base field.

The results of Q. Liu are stated and proved in the language of rigid analytic geometry. We believe that it is worth adapting them to the setting of Berkovich geometry and that this could lead to a different point of view on the sufficient conditions for algebraizablity. One may also wonder whether it is necessary to assume that the base field is algebraically closed to obtain an unconditional positive result. The case of a discretely valued base field (hence maximally complete but not algebraically closed) is, of course, particularly interesting.

6.5.2 Automorphisms of Mumford Curves

In this section, we use the uniformization of Mumford curves to study their groups of k-linear automorphisms. The fundamental result, proven by Mumford in [Mum72a, Corollary 4.12], is the following theorem. We include a proof of this fact that relies on the topology of Berkovich curves.

Theorem 6.5.8 *Let X be a k-analytic Mumford curve. Let $\Gamma \subset \mathrm{PGL}_2(k)$ be its fundamental group, and let $N := N_{\mathrm{PGL}_2(k)}(\Gamma)$ be the normalizer of Γ in $\mathrm{PGL}_2(k)$. Then, we have*

$$\mathrm{Aut}(X) \cong N/\Gamma.$$

Proof Let $p\colon O \to X$ be the universal cover of X provided by Theorem 6.5.3, and let $\sigma \in \mathrm{Aut}(X)$. Since p is locally an isomorphism of k-analytic curves, the automorphism σ can be lifted to an analytic automorphism $\widetilde{\sigma} \in \mathrm{Aut}(O)$ such that $p \circ \widetilde{\sigma} = \sigma \circ p$. By Lemma 6.5.1, $\widetilde{\sigma}$ extends uniquely to an automorphism of $\mathbb{P}_k^{1,\mathrm{an}}$, that is, an element $\tau \in \mathrm{PGL}_2(k)$. The automorphism τ has to normalize Γ: in fact, for any $\gamma \in \Gamma$, the element $\tau\gamma\tau^{-1} \in \mathrm{Aut}(O)$ induces the automorphism $\sigma\sigma^{-1} = \mathrm{id}$ on X. It follows that $\tau\gamma\tau^{-1} \in \Gamma$, so that $\tau \in N$.

Conversely, let $\tau \in N$. By definition, the limit set L of Γ is preserved by τ. It follows that τ induces an automorphism of $O = \mathbb{P}_k^{1,\mathrm{an}} - L$. Moreover, for each $\gamma \in \Gamma$ and each $x \in \mathbb{P}_k^{1,\mathrm{an}}$, we have

$$\tau(\gamma(x)) = (\tau\gamma\tau^{-1})(\tau(x)) \in \Gamma \cdot \tau(x).$$

It follows that τ descends to an automorphism of $X \simeq \Gamma\backslash O$. □

As was the case for the uniformization, Mumford's proof relies on non-trivial results in formal geometry. The Berkovich analytic proof turns out to be shorter and much less technical due to the fact that the uniformization of a Mumford curve can be interpreted as a universal cover of analytic spaces.

Recall from Remark 6.3.8 that the skeleton Σ_X of the Mumford curve X is a finite metric graph. We will denote by $\mathrm{Aut}(\Sigma_X)$ the group of isometric automorphisms of Σ_X. An interesting feature of the automorphism group of an analytic curve, which is immediate in the Berkovich setting, is the existence of a *restriction homomorphism*

$$\begin{aligned} \rho : \mathrm{Aut}(X) &\longrightarrow \mathrm{Aut}(\Sigma_X) \\ \sigma &\longmapsto \sigma_{|\Sigma_X}. \end{aligned}$$

Proposition 6.5.9 *Let X be a Mumford curve of genus at least 2. Then, the restriction homomorphism $\rho : \mathrm{Aut}(X) \to \mathrm{Aut}(\Sigma_X)$ is injective.*

Proof Let $\sigma \in \mathrm{Aut}(X)$ such that $\rho(\sigma) = \mathrm{id}$, that is, σ acts trivially on the skeleton Σ_X. Then, as in the proof of Theorem 6.5.8, one can lift σ to an automorphism of the universal cover $p : O \longrightarrow X$. By possibly composing this lifting with an element of the Schottky group, we can find a lifting $\widetilde{\sigma}$ that fixes a point x in the preimage $p^{-1}(\Sigma_X) \subset O$. Since σ fixes Σ_X pointwise, then $\widetilde{\sigma}$ fixes the fundamental domain in $p^{-1}(\Sigma_X)$ by the action of the Schottky group Γ_X containing x. By continuity of the action of Γ_X on $p^{-1}(\Sigma_X)$, the automorphism $\widetilde{\sigma}$ has to fix the whole $p^{-1}(\Sigma_X)$ pointwise. But then the corresponding element $\tau \in \mathrm{PGL}_2(k)$ obtained by extending $\widetilde{\sigma}$ thanks to Lemma 6.5.1(ii) has to fix the limit set of Γ_X, which is infinite when $g(X) \geq 2$. It follows that τ is the identity of $\mathrm{PGL}_2(k)$, hence that σ is the identify automorphism. □

Remark 6.5.10 The previous proposition can be proved also using algebraic methods as follows. The fact that $g(X) \geq 2$ implies that $\mathrm{Aut}(X)$ is a finite group. Then, for every $\sigma \in \mathrm{Aut}(X)$, $Y := X/\langle\sigma\rangle$ makes sense as a k-analytic curve, and the quotient map $f_\sigma : X \to Y$ is a ramified covering. Let us now suppose that $\rho(\sigma) = \mathrm{id}$. Then Y contains an isometric image of the graph Σ_X, whose cyclomatic number is $g(X)$, by Corollary 6.3.29. It follows from the definition of the genus that $g(Y) \geqslant g(X)$. We can now apply Riemann–Hurwitz formula to find that

$$2g(X) - 2 = \deg(f_\sigma)(2g(Y) - 2) + R,$$

where R is a positive quantity. Since $g(Y) \geqslant g(X) \geq 2$, we deduce that $\deg(f_\sigma) = 1$, hence $\sigma = \mathrm{id}$.

The proposition shows that $\mathrm{Aut}(\Sigma_X)$ controls $\mathrm{Aut}(X)$, but it is a very coarse bound when the genus is high. Much better bounds are known, as one can see in the examples below and in the first part of Appendix A.3, containing an outline of further results about automorphisms of Mumford curves, including the case of positive characteristic.

Example 6.5.11 Let X be a Mumford curve such that $\mathrm{Aut}(\Sigma_X) = \{1\}$. Then Proposition 6.5.9 ensures that X has no non-trivial automorphisms as well. Since, up to replacing k with a suitable field extension, every stable metric graph can be realized as the skeleton of a Mumford curve, one can build in this way plenty of examples of Mumford curves without automorphisms. For example, the graph of genus 3 in Fig. 6.4 below has a trivial automorphism group, as long as the edge lengths are generic enough, for example when all lengths are different.

This graph can be obtained by pairwise identifying the ends of a tree as in Fig. 6.5.

One can realize this tree inside $\mathbb{P}^{1,\mathrm{an}}_k$ as the skeleton of a fundamental domain under the action of a Schottky group in many ways. As an example, if $k = \mathbb{Q}_p$ with $p \geq 5$, a suitable Schottky group is obtained by carefully choosing the

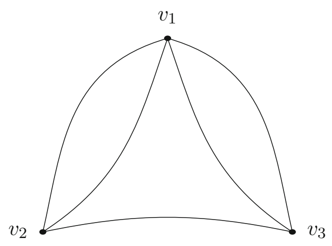

Fig. 6.4 The metric graph Σ_X has trivial group of automorphisms if the edge lengths are all different

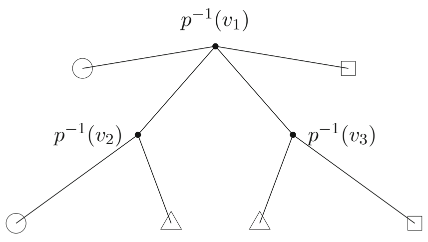

Fig. 6.5 The graph in the previous figure is obtained from its universal covering tree by pairwise gluing the ends of the finite sub-tree Σ_F. The gluing is made by identifying the ends that are marked with the same shape

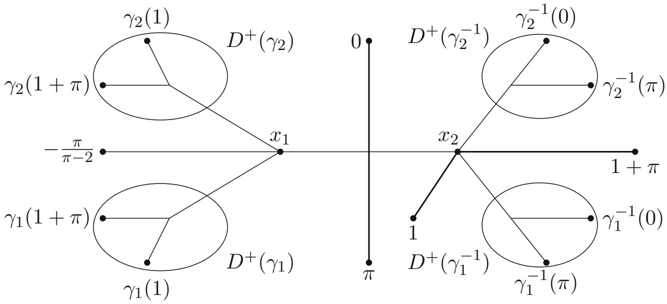

Fig. 6.6 The Schottky figure associated with (γ_1, γ_2)

Koebe coordinates that give rise to the desired skeleton. One can for instance pick $\Gamma = \langle M(0, \infty, p^3), M(1, 2, p^4), M(p, p-2, p^3)\rangle$ and verify that it gives rise to a fundamental domain whose skeleton is the tree in Fig. 6.5. As a consequence of Theorem 6.5.8, the normalizer of Γ in $PGL_2(k)$ is the group Γ itself.

Example 6.5.12 Assume that k is algebraically closed and that its residue characteristic is different from 2 and 3. Let π, ρ be elements of k satisfying $|\pi| < 1$, $\rho^3 = 1$ and $\rho \neq 1$. Fix the following elements of $\mathrm{PGL}_2(k)$:

$$a = \begin{bmatrix} -\pi & 0 \\ -2 & \pi \end{bmatrix},\ b = \begin{bmatrix} 1+\pi-\rho & (1+\pi)(\rho-1) \\ 1-\rho & (1+\pi)\rho - 1 \end{bmatrix}.$$

These elements are of finite order, respectively two and three. The fixed rigid points of a are 0 and π, while the fixed rigid points of b are 1 and $1+\pi$.

Thanks to our assumption that $\mathrm{char}(\widetilde{k}) \neq 2$, the transformation a acting on $\mathbb{P}^{1,\mathrm{an}}_k$ fixes the path joining 0 and π, and sends every open disc whose boundary point lies on this path to a disjoint open disc with the same boundary point. For example, the image by a of the disc $D^-(-\frac{\pi}{\pi-2}, 1)$ is the disc[2] $\mathbb{P}^{1,\mathrm{an}}_k - D^+\big(0, |\pi|\big)$, and vice versa.

The same happens for the action of b: the path joining 1 and $1+\pi$ is fixed, while any open disc with its boundary point on this path is sent to a disjoint open disc with the same boundary point. Since b is of order three, the orbit of such a disc consists of three disjoint discs. For example, the orbit of $D^-(0, 1)$ contains $b\big(D^-(0, 1)\big) = D^-\big(1 - \frac{\pi}{(1+\pi)\rho - 1}, 1\big)$ and $b^2\big(D^-(0, 1)\big) = D^-\big(1 - \frac{\pi}{(1+\pi)\rho^2 - 1}, 1\big)$.

Let us consider the elements $\gamma_1 := abab^2$ and $\gamma_2 := ab^2ab$. Using the geometry of a and b described above, one can check that the 4-tuple $\big(D^+(\gamma_1), D^+(\gamma_1^{-1}), D^+(\gamma_2), D^+(\gamma_2^{-1})\big)$ represented in Fig. 6.6 provides a Schottky figure adapted to (γ_1, γ_2).

Thanks to Proposition 6.4.24, the existence of a Schottky figure ensures that $\Gamma = \langle \gamma_1, \gamma_2 \rangle$ is a Schottky group of rank 2. Denote its limit set by L. By Theorem 6.4.18,

[2]Recall that on the projective line we consider also discs "centered in ∞" such as this one.

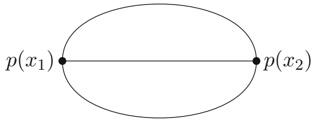

Fig. 6.7 The skeleton Σ_X of the Mumford curve uniformized by Γ

the quotient $X := \Gamma\backslash(\mathbb{P}_k^{1,\mathrm{an}} - L)$ makes sense as a k-analytic space and it is a Mumford curve of genus 2. Let $p\colon (\mathbb{P}_k^{1,\mathrm{an}} - L) \to X$ denote the universal cover. It also follows from Theorem 6.4.18 that the topology of X may be described quite explicitly from the action of Γ. We deduce in this way that the skeleton Σ_X of X is the metric graph represented in Fig. 6.7. By measuring the lengths of the paths joining the boundaries of the discs in the Schottky figure, one can verify that the three edges of Σ_X have equal lengths.

Let us now compute the automorphism group $\mathrm{Aut}(X)$. By Theorem 6.5.8, this can be done by computing the normalizer N of Γ in $\mathrm{PGL}_2(k)$. The elements a and b lie in N, since $\gamma_i a = a\gamma_i^{-1}$ for $i = 1, 2$ and $\gamma_1 b = b\gamma_2^{-1}$, but we can also find elements in N that do not belong to the subgroup generated by a and b. Let

$$c := \begin{bmatrix} 1+\pi & -\pi(1+\pi) \\ 2 & -(1+\pi) \end{bmatrix} \in \mathrm{PGL}_2(k).$$

A direct computation shows that the transformation c is such that $c^2 = \mathrm{id}$, $cac = a$ and $cbc = b^2$, so that c belongs to N. The group $N' = \langle a, b, c\rangle \subset \mathrm{PGL}_2(k)$ is then contained in N, and the quotient N'/Γ is isomorphic to the dihedral group D_6 of order 12. In fact, if we call α, β, γ the respective classes of a, b, c in N'/Γ, we have that $\alpha\beta = \beta\alpha$, and then $\langle\alpha, \beta\rangle$ is a cyclic group of order 6. However, the same computation above shows that γ does not commute with β. The group D_6 is also the automorphism group of the skeleton Σ_X, and so, by Proposition 6.5.9, we have $N = N'$ and the restriction homomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}(\Sigma_X)$ is an isomorphism.

Note that one can extract quite a lot of information from the study of the action of N on $\mathbb{P}_k^{1,\mathrm{an}}$. In this example, $\alpha \in \mathrm{Aut}(X)$ is an order 2 automorphism known as the *hyperelliptic involution*, since it induces a degree 2 cover of the projective line $\varphi : X \to \mathbb{P}_k^{1,\mathrm{an}}$. This last fact can be checked on the skeleton Σ_X by noting that $\alpha(p(x_1)) = p(x_2)$, and hence α has to switch the ends of every edge of Σ_X. As a result, the quotient $X/\langle\alpha\rangle$ is a contractible Mumford curve, and hence it is isomorphic to $\mathbb{P}_k^{1,\mathrm{an}}$.

This description of X as a cover of $\mathbb{P}_k^{1,\mathrm{an}}$ is helpful to compute an explicit equation for the smooth projective curve whose analytification is X. In fact, a genus 2 curve that is a double cover of the projective line can be realized as the smooth compactification of a plane curve of equation

$$y^2 = \prod_{i=1}^{6}(x - a_i),$$

where the $a_i \in k$ are the ramification points of the cover, and the involution defining the cover sends y to $-y$.

In order to find the a_i, we shall first compute the branch locus $B \subset X$ of the hyperelliptic cover. The fixed points of a are 0 and π, so the corresponding points $p(0)$, $p(\pi)$ are in B. The other branch points can be obtained by finding those $x \in \mathbb{P}^{1,\mathrm{an}}_k(k)$ satisfying the condition $\gamma_i(x) = a(x)$ for $i = 1, 2$. We have $\gamma_1(b(0)) = abab^2b(0) = aba(0) = a(b(0))$, and the same applies to $b(\pi)$, so the images by p of these two points are also in B. In the same way, we find that $\gamma_2(b^2(0)) = a(b^2(0))$ and $\gamma_2(b^2(\pi)) = a(b^2(\pi))$. We have found in this way that $B = \{p(0), p(b(0)), p(b^2(0)), p(\pi), p(b(\pi)), p(b^2(\pi))\}$.

To find the ramification locus, we have to compute $\varphi(B)$. Since $\langle\alpha\rangle$ is a normal subgroup of $\mathrm{Aut}(X)$, the element β acts as an automorphism of order 3 of $X/\langle\alpha\rangle \cong \mathbb{P}^{1,\mathrm{an}}_k$. Up to a change of coordinate of this projective line, we can suppose that the fixed points of β are 0 and ∞, so that β is the multiplication by a primititve third root of unity, and that the first ramification point is $a_1 = \varphi(p(0)) = 1$. Then, after possibly reordering them, the remaining ramification points are $a_2 = \rho$, $a_3 = \rho^2$ and a_4, ρa_4, $\rho^2 a_4$, with $|a_4 - 1| < 1$.

With a bit more effort, we can actually compute the value of a_4. To do this, notice that the function p is injective when restricted to the open fundamental domain

$$F^- = \mathbb{P}^{1,\mathrm{an}}_k - \big(D^+(\gamma_1) \cup D^+(\gamma_1^{-1}) \cup D^+(\gamma_2) \cup D^+(\gamma_2^{-1})\big).$$

If we set $F' = \varphi \circ p(F^-)$ we then have a two-fold cover $F^- \longrightarrow F' \subset \mathbb{P}^{1,\mathrm{an}}_k$ induced by $\varphi \circ p$, which can be explicitly written as a rational function $z \mapsto \frac{z^2}{(z-\pi)^2}$ (this function can be found by looking at the action of a on F^- explicitly). Note that F^- contains both the fixed rigid points of b, i.e. $1 + \pi$ and 1, and those of a, i.e. 0 and π. When we reparametrize the projective line on the target of φ to get the wanted equation, we are imposing the conditions $\alpha \circ p(1+\pi) \mapsto \infty$, $\alpha \circ p(1) \mapsto 0$ and $\alpha \circ p(0) \mapsto 1 = a_1$. These choices leave only one possibility for the ramification point a_4: it is $\left(\frac{1-\pi}{1+\pi}\right)^2$. We have now found the equation of the plane section of our Mumford curve: it is

$$y^2 = (x^3 - 1) \cdot \left(x^3 - \frac{(1-\pi)^6}{(1+\pi)^6}\right).$$

Note that $|a_4 - a_1| = \left|\frac{(1-\pi)^2}{(1+\pi)^2} - 1\right| = |\pi|$.

A different example of a hyperelliptic Mumford curve with a similar flavour is discussed in the expository paper [CK05], accompanied with figures and other applications of automorphisms of Mumford curves.

Example 6.5.13 The curve in Example 6.5.12 has the same automorphism group in every characteristic (different from 2 and 3). However, Mumford curves in positive characteristic have in general more automorphism than in characteristic 0.

An interesting class of examples are the so-called *Artin-Schreier-Mumford curves*, first introduced by Subrao in [Sub75]. We sketch here the main results and refer to [CKK10] for more detailed proofs of these facts. Let p be a prime, $q = p^e$ be a power of p, and $k = \mathbb{F}_q((t))$. Let X be the analytification of the curve defined inside $\mathbb{P}^1_k \times \mathbb{P}^1_k$ by the equation

$$(y^q - y)(x^q - x) = f(t) \quad \text{with} \quad f \in t\mathbb{F}_q[[t]].$$

This is an ordinary curve in characteristic $p > 0$ with many automorphisms, and for this reason has caught the attention of cryptographers and positive characteristic algebraic geometers alike. One way to study its automorphisms is to observe that X is a Mumford curve. A Schottky group attached to it can be constructed by fixing an element $v \in k$ and looking at the automorphisms of $\mathbb{P}^{1,\mathrm{an}}_k$ of the form

$$a_u = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}, b_u = \begin{bmatrix} v & 0 \\ u & v \end{bmatrix} \in \mathrm{PGL}_2(k),\ u \in \mathbb{F}_q^\times.$$

These transformations are all of order p, a_u represent translations by elements of $\mathbb{F}_q^\times$ and b_u their conjugates under the inversion $z \mapsto \frac{v}{z}$. The subgroup $\Gamma_v = \langle a_u^{-1} b_{u'}^{-1} a_u b_{u'} : (u, u') \in \mathbb{F}_q^{\times 2} \rangle$ of $\mathrm{PGL}_2(k)$ is a Schottky group of rank $(q-1)^2$, and for a certain value of v[3] it gives rise to the curve X by Schottky uniformization. The immediate consequence of this fact, is that X is a Mumford curve of genus $(q-1)^2$.

The group of automorphisms $\mathrm{Aut}(X)$ is isomorphic to a semi-direct product $(\mathbb{Z}/p\mathbb{Z})^{2e} \rtimes D_{q-1}$, and its action is easy to describe using the equation of the curve: the elementary abelian subgroup $(\mathbb{Z}/p\mathbb{Z})^{2e}$ consists of those automorphisms of the form $(x, y) \mapsto (x + \alpha, y + \beta)$ with $(\alpha, \beta) \in (\mathbb{F}_q)^2$, while the dihedral subgroup D_{q-1} is generated by $(x, y) \mapsto (y, x)$ and $(x, y) \mapsto (\gamma x, \gamma^{-1} y)$ for $\gamma \in \mathbb{F}_q^\times$. We deduce that the order of $\mathrm{Aut}(X)$ is $2(q-1)q^2$. In characteristic 0, it is not possible to have these many automorphisms, thanks to bounds by Hurwitz and Herrlich that would give rise to a contradiction (see Appendix A.3 for the precise statement of these bounds).

Acknowledgments We warmly thank Marco Maculan for his numerous comments on an earlier version of this text, and the anonymous referees for their remarks and corrections. The Appendix A.3 greatly benefited from insights and remarks by Jan Vonk and Henri Darmon.

While writing this text, the authors were supported by the ERC Starting Grant "TOSSIBERG" (grant agreement 637027). The second author was partially funded by a prize of the *Fondation des Treilles*. The *Fondation des Treilles* created by Anne Gruner-Schlumberger aims to foster the dialogue between sciences and arts in order to further research and creation. She welcomes researchers and creators in her domain of Treilles (Var, France).

[3]The relationship between v and $f(t)$ is not immediate, and it is the object of the paper [CKK10].

Appendix: Further Reading

The theory of Berkovich curves has several applications to numerous fields of mathematics, and uniformization plays a role in many of these. A complete description of these applications goes far beyond the scope of the present text, but we would like to provide the interested reader with some hints about the state of the art and where to find more details in the existing literature, as well as point out which simplifications adopted in this text are actually instances of a much richer theory.

A.1 Berkovich Spaces and their Skeleta

We provided a short introduction to the theory of Berkovich curves and their skeleta in Sect. 6.3.2 of this text.

The first discussion of this topic appears already in Chapter 4 of Berkovich's foundational book [Ber90]. In this context, the definition of the skeleton of a Berkovich curve X makes use of formal models and the semi-stable reduction theorem, that states that for the analytification of a smooth proper and geometrically irreducible algebraic curve over k, there exists a finite Galois extension K of k such that the base change X_K has a semi-stable formal model. Berkovich showed that the dual graph of the special fiber of any semi-stable formal model embeds in the curve X_K and that it is invariant by the action of the Galois group $\mathrm{Gal}(K/k)$ over X_K, which allows to define skeleta of X as quotients of skeleta of X_K. This construction is again found in A. Thuillier's thesis [Thu05], where it is exploited to define a theory of harmonic functions on Berkovich curves.

In Definition 6.3.3, we adopted another approach to the study of skeleta, via the use of triangulations. This was first introduced by Ducros in [Duc08] to study étale cohomology groups of Berkovich curves. In the case where k is algebraically closed, a comprehensive exposition of skeletons, retractions, and harmonic functions on non-Archimedean curves can be found in the paper [BPR14]. There, the authors are motivated by connections with tropical geometry, as, for a given algebraic variety over k, the skeletons of its analytification are tightly related to its tropicalization maps. Other than in the aforementioned paper, these connections are exposed in [Wer16], where the higher-dimensional cases are highlighted as well.

As for higher-dimensional spaces, Berkovich introduced skeleta in [Ber99]. They are simplicial sets onto which the spaces retract by deformation. They are constructed using semi-stable formal models and generalizations of them, so they are not known to exist in full generality, but Berkovich nonetheless managed to use them to prove that smooth spaces are locally contractible (hence admits universal covers).

The connections with tropical geometry have proven fruitful, among other things, to study finite covers of Berkovich curves $Y \to X$ over k. The general pattern is that these covers are controlled by combinatorial objects that are enhanced versions of compatible pairs (Σ_Y, Σ_X) of skeletons of the curves Y and X. Assume that k is algebraically closed. Whenever the degree of such a cover is coprime with the residue characteristic of k, the papers [ABBR15a] and [ABBR15b] give conditions on a pair (Σ_Y, Σ_X) to lift to a finite morphism of curves $Y \to X$. In the case of covers of degree divisible by the residue characteristic of k, the situation is still far from understood, but progress has been made thanks to the work of M. Temkin and his collaborators in the papers [CTT16, Tem17, BT20]. The main tool used in these works is the *different function*.

With regard to higher-dimensional varieties, a new approach to skeletons was proposed by E. Hrushosvksi and F. Loeser in [HL16] using techniques coming from model theory. They are able to define skeleta of analytifications of quasi-projective varieties and deduce the remarkable result that any such space has the homotopy type of a CW-complex.

In the specific case of curves over an algebraically closed base field, the paper [CKP18] uses triangulations in order to give a more concrete model-theoretic version of Berkovich curves (and morphisms between them). In particular, the authors manage to give an explicit description of definable subsets of curves and prove some tameness properties.

Without the assumption that k is algebraically closed, or rather that X has a semi-stable formal model over the valuation ring of k, the structure of analytic curves is much harder to grasp, due among other things to the difficulty of classifying virtual discs and virtual annuli. The curious reader will find much food for thought in the book by A. Ducros [Duc], which can nevertheless be of difficult reading for a first approach. If k is a discrete valuation field, a generalization of potential theory on Berkovich curves is provided in [BN16] thanks to a careful study of regular models, and the introduction of the notion of *weight function*. In regard to the problem of determining a minimal extension necessary for the existence of a semi-stable model, an approach via triangulations has been recently proposed in [FT19].

Finally, let us mention that we chose to introduce Berkovich curves as $\mathbb{A}^1$-like curves because we are convinced that this is a natural framework for studying uniformization, but the general theory is much richer, and contains many examples of Berkovich curves that are not $\mathbb{A}^1$-like.

A.2 Non-Archimedean Uniformization in Arithmetic Geometry

In the case of curves over the field of complex numbers, Schottky uniformization can be seen in the context of the classical uniformization theorem for Riemann surfaces, proven independently by P. Koebe and H. Poincaré in 1907. It states that every simply connected complex Riemann surface is conformally equivalent to the complex projective line, the complex affine line, or the Poincaré upper-half plane. As a consequence, the universal covering space of any Riemann surface X is one of these, and when X is compact, Koebe-Poincaré uniformization factors through the Schottky uniformization $\left(\mathbb{P}^{1,\text{an}}_{\mathbb{C}} - L\right) \to X$. A remarkable book on complex uniformization [dSG10] has been written by the group of mathematicians known under the collective name of Henri Paul de Saint-Gervais. It constitutes an excellent reference both on the historical and mathematical aspects of the subject.

In the non-Archimedean case, the history of uniformization is much more recent. The uniformization theory of elliptic curves over a non-Archimedean field $(k, |\cdot|)$ was the main motivation underlying J. Tate's introduction of rigid analytic geometry in the 1960s. Using his novel approach, Tate proved that every elliptic curve with split multiplicative reduction over k is analytically isomorphic to the multiplicative group $k^\times/q^{\mathbb{Z}}$ for some q in k with $0 < |q| < 1$. Tate's computations were known to experts, but remained unpublished until 1995, when they were presented in [Tat95] together with a discussion on further aspects of this theory, including automorphic functions, a classification of isogenies of Tate curves, and a brief mention of how to construct "universal" Tate curves over the ring $\mathbb{Z}[[q]][q^{-1}]$ using formal geometry. These formal curves appeared for the first time in the paper [DR73] by P. Deligne and M. Rapoport, who attributed it to M. Raynaud and called them *generalized elliptic curves*. In *loc. cit.* the authors exploited them to give a moduli-theoretic interpretation at the cusps of the modular curves $X_0(Np)$ with $p \nmid N$. Further reading in this direction include the foundational paper [KM85], that concerns the case of modular curves $X(Np^n)$ and [Con07], that provides a more contemporary perspective on generalized elliptic curves.

Interpreting the Schottky uniformization of Mumford curves of [Mum72a] as a higher genus generalization of Tate's theory inspired several novel arithmetic discoveries. One of the most important is the uniformization of Shimura curves, fundamental objects in arithmetic geometry that vastly generalize modular curves. In [Che76], I. Cherednik considered a Shimura curve $\mathcal{C}$ associated with a quaternion algebra B over $\mathbb{Q}$. For a prime p where B is ramified, he proved that the p-adic analytic curve $(\mathcal{C} \times_{\mathbb{Q}} \mathbb{Q}_p)^{\text{an}}$ can be obtained as a quotient of *Drinfeld p-adic halfplane* $\mathbb{P}^{1,\text{an}}_{\mathbb{Q}_p} - \mathbb{P}^{1,\text{an}}_k(\mathbb{Q}_p)$, by the action of a Schottky group. This Schottky group can be as a subgroup of a different quaternion algebra B' over $\mathbb{Q}$, constructed explicitly from B via a procedure known as *interchange of invariants*. The theory obtained in this way is classically referred to as *Cherednik-Drinfeld uniformization*, since V. Drinfeld gave a different proof of this result in [Dri76], building on a description of $\mathcal{C}$ as a moduli space of certain abelian varieties. The excellent paper [BC91] provides a detailed account of these constructions.

By generalizing Drinfeld's modular interpretation, the approach can be extended to some higher dimensional Shimura varieties, resulting in their description as quotients of the Drinfeld upper-half space via a uniformization map introduced independently by G. Mustafin [Mus78] and A. Kurihara [Kur80]. For a firsthand account of the development of this uniformization, we refer the reader to the book [RZ96] by M. Rapoport and T. Zink.

Non-Archimedean uniformization of Shimura varieties has remarkable consequences. First of all, it makes possible to find and describe integral models of Shimura varieties, since the property of being uniformizable imposes restrictions on the special fibers of such models. Furthermore, it gives a way to compute étale and ℓ-adic cohomology groups, as well as the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ on these, making it a powerful tool for studying Galois representations. All the aforementioned results were shown in the framework of formal and rigid geometry. However, more contemporary approaches to uniformization of Shimura varieties and Rapoport-Zink spaces make use of Berkovich spaces (see [Var98, JLV03]), or Huber adic geometry in the form of perfectoid spaces (see [SW13] and [Car19]). In particular, the perfectoid approach can be used to vastly generalize the uniformization of Shimura varieties and establish a theory of *local Shimura varieties*. This construction is exposed in the lecture notes [SW20] by P. Scholze and J. Weinstein.

Local and global uniformization of Shimura varieties are investigated in relation to period mappings, Gauss-Manin connections, and uniformizing differential equations in the book by Y. André [And03], where striking similarities between the complex and p-adic cases are highlighted. For more results about the relevance of Shimura varieties, not necessarily with regard to uniformization, we refer to [Mil05].

Finally, let us mention that Tate's uniformization of elliptic curves with split multiplicative reduction generalizes to abelian varieties. This is also a result of Mumford, contained in the paper [Mum72b], that can be regarded as a sequel to [Mum72a], since the underlying ideas are very similar. In this case, the uniformization theorem is formulated by stating that a totally degenerate abelian variety of dimension g over k is isomorphic to the quotient of the analytic torus $(\mathbb{G}_{m,k}^g)^{\mathrm{an}}$ by the action of a torsion-free subgroup of $(k^\times)^g$. This applies in particular to Jacobians of Mumford curves, a case surveyed in detail in the monograph [Lüt16]. We shall remark that Mumford's constructions are more general than their presentation in this text: they work not only over non-Archimedean fields, but more generally over fields of fractions of complete integrally closed noetherian rings of any dimension.

A.3 The Relevance of Mumford Curves

The uniformization theorem in the complex setting is a very powerful tool, and one of the main sources of analytic methods applied to the study of algebraic curves. This leads to the expectation that, in the non-Archimedean setting, Mumford curves can be more easily studied, turning out to be a good source of examples for testing certain conjectures. This is indeed the case for several topics in algebraic curves and their applications, as we could already sample in Sect. 6.5.2 on the subject of computing the group of automorphisms of curves.

This appendix is a good place to remark that Examples 6.5.12 and 6.5.13 in that section are instances of a much deeper theory. For a smooth projective algebraic curve C of genus $g \geq 2$ over a field of characteristic zero, the Hurwitz bound ensures that the finite group of automorphisms $\mathrm{Aut}(C)$ is of order at most $84(g-1)$. This bound is sharp: there exist curves of arbitrarily high genus whose automorphism groups attain it, the so-called *Hurwitz curves*. However, if we know that C is (the algebraization of) a Mumford curve, F. Herrlich proved a better bound in [Her80]. Namely, if we denote by p the residue characteristic of K, he showed that:

$$|\mathrm{Aut}(C)| \leq \begin{cases} 48(g-1) & p=2 \\ 24(g-1) & p=3 \\ 30(g-1) & p=5 \\ 12(g-1) & \text{otherwise.} \end{cases}$$

This result relies on the characterization of automorphism groups of Mumford curves as quotients N/Γ, where Γ is a Schottky group associated with C and N its normalizer in $\mathrm{PGL}_2(K)$ (see Theorem 6.5.8). One can show that the group N acts discontinuously on an infinite tree that contains the universal covering tree of the skeleton $\Sigma_{C^{\mathrm{an}}}$, and use Serre's theory of groups acting on trees to prove that N is an amalgam of finite groups. In his paper, Herrlich achieves the bounds above by classifying those amalgams that contain a Schottky group as a normal subgroup of finite index.

Over a field of characteristic $p > 0$, the Hurwitz bound is replaced by the Stichtenoth bound, stating that $|\mathrm{Aut}(C)| \leq 16g^4$, unless C is isomorphic to a Hermitian curve. When C is a Mumford curve, this bound can be improved in principle using Herrlich's strategy. However, this is not an easy task, as one has to overcome the much bigger difficulties that arise in positive characteristic. This has been achieved recently by M. Van der Put and H. Voskuil, who prove in [VvdP19, Theorem 8.7] that $|\mathrm{Aut}(C)| < \max\{12(g-1), g\sqrt{8g+1}+3\}$ except for three occurrences of (isomorphism classes of) X, which happen when $p=3$ and $g=6$. Moreover, in [VvdP19, Theorem 7.1] they show that the bound is achieved for any choice of the characteristic $p > 0$. The bound corrects and extends a bound given by G. Cornelissen, F. Kato and A. Kontogeorgis in [CKK01].

Another application of uniformization of Mumford curves is the *resolution of non-singularities* for hyperbolic curves[1] over $\overline{\mathbb{Q}}_p$. Given such a curve X, and a smooth point P of the special fiber of a semi-stable model of X, it is an open problem to find a finite étale cover $Y \longrightarrow X$ such that a whole irreducible component of the special fiber of the stable model of Y lies above P. Earlier versions of this problem were introduced and proved by S. Mochizuki [Moc96] and A. Tamagawa [Tam04], that showed connections with important problems in

[1] A hyperbolic curve in this context is a genus g curve with n marked points satisfying the inequality $2g-2+n > 0$.

anabelian geometry. The interest of the version proposed here is also motivated by anabelian geometry: F. Pop and J. Stix proved in [PS17] that any curve for which resolution of non-singularities holds satisfies also a valuative version of Grothendieck's section conjecture. In the paper [Lep13], E. Lepage uses Schottky uniformization in a Berkovich setting to show that resolution of non-singularities holds when X is a hyperbolic Mumford curve. His approach consists in studying μ_{p^n}-torsors of the universal cover of X, which are better understood since they can be studied using logarithmic differentials of rational functions. With this technique, he can show that there is a dense subset of type 2 points $\mathcal{V} \in X$, with the following property: every $x \in \mathcal{V}$ can be associated with a μ_{p^n}-torsor $\tau : Y \to X$ such that $\tau^{-1}(x)$ is a point of positive genus. This last condition ensures that the corresponding residue curve is an irreducible component of the stable model of Y.

Mumford curves have been also proven useful in purely analytic contexts, for instance to study potential theory and differential forms. Using the fact that all type 2 points in a Mumford curve are of genus 0, P. Jell and V. Wanner [JW18] are able to establish a result of Poincaré duality and compute the Betti numbers of the tropical Dolbeaut cohomology arising from the theory of bi-graded real valued differential forms developed in [CLD12].

Finally, let us mention that archimedean and non-archimedean Schottky uniformizations can be studied in a unified framework thanks to work of the authors [PT], where a moduli space $\mathcal{S}_g$ parametrizing Schottky groups of fixed rank g over all possible valued fields is constructed for every $g \geq 2$. This construction is performed in the framework of *Berkovich spaces over* $\mathbb{Z}$ developed in [Poi10, Poi13, LP]. More precisely, the space $\mathcal{S}_g$ is realized as an open, path-connected subspace of $\mathbb{A}_{\mathbb{Z}}^{3g-3,\mathrm{an}}$, it is endowed with a natural action of the group $\mathrm{Out}(F_g)$ of outer automorphisms of the free group, and exhibits interesting connections with other constructions of moduli spaces, in the frameworks of tropical geometry and geometric group theory. The space $\mathcal{S}_g$ seems to be ideal to study phenomena of degeneration of Schottky groups from archimedean to non-archimedean.

A different take on the interplay between archimedean and non-archimedean Schottky uniformizations is provided by Y. Manin's approach to Arakelov geometry. In the paper [Man91] several formulas for computing the Green function on a Riemann surface using Schottky uniformization and are explicitly inspired by Mumford's construction. These formulas involve the geodesics lengths in the hyperbolic handlebody uniformized by the Schottky group associated with such a surface, suggesting connections between hyperbolic geometry and non-archimedean analytic geometry. This result has been reinterpreted in term of noncommutative geometry by C. Consani and M. Marcolli [CM04] by replacing the Riemann surface with a noncommutative space that encodes certain properties of the archimedean Schottky uniformization. This noncommutative formalism has led to applications both in the non-archimedean world (see for example [CM03]) and in the archimedean one, for instance to Riemannian geometry in [CM08]. We think that the theory of Berkovich spaces could fit nicely in this picture, and it would be an interesting project to investigate the relations between noncommutative geometric objects related to Schottky uniformization (e.g. graph $C^{\star}$-algebras) and Mumford curves in the Berkovich setting.

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