The Multiplicative-Isolating Principle of Significantly Nonlinear Mathematical Models Creation

Rudolf Neydorf, Anatoly Gaiduk, and Nikita Gamayunov

Abstract A new principle of mathematical models creation for the technical and other objects which are characterized by essential nonlinearity of interdependence of variables is substantiated. It is shown that the use of special analytical functions with a number of unique properties for the multiplicative transformation of fragmented mathematical models that approximate certain parts of the modeled dependencies allows us to obtain new functions that are analytically isolated in their domain of definition. The analyticity of these functions makes it possible additively to combine them into a single analytical function that approximates the simulated dependence in the entire area of its description. In this case, the total approximation error of the mathematical model of the studied object does not exceed the largest error of fragmentary approximating functions. The unique properties of multiplicatively isolating functions provide an almost "seamless" additive union of fragments. This chapter is dedicated to the study and proof of these properties, which are illustrated by examples.

Keywords Nonlinear dependence · Mathematical model · Experimental data · Approximation · Analytical function · Multiplicative isolation

1 Introduction

Modern control systems for various processes and technical objects are usually created based on their mathematical models (MM). This approach, in particular, is

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A. G. Kravets et al. (eds.), *Cyber-Physical Systems: Modelling and Intelligent Control*, Studies in Systems, Decision and Control 338,
https://doi.org/10.1007/978-3-030-66077-2_3

characteristic when using analytical methods for the synthesis of control systems [1–4]. Currently, such analytical methods are widely used as linearization by feedback [5, 6], the passification method [7–9], the trajectory-positional approach [10, 11], the method of quasilinear models [12], and the Jordan controlled form method [13] and others. In these cases, the required control law is found by solving some systems of equations that are created on the basis of the requirements for the synthesized system taking into account the control object's MM properties. However, analytical methods of synthesis can be implemented if only the object MM is analytical, i.e. the functions of its equations are continuous and differentiable. This requirement is due to the fact that each analytical method for the control systems synthesis implies some mathematical transformations possibility of the MM equations of both objects and control systems.

On the other hand, the modern systems quality requirements have increased so that they can be satisfied only taking into account the nonlinearities that are inherent in almost all controlled objects. Therefore, the MM of real control objects are nonlinear [12–14]. Most often, these MMs are determined experimentally by applying some test actions to the object and fixing its reaction to these actions. The tables obtained as a result contain experimental point data on the dependence of the object's variables from the trial actions and other variables. Very often, these dependencies are very complex and cannot be described quite accurately by continuous analytical functions in the entire range of arguments. They have a pronounced fragmentary character, and neighboring fragments can have different slopes and curvature [14, 15].

Therefore, to obtain MM, these data are approximated at separate intervals by changes in variables by suitable analytical functions. However, the fragmented MM of the nonlinear object obtained in this way is not analytical, which does not allow the use of most of the known methods for the nonlinear controls synthesis.

An example of such a two-dimensional nonlinear dependence $f(x_1, x_2)$, having a fragmentary structure with respect to the variable x_1, is shown in Fig. 1. As seen here in the interval $x_1 \in [0 \div (4 - 0,333x_2)]$ derivative $\partial y/\partial x_1$ is positive, and when $x_1 \in [(4 - 0,333x_2) \div 6]$ this derivative is negative, i.e. at the boundary of these

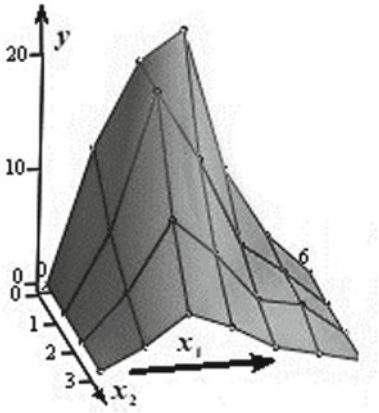

Fig. 1 Two-dimensional dependence $y = f(x_1, x_2)$

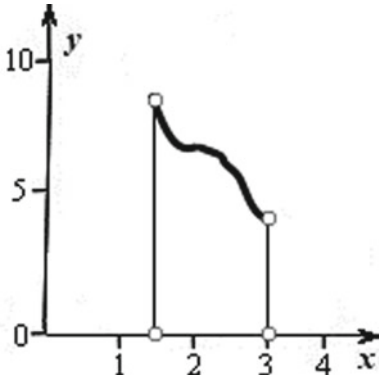

Fig. 2 Function fragment $f_0(x)$

intervals there is a "kink" of nonlinearity. In such cases, methods are used that can describe the "kink" with the necessary accuracy. The simplest and most effective is the piecewise (fragmentary) description [14], which provides any accuracy.

In Fig. 2 shows a fragmentary description $y = f_0(x)[1(x - 1.5) - 1(x - 3)]$ of the nonlinearity of one variable x by some function $f_0(x)$, but only on the interval $1.5 \leq x \leq 3$. Here, step function $1\,(x)$ is used, but it is not differentiable, therefore the MM is nonlinearity (Fig. 2) but is not analytic, since it has gaps in the derivatives on the points of the fragments joining, that excludes a possibility of analytical transformations of such models. The spline approximation also gives a fragmentary description, but it is also inconvenient for analytical transformations of MM [14, 15]. In addition to the "fragment-oriented" methods, methods are known that are oriented to the construction of analytical MMs: regression analysis [16], polynomial decompositions [17], methods of radial basis functions [18], and others. But they do not provide the necessary accuracy for approximating nonlinearities with kinks, gaps, and multi-extremums.

In order to overcome these difficulties, Professor R.A. Neydorf developed a multiplicative-isolating principle for the MM creation of essentially nonlinear controlled objects. The algorithmic basis of this principle is the original multiplicatively isolating function (MIF) developed by him. It allows one to obtain a common analytical MM of the controlled object using the Cut-Glue approximation method, which provides an additive "seamless" combination of fragments of experimental point data [12, 19, 20].

The aim of this work is to study the properties of MIF as the core of the multiplicative-isolating principle of creating essentially nonlinear MMs.

2 Multiplicative-Isolating Principle

The multiplicative-isolating principle is a fundamentally new approach to solving the problem of approximating experimental data, which, due to significant nonlinearity, need fragmentation to achieve the required approximation accuracy. It has two functionally different stages: the stage of preparation and primary processing of experimental data, as well as the stage of final processing with creation of the approximating MM. The experimental data array is split into fragments at the preparatory stage. Then each fragment is subjected to mathematical description by locally approximating functions. These functions are usually found in the class of analytic functions. The local approximation process is implemented using well-known classical methods [16–18].

At the stage of final processing, the experimental data undergo a multiplicative transformation using multiplicatively isolating functions. On the one hand, MIFs isolate the fragments created at the first stage, and on the other hand, they combine these fragments into a single analytical function. These operations are schematically

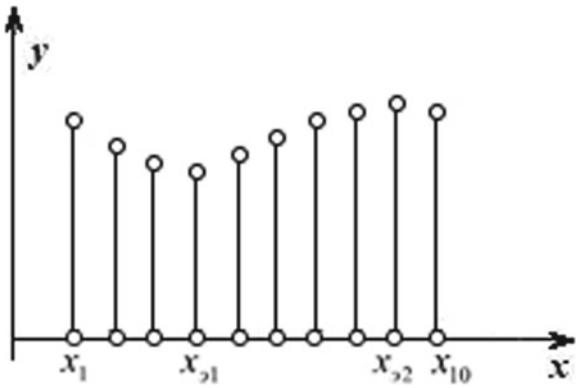

Fig. 3 Point array of experimental data

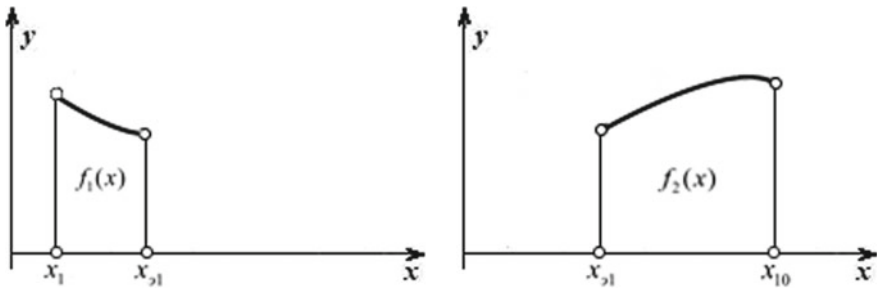

Fig. 4 Results of multiplicative transformation (1)

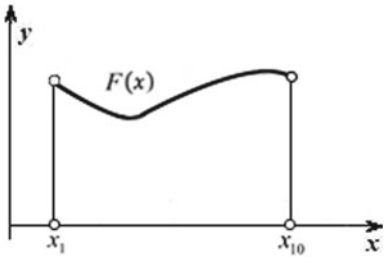

Fig. 5 The resulting single function $F(x)$

shown in Figs. 3, 4 and 5. In this case, two fragments are cut from the point experimental data as shown in Fig. 3. Each of these fragments is approximated by a suitable analytic function (Fig. 4), which are then combined (are glued) into a single continuous function (Fig. 5). These operations are an essential, distinctive feature of the multiplicative-isolating principle, which determines its approximating capabilities and uniqueness.

Thus, the processing of experimental data in accordance with the multiplicative-isolating principle consists in "cutting" them into separate fragments, local approximation of these fragments and then these fragments are glued into a single continuous function with analytical properties. Therefore, the practical implementation of this principle was called the "Cut-Glue Approximation" (CGA) method. Further we will consider in more detail the main stages of the multiplicative-isolating principle implementation.

The initial array of experimental data on a certain object, the model of which must be created, is represented by point values $y(x_1) - y(x_{10})$, shown in Fig. 3. The entire array of experimental data does not lend itself to sufficiently accurate approximation by known methods. This is caused by multi-extremums (minimum at the point $x_{э1}$ and maximum at the point $x_{э2}$), and also a rather sharp "kink" of nonlinearity at the point $x_{э1}$. Therefore, the array is cut, in this case, into two fragments. Values $y(x_1) - y(x_{э1})$ refer to the first fragment, and values $y(x_{э1}) - y(x_{10})$ to the second fragment.

Each of the shown fragments is described with a sufficiently high accuracy by the analytical locally approximating functions $\varphi_1(x)$ and $\varphi_2(x)$. This operation corresponds to the application of the well-known piecewise approach to the approximating nonlinear data problem [14, 16]. However, the paradigm of the multiplicative-isolating principle is to obtain the final result—MM, in the form of a function with analytical properties. This condition implies additional steps to combine (glue) the approximating functions of the selected fragments.

For this purpose, approximating functions of each individual fragment of experimental data is exposed to the isolating transformation. The essence of this transformation lies in the fact that the approximation created within the limits of the experimental data region limited by the fragment is preserved, and outside the fragment boundaries the multiplicatively isolating function provides practically zero value.

The result of the isolating transformation is new analytical functions—$f(x)$, which are formed from $\varphi(x)$ by multiplying each $\varphi(x)$ on a special function $E(x)$:

$$f_1(x) = \varphi_1(x)E_1(x), \quad f_2(x) = \varphi_2(x)E_2(x). \tag{1}$$

This operation determined the use of the term "multiplicatively". Figure 4 shows the functions $f_1(x)$ and $f_2(x)$, obtained as a result of applying the multiplicative operation (1).

Finally, the functions $f_1(x)$ и $f_2(x)$ are glued together into one analytical expression by simple summation:

$$F(x) = f_1(x) + f_2(x). \tag{2}$$

The result of their addition—function $F(x)$, is a single continuous curve, which is shown in Fig. 5.

As can be seen from expressions (1) and (2), the function $E_i(x)$ performs two operations simultaneously: (1) multiplicatively selects each fragment $\varphi(x)$, keeping the values of the approximating function in its definition domain; (2) multiplicatively "zeroes out" all values of this locally approximating function $\varphi(x)$ outside the fragment, forming new multiplicatively transformed functions $f(x)$. This provides the actual fragment "isolation" $\varphi(x)$ in the experimental data array. Therefore the function $E_i(x)$ from (1) is called the multiplicatively isolating function.

The transformed functions (1) are independent from each other and from other fragments. Thus, the operation addition of the multiplicatively transformed functions $f_1(x)$ and $f_2(x)$ (2) becomes correct both in the mathematical sense and in the sense of the multiplicative-isolating principle's paradigm.

Thus, the multiplicatively isolating function (MIF) is the most essential element of the multiplicative-isolating principle implementation, so we pass on to a detailed consideration of its properties.

3 Properties of the Multiplicatively Isolating Function

This function should have the following ideal properties as a means of implementing the multiplicative-isolating principle. First, it should have a value ideally equal to 1, on the segment of its argument $[x_{i-1}, \ x_i]$, i.e.

$$\forall x \in \left[x_{i-1}, \ x_i\right] \rightarrow E_i^I(x) = 1. \tag{3}$$

Secondly, in the rest of the its definition area, the condition of its values (ideally, exact zeroing) tending to zero must be fulfilled, i.e.

$$\forall x \ : \ x_{i-1} > x > x_i \rightarrow E_i^I(x) = 0. \tag{4}$$

Function $E_i(x)$ at the interval boundary points x_{i-1} and x_i should ideally have an average value between its minimum and maximum values throughout the fragment, i.e. 0.5. Then, when adding and at the edge points of the segments, values equal to

1 will be obtained. Therefore, the function $E_i^I(x)$ must, in addition to the argument x, contain additional parameters—area coordinates to be cut out, here called the "fragment". In this way

$$E_i^I(x, \ldots) = E_i^I(x, x_l, x_r), \tag{5}$$

where x_l, x_r are the left and right fragment boundaries, "cut out" from the experimental data array. Accordingly, function $E_i^I(x)$ (5) should ideally be described by the following conditional-logical expression:

$$E_i^I(x, x_l, x_r) = \begin{cases} 1, & \forall\, x \in (x_l, x_r); \\ 0.5, & \forall\, x = x_l, x_r; \\ 0, & \forall\, x_l > x > x_r. \end{cases} \tag{6}$$

The real function $E_i(x)$ was developed and presented in works [13, 19, 20] and many others. In the general case, it is described by the following expression:

$$E_i(x, S) = \frac{\left[x - x_l + \sqrt{(x - x_l)^2 + \varepsilon_l^2}\right] \cdot \left[x_r - x + \sqrt{(x_r - x)^2 + \varepsilon_r^2}\right]}{4 \cdot \left[\sqrt{(x - x_l)^2 + \varepsilon_l^2}\right] \cdot \left[\sqrt{(x_r - x)^2 + \varepsilon_r^2}\right]}. \tag{7}$$

The main argument of function (7) is the coordinate variable x, and the set S contains parametric settings that provide the functional of the multiplicative-isolating principle.

In the one-dimensional case (x—scalar) MIF (7) contains the minimum number of parametric settings required for multiplicative processing $S = \left\{s_i | i = \overline{1,4}\right\}$. Here $s_1 = x_l$, $s_2 = x_r$—left and right boundaries of the approximated fragment; $s_3 = \varepsilon_l$, $s_4 = \varepsilon_r$—parameters of the experimental data fragment left and right boundaries approximation error, respectively.

Settings x_l and x_r ensure the formation of the one-dimensional fragment boundaries when it is "cut out", and $\varepsilon_l = \varepsilon_r$—the fulfillment of the conditions for the MIF existence [19]. The study of function (7) as a multiparameter dependence shows that it really has all the properties important for the multiplicative-isolating principle implementation. It should be noted that, provided $\varepsilon_l = \varepsilon_r$, function (7) has symmetric left and right fronts of the "cut out" fragment, therefore the function $E_i(x, S)$ with $\varepsilon_l = \varepsilon_r$ is called "symmetric". Accordingly, function (7) with $\varepsilon_l \neq \varepsilon_r$ is called "asymmetric" [19, 20].

Let us consider the main features and properties of multiplicatively isolating function (7):

(1) MIF (7) in the interval (x_l, x_r) takes values close to one. Moreover, the proximity of its values to one is effectively adjusted by the parameters ε_l, ε_r. In ranges $(-\infty, x_l)$ and (x_r, ∞) MIF (7) takes on values as close to zero as desired.

Moreover, both the proximity of the function value to zero and the proximity to the edge points are also adjusted by the parameters $\varepsilon_l, \varepsilon_r$.

This MIF property is illustrated in Fig. 6, which shows the MIF (7) graph at $x_l = 14, x_r = 21$ and at $\varepsilon_l = \varepsilon_r = \varepsilon$. It is clearly seen: the smaller value of ε, the MIF values in the interval $x \in (x_l, \ x_r)$ closer to 1, and if $x \notin (x_l, \ x_r)$, then the values of $E_i(x, x_l, x_r, \varepsilon)$ (7) are closer to zero. In this case, the graph of function $E_i(x, x_l, x_r, \varepsilon)$ (7) is symmetrical about point $x_m^s = 17.5$, which in this case is the midpoint of the interval $(x_l, \ x_r)$.

(2) Function (7) symmetric at $\varepsilon_l = \varepsilon_r$ about the middle of the interval $(x_l, \ x_r)$ and has a single maximum at point $x_m^s = (x_r + x_l)/2$ [19, 20]. This is also clearly seen in Fig. 6, where point $x_m^s = 17.5$.

(3) At boundary points $x_l, \ x_r$ MIF (7) takes a value from 0.25 to 0.5, which, in turn, depends on the width of the fragment $|x_r - x_l|$ and value $\varepsilon_l, \varepsilon_r$ [20].

(4) Function (7) of the asymmetric MIF has a single maximum at point x_m^a, which is shifted to boundary x_l or x_r, which has a smaller parameter ε_l or ε_r. This MIF property is illustrated in Fig. 7 by graphs $E_1(x, S)$ and $E_2(x, S)$ where $x_l = -0.5, \ x_r = 0.5$ and $\varepsilon_{l1} = 0.1, \ \varepsilon_{r1} = 1.0, \varepsilon_{l2} = 0.3, \ \varepsilon_{r2} = 0.01$.

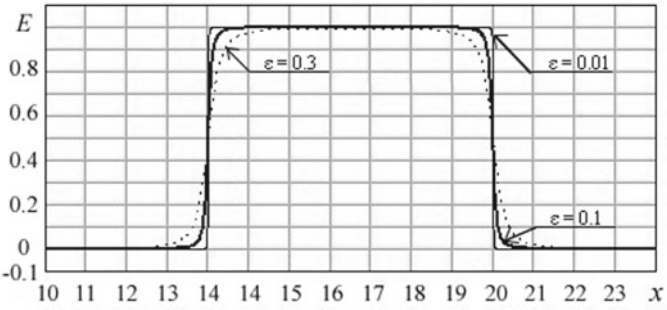

Fig. 6 Symmetric MIF with different values ε

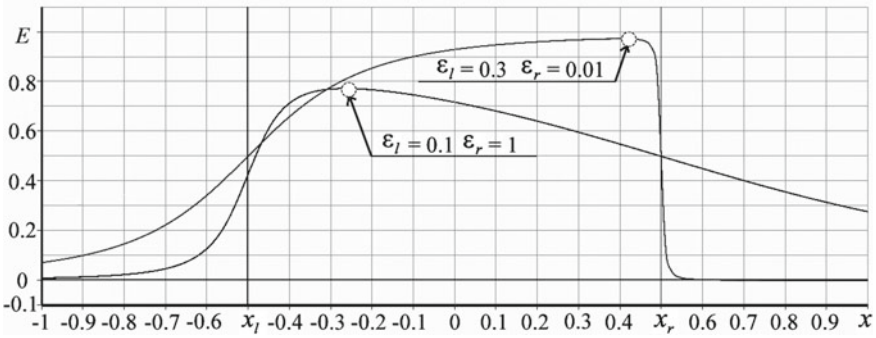

Fig. 7 Asymmetric MIFs

As you can see in Fig. 7, the MIF curves are essentially non-symmetric. The extremum point x_m^a of function $E_1(x, S)$ really shifts towards a lower value ε_l or ε_r. So, if $\varepsilon_{l1} = 0.1$, $\varepsilon_{r1} = 1.0$, then extremum $E_1(x, S)$ corresponds to the point $x_m^a = -0.265$, if $\varepsilon_{l2} = 0.3$, $\varepsilon_{r1} = 0.01$, then the extremum corresponds to point $x_m^a = 0.403$. Extreme function value $E(x, S)$ becomes much less than 1 with an increase in the average value $(\varepsilon_l + \varepsilon_r)/2$ and approaches to 1 as this value decreases.

(5) MIF (7) is infinitely differentiable, like any radical-fractional function.

Examples of creating mathematical models based on the multiplicative-isolating principle can be found in [12, 19] and others.

4 Conclusion

The multiplicative-isolating principle of creation of mathematical nonlinear models based on experimental data is an effective approach that has no analogues in world practice. A practical implementation of this principle is the Cut-Glue approximation method, in which the fundamental properties of the multiplicatively isolating functions are used. This method is perspective and, that is important, an effective way of the mathematical description of experimental data, and also the subsequent creation of analytical models of significantly nonlinear objects. The features of the multiplicative-isolating principle are considered above allow to solve the problems of minimization of approximation error and increase in accuracy of the description of experimental data.

Acknowledgements Chapter is prepared on the results of the projects creation with support of the RFBR grants No. 18-08-01178\20 in DGTU and No. 19-08-01226 in LLC "SRI MVUS".

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