Awareness Logic: A Kripke-Based Rendition of the Heifetz-Meier-Schipper Model

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Abstract. Heifetz, Meier & Schipper (HMS) present a lattice model of awareness. The HMS model is syntax-free, which precludes the simple option to rely on formal language to induce lattices, and represents uncertainty and unawareness with one entangled construct, making it difficult to assess the properties of either. Here, we present a model based on a lattice of Kripke models, induced by atom subset inclusion, in which uncertainty and unawareness are separate. We show the models to be equivalent by defining transformations between them which preserve formula satisfaction, and obtain completeness through our and HMS' results.

1 Introduction

Awareness has been studied with vigor in logic and game theory since its first formal treatment by Halpern and Fagin in [8]. In these fields, awareness is added as a complement to uncertainty in models for knowledge and rational interaction. In short, where uncertainty concerns an agent's ability to distinguish possible states of the world based on its available information, awareness concerns the agent's ability to even contemplate aspects of a state, where such inability stems from the *unawareness* of the concepts that constitute said aspects. Thereby, models that include awareness avoid problems of logical omniscience (at least partially) and allows modeling game theoretic scenarios where the possibility of some action may come as an utter surprise.

To model awareness, the seminal [8] introduces the Logic of General Awareness (LGA), taking a syntax-based approach: an agent a's awareness in state w is given by an *awareness function* assigning (a, w) a set of formulas. This approach has since been inherited by a multitude of models.

In contrast, Heifetz, Meier and Schipper (HMS) construct a syntax-free framework [15], which is the main topic of this paper. In their *unawareness frames*, both "atomic" and epistemic events are defined without any appeal to atomic propositions or other syntax.

The backbone of an unawareness frame is a complete lattice of state-spaces $(\mathcal{S}, \preceq)$, with the intuition that the higher a space is, the richer the "vocabulary"

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it has to describe its states. Since the approach is syntax-free, this intuition is not modeled using a formal language. It is represented using $\preceq$ and a family of maps $r_S^{S'}$ which projects state-space S' down to S, with $r_S^{S'}(s)$ interpreted as the representation of s in the more limited vocabulary available in S. Uncertainty and unawareness are represented *jointly* by a *possibility correspondence* Π_a for each $a \in Ag$, which maps a state weakly downwards to the set of states the agent considers possible. If the mapped-to space is strictly less expressive, this represents that the agent does not have full awareness of the mapped-from state.

That HMS keep their model syntax-free is motivated in part by its applicability among economists [15, p. 79]. We think their lattice-based conceptualization of awareness is both elegant, interesting and intuitive—but we also find its formalization cumbersome. Exactly the choice to go fully syntax-free robs the model of the option to rely on formal language to induce lattices and to specify events, resulting in constructions which we find less than very easy to follow. This may, of course, be an artifact of us being accustomed to non-syntax-free models used widely in epistemic logic.

Another artifact of our familiarity with epistemic logic models is that we find HMS' joint definition of uncertainty and unawareness difficult to relate to other formalizations of knowledge. When HMS propose properties of their Π_a maps, it is not clear to us which aspects concern knowledge and which concern awareness. They merge two dimensions which, to us, would be clearer if left separated.[1]

With these two motivations, this paper proposes a non-syntax-free, Kripke model-based rendition of the HMS model. Roughly, we suggest to start from a Kripke model $\mathtt{K}$ for a set of atoms At, spawn a lattice containing restrictions of $\mathtt{K}$ to subsets of At, and finally add maps π_a on the lattice that take a world to a copy of itself in a restricted model. This keeps the epistemic and awareness dimensions separate: accessibility relations R_a of $\mathtt{K}$ encode epistemics while maps π_a encode awareness. We show that under three assumptions on π_a and when each R_a is an equivalence relation, the result is equivalent to the HMS model, in the sense that the two satisfy the same formulas of the language of knowledge and awareness, defined below.

Defining an equivalent model, we do not aim to generalize that of HMS, but we do include an additional perspective. [15,20] argue that the HMS model allows agents to reason about their unawareness, as possibility correspondences Π_a provide them a subjective perspective, while LGA-based approaches only present an outside perspective, as the full model must be taken into account when assigning knowledge and awareness.[2] Oppositely, Halpern and Rêgo [13] point out that the HMS model includes no objective state, and so no outside

[1] As a reviewer points out, then HMS take *explicit* knowledge as foundational, and derive awareness from it. This makes the one-dimensional representation justified, if not even desirable. In contrast, epistemic logic models are standardly interpreted as taking *implicit* knowledge as foundational. We think along the second line, and add awareness as a second dimension. We are not taking a stand on whether one interpretation is superior, but provide results to move between them.

[2] [13] argues that this boils down to a difference in philosophical interpretation.

perspective. The present model has both: the starting Kripke model provides an outsider perspective on agents' knowledge, while the submodel obtained by following π_a presents the subjective perspective. We remark further on this below.

The paper progresses as follows. Sects. 2 and 3 present respectively the HMS model and our rendition. Sections 4 and 5 contain our main technical results: Sect. 4 introduces transformations between the two models classes, while Sect. 5 shows that they preserve formula satisfaction. Section 6 presents a logic due to HMS [14], and shows, as a corollary to our results, that it is complete with respect to our rendition. Section 7 holds concluding remarks.

Throughout the paper, we assume that Ag is a finite, non-empty set of agents, and that At is a countable, non-empty set of atoms.

2 The HMS Model

This section presents HMS unawareness frames [15], their syntax-free notions of knowledge and awareness, and their augmentation with HMS valuations, producing HMS models [14]. For context, the HMS model is a multi-agent generalization of the Modica-Rustichini model [19] which is equivalent to Halpern's model in [10], generalized by Halpern and Rêgo to multiple agents [13], resulting in a model equivalent to the HMS model, cf. [14]. See [20] for an extensive review.

The following definition introduces the basic structure underlying the HMS model, as well as the properties of the Π_a map that controls the to-be-defined notions of knowledge and awareness. The properties are described after Definition 1. Following Definition 4 of HMS models, Fig. 1 illustrates a full HMS model, including its unawareness frame.

Definition 1. *An **unawareness frame** is a tuple* $\mathsf{F} = (\mathcal{S}, \preceq, \mathcal{R}, \Pi)$ *where* $(\mathcal{S}, \preceq)$ *is a complete lattice with* $\mathcal{S} = \{S, S', ...\}$ *a set of disjoint, non-empty **state-spaces*** $S = \{s, s', ...\}$ *s.t.* $S \preceq S'$ *implies* $|S| \leq |S'|$*. Let* $\Omega_\mathsf{F} := \bigcup_{S \in \mathcal{S}} S$ *be the disjoint union of state-spaces in* $\mathcal{S}$*. For* $X \subseteq \Omega_\mathsf{F}$*, let* $S(X)$ *be the state-space containing* X*, if such exists (else* $S(X)$ *is undefined). Let* $S(s)$ *be* $S(\{s\})$*.*

$\mathcal{R} = \{r_S^{S'} : S, S' \in \mathcal{S}, S \preceq S'\}$ *is a family of **projections*** $r_S^{S'} : S' \to S$*. Each* $r_S^{S'}$ *is surjective,* r_S^S *is* Id*, and* $S \preceq S' \preceq S''$ *implies commutativity:* $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$*. Denote* $r_S^T(w)$ *also by* w_S*.*

$D^\uparrow = \bigcup_{S' \succeq S} (r_S^{S'})^{-1}(D)$ *is the **upwards closure** of* $D \subseteq S \in \mathcal{S}$*.*[3]

Π *assigns each* $a \in Ag$ *a **possibility correspondence*** $\Pi_a : \Omega_\mathsf{F} \to 2^{\Omega_\mathsf{F}}$ *satisfying*

Conf ***(Confinement)*** *If* $w \in S'$*, then* $\Pi_a(w) \subseteq S$ *for some* $S \preceq S'$*.*
Gref ***(Generalized Reflexivity)*** $w \in (\Pi_a(w))^\uparrow$ *for every* $w \in \Omega_\mathsf{F}$*.*
Stat ***(Stationarity)*** $w' \in \Pi_a(w)$ *implies* $\Pi_a(w') = \Pi_a(w)$*.*
PPI ***(Projections Preserve Ignorance)*** *If* $w \in S'$ *and* $S \preceq S'$*, then* $(\Pi_a(w))^\uparrow \subseteq (\Pi_a(r_S^{S'}(w)))^\uparrow$*.*

[3] To avoid confusion, note that for $d \in S$, $(r_S^{S'})^{-1}(d) = \{s' \in S' : r_S^{S'}(s') = d\}$ and for $D \subseteq S$, $(r_S^{S'})^{-1}(D) = \bigcup_{d \in D} (r_S^{S'})^{-1}(d)$.

PPK ***(Projections Preserve Knowledge)*** *If $S \preceq S' \preceq S''$, $w \in S''$ and $\Pi_a(w) \subseteq S'$, then $r^{S'}_S(\Pi_a(w)) = \Pi_a(r^{S''}_S(w))$.*

Jointly call these five properties of Π_a the ***HMS properties****.*

Conf ensures that agents only consider possibilities within one fixed "vocabulary"; Gref induces factivity of knowledge and Stat yields introspection for knowledge and awareness. PPI entails that at down-projected states, agents neither "miraculously" know or become aware of something new, while PPK implies that at down-projected states, the agent can still "recall" all events she knew before, if they are still expressible. Jointly PPI and PPK imply that agents preserve awareness of all events at down-projected states, if they are still expressible.

Remark 2. Unawareness frames include no objective perspective, as agents do not—unless they are fully aware—have a range of uncertainty defined for the maximal state-space. Taking the maximal state-space to contain a designated 'actual world' and as providing a full and objective description of states, one can still not evaluate agents "true" uncertainty/implicit knowledge. See e.g. Fig. 1 below: In $(\neg i, \ell)$, the dashed agent's "true" uncertainty about ℓ is not determined.

2.1 Syntax-Free Unawareness

Unawareness frames provide sufficient structure to define syntax-free notions of knowledge and awareness. These are defined directly as events on Ω_{F}.

Definition 3. *Let $\mathsf{F} = (\mathcal{S}, \preceq, \mathcal{R}, \Pi)$ be an unawareness frame. An* ***event*** *in F is any pair $(D^{\uparrow}, S)$ with $D \subseteq S \in \mathcal{S}$ with S also denoted $S(D^{\uparrow})$. Let Σ_{F} be the set of events of F.*

The ***negation*** *of the event $(D^{\uparrow}, S)$ is $\neg(D^{\uparrow}, S) = ((S \setminus D)^{\uparrow}, S)$.*

The ***conjunction*** *of events $\{(D_i^{\uparrow}, S_i)\}_{i \in I}$ is $((\bigcap_{i\in I} D_i^{\uparrow}), \sup_{i \in I} S_i)$.*

The events that a ***knows*** *event $(D^{\uparrow}, S)$ and where a is* ***aware*** *of it are*

$$\boldsymbol{K}_a((D^{\uparrow}, S)) = \begin{cases} (\{w \in \Omega_{\mathsf{F}} \colon \Pi_a(w) \subseteq D^{\uparrow}\}, S(D)) & \text{if } \exists w \in \Omega_{\mathsf{F}}. \Pi_a(w) \subseteq D^{\uparrow} \\ (\emptyset, S(D)) & \text{else} \end{cases}$$

$$\boldsymbol{A}_a((D^{\uparrow}, S)) = \begin{cases} (\{w \in \Omega_{\mathsf{F}} \colon \Pi_a(w) \subseteq S(D^{\uparrow})^{\uparrow}\}, S(D)) & \text{if } \exists w \in \Omega_{\mathsf{F}}. \Pi_a(w) \subseteq S(D^{\uparrow})^{\uparrow} \\ (\emptyset, S(D)) & \text{else} \end{cases}$$

Negation, conjunction, knowledge and awareness events are well-defined [15,20]. To illustrate the definitions, some intuitions behind them: i) an event modeled as a pair $(D^{\uparrow}, S)$ captures that a) if the event is expressible in S, then it is also expressible in any $S' \succeq S$, hence $D^{\uparrow}$ is the set of all states where the event is expressible and occurs, and b) the event is expressible in the"vocabulary" of S, but not the "vocabulary" of lower state-spaces: $D \subseteq S$ are the states with the lowest "vocabulary" where the event is expressible and occurs. [20] remarks that for $(D^{\uparrow}, S)$, if $D \neq \emptyset$, then S is uniquely determined by $D^{\uparrow}$. ii) Events are given a non-binary understanding: an event $(D^{\uparrow}, S)$ and it's negation does not partition Ω_{F}, as $s \in S' \prec S$ is in neither, but they do partition every $S'' \succeq S$. *iii*) Conjunction defined using supremum captures that the state-space required to express the conjunction of two events is the least expressive state-space that can express both events. *iv*) Knowledge events are essentially defined as in Aumann structures/state-space models: the agent knows an event if its "information cell" is a subset of the event's states. *v*) Awareness events captures that "*an agent is aware of an event if she considers possible states in which this event is "expressible"*." [20, p. 97]

2.2 HMS Models

Though unawareness frames provide a syntax-free framework adequate for defining awareness, HMS [14] use them as a semantics for a formal language in order to identify their logic. The language and logic are topics of Sects. 5 and 6.

Instead, the models we will later define are not syntax-free. As Kripke models, they include a valuation of atomic propositions. Therefore, they do not corre-

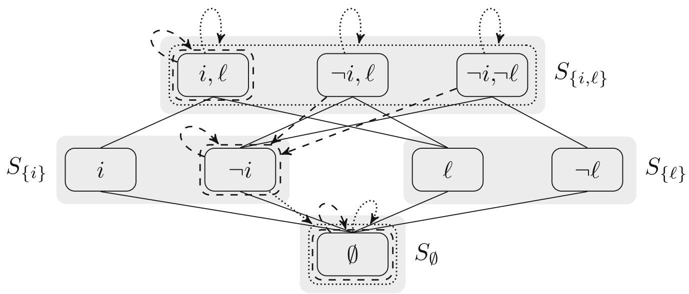

Fig. 1. An HMS model with four state-spaces (gray rectangles), ordered spatially as a lattice. States (smallest rectangles) are labeled with their true literals, over the set $At = \{i, \ell\}$. Thin lines between states show projections. There are two possibility correspondences (dashed and dotted): arrow-to-rectangle shows a mapping from state to set (information cell). Omitted arrows go to $S_{\emptyset}$ and are irrelevant to the story. **Story:** Buyer (dashed) and Owner (dotted) consider trading a firm, the price influenced by whether i (a value-raising innovation) and ℓ (a value-lowering lawsuit) occurs. Assume both occur and take (i, ℓ) as actual. Then Buyer has full information, while Owner has factual uncertainty and uncertainty about Buyer's awareness and higher-order information, ultimately considering it possible that Buyer holds Owner fully unaware. *In detail:* Buyer's (i, ℓ) information cell has both i and ℓ defined (and is also singleton), so Buyer is aware of them (and also knows everything). Owner is also aware of i and ℓ, but their (i, ℓ) information cell contains also $\neg i$ and $\neg \ell$ states, so Owner knows neither. Owner is also uncertain about Buyer's information: Owner knows that either Buyer knows i and ℓ (cf. Buyer's (i, ℓ) information cell), or Buyer knows $\neg i$, but is unaware of ℓ (cf. the dashed arrows from $\neg i$ states to the less expressive state space $S_{\{i\}}$) and then only holds it possible that Owner is unaware of both i and ℓ (cf. the dotted map to $S_{\emptyset}$). See also Remark 6 concerning $S_{\{\ell\}}$.

spond to unawareness frames directly, but to the models that result by augmenting such frames with valuations. To compare the two model classes, we define such valuations here, postponing HMS syntax and semantics to Sect. 5. Figure 1 illustrates an HMS model, using an example inspired by [15, p. 87]

Definition 4. *Let* $\mathsf{F} = (\mathcal{S}, \preceq, \mathcal{R}, \Pi)$ *be an unawareness frame with events* Σ_{F}*. An* ***HMS valuation*** *for* At *and* F *is a map* $V_{\mathsf{M}} : At \to \Sigma_{\mathsf{F}}$*, assigning to every atom from* At *an event in* F*. An* ***HMS model*** *is an unawareness frame augmented with an HMS valuation, denoted* $\mathsf{M} = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_{\mathsf{M}})$*.*

Remark 5. HMS valuations only partially respect the intuitive interpretation of state-spaces lattices, where $S \preceq S'$ represents that S' is at least as expressive as S. If $S \preceq S'$, then $p \in At$ having defined truth value at S entails that it has defined truth value at S', but if S is strictly less expressive than S', then this does not entail that there is some atom q with defined truth value in S', but undefined truth value in S. Hence, there can exist two spaces defined for the same set of atoms, but where one is still "strictly more expressive" than the other.

Remark 6. Concerning Fig. 1, then the state-space $S_{\{\ell\}}$ is, in a sense, redundant: its presence does not affect the knowledge or awareness of agents in the state (i, ℓ), and it presence is not required by definition. This stands in contrast with the corresponding Kripke lattice model in Fig. 2, cf. Remark 12.

3 Kripke Lattice Models

The models for awareness we construct starts from Kripke models:

Definition 7. *A* ***Kripke model*** *for* $At' \subseteq At$ *is a tuple* $\mathtt{K} = (W, R, V)$ *where* W *is a non-empty set of worlds,* $R : Ag \to \mathcal{P}(W^2)$ *assigns to each agent* $a \in Ag$ *an accessibility relation denoted* R_a*, and* $V : At' \to \mathcal{P}(W)$ *is a valuation.*

The ***information cell*** *of* $a \in Ag$ *at* $w \in W$ *is* $I_a(w) = \{v \in W : wR_av\}$*.*

The term 'information cell' hints at an epistemic interpretation. For generality, R may assign non-equivalence relations. Some results explicitly assume otherwise.

As counterpart to the HMS state-space lattice, we build a lattice of restricted models. The below definition of the set of worlds W_X ensures that for any $X, Y \subseteq At$, $X \neq Y$, the sets W_X and W_Y are disjoint, mimicking the same requirement for state-spaces. In the restriction $\mathtt{K}_X$ of $\mathtt{K}$, it is required that $(w_X, v_X) \in R_{aX}$ iff $(w, v) \in R_a$. Each direction bears similarity to an HMS property: left-to-right to PPK and right-to-left to PPI. They also remind us, resp., of the *No Miracles* and *Perfect Recall* properties from Epistemic Temporal Logic, cf. e.g., [3,17].

Definition 8. *Let* $\mathtt{K} = (W, R, V)$ *be a Kripke model for* At*. The* ***restriction*** *of* $\mathtt{K}$ *to* $X \subseteq At$ *is the Kripke model* $\mathtt{K}_X = (W_X, R_X, V_X)$ *for* X *where*

$W_X = \{w_X : w \in W\}$ *where* w_X *is the ordered pair* (w, X)*,*

$R_{Xa} = \{(w_X, v_X) : (w, v) \in R_a\}$ *and*

$V_X : X \to \mathcal{P}(W_X)$ *such that, for all* $p \in X$*,* $w_X \in V_X(p)$ *iff* $w \in V(p)$*.*

For the R_{Xa} *information cell of* a *at* w_X*, write* $I_a(w_X)$*.*

To construct a lattice of restricted models, we simply order them in accordance with subset inclusion of the atoms. This produces a complete lattice.

Definition 9. *Let* $\mathtt{K}$ *be a Kripke model for At. The* ***restriction lattice*** *of* $\mathtt{K}$ *is* $(\mathcal{K}(\mathtt{K}), \trianglelefteq)$ *where* $\mathcal{K}(\mathtt{K}) = \{\mathtt{K}_X\}_{X \subseteq At}$ *is the set of restrictions of* $\mathtt{K}$*, and* $\mathtt{K}_X \trianglelefteq \mathtt{K}_Y$ *iff* $X \subseteq Y$*.*

Projections in unawareness frames are informally interpreted as mapping states to alternates of themselves in less expressive spaces. Restriction lattices offer the same, but implemented w.r.t. At: if $Y \subseteq X \subseteq At$, then w_Y is the alternate of w_X formally described by the smaller vocabulary of atoms, Y.

The accessibility relations of the Kripke models in a restriction lattice accounts for the epistemic dimension of the HMS possibility correspondence Π_a. For the awareness dimension, each agent $a \in Ag$ is assigned an *awareness map* π_a that maps a world w_X down to $\pi_a(w_X) = w_Y$ for some $Y \subseteq X$. We think of $\pi_a(w_X)$ as a's *awareness image* of w_X—i.e., w_X as it occurs to a given her (un)awareness; the submodel from $\pi_a(w_X)$ is thus a's subjective perspective.

In the following definition, we introduce three properties of awareness maps, which we will assume. Intuitions follow the definition.

Definition 10. *With* $\mathsf{L} = (\mathcal{K}(\mathtt{K}), \trianglelefteq)$ *a restriction lattice, let* $\Omega_{\mathsf{L}} = \bigcup \mathcal{K}(\mathtt{K})$ *and let* π *assign to each agent* $a \in Ag$ *an* ***awareness map*** $\pi_a : \Omega_{\mathsf{L}} \to \Omega_{\mathsf{L}}$ *satisfying*

D *(**Downwards**) For all* $w_X \in \Omega_{\mathsf{L}}$*,* $\pi_a(w_X) = w_Y$ *for some* $Y \subseteq X$*.*
II *(**Introspective Idempotence**) If* $\pi_a(w_X) = w_Y$*, then for all* $v_Y \in I_a(w_Y)$*,* $\pi_a(v_Y) = u_Y$ *for some* $u_Y \in I_a(w_Y)$*.*
NS *(**No Surprises**) If* $\pi_a(w_X) = w_Z$*, then for all* $Y \subseteq X$*,* $\pi_a(w_Y) = w_{Y \cap Z}$*.*

Call $\mathsf{K} = (\mathcal{K}(\mathtt{K}), \trianglelefteq, \pi)$ *the* ***Kripke lattice model*** *of* $\mathtt{K}$*.*

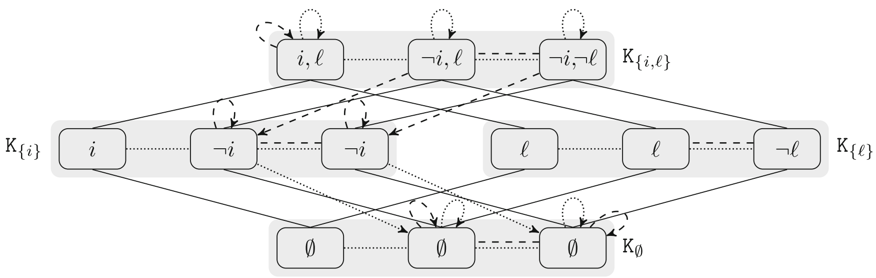

Fig. 2. A Kripke lattice model of the Fig. 1 example. See four restrictions (gray rectangles), ordered spatially as a lattice. States (smallest rectangles) are labeled with their true literals, over the set $At = \{i, \ell\}$. Horizontal dashed and dotted lines *inside restrictions* represent Buyer and Owner's accessibility relations (omitted are links obtainable by reflexive-transitive closure), while dotted and dashed arrows *between restrictions* represent their awareness maps (some arrows are omitted: they go to states' alternates in $\mathtt{K}_\emptyset$, and are irrelevant from (i, l)). Thin lines connect states with their alternate in lower restrictions. See also Remark 12 concerning $\mathtt{K}_{\{\ell\}}$.

D ensures that an agent's awareness image of a world is a restricted representation of that same world. Hence the awareness image does not conflate worlds, and does not allow the agent to be aware of a more expressive vocabulary than that which describes the world she views from. With II and accessibility assumed reflexive, it entails that π_a is idempotent: for all w_X, $\pi_a(\pi_a(w_X)) = \pi_a(w_X)$. Alone, II states that in her awareness image, the agent knows, and is aware of, the atoms that she is aware of. Given that accessibility is distributed by inheritance through the Kripke models in restriction lattices, the property implies that the same holds for every such model. NS guarantees that awareness remains "consistent" down the lattice, so that awareness of an atom does not appear or disappear without reason. Consider the consequent $\pi_a(w_Y) = w_{Y \cap Z}$ and its two subcases $\pi_a(w_Y) = w_{Y^*}$ with $Y^* \subseteq Y \cap Z$ and $Y^* \supseteq Y \cap Z$. Colloquially, the first states that if atoms are removed from the description of the world from which the agent views, then they are also removed from her awareness. Oppositely, the second states that if atoms are removed from the description of the world from which the agent views, then no more than these should be removed from her awareness. Jointly, no awareness should "miraculously" appear, and all awareness should be "recalled".[4]

Remark 11. Contrary to HMS models (cf. Remark 2), Kripke lattice models have an objective perspective: designating an 'actual world' in $\mathtt{K}_{At}$ allows one to check agents' uncertainty about the possible states of the world described by the maximal language, i.e., from $\mathtt{K}_{At}$ we can read off their "actual implicit knowledge". See e.g. Fig. 2: In the $(\neg i, \ell)$ state, the dashed agent's "true" uncertainty about ℓ *is* determined, contrary to the same state in the HMS model of Fig. 1.

Remark 12. In Remark 6, we mentioned that the HMS state-space $S_{\{\ell\}}$ of Fig. 1 is redundant. Similarly, $\mathtt{K}_{\{\ell\}}$ is redundant in Fig. 2 (from (i, ℓ), $\mathtt{K}_{\{\ell\}}$ is unreachable.) However, contrary to the HMS case, it is here required by definition, as a restriction lattice contains all restrictions of the original Kripke model. For simplicity of constructions, we have not here attempted to prune away redundant restrictions. A more general model class may be obtained by letting models be based on sub-orders of the restriction lattice. See also the concluding remarks.

4 Moving Between HMS Models and Kripke Lattices

To clarify the relationship between HMS models and Kripke lattice models, we introduce transformations between the two model classes, showing that a model from one class encodes the structure of a model from the other. The core idea is to think of a possibility correspondence Π_a as the composition of I_a and π_a: $\Pi_a(w)$ is the information cell of the awareness image of w.

The propositions of this section show that the transformations produce models of the desired class. Additionally, their proofs shed partial light on the relationship between the HMS properties and those assumed for awareness maps π_a and accessibility relations R_a: we discuss this shortly in the concluding remarks.

[4] Again, we are reminded of *No Miracles* and *Perfect Recall*.

4.1 From HMS Models to Kripke Lattice Models

Moving from HMS models to Kripke lattice models requires a somewhat involved construction as it must tease apart unawareness and uncertainty from the possibility correspondences, and track the distribution of atoms and their relationship to awareness. For an example, then the Kripke lattice model in Fig. 2 is the HMS model of Fig. 1 transformed.

Definition 13. *Let* $\mathsf{M} = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_{\mathsf{M}})$ *be an HMS model with maximal state-space* T. *For any* $O \subseteq \Omega_{\mathsf{M}}$, *let* $At(O) = \{p \in At \colon O \subseteq V_M(p) \cup \neg V_M(p)\}$.[5]
The L***-transform model of*** M *is* $L(\mathsf{M}) = (\mathcal{K}(\mathtt{K}), \trianglelefteq, \pi)$ *where the Kripke model* $\mathtt{K} = (W, R, V)$ *for* At *given by*
$W = T$;
R *maps each* $a \in Ag$ *to* $R_a \subseteq W^2$ *s.t.* $(w, v) \in R_a$ *iff* $r^T_{S(\Pi_a(w))}(v) \in \Pi_a(w)$;
$V : At \to \mathcal{P}(W)$, *defined by* $V(p) \ni w$ *iff* $w \in V_{\mathsf{M}}(p)$, *for every* $p \in At$;
π *assigns each* $a \in Ag$ *a map* $\pi_a : \Omega_{L(\mathsf{M})} \to \Omega_{L(\mathsf{M})}$ *s.t. for all* $w_X \in \Omega_{L(\mathsf{M})}$,
$\pi_a(w_X) = w_Y$ *where* $Y = At(S_Y)$ *for the* $S_Y \in \mathcal{S}$ *with* $S_Y \supseteq \Pi_a(r^T_{S_X}(w))$
where $S_X = \min\{S \in \mathcal{S} \colon At(S) = X\}$.

The ***state correspondence*** *between* M *and* $L(\mathsf{M})$ *is the map* $\ell : \Omega_{\mathsf{M}} \to 2^{\Omega_{L(\mathsf{M})}}$ *s.t. for all* $s \in \Omega_{\mathsf{M}}$
$\ell(s) = \{w_X \in W_X \colon w \in (r^T_{S(s)})^{-1}(s) \text{ for } X = At(S(s))\}$.

Intuitively, in the L-transform model, a world $v \in W$ is accessible from a world $w \in W$ for an agent if, and only if, v's restriction to the agent's vocabulary at w is one of the possibilities she entertains.[6] In addition, the awareness map π_a of agent a relates a world w_X to its less expressive counterpart w_Y if, and only if, Y is the vocabulary agent a adopts when describing what she considers possible.

Remark 14. The L-transform model $L(\mathsf{M})$ of M is well-defined as the object $\mathtt{K} = (W, R, V)$ is in fact a Kripke model for At: *i*) By def. of HMS models, $W = T \in \mathcal{S}$ is non-empty; *ii*) for each a, $R_a \subseteq W^2$ is well-defined: if $w \in T = W$, then by Conf, $\Pi_a(w) \subseteq S$, for some $S \in \mathcal{S}$. Hence, $U = \{v \in T \colon r^T_S(v) \in \Pi_a(w)\}$ is well-defined, and so is $\{(w, v) \in T^2 \colon v \in U\} = R_a$; *iii*) As V_{M} is an HMS valuation $V_{\mathsf{M}} : At \to \Sigma$ for At, clearly V is valuation for At. Hence $\mathtt{K} = (W, R, V)$ is a Kripke model for At.

Remark 15. The min used in defining S_X is due to the issue of Remark 5.

Remark 16. The state correspondence map ℓ is also well-defined. That it maps each state in Ω_{M} to a *set* of worlds in $\Omega_{L(\mathsf{M})}$ points to a construction difference between HMS models and Kripke lattice models: in the former, the downwards projections of two states may 'merge' them, so state-spaces may shrink when moving down the lattice; in the latter, distinct worlds remain distinct, so all world sets in a restriction lattice share cardinality.

5 $At(O)$ contains the atoms that have a defined truth value in every $s \in O$.
6 We thank a reviewer for this wording.

As unawareness and uncertainty are separated in Kripke lattice models, we show two results about L-transforms. The first shows that the Conf, Stat and PPK entail that π_a assigns awareness maps, and the second that the five HMS properties entail that R assigns equivalence relations. In showing the first, we make use of the following lemma, which intuitively shows that the information cell of an agent contains a state described with a certain vocabulary if, and only if, the agent considers possible the corresponding state described with the same vocabulary:

Lemma 17. *For every $w_Y \in \Omega_{\mathsf{K}}$, if $\Pi_a(w) \subseteq S$ and $At(S) = Y$, then $v_Y \in I_a(w_Y)$ iff $v_S \in \Pi_a(w)$.*

Proof. Let $w_Y \in \Omega_{L(\mathsf{M})}$. Consider the respective $w \in T = W$ and let $\Pi_a(w) \subseteq S$, with $At(S) = Y$. Assume that $v_Y \in I_a(w_Y)$. This is the case iff (def. of I_a) $(w_Y, v_Y) \in R_{Ya}$ iff (def. of restriction lattice) $(w, v) \in R_a$ iff (Definition 13) $v_S \in \Pi_a(w)$.

Proposition 18. *For any HMS model M, its L-transform $L(\mathsf{M})$ is a Kripke lattice model.*

Proof. Let $\mathsf{M} = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_{\mathsf{M}})$ be an HMS model with maximal state-space T. We show that $L(\mathsf{M}) = (\mathcal{K}(\mathtt{K}), \trianglelefteq, \pi)$ is a Kripke lattice model by showing that π_a satisfies the three properties of an awareness map:

D: Consider an arbitrary $w_X \in \Omega_{L(\mathsf{M})}$. By def. of L-transform, $X = At(S)$ for some $S \in \mathcal{S}$. Let $S_X = \min\{S \in \mathcal{S} \colon At(S) = X\}$. If $w_X \in W_X$ then for some $w \in W = T$, $w_{S_X} \in S_X$. By Conf, $\Pi_a(w_{S_X}) \subseteq S_Y$, for some $S_Y \preceq S_X$. Let $Y = At(S_Y)$. Then, by def. of π_a, $\pi_a(w_X) = w_Y$ and $Y \subseteq X$.

II: Let $\pi_a(w_X) = w_Y$. By def. of π_a, it holds that $\Pi_a(r^T_{S_X}(w)) \subseteq S_Y$ with $At(S_Y) = Y$ and $S_X = \min\{S \in \mathcal{S} \colon At(S) = X\}$. For a contradiction, suppose there exists a $v_Y \in I_a(w_Y)$ s.t. for all $u_Y \in I_a(w_Y)$, $\pi_a(v_Y) \neq u_Y$. Then $\pi_a(v_Y) = t_Z$ for some $Z \subseteq Y$ and $t_Z \notin I_a(w_Y)$. By def. of π_a, $\pi_a(v_Y) = t_Z$ iff $\Pi_a(r^T_{S_Y}(v)) \subseteq S_Z$, where $Z = At(S_Z)$. Then, by Lemma 17, $t_Z \in I_a(v_Z)$ iff $t_{S_Z} \in \Pi_a(r^T_{S_Y}(v))$. Moreover, as $\Pi_a(r^T_{S_X}(w)) \subseteq S_Y$ and $At(S_X) = X$, by Lemma 17, it also follows that $v_Y \in I_a(w_Y)$ iff $v_{S_Y} \in \Pi_a(r^T_{S_X}(w))$. Since $v_Y \in I_a(w_Y)$ then $v_{S_Y} \in \Pi_a(r^T_{S_X}(w))$. Hence, by Stat, $\Pi_a(r^T_{S_X}(w)) = \Pi_a(r^T_{S_Y}(v))$, which implies $t_{S_Z} \in \Pi_a(r^T_{S_X}(w))$. But then $t_Z \in I_a(v_Z)$, contradicting the assumption that $t_Z \notin I_a(w_Y)$. Thus, for all $v_Y \in I_a(w_Y)$, $\pi_a(v_Y) = u_Y$ for some $u_Y \in I_a(w_Y)$.

NS: Let $\pi_a(w_X) = w_Y$. By D (cf. item 1. above), $Y \subseteq X$. Consider an arbitrary $Z \subseteq X$. We have two cases: either *i*) $Z \subseteq Y$ or *ii*) $Y \subseteq Z$. *i*): then $Z \subseteq Y \subseteq X$. Let $Z = At(S_Z)$, $Y = At(S_Y)$, and $X = At(S_X)$. Then $S_Z \preceq S_Y \preceq S_X$. By PPK, $\left(\Pi_a(r^T_{S_X}(w))\right)_Z = \Pi_a(r^T_{S_Z}(w))$. As $\pi_a(w_X) = w_Y$, by def. of π_a, $\Pi_a(r^T_{S_X}(w)) \subseteq S_Y$. Then $\left(\Pi_a(r^T_{S_X}(w))\right)_Z = r^{S_Y}_{S_Z}\left(\Pi_a(r^T_{S_X}(w))\right) \subseteq S_Z$. Hence $\Pi_a(r^T_{S_Z}(w)) \subseteq S_Z$, and by def. of π_a, $\pi_a(w_Z) = w_Z$. As $Z \subseteq Y$, $\pi_a(w_Z) = w_Z = w_{Z\cap Y}$. *ii*): then $Y \subseteq Z \subseteq X$. By analogous reasoning, we have $\pi_a(w_Y) = w_Y = w_{Y\cap Z}$ as $Y \subseteq Z$. We can conclude that if $\pi_a(w_X) = w_Y$, then for all $Z \subseteq X$, $\pi_a(w_Z) = w_{Z\cap X}$.

Proposition 19. *If $L(\mathsf{M}) = (\mathcal{K}(\mathtt{K} = (W, R, V)), \trianglelefteq, \pi)$ is the L-transform of an HMS model M, then for every $a \in Ag$, R_a is an equivalence relation.*

Proof. Let $\mathsf{M} = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_{\mathsf{M}})$ have maximal state-space T.

Reflexivity: Let $w \in T$ and $\Pi_a(w) \subseteq S$, for some $S \in \mathcal{S}$. By def. of upwards closure, $(\Pi_a(w))^{\uparrow} = \bigcup_{S' \succeq S}(r_S^{S'})^{-1}(\Pi_a(w))$, and by Gref, $w \in (\Pi_a(w))^{\uparrow} = \bigcup_{S' \succeq S}(r_S^{S'})^{-1}(\Pi_a(w))$. Since $T \succeq S$, then $r_S^T(w) \in \Pi_a(w)$. Thus, $(w, w) \in R_a$, by def. L-transform. By def. of restriction lattices, this holds for all $A \subseteq At$, i.e. $(w_A, w_A) \in R_{Aa}$.

Transitivity: Let w, v, u be in T. By Conf, there are $S, S' \in \mathcal{S}$ such that $\Pi_a(w) \subseteq S$ and $\Pi_a(v) \subseteq S'$. Assume that $(w, v) \in R_a$ and $(v, u) \in R_a$. By def. of R_a, then $r_S^T(v) \in \Pi_a(w)$ and $r_{S'}^T(u) \in \Pi_a(v)$. By Stat, $\Pi_a(w) = \Pi_a(r_S^T(v))$ and $\Pi_a(v) = \Pi_a(r_{S'}^T(u))$. As $v \in T$ and $S \preceq T$, by PPI, $\Pi_a(v)^{\uparrow} \subseteq \Pi_a(r_S^T(v))^{\uparrow} = \Pi_a(w)^{\uparrow}$. Hence, as $r_{S'}^T(u) \in \Pi_a(v)^{\uparrow}$, also $r_{S'}^T(u) \in \Pi_a(w)^{\uparrow}$. By def. of upwards closure, $r_S^T(u) \in \Pi_a(w)$. Finally, $(w, u) \in R_a$ by def. of R_a.

Symmetry: Let $w, v \in T$ be in T. Assume that $(w, v) \in R_a$. By Conf, there are $S, S' \in \mathcal{S}$ such that $\Pi_a(w) \subseteq S$ and $\Pi_a(v) \subseteq S'$. Then $r_S^T(v) \in \Pi_a(w)$ (def. of L-transform), and by Stat, $\Pi_a(w) = \Pi_a(r_S^T(v))$. As $v \in T$ and $T \succeq S$, by PPI, by $\Pi_a(v)^{\uparrow} \subseteq \Pi_a(r_S^T(v))^{\uparrow}$. Then, by def. of upwards closure, $T \succeq S' \succeq S$. As $v \in T$, by PPK, $r_S^{S'}(\Pi_a(v)) = \Pi_a(r_S^T(v))$. By Gref, $x \in \Pi_a(w)^{\uparrow}$, and since $\Pi_a(w) \subseteq S$ then $r_S^T(w) \in \Pi_a(w)$, by def. of upward closure. Then $r_S^T(w) \in \Pi_a(w) = \Pi_a(r_S^T(v)) = r_S^{S'}(\Pi_a(v))$. So $r_S^T(w) \in r_S^{S'}(\Pi_a(v))$, i.e. $r_{S'}^T(w) \in \Pi_a(v)$, by def. of r. Hence, $(v, w) \in R_a$, by def. of R_a.

4.2 From Kripke Lattice Models to HMS Models

Moving from Kripke lattice models to HMS models requires a less involved construction, as the restriction lattice almost encode projections, and unawareness and uncertainty are simply composed to form possibility correspondences:

Definition 20. *Let $\mathsf{K} = (\mathcal{K}(\mathtt{K} = (W, R, V)), \trianglelefteq, \pi)$ be a Kripke lattice model for At. The H-**transform** of K is $H(\mathsf{K}) = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_{H(\mathsf{K})})$ where*
$\mathcal{S} = \{W_X \subseteq \Omega_{\mathsf{K}} : \mathtt{K}_X \in \mathcal{K}(\mathtt{K})\}$;
$W_X \preceq W_Y$ iff $\mathtt{K}_X \trianglelefteq \mathtt{K}_Y$;
$\mathcal{R} = \{r_{W_Y}^{W_X} : r_{W_Y}^{W_X}(w_X) = w_Y$ for all $w \in W$, and all $X, Y \subseteq At\}$;
$\Pi = \{\Pi_a \in (2^{\Omega_{\mathsf{K}}})^{\Omega_{\mathsf{K}}} : \Pi_a(w_X) = I_a(\pi_a(w_X))$ for all $w \in W, X \subseteq At, a \in Ag\}$;
$V_{H(\mathsf{K})}(p) = \{w_X \in \Omega_{\mathsf{K}} : X \ni p$ and $w_X \in V_X(p)\}$ for all $p \in At$.

As HMS models lump together unawareness and uncertainty, we show only one result in this direction:

Proposition 21. *For any Kripke lattice model $\mathsf{K} = (\mathcal{K}(\mathtt{K} = (W, R, V)), \trianglelefteq, \pi)$ s.t. R assigns equivalence relations, the H-transform $H(\mathsf{K})$ is an HMS model.*

Proof. Let K be as stated and let $H(\mathsf{K}) = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_{H(\mathsf{K})})$ be its H-transform.

$\mathcal{S} = \{W_X, W_Y, ...\}$ is composed of non-empty disjoint sets by construction and $(\mathcal{S}, \preceq)$ is a complete lattice as $(\mathcal{K}(\mathtt{K}), \trianglelefteq)$ is so. $\mathcal{R}$ is clearly a family of well-defined, surjective and commutative projections. As Π assigns to each $a \in Ag$, $\Pi_a(w_X) = I_a(\pi_a(w_X))$, for all $w \in W$, $X \subseteq At$, it assigns a a map $\Pi_a : \Omega_{H(\mathsf{K})} \rightarrow 2^{\Omega_{H(\mathsf{K})}}$, which is a possibility correspondence as it satisfies the HMS properties:

Conf: For $w_X \in W_X$, $\Pi_a(w_X) = I_a(\pi_a(w_X))$, by Definition 20. By D, $\pi_a(w_X) = w_Y$ for some $Y \subseteq X$, and $I_a(\pi_a(w_X)) = I_a(w_Y)$. So, $\Pi_a(w_X) \subseteq W_Y$ for some $Y \subseteq X$.

Gref: Let $w_X \in \Omega_{\mathsf{K}}$, $X \subseteq At$. By D, $\pi_a(w_X) = w_Y$ for some $Y \subseteq X$. By def. of Π_a and I_a, $\Pi_a(w_X) = I_a(w_Y) = \{v_Y \in \Omega_{\mathsf{K}} : (w_Y, v_Y) \in R_{Ya}\}$. Hence $\Pi_a(w_X) \subseteq W_Y$. By def. of upward closure, $(\Pi_a(w_X))^{\uparrow} = (I_a(w_Y))^{\uparrow} = \{u_Z \in \Omega_{\mathsf{K}} : Y \subseteq Z$ and $u_Y \in \{v_Y \in \Omega_{\mathsf{K}} : (w_Y, v_Y) \in R_{Ya}\}\}$, with the last identity given by the def. of $r^{W_Z}_{W_Y}$. As R_a is an equivalence relation, so is R_{Ya}, by def. So $w_Y \in \{v_Y \in \Omega_{\mathsf{K}} : (w_Y, v_Y) \in R_{Ya}\}$, and since $Y \subseteq X$, then $w_X \in (\Pi_a(w_X))^{\uparrow}$.

Stat: For $w_X \in \Omega_{\mathsf{K}}$, assume $v \in \Pi_a(w_X) = I_a(\pi_a(w_X))$. By D, $v \in I_a(w_Y)$, for some $Y \subseteq X$. With R_{Ya} an equivalence relation, $v \in I_a(w_Y)$ iff $w_Y \in I_a(v)$, i.e., $I_a(v) = I_a(w_B)$. II and D entails that for all $u_Y \in I_a(w_Y)$, $\pi_a(u_Y) = u_Y$, so $\pi_a(v) = v$. Therefore $\Pi_a(v) = I_a(\pi_a(v)) = I_a(v) = I_a(w_Y) = I_a(\pi_a(w_X)) = \Pi_a(w_X)$. Thus, if $v \in \Pi_a(w_X)$, then $\Pi_a(v) = \Pi_a(w_X)$.

PPI: Let $w_X \in W_X$ and $W_Y \preceq W_X$, i.e. $Y \subseteq X \subseteq At$. Let $q_Q \in (\Pi_a(w_X))^{\uparrow}$ with $Q \subseteq At$. By def. of Π_a and D, $\Pi_a(w_X) = I_a(\pi_a(w_X)) = I_a(w_Z)$ for some $Z \subseteq X$. By def. of upwards closure, it follows that $q_Z \in I_a(w_Z) = \Pi_a(w_X)$. Now let $\pi_a(w_Y) = w_P$ for some $P \subseteq Y$. Then, by NS, $P = Z \cap Y$, so $P \subseteq Z$. As $q_Z \in I_a(w_Z)$, then $q_P \in I_a(w_P) = I_a(\pi_a(w_Y)) = \Pi_a(w_Y)$, by def. of restriction lattice. Since $q_Q \in (\Pi_a(w_X))^{\uparrow} = (I_a(w_Z))^{\uparrow}$, then $Z \subseteq Q$. It follows that $P \subseteq Z \subseteq Q$, which implies $q_Q \in (\Pi_a(w_Y))^{\uparrow}$. Hence, if $q_Q \in (\Pi_a(w_X))^{\uparrow}$, then $q_Q \in (\Pi_a(w_Y))^{\uparrow}$, i.e., $(\Pi_a(w_X))^{\uparrow} \subseteq (\Pi_a(w_Y))^{\uparrow}$.

PPK: Suppose that $W_Z \preceq W_Y \preceq W_X$, $w_X \in W_X$ and $\Pi_a(w_X) \subseteq W_Y$, i.e. $\Pi_a(w_X) = I_a(w_Y)$ and $\pi_a(w_X) = w_Y$. As $Z \subseteq Y \subseteq X$, NS implies $\pi_a(w_Z) = w_{Z \cap Y} = w_Z$. Hence, $\Pi_a(w_Z) = I_a(w_Z) \subseteq W_Z$. Hence PPK is established if $(I_a(w_Y))_Z = I_a(w_Z)$. As $(I_a(w_Y))_Z = \{x_Z \in \Omega_{\mathsf{K}} : x_Y \in I_a(w_Y)\}$, then clearly $(I_a(w_Y))_Z = I_a(w_Z)$. Thus, $(\Pi_a(w_X))_Z = \Pi_a(w_Z)$.

Finally, $V_{H(\mathsf{K})}$ is an HMS valuation as for each $p \in At$, $V_{H(\mathsf{K})}(p)$ is an event $(D^{\uparrow}, S)$ with $D = \{w_{\{p\}} \in W_{\{p\}} : w_{\{p\}} \in V_{\{p\}}(p)\}$ and $S = W_{\{p\}}$.

5 Language for Awareness and Model Equivalence

Multiple languages for knowledge and awareness exist. The Logic of General Awareness (LGA, [8]) takes implicit knowledge and awareness as primitives, and define explicit knowledge as 'implicit knowledge $\wedge$ awareness'; other combinations are discussed in [4]. Variations of LGA include quantification over objects [5], formulas [1,11,12], and even unawareness [7], alternative operators informed through cognitive science [2], and dynamic extensions [4,7,9,16].

HMS [14] follow instead Modica and Rustichini [18,19] and take explicit knowledge as primitive and awareness as defined: an agent is aware of φ iff she either explicitly knows φ, or explicitly knows that she does not explicitly know φ.

Definition 22. *Let Ag be a finite, non-empty set of agents and At a countable, non-empty set of atoms. With $a \in Ag$ and $p \in At$, define the language $\mathcal{L}$ by*

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\varphi$$

and define $A_a\varphi := K_a\varphi \vee K_a\neg K_a\varphi$.

Let $At(\varphi) = \{p \in At\colon p \text{ is a subformula of } \varphi\}$, for all $\varphi \in \mathcal{L}$.

5.1 HMS Models as a Semantics

The satisfaction of formulas over HMS models is defined as follows. The semantics are three-valued, so formulas may have undefined truth value: there may exist a $w \in \Omega_{\mathsf{M}}$ such that neither $\mathsf{M}, w \vDash \varphi$ nor $\mathsf{M}, w \vDash \neg\varphi$. This happens if and only if φ contains atoms with undefined truth value in w.

Definition 23. *Let $\mathsf{M} = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_{\mathsf{M}})$ be an HMS model and let $w \in \Omega_{\mathsf{M}}$. Satisfaction of $\mathcal{L}$ formulas is given by*

$$\begin{array}{llll} \mathsf{M}, w \vDash \top & \text{for all } w \in \Omega_{\mathsf{M}} & & \\ \mathsf{M}, w \vDash p & \text{iff } w \in V_{\mathsf{M}}(p) & \mathsf{M}, w \vDash \varphi \wedge \psi & \text{iff } w \in [\![\varphi]\!] \cap [\![\psi]\!] \\ \mathsf{M}, w \vDash \neg\varphi & \text{iff } w \in \neg[\![\varphi]\!] & \mathsf{M}, w \vDash K_a\varphi & \text{iff } w \in \boldsymbol{K}_a([\![\varphi]\!]) \end{array}$$

where $[\![\varphi]\!] = \{v \in \Omega_{\mathsf{M}}\colon \mathsf{M}, v \vDash \varphi\}$ for all $\varphi \in \mathcal{L}$.

With the HMS semantics being three-valued, they adopt a non-standard notion of validity which requires only that a formula be always satisfied *if its has a defined truth value*. The below is equivalent to the definition in [14], but is stated so that it also works for Kripke lattice models:

Definition 24. *A formula $\varphi \in \mathcal{L}$ is valid over a class of models $\boldsymbol{C}$ iff for all models $M \in \boldsymbol{C}$, for all states w of M which satisfy p or $\neg p$ for all $p \in At(\varphi)$, w also satisfies φ.*

5.2 Kripke Lattice Models as a Semantics

We define semantics for $\mathcal{L}$ over Kripke lattice models. Like the HMS semantics, the semantics are three-valued, as it is possible that a pointed Kripke lattice model (M, w_X) satisfies neither φ nor $\neg\varphi$. This happens exactly when φ contains atoms not in X.

Definition 25. *Let $\mathsf{K} = (\mathcal{K}(K = (W, R, V)), \trianglelefteq, \pi)$ be a Kripke lattice model with $w_X \in \Omega_{\mathsf{K}}$. Satisfaction of $\mathcal{L}$ formulas is given by*

$$\begin{array}{llll} \mathsf{K}, w_X \Vdash \top & & \text{for all } w_X \in \Omega_{\mathsf{K}} & \\ \mathsf{K}, w_X \Vdash p & \text{iff} & w_X \in V_X(p) & \text{and } p \in X \\ \mathsf{K}, w_X \Vdash \neg\varphi & \text{iff} & \text{not } \mathsf{K}, w_X \Vdash \varphi & \text{and } At(\varphi) \subseteq X \\ \mathsf{K}, w_X \Vdash \varphi \wedge \psi & \text{iff} & \mathsf{K}, w_X \Vdash \varphi \text{ and } \mathsf{K}, w_X \Vdash \psi & \text{and } At(\varphi \wedge \psi) \subseteq X \\ \mathsf{K}, w_X \Vdash K_a\varphi & \text{iff} & \pi_a(w_X) R_{Ya} v_Y \text{ implies } \mathsf{K}, v_Y \Vdash \varphi, & \\ & & \text{for } Y \subseteq At \text{ s.t. } \pi_a(w_X) \in W_Y & \text{and } At(\varphi) \in X \end{array}$$

5.3 The Equivalence of HMS and Kripke Lattice Models

L- and H-transforms not only produce models of the correct class, but also preserve finer details, as any model and its transform satisfy the same formulas.

Proposition 26. *For any HMS model* M *with* L*-transform* $L(\mathsf{M})$*, for all* $\varphi \in \mathcal{L}$*, for all* $w \in \Omega_{\mathsf{M}}$*, and for all* $v \in \ell(w)$*,* $\mathsf{M}, w \vDash \varphi$ *iff* $L(\mathsf{M}), v \Vdash \varphi$*.*

Proof. Let Σ_{M} be the events of $\mathsf{M} = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_{\mathsf{M}})$ with maximal state-space T and let $L(\mathsf{M}) = (\mathcal{K}(\mathtt{K} = (W, R, V)), \trianglelefteq, \pi)$. The proof is by induction on formula complexity. Let $\varphi \in \mathcal{L}$ and let $w \in \Omega_{\mathsf{M}}$ with $At(S(w)) = X$.

Base: i) $\varphi := p \in At$ or *ii)* $\varphi := \top$. *i)* $\mathsf{M}, w \vDash p$ iff $w \in V_{\mathsf{M}}(p)$. As $V_{\mathsf{M}}(p) \in \Sigma_{\mathsf{M}}$, $(r^T_{S(w)})^{-1}(w) \subseteq V_{\mathsf{M}}(p)$. By def. of $L(\mathsf{M})$, if $v \in T = W$, then $v \in V_{\mathsf{M}}(p)$ iff $v \in V(p)$, so $v \in (r^T_{S(w)})^{-1}(w)$ iff $v \in V(p)$ iff $v_X \in V_X(p)$, with $p \in X$ (def. of Kripke lattice models). Hence, by def. of ℓ, $v \in \ell(w) = \{u_X \in W_X : u \in (r^T_{S(w)})^{-1}(w) \text{ for } X = At(S(w))\}$ iff $v \in V_X(p)$, i.e., iff $L(M), v \Vdash p$ for all $v \in \ell(w)$. *ii)* is trivial.

Step. Assume $\psi, \chi \in \mathcal{L}$ satisfy Proposition 26.

$\varphi := \neg\psi$. There are two cases: *i)* $At(\psi) \subseteq At(S(w))$ or *ii)* $At(\psi) \nsubseteq At(S(w))$. *i)* $\mathsf{M}, w \vDash \neg\psi$ iff (def. of $\vDash$) $w \in \neg[\![\psi]\!]$ iff (def. of V_{M}) $(r^T_{S(w)})^{-1}(w) \subseteq \neg[\![\psi]\!]$ iff (def. of $[\![\psi]\!]$) for all $v \in (r^T_{S(w)})^{-1}(w)$, $\mathsf{M}, v \nvDash \psi$ iff (Definition 13) for all $v \in (r^T_{S(w)})^{-1}(w)$, not $L(\mathsf{M}), v \Vdash \psi$ iff (def. of $\ell(w)$) for all $v_X \in \ell(w)$, not $L(\mathsf{M}), v_X \Vdash \psi$, with $At(\psi) \subseteq X$ iff (def. of $\Vdash$) for all $v_X \in \ell(w)$, $L(\mathsf{M}), v_X \Vdash \neg\psi$. *ii)* is trivial: φ is undefined in (M, w) iff it is so in $(L(\mathsf{M}), w_X)$.

$\varphi := \psi \wedge \chi$. The case follows by tracing *iff*s through the definitions of $\vDash$, V_{M}, $[\![\cdot]\!]$, $\left(r^T_{S(w)}\right)^{-1}$, L-transform, ℓ, and $\Vdash$.

$\varphi := K_a\psi$. $\mathsf{M}, w \vDash K_a\psi$ iff (def. of $\vDash$) $w \in \boldsymbol{K}_a([\![\psi]\!])$ iff (def. of $\boldsymbol{K}_a$) $\Pi_a(w) \subseteq [\![\psi]\!]$. Let $\Pi_a(w) \subseteq S$, for some $S \in \mathcal{S}$, and let $X = At(S(w))$ and $Y = At(S)$. Then $v_S \in \Pi_a(w) \subseteq [\![\psi]\!]$ iff (def. of $[\![\psi]\!]$) for all $v_S \in \Pi_a(w)$, $\mathsf{M}, v_S \vDash \psi$ iff (def. of V_{M}) for all $(r^T_S)^{-1}(v_S)$ with $v_S \in \Pi_a(w)$, $\mathsf{M}, v_T \vDash \psi$ iff (def. of L-transform) for all v_{At} with $r^T_S(v) \in \Pi_a(w)$, $L(\mathsf{M}), v_{At} \Vdash \psi$ and $At(\psi) \subseteq At$ iff (def. of L-transform) for all v_{At} with $(w_{At}, v_{At}) \in R_{Ata}$, $L(\mathsf{M}), v_{At} \Vdash \psi$ and $At(\psi) \subseteq At$ iff (def. of restriction lattice) for all v_Y with $(w_Y, v_Y) \in R_{Ya}$, $L(\mathsf{M}), v_Y \Vdash \psi$ and $At(\psi) \subseteq Y$ iff (def. of π_a and $\pi_a(w_X) = w_Y$), for all v_Y with $(\pi_a(w_X), v_Y) \in R_{Ya}$, $L(\mathsf{M}), v_Y \Vdash \psi$ and $At(\psi) \subseteq Y$ iff (def. of $\Vdash$) $L(\mathsf{M}), w_X \Vdash K_a\psi$ and $At(\psi) \subseteq Y$.

Proposition 27. *For any Kripke lattice model* K *with* H*-transform* $H(\mathsf{K})$*, for all* $\varphi \in \mathcal{L}$*, for all* $w_X \in \Omega_{\mathsf{K}}$*,* $\mathsf{K}, w_X \Vdash \varphi$ *iff* $H(\mathsf{K}), w_X \vDash \varphi$*.*

Proof. Let $\mathsf{K} = (\mathcal{K}(K = (W, R, V)), \trianglelefteq, \pi)$ with $w_X \in \Omega_{\mathsf{K}}$, $\pi_a(w_X) \in W_Y$ with $Y \subseteq At$, and let $H(\mathsf{K}) = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_{H(\mathsf{K})})$. Let $\varphi \in \mathcal{L}$ and proceed by induction on formula complexity.

Base: i) $\varphi := p \in At$ or *ii)* $\varphi := \top$. *i)* $\mathsf{K}, w_X \Vdash p$ iff (def. of $\Vdash$) $w_X \in V_X(p)$ with $p \in X$ iff (def. of H-transform) $w_X \in V_{H(\mathsf{K})}(p)$ iff (def. of $\vDash$) $H(\mathsf{K}), w_X \vDash p$. *ii)* is trivial.

Step. Assume $\psi, \chi \in \mathcal{L}$ satisfy Proposition 27.

$\varphi := \neg\psi$. There are two cases: *i)* $At(\psi) \subseteq X$ or *ii)* $At(\psi) \nsubseteq X$. *i)* $\mathsf{K}, w_X \Vdash \neg\psi$ iff (def. of $\Vdash$) not $\mathsf{K}, w_X \Vdash \psi$ iff (def. of $[\![\psi]\!]$) $w_X \notin [\![\psi]\!]$ iff (def. of $[\![\psi]\!]$ and $At(\psi) \subseteq X$) $w_X \in \neg[\![\psi]\!]$ iff (def. of $\vDash$) $H(\mathsf{K}), w_X \vDash \neg\psi$. *ii)* is trivial: φ is undefined in (K, w_X) iff it is so in $(H(\mathsf{M}), w_X)$.

$\varphi := \psi \wedge \chi$. The case follows by tracing *iff*s through the definitions of $\Vdash$, H-transform, and $\Vdash$.

$\varphi := K_a\psi$. $\mathsf{K}, w_X \Vdash K_a\psi$ iff (def. of $\Vdash$) $\pi_a(w_X) R_{Ya} v_Y$ implies $\mathsf{K}, v_Y \Vdash \varphi$ iff (def. of π_a, i.e. $\pi_a(w_X) = w_Y$ and def. of I_a), for all v_Y s.t. $(w_Y, v_Y) \in R_{Ya}$, i.e. for all $v_Y \in I_a(w_Y)$, $\mathsf{K}, v_Y \Vdash \varphi$ iff (def. of Π_a, i.e. $\Pi_a(w_X) = I_a(\pi_a(w_X) = I_a(w_Y))$ $\Pi_a(w_X) \subseteq [\![\psi]\!]$ iff (def. of $\boldsymbol{K}_a$) $w \in \boldsymbol{K}_a([\![\psi]\!])$ iff (def. of $\vDash$) $H(\mathsf{K}), w_X \vDash K_a\psi$.

6 The HMS Logic of Kripke Lattice Models with Equivalence Relations

As we may transition back-and-forth between HMS models and Kripke lattice models with equivalence relations in a manner that preserve satisfaction of formula of $\mathcal{L}$, soundness and completeness of a $\mathcal{L}$-logic is also transferable between the model classes. We thereby show such results for Kripke lattice models with equivalence relations as a corollary to results by HMS [14].

Definition 28. *The logic Λ_{HMS} is the smallest set of $\mathcal{L}$ formulas that contain the axioms in, and is closed under the inference rules of,* Table 1.

Table 1. Axioms and inference rules of the HMS logic of unawareness, Λ_{HMS}.

All substitution instances of propositional logic, including the formula $\top$	
$A_a\neg\varphi \leftrightarrow A_a\varphi$	(Symmetry)
$A_a(\varphi \wedge \psi) \leftrightarrow A_a\varphi \wedge A_a\psi$	(Awareness Conjunction)
$A_a\varphi \leftrightarrow A_a K_b\varphi$, for all $b \in Ag$	(Awareness Knowledge Reflection)
$K_a\varphi \rightarrow \varphi$	(T, Axiom of Truth)
$K_a\varphi \rightarrow K_a K_a\varphi$	(4, Positive Introspection Axiom)
From φ and $\varphi \rightarrow \psi$, infer ψ	(Modus Ponens)
For $\varphi_1, \varphi_2, ..., \varphi_n, \varphi$ that satisfy $At(\varphi) \subseteq \bigcup_{i=1}^n At(\varphi_i)$, from $\bigwedge_{i=1}^n \varphi_i \rightarrow \varphi$, infer $\bigwedge_{i=1}^n K_a\varphi_i \rightarrow K_a\varphi$	(RK-Inference)

As the L-transform of an HMS model has equivalence relations, one may be surprised by the lack of the standard negative introspection axiom $5 : (\neg K_a\varphi \rightarrow K_a\neg K_a\varphi)$ among the axioms of Λ_{HMS}. However, including 5 would make collapse awareness [18]. In [14], HMS remarks that Λ_{HMS} imply the weakened version $K_a\neg K_a\neg K_a\varphi \rightarrow (K_a\varphi \vee K_a\neg K_a\varphi)$, which by the Modici-Rustichini definition of awareness is $K_a\neg K_a\neg K_a\varphi \rightarrow A_a\varphi$. Defining unawareness by $U_a\varphi := \neg A_a\varphi$, this again equates $U_a\varphi \rightarrow \neg K_a\neg K_a\neg K_a\varphi$. Additionally, HMS notes that if φ is a theorem, then $A_a\varphi \rightarrow K_a\varphi$ is a theorem, that 4 implies introspection of awareness ($A_a\varphi \rightarrow K_aA_a\varphi$), while Λ_{HMS} entails that *awareness is generated by primitives propositions*, i.e., that $A_a\varphi \leftrightarrow \bigwedge_{p\in At(\varphi)} A_ap$ is a theorem. The latter two properties entails that HMS awareness is *propositionally determined*, in the terminology of [13].

Using the above given notion of validity and standard notions of proof, soundness and strong completeness, HMS [14] state that, as standard,

Lemma 29. *The logic Λ_{HMS} is strongly complete with respect to a class of structures $\mathfrak{S}$ iff every set of Λ_{HMS} consistent formulas is satisfied in some $\mathfrak{s} \in \mathfrak{S}$.*

Let $\boldsymbol{M}$ be the class of HMS modes. Using a canonical model, HMS show:

Theorem 30 ([14])**.** *Λ_{HMS} is sound and strongly complete with respect to $\boldsymbol{M}$.*

Let $\boldsymbol{KLM}_{EQ}$ be the class of Kripke lattice models where all accessibility relations are equivalence relations. As a corollary to Theorem 30 and our transformation and equivalence results, we obtain

Theorem 31. *Λ_{HMS} is sound and strongly complete with respect to $\boldsymbol{KLM}_{EQ}$.*

Proof. Soundness: The axioms of Λ_{HMS} are valid $\boldsymbol{KLM}_{EQ}$. We show the contrapositive. Let $\varphi \in \mathcal{L}$. If φ is not valid in $\boldsymbol{KLM}_{EQ}$, then for some $\mathsf{K} \in \boldsymbol{KLM}_{EQ}$ and some w from K, $\mathsf{K}, w \Vdash \neg\varphi$. Then its H-transform $H(\mathsf{K})$ is an HMS model cf. Proposition 21, and $H(\mathsf{K}), w \vDash \neg\varphi$ cf. Proposition 27. Hence φ is not valid in the class of HMS models. The same reasoning implies that the Λ_{HMS} inference rules preserve validity.

Completeness: Assume $\Phi \subseteq \mathcal{L}$ is a consistent set, and let $\mathfrak{M}$ be the canonical model of HMS, with $\mathfrak{w}$ a state in $\mathfrak{M}$ that satisfies Φ. This exists, cf. [14]. By Propositions 18 and 19, $L(\mathfrak{M})$ is in $\boldsymbol{KLM}_{EQ}$. By Proposition 26, for all $v \in \ell(\mathfrak{w})$, $L(\mathfrak{M}), v \Vdash \Phi$. By Lemma 29, Λ_{HMS} is thus strongly complete w.r.t. $\boldsymbol{KLM}_{EQ}$.

7 Concluding Remarks

This paper has presented a Kripke model-based rendition of the HMS model of awareness, and shown the two model classes equally general w.r.t. $\mathcal{L}$, by defining transformations between the two that preserve formula satisfaction. A corollary to this result is completeness of the HMS logic for the introduced model class.

There are several issues we would like to study in future work:

In recasting the HMS model, we teased apart the epistemic and awareness dimensions merged in the HMS possibility correspondences, and Propositions 18, 19 and 21 about L- and H-transforms show that the HMS properties are satisfied iff each π_a satisfies D, II and NS, and each R_a is an equivalence relation. For a more fine-grained property correspondence, the propositions' proofs show that each property of one model is entailed by a strict subset of the properties of the other. In some cases, the picture emerging is fairly clear: e.g., HMS' Conf is shown only using the restrictions lattice construction (RLC) plus D and *vice versa*; PPK uses only NS and RLC, while PPK and Conf entail NS. In other cases, the picture is more murky, e.g., when we use Stat, PPI and PPK to show the seemingly simple symmetry of R_a. We think it would be interesting to decompose properties on both sides to see if clearer relationships arise.

There are two issues with redundant states in Kripke lattice models. One concerns redundant restrictions, cf. Remark 12, which may be solved by working with a more general model class, where models may also be based on sub-orders of the restriction lattice. A second one concerns redundant states. For example, in Fig. 2, $\mathtt{K}_\emptyset$ contains three 'identical' states where no atoms have defined truth values—$\mathtt{K}_\emptyset$ is bisimilar to a one-state Kripke model. As bisimulation contracting each $\mathtt{K}_X$ may collapse states from which awareness maps differ, one must define a notion of bisimulation that takes awareness maps into consideration (notions of bisimulation for other awareness models exists, e.g. [6]). Together with a more general modal class definition, this could hopefully solve the redundancy issues.

Though [13,14,20] provide comparisons of the HMS and LGA [8,13] models, we would like to make a direct comparison with the latter to understand Kripke lattice models from an awareness function perspective. It would then be natural to use the LGA language with awareness and implicit knowledge as primitives over Kripke lattice models, which is possible as they include objective states.

The HMS logic is complete for HMS models and Kripke lattice models with equivalence relations. [13] prove completeness for HMS models using a standard validity notion, a 'φ is at least as expressive as ψ' operator and variants of axioms T, 4 and 5. We are very interested in considering this system and its weaker variants for Kripke lattice models, also with less assumptions on the relations.

Finally, issues of dynamics spring forth: first, whether existing awareness dynamics may be understood on Kripke lattice models; second, whether DEL action models may be applied lattice-wide with reasonable results; and third, whether the π_a maps may be thought in dynamic terms, as they map between models.

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