Simpler Completeness Proofs for Modal Logics with Intersection

Yì N. Wáng[1(✉)] and Thomas Ågotnes[2,3]

[1] Department of Philosophy (Zhuhai), Sun Yat-sen University, Zhuhai, China
ynw@xixilogic.org
[2] University of Bergen, Bergen, Norway
[3] Southwest University, Chongqing, China
Thomas.Agotnes@uib.no

Abstract. There has been a significant interest in modal logics with intersection, prominent examples including epistemic and doxastic logics with distributed knowledge, propositional dynamic logic with intersection, and description logics with concept intersection. Completeness proofs for such logics tend to be complicated, in particular on model classes such as S5 used, e.g., in standard epistemic logic, mainly due to the undefinability of intersection of modalities in standard modal logic. A standard proof method for the S5 case uses an "unraveling-folding" technique to achieve a treelike model to deal with the problem of undefinability. This method, however, is not easily adapted to other logics, due to its reliance on S5 in a number of steps. In this paper we demonstrate a simpler and more general proof technique by building a treelike canonical model directly, which avoids the complications in the processes of unraveling and folding. We illustrate the technique by showing completeness of the normal modal logics K, D, T, B, S4 and S5 extended with intersection modalities. Furthermore, these treelike canonical models are compatible with Fischer-Ladner-style closures, and we combine the methods to show the completeness of the mentioned logics further extended with transitive closure of union modalities known from PDL or epistemic logic. Some of these completeness results are new.

Keywords: Modal logic · Intersection modality · Transitive closure of union modality · Completeness · Epistemic logic · Distributed knowledge

1 Introduction

Intersection plays a role in several areas of modal logic, including epistemic logics with distributed knowledge [11,15], propositional dynamic logic with intersection of programs [13], description logics with concept intersection [3,4], and coalition logic [1]. It is well-known that relational intersection in Kripke models is not modally definable and that standard logics with intersection are not canonical (cf., e.g., [14]).

M. A. Martins and I. Sedlár (Eds.): DaLí 2020, LNCS 12569, pp. 259–276, 2020.
https://doi.org/10.1007/978-3-030-65840-3_16

A method for proving completeness for certain modal logics with intersection was developed in [11,12,14–16] for various (static) epistemic logics with distributed knowledge, and later explicated and extended in [17–19] as the *unraveling-folding method* which is applicable to various static or dynamic epistemic S5 logics with distributed knowledge with or without common knowledge.

Let us take a closer look at this technique for epistemic logic with distributed knowledge (S5D). It is known that the canonical S5 model built in the standard way is not a model for the classical axiomatization for this logic. This is because the accessibility relation R_G (where G is a set) that is (implicitly) used to interpret the intersection (distributed knowledge) modality is not necessarily the intersection of individual accessibility relations R_a ($a \in G$). In the canonical S5 model we can ensure that $R_G \subseteq \bigcap_{a\in G} R_a$, but not that $R_G \supseteq \bigcap_{a\in G} R_a$.

The unraveling-folding method is carried out in the following way. A *pre-model* is a standard S5 model where R_G is treated as a primitive relation for each group G. A *pseudo model* is a pre-model satisfying the following two constraints:

1. $R_{\{a\}} = R_a$ for every agent a, and
2. $R_G \subseteq \bigcap_{a\in G} R_a$ for every agent a and group G

A (proper) S5D model is then a pseudo model that satisfies also a third constraint:

3. $R_G \supseteq \bigcap_{a\in G} R_a$ for every agent a and group G

A canonical pseudo model can be truth-preservingly translated to a *treelike* pre-model using an *unraveling* technique, and then *folded* to an S5D model while also preserving the truth of all formulas (for details of the two processes see [18]). Completeness is achieved by first building a canonical pseudo model for a given consistent set Φ of formulas, and then translating it to an S5D model for Φ using the unraveling-folding method.

There are many subtleties not mentioned in this simplified overview, which in particular makes the method cumbersome to adapt to extensions of basic epistemic logic or to non-S5 based logics.

In this paper we demonstrate a simpler way to prove completeness for modal logics with intersection. Since we know that a treelike model typically works for such logics, the idea is to build a treelike model *directly* for a given consistent set of formulas. We call such a model a *standard model*. This eliminates having to deal with the details of the unraveling and folding processes, and dramatically simplifies proofs.

We illustrate the technique by building the standard model for each of the modal logics, K, D, T, B, S4 and S5, extended with intersection. We furthermore demonstrate that the method is useful by showing that it is compatible with finitary methods based on Fischer-Ladner-style closures, and introduce finitary standard models for the mentioned logics further extended with the transitive closure of the union, used in, e.g., PDL and epistemic logic (common knowledge), as well. Some of these completeness results have been stated in the literature before, often without proof.

The rest of the paper is structured as follows. In the next section we introduce basic definitions and conventions. In Sect. 3 we give a taste of the proof technique by demonstrating it on a well-known case: $\mathbf{S5}^{\cap}$ with intersection. The reader who wants to immediately see what the technique looks like can jump directly to that section. In Sect. 4 we systematically consider a class of well-known modal logics extended with intersection. For each of them we introduce an axiomatization and show its completeness. We then extend the logics, proofs and results further with a modality for the transitive closure of union in Sect. refsec:logiccd. We conclude in Sect. 6.

2 Preliminaries

In this paper we study modal logics over multi-modal languages with countably many standard unary modal operators: $\Box_0$, $\Box_1$, $\Box_2$, etc. On top of these we focus on two types of modal operators, each *indexed* by a finite nonempty set I of natural numbers:

- *Intersection modalities*, denoted $\cap_I$;
- *Transitive closure of union modalities*, henceforth referred to as *union*$^+$ *modalities* for brevity, denoted $\uplus_I$.

We mention some applications of these modalities below.

The languages are parameterized by a countably infinite set PR of propositions, and a countable set $\mathcal{I}$ of primitive types. A finite non-empty subset $I \subseteq \mathcal{I}$ is called an *Index*. We are interested in the following languages.

Definition 1 (languages).

$$
\begin{array}{ll}
(\mathcal{L}) & \varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid \Box_i\varphi \\
(\mathcal{L}^{\cap}) & \varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid \Box_i\varphi \mid \cap_I\varphi \\
(\mathcal{L}^{\cap\uplus}) & \varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid \Box_i\varphi \mid \cap_I\varphi \mid \uplus_I\varphi
\end{array}
$$

where $p \in$ PR, $i \in \mathcal{I}$ and I is an index. Boolean connectives are defined as usual.

A Kripke model M (over PR and $\mathcal{I}$) is a triple (S, R, V), where S is a nonempty set of states, $R : \mathcal{I} \to \wp(S \times S)$ assigns to every modality $\Box_i$ a binary relation R_i on S, and $V :$ PR $\to S$ is a valuation which associates with every propositional variable a set of states where it is true.

Definition 2 (satisfaction). *For a given formula α, the* truth *of it in, or its* satisfaction *by, a model $M = (S, R, V)$ with a designated state s, denoted $M, s \models \alpha$, is defined inductively as follows.*

$$
\begin{array}{lll}
M, s \models p & \textit{iff} & s \in V(p) \\
M, s \models \neg\varphi & \textit{iff} & \textit{not } (M, s) \models \varphi \\
M, s \models (\varphi \to \psi) & \textit{iff} & M, s \models \varphi \textit{ implies } M, s \models \psi \\
M, s \models \Box_i\varphi & \textit{iff} & \textit{for all } t \in S, \textit{ if } (s, t) \in R_i \textit{ then } M, t \models \varphi \\
M, s \models \cap_I\varphi & \textit{iff} & \textit{for all } t \in S, \textit{ if } (s, t) \in \bigcap_{i \in I} R_i \textit{ then } M, t \models \varphi \\
M, s \models \uplus_I\varphi & \textit{iff} & \textit{for all } t \in S, \textit{ if } (s, t) \in \biguplus_{i \in I} R_i \textit{ then } M, t \models \varphi
\end{array}
$$

where[1] $\biguplus_{i\in I} R_i$ *is the transitive closure of* $\bigcup_{i\in I} R_i$.

Thus, the *intersection* modalities are interpreted by taking the intersection, and the *union*$^+$ modalities by taking the transitive closure of the union.

Given a formula φ and a class $\mathscr{C}$ of models, we say φ is *valid* ($\models \varphi$) in $\mathscr{C}$ iff φ is true in all states in all models of $\mathscr{C}$. A formula φ is a *logical consequence* of a set of formulas Φ ($\Phi \models \varphi$) if φ is true in a given state in a given model whenever all formulas in Φ are. We are interested in certain classes of models, in particular those defined by well-known *frame conditions*. In this paper we are going to focus on some of the most well known frame conditions (see, e.g., [9]). These are *seriality*, *reflexivity*, *symmetry*, *transitivity* and *Euclidicity*. It is well known that these frame conditions are characterized by the formulas D ($\Box_i\varphi \to \neg\Box_i\neg\varphi$), T ($\Box_i\varphi \to \varphi$), B ($\neg\varphi \to \Box_i\neg\Box_i\varphi$), 4 ($\Box_i\varphi \to \Box_i\Box_i\varphi$) and 5 ($\neg\Box_i\varphi \to \Box_i\neg\Box_i\varphi$), respectively. With respect to different combinations of these frame conditions, normal modal logics K, D (also known as KD), ¡(also known as KT), B (also known as KTB), S4 (also known as KT4) and S5 (also known as KT5) based on the language $\mathcal{L}$ are well studied in the literature. We shall refer an "S5 model" to a Kripke model in which the binary relation is an equivalence relation, and likewise for a D, T, B or S4 model.

In this paper we will focus on the corresponding logics over the languages $\mathcal{L}^{\cap}$ and $\mathcal{L}^{\cap\uplus}$, and they will be named in a comprehensive way as follows:

$$\begin{array}{l} \mathrm{K}^{\cap},\ \mathrm{D}^{\cap},\ \mathrm{T}^{\cap},\ \mathrm{B}^{\cap},\ \mathrm{S4}^{\cap},\ \mathrm{S5}^{\cap}, \\ \mathrm{K}^{\cap\uplus}, \mathrm{D}^{\cap\uplus}, \mathrm{T}^{\cap\uplus}, \mathrm{B}^{\cap\uplus}, \mathrm{S4}^{\cap\uplus}, \mathrm{S5}^{\cap\uplus}. \end{array}$$

There are well known applications of these logics, for example are $\mathrm{S5}^{\cap}$ and $\mathrm{S5}^{\cap\uplus}$ (under the restriction that $\mathcal{I}$ is finite) well known as S5D (multi-agent S5 with distributed knowledge) and S5CD (multi-agent S5 with distributed and common knowledge) respectively in the area of epistemic logic. The logics $\mathrm{K}^{\cap}$ and $\mathrm{S4}^{\cap}$ are known as $\mathcal{ALC}(\cap)$ (i.e., $\mathcal{ALC}$ with role intersection) and $\mathcal{S}(\cap)$ (where $\mathcal{S}$ is $\mathcal{ALC}$ with role transitivity) respectively in the area of description logic [3,4].[2] The logic $\mathrm{K}^{\cap\uplus}$ is close to propositional dynamic logic with intersection (IPDL) [13] or the description logic $\mathcal{ALC}(\cap, \cup, *)$, and similarly, $\mathrm{S4}^{\cap\uplus}$ is close to $\mathcal{S}(\cap, \cup, *)$.[3]

1 Although the symbol $\biguplus$ is sometimes used for disjoint union, we repurpose it here for transitive closure of the union.

2 The subscript i of a unary modal operator $\Box_i$ typically stands for an agent in epistemic logic or a role in description logic. In epistemic logic, a finite number of agents is assumed, and the intersection modality (i.e., a distributed knowledge operator) is an arbitrary intersection over a finite domain. In description logic, the number of roles are typically unbounded, but the intersection is binary, which is in effect equivalent to finite intersection.

3 There are two major differences however. First, the Kleene star in both logics are the reflexive-transitive closure, and we consider the transitive closure which is denoted by a "+" in the symbol $\uplus$. Second, $\uplus_I$ is a compound modality (union and then take the transitive closure), while in those logics the Kleene star is separated from the union, and as a result, the Kleene star applies to the intersection as well, which we do not consider here.

The minimal logic K can be axiomatized by the system **K** composed of the following axiom (schemes) and rules (where $\varphi, \psi \in \mathcal{L}$ and $i \in \mathcal{I}$):

(PC) all instances of all propositional tautologies
(MP) from $(\varphi \to \psi)$ and φ infer ψ
(K) $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$
(N) from φ infer $\Box_i\varphi$.

Axiomatizations for D, T, B , S4 and S5, which are named **D**, **T**, **B**, **S4** and **S5** respectively, can be obtained by adding characterization axioms to **K**. In more detail, $\mathbf{D} = \mathbf{K} \oplus \mathrm{D}$, $\mathbf{T} = \mathbf{K} \oplus \mathrm{T}$, $\mathbf{B} = \mathbf{T} \oplus \mathrm{B}$, $\mathbf{S4} = \mathbf{T} \oplus 4$ and $\mathbf{S5} = \mathbf{T} \oplus 5$, where the symbol $\oplus$ means combining the axioms and rules of the two parts. Details can be found in standard modal logic textbooks (see, e.g., [8,9]). Given an axiomatization **L**, we use "$\vdash_{\mathbf{L}} \varphi$" to denote that φ is derivable in **L**, and when Φ is a set of formulas "$\Phi \vdash_{\mathbf{L}} \varphi$" means that $\vdash_{\mathbf{L}} (\psi_0 \wedge \cdots \wedge \psi_n) \to \varphi$ for some $\psi_0, \ldots, \psi_n \in \Phi$.

A logic extended with the intersection modality is typically axiomatized by adding axioms and rules to the corresponding logic without intersection. The axioms and rules to be added are in total called the *characterization of intersection*, and depends on which logic we are dealing with. Similarly we can define the *characterization of the transitive closure of union*, which can be made independent to the concrete logic (will be made clearer in Sect. 5).

Characterizations of intersection and transitive closure of union can be found in the literature for some of the logics, including $\mathrm{K}^{\cap}$, $\mathrm{T}^{\cap}$, $\mathrm{S4}^{\cap}$, $\mathrm{S5}^{\cap}$ and $\mathrm{S5}^{\cap\uplus}$ in epistemic logic (see [11,15,17]). For base logic S5, intersection in $\mathbf{S5}^{\cap}$ and $\mathbf{S5}^{\cap\uplus}$ is characterized by the following axioms and rules:

- (K$\cap$) $\cap_I(\varphi \to \psi) \to (\cap_I\varphi \to \cap_I\psi)$
- (D$\cap$) $\cap_I\varphi \to \neg \cap_I \neg\varphi$
- (T$\cap$) $\cap_I\varphi \to \varphi$
- (4$\cap$) $\cap_I\varphi \to \cap_I\cap_I\varphi$
- (B$\cap$) $\neg\varphi \to \cap_I\neg\cap_I\varphi$
- (5$\cap$) $\neg\cap_I\varphi \to \cap_I\neg\cap_I\varphi$
- (N$\cap$) from φ infer $\cap_I\varphi$
- ($\cap$1) $\Box_i\varphi \leftrightarrow \cap_{\{i\}}\varphi$
- ($\cap$2) $\cap_I\varphi \to \cap_J\varphi$, if $I \subseteq J$

Transitive closure of union in $\mathbf{S5}^{\cap\uplus}$ is characterized by the following:

- (K$\uplus$) $\uplus_I(\varphi \to \psi) \to (\uplus_I\varphi \to \uplus_I\psi)$
- (D$\uplus$) $\uplus_I\varphi \to \neg\uplus_I\neg\varphi$
- (T$\uplus$) $\uplus_I\varphi \to \varphi$
- (4$\uplus$) $\uplus_I\varphi \to \uplus_I\uplus_I\varphi$
- (B$\uplus$) $\neg\varphi \to \uplus_I\neg\uplus_I\varphi$
- (5$\uplus$) $\neg\uplus_I\varphi \to \uplus_I\neg\uplus_I\varphi$
- (N$\uplus$) from φ infer $\uplus_I\varphi$
- ($\uplus$1) $\uplus_I\varphi \to \Box_i(\varphi \wedge \uplus_I\varphi)$, if $i \in I$
- ($\uplus$2) from $\varphi \to \bigwedge_{i \in I} \Box_i(\varphi \wedge \psi)$ infer $\varphi \to \uplus_I\psi$

It is known that the axiomatization $\mathbf{S5}^{\cap} = \mathbf{S5} \oplus \{\mathrm{K}\cap, \mathrm{T}\cap, 5\cap, \cap 1, \cap 2\}$ is sound and complete[4] for the logic $\mathrm{S5}^{\cap}$, and $\mathbf{S5}^{\cap\uplus} = \mathbf{S5}^{\cap} \oplus \{\mathrm{K}\uplus, \uplus 1, \uplus 2\}$ is sound and complete[5] for the logic $\mathrm{S5}^{\cap\uplus}$ (see, e.g., [11]), in the case that $\mathcal{I}$ is finite. However, since the intersection and union$^+$ modalities are interpreted as operations over relations for standard box operators, their properties change in accordance with those for standard boxes. As a result, the characterization axioms and rules vary for weaker logics. We shall look into this in the following sections. First we define some basic terminology that will be useful.

Definition 3 (paths, (proper) initial segments, rest, tail). *Given a model $M = (S, R, V)$, a* path *of M is a finite nonempty sequence $\langle s_0, I_1, \ldots, I_n, s_n\rangle$ where: (i) $s_0, \ldots, s_n \in S$, (ii) $I_1, \ldots, I_n$ are indices, and (iii) for all $x = 1, \ldots, n$, $(s_{x-1}, s_x) \in \bigcap_{i \in I_x} R_i$.*

For paths $s = \langle s_0, I_1, \ldots, I_m, s_m\rangle$ and $t = \langle t_0, J_1, \ldots, J_n, t_n\rangle$ of a model,

- *We say that s is an* initial segment *of t, denoted $s \preceq t$, if $m \le n$, $s_x = t_x$ for all $x = 0, \ldots, m$, and $I_y = J_y$ for all $y = 1, \ldots, m$, and then we say that t extends s with $\langle J_{m+1}, t_{m+1}, \ldots, J_n, t_n\rangle$;*
- *We say s is a* proper initial segment *of t, denoted $s \prec t$, if the former is an initial segment of the latter and $m < n$;*
- *We write* $\mathsf{tail}(s)$ *for s_m, and similarly* $\mathsf{tail}(t)$ *for t_n;*
- *When s is an initial segment of t, we write $t \setminus s$ to stand for the path $\langle t_m, J_{m+1}, \ldots, J_n, t_n\rangle$. Note that* $\mathsf{tail}(s)$ *is kept in $t \setminus s$, and when $s = t$, we have $t \setminus s = \langle t_n\rangle$.*

Given a natural number i, a path $s = \langle s_0, I_1, \ldots, I_n, s_n\rangle$ is called:

- *An* i-path*, if i appears in all the indices of the path, i.e., $i \in \bigcap_{x=1}^{n} I_x$ (note that a path of length 1, such as $\langle s_0\rangle$, is trivially an i-path).*
- *An* I-path*, where I is an index, if $I \subseteq \bigcap_{x=1}^{n} I_x$.*

3 A Simple Completeness Proof for $\mathrm{S5}^{\cap}$

To illustrate the new technique we now give a proof, omitting some details, for the particular case of $\mathbf{S5}^{\cap}$, assuming familiarity with the canonical model method for classical modal logics. In the next section we demonstrate the generality of the technique and provide all details.

Let MCS be the set of all maximal $\mathbf{S5}^{\cap}$-consistent sets of $\mathcal{L}^{\cap}$-formulas. For a given index I, the *canonical relation* $\rhd_I$ is a binary relation on MCS, such that $\Phi \rhd_I \Psi$ iff for all φ, $\cap_I \varphi \in \Phi$ implies $\varphi \in \Psi$. It is easy to see that $\rhd_I$ is an equivalence relation. A *canonical path* is a sequence $\langle \Phi_0, I_1, \ldots, I_n, \Phi_n\rangle$ such that: (i) $\Phi_0, \ldots, \Phi_n \in$ MCS, (ii) $I_1, \ldots, I_n$ are indices, and (iii) for all $x = 1, \ldots, n$, $(s_{x-1}, s_x) \in \rhd_{I_x}$. We use similar terminology and notation for canonical paths as for paths in a model (Definition 3).

[4] $\mathrm{D}\cap$, $4\cap$, $\mathrm{B}\cap$ and $\mathrm{N}\cap$ are not needed in the sense that they are derivable.

[5] $\mathrm{D}\uplus$, $\mathrm{T}\uplus$, $4\uplus$, $\mathrm{B}\uplus$, $5\uplus$ and $\mathrm{N}\uplus$ are not needed in the sense that they are derivable.

Definition 4. *The* standard model *for* $\boldsymbol{S5}^{\cap}$ *is a tuple* $\mathsf{M} = (\mathsf{S}, \mathsf{R}, \mathsf{V})$ *such that:*

- S *is the set of all canonical paths;*
- *For all* $i \in \mathcal{I}$, $\mathsf{R}_i \subseteq \mathsf{S} \times \mathsf{S}$ *such that* $(s,t) \in \mathsf{R}_i$ *iff (i)* s *and* t *have a common initial segment* u*, and (ii) both* $s \setminus u$ *and* $t \setminus u$ *are* i*-paths.*
- *For any propositional variable* p, $\mathsf{V}(p) = \{s \in \mathsf{S} \mid p \in \mathsf{tail}(s)\}$.

Lemma 5. *The* standard model *for* $\boldsymbol{S5}^{\cap}$ *is an S5 model.*

Proof. An easy verification of the definition of the standard model.

Lemma 6 (truth). *For any* $\varphi \in \mathcal{L}^{\cap}$ *and a state* s *of* M, $\mathsf{M}, s \models \varphi$ *iff* $\varphi \in \mathsf{tail}(s)$.

Proof. By induction on φ. The atomic and Boolean cases are easy to show. Interesting cases are for the modalities $\Box_i$ ($i \in \mathcal{I}$) and $\cap_I$ (I is an index), the former following easily from the latter.

$$\begin{aligned} & \mathsf{M}, s \models \cap_I \psi \\ \Leftrightarrow\ & \text{for all } t, \text{ if } (s,t) \in \textstyle\bigcap_{i\in I} \mathsf{R}_i \text{ then } \mathsf{M}^{\mathrm{L}}, t \models \psi \\ \Leftrightarrow\ & \text{for all } t, \text{ if } (s,t) \in \textstyle\bigcap_{i\in I} \mathsf{R}_i \text{ then } \psi \in \mathsf{tail}(t) \Leftrightarrow\ \cap_I \psi \in \mathsf{tail}(s) \end{aligned}$$

where the last step needs an argument.

Suppose $\cap_I \psi \notin \mathsf{tail}(s)$, we get $\neg \cap_I \psi \in \mathsf{tail}(s)$. Let $\Phi^- = \{\neg\psi\} \cup \{\psi' \mid \cap_I \psi' \in \mathsf{tail}(s)\}$. We can show that Φ^- is $\mathbf{S5}^{\cap}$-consistent just as in a classical proof of the existence lemma. Use the Lindenbaum construction to extend Φ^- into $\Phi \in$ MCS. Since $\neg\psi \in \Phi$, $\psi \notin \Phi$. Let t be s extended with $\langle I, \Phi \rangle$. Clearly, $\psi \notin \mathsf{tail}(t)$ and $(s,t) \in \bigcap_{i\in I} \mathsf{R}_i$ (since $s\, \mathsf{R}_i\, t$ for all $i \in I$).

Suppose $\cap_I \psi \in \mathsf{tail}(s)$ and assume towards a contradiction that there is a state t such that $(s,t) \in \bigcap_{i\in I} \mathsf{R}_i$ and $\psi \notin \mathsf{tail}(t)$. By definition, s and t have a common initial segment u, and $s \setminus u$ and $t \setminus u$ are both I-paths. There are three cases: (i) $s \preceq t$, (ii) $t \preceq s$, and (iii) s and t fork (i.e., neither (i) or (ii)). Since $\rhd_I$ is an equivalence relation, in all cases it is easy to verify that $\mathsf{tail}(s) \rhd_I \mathsf{tail}(t)$.

Theorem 7. $\boldsymbol{S5}^{\cap}$ *is a strongly complete axiomatization of* $S5^{\cap}$.

4 Logics over $\mathcal{L}^{\cap}$

In this section we study the logics over the language $\mathcal{L}^{\cap}$, namely, $K^{\cap}$, $D^{\cap}$, $T^{\cap}$, $B^{\cap}$, $S4^{\cap}$ and $S5^{\cap}$, which means that in this section a "formula" stands for a formula of $\mathcal{L}^{\cap}$, and a "logic" without further explanation refers to one of the six. We shall provide a general method for proving completeness for these logics.

The axiomatization **L** we will provide for a logic L is an extension of the axiomatization for the corresponding logic without intersection, with the characterization of intersection. The characterization of intersection depends on the frame conditions. For a given class of models, the characterization of intersection is listed below:

$$\begin{array}{ll}\mathbf{Int(K)} &= \{\mathrm{K}\cap, \cap 1, \cap 2\}\\ \mathbf{Int(D)} &= \{\mathrm{K}\cap, \cap 1, \cap 2\}\\ \mathbf{Int(T)} &= \{\mathrm{K}\cap, \mathrm{T}\cap, \cap 1, \cap 2\}\\ \mathbf{Int(B)} &= \{\mathrm{K}\cap, \mathrm{T}\cap, \mathrm{B}\cap, \cap 1, \cap 2\}\\ \mathbf{Int(S4)} &= \{\mathrm{K}\cap, \mathrm{T}\cap, 4\cap, \cap 1, \cap 2\}\\ \mathbf{Int(S5)} &= \{\mathrm{K}\cap, \mathrm{T}\cap, 5\cap, \cap 1, \cap 2\},\end{array}$$

where **Int(K)** is the characterization of intersection for the class of all models, **Int(D)** for the class of all D models, **Int(T)** for the class of all T models, and so on. Note that D$\cap$ is not included in **Int(D)**: it is in fact invalid in $\mathrm{D}^\cap$ [2].

By adding the characterization of intersection to the axiomatization of a logic, we get an axiomatization for the corresponding logic over $\mathcal{L}^\cap$. To be precise, we list the axiomatizations as follows:

$$\begin{array}{rl}\mathbf{K}^\cap &= \mathbf{K} \oplus \mathbf{Int(K)}\\ \mathbf{D}^\cap &= \mathbf{D} \oplus \mathbf{Int(D)}\\ \mathbf{T}^\cap &= \mathbf{T} \oplus \mathbf{Int(T)}\\ \mathbf{B}^\cap &= \mathbf{B} \oplus \mathbf{Int(B)}\\ \mathbf{S4}^\cap &= \mathbf{S4} \oplus \mathbf{Int(S4)}\\ \mathbf{S5}^\cap &= \mathbf{S5} \oplus \mathbf{Int(S5)}.\end{array}$$

It is not hard to verify that all the above axiomatizations are sound in their corresponding logics, respectively.

Some of the above axiomatizations, in particular, $\mathbf{K}^\cap$, $\mathbf{T}^\cap$, $\mathbf{S4}^\cap$ and $\mathbf{S5}^\cap$, are given in [11]. An outline of a completeness proof is also found there, without details. Similarly, equivalent axiomatizations for some of the cases are also found in [5], with proof of completeness only for the $\mathbf{K}^\cap$ case. For logics extending $\mathbf{K}^\cap$ detailed proofs can be found for certain cases, such as the $\mathbf{S5}^\cap$ with only a single intersection modality for the set of all agents (which is assumed to be finite) [10]. A more general and detailed proof based on this technique for the S5 case can be found in [18] (still for the S5 case). The proof goes through an unraveling-folding procedure, mentioned in the introduction. Due to the subtleties in the unraveling and folding processes, it is difficult to apply this technique directly to new logics, as it has to be adapted from the beginning (for example, even the definition of a *path* depends on the underlying logic) through several steps all the way to the very end of the procedure.

We introduce a simpler method for proving completeness, that can easily be adapted to a range of different logics. This is a relatively straightforward variant of the canonical model method. For each of the logics L mentioned above, with corresponding axiomatization **L**, we show that **L** is a complete axiomatization of L, which is equivalent to finding an L model for every **L**-consistent set of formulas. The model we are going to build is called a *standard model.*

Let $\mathrm{MCS}^{\mathbf{L}}$ be the set of all maximal **L**-consistent sets of $\mathcal{L}^\cap$-formulas.[6] Given **L**, given an index I, we shall write $\rhd_I$ to stand for the binary relation on $\mathrm{MCS}^{\mathbf{L}}$, such that $\Phi \rhd_I \Psi$ iff for all φ, $\cap_I \varphi \in \Phi$ implies $\varphi \in \Psi$. This type of relations is

[6] We refer to a modal logic textbook, say [8], for a definition of a *(maximal) consistent set of formulas.*

typically used in the definition of a canonical model, and are sometimes called *canonical relations*. We easily get the following proposition.

Proposition 8. *For any index I, the canonical relation $\rhd_I$ on $MCS^{\boldsymbol{L}}$ is:*

1. *serial, if I is singleton and $\boldsymbol{L}$ is $\boldsymbol{D}^{\cap}$;*
2. *reflexive, if $\boldsymbol{L}$ is $\boldsymbol{T}^{\cap}$;*
3. *reflexive and symmetric, if $\boldsymbol{L}$ is $\boldsymbol{B}^{\cap}$;*
4. *reflexive and transitive, if $\boldsymbol{L}$ is $\boldsymbol{S4}^{\cap}$;*
5. *an equivalence relation, if $\boldsymbol{L}$ is $\boldsymbol{S5}^{\cap}$;*
6. *s.t. $\rhd_J \subseteq \rhd_I$, for any index $J \supseteq I$.*

Definition 9 (canonical paths). *Given an axiomatization $\boldsymbol{L}$, a* canonical path *for $\boldsymbol{L}$ is a sequence $\langle \Phi_0, I_1, \ldots, I_n, \Phi_n \rangle$ such that:*

(i) $\Phi_0, \ldots, \Phi_n \in MCS^{\boldsymbol{L}}$,
(ii) $I_1, \ldots, I_n$ *are indices, and*
(iii) for all $x = 1, \ldots, n$, $(\Phi_{x-1}, \Phi_x) \in \rhd_{I_x}$.

Initial segments, $\mathsf{tail}(s)$*, ("canonical") i-paths, I-paths, and so on, are defined exactly like for paths in a model (Definition 3).*

The *standard models* we will define for these logics are a bit different from the canonical model for a standard modal logic. As mentioned existing proofs are based on transforming the canonical model to a treelike model. We will construct a treelike model directly: in the standard model for a logic L, a state will be a canonical path for **L**. However, the binary relations in a standard model is dependent on the concrete logic we focus on. We now first define these binary relations and then introduce the definition of a standard model.

Definition 10 (standard relations). *Given a logic L with its axiomatization $\boldsymbol{L}$, we define R^L as follows. For any $i \in \mathcal{I}$, R^L_i is the binary relation on the set of canonical paths for $\boldsymbol{L}$, called the* standard relation for i*, such that:*

- *When L is $K^{\cap}$ or $D^{\cap}$: for all canonical paths s and t for $\boldsymbol{L}$, $(s,t) \in \mathsf{R}^L_i$ iff t extends s with $\langle I, \Phi \rangle$ for some $I \ni i$ and $\Phi \in MCS^{\boldsymbol{L}}$;*
- *When L is $T^{\cap}$: for all canonical paths s and t for $\boldsymbol{T}^{\cap}$, $(s,t) \in \mathsf{R}^{T^{\cap}}_i$ iff $t = s$ or t extends s with $\langle I, \Phi \rangle$ for some $I \ni i$ and $\Phi \in MCS^{\boldsymbol{T}^{\cap}}$;*
- *When L is $B^{\cap}$: for all canonical paths s and t for $\boldsymbol{B}^{\cap}$, $(s,t) \in \mathsf{R}^{B^{\cap}}_i$ iff one of the following holds for some $I \ni i$ and $\Phi \in MCS^{\boldsymbol{B}^{\cap}}$:*
  (i) $t = s$
  (ii) s *extends* t *with* $\langle I, \Phi \rangle$
  (iii) t *extends* s *with* $\langle I, \Phi \rangle$;
- *When L is $S4^{\cap}$: for all canonical paths s and t for $\boldsymbol{S4}^{\cap}$, $(s,t) \in \mathsf{R}^{S4^{\cap}}_i$ iff s is an initial segment of t and $t \setminus s$ is a canonical i-path;*
- *When L is $S5^{\cap}$: for all canonical paths s and t for $\boldsymbol{S5}^{\cap}$, $(s,t) \in \mathsf{R}^{S5^{\cap}}_i$ iff*
  (i) s *and* t *have a common initial segment* u*, and*
  (ii) both $s \setminus u$ *and* $t \setminus u$ *are canonical i-paths.*

Definition 11 (standard models). *Given a logic L, the* standard model *for L is a tuple* $\mathsf{M}^L = (\mathsf{S}, \mathsf{R}, \mathsf{V})$ *such that:*

- S *is the set of all canonical paths for* ***L****;*
- $\mathsf{R} = \mathsf{R}^L$*;*
- *For any propositional variable p,* $\mathsf{V}(p) = \{s \in \mathsf{S} \mid p \in \mathsf{tail}(s)\}$*.*

Lemma 12 (standardness). *The following hold:*

1. $\mathsf{M}^{K^\cap}$ *is a Kripke model;*
2. $\mathsf{M}^{D^\cap}$ *is a D model;*
3. $\mathsf{M}^{T^\cap}$ *is a T model;*
4. $\mathsf{M}^{B^\cap}$ *is a B model;*
5. $\mathsf{M}^{S4^\cap}$ *is an S4 model;*
6. $\mathsf{M}^{S5^\cap}$ *is an S5 model.*

Lemma 13 (existence). *For any logic L, state s of* M^L*, and index I, if* $\cap_I\varphi \notin \mathsf{tail}(s)$ *then there is a state t of* M^L *such that* $(s,t) \in \bigcap_{i\in I} \mathsf{R}_i^L$ *and* $\varphi \notin \mathsf{tail}(t)$*.*

Proof. Let s be a state of M^{L} and $\cap_I\varphi \notin \mathsf{tail}(s)$. So $\neg \cap_I \varphi \in \mathsf{tail}(s)$. Consider the set $\Phi^- = \{\neg\varphi\} \cup \{\psi \mid \cap_I\psi \in \mathsf{tail}(s)\}$. We can show Φ^- is $\mathbf{L}$ consistent just as in a classical proof of the existence lemma (see, e.g., [8]). We can then extend it into a maximal consistent set Φ of formulas using the Lindenbaum construction. Since $\neg\varphi \in \Phi$, $\varphi \notin \Phi$. Let t be s extended with $\langle I, \Phi\rangle$. By definition it is clear that $\varphi \notin \mathsf{tail}(t)$ and for all L, $(s,t) \in \bigcap_{i\in I} \mathsf{R}_i^{\mathrm{L}}$ (since $s\, \mathsf{R}_i^{\mathrm{L}}\, t$ for all $i \in I$).

Lemma 14 (truth). *Given a logic L, a formula* φ*, and a state s of* M^L*, it holds that:* $\mathsf{M}^L, s \models \varphi$ *if and only if* $\varphi \in \mathsf{tail}(s)$*.*

Proof. The proof is by induction on φ. The atomic case is by definition. Boolean cases are easy to show. Interesting cases are for the modalities $\Box_i$ ($i \in \mathcal{I}$) and $\cap_I$ (I is an index). We start with the case for $\cap_I\psi$.

$$
\begin{aligned}
& \mathsf{M}^{\mathrm{L}}, s \models \cap_I\psi \\
\Leftrightarrow\ & \text{for all } t, \text{ if } (s,t) \in \textstyle\bigcap_{i\in I} \mathsf{R}_i^{\mathrm{L}} \text{ then } \mathsf{M}^{\mathrm{L}}, t \models \psi \\
\Leftrightarrow\ & \text{for all } t, \text{ if } (s,t) \in \textstyle\bigcap_{i\in I} \mathsf{R}_i^{\mathrm{L}} \text{ then } \psi \in \mathsf{tail}(t) \\
\Rightarrow\ & \cap_I\psi \in \mathsf{tail}(s) \qquad\qquad \text{(existence lemma)}
\end{aligned}
$$

For the converse of the last step, suppose $\cap_I\psi \in \mathsf{tail}(s)$ and assume towards a contradiction that there is a state t such that $(s,t) \in \bigcap_{i\in I} \mathsf{R}_i^{\mathrm{L}}$ and $\psi \notin \mathsf{tail}(t)$.

- If L is $\mathrm{K}^\cap$ or $\mathrm{D}^\cap$, it must be that t extends s with $\langle J, \Phi\rangle$ for $J \supseteq I$ and $\Phi \in \mathrm{MCS}^{\mathbf{L}}$. By definition $\mathsf{tail}(s) \rhd_J \mathsf{tail}(t)$, and by Proposition 8.6, we have $\mathsf{tail}(s) \rhd_I \mathsf{tail}(t)$. Therefore $\psi \in \mathsf{tail}(t)$, which leads to a contradiction.
- If L is $\mathrm{T}^\cap$, we face an extra case compared with the above, namely $s = t$. A contradiction can be reached by applying the axiom T$\cap$.

- If L is $B^\cap$, then (i) $t = s$ or (ii) $s = \langle t, J, \Phi\rangle$ or (iii) $t = \langle s, J, \Phi\rangle$ where $J \supseteq I$ and $\Phi \in \mathrm{MCS}^{\mathbf{B}}$. Case (i) can be shown similarly to the case when L is $T^\cap$, and case (iii) to the case when L is $K^\cap$ or $D^\cap$. For case (ii), it is important to observe that $\rhd_I$ is symmetric (Proposition 8.3) and $\rhd_J \subseteq \rhd_I$ (Proposition 8.6).
- If L is $S4^\cap$, s must be an initial segment of t and $t \setminus s$ is an I-path. We get $\mathsf{tail}(s) \rhd_I \mathsf{tail}(t)$ by Proposition 8.6 and the reflexivity and transitivity of $\rhd_I$ (Proposition 8.4). Therefore $\psi \in \mathsf{tail}(t)$ which leads to a contradiction.
- If L is $S5^\cap$, s and t have a common initial segment u, and $s \setminus u$ and $t \setminus u$ are both I-paths. When one of s and t is an initial segment of the other, it can be shown like in the case when L is $S4^\cap$. The interesting case is when s and t really fork, in this case we can show both $\mathsf{tail}(s) \rhd_I \mathsf{tail}(u)$ and $\mathsf{tail}(u) \rhd_I \mathsf{tail}(t)$ by transitivity and symmetry of $\rhd_I$ (Proposition 8.5) and Proposition 8.6, so that $\mathsf{tail}(s) \rhd_I \mathsf{tail}(t)$. Then $\psi \in \mathsf{tail}(t)$, which leads to a contradiction.

Finally, the case for $\Box_i\psi$: $\mathsf{M}^{\mathrm{L}}, s \models \Box_i\psi \iff \mathsf{M}^{\mathrm{L}}, s \models \cap_{\{i\}}\psi$ (validity of $\cap 1$) $\iff \cap_{\{i\}}\psi \in \mathsf{tail}(s)$ (special case of $\cap_I\psi$) $\iff \Box_i\psi \in \mathsf{tail}(s)$ (axiom $\cap 1$).

Theorem 15 (strong completeness). *Given* $L \in \{K^\cap, D^\cap, T^\cap, B^\cap, S4^\cap, S5^\cap\}$ *and its axiomatization* $\boldsymbol{L}$*, for any* $\Phi \subseteq \mathcal{L}^\cap$ *and* $\varphi \in \mathcal{L}^\cap$*, if* $\Phi \models \varphi$*, then* $\Phi \vdash_{\boldsymbol{L}} \varphi$*.*

Proof. Suppose $\Phi \nvdash_{\mathbf{L}} \varphi$. It follows that $\Phi \cup \{\neg\varphi\}$ is $\mathbf{L}$ consistent. Extend it to be a maximal consistent set Ψ, then $\langle\Psi\rangle$ is a canonical path. By the truth lemma, for any formula ψ, we have $\mathsf{M}, \langle\Psi\rangle \models \psi$ iff $\psi \in \Psi$. It follows that Ψ is satisfiable, which leads to $\Phi \not\models \varphi$.

5 Logics over $\mathcal{L}^{\cap\uplus}$

In this section we study the logics with both the intersection and union$^+$ modalities. The language is set to be $\mathcal{L}^{\cap\uplus}$ in this section, and by a "logic" without further explanation we mean one of $K^{\cap\uplus}$, $D^{\cap\uplus}$, $T^{\cap\uplus}$, $B^{\cap\uplus}$, $S4^{\cap\uplus}$ or $S5^{\cap\uplus}$.

Compared with the characterization of intersection, that of transitive closure of union is more straightforward:

$$\mathbf{Un(K)} = \mathbf{Un(D)} = \mathbf{Un(T)} = \mathbf{Un(B)} = \mathbf{Un(S4)} = \mathbf{Un(S5)} = \{\mathrm{K}\uplus, \uplus 1, \uplus 2\}.$$

These axioms are not new, see, e.g., [11], although as far as we know they have not been studied in combination with D and B in the literature. For simplicity we write **Un** for this set of axioms. Additional validities for union$^+$ corresponding to specific frame conditions can be derived in specific logic systems. For instance, $\mathrm{D}\uplus$ is a theorem of $\mathbf{D} \oplus \mathbf{Un}$.

By adding to the axiomatization of a logic over $\mathcal{L}^\cap$ the characterization of union$^+$, we get a sound axiomatization for the corresponding logic over $\mathcal{L}^{\cap\uplus}$. To be precise, we list the axiomatizations as follows:

$$\begin{aligned}
\mathbf{K}^{\cap\uplus} &= \mathbf{K}^{\cap} \oplus \mathbf{Un}\\
\mathbf{D}^{\cap\uplus} &= \mathbf{D}^{\cap} \oplus \mathbf{Un}\\
\mathbf{T}^{\cap\uplus} &= \mathbf{T}^{\cap} \oplus \mathbf{Un}\\
\mathbf{B}^{\cap\uplus} &= \mathbf{B}^{\cap} \oplus \mathbf{Un}\\
\mathbf{S4}^{\cap\uplus} &= \mathbf{S4}^{\cap} \oplus \mathbf{Un}\\
\mathbf{S5}^{\cap\uplus} &= \mathbf{S5}^{\cap} \oplus \mathbf{Un}.
\end{aligned}$$

It is well known that logics with both a basic modality and a modality for the transitive closure of the basic modality is not semantically *compact*; we will thus be concerned only with *weak* rather than *strong* completeness in this section. We will make extensive references to the names of logics and axiomatizations, and for simplicity we shall call a tuple $\sigma = (\mathrm{L}, \mathbf{L}, \alpha, \iota)$ a *signature*, when L is one of the logics $\mathrm{K}^{\cap\uplus}$, $\mathrm{D}^{\cap\uplus}$, $\mathrm{T}^{\cap\uplus}$, $\mathrm{B}^{\cap\uplus}$, $\mathrm{S4}^{\cap\uplus}$ and $\mathrm{S5}^{\cap\uplus}$, $\mathbf{L}$ is the corresponding axiomatization for L, α is a formula of $\mathcal{L}^{\cap\uplus}$, and ι is an index such that (i) $i \in \iota$ for every $\Box_i$ occurring in α, and (ii) every index occurring in α is a subset of ι.

Definition 16 (closure). *Given a signature $\sigma = (L, \boldsymbol{L}, \alpha, \iota)$, the σ-*closure*, denoted $cl(\sigma)$, is the minimal set of formulas satisfying the following conditions:*

1. *$\alpha \in cl(\sigma)$;*
2. *If $\varphi \in cl(\sigma)$, then all the subformulas of φ are also in $cl(\sigma)$;*
3. *If φ does not start with a negation symbol and $\varphi \in cl(\sigma)$, then $\neg\varphi \in cl(\sigma)$;*
4. *For any $i \in \iota$,*
   (i) If $\cap_{\{i\}}\varphi \in cl(\sigma)$ then $\Box_i\varphi \in cl(\sigma)$, and
   (ii) If $\Box_i\varphi \in cl(\sigma)$ then $\cap_{\{i\}}\varphi \in cl(\sigma)$;
5. *For indices I and J with $I \subset J \subseteq \iota$, if $\cap_I\varphi \in cl(\sigma)$ then $\cap_J\varphi \in cl(\sigma)$;*
6. *For indices $I, J \subseteq \iota$, if $\uplus_I\varphi \in cl(\sigma)$ and $I \cap J \neq \emptyset$ then $\cap_J\uplus_I\varphi \in cl(\sigma)$.*[7]

It is not hard to verify that $cl(\sigma)$ is finite and nonempty for any signature σ. Given $\sigma = (\mathrm{L}, \mathbf{L}, \alpha, \iota)$, a set of formulas is said to be *maximal* $\mathbf{L}$*-consistent in* $cl(\sigma)$, if it is (i) a subset of $cl(\sigma)$, (ii) $\mathbf{L}$-consistent and (iii) maximal in $cl(\sigma)$ (i.e., any proper superset which is a subset of $cl(\sigma)$ is inconsistent). We write MCS^σ for the set of all maximal $\mathbf{L}$-consistent sets of formulas in $cl(\sigma)$.

Now we adapt the canonical relations to the finitary case. Given a signature σ and an index I, we may try to define a canonical relation $\rhd_I$ to be a binary relation on MCS^σ, such that $\Phi \rhd_I \Psi$ iff for all φ, $\cap_I\varphi \in \Phi$ implies $\varphi \in \Psi$, like we did for the logics over $\mathcal{L}^{\cap}$. But there are subtleties here. For example, transitivity may be lost for $\mathrm{S4}^{\cap\uplus}$, if $\cap_I\varphi \in \Phi$ but $\cap_I\cap_I\varphi \notin \Phi$ in case the latter is not included in the closure. We introduce the formal definition below.

Definition 17 (finitary canonical relation). *For a signature $\sigma = (L, \boldsymbol{L}, \alpha, \iota)$ and an index $I \subseteq \iota$, the* canonical relation *$\rhd_I$ for σ is the binary relation on MCS^σ, such that the following hold for all $\Phi, \Psi \in MCS^\sigma$:*

[7] This is the place where the use of ι is essential to make sure that a closure is finite.

- *If L is* $K^{\cap\uplus}$, $D^{\cap\uplus}$ *or* $T^{\cap\uplus}$: $\Phi \rhd_I \Psi$ *iff* $\{\varphi \mid \cap_I\varphi \in \Phi\} \subseteq \Psi$;
- *If L is* $B^{\cap\uplus}$: $\Phi \rhd_I \Psi$ *iff* $\{\varphi \mid \cap_I\varphi \in \Phi\} \subseteq \Psi$ *and* $\{\varphi \mid \cap_I\varphi \in \Psi\} \subseteq \Phi$;
- *If L is* $S4^{\cap\uplus}$: $\Phi \rhd_I \Psi$ *iff* $\{\cap_I\varphi \mid \cap_I\varphi \in \Phi\} \subseteq \{\cap_I\varphi \mid \cap_I\varphi \in \Psi\}$;
- *If L is* $S5^{\cap\uplus}$: $\Phi \rhd_I \Psi$ *iff* $\{\cap_I\varphi \mid \cap_I\varphi \in \Psi\} = \{\cap_I\varphi \mid \cap_I\varphi \in \Psi\}$.

Note that for all the logics, from $\Phi \rhd_I \Psi$ we still get that $\cap_I\varphi \in \Phi$ implies $\varphi \in \Psi$, as the criteria above are at least not weaker. We get the following proposition that is similar to Proposition 8.

Proposition 18. *For any signature* $\sigma = (L, \boldsymbol{L}, \alpha, \iota)$ *and any index* $I \subseteq \iota$, *the canonical relation* $\rhd_I$ *for* σ *is:*

1. *Serial, if* I *is singleton and* $\boldsymbol{L}$ *is* $\boldsymbol{D}^{\cap\uplus}$;
2. *Reflexive, if* $\boldsymbol{L}$ *is* $\boldsymbol{T}^{\cap\uplus}$;
3. *Reflexive and symmetric, if* $\boldsymbol{L}$ *is* $\boldsymbol{B}^{\cap\uplus}$;
4. *Reflexive and transitive, if* $\boldsymbol{L}$ *is* $\boldsymbol{S4}^{\cap\uplus}$;
5. *An equivalence relation, if* $\boldsymbol{L}$ *is* $\boldsymbol{S5}^{\cap\uplus}$.
6. $\rhd_J \subseteq \rhd_I$, *for any index* J *such that* $I \subseteq J \subseteq \iota$.

Proof. For seriality when $I = \{i\}$: given $\Phi \in \mathrm{MCS}^\sigma$ and a formula φ such that $\cap_{\{i\}}\varphi \in \Phi$, it suffices to show the existence of a $\Psi \in \mathrm{MCS}^\sigma$ such that $\varphi \in \Psi$. This is easy, take φ and extend it to be **L**-maximal in $cl(\sigma)$ (note that $\varphi \in cl(\sigma)$).

For reflexivity, we make use of the axiom T$\cap$ and the fact that $cl(\sigma)$ is closed under subformulas.

For the combinations of frame conditions for $\mathbf{B}^{\cap\uplus}$, $\mathbf{S4}^{\cap\uplus}$ and $\mathbf{S5}^{\cap\uplus}$, we can see that they are enforced by the definition of the canonical relation.

Definition 19 (finitary canonical paths). *Given a signature* $\sigma = (L, \boldsymbol{L}, \alpha, \iota)$, *a* canonical path *for* $\boldsymbol{L}$ *in* $cl(\sigma)$ *is a sequence* $\langle \Phi_0, I_1, \ldots, I_n, \Phi_n\rangle$ *such that:*

(i) $\Phi_0, \ldots, \Phi_n \in MCS^\sigma$,
(ii) $I_1, \ldots, I_n \subseteq \iota$ *are indices, and*
(iii) *for all* $x = 1, \ldots, n$, $(\Phi_{x-1}, \Phi_x) \in \rhd_{I_x}$.

Initial segments, tails of paths, ("canonical") i-paths, I-paths, and so on, are defined like for paths in a model (Definition 3).

Definition 20 (standard relation). *Given a signature* $\sigma = (L, \boldsymbol{L}, \alpha, \iota)$, *for any* $i \in \iota$, *the* standard relation R_i^σ *is the binary relation on the canonical paths for* $\boldsymbol{L}$ *in* $cl(\sigma)$, *such that:*

- *If L is* $K^{\cap\uplus}$ *or* $D^{\cap\uplus}$: *for all canonical paths s and t for* $\boldsymbol{L}$ *in* $cl(\sigma)$, $(s,t) \in \mathsf{R}_i^\sigma$ *iff t extends s with* $\langle I, \Phi\rangle$ *for* $\Phi \in MCS^\sigma$ *and some index I such that* $i \in I \subseteq \iota$;
- *If L is* $T^{\cap\uplus}$: *for all canonical paths s and t for* $\boldsymbol{T}^{\cap\uplus}$ *in* $cl(\sigma)$, $(s,t) \in \mathsf{R}_i^\sigma$ *iff* $t = s$ *or t extends s with* $\langle I, \Phi\rangle$ *for* $\Phi \in MCS^\sigma$ *and some index I s.t.* $i \in I \subseteq \iota$;
- *If L is* $B^{\cap\uplus}$: *for all canonical paths s and t for* $\boldsymbol{B}^{\cap\uplus}$ *in* $cl(\sigma)$, $(s,t) \in \mathsf{R}_i^\sigma$ *iff (i)* $t = s$ *or (ii)* $s = \langle t, I, \Phi\rangle$ *or (iii)* $t = \langle s, I, \Phi\rangle$ *for* $\Phi \in MCS^\sigma$ *and some index I such that* $i \in I \subseteq \iota$;

- *If L is $S4^{\cap\uplus}$: for all canonical paths s and t for $\boldsymbol{S4}^{\cap\uplus}$ in $cl(\sigma)$, $(s,t) \in \mathsf{R}_i^\sigma$ iff s is an initial segment of t and $t \setminus s$ is a canonical i-path;*
- *If L is $S5^{\cap\uplus}$: for all canonical paths s and t for $\boldsymbol{S5}^{\cap\uplus}$ in $cl(\sigma)$, $(s,t) \in \mathsf{R}_i^\sigma$ iff (i) s and t have a common initial segment u, and (ii) both $s \setminus u$ and $t \setminus u$ are canonical i-paths.*

Definition 21 (finitary standard models). *Given a signature $\sigma = (L, \boldsymbol{L}, \alpha, \iota)$, the* standard model *for σ is a tuple $\mathsf{M}^\sigma = (\mathsf{S}, \mathsf{R}, \mathsf{V})$ such that:*

- *S is the set of all canonical paths for $\boldsymbol{L}$ in $cl(\sigma)$.*
- *$\mathsf{R}_i = \mathsf{R}_i^\sigma$.*
- *For any propositional variable p, $\mathsf{V}(p) = \{s \in \mathsf{S} \mid p \in \mathsf{tail}(s)\}$.*

Lemma 22 (standardness). *For any signature $\sigma = (L, \boldsymbol{L}, \alpha, \iota)$,*

1. *M^σ is a Kripke model;*
2. *M^σ is a D model when $L = D^{\cap\uplus}$ and $\boldsymbol{L} = \boldsymbol{D}^{\cap\uplus}$;*
3. *M^σ is a T model when $L = T^{\cap\uplus}$ and $\boldsymbol{L} = \boldsymbol{T}^{\cap\uplus}$;*
4. *M^σ is a B model when $L = B^{\cap\uplus}$ and $\boldsymbol{L} = \boldsymbol{B}^{\cap\uplus}$;*
5. *M^σ is an S4 model when $L = S4^{\cap\uplus}$ and $\boldsymbol{L} = \boldsymbol{S4}^{\cap\uplus}$;*
6. *M^σ is an S5 model when $L = S5^{\cap\uplus}$ and $\boldsymbol{L} = \boldsymbol{S5}^{\cap\uplus}$.*

Lemma 23 (existence). *For any signature σ, any state s of M^σ, and any index $I \subseteq \iota$,*

1. *Given $\cap_I \varphi \in cl(\sigma)$, if $\cap_I \varphi \notin \mathsf{tail}(s)$, then there is a state t of M^σ such that $(s,t) \in \bigcap_{i \in I} \mathsf{R}_i^\sigma$ and $\varphi \notin \mathsf{tail}(t)$.*
2. *Given $\uplus_I \varphi \in cl(\sigma)$, if $\uplus_I \varphi \notin \mathsf{tail}(s)$, then there is a state t of M^σ such that $(s,t) \in \biguplus_{i \in I} \mathsf{R}_i^\sigma$ and $\varphi \notin \mathsf{tail}(t)$.*

Proof. Let $\sigma = (\mathrm{L}, \mathbf{L}, \alpha, \iota)$ and s be a state of M^σ.

(1) Let $\cap_I \varphi \notin \mathsf{tail}(s)$. So $\neg \cap_I \varphi \in \mathsf{tail}(s)$. Consider the set $\Phi^- = \{-\varphi\} \cup \{\psi \mid \cap_I \psi \in \mathsf{tail}(s)\}$ (where $-\varphi$ is ψ if $\varphi = \neg\psi$, and is $\neg\varphi$ if φ does not start with a negation symbol). Clearly $\Phi^- \subseteq cl(\sigma)$ and it is not hard to show that it is $\mathbf{L}$ consistent. We can then extend it into a maximal consistent set Φ of formulas in $cl(\sigma)$. Since $-\varphi \in \Phi$, $\varphi \notin \Phi$. Let t be s extended with $\langle I, \Phi \rangle$. By definition it is clear that $\varphi \notin \mathsf{tail}(t)$ and $(s,t) \in \bigcap_{i \in I} \mathsf{R}_i^\sigma$ (since $s \, \mathsf{R}_i^\sigma \, t$ for all $i \in I$).

(2) Let $\mathcal{P}$ be the property on the states of M^σ such that for any s, $s \in \mathcal{P}$ iff for any t, if $(s,t) \in \biguplus_{i \in I} \mathsf{R}_i^\sigma$ then $\varphi \in \mathsf{tail}(t)$. The equivalent condition is that for any state s_0 of M^σ, $s_0 \in \mathcal{P}$ iff $\varphi \in \mathsf{tail}(s_n)$ holds for any path $\langle s_0, \{i_0\}, \ldots, \{i_{n-1}\}, s_n \rangle$ of M^σ with $\{i_0, \ldots, i_{n-1}\} \subseteq I$. Let $\psi = \bigvee_{s \in \mathcal{P}} \widehat{\mathsf{tail}(s)}$ (where $\widehat{\mathsf{tail}(s)}$ is the conjunction of all formulas in $\mathsf{tail}(s)$). We get the following:

(a) For any $i \in I$, $\vdash_{\mathbf{L}} \psi \to \Box_i \varphi$. First observe that for every $s_0 \in \mathcal{P}$, any path $\langle s_0, \{i_0\}, \ldots, \{i_{n-1}\}, s_n \rangle$ as described above is such that $\varphi \in \mathsf{tail}(s_n)$. As a special case, for any state s_1, if $\langle s_0, \{i\}, s_1 \rangle$ is a path, namely $\mathsf{tail}(s_0) \rhd_{\{i\}} \mathsf{tail}(s_1)$, then $\varphi \in \mathsf{tail}(s_1)$. It follows that $\Box_i \varphi \in \mathsf{tail}(s_0)$ (for otherwise it violates the first clause; just treat $\Box_i$ to be $\cap_{\{i\}}$). This means that $\Box_i \varphi$ is a conjunct of every disjunct of ψ, and so $\vdash_{\mathbf{L}} \psi \to \Box_i \varphi$.

(b) For any $i \in I$, $\vdash_{\mathbf{L}} \psi \rightarrow \Box_i\psi$. Suppose towards a contradiction that $\psi \wedge \neg\Box_i\psi$ is consistent. There must be a disjunct of ψ, say $\widehat{\mathsf{tail}(t)}$ (with $t \in \mathcal{P}$), such that $\widehat{\mathsf{tail}(t)} \wedge \neg\Box_i\psi$ is consistent. By properties of MCS^σ we have $\vdash_{\mathbf{L}} \bigvee\{\widehat{\Phi} \mid \Phi \in \mathrm{MCS}^\sigma\}$ (similarly $\widehat{\Phi}$ is the conjunction of formulas in Φ). So there must be $\Phi \in \mathrm{MCS}^\sigma \setminus \{\mathsf{tail}(s) \mid s \in \mathcal{P}\}$ such that $\widehat{\mathsf{tail}(t)} \wedge \neg\Box_i\neg\widehat{\Phi}$ is consistent. It follows that $\mathsf{tail}(t) \rhd_{\{i\}} \Phi$. The path u which extends t with $\langle\{i\}, \Phi\rangle$ is such that $(t, u) \in \mathsf{R}^\sigma_{\{i\}}$. Since $t \in \mathcal{P}$, we have $u \in \mathcal{P}$ as well. However, this conflicts with the fact that $\Phi \notin \{\mathsf{tail}(s) \mid s \in \mathcal{P}\}$.

Now suppose $s \in \mathcal{P}$, and we must show $\uplus_I\varphi \in \mathsf{tail}(s)$. By (a) and (b), $\vdash_{\mathbf{L}} \psi \rightarrow \bigwedge_{i\in I} \Box_i(\psi \wedge \varphi)$, and then by $\uplus$2 we have $\vdash_{\mathbf{L}} \psi \rightarrow \uplus_I\varphi$. Let $\xi = \widehat{\mathsf{tail}(s)}$. It follows that $\vdash_{\mathbf{L}} \xi \rightarrow \psi$, as ξ is one of the disjuncts of ψ. We get $\vdash_{\mathbf{L}} \xi \rightarrow \uplus_I\varphi$, and so $\uplus_I\varphi \in \mathsf{tail}(s)$ for $\mathsf{tail}(s)$ is consistent.

Lemma 24 (truth). *Given a signature σ, a formula $\varphi \in cl(\sigma)$, and a state s of M^σ, it holds that: $\mathsf{M}^\sigma, s \models \varphi$ iff $\varphi \in \mathsf{tail}(s)$.*

Proof. The proof is by induction on φ. The atomic and Boolean cases are easy to show. The cases for the modalities $\Box_i$ ($i \in \mathcal{I}$) and $\cap_I$ (I is an index) are not much different from those of the proof of Lemma 14 (we need to be careful with the closure, however; just note that all the i's and I's used here are bounded by an ι). Here we detail the case for $\uplus_I\psi$.

$$\begin{aligned}
&\mathsf{M}^\sigma, s \models \uplus_I\psi \\
\Leftrightarrow\ &\text{for all } t, \text{ if } (s,t) \in \biguplus_{i\in I} \mathsf{R}^\sigma_i \text{ then } \mathsf{M}^\sigma, t \models \psi \\
\Leftrightarrow\ &\text{for all } t, \text{ if } (s,t) \in \biguplus_{i\in I} \mathsf{R}^\sigma_i \text{ then } \psi \in \mathsf{tail}(t) \\
\Rightarrow\ &\uplus_I\psi \in \mathsf{tail}(s) \qquad\qquad\qquad\qquad \text{(existence lemma)}
\end{aligned}$$

For the converse direction of the last step, suppose $\uplus_I\psi \in \mathsf{tail}(s)$ and towards a contradiction that there is a state t such that $(s, t) \in \biguplus_{i\in I} \mathsf{R}^\sigma_i$ and $\psi \notin \mathsf{tail}(t)$. So there is a path $\langle s_0, \{i_0\}, \ldots, \{i_{n-1}\}, s_n\rangle$ of M^σ such that $\{i_0, \ldots, i_{n-1}\} \subseteq I$, $s = s_0$ and $t = s_n$.

- If L is $\mathrm{K}^{\cap\uplus}$ or $\mathrm{D}^{\cap\uplus}$, it must be that t extends s with $\langle J_0, \Phi_1, \ldots, J_{n-1}, \Phi_n\rangle$ where $\psi \notin \Phi_n$ and for each x, $i_x \in J_x$ and $\Phi_x \in \mathrm{MCS}^\sigma$. By definition $\mathsf{tail}(s_0) \rhd_{J_0} \Phi_1 \rhd_{J_1} \cdots \rhd_{J_{n-1}} \Phi_n$. By the axioms $\uplus$1, $\cap$1 and $\cap$2 we can get $\vdash_{\mathbf{L}} \uplus_I\psi \rightarrow \cap_{J_0}\uplus_I\psi$, and $\uplus_I\psi \in \Phi_1$ for $\cap_{J_0}\uplus_I\psi \in cl(\sigma)$. Doing this recursively, we get $\uplus_I\psi \in \Phi_n$ and so $\psi \in \Phi_n$ by T$\uplus$, which contradicts $\psi \notin \mathsf{tail}(t)$.
- If L is $\mathrm{T}^{\cap\uplus}$, we face an extra case compared with the above, namely $s = t$. A contradiction can be achieved by applying the axiom T$\uplus$.
- If L is $\mathrm{B}^{\cap\uplus}$, there are three cases: (i) $s_{x+1} = s_x$ or (ii) $s_x = \langle s_{x+1}, J, \Phi\rangle$ or (iii) $s_{x+1} = \langle s_x, J, \Phi\rangle$ where $J \supseteq I$ and $\Phi \in \mathrm{MCS}^\sigma$. In all cases, by similar reasoning to the above (for case (ii) we use the symmetric condition for $\rhd_I$), we can show that $\psi \in \mathsf{tail}(s_{x+1})$ given $\uplus_I\psi \in \mathsf{tail}(s_x)$, and then reach a contradiction similarly.

– If L is $S4^{\cap\uplus}$, s_x $(0 \leq x < n)$ must be an initial segment of s_{x+1} and $s_{x+1} \setminus s_x$ is a finitary canonical i_x-path ($i_x \in I$). By the axioms $\uplus$1 and $\cap$1, $\vdash_{\mathbf{S4}^{\cap\uplus}} \uplus_I \psi \to \cap_J \uplus_I \psi$ (for all $i_x \in J \subseteq \iota$). So we get $\cap_J \uplus_I \psi \in \mathsf{tail}(s_{x+1})$ (we use T$\uplus$ in the case when $s = t$). Recursively carrying this out, we get $\cap_J \uplus_I \psi \in \mathsf{tail}(t)$, and so $\psi \in \mathsf{tail}(t)$ which leads to a contradiction.
– If L is $S5^{\cap\uplus}$, then s_x and s_{x+1} have a common initial segment u, and $s_x \setminus u$ and $s_{x+1} \setminus u$ are both finitary canonical i_x-paths. Since $\vdash_{\mathbf{S5}^{\cap\uplus}} \uplus_I \psi \to \cap_J \uplus_I \psi$ (for all $i_x \in J \subseteq \iota$), $\cap_J \uplus_I \psi \in \mathsf{tail}(s_0)$, and by the definition of $\rhd$, $\cap_J \uplus_I \psi \in \mathsf{tail}(s_x)$, so $\psi \in \mathsf{tail}(t)$ which leads to a contradiction as well.

Theorem 25 (weak completeness). *Let* $\boldsymbol{L}$ *be the corresponding axiomatization introduced for a logic* $L \in \{K^{\cap\uplus}, D^{\cap\uplus}, T^{\cap\uplus}, B^{\cap\uplus}, S4^{\cap\uplus}, S5^{\cap\uplus}\}$*. For any* $\varphi \in \mathcal{L}^{\cap\uplus}$*, if* $\models \varphi$*, then* $\vdash_{\boldsymbol{L}} \varphi$*.*

Proof. Suppose $\nvdash_{\mathbf{L}} \varphi$. It follows that $\{\neg\varphi\}$ is $\mathbf{L}$ consistent. Extend it to be a maximal consistent set Φ in $cl((\mathrm{L}, \mathbf{L}, \neg\varphi, \iota))$ with ι including $\{i \mid \Box_i \text{ occurs in } \varphi\}$ and all the indices occurring in φ, then $\langle\Phi\rangle$ is a canonical path for $\mathbf{L}$ in $cl((\mathrm{L}, \mathbf{L}, \neg\varphi, \iota))$. By the truth lemma, for any formula ψ in the closure, we have $\mathsf{M}^{(\mathrm{L}, \mathbf{L}, \neg\varphi, \iota)}, \langle\Phi\rangle \models \psi$ iff $\psi \in \Phi$. It follows that Φ is satisfiable, which leads to $\not\models \varphi$.

6 Discussion

We focused mainly on the completeness proof for the modal logics, K, D, T, B, S4 and S5, extended with intersection and with or without the transitive closure of union. For some of these logics proofs of completeness using the unraveling-folding technique exist in the literature, for some no or only partial proofs exist. We have to omit details here, but the method can also be directly applied to many other canonical multi-modal logics with the intersection modality, including popular systems of epistemic and doxastic logics such as S4.2, S4.3, S4.4 – we have in fact already applied successfully for the KD45 case.[8] By avoiding the model translation processes used in the unraveling-folding method and building a standard model directly, the proofs we present are dramatically simpler than those found in the literature for special cases. We believe that the readers who are familiar with the canonical model method for completeness proofs of modal logics will find the proofs very familiar and straightforward.

While our approach is inspired by simplifying the existing proof technique, the standard model we build is not identical to the model produced by the unraveling-folding processes: it is simpler because we do not have to use so-called reductions of paths. We emphasize, however, that the unraveling-folding method was still important for us to arrive at this proof technique: it explains why we take (finitary) canonical paths to be the states of the standard model. Further work that could be interesting is to show possible bisimilarity of the model we build to that by the unraveling-folding processes.

8 In an extension of [2], to appear.

It is worth mentioning that our results are slightly more general than most existing proofs from the literature on distributed knowledge in that it allows a (countably) infinite set of boxes. This slightly complicates the proofs in the cases with transitive closure of the union, requiring the use of the σ signatures.

Finally, as mentioned the full language of PDL with intersection (IPDL) is more general than the languages we have considered here: it allows, e.g., transitive closure of intersections. While there are complete axiomatizations of IPDL with infinitary and/or unorthodox inference rules [7], and complete axiomatizations with finitary orthodox rules of iteration-free IPDL [6], finitary orthodox axiomatization of full IPDL is a long-standing open problem. Perhaps the technique presented in this paper could help shed some new light on that problem.

Acknowledgments. We thank the anonymous reviewers for very detailed and useful comments and suggestions. Yì N. Wáng acknowledges funding support by the National Social Science Foundation of China (Grant No. 16CZX048, 18ZDA290), and the Fundamental Research Funds for the Central Universities of China.

References

1. Ågotnes, T., Alechina, N.: Embedding coalition logic in the minimal normal multimodal logic with intersection. In: Ju, S., Liu, H., Ono, H. (eds.) Modality, Semantics and Interpretations. LASLL, pp. 1–22. Springer, Heidelberg (2015). https://doi.org/10.1007/978-3-662-47197-5_1
2. Ågotnes, T., Wáng, Y.N.: Group belief. In: Dastani, M., Dong, H.,van der Torre, L. (eds.) Logic and Argumentation, pp. 3–21. Springer, Cham (2020)
3. Baader, F., Calvanese, D., McGuinness, D.L., Nardi, D., Patel-Schneider, P.F.: The Description Logic Handbook, 2nd edn. Cambridge University Press, Cambridge (2017)
4. Baader, F., Horrocks, I., Lutz, C., Sattler, U.: An Introduction to Description Logic. Cambridge University Press, Cambridge (2017)
5. Balbiani, P.: Axiomatization of logics based on Kripke models with relative accessibility relations. In: Orłowska, E. (ed.) Incomplete Information: Rough Set Analysis, pp. 553–578. Physica-Verlag HD, Heidelberg (1998). https://doi.org/10.1007/978-3-7908-1888-8_17
6. Balbiani, P.: Eliminating unorthodox derivation rules in an axiom system for iteration-free PDL with intersection. Fundamenta Informaticae **56**(3), 211–242 (2003)
7. Balbiani, P., Vakarelov, D.: PDL with intersection of programs: a complete axiomatization. J. Appl. Non-classical Logics **13**(3–4), 231–276 (2003)
8. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic. Cambridge University Press, Cambridge (2001)
9. Chellas, B.F.: Modal Logic: An Introduction. Cambridge University Press, Cambridge (1980)
10. Fagin, R., Halpern, J.Y., Vardi, M.Y.: What can machines know? On the properties of knowledge in distributed systems. J. ACM **39**(2), 328–376 (1992)
11. Fagin, R., Halpern, J.Y., Moses, Y., Vardi, M.Y.: Reasoning about Knowledge. The MIT Press, Cambridge (1995)

12. Halpern, J.Y., Moses, Y.: A guide to completeness and complexity for modal logics of knowledge and belief. Artif. Intell. **54**(3), 319–379 (1992)
13. Harel, D., Kozen, D., Tiuryn, J.: Dynamic Logic. The MIT Press, Cambridge (2000)
14. van der Hoek, W., Meyer, J.J.C.: Making some issues of implicit knowledge explicit. Int. J. Found. Comput. Sci. **3**(2), 193–224 (1992)
15. Meyer, J.J.C., van der Hoek, W.: Epistemic Logic for AI and Computer Science. Cambridge University Press, Cambridge (1995)
16. Sahlqvist, H.: Completeness and correspondence in the first and second order semantics for modal logic. In: Kanger, S. (ed.) Proceedings of the Third Scandinavian Logic Symposium, Studies in Logic and the Foundations of Mathematics, vol. 82, pp. 110–143. Elsevier (1975)
17. Wáng, Y.N.: Logical Dynamics of Group Knowledge and Subset Spaces. Ph.D. thesis, University of Bergen (2013)
18. Wáng, Y.N., Ågotnes, T.: Public announcement logic with distributed knowledge: expressivity, completeness and complexity. Synthese **190**, 135–162 (2013). https://doi.org/10.1007/s11229-012-0243-3
19. Wáng, Y.N., Ågotnes, T.: Relativized common knowledge for dynamic epistemic logic. J. Appl. Logic **13**(3), 370–393 (2015)