Research in Mathematics Education

Series Editors: Jinfa Cai · James A. Middleton

Charles Hohensee
Joanne Lobato *Editors*

Transfer of Learning

Progressive Perspectives for Mathematics Education and Related Fields

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Research in Mathematics Education

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Editors

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Progressive Perspectives for Mathematics Education and Related Fields

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Foreword

Helping learners develop understanding and skill in one context and then even store them, let alone apply them in another context, has been an enduring goal for decades. Recently, however, the study of transfer has transformed, becoming increasingly rigorous and useful for the improvement of mathematics learning experiences. This is one of the books in the *Research in Mathematics Education* series. Charles Hohensee and Joanne Lobato, the editors of this volume, provide us with a comprehensive look at transfer in mathematics education.

This is the first book in mathematics education research that addresses transfer. The chapters cover diverse approaches ranging from embodied cognition to more conventional assessment of near and far transfer and to sociocultural approaches examining the interaction of tools, goals, and actors in classroom contexts. Philosophically, this volume is eclectic. Transfer of learning is seen by the collective of authors as too important to pigeonhole into a single, narrow perspective. That is one of the delights of this book: If one can somehow utilize knowledge or practices learned in one place and time in another place and time, that is transfer. How transfer occurs, what aspects of a learning situation are transferable, and under what conditions teachers or curriculum designers may impact transfer are questions that each of the authors deals with from within their own theoretical framework. Six different but overlapping traditions interweave throughout the chapters, sometimes competing and sometimes complementing each other.

The extended discussions of transfer between mathematics and other science, technology, engineering, and mathematics (STEM) subject matter, we feel, will be of special interest to researchers and practitioners. The work presented here can guide the simultaneous design and planning of learning experiences in K-12 STEM courses. Additionally, the "so what" question regarding transfer is effectively addressed in this volume through several chapters examining transfer to and from out-of-school settings. This is a unique contribution to mathematics education at this time, marking this volume as a key resource for researchers and practitioners who seek to understand what about school mathematics is not only applicable but is actually applied by learners in their own lives beyond the bounds of their mathematics classroom.

Our intent for this series is to publish the latest research in the field in a timely fashion. This design is particularly geared towards highlighting the work of promising graduate students and junior faculty working in conjunction with senior scholars. The audience for this monograph series consists of those at the intersection of researchers and mathematics education leaders—people who need the highest quality research, methodological rigor, and potentially transformative implications ready at hand to help them make decisions regarding the improvement of teaching, learning, policy, and practice. With this vision, our mission for this book series is:

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We are grateful for the support of Melissa James from Springer in developing and publishing this book series as well as supporting the publication of this volume.
We thank the editors (Hohensee and Lobato) and all of the authors who have contributed to this innovative and comprehensive volume!

University of Delaware, Newark, DE, USA Jinfa Cai
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Preface

With this book, we have aggregated a number of progressive perspectives on the transfer of learning in the context of mathematics education and related fields. The book is part of Springer's growing Research in Mathematics Education monograph series, which is composed of thematic volumes of peer-reviewed, high-quality contributions on timely topics.

The publication of this book is particularly timely because, over the past 20 years, a new generation of transfer researchers have emerged that have been developing progressive perspectives and using them to frame empirical studies in STEM education research. The development of these progressive perspectives was in reaction to the rash of criticism of traditional transfer research. The progressive perspectives represented in the chapters of this book implicitly and explicitly address many of those criticisms.

A number of factors motivated us to embark on this edited volume on the transfer of learning. First, despite the negative critiques of traditional transfer research, we view the underlying phenomenon of transfer to be of critical importance for mathematics teaching and learning. Second, we perceived a need to bring together into a single volume recent efforts from researchers whose work could usefully inform future directions for transfer research in the domain of mathematics education. Third, we felt the time was right to bring together interdisciplinary contributions with links to mathematics education as a way to stimulate dialogue about transfer across disciplines.

It is our hope that the chapters in this book will be useful to those researchers who principally focus on transfer, as well as to those who do not typically focus on transfer but who find ideas contained in these chapters relevant to their work. To that end, we have tried to achieve a balance between theoretical chapters and those that are empirically based. We have also included authors from many different countries in order to provide an intriguing range of perspectives. Thus, we feel the book is well positioned to generate new and renewed excitement for transfer research and to motivate the field of mathematics education to focus more efforts on understanding this enduring and important topic.

Newark, DE, USA — Charles Hohensee
San Diego, CA, USA — Joanne Lobato

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About the Authors

Charles Hohensee is an Associate Professor in the School of Education at the University of Delaware. He received his Ph.D. in Mathematics Education from San Diego State University in 2011. Hohensee's research examines the phenomenon of *backward transfer*, which is about how learning something new influences a learner's prior knowledge. Hohensee has published a number of articles on transfer and backward transfer.

Joanne Lobato is a Professor of Mathematics Education in the Department of Mathematics and Statistics at San Diego State University. Lobato has a long-standing interest in the transfer of learning, having developed the actor-oriented transfer perspective and served as the transfer strand editor for the *Journal of the Learning Sciences*. She also conducts research on learning from dialogic videos, student noticing, and relationships between teaching practices and student learning. Lobato is an Associate Editor for *Mathematical Thinking and Learning*.

Part I
Progressive Theoretical Perspectives of Transfer

Chapter 1
Current Conceptualizations of the Transfer of Learning and Their Use in STEM Education Research

Joanne Lobato and Charles Hohensee

> We believe that the metaphor underlying transfer—namely, of transporting knowledge from one concrete situation to another—is fundamentally flawed… Our goal is to recommend not an "improved version" of transfer, but rather the abandonment altogether of "transfer" as a view of how learning takes place. (Carraher & Schliemann, 2002, p. 20)

> We believe that the distinction between acquiring knowledge and applying it [transfer] is inappropriate for education. (Hiebert et al., 1996, p. 14)

> A persistent follower of the PM [participation metaphor] must realize, sooner or later, that from a purely analytical point of view, the metaphorical message of the notion of transfer does not fit into PM-generated conceptual frameworks. (Sfard, 1998, p. 9)

As these epigraphs illustrate, 20–25 years ago, mathematics education research largely turned away from transfer as a viable conceptual construct, and consequently, away from conducting and publishing transfer studies. In contrast, in the past 10 years, there has been a marked upsurge in publications on the transfer of learning in mathematics education research specifically and STEM education research more broadly. Such studies have been grounded in progressive perspectives on transfer rather than in the traditional perspective. This chapter begins with a brief account of this evolution, from rejection of the traditional transfer approach to the development and use of progressive transfer perspectives. In the main body of the chapter, we present the key features of six progressive perspectives on the transfer of learning, using examples of their recent use in STEM education research. Finally, we end with a discussion of the motivation for and organization of this book.

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1.1 The Emergence of Progressive Transfer Perspectives

1.1.1 Traditional Transfer Perspective and Critiques

By the traditional transfer perspective, we refer broadly to the family of approaches that emerged during the cognitive revolution of the last half of the twentieth century and came to dominate transfer research (e.g., by Bassok & Holyoak, 1993; Gentner, 1983, 1989; Ross, 1984; Singley & Anderson, 1989). Although different strands exist within this perspective, they share multiple features. First, transfer is defined as the application of knowledge or skills learned in one situation to a new or varied context (Alexander & Murphy, 1999; Bransford, Brown, & Cocking, 2000). Second, the formation of sufficiently abstract representations is a necessary condition for transfer, where abstraction is conceived as a process of decontextualization (Fuchs et al., 2003; Gentner, 1983; Reeves & Weisberg, 1994). Third, the occurrence of transfer is attributed to the psychological invariance of symbolic mental representations. Specifically, transfer occurs if the representations that people construct of initial learning and transfer situations are identical, overlap, or can be related via mapping (Anderson, Corbett, Koedinger, & Pelletier, 1995; Gentner, Loewenstein, & Thompson, 2003; Gick & Holyoak, 1983, 1987; Novick, 1988; Reed, 1993; Sternberg & Frensch, 1993).

Methodologically, traditional transfer studies typically present subjects with a sequence of tasks that share some structural features (e.g., a common solution approach or shared principle) but have different surface forms (e.g., different word problem contexts), according to an expert's knowledge of the topic. Subjects are then taught some solution strategy, principle, or procedure with the initial learning task. If the subjects perform better on a transfer task than a control group (who receive the transfer task but no learning tasks), then transfer is said to have occurred (Singley & Anderson, 1989). Some researchers have made adaptations to this basic approach by using multiple measures to capture the transfer of learning (e.g., Chen & Klahr, 1999) or verbal protocol methods to examine solution procedures (e.g., Bassok & Holyok, 1989; Nokes, 2009), though, according to Novick (1988), most traditional transfer studies rely primarily on performance measures. [For a more nuanced discussion of differences among cognitivist perspectives and a historical account of the linkages between cognitivist perspectives and Thorndike's (1906) associationist transfer theory of common elements see Cox (1997) and Lobato (2006, 2012).]

The traditional transfer perspective encountered a rash of criticism beginning in the mid-1980s as situated cognition and socio-cultural perspectives on learning became popular (Gruber, Law, Mandl, & Renkl, 1996; Laboratory of Comparative Human Cognition, 1983; Lave, 1988; Lerman, 1999; Packer, 2001). We briefly review three critiques of the theoretical and methodological roots of transfer. First, the traditional transfer perspective is rooted in a conception of knowledge as tools that can be acquired in one situation and transported unchanged to another situation (Greeno, 1997; Packer, 2001). The tools are assumed to be independent of the situations in which they are used. As Lave (1988) put it, "the beneficial cognitive consequences of decontextualized learning, freeing oneself from experience" are seen as "a condition for generalization about experience" (p. 41). However, from a situated perspective, the notion of detaching from concrete experience is problematic because knowledge cannot be isolated from practice and meaningfully studied (Hall, 1996; van Oers, 1998). Second, the focus on the invariance of mental representations as a transfer mechanism is severely limited by ignoring the contribution of the environment, artifacts, and other people to the organization and support of the generalization of learning (Beach, 1995, 1999; Guberman & Greenfield, 1991; Pea, 1987). Finally, traditional transfer studies privilege the perspective of the observer and rely on models of expert performance, accepting as evidence of transfer only specific correspondences defined a priori as being the "right" mappings (Evans, 1998; Lobato, 2003). Consequently, transfer experiments can become what Lave (1988) called an "unnatural, laboratory game in which the task becomes to get the subject to match the experimenter's expectations," rather than an investigation of the "processes employed as people naturally bring their knowledge to bear on novel problems" (p. 20).

1.1.2 Response to Critiques in STEM Education Research

In the wake of these critiques, transfer fell out as an important area of research in mathematics education. Carraher and Schliemann (2002) advocated abandoning transfer as a research construct because of the deep association of transfer with what they considered faulty conceptual roots. Lave, a social anthropologist, whose work extended to mathematics education, also recommended moving away from the transfer construct. For example, in a study of the mathematics used by adult grocery shoppers, Lave (1988) concluded that the shoppers did not transfer relevant school mathematics. Although she acknowledged the existence of "continuity of activity across situations," she quickly added that "learning transfer is not the central source of continuity" (p. 186). Other researchers adopted the view that learning and transfer are conceptually indistinguishable, thus negating the need to devote special attention to transfer (e.g., Campione, Shapiro, & Brown, 1995; Hammer, Elby, Scherr, & Redish, 2005).

However, the underlying phenomenon that was narrowly and imperfectly captured by the construct of transfer remains important in mathematics teaching and learning. For example, whenever math teachers are faced with the task of constructing an exam, they have to make decisions about whether to repeat tasks presented in the instructional unit or whether it's "fair" to include novel tasks—a decision that seems to draw upon assumptions about transfer, not just learning. Similarly, researchers conducting an evaluation of an innovative instructional treatment need to decide how closely to pair assessment items with instructional activities. Researchers operating from a Realistic Mathematics Education perspective may wonder how activities grounded in real-world contexts transfer to abstract domains (e.g., Stephan & Akyuz, 2012). Even in critiques of transfer, researchers acknowledge that "any new conceptualization—thus, any learning—is only possible thanks to our ability to transfer existing conceptual schemes into new contexts" (Sfard, 1998, p. 9). To resolve this tension between the avoidance of transfer and the necessity of transfer, Lerman (2000), in his work on the "social turn" in mathematics education research, argued that "the notion of transfer of knowledge, present as decontextualized mental objects in the minds of individuals, from one situation to another, becomes untenable but at the very least requires reformulation" (p. 25). We turn next to the development of a number of such reformulations of transfer.

1.1.3 Development and Uptake of Progressive Perspectives

From 1993 to 2006, several progressive perspectives on the transfer of learning emerged. In the next section of this chapter, we present the following six theoretical perspectives: (a) *preparation for future learning* (Bransford & Schwartz,1999); (b) *actor-oriented transfer* (Lobato, 1996), (c) *transfer in pieces* (Wagner, 2006); (d) *expansive framing* (Engle, 2006); (e) *consequential transitions* (Beach, 1999); and (f) an *activity-theoretic perspective* (Tuomi-Gröhn & Engeström, 2003). Another notable contribution is the reformulation of transfer from the lens of situated cognition, developed by James Greeno, referred to as the affordances-and-constraints perspective (Greeno, 1997; Greeno, Smith, & Moore, 1993). Although this approach was never fully developed, and Greeno later shifted from using the term "transfer" to "productivity" (Hatano & Greeno, 1999, p. 647), his significant contributions influenced the development of the actor-oriented transfer perspective and the expansive-framing perspective. During this same time period, The National Science Foundation funded two transfer conferences: the first supported by the Social, Behavioral, and Economic Sciences Directorate (Mestre, 2003), and the second supported by the Education and Human Resources Directorate (Lobato, 2004). Thus, when the *Journal of the Learning Sciences* sponsored a transfer strand in 2006, the time seemed ripe to attract empirical papers grounded in progressive perspectives on transfer and theoretical papers that further developed alternative approaches to transfer. The first author of this chapter, who served as the strand editor, was surprised to find that few empirical studies using the emerging progressive perspectives were submitted, while other submissions were grounded unquestioningly in the traditional transfer perspective.

Three factors likely contributed to what seemed like a slow proliferation of ideas from progressive perspectives on transfer. First, reformulating transfer is not simply a matter of offering a new definition of transfer. A network of related constructs need to be re-imagined. Engle (2012), arguing that by 2012 the field was seeing a resurgence of transfer research, attributed that resurgence to: (a) the treatment of transfer as a complex, multifaceted social and cognitive phenomenon, rather than a simple, unitary construct, (b) the articulation of new processes that mediate transfer, and (c) a shift in perspective from expert models to an understanding of the "diverse and often unanticipated ways in which students make use of prior learning" (p. 348).

Second, because the traditional transfer perspective was solidly rooted in information processing, and progressive perspectives largely emerged from situated and socio-cultural perspectives, there were associated difficulties, resistance, and misunderstandings that often result from changing well-established constructs. This can be seen in a lively exchange published by the *Educational Researcher* between advocates of the traditional transfer perspective (Anderson, Reder, & Simon, 1996, 1997) and an advocate of a situated perspective on transfer (Greeno, 1997). Specifically, Anderson et al. (1997) casually dismissed any discrepancies between the two approaches as differences in form and not substance rather than acknowledging that each held a different set of theoretical assumptions and commitments. Finally, while the methods used in the traditional transfer perspective were well established, methods appropriate for progressive perspectives had to be formulated (e.g., Lobato, 2008a; Schwartz & Martin, 2004).

In the past 10 years, the situation has changed. There has been a marked upsurge in publications on the transfer of learning in math education research specifically and STEM education research more broadly. We conducted an informal search for articles published between 2008 and 2019 in mathematics education journals (with a less thorough search in science education journals) that were grounded in progressive perspectives on the transfer of learning. Even with this non-comprehensive search, we found 65 articles, published by a variety of STEM education researchers. We concluded that something had shifted in the field. Perhaps adequate time had finally passed for progressive transfer perspectives to be developed sufficiently for wider implementation. We turn next to a presentation of the six major progressive perspectives that we found in these articles, with illustrations of their use from a subset of the 65 articles.

1.2 Six Progressive Perspectives Used in STEM Education

1.2.1 Preparation for Future Learning

The *preparation for future learning* (PFL) perspective on transfer (developed by Bransford & Schwartz, 1999) responds to the critique that the traditional transfer approach ignores real-world conditions that people can often exploit, such as seeking additional learning resources and having opportunities to obtain feedback. Traditional tests for transfer typically take place in environments where people do not have access to information sources other than what they have learned previously. In contrast, the PFL approach examines how an instructional experience (such as investigating a set of contrasting cases) prepares people to benefit from a learning opportunity. In articulating the PFL perspective, Bransford and Schwartz (1999) point to a study by Singley and Anderson in which there appeared to be no transfer from learning one text editor to another, using a traditional test of transfer. However, the benefits of the prior experiences with a text editor were evident several days into learning the second program. In sum, the transfer of prior knowledge may not be apparent until people have been given the opportunity to learn new information.

Key Features Schwartz and colleagues have developed a methodological approach utilized in PFL studies, which they call the *double transfer paradigm* (Schwartz, Bransford, & Sears, 2005; Schwartz & Martin, 2004). Students are assigned one of two instructional treatments. One of the treatments is conceived as a preparatory activity and may focus on inventing a method during problem solving (Schwartz & Martin, 2004), using contrasting cases (Roelle & Berthold, 2015), or having a hands-on experience (for instance, in a science museum; Watson, 2010). The other treatment (which serves as a control) is usually a more traditional teaching experience (such as lecture followed by practice). Half of the students from both treatments are then given access to an additional learning resource, such as a sample worked problem or a lecture, followed by a request to solve a target transfer problem. The other half of the students in each treatment solve the target transfer problem directly without access to the learning resource. The researchers then look both at what people *transfer in* from the instructional treatment to learn from the resource and what they *transfer out* to solve a target problem.

For example, Schwartz and Martin (2004) used the double transfer paradigm with Grade 9 Algebra 1 students learning about the statistics concept of standardization. The students were assigned to two treatments—*invention* versus *tell-and-practice*. Students in the invention treatment engaged in problem solving to invent their own ways to compare two exceptional scores from different distributions and decide which was better. The tell-and-practice group was taught a visual method for determining standardization and then asked to use that method on a practice task. Half of the students from each treatment group were given the common learning resource of a worked example for a task from the targeted domain. Then all students were given a transfer task. The results showed that the students from the invention treatment, who also received the learning resource, were the only group to perform well on the transfer task. This is despite the fact that the students struggled with the invention activity and did not complete it successfully. Additionally, the students from both treatment groups performed about the same on the transfer task, when they did not have access to the learning resource. Schwartz and Martin (2004) hypothesized that the students in the invention treatment were more likely to notice important dimensions of the standardization concept (such as range and number of observations) than the students in the tell-and-practice treatment and then use this knowledge to learn more deeply from the worked example.

Purpose and Uses Although not all PFL studies utilize the double transfer methodological design, many focus on the nature of the preparatory activity, the transferring in to the common learning resource, and the links between the two experiences. For example, Vahey and colleagues extended the PFL approach to design a series of interdisciplinary experiences for middle school students related to the targeted mathematical content of proportional reasoning (Swan et al., 2013; Vahey et al., 2012). Students first engaged with a complex, real-world water allocation problem involving countries from the Middle East in their social studies class, before receiving more formal introduction to proportions in math class, followed by opportunities to transfer out that knowledge to activities in their science and language arts classes. While the preparatory water allocation problem was messy and frustrating for students, it appeared to direct their attention to key dimensions of the situation (such as the importance of attending to more than one quantity when making decisions in a proportional situation), which then shaped what was learned about proportionality in the math classroom.

In a second example, this one with U.S. prospective math teachers, Jacobson (2017) drew upon the PFL perspective to compare different types of early field experiences on the common learning resource of teacher education coursework. *Instruction-focused* field experiences included opportunities for prospective teachers to teach, whereas *exploration-focused* field experiences focused on observing or interviewing students but did not include teaching. Participating in early, instruction-focused field experiences was positively related to outcome measures for the teacher education courses (i.e., mathematical knowledge for teaching and beliefs about active-learning and math-as-inquiry), which was not the case for exploration-focused field experiences. Jacobson concluded,

> Rather than being merely a context for practicing what has already been learned, field experience—especially early instruction-focused field experience—may prepare prospective teachers to learn mathematics and develop beliefs about mathematics (i.e., gain applicative knowledge) from learning opportunities such as concurrent and subsequent university coursework and from the resources available during student teaching. (p. 181)

1.2.2 Actor-Oriented Transfer Perspective

From the *actor-oriented transfer* (AOT) perspective, the conceptualization of transfer shifts from what MacKay (1969) calls an *observer's* (expert's) viewpoint to an *actor's* (learner's) viewpoint (Lobato, 2003). By adopting an actor's perspective on transfer, one seeks to understand the ways in which people generalize their learning experiences beyond the conditions of initial learning, by looking for evidence of the influence of prior experiences on actors' activity in novel situations, rather than predetermining what counts as transfer using models of expert performance (Lobato, 2012). A researcher operating from the AOT perspective does not measure transfer against a particular cognitive or behavioral target but rather investigates instances in which the students' prior experiences shape their activity in the transfer situation, even if the result is non-normative or incorrect performance. Consequently, several studies have demonstrated instances in which students provided little or no evidence of transfer from a traditional perspective; however, when the data were re-analyzed from an AOT perspective, evidence was found that students had constructed relationships between previous learning activities and new situations, and that these perceived relationships influenced students' engagement in the new situations (Cui, 2006; Karakok, 2009; Lobato, 2008b; Thompson, 2011).

Key Features Because AOT research assumes that people regularly generalize their learning experiences, the research question shifts from whether or not transfer occurred to an investigation of the interpretative nature of the connections that people construct between learning and transfer situations, guided by aspects of the situations that they find personally salient (Lobato, 2008a). Consequently, the research methods are typically qualitative in nature, drawing upon interview or observational data and using coding methods that identify the personal, and often surprising, interpretations and connections constructed by actors (Lobato & Siebert, 2002). For example, Roorda, Vos, and Goedhart (2015) conducted a 2-year longitudinal study of high school students' transfer of learning experiences related to instantaneous rates of change from both mathematics and physics classes to novel tasks in a series of interviews. Their analysis identified the particular ideas, language, and procedures from the math and physics classes that appeared to influence the students' work on the interview tasks. Similarly, Nagle, Casey, and Moore-Russo (2017) revealed the specific ways in which Grade 8 students connected their ideas about slope and covariational reasoning to novel statistics tasks in which they were asked to place the line of best fit informally.

In moving to explanatory accounts of the occurrence of transfer, the AOT perspective treats transfer as a distributed phenomenon across individual cognition, social interactions, material resources, and normed practices. For example, in our own work, we posited *noticing* as a multi-faceted transfer process (Lobato, Hohensee, & Rhodehamel, 2013; Lobato, Rhodehamel, & Hohensee, 2012). Specifically, we offered an explanatory account of the occurrence of transfer in a classroom-based study by coordinating the particular mathematical features that individuals attended to, with the social organization of that noticing through discourse practices and the nature of mathematical activity.

Purpose and Uses The AOT perspective is particularly useful within the context of design-based research, where information about the often surprising ways in which people generalize their learning experiences and interpret transfer situations, can usefully inform and improve the design of the instructional environment (Lobato, 2003, 2008a). For example, Johnson, McClintock, and Hornbein (2017) designed two dynamic computer environments to explore the transfer of covariational reasoning from activities set in a Ferris wheel context to a bottle-filling context. Their investigation revealed the transfer of covariational reasoning involving quantities that the students conceived as measureable. It also illuminated the increased complexity of the bottle-filling context, namely that students could perceive that liquid was accumulating in a container without conceiving of an attribute in the situation that could be measured. In turn, the information that was gained informed subsequent design and instructional responses, as indicated in the follow-up chapter by Johnson and colleagues in this volume.

The AOT perspective was originally developed to model students' generalizations of their subject-matter learning experiences in school or design-based research instructional sessions. It has been extended in several ways, including the investigation of task-to-task transfer via written problem-solving activities outside of school (Mamolo & Zazkis, 2012) and teaching interviews (Lockwood, 2011). The AOT perspective has also been used in research on teachers. For example, Penuel, Phillips, and Harris (2014) examined teachers' curriculum implementation through an AOT lens. The analysis focused on the teachers' differing interpretations of the goals and guidance embedded in the materials for a curricular unit and how those perceptions were related to patterns of enactment. Similarly, Sinha et al. (2013) examined how a group of elementary teachers tackled new curricular units in their school after working with a research team on an initial reform-oriented unit.

1.2.3 Transfer-in-Pieces Perspective

The *transfer-in-pieces* perspective is a progressive perspective on transfer attributed to Joseph Wagner (2006, 2010). According to Wagner (2006), transfer is conceptualized as "the incremental growth, systematization, and organization of knowledge resources that only gradually extend the span of situations in which a concept is perceived as applicable" (p. 10). This incremental-growth perspective on transfer is progressive because it contrasts with the traditional view that transfer is the "all-or-nothing transportation of an abstract knowledge structure across situations" (p. 10).

Central to this perspective is the notion of *concept projections* (diSessa & Wagner, 2005), which are particular knowledge resources that allow the knower to interpret a situation's affordances in a meaningful way. For example, a concept projection that young children may have is to recognize situations that involve equal sharing as being about division. A second concept projection is to recognize situations that involve removing equal-sized groups as being about division as well. Forming and connecting concept projections allows an individual to see the "same thing" across multiple problems (in this case division), which counts as transfer from this perspective and results in the individual developing a more robust generalizable concept. That is, coming to recognize a concept in different contextual situations is a form of transfer that depends upon the individual connecting multiple concept projections (Wagner, 2010).

Key Features To explain key features of the transfer-in-pieces perspective on transfer, we first must describe features of the *knowledge-in-pieces framework* for how knowledge develops (diSessa, 1993), because it is on that framework that the transfer-in-pieces perspective is based. Then, we explain why those features are relevant to conceptualizing transfer.

Knowledge-in-Pieces Framework A core principle underlying the knowledge-in-pieces framework, which was initially developed through science education research, is that understandings of concepts are fundamentally based on the ways individuals derive information from the world (diSessa, 1993). For example, how well a student understands linear functions will be largely determined by the ways the student gathers information from the world about dependency relationships, rates, speed, steepness, and so on. Moreover, the origins of knowledge are based on intuitive, unsystematically-collected information from the world, and individuals' knowledge advances as they develop more systematic ways to derive that information.

According to diSessa and Sherin (1998), two important interrelated knowledge resources work together to derive and interpret information from the world, namely *readout strategies* and *causal nets*. Readout strategies refer to the set of strategies that individuals employ to determine what to focus on and subsequently notice about the world (i.e., what to notice about a particular perceptual or conceptual field). Causal nets then refer to the set of inferences individuals can make about the information collected by the readout strategies. In other words, readout strategies are used to gather information whereas causal nets are used to interpret that information. As these knowledge resources become more systematic, the associated knowledge develops.

Applying Knowledge-in-Pieces to Transfer-in-Pieces Wagner's (2006) conceptualization of transfer was based on the ideas described above. Specifically, Wagner argued that readout strategies and causal nets are processes that individuals use, not only to gather information about the world, but also to make decisions about when transfer is appropriate. When a person encounters a new context in the pursuit of particular goals, readout strategies will guide what gets attended to and noticed in the new situation, causal nets will be used to make inferences about what was noticed, and knowledge resources that were useful in prior activities related to those goals will become available. As readout strategies and causal nets become more systematic and organized, transfer of particular knowledge is more likely to occur in a greater span of novel contexts.

Purpose and Uses One purpose of Wagner's (2006) progressive perspective is to address the apparent contradiction that instances of transfer are rare in empirical studies conducted from a traditional perspective (Detterman, 1993); yet it is widely held that transfer is pervasive in everyday life (Brown, 1989). From Wagner's transfer-in-pieces perspective, the reason transfer has been difficult to find empirically is because researchers aligned with the traditional perspective mistakenly look only for an all-at-once phenomenon. In contrast, research guided by a transfer-in-pieces perspective looks "for incremental indications of transfer" (p. 40), and "trace[s] the development" (p. 13) of transfer.

A second purpose of Wagner's perspective on transfer is to offer a new way to conceptualize the mechanisms underlying transfer. The traditional perspective "locate[s] the mechanism of transfer in the construction or induction of schemata represented at appropriate levels of abstraction" (Wagner, 2006, p. 64). However, Wagner (2006) presented a case study in which a student's ability to articulate an abstraction came after, rather than before, he transferred his knowledge to a new context. Therefore, constructing an abstraction cannot be solely driving transfer. Instead, Wagner explained the mechanisms of transfer in terms of readout strategies and causal nets. Specifically, Wagner (2006) used the same case study to track the incremental development of knowledge resources that enabled the undergraduate student to gradually transfer his developing knowledge of the law of large numbers to a wider array of contexts. As described by Wagner:

> [The student] took different ideas initially applicable only in isolated contexts …. The isolated contexts to which they applied individually grew incrementally into a larger family of situations perceived by [the student] to be alike, in that they all offered affordances for the ideas in the common frame. (p. 63)

Had this study been conducted using a traditional perspective, transfer would likely not have been observed because it happened gradually, rather than all at once.

1.2.4 Expansive Framing

The *expansive-framing* perspective on transfer, attributed to Randi Engle responds to the critique that the focus on cognitive mechanisms from a traditional transfer perspective has failed to acknowledge the contribution of social interactions, language, and cultural artifacts, to the occurrence of transfer (Engle, 2006; Engle, Lam, Meyer, & Nix, 2012; Engle, Nguyen, & Mendelson, 2011). The construct of *framing*, first offered by Bateson (1955/ 1972) and later developed by Goffman (1974), refers to what sense participants have of the nature of a given activity. For example, a lesson on quadratic functions may be framed as something useful only for the next exam, or it may be framed as being useful for understanding real-world phenomena, such as the acceleration of a car. Engle referred to the latter as an example of *expansive framing* and advanced the hypothesis that transfer is more likely to occur to the extent that learning and transfer contexts have been framed to create intercontextuality. When a high degree of intercontextuality occurs, the content established during learning is considered relevant to the likely transfer situations.

Key Features Engle et al. (2011) offered a framework of five types of expansive framing that are productive for transfer. The first three types focus on different aspects of the setting—time, place, and participants. The first type refers to the framing of a learning activity as being temporally connected with ongoing or future activity (versus being an isolated event). Second, lessons can be framed as being relevant to activity that occurs in other places, such as in a profession. Third, the learning activities can be framed as being relevant to a larger community beyond the classroom. The fourth type of expansive framing involves the topic that is being learned. The content of individual lessons can be framed as being connected to each other and part of a larger whole (e.g., graphs, equations and tables framed as representations of functions). Finally, the last type of framing involves how participants are positioned relative to the creation of knowledge in the field. In expansive framing, students are positioned as being capable of authoring their own ideas and are asked to revoice and credit other students with authorship (rather than framing explanation and revoicing as elaboration only of the textbook's ideas). Research from the expansive-framing perspective not only identifies teacher actions or features of instructional materials but also the aspects of expansiveness that appear to be appropriated or perceived by students (Lam, Mendelson, Meyer, & Goldwasser, 2014).

To test their hypotheses about the relationship between expansive framing and transfer, Engle et al. (2011) designed a tutoring experiment with two framing conditions (expansive versus bounded) using 28 high school biology students. Each student participated individually in 3–4 hours of tutoring on the cardiovascular system over two sessions, preceded by a pre-test and followed by a survey (to assess how students perceived the framing) and a post-test with transfer tasks about the respiratory system. The expansive-framing treatment attempted to address all 5 types of expansive framing. According to the survey, students generally perceived the intended differences in framing by condition, with the framing of time and authorship role being the most salient to them. On the measures of transfer, the students in the expansive-framing condition were more likely to transfer facts, a conceptual principle (the differential pressure principle), and a strategy (drawing diagrams) than those in the bounded condition.

Purpose and Uses The expansive-framing perspective first emerged in response to the inadequacy of traditional transfer processes to account fully for instances of transfer in a particular classroom setting. Specifically, Engle (2006) initially attempted to explain the observation that a group of fifth-grade students transferred graded and multi-causal arguments from a learning context (i.e., explaining whale endangerment) to a novel context (i.e., explaining the endangerment of another species) through cognitive modeling. She found that analogical mapping and the construction of abstract mental representations explained some but not all of the transfer findings. That is when she turned to framing.

Since that time, the expansive-framing perspective has been extended in at least three ways. Becherer (2015) used qualitative, rather than quantitative, methods to relate differing framing moves across two classrooms to different types of transfer (routine versus adaptive). Hickey, Chartrand, and Andrews (2020) built upon expansive framing to generate an assessment framework that embeds expansively-framed engagement within multiple levels of increasingly formal assessments. In contrast, Zuiker (2014) combined Beach's (1999) conception that transfer is about making transitions with Engle's transfer process of expansive framing.

1.2.5 Consequential-Transitions Perspective

The *consequential-transitions* perspective is a progressive conceptualization of transfer that originated with King Beach (1999). Instead of the traditional conceptualization that transfer is the use of prior knowledge to solve novel problems, Beach reconceptualized transfer more broadly as when individuals are faced with making *transitions* to accommodate changing relations between themselves and social activities. According to this perspective, transfer is described as the "continuity and transformation of knowledge, skill, and identity across various forms of social organization" and as involving "multiple interrelated processes rather than a single general procedure" (p. 112). Beach viewed these transitions as *consequential* to the individual because they may involve struggle and affect one's social position. An example of a consequential transition would be when students are faced with learning about algebra after years of learning arithmetic. Although Beach described transfer of learning in terms of consequential transitions, he also viewed consequential transitions as encompassing generalization that extends beyond the transfer of learning.

Key Features There are four types of consequential transitions, (a) lateral, (b) collateral, (c) encompassing, and (d) mediational. *Lateral consequential transitions* occur when individuals move in a single direction from one social activity to another. This type of transition is the least complex of the four types and the most closely associated with the traditional conceptualization of transfer. For example, Nepali high school students experienced a lateral transition when becoming shopkeepers (Beach, 1999). During this one-way transition (i.e., they did not subsequently return to school), the students were faced with transforming their knowledge of school mathematics for use in the practices of shopkeeping.

Collateral consequential transitions occur when individuals move back and forth between activities (i.e., these transitions are multi-directional). They are more common than lateral transitions but also more complex. For example, the Nepali shopkeepers who were living in the same village as the Nepali students described above, experienced a collateral transition when they went back to school to take adult education evening classes (Beach, 1999). In contrast to the students whose transition was in a single direction, these shopkeepers experienced a transition that moved back and forth between the mathematics activities associated with running their shops during the day and the arithmetic activities they engaged in during the evening classes.

Encompassing consequential transitions occur when individuals participate in an activity that is itself changing. This type of transition can be generational in nature (i.e., it can be particularly challenging for older generations to adapt to changes in social activities created by younger generations). For example, conventional machinists, who were accustomed to manual machining, experienced an encompassing transition when faced with having to adapt to computerized machining (Beach, 1999).

Mediational consequential transitions occur when individuals learn to participate in activities, typically educational, that simulate the actual activity. These types of transitions serve as bridges between "where the participants are currently and where they are going" (p. 118). For example, part-time actors experienced a mediational transition when attending bartending classes at a vocational school (Beach, 1999). These individuals were learning to participate in activities that approximated bartending in a restaurant. However, the activities did not constitute full-fledged bartending because, for example, the individuals were still learning to shift away from consulting written directions to make drinks.

Purpose and Uses We outline three purposes of this progressive perspective. First, Beach's consequential-transitions perspective conceives of and examines transfer as a set of interrelated psychological *and* social processes. In contrast, the traditional perspective conceives of transfer singularly as a psychological process. Second, the consequential-transitions perspective accounts for the *context* of transfer (i.e., the social activities serve as the context), whereas the traditional perspective accounts for how knowledge becomes increasingly decontextualized. Third, the consequential-transitions perspective captures the effects of transfer on individuals' identities and their social position, as well as the concomitant struggles involved. Conversely, the traditional conceptualization considers transfer in a way that ignores issues of identity and social positioning.

Two progressive transfer studies that have made use of Beach's consequential-transitions perspective are Jackson (2011) and Hohensee and Suppa (2020). Jackson used *collateral transitions* to examine a child's back and forth transition between doing mathematics at school and at home. This lens afforded an examination of transfer that foregrounded the setting and that revealed the complexities of transferring knowledge between settings. A traditional perspective would not have afforded these insights. Jackson has a follow-up chapter in this book.

Hohensee and Suppa (2020) used *encompassing transitions* as the lens. They examined prospective teachers' experiences with learning about early algebra in a teacher preparation course after the prospective teachers had already learned about regular algebra in high school. This lens was used because the prospective teachers felt as if algebra was being changed on them, and they were faced with adapting to those changes. Results revealed ways the prospective teachers struggled with making this transition.

1.2.6 Activity-Theoretic Perspective

The *activity-theoretic perspective* attributed to Yrjö Engeström (e.g., Engeström & Sannino, 2010; Tuomi-Gröhn & Engeström, 2003) is a progressive take on transfer that is rooted in activity theory (Engeström, 2001). Instead of the traditional conceptualization that transfer is an individual cognitive process (Detterman, 1993), Engeström's activity-theoretic perspective reconceptualizes transfer as a collective process that happens within social activity systems. Furthermore, according to this perspective, transfer is conceptualized as occurring on two dimensions. First, it involves expansion of a social activity within a social system, what Tuomi-Gröhn and Engeström (2003) referred to as a "transformation in collective activity systems and institutions (e.g., schools and workplaces)" (p. 30). Second, there is a proliferation of the newly expanded activity to other social activity systems, for example, by "recruiting a growing number of participants in the transformation effort" (p. 31). An essential difference between Engeström's activity-theoretic perspective and Beach's (1999) consequential transitions perspective is that the former is about organizations creating change within social systems, whereas the latter is about individuals adapting to changes within social systems.

Key Features An important feature of this activity-theoretic perspective is that transfer is a collective process that involves a cycle of seven strategic actions. These actions, in the order in which they occur, are: *questioning*, *analyzing*, *modeling*, *examining the model*, *implementing*, *consolidating and proliferating*, and *evaluating* (Tuomi-Gröhn & Engeström, 2003). The cycle begins when members of an organization *question* (or criticize, reject and/or have conflicting points of view about) an existing social practice (Engeström & Sannino, 2010). This action serves as the trigger for the transfer process. For example, in a study by Engeström (2009), students began to question why their school did not provide them access to computers during recess.

The second action the organization engages in is an *analysis* of the question. The analysis may include an examination of the origins and history of the social practice in question to identify causes, or the "inner systemic relations" of the practice to identify explanatory mechanisms (Engeström & Sannino, 2010, p. 7). For example, the teachers who were considering making computers available to students during recess, intensely debated the idea among themselves and then consulted another school that had been providing their students access to computers about how their students were interacting with the computers.

The third and fourth actions, *modeling* and *examining the model*, involve developing a representation of past and present issues raised during questioning, as well as a future vision for that practice that addresses the issues, and then making the model publicly sharable and scrutinizable. In the computers-in-school example, a subcommittee of teachers created a model for putting computers in school hallways by reconceptualizing the school as a work environment for both students and teachers (Engeström, 2009). The model was then debated among the teachers.

The final three actions, *implementing*, *consolidating/proliferating*, and *evaluating*, are respectively, when the organization puts the model into practice, when the implemented model is used to influence other social practices, and when the organization monitors and reflects upon the newly implemented ideas. It is during these three actions that the two types of transfer described previously occur. Specifically, during implementation, there is the "transfer of new models into practice," and during proliferation, there is the "transfer of local innovations and new forms of practice into other activity systems and organizations" (Tuomi-Gröhn & Engeström, 2003, p. 32). In the school example, the computers-in-hallways idea was eventually implemented as part of an effort to make the physical environment more pleasant (Engeström, 2009).

Purpose and Uses One purpose for this activity-theoretic perspective on transfer is to capture types of transfer that occur at the organizational level rather than at the individual level (i.e., "collective developmental transfer;" Tuomi-Gröhn & Engeström, 2003, p. 34). Second, this progressive perspective captures transfer in complex activity systems, such as workplaces and schools. Third, this perspective accounts for transfer that is "not triggered by an instructor giving a task to be learned … [but] when some practitioners reject the given wisdom and begin to question it" (p. 32).

Several studies in mathematics education have made use of Engeström's activity-theoretic perspective on transfer. Tomaz and David (2015) used this perspective to examine how students working on a project came to modify particular mathematical activities they had been taught regarding proportional reasoning. Tomaz and David have a follow-up chapter in this book. Additionally, FitzSimons (2003) used an activity-theoretic lens to understand an adult mathematics learner as she transferred her school-based mathematics learning across a school-home boundary to help her children with their mathematics homework.

1.3 Motivation for and Organization of This Book

We view this point in the history of transfer research as an opportune time for a book to be published on progressive perspectives on the transfer of learning. The six progressive perspectives that we reviewed in the previous section provide a well-developed foundation for additional theoretical contributions. The renewed interest in transfer research can serve as a catalyst to broaden the use of progressive transfer perspectives among mathematics education researchers, as well as among researchers in related fields, and particularly among those who might otherwise not have considered a focus on transfer.

Consequently we had three main goals when we embarked on this venture of bringing contributors together for this book. First, we wanted to provide a venue to showcase and aggregate leading-edge research on the transfer of learning from progressive perspectives. Second, we hoped to establish transfer as a valued subfield of research within mathematics and science education research. Third, we anticipated that this book could provide researchers with a foundation for forging a path for future transfer research. The collection of theoretical and empirical chapters that comprise this book represent an exciting array of progressive perspectives on transfer that could set a course for how transfer research moves forward.

The book has been organized into four parts. Part I is comprised of six chapters, including this chapter (i.e., Chaps. 1, 2, 3, 4, 5 and 6), that theoretically explore progressive perspectives on transfer. Nathan and Alibali theorize about transfer from an embodied and distributed perspective. Johnson, McClintock, and Gardner's account of transfer interweaves theories about AOT, variation (Marton, 2006), and quantitative reasoning (Thompson, 2011). Hohensee argues for theory development about an extension of AOT called backward transfer. Karakok discusses potential parallels and associations between AOT and mathematical creativity. Finally, Danish, Saleh, Gomoll, Sigley, and Hmelo-Silver use an activity-theoretic approach to theorize about how the object of students' shared activities helps determine which mathematical tools students see as applicable for new activities.

Part II is comprised of five chapters (i.e., Chaps. 7, 8, 9, 10 and 11) that examine transfer empirically as it occurs in STEM classrooms. Moore uses AOT to examine how the meanings that pre-service secondary teachers constructed for particular graphs influenced their thinking about other graphs. Lockwood and Reed also use AOT and explore the ways an undergraduate's thinking on a particular combinatorial problem influenced his thinking on other problems. Michelsen draws on the expansive-framing perspective to investigate intercontextuality between tenth-grade students' mathematics and biology classes. Tomaz and David employ Engeström's activity-theoretic perspective to consider the boundary-crossing of seventh-graders when they were studying a common topic across three content areas. Finally, Grover draws upon the PFL approach to transfer, as well as the expansive-framing perspective, to look at how middle school students learned text-based computer programming after learning block-based programming.

Part III is comprised of four chapters (i.e., Chaps. 12, 13, 14 and 15) that empirically examine transfer when it occurs, in whole or in part, outside of school settings. Jackson tracks two 10-year-old students' mathematical activities at home and in school to illustrate how conceptualizations of transfer can be informed by ethnographic accounts of learning. Pugh, Bergstrom, Olson, and Kriescher present their *transformative experience* perspective on transfer and extend it to include the idea of motivation to account for how students applied school-based learning in out-of-school contexts. Billett examines how individuals adapted what they learned in school and other social settings to occupational contexts. Finally, Triantafillou and Potari use Engeström's activity-theoretic perspective, along with objectification theory (Radford, 2008), to look at how engineering students applied school-based knowledge to their apprenticeship.

Finally, Part IV is comprised of three chapters (i.e., Chaps. 16, 17 and 18) that examine how transfer relates to teaching and researching. Diamond investigates what teachers believe about how to support students in transferring their learning. Mamolo uses the AOT lens to explore how a prospective teacher's own K–12 experiences influenced their responses during scripted role playing. Finally, Evans tackles the transferability of research findings by examining different aspects of the context in which research occurs.

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Chapter 2
An Embodied Theory of Transfer of Mathematical Learning

Mitchell J. Nathan and Martha W. Alibali

In the brief photo transcript shown in Table 2.1, below, taken from a high school engineering lesson, we encounter a critical educational challenge: In the rich sensory stream of spoken words and metaphors, written symbols, diagrams and sketches, gestures, simulations, and actions on objects, all of which occur in multiple venues such as the classroom and machine shop, how do learners perceive and construct for themselves a connected meaning of a concept such as *theta*, the angle of ascent of a projectile? The answer, we argue, depends on a theory of transfer that is *embodied*: The concept is depicted and comprehended in terms of actions, gestures, spatial metaphors, and other body-based resources; embedded in various specific physical and social settings; extended across multiple modalities, material resources and participants; and enacted through the actual or simulated interplay of perception and action among students and their teachers.

Project-based learning (PBL) environments, such as those common to problem-based and other science, technology, engineering, mathematics (STEM) education settings, offer a rich stream of activities and experiences that are intended to ground students' understanding of important mathematical ideas and to motivate the relevance of these ideas across a range of content and contexts. In so doing, success in PBL settings requires learners to construct a concept—such as THETA—and follow it across a multitude of modal forms and contexts while recognizing it as *invariant*. Understanding what is required of students to establish, perceive, maintain, and express such *invariant relations* across such environmental and perceptual variability motivates an embodied theory of transfer of mathematical knowledge.

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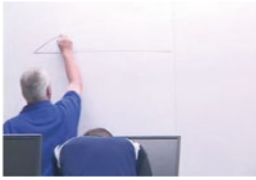
Table 2.1 Photo Transcript 1: Day 1

Teacher (to class): What happens when the, when we **project something through the air [1]**, is we end up with **something like [2] this** depending upon the angle here, which is *theta*. And, this is our range. And basically, what we have, is, we're working with vectors here. So, we end up with, some vectors that look like this and we call this, vector V_x and V … V_y. And we can say that, V_y we're gonna start with V_y here. This distance, this, right here. So, we're gonna start with, V_y, equals V_o, sine, of *theta*. Student: Mr. [Name], what's V_o?	 [1]  [2]

Note. Bold text and indices align with images

In this chapter, we first argue that the processes involved in establishing and maintaining cohesion of invariant relations during PBL are not readily described by classic accounts of transfer. We hypothesize instead that the processes involved in transfer of mathematical ideas throughout complex learning settings are necessarily embodied, and we consider the assumptions that form the basis of an embodied account of transfer as mapping of modes of perceiving and acting to achieve cohesion across contexts. We then use this embodied framework to illustrate successful and unsuccessful transfer in PBL settings. From this, we propose that transfer processes are necessarily embodied and socially mediated, in that they are grounded in the actions on and perceptions of the material world in which they are embedded and they are extended across multiple actors, typically learners and their teachers. These elements come together in an embodied theory of transfer. In the final section, we discuss the implications of embodied transfer for educational practice and identify important future areas for research.

2.1 Limitations of Classic Approaches of Transfer

Transfer can be defined as the application and extension of learned mathematical ideas beyond the context in which they were originally learned. Transfer has a long history in educational psychology (Bransford & Schwartz, 1999; Woodworth & Thorndike, 1901). Indeed, the enterprise of a liberal arts education is predicated on the notion of transfer and on the idea that learning general topics and principles will provide guidance for addressing the social and scientific issues facing the next generation of leaders, scholars, artisans, and others.

At the heart of classical models of transfer lies the notion of *common elements*, wherein the transfer of skilled performance is modeled as reapplication of previously learned actions that follow from an expert's assessment of the degree of overlap of environmental conditions that may be readily observed (near transfer) or that are only apparent at a deeper, structural level (far transfer; Singley & Anderson, 1989; Taatgen, 2013). From this theoretical perspective, abstracted, rule-like condition-action processes are antecedent to successful transfer. In this sense, perceptual richness is antithetical to transfer because it works against the formation of abstractions and their reapplication (e.g., Kaminski, Sloutsky, & Heckler, 2013).

Classical accounts of transfer fall short at explaining PBL and the learning that occurs in complex settings in several respects. First, the common elements that are the signature of classical accounts of transfer are often identified by experts, rather than generated from the learner's perspective (Lobato, 2003, 2006). Thus, it is not clear whether learners are aware of them and actually transferring on the basis of those common elements. Second, classical accounts are founded on analyses of simple stimuli, for which identifying common elements is relatively straightforward. This is not the case in many PBL settings, in which a single curriculum unit can extend over long periods of time in multiple spaces; can include many participants; and can engage a variety of objects, technological resources, and notational systems (Kozma, 2003). Third, classical accounts foreground learners' transfer processes while marginalizing (or neglecting entirely) the pedagogical processes enacted by teachers that establish the contexts in which transfer takes place and that support processes of transfer.

A primary issue for students in PBL is having a cohesive experience so that the various elements of the learning environment are experienced as connected and meaningful (Nathan, Wolfgram, Srisurichan, Walkington, & Alibali, 2017). *Cohesion* is the quality of unity or relatedness of ideas and experiences. It is commonly operationalized in terms of the degree to which ideas in a complex text are interconnected, even as one moves across clauses and sentences (McNamara, Graesser, Cai, & Kulikowich, 2011). As used here, producing cohesion refers to forming and maintaining connections among the many disparate elements of the learning environment that might otherwise serve as obstacles to transfer. For engagement and learning to take place in PBL settings, cohesion of invariant relations is paramount because ideas are presented in a variety of forms and settings. However, this process has been neglected in classical accounts of transfer.

2.2 Transfer as Embodied: Underlying Assumptions

Numerous challenges and alternatives to the classical theory of transfer have been raised, addressing the reductionist basis of transfer and the insensitivity of the classical theory to situated context (e.g., Detterman & Sternberg, 1993) and culture (e.g., Scribner & Cole, 1981). For example, the situative perspective (Greeno, Smith, & Moore, 1993) privileges participation across contexts over the reapplication of knowledge in assessing transfer. The actor-oriented transfer perspective (Lobato, 2003, 2008) considers generalized behaviors based on the agent's perspective of what is similar across familiar and novel settings. An account based on learners' "episodic feelings" integrates cognition, emotion, and bodily experiences in explaining patterns of transfer (Nemirovsky, 2011). These alternative frameworks share a view of the learner as an embedded, engaged, embodied actor and allow for the world to "seep in" to the cognitive realm that was previously theorized as isolated from the material realm.

With advancements in theories of embodied cognition, there is now a sufficient conceptual and empirical foundation for articulating an embodied theory of transfer of mathematical ideas. Indeed, we argue that an embodied perspective is necessary to account for learners' transfer in PBL settings and other complex learning environments, and it can also help explain instances of unsuccessful transfer. A related idea was presented by Goldstone, Landy, and Son (2008), who theorized that learning grounded in perception and interaction supports generation of transferable knowledge. They demonstrate successful transfer on tasks such as solving symbolic algebra equations and understanding the cross-domain application of deep principles of complex systems performance. Based on their analyses of these examples, they propose that perceptual knowledge transfers

> to new scenarios and transports across domains, most often proceeding not through acquiring and applying symbolic formalisms but rather through modifying automatically perceived similarities between scenarios by training one's perceptual interpretations. (p. 329)

This account of the role of perception and interaction in transfer is promising for a broad account of transfer to complex, collaborative, multimodal learning contexts.

There are several assumptions at the core of our embodied theory of transfer. The first assumption is that the cognitive system is a *predictive architecture*. Rather than passively waiting for input to act, humans are continually anticipating the next events in the stream of sensory input and are already poised to respond. In this sense, transfer is the default mode—no two environmental stimuli are identical, and, regardless, body states are never fixed in time. Whether transfer is deemed successful is often a function of experts' expectations for what should be transferred, rather than whether any form of transfer took place for the learner.

Second, there is reciprocity between cognitive states and actions, such that actions (arm movements performed by a student, for example) can drive the system into related cognitive states through the process of *action-cognition transduction* (Nathan, 2017; Nathan & Walkington, 2017). Transduction provides an account for how systems can operate in "forward" and "reverse" directions, a common property of many physical and biological systems. For example, with cognition driving action in the "forward" direction, a student may spontaneously extend her arms in a mathematically relevant manner to assist her in reasoning about a property of triangles. Students can also be prompted to extend their arms in either a mathematically relevant or irrelevant manner by having them touch locations on an interactive whiteboard. Nathan and colleagues (Nathan et al., 2014) investigated the hypothesis that mathematically relevant movements would drive the cognition-action system in the "reverse" direction and activate the appropriate conceptual reasoning for the task, but mathematically irrelevant movements would not assist the student. In support of this hypothesis, they found the mathematically relevant movements improved mathematical proof production, even though participants reported making no connection between the mathematics and the directed movements, and mathematically irrelevant movements did not.

Transduction recognizes that actions can drive the system to certain cognitive states using many of the same pathways that enable cognitive processes to elicit actions. Transduction plays an integral role in explaining successful and unsuccessful transfer. It explains, for example, how the execution of previous modes of perceiving and acting, activated by familiar contextual cues or expectations of the predictive architecture, can activate inappropriate concepts, leading to unsuccessful transfer to new task demands. Our focus on transduction reflects the empirically supported view that the coupling between cognition and action involves rich, multidirectional pathways (e.g., Abrahamson & Trninic, 2015; Nathan et al., 2014; Thomas, 2013)—richer than those that are typically described in classical information processing theory, which generally acknowledges only a unidirectional pathway via which cognition drives actions.

Third, people do not come to know the world as a verbatim sensorial record of an objective external world; instead, people are driven to make sense of their experiences, and meaning is constructed through the continuous interplay of social, cognitive, motoric, and perceptual processes of a highly dynamic, self-regulating organism, in what is often referred to as the *perception-action loop* (cf. Neisser's (1976) "perceptual cycle" as being central to everyday cognition). People construct mathematical meanings by coordinating situated perceptual and motor behaviors with the behaviors of mathematical objects (Abrahamson & Sánchez-García, 2016). Thus, the world we can know depends in part on the ways in which we can interact with it, physically and perceptually (Varela, Thompson, & Rosch, 1991). Meaning making also depends on establishing and maintaining common ground among interlocutors (e.g., H. H. Clark & Schaefer, 1989; Nathan, Alibali, & Church, 2017). Embodied processes are crucial for efforts to manage common ground in pedagogical contexts, where teachers regularly strive to foster common ground by using indexical speech and linking gestures (e.g., Alibali et al., 2014; Alibali, Nathan, Boncoddo, & Pier, 2019).

Fourth, mathematical ideas are *embodied* and *tangible* (Hall & Nemirovsky, 2012), and they can be expressed in metaphorical speech (Lakoff & Núñez, 2000), gestures and simulated actions (Hostetter & Alibali, 2008, 2019), diagrams and inscriptions (de Freitas & Sinclair, 2014), and physical objects (Martin & Schwartz, 2005). Importantly, mathematical ideas in different modalities may be linked via speech, gestures, and action (Goodwin, 2013), creating a rich multimodal experience that is a signature of PBL and that serves to ground the meanings of the referents.

Fifth, cognition is *extended* beyond the individual actor's brain such that task-relevant knowledge is grounded and distributed across actors, objects, and space (A. Clark & Chalmers, 1998). One example is cognitive offloading, wherein actors "use the world as its own model" (Brooks, 1991, p. 139) rather than depend on symbolic representations of the world and symbol-manipulation operations on those representations, which are the hallmark of traditional transfer (e.g., Lave, Murtaugh, & de la Rocha, 1984).

Sixth, because transfer is *embedded* in the situations in which activity unfolds, teachers and students are each engaged in transfer, and they serve as actors in exchanges that are situated in particular learning contexts. In many cases, teachers and curriculum developers have thoughtfully designed specific contextual supports for transfer; in other cases, teachers generate such supports "on the spot."

Finally, conceptual development naturally follows a process of *progressive formalization* (Romberg, 2001), which can be instantiated in the pedagogical practice of *concreteness fading* (Fyfe, McNeil, Son, & Goldstone, 2014). Concreteness fading is a developmentally informed approach to instruction that recognizes the importance of initial physical interactions (enactive processes) for early sense making about new concepts. This physical interaction creates the preconditions that support the emergence of perceptually based representations, and the eventual construction of abstract symbols, as physical and perceptual qualities are explicitly faded. Many educational approaches neglect this progression and instead follow the *formalisms first* approach to instruction (Nathan, 2012), wherein mathematical ideas are initially introduced in their most formal, symbolic, decontextualized form and only later grounded and applied. The conventional rationale is that the perceptual sparseness of abstract symbols benefits learners by reducing perceptual distraction (e.g., Kaminski et al., 2013). However, novices often flounder with early presentation of decontextualized symbols (Nathan, 2012). Experimental comparisons reveal benefits of concreteness-fading instruction over formalisms-first instruction for a wide range of mathematical concepts spanning elementary arithmetic, middle school and secondary level algebra, and postsecondary systems-theory concepts (Fyfe et al., 2014). Concreteness fading is especially well suited for fostering key STEM education principles in design- and product-oriented collaboration, as commonly implemented in PBL settings.

2.3 Transfer: Mapping of Invariant Relations to Achieve Cohesion

From an embodied perspective, the crux of transfer is establishing cohesion across contexts and physical instantiations, such that modes of perceiving and acting appropriate for engaging with a mathematical relation in one context (i.e., with a particular object or representation) also meaningfully apply in another context. In past work (Nathan et al., 2013; Nathan, Wolfgram, et al., 2017), we identified the significant challenges that students faced as they developed, to varying degrees, the skills for noticing and acting on similarities of different materials, labels, ecological contexts, iconic representations, and symbolic notations by virtue of their shared invariant mathematical relations. For example, in the excerpt of an engineering lesson presented at the outset of this chapter, the notion of *angle of ascent* (THETA) is depicted in a variety of *modal forms,* including speech, symbols, gestures, and diagrams (and, later, in a working physical device); however, a single invariant mathematical relation underlies all of these forms. To establish cohesion, various modal forms must be regarded by students as similar in terms of their perceptions and actions.

Critically, these differing modal forms vary in the actions they afford. Following J. J. Gibson (1966, 1979/2014), we define an *affordance* as the complementary relation between an object (which we take to include symbolic and material objects that are physical or imagined) and an actor who engages with that object. In an engineering lesson, for example, a physical device may afford grasping and holding, whereas the symbolic expression that mathematically models the behavior of that device does not. Thus, the processes of perceiving and acting that apply to one modal form may not apply to another modal form. For cohesion to be produced, the perceptions and actions applied to one modal form as it is manifest in one context must evoke in the actor a connection to a related modal form, which may be encountered in the same or in a different context.

A striking example of cohesion production is provided by Alibali and Nathan (2007) in an early algebra lesson for sixth-grade students. The teacher sought to connect a drawing of a pan balance scale (the initial modal form) that had an arrangement of blocks placed on the two sides to a symbolic equation (the second modal form) that represented that configuration of the balance scale with literal symbols and arithmetic operators. In the first such arrangement, two spheres on the left pan exactly balanced the sphere and two cylindrical blocks on the right pan.

The teacher emphasized that simultaneously removing the same type of block from the two sides of the balance scale corresponded to subtracting the same value from both sides of the equation, thus establishing the mapping between the balance scale and the equation and extending the original action that applies to the pan balance (object removal) to algebraic manipulation (symbolic subtraction). This provides a clear example of how embodied processes support transfer by depicting the ways these lifting actions can be applied first in a primary context (pan balance) to a second context (symbolic equation). It also shows how a teacher simulates the lifting of two literal symbols simultaneously from each side of the equation as a way to maintain cohesion when shifting modal forms from objects on a balance scale to an equation.

For a learner to exhibit transfer of knowledge across different contexts, a *mapping* between the actions afforded by the modal forms in each context must be made to establish cohesion. Mapping may be spontaneous or require instructional support. Lobato and colleagues (e.g., Lobato, 2003; Lobato, Ellis, & Muñoz, 2003) highlight ways the educational environment can be structured to orient learners' attention to such mappings, and they refer to such practices as *focusing phenomena*.

Evidence that a mapping has been formed may then be revealed in learners' later behaviors. For example, we may observe students tilting a ballistic device (e.g., a catapult) to launch a projectile at a particular angle in a way that is fundamentally similar to solving the range equation for a particular value of THETA. That is, the device acts as a "range function" that "computes" the landing distance of an object given (virtually) any input angle, which is achieved by tilting the launch pad. We consider evidence of such a mapping later in this chapter.

In brief, we argue that transfer occurs when learners and teachers establish cohesion of their experiences by mapping modes of perceiving and acting that they successfully used in one context to a new context. Learners and teachers *express* that cohesion across contexts in a variety of ways, principally through speech, gestures, and actions, including simulated actions.

Note that mapping supports identification of invariant relations by juxtaposing contexts that afford corresponding modes of perceiving and acting. Importantly, this identification and mapping may be implicit or explicit for the learner. This view of transfer differs from classical theories that rely on extracting common knowledge structures or rules with generalized conditions for application.

Transfer, by this account, centers on two distinct but related processes: *constructing a mapping of an invariant relation across contextualized modal forms* and *expressing cohesion* established by that mapping, as indicated by various behaviors, as described below. We consider each of these processes in turn.

2.3.1 Mapping as a Mechanism for Cohesion

We posit that mapping is a mechanism for establishing cohesion. Mapping can be aided by the focusing "moves" made by teachers, parents, curriculum designers, and knowledgeable others who already apprehend connections, and it can be supported by contextual cues, such as spatial alignment, labeling, and deictic gestures. Mapping can also be managed by learners who regulate their own environments to provide helpful contextual supports, such as placing information side by side.

Mapping involves constructing a relation between two (or more) objects, inscriptions, or ideas. We argue that there are multiple mechanisms by which mapping may occur. In some cases, learners may engage in explicit analogical mapping. For example, a child might reason about fraction division by explicitly mapping elements of a given fraction-division problem to elements in a whole-number division problem, saying, "6 ÷ ¼. Well, if I was doing 6 divided by 2, I would make groups of 2. So, 6 ÷ ¼, I'm going to make groups of ¼." In other cases, learners may perform mapping in a more implicit way, via *relational priming*, a process by which exposure to some task or situation primes a relation that can then be recognized or used in a novel task or situation (Day & Goldstone, 2011; Leech, Mareschal, & Cooper, 2008; Sidney & Thompson, 2019). For example, after modeling whole-number division problems with cubes— by forming groups the size of the divisor—a learner might enact the same relation to model a fraction-division problem because that relation (forming groups) was primed in the initial task (Sidney & Alibali, 2017). Another means of forming the mapping is through *conceptual metaphor* (Lakoff & Núñez, 2000), where one idea, such as arithmetic, is referred to in terms of another idea, such as object collection. As in this example, the second domain (the *target domain*) is more familiar and more concrete than the first, *source domain*. Conceptual metaphors are grounding, inference-preserving cross-domain mappings. Using conceptual metaphor, the inferential structure of one conceptual domain (say, whole numbers) is used to reason about another (say, fractions). In still other cases, learners may map relations via *conceptual blending* (Fauconnier & Turner, 1998, 2008), a mechanism by which people link two ideas that share structure, and "project selectively from those inputs into a novel 'blended' mental space, which then dynamically develops emergent structure" (Fauconnier, 2000, p. 2495). All of these forms of mapping—analogical mapping, relational priming, conceptual metaphors, and conceptual blends—forge correspondences, and these correspondences may afford engaging in corresponding modes of perceiving and acting.

Because transfer involves mapping modes of perceiving and acting from one context or representation to another to produce cohesion, we assert that pedagogical moves that support mapping are integral to transfer. Indeed, teachers engage in many practices, both planned and spontaneous (Alibali et al., 2014; Nathan, Wolfgram, et al., 2017), that highlight invariant relations across contexts, representations, and material forms. In subsequent sections, we highlight several distinct mapping practices that teachers use, both in ordinary mathematics instruction and in PBL settings, including *projecting* invariant relations across time and space and *coordinating* representations using techniques such as consistent labeling, linking gestures, and gestural catchments (Nathan et al., 2013).

2.3.2 Expression of the Mapping

If, indeed, this mapping of modal-specific ways of perceiving and acting is at the heart of transfer, it will be expressed—at least in some cases—in learners' behaviors. Learners may, for example, appropriate actions or ways of thinking applied in one context for use in another, and they may make mappings (either implicit or explicit) between the contexts. Some aspects of learners' behaviors in the novel context—their language, gestures, or actions—may reveal the mapping of modal-specific forms of perceiving and acting from a prior context (Donovan et al., 2014).

Learners' behaviors in different contexts often involve different sorts of actions, and their gestures in novel contexts may reveal activation of action patterns that they have produced in other contexts (Donovan et al., 2014). Learners may produce gestures in novel contexts that are similar in form to actions they produced in previous contexts. This repetition of gesture form—termed a gestural "catchment" by McNeill (2000)—is thought to reveal cohesion in speakers' thinking. Gestural catchments may reveal implicit or explicit mappings between contexts, representations, or material forms (Donovan, Brown, & Alibali, 2021).

Mapping often involves forming a conceptual blend, and such blends can be expressed in many ways (Fauconnier & Turner, 2008; Williams, 2008). When conceptual blends are established in classroom settings, the physical context typically offers a material anchor for the blend. Thus, the blends observed in PBL settings are often *grounded blends* (Liddell, 1998) that include elements of the immediate, physical environment. For example, a student may mount a protractor on a catapult arm and rewrite the angular measures as distances to the target, thus using a material anchor to blend angular measure with projectile motion using trigonometry and the laws of kinematics. The actions that the student previously applied to the original artifact (such as adjusting the angle of the protractor) can support new, inferential actions, such as retargeting based on lineal measure, which extend the student's repertoire of actions into the space of the new conceptual blend (Williams, 2008).

The earlier example of a teacher simulating the lifting of the same symbols off two sides of an equation, much as one lifts the same objects off two sides of a balance scale, is one such conceptual blend. Here, we can see how the mapping is formed. In this conceptual blend, the equation is treated as a pan balance and the adding and removing of objects to maintain balance maps to the manipulation of terms in the equation to maintain equivalence. Further, the teacher expressed this mapping explicitly in speech, noting that she wanted to "take a sphere off of each side" but saying that "instead of taking it off the pans, I'm going to take it off this equation." Thus, she identified the invariant relation of maintaining equivalence, performed the mapping of the pan balance to the equation with an explicit verbal link, and expressed cohesion across the modal forms through the reapplication of gestures that depicted the same actions. This mapping is illustrated in Fig. 2.1.

Other features of the teacher's speech also manifest her effort to align the diagram and the equation. For example, she used the same pronoun to refer to the sphere pictured in the diagram and the symbol *s* in the equation: "Instead of taking *it* off the pans, I'm going to take *it* off this equation" (emphases added), thus highlighting that the two inscriptions refer to the same quantity. She also used the same verb—*taking off*—to refer to removing a sphere from each side of the pan balance and subtracting *s* from each side of the equation. Thus, she highlighted the correspondence of these actions using a common label.

The teacher also expressed the correspondence between the pan balance and the equation in her gestures. She used a grasping gesture with both hands to gesturally depict taking the blocks off the two sides of the scale—a simulated action (Hostetter & Alibali, 2008, 2019) over the drawing of the scale. She then produced this same grasping handshape over the corresponding symbols in the equation to refer to subtracting values from the two sides of the equation. With this gestural catchment, the teacher sought to communicate the invariant relation of equivalence as "remove the

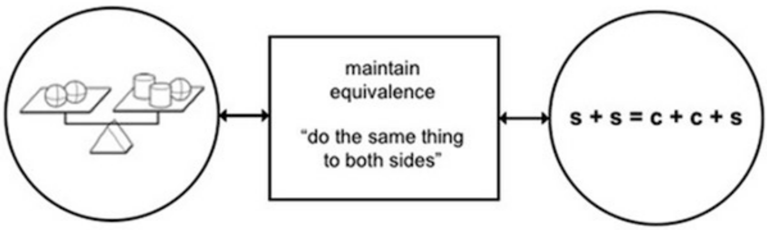


Fig. 2.1 The math teacher identified an invariant relation of maintaining equivalence and performed the mapping of the pan balance to the equation with an explicit verbal link and repeated gestures that depicted actions

same quantity from both sides" as it applied both to the physical pan balance depicted in the drawing and to the symbolic equation.

Note that this teacher *simulated* the action of "grasping objects" over both the diagrammatic and the symbolic representations, even though neither of these two inscriptions (diagram and equation) would afford this physical action. Both are two-dimensional representations, so their elements cannot be grasped or picked up. Importantly, however, the teacher's hands were configured as if actually grasping objects, and, in this way, her gesture evoked the physical objects that were represented symbolically in the diagram and the equation. Thus, in this simulated action, the teacher expressed a set of analogical relationships among the physical situation—which would afford such action—and the two inscriptions.

Thus, this conceptual blend was expressed in a range of ways: via an explicit verbal link, via common labels for related elements, and via a gestural catchment of the same simulated action performed in both spaces. The blend was grounded both in the two inscriptions, which were physically present, and in the (absent) physical objects that were evoked by the configuration and motion of the teacher's hands in real space (cf. Liddell, 1998). Using speech and gestures in these ways, the teacher organized corresponding elements of different representations with reference to one another, linking them together multimodally, in an effort to help students apprehend their connections.

This example also illustrates the centrality of the teacher in our theory of transfer. Teachers use a range of verbal and gestural techniques to support students in identifying the invariant relations and making the relevant mappings across contexts, representations, and material forms to establish cohesion (Alibali et al., 2014; Nathan, Wolfgram, et al., 2017). This is why we claim that the pedagogically designed actions of teachers—as well as parents, collaborators, and curriculum developers—are an integral part of transfer when viewed from an embodied perspective. We further suggest that expressing cohesion in the various ways described here is productive for learners' thinking, in the sense that it affirms, strengthens, and reifies the mappings across modal forms that have been established. It also serves as an effective means of communicating these mappings to others during collaboration or instruction.

2.4 Illustrating Embodied Transfer in a PBL Context

In this section, we provide examples from a PBL engineering classroom that demonstrate the power of an embodied theory of transfer to account for both successful and unsuccessful transfer. The examples also illustrate how a teacher's pedagogical moves foster cohesion for students in the PBL classroom and are thus a necessary part of an embodied account of transfer. The examples show how successful transfer arises by establishing this cohesion, whereas unsuccessful transfer occurs when learners' actions remain overly restricted to earlier modes of perceiving and acting.

2.4.1 The Three Central Elements when Analyzing Transfer from an Embodied Perspective

The accompanying examples illustrate the complex process of transfer that students and teachers face in the PBL classroom. Photo Transcript 1 (Table 2.2, which includes the excerpt from the chapter introduction) is taken from an engineering class in a U.S. Midwestern urban high school in late spring, near the end of the school year. This excerpt sets the PBL design challenge to build a ballistic device that can make a projectile hit a basket at some location, undisclosed until the last moment, with successful engineering based on the underlying math and physics of projectile motion. Even the open lecture, which focuses on trigonometry and kinematics, is rich with embodied methods of grounding the target invariant relation and other associated mathematical ideas and helping to foster cohesion as these ideas are manifest in multiple modalities, including symbols, drawings, words, wood, and the teacher's gestures.

We distinguish between the authentic classroom learning experience in which the students and teacher are embedded and the analytic process that is undertaken by researchers who study these classroom events. In terms of analysis, there are three central elements of transfer. First, it is critical for the analyst to *identify the invariant relation* that is central to the curriculum design and threaded throughout the modal forms. For this multiday unit, for example, the invariant is THETA, highlighted by the teacher on Day 1 and labelled as the "angle of projection." Second, the analyst must *describe the mapping* of the invariant relation across the range of modal forms used in the series of lessons. Third, the analyst must be able to describe how this mapping is expressed by the teacher and the students in the learning environment.

Separately from the analytic concerns of researchers, for learners to experience a sense of cohesion across the various modal forms and contexts that are the hallmark of the project-based curriculum, they must construct for themselves the mapping of modes of perceiving and acting that, optimistically, will apply across contexts. The mappings that are part of the expert model of transfer are important for the curriculum design, and may be shared in teacher supplementary materials, but they often remain implicit to the students (Prevost et al., 2014). Learners act on the new modal forms (e.g., their design sketches, mathematical models, and machined devices) in accordance with their constructed mappings. Learners' actions may operate in accordance with the expert model, indicating effective near and far transfer, as will be seen in Photo Transcript 2 (Table 2.3). Alternatively, learners' actions may be applied to subsequent modal forms in ways that do not align with the conceptual structure of the invariant relation, leading to "false transfer," as illustrated in Photo Transcript 3 (Table 2.4).

In the examples that follow, THETA is most commonly invoked by the teacher and by several of the students in a gesture of a flat hand posed at a fixed angle or of a flat hand pivoted at the wrist to refer to the range of angular values that THETA can take. The repeated expression of this idea in gesture makes up a gestural catchment, which reinforces cohesion across different manifestations of the invariant relation. Further, THETA is also evident in the design sketches created by the students and in the material devices that students build as they strive to create a ballistic device that can be adjusted "on demand" to enable a projectile to precisely hit a desired target.

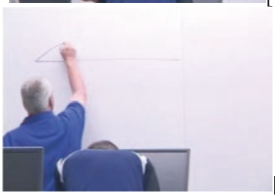
An invariant relation across modal contexts This first photo transcript demonstrates (a) identification of the invariant relation and (b) the ways a teacher uses pedagogical actions to highlight for students the mapping of the invariant relation across multiple modal forms.

Table 2.2 Photo Transcript 1: Day 1

Line	Transcript	Photo
1	T: I had given you an assignment to start working on a ballistic device that will throw a ping pong ball.	
2	T: And we had some constraints with that, um on a handout that I gave you. Particular constraints.	
3	T: What I wanna do today, is talk about, **the angle of the projection [1]**, that we shoot this, fire our ping pong ball and the **distance [2]** it'll go.	 [1]  [2]
4	T: And kinda mathematically determine what's the **best angle [3]** to get the maximum range, given a set velocity, of that we're firing this thing, okay?	 [3]
5	T: So we know that we can change the distance.	
6	T: What are some of the ways that we can change the distance, if we're shootin' a ping pong ball out of a device? [Name]?	
7	S: Angle of like, the ball.	
8	T: Okay. **Angle of projection. [4]**	 [4]

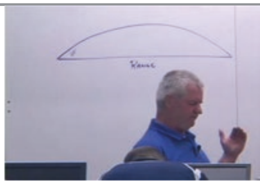
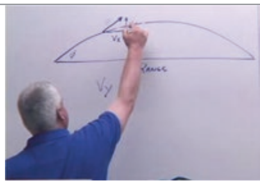
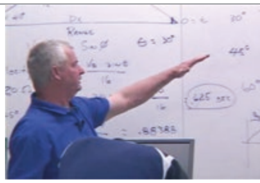
(continued)

Table 2.2 (continued)

Line	Transcript	Photo
9	T: That's gonna have an effect on it, right? What else?	
10	S: Velocity.	
11	T: Velocity. Which is, the speed in a certain, **in a set direction [5]** that we wanna go, 'kay.	 [5]
12	T: Those basically are the two elements that are gonna **affect the range [6].**	 [6]
	[Omitted portion]	
13	T: Alright so up here on the board, I want you to follow along, this is definitely a little bit complicated but I think we can get a handle on it.	
14	T: We're gonna—we're gonna look at two aspects of this.	
15	T: One, we're gonna look at the angle that affects our range.	
16	T: And once we pick, a-a-and then after we select an angle, we're also gonna calculate the range that we can get by, with those different angles.	
17	T: So let's look at how this works.	
18	T: First of all, put this over here, so draw it along with me.	
19	T: What happens when the, when we **project something through the air [7]**, is we end up with **something like [8] this** depending upon the angle here, which is *theta*.	 [7]  [8]
20	T: And, this is our range.	

(continued)

Table 2.2 (continued)

Line	Transcript	Photo
21	T: And basically, what we have, is, **we're working with vectors here [9]**.	 [9]
22	T: So we end up with, **some vectors that look like this and we call this, vector V_x and V, V_y[10].**	 [10]
23	T: And we can say, that, V_y we're gonna start with V_y here.	
24	T: This distance this, right here.	
25	T: So we're gonna start with, V_y, equals V_o, sine, of *theta*.	
26	S: Mr. [Name], what's V_o?	
27	T: Actually V_o is going to be the velocity. 'Kay. Good question.	
	[Omitted portion]	
28	T: 'Kay, now to relate this to our project, I'm actually gonna give you a distance and I'm gonna say "okay we're gonna send, we're gonna set the basket fifteen feet away,"	
29	T: but whatever distance that is, I'm gonna decide that at the time.	
30	T: We're gonna set the, the basket so many feet away and you have to try to hit it.	
31	T: So by doing some calculations on, what you're, um, ballistic device fires, you can kinda set your angle hopefully to get, to get that distance.	
	[Omitted portion]	
32	T: Well what I want you to do is after you, **assemble your ballistic device, I actually want you to be able to gauge these angles on the device [11]**	 [11]
33	T: and maybe we can stick an angle gauge in there somehow to check these angles	

(continued)

Table 2.2 (continued)

Line	Transcript	Photo
34	T: and you **determine at thirty degrees [12] what's your distance look like**.	 [12]
35	T: **At forty-five degrees [13] what's your distance look like [14].**	 [13]  [14]
36	T: At s-, at our range and at sixty, you know and so forth, get an idea of what your range is	
37	T: so that morning when we go down to the gym and we set this up and I throw a number at you	
38	T: which will be, it'll be somewhere between ten and twenty.	
39	T: So you're gonna have to try to design, you're gonna have to design your device to be able to fit within that parameter, constraints.	

We now analyze how the conditions for transfer are established by the teacher in this setting through his pedagogical actions. Our analysis of transfer in PBL settings rests on three analytic actions: (a) identify the invariant relation; (b) describe one or more mappings; and (c) document how participants in the learning environment express those mappings. Photo Transcript 1 illustrates the first two of these, with the mapping as a conceptual blend. The third component—how both the students and the teacher express those mappings using language, gesture, and action—is illustrated in Photo Transcripts 2 and 3.

The invariant relation is called out by the teacher as part of his presentation in Photo Transcript 1, Line 3, "talk about, the angle of the projection that we shoot this, fire our ping pong ball and the distance it'll go." Later, the angle of projection is referred to as "theta" by the teacher.

Describing the mapping of THETA involves identifying the relations among its various manifestations such that these seemingly dissimilar manifestations can be perceived as similar (Lobato, 2003). Our analysis reveals seven manifestations in all:

- as the measure of the sweep of an arm and hand to depict sample angular values (Line 3; photo [1]);
- as a drawn angle where the arc of the projectile meets the ground or baseline elevation (Line 19; photo [8]);
- as a Greek symbol (Line 21), called "theta," first written as the Greek letter Phi (φ) (photo [9]) and then later written as the Greek letter θ (photo [11]);
- in drawings and gestures that specify THETA as the direction of V_o, the initial velocity vector of the projectile that is related trigonometrically to component vectors V_x and V_y (Lines 21–23, photo [10]);
- as an equation parameter for computing velocity and range (Lines 23–25);
- as a physically manipulable quantity on the device students build ("you can kinda set your angle hopefully to get, to get that distance"; Line 31),
- as the reading from an angular measurement instrument (e.g., protractor; "I actually want you to be able to gauge these angles on the device and maybe we can stick an angle gauge in there somehow to check these angles"; Lines 32–34, photos [11] and [12]).

The intended result is a conceptual blend in which the manifestations of THETA are linked to one another in a cohesive network. Figure 2.2 presents a snapshot from the classroom depicting this network structure for THETA that, at that point in the lesson, is manifest in trigonometric relations, kinematics equations and diagrams, and gestures. Figure 2.3 illustrates the network of modal forms of THETA used throughout the unit.

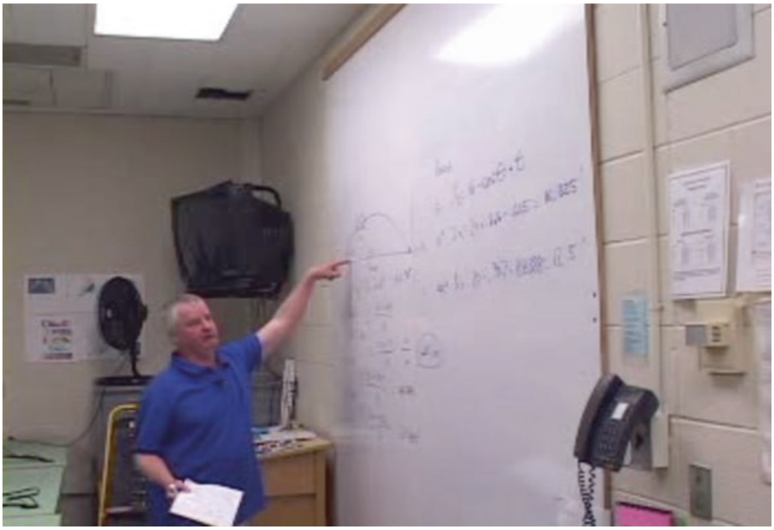


Fig. 2.2 Image of the whiteboard showing different manifestations of THETA

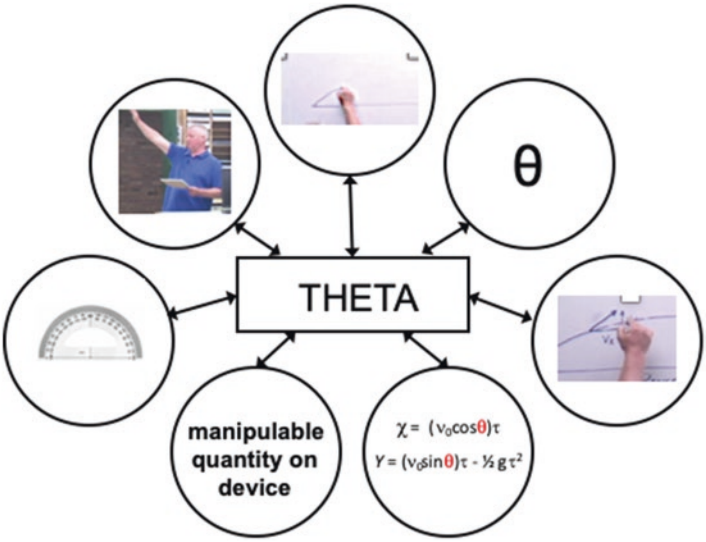


Fig. 2.3 The network of modal forms of THETA used throughout the unit

Successful transfer exhibited by students via gestural catchment The third element of analyzing transfer within an embodied framework is explicating ways those in the learning environment express cohesion. One expression of cohesion is illustrated in Photo Transcript 2 (Table 2.3), in which the teacher interacts directly with students who have been working in project design teams. To foreshadow, Photo Transcript 2 shows that at least two of the students express the cohesion of the invariant relation across two different instructional contexts: the formal lecture on kinematics given by the teacher, which involves a whole-class participation structure, and interactions that take place in the machine shop setting, which involve a small-group participation structure, which is the focus of the transcript. Here we observe the ways in which participants use body-based resources in several ways: to express the mathematical role of THETA that was depicted in the lecture; as it was drawn in their design sketch; as a measured and variable quantity; and in terms of its functional role for the project, which aims to control the trajectory of the projectile.

At the beginning of Photo Transcript 2, we observe two students (talking over one another) in a group of four express to the teacher how the design sketch they have drawn provides adjustments to the angle of projection (which they call at points “the elevation” and “different angles”) and a way to fix the angle of projection.

Student 1 notes (Line 8, photos [1] and [2]), “That’ll allow you to unscrew it, move it up and down,” and Student 2 concurs (Line 9). Especially notable is the gesture produced by Student 1 as he describes “move it up and down.” This gesture imitates the hand movement that the teacher previously used during the lecture to designate the many values THETA can take on, thus forming a gestural catchment.

Table 2.3 Photo Transcript 2: Day 1

Line	Transcript	Photo
1	T: Let's check, you guys. Where are you at?	
2	T: [Name] and [Name], what do we have here?	
3	S1: We got a, uh, thingy that works.	
4	T: Explain what you have goin' on here.	
5	T: 'Kay, so that is, where's your sheet with your constraints on it?	
	[Omitted portion]	
6	S1: Just a piece of wood to hold onto it.	
7	S1: Locking screw right there.	
8	S1: That'll allow you to **unscrew it, move it up and down (performs gesture three times in quick succession) [1] [2].**	 [1]  [2]
9	S2: (At the same time) Yeah.	
10	S1: And then **tighten it at whatever elevation you want [3]**.	 [3]
11	S2: **Different, different angles [4].**	 [4]
12	S1: A protractor sitting here. With a string with a weight on it.	
13	S1: So as you tip it, it'll, that'll tell you what degree you're tipping it.	
14	T: (At the same time) Oh! I like that. That's nice.	
15	S1: So that **tells you what degree so we can figure that out [5]**.	 [5]

The student presents an upright (left) hand with flat palm and proceeds to bend at the wrist and bring the hand back to upright three times in a couple of seconds, each time maintaining a somewhat flat palm. We regard this catchment as evidence that this student apprehends how THETA is manifest in the design sketch and that it aligns with the teacher's description.

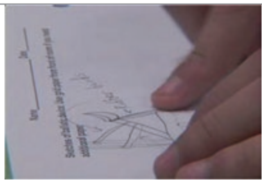
Student 1 continues, "And then tighten it at whatever elevation you want" (Line 10, photo [3]). He depicts this action by moving his right hand up to be near the left and making a motion typical for tightening a screw. Student 2 further immediately elaborates, "Different angles" (Line 11, photo [4]). In so doing, he, too, makes a gesture for the angle THETA twice in quick succession. This gesture repeats the gesture produced by the teacher during lecture and the gesture produced by Student 1 moments earlier, thus continuing to build the gestural catchment and providing further evidence that Student 2 also constructed a cohesive account of THETA as it relates to their design. The students further demonstrate their understanding as reflected in their method of measuring the angle of projection with the clever use of a weighted string moving across a protractor that is mounted on the device (Lines 12–15, photo [5]).

In brief, Photo Transcript 2 demonstrates how students express cohesion in this PBL activity through a gestural catchment and through connecting language directed at their design sketch (which provides a material anchor of one manifestation of THETA), the mathematics of THETA, and the angular measurement device. This excerpt also illustrates that the teacher contributes to transfer by using brief but important prompts. But it is the activity structure as a whole that really provides the mapping of the invariant relation across contexts by forging connections between the hands-on design project and the mathematics and physics presentation.

Unsuccessful transfer as inappropriate mapping of the invariant relation In contrast to Photo Transcript 2, which illustrates successful transfer, Photo Transcript 3 (Table 2.4) involves students who latch onto the wrong adjustable feature, so their design varies the initial velocity but not the angle of projection. The students' expressions of the mapping reveal this to be their constructed understanding, rather than a process of directly perceiving the invariant relation as labelled by the teacher. During this excerpt, the teacher recognizes that the students' actions reveal that their thinking and design is based on the incorrect mapping of the angle THETA to their device, which is contributing to unsuccessful transfer. In response, the teacher attempts to repair the mapping by reinstating the gestural catchment and making an explicit, direct mapping between the part of the device that could instantiate THETA and the mathematical inscriptions that model the influence of THETA on projectile motion that were previously written on the board.

The exchange in Photo Transcript 3 shows how transfer can be thwarted when students construct an inappropriate mapping for the target invariant relation. The teacher provides a rich prompt (Line 1), asking, "How are you going to change the angle of your trajectory?" invoking the gestural catchment that has come to signify THETA (photo [1]). The students have designed a catapult that includes rubber bands that can be set at different points before their release, altering the tension and therefore the speed with which the catapult arm will release. The students see the

Table 2.4 Photo Transcript 3 (Unsuccessful Transfer): Day 2

Line	Transcript	Photo
1	T: Alright now let me ask a question regarding **how are you going to change the angle of your trajectory [1]?**	 [1]
	[Omitted portion]	
2	S2: Right there.	
3	S1: We'll have this rubber band here, pull it down here.	
4	S1: And so we have several spokes here so the further we pull it down and attach it, that, **that changes the angle for us [2].**	 [2]
5	T: Well I'm wondering if the further you, pull your rubber band down–	
6	S1: Mmhm.	
7	T: –is gonna affect your, **velocity, more than your angle [3].**	 [3]
8	S2: [At the same time] Yeah it's, well no, this is the velocity	
9	S2: but what we're sayin' is that this is **how hard it pulls, but then right here [4]**, where it, where it, **where the fulcrum is like this actually you can tilt it [5].**	 [4]  [5]
10	S2: [At the same time] The rubber bands control the tension but the placement is what really controls...	
11	S2: Like. See what we're saying?	

(continued)

Table 2.4 (continued)

Line	Transcript	Photo
12	T: So it's, it, okay so, **if I could, suggest, I think that [6]**, you might be able to adjust your angle by, by having some type, by controlling where this stops.	 [6]
13	S1: Yeah.	
14	T: But that's probably also gonna affect your, maybe affect your velocity.	
15	T: What I'm saying is, either that or else you have to tip the whole thing.	
16	S2: No, we don't.	
17	S2: That's why, 'cause the two sides stay put but then the top part can, tilt, right there.	
18	T: Okay.	
19	S2: [At the same time] So the fulcrum can change positions, basically.	
20	T: Alright. So I think maybe what you need to do is is, take into consideration what I just said about–	
21	S1: Yeah.	
22	T: –**being able to control the angle [7].**	 [7]
23	T: **That's why we did everything we did here [8]–**	 [8]
24	S1: Mmhm.	
25	T: –with the math. Because we wanna–	
26	S1: (At the same time) The math yeah.	
27	T: –be able to adjust the angle of the trajectory.	
28	T: I would try to keep, the velocity, the same, consistent, throughout the whole, every test that you do that that's consistent	

(continued)

Table 2.4 (continued)

Line	Transcript	Photo
29	T: and so all you're gonna change once you, one you decide what that velocity has to be, **all you're gonna change is your angle [9].**	 [9]
30	S1: Yeah.	
31	T: Okay?	
32	S1: Mmhm.	
33	T: I don't really want you to use the tension on the rubber bands, as, the only control.	
34	**T: I want you to have an angle adjustment [10].**	 [10]

different positions of the rubber bands as taking different angles (see Fig. 2.4), which they predict will alter the angle of projection: "So the further we pull it down and attach it, that, that changes the angle for us" (Lines 3–4, photo [2]). In response, the teacher rightly observes (Line 5) that the catapult arm will release at the same angle regardless of the placement of the rubber band, but the change in tension will affect the initial speed of the projectile. The teacher points to the design sketch to help clarify his critique (photo [3]).

The students do not pick up on this critique but offer a defense (Lines 8–11), "See what we're saying?" This suggests that the students are not merely misinterpreting the theory or misreading their own design sketch. The second student speaker (Lines 8–9) offers this account, "Well, no, this is the velocity, but what we're sayin' is that this is how hard it pulls, but then right here, where it, where it, where the fulcrum is like this actually you can tilt it" and demonstrates this idea in photos [4] and [5].

A reasonable interpretation is that the students operate with a preexisting "ontological coherence" (Slotta & Chi, 2006) for velocity exclusively as a scalar measure of speed of the projectile, which interferes with their adoption of a new conceptualization of velocity as a vector quantity (i.e., V_o) that includes both speed and direction. Prior ontological commitments of this sort are notoriously difficult to alter. Here we observe such a case from two students in defense of their design when the first student says (Line 17), "That's why, 'cause the two sides stay put but then the top part can, tilt, right there," and the second (overlapping) says (Line 19), "So the fulcrum can change positions, basically." In neither case, however, will this design provide the control of the angle of projection that the project requires.

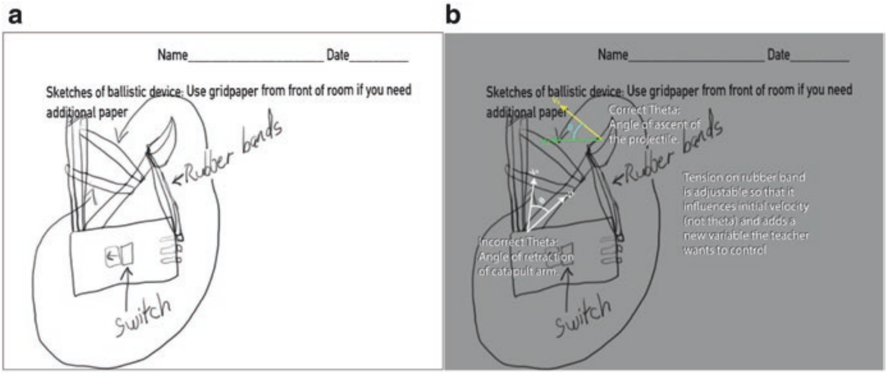


Fig. 2.4 (**a**) One group's original design sketch with (**b**) the vectors and angles added that label the correct and incorrect matches to THETA

An interesting part of this exchange comes when the teacher identifies the break in cohesion. By way of repair, he offers two mapping acts. First, he reinvokes the THETA gesture but this time does so in the same plane as the paper design sketch (photo [7]) while saying (Line 22) "being able to control the angle." In this way, the teacher connects the variation of the angle to the students' design sketch. Second, the teacher makes explicit reference, with speech, gesture, and upturned eye gaze (photo [8]), to the mathematical derivation still on the whiteboard at the front of the room and starts out saying (Lines 23–29), "that's why we did everything we did here with the math," and ends with, "all you're gonna change is your angle." The students acknowledge this midway and repeat (Line 26), "The math, yeah," but they seem disappointed by the teacher's reaction to their design.

2.5 Reflections on an Embodied Theory of Transfer

In this chapter, we have advanced the argument that transfer is fundamentally an embodied process. This is made especially evident when studying PBL settings. Learning and teaching in PBL settings are embedded in rich, multimodal contexts where content knowledge and information are often extended across a variety of semantic resources, including objects, inscriptions, and other actors. We assume that learners and teachers have a natural drive for cohesion in the learning experience—learners, to experience continuity, and teachers, to provide a meaningful and engaging learning environment in which their students achieve the desired understandings. We observe that both teachers and learners engage embodied processes as they map invariant relations across various modal forms. This mapping enables agents in educational settings to apply prior modes of perceiving and acting to new contexts and to create movements that will activate those invariant relations through transduction. Mapping may be explicit, as in analogical mapping; implicit, as in the case of priming relational structures; or some combination, as may be seen with conceptual blends. Teachers and students express cohesion by connecting different contexts and different modal forms via speech, actions, and gestures, as when a teacher simulates picking up symbols simultaneously from both sides of an equation or when a student invokes a gestural catchment to indicate how a structural property of a device enacts the relationship depicted in a mathematical model. We now consider some notable aspects of the proposed theory, implications for educational practice, and open research questions that may advance understanding of transfer.

We have argued that there are three core elements to embodied transfer: (a) identifying the central invariant relation that is manifest in multiple contexts, representations, or modalities; (b) mapping that relation across those contexts, representations, or modalities; and (c) expressing cohesion across the disparate manifestations of that invariant relation. We view the order of these three elements as somewhat fluid. Mapping across contexts—performed by a teacher, for example—might precede a student's awareness of the central invariant relation. The mapping can provide a means for comparison that enables the learner to perceive connections between contexts and inscriptions, as when students experience that they are performing similar actions in ontologically different contexts. The actions performed in the new context can activate common cognitive states through transduction, which then help the student to notice the invariant relation in the new context, thereby enabling mapping across the contexts. Expression can also play a role in making implicit mappings more explicit for the learner, as when students' reflections on their motoric behaviors bring these relations into conscious awareness. This may be one reason why self-explanation is a powerful mechanism for promoting transfer (see, e.g., Rittle-Johnson, 2006).

An important assumption of an embodied theory of transfer is that transfer operates within a predictive architecture and a set of feedforward mechanisms that ready the system to act. Consequently, transfer is not an occasional process but a continual one. A system always looking to act will also activate cognitive states in accord with its actions. This offers a theoretical basis for understanding near and far positive transfer as well as negative transfer. In this framework, near transfer is especially likely when modes of perceiving and acting from an earlier context are activated and readily apply in a new context. The teacher simulating lifting the drawn objects off of the drawn pan balance is one such case, given that these affordances for a physical pan balance would normally apply. We describe as *far transfer* those cases in which the earlier modes of perceiving and acting are not directly applicable and that require some modification and some enhanced mapping support to establish correspondences. Negative transfer is expected when the mapping is salient but the associated modes of perceiving and acting are no longer relevant. One example is the "add all the numbers" error commonly made by elementary and middle school students solving mathematical equivalence problems (e.g., offering "15" as a solution for a problem such as $3 + 4 + 5 = 3 + \_\_$; Knuth, Stephens, McNeil, & Alibali, 2006; McNeil, 2014; Perry, Church, & Goldin-Meadow, 1988).

Our proposal also raises the issue of *false transfer*, which may occur when actors apply modes of perceiving and acting that they expect to be applicable but that match only at a surface level and which therefore do not yield successful transfer (in terms of experts' expectations). The persistence of false transfer in the face of feedback may be due to students' prior ontological commitments that offer strong matches to the current circumstances (Chi, Roscoe, Slotta, Roy, & Chase, 2012; Slotta & Chi, 2006). One example is treating velocity as a scalar measure of speed in a design project that requires that velocity be treated as a vector quantity specifying both speed and direction. The activation of inappropriate modes of perceiving and acting can help explain why tasks that share surface structure but different invariant relations so readily lead to false transfer.

As these classroom examples make clear, transfer is an embedded process, situated in a particular physical and sociocultural learning context. PBL is also an extended process such that multiple actors (often a teacher and students) are engaged in transfer, mapping invariant relations across modal forms. The contributions of both teachers and learners to transfer suggest that transfer is a fundamentally social activity (Lobato, 2006). This view suggests several powerful ways to promote transfer, particularly in complex learning environments. In past work (Nathan, Wolfgram, et al., 2017), we documented some of the key processes that teachers draw on to foster cohesion across representations, contexts, and settings: Teachers actively bridge ecological shifts when learning takes place in different ecological contexts (such as the classroom and the machine shop), and teachers check that their students are aware of the continuity they strive for; teachers coordinate ideas across different spaces using common labels, thoughtful juxtaposition, gestural catchments, and deixis in both speech and gesture; and they project invariant relations forward and backward in time to promote temporal continuity. Our position is that these pedagogical processes are integral to transfer. Excluding the teacher from a theory of transfer risks creating a theory that is unable to account for transfer as it occurs in authentic settings.

Our theory also highlights the importance of understanding the fine structure of the ways in which teachers and students express cohesion. In this regard, we draw on Goodwin's (2013) observation that speakers commonly *layer* semiotic fields one upon another during discourse, a process he termed *lamination.* In our view, teachers and students may laminate different representations together—that is, layer them together in space or time using language, gesture, or action—thereby fusing them conceptually. For example, consider the teacher (described earlier) who produced the same gesture of removing objects from two sides over a drawing of a pan balance and then over a symbolic equation representing the state of the pan balance. With this catchment gesture, the teacher laminates together the pan balance and the equation. She organizes elements of these manifestations of the invariant relation with respect to one another and uses gestures to express their correspondences.

An embodied account of transfer can also provide insight into why certain instructional approaches have proven effective. The proposed theory naturally explains the success of instructional approaches that bring actions in target contexts into close alignment with actions in the original source context. For example, bridging instruction (Nathan, Stephens, Masarik, Alibali, & Koedinger, 2002) relies on mapping students' invented strategies for algebraic reasoning to those that experts have identified as important for achieving curricular goals. Concreteness fading (Fyfe et al., 2014) helps learners to ground formal notations in terms of familiar modes of perceiving and acting, applying the resulting actions to a broader range of contexts.

An embodied account of transfer also has implications for assessment practices. Regarding formative assessment, it is well documented that learners sometimes exhibit ways of thinking in actions and gestures even before they have explicit awareness of their new understanding or before they have constructed verbal accounts of their new ways of thinking (Church & Goldin-Meadow, 1986; Goldin-Meadow, Alibali, & Church, 1993). Teachers who notice these nonverbal expressions can more accurately model students' conceptual development and can be responsive with their own pedagogical actions. Even untrained adults generate more accurate descriptions of children's understandings when they attend to children's gestures along with their verbal utterances (Goldin-Meadow, Wein, & Chang, 1992). Improving teachers' skills for noticing students' gestures can greatly enhance teaching and learning (Roth, 2001).

Summative assessment is generally more evaluative, taking place at the end of a major curricular unit. Summative assessment practices are dominated by students' verbalizable knowledge, often excluding learners' embodied forms of expression and therefore underestimating student knowledge. Further, assessment methods using computer keyboards can interfere with body-based forms of expression and can even impair students' thinking (Nathan & Martinez, 2015).

An embodied account of transfer raises several important questions for future research. First, what kinds of discourse practices contribute to students' identification and mapping of invariant relations across contexts? For example, to what extent are instructional practices such as using common labels or producing gestural catchments valuable for supporting students' mapping across contexts? Relatedly, which discourse practices help learners progress from an implicit, action-based understanding of invariant relations to explicit, verbalizable knowledge?

Second, do effective approaches to mapping depend on the target concept or on the age, prior knowledge, or cognitive skill of the learner? It is possible that some learners may benefit from more explicit mapping, whereas others may do better with more implicit approaches. These individual differences, in turn, may be due to differences in learners' prior knowledge or in their patterns of cognitive skills.

Third, what are the consequences of variations in mapping practices or variations in expressing cohesion? For example, do some types of mapping lead to more durable knowledge or to greater gains in students' conceptual understanding of the target mathematical concepts? Does expressing cohesion in gestures or speech help learners to stabilize that knowledge and make it more explicit? These questions raise further issues about underlying mechanisms, which can be construed at a variety of different grain sizes. One potentially fruitful level of analysis involves considering the management of attention in social interactions that focus on transfer. How do teachers' mapping practices affect students' attention to aspects of the context or to features of the particular representations being linked? More generally, how do contextual supports and social guidance of transfer influence learners' attention, and how is attention involved in identifying invariant relations and in mapping across contexts?

Finally, given that our account has emphasized the social aspects of transfer, how do dimensions of social relationships, such as warmth, respect, and power, affect patterns of transfer? For example, are students especially likely to attend to novel mappings expressed by social partners who display respect for their ideas and concern for their learning (Gutiérrez, Brown, & Alibali, 2018)? How does the history of a social relationship affect the negotiation of transfer by individuals in that relationship?

Although there are many questions yet to be addressed, we believe that an embodied perspective yields a novel and valuable conceptualization of transfer. There is increasing awareness among both scholars and practitioners of the embodied nature of cognition (e.g., Barsalou, 2008; Glenberg, 1997; Rosenfeld, 2016; Wilson, 2002). In our view, an embodied perspective on transfer is necessary because transfer occurs in a rich physical and social world. By focusing on invariant relations, how they are mapped across contexts, and how cohesion across contexts and across modalities is expressed and negotiated, we open new avenues of inquiry, and these avenues promise to shed light on transfer as it occurs in PBL settings and other complex learning contexts.

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Chapter 3
Opening Possibilities: An Approach for Investigating Students' Transfer of Mathematical Reasoning

Heather Lynn Johnson, Evan McClintock, and Amber Gardner

How might teachers and researchers engender students' mathematical reasoning across a range of situations? Or, put another way, how might students' transfer of mathematical reasoning be promoted? What counts as transfer of mathematical reasoning? And what might serve as evidence of such transfer?

Researchers' views of transfer afford what constitutes evidence of transfer (Lobato, 2003, 2008, 2012) as well as the scope of what counts as possible to be transferred. We view transfer as something more than the application of a procedure from one situation to another (Lobato, 2003), meaning that students can engage in transfer even if they do not accurately apply a procedure across different situations. To weigh what could serve as evidence of transfer, we navigate tensions between our own researcher perspectives and students' perspectives. Hence, we draw on actor-oriented transfer (AOT) theory (Lobato, 2003, 2008, 2012), in which Lobato problematizes the perspectives that researchers employ when investigating students' transfer.

To locally integrate theories (Bikner-Ahsbahs & Prediger, 2010), researchers extend beyond combining or coordinating theories to explain empirical phenomena to build new theories and approaches. We draw on three theories to investigate students' transfer of mathematical reasoning: Lobato's theory of AOT (Lobato, 2003, 2008, 2012), Marton's variation theory (Kullberg, Runesson Kempe, & Marton, 2017; Marton, 2015), and Thompson's theory of quantitative reasoning (Thompson, 1994, 2002, 2011; Thompson & Carlson, 2017). In each of their theories, these scholars distinguish between the perspectives of students and those of the researchers. Lobato (2003) centered the student perspective when expanding the scope of what could count as evidence of transfer. Marton (2015) distinguished between

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adults' and children's perspectives, explaining that adults cannot expect that by showing and telling children something they as adults discern that they will necessitate children's discernment. Thompson (1994) argued that a quantity is something more than a label for a unit (e.g., 5 feet), explaining that quantities depend on individuals' conceptions of attributes of objects. By integrating these theories, we center the student perspective in our investigation of students' transfer.

The opening possibilities approach stems from Johnson's program of research, consisting of iterative design experiments (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), in which Johnson led fine-grained investigations of secondary students' reasoning related to rate and function. With this approach, we aim to open possibilities for researchers to investigate students' transfer and for students to engage in mathematical reasoning. By focusing on students' transfer of mathematical reasoning (e.g., Johnson, McClintock, & Hornbein, 2017), researchers can extend the objects of their transfer study. By integrating different theoretical perspectives (Lobato, 2003; Marton, 2015; Thompson, 2011), researchers can expand how they theorize transfer. By linking theory and method in a way that mutually informs, rather than prescribes, the other (Chan & Clarke, 2019), researchers can broaden methods for transfer study. To demonstrate the viability of this approach, we provide an empirical example of a secondary student's transfer of a particular form of mathematical reasoning, covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson & Carlson, 2017). We conclude with implications for the design of transfer studies.

3.1 Theoretical Background: Students' Transfer, Discernment, and Reasoning

In integrating theories, we bring together different assumptions. First, researchers' focus on students' perspectives impacts claims of what can constitute evidence of transfer (Lobato, 2003, 2008). Second, students' discernment plays a role in their transfer, and students discern both difference and similarity (Marton, 2006). Third, the object of students' transfer can extend beyond knowledge of mathematical concepts to include forms of mathematical reasoning (Johnson, McClintock, & Hornbein, 2017).

3.1.1 Transfer and Discernment

From an AOT perspective, transfer is generalization rather than application (Lobato, 2003, 2008). In other words, transfer is something other than the accurate application of a solution method across situations. Lobato (2008) defined transfer as "the generalization of learning, which also can be understood as the influence of a learner's prior activities on his or her activity in novel situations" (p. 169). Hence, students transfer their mathematical reasoning when they generalize some form of reasoning from one situation to a novel one. For example, consider two situations: a Cannon Man, flying up into the air, then parachuting back down, and a Toy Car, moving along a curved path, with a stationary object nearby. In each situation, students can sketch a Cartesian graph to represent a relationship between attributes: Cannon Man's height from the ground and his total distance traveled and the Toy Car's distance from the stationary object and its total distance traveled. Even if students do not sketch accurate graphs in either situation, they may still transfer reasoning from the Cannon Man to the Toy Car. To gather evidence of students' transfer, researchers employing an AOT perspective scour data for relationships of sameness that students may construct (Lobato, 2003, 2008). For example, students may recognize that the total distance traveled continues to increase in both situations. Although there has been a focus on sameness, Lobato (2008) acknowledged the possibility for researchers' AOT analysis methods to include attention to difference.

For a given graph in a Cartesian coordinate system, some students attend to attributes represented on the axes, whereas other students attend to only a trace in the plane. However, it is important for each and every student to attend to graph attributes. Employing Marton's variation theory (Kullberg et al., 2017; Marton, 2015), designers can develop task sequences to provide opportunities for students to *discern* particular aspects of graphs. Discernment involves more than noticing. It implies separation of an object's features from the object itself (Marton, 2015). For example, to discern attributes represented on graph axes, students would separate those attributes from other aspects of a graph.

Through systematic variation, designers can engender opportunities for students' discernment (Kullberg et al., 2017; Marton, 2006, 2015); in the task sequences, difference (contrast) should precede sameness (generalization). Systemic variation necessitates patterns of variation and invariance. For example, suppose researchers intend Cartesian graphs to be an object of learning for students. In the first task, students can encounter different kinds of graphs (contrast) so that students may discern graphs as an object and Cartesian graphs as a dimension of variation of the broader object of graphs. The relationship between variables would remain invariant, and the type of graph would vary. In a subsequent task, students can encounter different kinds of Cartesian graphs (generalization). Now, the type of graph (Cartesian) would remain invariant, and the relationship between variables would vary. Notably, the object of learning is the first thing varied (the type of graph), then characteristics of the object of learning (relationships between variables), so that students may discern which aspects of Cartesian graphs are necessary and which aspects are optional.

Researchers can employ variation theory in their study of transfer. Broadly, Marton (2006) defined transfer as being "about how what is learned in one situation affects or influences what the learner is capable of doing in another situation" (p. 499). Summarizing results of different studies, Marton (2006) argued that students' discernment of both difference and sameness contributes to their transfer. To illustrate, consider Marton's (2006) example of the Cantonese spoken language, which includes both sound and tone. Suppose a student hears two Cantonese words in succession, both with the same sound but with different tones; this can provide the student an opportunity to discern, or separate, the tone from the sound, not only in the second word but also in the first. This kind of discernment can also apply to the Cannon Man and Toy Car situations. For example, a student may discern, or separate, the difference in literal movement of each object from the object's total distance traveled. Hence, it is possible for the discernment of difference (e.g., the difference in tone or literal movement) to be what a learner transfers from one situation to another.

Both Marton and Lobato used the term *generalization*. We interpret their uses of the term to be compatible but not synonymous. Lobato used generalization in a broader sense, whereas Marton used generalization to address a specific kind of variation. We view Lobato's explanation of transfer as "generalization of learning" to be consistent with Marton's definition of transfer—that is, the influence of one situation on a new situation. Marton employed generalization to refer to a pattern of sameness in task sequences, which should follow patterns of difference (contrast). For example, suppose a teacher intends to develop a task sequence for students to discern, or separate, the attribute of "increasing" on a graph. The teacher would begin with contrast, for instance providing students graphs that increase, decrease, and remain constant. Then the teacher would follow with generalization, for instance providing students with graphs having different kinds of increases (e.g., linear, quadratic, exponential). Integrating theories, we aim to illustrate how difference can play a role in the generalization of learning, or transfer, from an AOT perspective.

3.1.2 Discernment and Reasoning

In the theory of quantitative reasoning (Thompson, 1994, 2011; Thompson & Carlson, 2017), Thompson focuses on students' conceptions of attributes, which may be involved in problem situations or represented in graphs. Whether an attribute is also a quantity depends on the students' perspectives rather than the observers' perspectives. When a student conceives of some attribute as being possible to measure, then that attribute is a *quantity* for the student. For example, an observer may conceive of how it could be possible to measure a toy car's distance from a stationary object, yet students may wonder where to even look for, let alone measure, such a distance. Thompson's theory centers students' conceptions of possibilities for measurement (e.g., using a string to measure the distance between two objects) rather than on their end results of measurements (e.g., exactly how far the toy car is from the stationary object at a given moment). Therefore, students can engage in quantitative reasoning without applying particular procedures or determining certain results. Integrating theories, we explain a particular kind of discernment, a conception of graph attributes as being possible to measure, that we aim to promote in students.

3.2 The Opening Possibilities Approach

The opening possibilities approach, shown in Fig. 3.1, links theory and method to investigate students' transfer of mathematical reasoning. Theory and method are positioned across from each other to represent a complementary, rather than hierarchical, relationship between them. The double-headed arrow in the center shows that theory and method mutually inform, rather than prescribe, the other. Three overarching questions guide the approach: What counts as students' transfer of mathematical reasoning? How can researchers engender students' transfer of mathematical reasoning? What constitutes evidence of students' transfer of mathematical reasoning? In response, researchers may draw on a range of theories and methods which in turn afford and constrain their design decisions, data collection, and data analysis.

3.2.1 What Counts as Transfer of Students' Mathematical Reasoning?

How researchers theorize students' mathematical reasoning influences what counts as evidence of students' reasoning. By a student's mathematical reasoning, we mean purposeful thinking in action occurring in a setting that constitutes mathematics for the student. With Thompson's theory of quantitative reasoning, we focus on students' conceptions of what may be possible to measure rather than on end results obtained from measurement. Bringing together Lobato's AOT theory and Marton's variation theory, by transfer of students' mathematical reasoning, we mean how students' mathematical reasoning in prior situations influences their mathematical reasoning in new situations. Through our methods, we aim to infer students' reasoning (and transfer of reasoning) based on their observable behaviors. To gather evidence of students' engagement in the intended mathematical reasoning, we focus on students' conceptions as they are engaging with task sequences rather than on their end results.

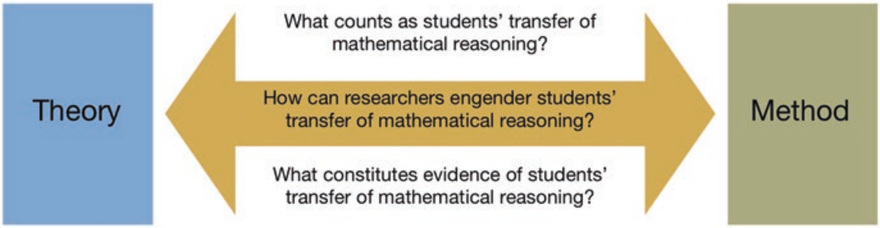

Fig. 3.1 The opening possibilities approach

3.2.2 How Can Researchers Engender Students' Transfer of Mathematical Reasoning?

We aim to promote students' engagement in mathematical reasoning rather than answer finding. Hence, we design task sequences in which we work to engineer opportunities for students' reasoning writ large as well as opportunities for students to engage in mathematical reasoning. Our stance on students' reasoning influences our assumptions about the viability of their reasoning, which in turn influences our methods. First, we assume that students working on a task may have goals for the task that are different from our own (Johnson, Coles, & Clarke, 2017). Second, we acknowledge that the reasoning we intend may be different from the reasoning that students engage in during task sequences. Third, we assume that students' reasoning is viable and productive, regardless of its form. In our methods, we do not seek to "fix" students' reasoning. Rather, we seek to understand and engender students' mathematical reasoning in its many forms.

3.2.3 What Constitutes Evidence of Students' Transfer of Mathematical Reasoning?

We view students as experts in their own mathematical reasoning, and thereby our role as researchers is to elicit and explain that reasoning. To gather evidence of students' transfer of mathematical reasoning, we build from four criteria put forth by Lobato (2008). First, students demonstrate a change in their reasoning from one task to another. Second, prior to the task sequences, students demonstrate limited evidence of the intended reasoning. Third, students' reasoning on earlier tasks influences their reasoning on later tasks. Fourth, students' change in reasoning is something other than a spontaneous occurrence. When analyzing for evidence of influence of students' reasoning, from earlier tasks to later tasks, we consider both contrast and generalization (Marton, 2006). That is, we take as evidence of transfer not only students' perspectives of how tasks are similar but also how they perceive those tasks to be different.

3.3 Opening Possibilities for Students' Covariational Reasoning

To operationalize the opening possibilities approach, we address a particular form of mathematical reasoning, covariational reasoning (Carlson et al., 2002; Thompson & Carlson, 2017). Not confined to a single area of mathematics, covariational reasoning transcends different mathematical concepts, including the gatekeeping concepts of rate and function.

3.3.1 What Counts as Transfer of Students' Covariational Reasoning?

When students engage in covariational reasoning, they can form and interpret relationships between attributes which they conceive to be capable of varying and possible to measure. In other words, covariational reasoning involves both students' conceptions of attributes and their conceptions of a relationship between those attributes (Carlson et al., 2002; Thompson & Carlson, 2017). To illustrate, in the Toy Car situation, a student may conceive of varying lengths of a stretchable cord connecting the car to a stationary object and a trace of the distance traveled as the car moves along its path. Furthermore, that student may conceive of a relationship between the cord length and distance traveled: The cord could start off longer, then shorten, while the toy car's total distance traveled keeps increasing. By transfer of students' covariational reasoning, we mean how students' covariational reasoning in one situation (e.g., the Cannon Man) influences their covariational reasoning in a new situation (e.g., the Toy Car).

3.3.2 How Can Researchers Engender Students' Transfer of Students' Covariational Reasoning?

We view tasks to be more than a problem statement. Tasks encompass the intentions of those designing, implementing, and interacting with the tasks as well as physical materials (Johnson, Coles, & Clarke, 2017). Our task sequences comprise students' sketching and interpreting Cartesian graphs, which means that we address both students' covariational reasoning and their conceptions of graphs themselves. By incorporating patterns of difference and sameness, we intend to provide opportunities for students to discern necessary aspects of graphs from optional ones. For example, even though the Cannon Man flies up and down while the Toy Car moves along a path, the total distance traveled for both continues to increase. If students were to only experience one kind of motion, they might not have sufficient opportunities to separate the literal motion of the objects from a measurable attribute of the objects, such as their total distance traveled.

3.3.3 What Constitutes Evidence of Students' Transfer of Covariational Reasoning?

To gather evidence of students' covariational reasoning, we infer students' conceptions based on their observable behavior. We examine students' work when sketching Cartesian graphs because sketching graphs can provide students opportunities to represent relationships between attributes. We focus on students' process of sketching graphs rather than on assessing the accuracy of their resulting graphs. Although students may engage in covariational reasoning when doing things other than graph sketching, we have found instances of students' graph sketching to offer compelling evidence of their covariational reasoning. Yet, students' difficulties or facilities with graphs can present challenges when analyzing for reasoning. Integrating different theories affords us opportunities to explain students' discernment of graph attributes in conjunction with their transfer of covariational reasoning.

3.4 The Promise of Opening Possibilities: An Instantiation of the Approach

To demonstrate the promise of the opening possibilities approach, we report data from a larger study in which Johnson conducted a set of three individual, task-based interviews (Goldin, 2000) with each of 13 secondary students to investigate their covariational reasoning and conceptions of graphs. We report data from one of those students, Aisha, who demonstrated transfer of covariational reasoning. To contextualize the data, we explain the design of our task sequences and our methods for data analysis. With this instantiation of the opening possibilities approach, we build on Johnson and colleagues' earlier investigation of a secondary student's transfer of covariational reasoning (Johnson, McClintock, & Hornbein, 2017).

3.4.1 The Task Sequences

We implemented three task sequences, each with a different background: a Ferris Wheel, a Cannon Man, and a Toy Car, respectively. Across the task sequences, students explored different situations, then sketched one or more Cartesian graphs to represent a relationship between attributes in a situation given in an animation. We adapted the Ferris Wheel task sequence from Johnson and colleagues' earlier research (Johnson, McClintock, & Hornbein, 2017). We developed the Cannon Man and Toy Car task sequences in Desmos, a freely available digital mathematics tool, in collaboration with Meyer, the chief academic officer of Desmos.

The Ferris Wheel task sequence incorporated three key elements. First, students manipulated an online interactive of a turning Ferris wheel. Second, students sketched a single graph representing a relationship between a Ferris wheel cart's height from the ground and its total distance traveled around the wheel for one revolution of a Ferris wheel. Third, students interpreted a replica of another student's graph, explaining how they thought that student may have been thinking when sketching the graph.

The Cannon Man and Toy Car task sequences each incorporated six key elements (Johnson, McClintock, & Gardner, 2020). First, students viewed a video animation, then discussed how it could be possible to measure different attributes in the situation (e.g., Cannon Man's height from the ground and his total distance traveled). Second, students explored variation in each of the individual attributes by manipulating dynamic segments on the horizontal and vertical axes. Figure 3.2 shows a dynamic segment in the Cannon Man task sequence. Third, students sketched a graph to represent a relationship between attributes, then viewed a computer-generated graph. Fourth, students re-explored variation in each of the individual attributes, with the attributes represented on different axes. Fifth, students sketched a new graph to represent the same relationship between attributes, then viewed a computer-generated graph. Figures 3.3 and 3.4 show the two different computer-generated graphs in the Cannon Man and Toy Car task sequences, respectively. Sixth, students responded to questions about relationships represented by both graphs.

We integrated Thompson's theory of quantitative reasoning and Marton's variation theory in our design of the Cannon Man and Toy Car task sequences. First, students could vary each attribute individually, then both attributes together. With the dynamic segments (e.g., Fig. 3.2), we operationalized Thompson's recommendation that students use their fingers as tools to represent variation in individual attributes (Thompson, 2002). Furthermore, the design afforded opportunities for students to discern each graph axis as representing variation in a single attribute (Marton's variation theory). After manipulating individual attributes, students sketched a graph to represent a relationship between attributes.

Second, students repeated the process for a new Cartesian plane with the same attributes represented on different axes. This design choice was not a novelty; Moore and colleagues also leveraged this design move (Moore, Silverman, Paoletti, & LaForest, 2014; Moore, Stevens, Paoletti, Hobson, & Liang, 2019). Our theoretical underpinning for this design choice rests in Marton's variation theory. With the new graph, we incorporated contrast. The relationship between variables in the Cannon Man task sequence remained invariant; only the graph was different. With

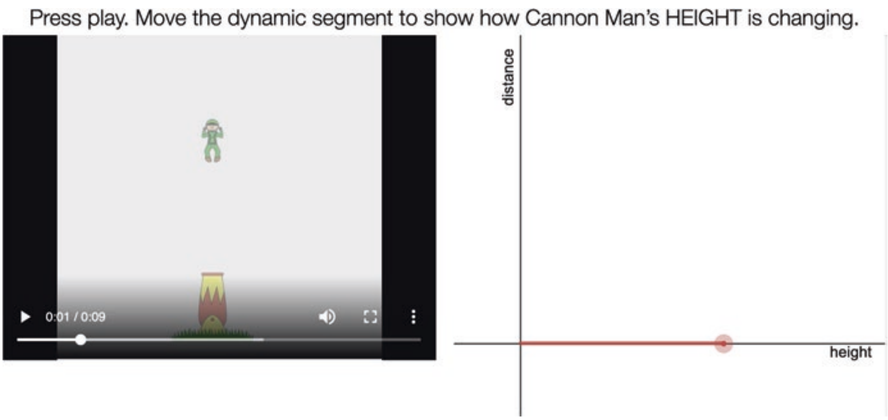

Fig. 3.2 A dynamic segment in the Cannon Man task sequence

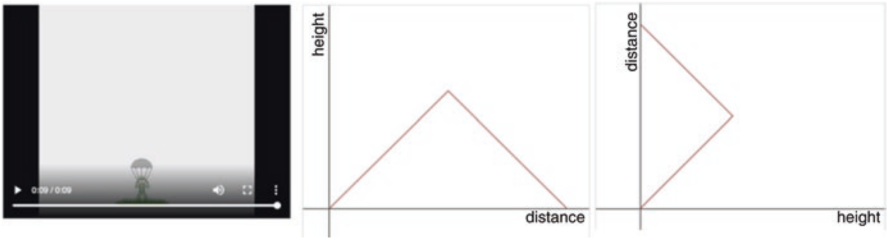

Fig. 3.3 Two different graphs in the Cannon Man task sequence

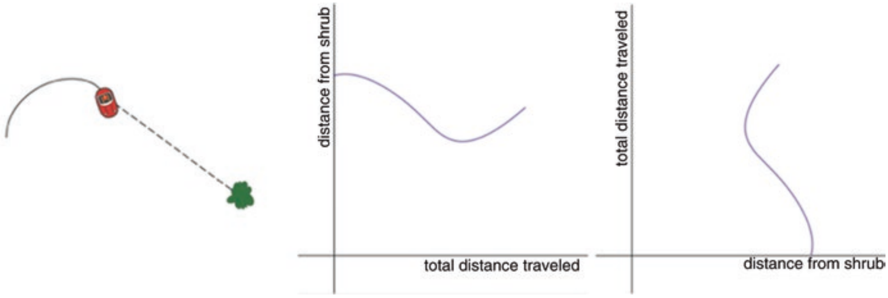

Fig. 3.4 Two different graphs in the Toy Car task sequence

this move, we intended to provide opportunities for students to discern a Cartesian plane as separable from a specific instance of a Cartesian graph.

We designed the first and second patterns of variation and invariance against a single background (the Cannon Man). Next, we engaged in generalization, per Marton's variation theory, repeating those patterns against a new background (the Toy Car). In the video animation (the first element of the task sequence), the literal motion of the Toy Car was different from the literal motion of the Cannon Man. For example, the Toy Car moved along a curved path, but Cannon Man moved up and down. We intended this difference to provide students opportunities to discern what was necessary (e.g., direction of variation in attributes) from what was optional (e.g., literal motion of objects). Across both task sequences, we kept the kind of attributes invariant because we anticipated it would be less difficult for students to conceive of measuring length attributes (e.g., height, distance) than other kinds of attributes, such as area or volume (see also Johnson, McClintock, & Hornbein, 2017).

3.4.2 Data Analysis Methods

To claim that students transferred their covariational reasoning, we first provide evidence of students' engagement in covariational reasoning within and across tasks (Thompson's theory of quantitative reasoning). Second, we identify differences and commonalities that students discerned across tasks (Marton's variation theory). Third, we demonstrate that students meet Lobato's (2008) four criteria for evidence of transfer from an AOT perspective.

Covariational reasoning Our analysis focused on two areas: students' conceptions of attributes as possible to measure and capable of varying and students' conceptions of relationships between those attributes. The framework put forth by Thompson and Carlson (2017) provided fine-grained distinctions regarding different levels of students' covariational reasoning. We gathered evidence of the presence of covariational reasoning rather than distinguishing between different levels of covariational reasoning. As a litmus test for covariational reasoning, we identified the level that Thompson and Carlson (2017) termed gross coordination, in which students conceive of a relationship as a loose joining of two attributes. To illustrate, to claim a student engaged in covariational reasoning in the Toy Car situation, we drew on two pieces of evidence. First, the student conceived of both distance attributes as capable of varying and possible to measure; for example, the student could separate a distance attribute from the situation itself (possible to measure) and show or explain how that distance could vary beyond just describing literal motion of an object (capable of varying). Second, the student conceived of a loose joining of those distances, for example, by showing or explaining how those different distances could vary together (e.g., one distance increased and decreased while the other distance continued to increase).

Transfer of covariational reasoning Our analysis focused on students' discernment of difference and sameness, and students' evidence of engagement in transfer, from an AOT perspective. Drawing on Marton's theory, we analyzed students' discernment when they encountered what we intended to be instances of contrast and generalization. For example, we examined how students discerned attributes represented on each graph axis (a necessary aspect) or the differences in literal motion between the Cannon Man and the Toy Car (an optional aspect). We specified the four criteria put forth by Lobato (2008) for our task sequence. First, students demonstrated a change in reasoning from the Ferris Wheel task sequence (first interview) to the Toy Car task sequence (third interview). Second, in the Ferris Wheel task sequence, students demonstrated limited evidence of covariational reasoning. Third, students' reasoning during the Cannon Man task sequence (second interview) influenced their reasoning during the Toy Car task sequence (third interview). Fourth, students' change in reasoning resulted from their work on interview tasks, and it was not just a spontaneous occurrence.

3.4.3 Empirical Evidence: Aisha's Engagement with the Task Sequences

Aisha attended a high-performing suburban high school in the metropolitan area of a large U.S. city, with just over half of the student population identifying as students of color. Aisha was near the end of ninth grade (about 15 years old) and enrolled in an Algebra I course, which was typical for students in a general college-preparatory track at her school. Aisha's interviews spanned a 2-week time frame, with at least 1 day between; interviews occurred during the school day when she had a free period. She engaged with one task sequence in each interview: Ferris Wheel, Cannon Man, and Toy Car, in that order, working on a tablet (an iPad), with paper and pencil available.

We begin with transcripts and description from each of the task sequences, across the three interviews, followed by our analysis within and across tasks. Figure 3.5 shows some of the graphs that Aisha drew during the interviews. Aisha's Ferris Wheel graph is shown in Fig. 3.5 (left). The Cannon Man and Toy Car graphs, shown in Fig. 3.5 (middle, right), are the second Cartesian graphs that Aisha drew in the task sequence (graphs that we intended to provide contrast per Marton's variation theory).

Ferris Wheel Aisha sketched a graph relating a Ferris wheel cart's height from the ground and total distance traveled around one revolution of the Ferris wheel. While sketching, Aisha explained why she drew the graph in the manner that she did.

Aisha: I feel like the height would be more like the line [sketches a line; Fig. 3.5, left]. Distance would be more like the rise and run of the situation [sketches small segments; Fig. 3.5, left]. Cause you're using the rise and run to find the line, and you need to use the distance to find the height.

Cannon Man Aisha sketched a graph relating Cannon Man's height from the ground and total distance traveled, with the height represented on the horizontal axis

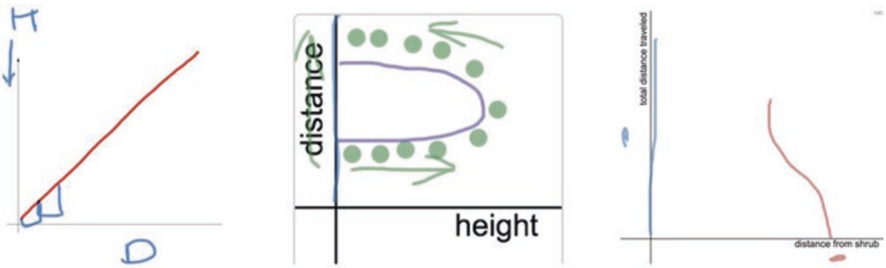

Fig. 3.5 Aisha's Ferris Wheel, Cannon Man, and Toy Car graphs, respectively

and the distance on the vertical axis. Next, Johnson asked Aisha to explain how the graph showed both Cannon Man's height and distance.

Johnson: Can you show me how you see the height increasing and decreasing in this purple graph? [Points to the curved graph Aisha drew; Fig. 3.5, middle]
Aisha: It's [the height's] increasing here, since it's [the graph's] backwards in my opinion [Sketches green dots, beginning on bottom left near the vertical axis, then moving outward; Fig. 3.5, middle]. Decreasing here [Continues to sketch green dots until getting close to the vertical axis, adding arrows after sketching dots; Fig. 3.5, middle].
Johnson: How is the distance changing?
Aisha: [Turns iPad so that the vertical axis is horizontal. Draws arrow parallel to vertical axis; Fig. 3.5, middle.] That way. Continues to get bigger.

Toy Car Before sketching the graph shown in Fig. 3.5 (right), Aisha spontaneously stated that the Toy Car's distance traveled was the "same as the Cannon Man." Following up, Johnson asked Aisha to explain how those different distances could possibly be the same.

Johnson: So, you said the total distance traveled is like the Cannon Man. Why is that like the Cannon Man again? Cause Cannon Man goes up and down, and this one moves around. How are those things the same?
Aisha: Just because Cannon Man is coming back down, doesn't mean his distance is going down. His distance is still rising.

To explore change in the Toy Car's total distance traveled and the Toy Car's distance from the shrub, Aisha manipulated dynamic segments located on the vertical and horizontal axes, respectively. For the total distance, Aisha began at the origin, continually moving the segment up along the vertical axis. She explained: "I moved it up. It continuously went up, because the distance doesn't decrease. The total distance traveled doesn't decrease." For the distance from the shrub, Aisha began to the right of the origin, initially moving the segment to the left, and then to the right, along the vertical axis. She explained:

> I moved it [the segment] to the left, because it [the Toy Car] was getting closer to the shrub. Then, when it [the Toy Car] started to turn, I started to move it [the segment] back up to the right, because it [the Toy Car] was getting closer to the shrub.

Next, Aisha sketched the graph shown in Fig. 3.5 (right). After viewing the computer-generated graph, Aisha stated what she thought the curved graph represented. Aisha stated: "This [moving her finger from left to right along the horizontal axis] is tracking the distance from the shrub, and this [moving her finger along the curved graph, beginning near the horizontal axis] is also tracking the distance."

3.4.4 Analysis: Aisha's Reasoning Within and Across Tasks

Within tasks: The Ferris Wheel task sequence Before sketching a graph, Johnson asked Aisha to explain how she might use a string to measure the Ferris wheel cart's height from the ground and total distance traveled. Appealing to a nonstandard unit, such as a string, was a typical move by Johnson to encourage students to do something other than try to find an answer. For the height, Aisha told Johnson that she would tie the string to the Ferris wheel cart, then drop it down to the ground. For the distance, Aisha said that she would start at the base of the Ferris wheel and then just "go around," moving her finger counterclockwise around the wheel until she ended up back at the base. Aisha's actions demonstrated that she could conceive of the height and distance as attributes possible to measure, or as quantities, per Thompson's theory.

When sketching a graph, Aisha treated height and distance as inputs and outputs, explaining how one might use a formula or rule to determine one amount (height) given another amount (distance). Aisha included both height and distance in a single graph and labeled the axes, but the height and distance were juxtaposed as individual parts of a line graph. A loose joining of attributes would give evidence of covariational reasoning at the gross coordination level. However, Aisha had yet to demonstrate if she could conceive of a relationship between different values of the attributes (e.g., when the cart is this far off the ground, the cart would have traveled this much distance) or even of those attributes as varying together (e.g., the cart's height increased and decreased while the cart's distance traveled continued to increase). Per Thompson's theory, Aisha demonstrated limited evidence of the object of transfer (covariational reasoning). Hence, per Lobato's (2008) criteria, if Aisha were to demonstrate covariational reasoning during a subsequent task sequence, an argument for transfer could be built.

Within tasks: The Cannon Man task sequence The interview began with Johnson telling Aisha to view the video animation, then explain what she thought she might be able to measure in the situation. With this question, Johnson intended to investigate what attributes students might discern on their own. Aisha came up with two attributes: the distance from when the parachute deploys and how high Cannon Man gets in the air, both of which she interpreted in relationship to the ground. To encourage Aisha to talk more about how she might measure the attributes, Johnson asked Aisha how the height was changing. Aisha said that she could measure Cannon Man's height using feet, and there would be more feet when Cannon Man was higher in the air. If a student did not spontaneously identify one of the intended attributes, Johnson would introduce that attribute; here, it was total distance traveled. Aisha said that she thought of it the same way as the height—the further Cannon Man is in the air, the more feet he would have. Johnson then suggested that Aisha think of the total distance as a round trip. With such a move, Johnson intended to give students opportunities to extend beyond their initial impressions of attributes. Aisha responded by explaining that the distance would keep getting bigger and that you could find it by doubling the distance from the ground to Cannon Man's highest point (which she called the "vertex"). Again, in this task sequence, Aisha provided evidence that she conceived of the different attributes as possible to measure (quantities, per Thompson's theory).

Unlike in the Ferris Wheel task sequence, Aisha demonstrated evidence of covariational reasoning in the Cannon Man task sequence. This happened when Aisha sketched the second graph (Fig. 3.5, middle). When annotating the graph that she drew in the Cannon Man task sequence (Fig. 3.5, middle), Aisha explained how she showed the height to be both increasing and decreasing as well as the distance to be increasing. Taken together with earlier evidence of her conceptions of the attributes as being possible to measure, Aisha's loose joining of the varying attributes demonstrates evidence of her covariational reasoning at the gross coordination level, per Thompson's theory. Building our case for Aisha's transfer, per Lobato's (2008) criteria, Aisha demonstrated a change in reasoning from the Ferris Wheel to the Cannon Man.

Aisha's engagement in covariational reasoning occurred not with her first graph but with her second. Per Marton's variation theory, we designed the second graph as contrast so that students could have an opportunity to discern the Cartesian plane itself as being separate from the particular graph being sketched. Aisha discerned the representation of the total distance traveled on a Cartesian plane in the second graph, stating: "I imagine the distance on the ground, which I can't do." In sketching her second graph (Fig. 3.5, middle), Aisha demonstrated that she discerned necessary aspects of Cartesian graphs (that axes represent measurable attributes) from optional aspects (that the location of an attribute on a graph axis matches the literal orientation of the attribute in a situation). By designing task sequences to promote students' discernment of difference in the Cartesian plane, we aimed to engineer opportunities for students to engage in covariational reasoning, and Aisha's actions pointed to the viability of this design move.

Within tasks: The Toy Car task sequence As did the Cannon Man interview, the Toy Car interview began with Aisha identifying "the distance the car drove" as an attribute. Aisha was not sure how she might measure it, so Johnson asked her to sketch the path that she saw the car taking. As in the Cannon Man interview, Johnson asked Aisha how the attribute was changing. Aisha said that it would keep increasing, if one were thinking about the distance the car was going, and not from the start to the end because the car's ending point is close to the starting point. Next, Johnson introduced the attribute of the distance from the shrub and asked Aisha how she saw that attribute changing, to which Aisha responded that the car went "closer to" and then "further from" the shrub, moving her finger along the path of the car. To investigate how Aisha might separate the attribute of the distance from the shrub from the literal motion of the car, Johnson asked Aisha to draw where she saw the distance. Aisha sketched dotted lines from the car's starting point to the shrub and from the car's ending point to the shrub. At this point, Aisha had not seen the dotted line image shown in Fig. 3.4; she had only seen the video animation of the moving car, which had no annotations for distance. As she did in the Cannon Man task sequence, Aisha provided evidence that she conceived of the different attributes as possible to measure (quantities, per Thompson's theory).

Aisha demonstrated covariational reasoning during the Toy Car task sequence but, as had happened in the Cannon Man task sequence, it was not until she sketched the second graph. When sketching that graph (Fig. 3.5, right), Aisha accounted for both the increase in the total distance and the increase and decrease in the distance from the shrub. As with Cannon Man, Aisha identified the segment along the vertical axis as tracking the total distance traveled, which continually increased, and the trace in the plane as tracking the attribute that both increased and decreased. She found the vertical dynamic segment (Fig. 3.5, right) to be necessary to "show" the total distance traveled. Hence, her representation of the joined attributes entailed two connected inscriptions, the dynamic segment and the trace. Building our case for transfer, per Lobato's (2008) criteria, Aisha's reasoning on the Cannon Man task influenced her reasoning on the Toy Car task. In both tasks, she conceived of the total distance traveled to be continually increasing, and she represented that increase by sketching a segment along the vertical axis, beginning at the origin, and extending upward.

Across tasks: From the Cannon Man to the Toy Car We draw further evidence of transfer from Aisha's spontaneous utterance of a sameness that she identified across the Toy Car and Cannon Man task sequences. When working on the Toy Car task, without prompting, Aisha spontaneously stated that she thought an attribute—total distance—was "the same" in both the Toy Car and the Cannon Man tasks. We contend that Aisha's discernment of differences across the task situations contributed to her spontaneous identification of this sameness. Per Marton's theory, we incorporated contrast across the Toy Car and Cannon Man task situations, with difference in the literal motion of each object (Cannon Man moved up and down, whereas the Toy Car moved in a curved path). We did not assume that our design alone would be sufficient to ensure students' discernment; we provided conditions under which discernment might occur. Aisha evidenced such discernment as she separated the direction of the literal motion of each object from the variation in an attribute (total distance) in each situation. For example, Aisha moved the dynamic segment representing the Toy Car's total distance traveled to show that the distance continued to increase, despite the Toy Car moving along a curved path. Consistent with our intent, Aisha distinguished necessary attributes (e.g., continual increase in total distance traveled) from optional aspects (the literal motion of the objects). Drawing on the corpus of evidence, we claim that Aisha transferred her covariational reasoning from the Cannon Man task sequence to the Toy Car task sequence, and her discernment of differences in the literal motion of each object played a role in that transfer.

3.5 Discussion

3.5.1 What Is Possible to Transfer?

With the opening possibilities approach, we aim to expand objects of transfer study. In Lobato's investigation of transfer from an AOT perspective, the focus was on students' transfer of mathematical concepts, such as slope (e.g., Lobato, 2003, 2008, 2012). We demonstrate how the object of transfer can be a form of mathematical reasoning, which can transcend different mathematical concepts. In our application of this approach to students' covariational reasoning, we leave open possibilities for concepts that researchers may address. For example, researchers may engender students' covariational reasoning to develop students' understanding of function writ large or even inverse function more specifically. In our approach, we center students' mathematical reasoning as something that is more than just a process whose value rests in its service to students' development of understanding of mathematical concepts. As a result, we expand what can count as mathematics and, in turn, what can be transferred.

3.5.2 Integrating Theories to Open Possibilities: Reasoning, Discernment, and Transfer

We open possibilities for investigating students' covariational reasoning when interpreting and sketching Cartesian graphs, which are ubiquitous in students' math courses. To address both students' covariational reasoning and their conceptions of graphs, we have drawn on theories that explain students' reasoning (Thompson's theory) and discernment (Marton's theory). Researchers have found that Cartesian graphs may mitigate opportunities for covariational reasoning; university students and prospective teachers may not demonstrate covariational reasoning when sketching graphs despite evidence suggesting their engagement in covariational reasoning in situations not involving graphs (Carlson et al., 2002; Moore et al., 2019). One response to such findings can be to question the potential for researchers and teachers to leverage Cartesian graphs to engender students' covariational reasoning. We take a different stance, provided that students also have opportunities to conceive of graphs as representing relationships between quantities. Integrating theories has afforded our creation of such opportunities, with Marton's variation theory being instrumental in this work. By incorporating contrast and generalization in our task sequences, we have made efforts to problematize aspects of Cartesian graphs as dimensions of variation, and empirical evidence points to the viability of such design.

Our empirical work has focused on secondary students' covariational reasoning, but this design can be applicable to university students or even younger students. By engineering opportunities for students' reasoning in a familiar setting (a Cartesian graph) without specifying a particular mathematical concept, we create room for students to engage in reasoning that may be different from what they have done in previous math courses or in their work with graphs. Furthermore, we connect graphs to situations, such as the Toy Car, so that students can have opportunities to conceive of graphs as representing measurable attributes of events that could occur in the world. Too often, students experience mathematics as a game with rules determined by people in authority (Gutiérrez, 2013) rather than as an opportunity to engage in reasoning and thinking to quantify their world in ways that make sense to them. If students expect that we intend for them to arrive at particular answers or demonstrate their knowledge of certain procedures (even if that was not our intent), the reasoning students demonstrate can be quite different from the reasoning we intend to promote, even if students are capable of demonstrating the intended reasoning. We view our focus on covariational reasoning and Cartesian graphs as one of many avenues for the opening possibilities approach. In future studies, researchers may investigate different forms of reasoning in other situations, such as geometric reasoning in dynamic geometry platforms.

Integrating theories has afforded our articulation of a role of difference, as well as sameness, in investigating students' transfer of mathematical reasoning from an AOT perspective. Again, Marton's variation theory has been crucial in this work. Designing for contrast and generalization has opened possibilities for us to scour the data for differences and similarities that students construe between situations as well as for students to distinguish between necessary and optional aspects of the situations. In Aisha's case, we opened opportunities for her to discern physical characteristics of the situation as optional and measurable attributes as necessary (e.g., the total distance of both Cannon Man and Toy Car continuing to increase despite differences in their literal motion), and this discernment played a role in her transfer of covariational reasoning. The objects of students' covariational reasoning are more than observable things students might notice (e.g., the literal movement of a toy car); they are measurable attributes of situations (e.g., a toy car's distance from a stationary object). However, it can be difficult for students to even conceive of situations as having measurable attributes. When integrating theories, we layer different explanations to guide our larger aim. Thompson's theory explains a form of students' reasoning to promote; Marton's theory provides guidance for design choices to engineer opportunities for students to discern measurable attributes of the situations to foster students' engagement in the intended reasoning. In future studies, researchers can investigate how designing for contrast and generalization, to promote discernment of difference, may afford students' transfer of other forms of reasoning.

3.5.3 Expanding Design Possibilities for Transfer Studies

Through the opening possibilities approach, we work to expand design possibilities for investigating students' transfer to extend beyond pre-post designs. Lobato (2008) has distinguished between tasks implemented during a design experiment study and tasks implemented in pre- or post-interviews. To provide evidence of transfer from an AOT perspective, researchers demonstrate that students' conceptions changed from tasks in a pre-interview to tasks in a post-interview and that students' work during the design experiment tasks has influenced their changed conceptions. Rather than separating design-experiment tasks from post-interview tasks, we illustrate how a student can transfer mathematical reasoning from one design experiment task to another, similar to how Marton (2006) described the possibility for students to transfer their discernment of tone from sound when hearing Cantonese words in succession.

We concur with Cobb's (2007) appeal for theory expansion rather than replacement. With the design expansion we propose, we intend to open new possibilities for investigations from an AOT perspective, in particular, by foregrounding roles of difference and similarity. Across the Cannon Man and Toy Car task sequences, we have designed for contrast and generalization and subsequently have analyzed for both difference and similarity. Integrating Marton's variation theory with an AOT perspective has afforded us this possibility. In turn, we have then been able to analyze for students' transfer of reasoning within the design experiment tasks themselves rather than examining students' reasoning on a separate set of transfer tasks, as was done in an earlier study (Johnson, McClintock, & Hornbein, 2017).

3.6 Conclusion

With theory integration comes responsibility, including the consideration of the epistemological roots of different theories (Bikner-Ahsbahs & Prediger, 2010). Such responsibility is both a limitation and an affordance of the approach, because each theory needs to be weighed in light of the other(s). Integrating theories is a purposeful choice so that researchers can explain phenomena that extend beyond the bounds of a single theory. We have integrated theories specific to reasoning and transfer (Thompson's and Lobato's theories, respectively) with a theory that addresses discernment of different content and extends beyond transfer (Marton's theory). The *grain size* (Watson, 2016) of the theories differ, with two being more domain specific and one being broader. However, we have not imposed a hierarchy of theories onto our analysis; instead, we have layered analytic techniques from each theory. To guide our choices, we have drawn on scholars' assumptions of distinctions between researchers' and students' perspectives and have articulated how those assumptions have influenced our work.

With opening possibilities, we offer an approach to navigate complexities in researchers' investigations of students' transfer of mathematical reasoning. Although our focus is on transfer, we can conceive of the guiding questions as applicable to the broader work of research. Researchers can examine what counts as their object of investigation, how they may engender the study of that object, and what may constitute evidence of the objects of study. Assuming that theory and method mutually inform each other, our approach affords the integration of different theories to embrace, rather than reduce, complexities. Through this approach, we expand design possibilities for investigating students' transfer, acknowledging a symbiotic relationship between the theories that we integrate and the contributions that those theories and methods make possible.

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Chapter 4
A Case for Theory Development About Backward Transfer

Charles Hohensee

The body of mathematics education research that has examined transfer of learning from a constructivist *complex-systems view* of knowledge development suffers from what I perceive as a research imbalance. A complex-systems view, as defined by Smith, diSessa, and Roschelle (1993), is the view that knowledge is composed of "numerous elements and complex substructure that may gradually change, in bits and pieces and in different ways" (p. 148). Incidentally, this view emerged in reaction to the more widely held view that knowledge is composed of "separable independent units" (Smith et al., 1993, p. 125). The research imbalance to which I refer is that most transfer of learning research that assumes a complex-systems view has focused on how prior knowledge within the system is applied to new contexts (e.g., Wagner, 2010) or how it influences new learning (e.g., Bransford & Schwartz, 1999), without equal research attention being given to the transfer of learning in the other direction, namely in the direction of how new learning influences prior knowledge. The former is often referred to as *forward transfer* (Gentner, Loewenstein, & Thompson, 2004), and the latter I refer to as *backward transfer* (Hohensee, 2014). The research imbalance in favor of forward transfer is inconsistent with a complex-systems view of knowledge development because complex systems have multidirectional interrelationships.

This imbalance within transfer of learning research could be having undesirable consequences for the field of mathematics education. For instance, without transfer of learning research in both directions (i.e., forward and backward), the field of mathematics education will, at best, develop only partial understandings of how learning transfers within a complex system of knowledge. Second, because of the imbalance, there may exist untapped pathways to improving mathematics learning that will not be explored.

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An important step in establishing greater balance would be for researchers to engage in theory development about backward transfer in mathematics education. Theory development about backward transfer in mathematics education would help to legitimize and raise awareness of transfer of learning in the less-studied backward direction. A developed theory about backward transfer could also unify, under a common theory, the limited number of studies that, to date, have reported effects that resemble backward transfer but that have not yet been labeled as such.[1] Unifying these studies under a common theory of backward transfer would also help to establish a critical mass of findings about backward transfer and generate momentum for more research in this area. Finally, theory about backward transfer would allow for comparisons between transfer of learning in the two directions and could eventually lead to a unified theory that explains transfer in both directions.

In this chapter, I make a case, within the field of mathematics education, for the need for theory development about backward transfer. I begin by presenting my conceptualization of backward transfer. Then, I outline several reasons for theory development in educational research more broadly, and why these reasons are applicable to backward transfer in mathematics education. Next, I explain the process I went through to search the literature for prior research on backward transfer and provide an overview of prior research and the state of theory development about backward transfer in mathematics education and related fields. Finally, I present several aspects of theory development for backward transfer in mathematics education that I view as most pressing.

4.1 Conceptualization of Backward Transfer

Backward transfer, as I conceive it, is an extension of Lobato's (2008) definition of transfer, which is that transfer is "the influence of a learner's prior activities on his or her activity in novel situations" (p. 169) and "the processes by which people generalize their learning experiences, regardless of whether the personal relations of similarity that people form across situations lead to correct performance" (p. 168). Note that within Lobato's definition, the influence is in the forward direction *from* a learner's prior activities *to* a learner's activities in novel situations. I extended this definition in the other direction to include influences that activities in a new or novel situation might have on learners' prior activities regardless of whether the influences lead to correct performance. The specific definition I use for backward transfer is the following: Backward transfer is the influence that learning experiences about a new topic have on learners' prior ways of reasoning about an initial topic (Hohensee, 2014).

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[1] Note that in addition to the limited number of studies, as described in this chapter, that report backward transfer effects, there is a sizeable body of language-learning research on backward transfer that will not be considered because those studies focus on production and comprehension of language rather than on cognition.

My conceptualization of backward transfer is grounded in a complex-systems view of knowledge development. According to this view, "Learning a domain of elementary mathematics or science may entail changes of massive scope. New elements may gradually come to play central roles as core knowledge, creating very large ripple effects through the system" (Smith et al., 1993, p. 148). It is the ripple effects through the complex knowledge system during learning something new that I think of as potential backward transfer effects.

Note that in my definition of backward transfer, I referred to *ways of reasoning* rather than to underlying mental structures, such as *conceptions*, *knowledge, schemes*, and so on. At this early stage of theory development, claiming that learners' conceptions, knowledge, or schemes are being influenced during backward transfer seemed overly strong. My rationale for using ways of reasoning was because it seemed more conservative to claim that ways of reasoning are being influenced during backward transfer. Gravemeijer (2004) and McClain, Cobb, and Gravemeijer (2000) used ways of reasoning to refer to "mathematical activities that students engage in while solving, explaining, justifying, identifying, and so on" (Hohensee, 2014). It is these activities, rather than the underlying mental structures, that I focus my attention on when studying backward transfer.

To illustrate backward transfer, consider Alan, a 10th-grade student who participated in a 12-lesson instructional unit on quadratic functions in his regular algebra class (Hohensee, Willoughby, & Gartland, 2020). Before and after the unit, my research team and I gave Alan several problems about linear functions to examine if and how his prior ways of reasoning about linear functions had been influenced by his new learning experiences with quadratic functions. One of the problems we gave Alan before the unit on quadratic functions involved a picture of a plant as it grew at a steady rate over 4 days. Alan was asked to find the height on Day 17 (see Fig. 4.1).

To solve this problem, Alan created a table for days and heights and continued the pattern until he arrived at Day 17 and the correct height.

After the quadratic function unit (i.e., approximately 4 weeks later), we gave Alan a similar problem. The problem involved four snapshots of a container as it filled with rainwater at a constant rate over 4 hours. Alan was asked to find the height of the water after 11 hours (see Fig. 4.2).

To solve this problem, Alan did not create a table, but instead divided each height in the picture by the associated number of hours. Because each quotient was approximately 3, Alan decided to multiply Hour 11 by 3 to find the height, but this was an incorrect answer. Clearly, Alan's reasoning had changed. It was our hypothesis that Alan's prior ways of reasoning about the first problem were influenced by his participation in the new learning experiences about quadratic functions. In other words, we hypothesized that this was an instance of backward transfer.

Throughout the remainder of this chapter, I present a case for why more theory development about backward transfer is warranted. As part of making this case, I summarize published studies that have reported what I categorize under the umbrella of backward transfer effects. Although these studies are limited in number, they

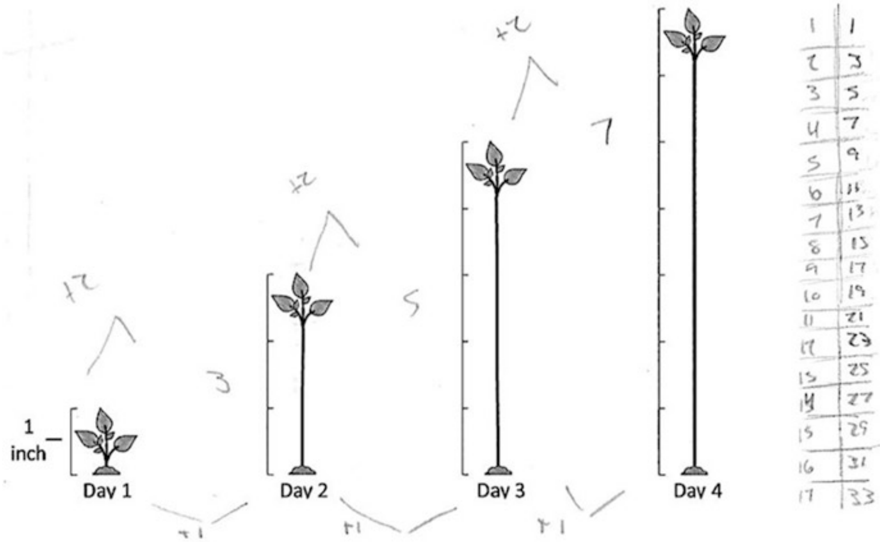

Fig. 4.1 Alan's response to the plant problem before the quadratic function unit

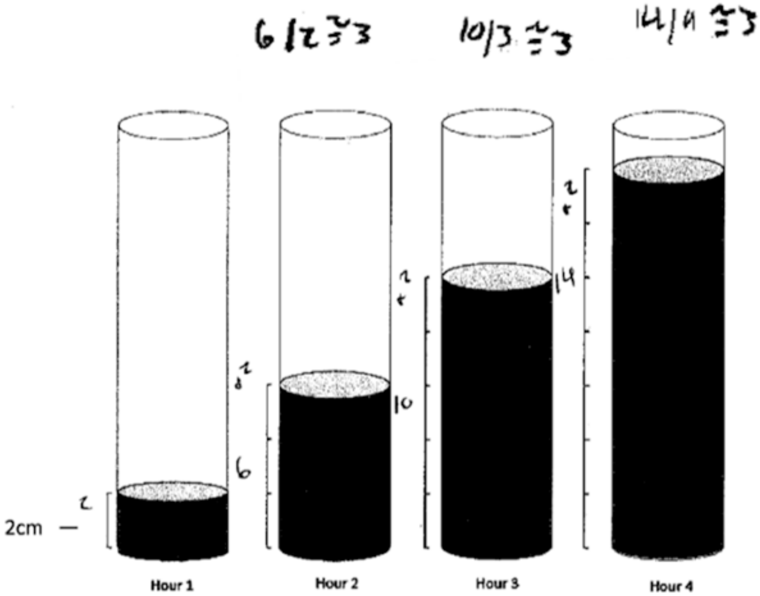

Fig. 4.2 Alan's response to the rainwater problem after the quadratic function unit

potentially represent the tip of the iceberg in terms of new insights that research on backward transfer could generate about how students learn mathematics.

4.2 Reasons for Theory Development in Educational Research Applied to Research on Backward Transfer

As stated above, theory about backward transfer would legitimize and raise awareness of backward transfer, unify the limited studies on cognition that have reported effects resembling backward transfer, and allow for comparisons between forward and backward transfer. In addition to these reasons for theory development, diSessa (1991) provided five further reasons for more theory development in educational research, which also apply to research on backward transfer.

First, more theory development is needed in educational research because theories take time to develop, on the scale of decades or generations (diSessa, 1991). This is an important consideration for theory about backward transfer because backward transfer has only recently been introduced to the field of mathematics education (i.e., Hohensee, 2014). And, even if theory about backward transfer had been in active development right from its introduction to this field, the time for theory development would still be far shorter than the decades or generations recommended by diSessa (1991). As a comparison, forward transfer theory development has been ongoing at least since the time of Thorndike (circa 1920), and much more developed theory about forward transfer exists (e.g., Beach's theory of *consequential transitions*, 1999; Bransford & Schwartz's theory of *preparation for future learning*, 1999; Engle's theory of *transfer as framing*, 2006; Greeno, Moore, & Smith's theory of *transfer of situated learning*, 1993; Lobato's theory of *actor-oriented transfer*, 2012; and Wagner's theory of *transfer in pieces*, 2010).

Second, more theory development is needed because theory is a richly interconnected collection of ideas (diSessa, 1991). With respect to backward transfer, only a few of the connections of ideas have been explored thus far and many other potential connections have yet to be examined. For example, in mathematics education to date, only one underlying mechanism has connected backward to forward transfer, namely the process of *student noticing* (Hohensee, 2016; Lobato, Rhodehamel, & Hohensee, 2012). The kinds of connections of ideas about backward transfer that require exploration and development will be provided later in this chapter.

Third, theory development is needed because generalities are not stumbled upon but emerge through theory development (diSessa, 1991). Results from my literature review of mathematics and science education research support this point by revealing that few mathematics and science education studies have stumbled upon backward transfer effects. Rare exceptions include Arzi, Ben-Zvi, and Ganiel (1985), Macgregor and Stacey (1997), Rebello et al. (2005), and Van Dooren, De Bock, Hessels, Janssens, and Verschaffel (2004). This underrepresentation of backward transfer effects in the literature is likely, at least partially, due to the lack of exposure of backward transfer in the field. However, this underrepresentation also suggests that backward transfer may be difficult to detect. This is akin to the difficult-to-detect sub-atomic neutrinos that diSessa (1991) argued required theory before detection was possible. Thus, theory development of backward transfer could inform the design of more sensitive and precise measures of backward transfer effects.

Fourth, theory is needed for there to be data (diSessa, 1991). This reason is highly relevant to theory development about backward transfer. Because backward transfer theory is still in its infancy, backward transfer researchers are currently limited in the kinds and quality of data about backward transfer that can currently be collected and analyzed. DiSessa (1991) stated that without theory, the "whole rationale for the experiment and set of observations would not exist, nor would the fabric of reasoning that makes the observations informative" (p. 225) and that "we can sometimes judge the quality of theory by the quality of its data" (p. 226). By developing theory, the kinds of backward transfer data researchers would be able to collect would expand and the quality of that data would improve.

Fifth, theory development is needed because respectable theory transcends common sense (diSessa, 1991). Views about backward transfer that are based on common sense might create a misleading picture of what backward transfer is and cast doubt on whether backward transfer research is a worthwhile pursuit. For example, diSessa pointed out that common sense could lead someone to doubt the existence of Newtonian forces. Similarly, I have encountered a number of common-sense views that reflect doubts about the existence of backward transfer. One common-sense idea expressed to me was that what I interpret as backward transfer can be explained away as nothing more than students being swayed by what they have been studying most recently. Another common-sense notion I have encountered is that backward transfer only happens because students failed to establish the required clear understanding of a concept when they had the opportunity to do so. A third common-sense explanation I have encountered is that what I call backward transfer is nothing more than students continuing to develop their prior knowledge. Without theory development, common-sense explanations of backward transfer may too conveniently be used to explain away or cast doubt on what is actually occurring and perpetuate the imbalance in mathematics and science education research described above.

4.3 Literature Search for Research on Backward Transfer

My literature search for prior research on backward transfer began over 10 years ago in 2009. Since then I have, on an ongoing basis, scoured mathematics education research in particular, and education research in general, as well as cognitive science and psychology research, for articles on backward transfer and related constructs. I have searched using numerous keywords, including the following: *backward transfer*, *backward learning*, *backward knowledge*, *reverse learning*, *reverse transfer*, *reverse knowledge*, *inverted learning*, *inverted transfer*, *inverted knowledge*, *retrospective transfer*, *retrospective learning*, *two-way learning*, and *two-way transfer*. In addition to my own literature search, I have consulted numerous social-science researchers for suggestions about potential sources of research on backward transfer.

This decade-long search has revealed four terms that have been used to represent what I define as backward transfer. By far the most commonly used term is *backward transfer*. However, this term is primarily found in linguistics research (e.g., Cook, 2003), and no uses of this term were found in mathematics education research except those that I published or that have drawn upon my work (e.g., Moore, 2012; Young, 2015). I additionally identified the use of three other terms that align with what I define as backward transfer, namely *transfer backward* (Gentner et al., 2004), *retrospective transfer* (Marton, 2006), and *met-afters* (Lima & Tall, 2008), all of which will be explained later in this chapter.

4.4 Overview of Research and the Current State of Theory About Backward Transfer

Prior research and the current state of theory about backward transfer comes from a limited number of studies on cognition. Of the research that has reported on backward transfer effects explicitly, or that I interpreted as backward transfer effects but that were referred to with other labels, three categories of theories exist: (a) theories that explain unproductive backward transfer effects in terms of interference and overgeneralization, (b) theories that explain productive backward transfer effects in terms of specific changes to cognitive structures, and (c) theories that explain both productive and unproductive backward transfer effects in terms of attention and noticing. I define *productive* backward transfer effects as when new learning enhances, clarifies, or deepens prior ways of reasoning and *unproductive* effects as when new learning muddles, distorts, or disrupts aspects of prior ways of reasoning. Next, I present the mathematics and science education studies that have reported backward transfer effects and I critique the theories that were used to explain those effects.

4.4.1 Theories That Explain Unproductive Backward Transfer Effects

During my literature search for studies on backward transfer effects, I identified three studies that reported what I would interpret as unproductive backward transfer effects: Macgregor and Stacey (1997) examined students' understandings of algebra symbols, Van Dooren et al. (2004) focused on student learning of proportional and non-proportional relationships, and Lima and Tall (2008) investigated students' problem-solving strategies for linear and quadratic equations.[2]

Macgregor and Stacey's (1997) study on students' understandings of algebra symbols. The topic of study in Macgregor and Stacey (1997) was how students interpret and use algebra symbols. The research goal was to explain the roots of students' misinterpretations of algebra letters and unclosed algebra expressions. Previously, Hart (1981) had shown that students' progress in algebra (or lack thereof) could be only partially explained with IQ and cognitive-development levels. Macgregor and Stacey conducted their study to identify additional factors that help explain students' misinterpretations of algebra symbols.

For their study, Macgregor and Stacey (1997) gave a paper-and-pencil pre-assessment about interpretations of algebra letters to 11- and 12-year-olds, who had not yet received any algebra instruction ($n = 42$), and gave several items from the same assessment to 11- to 15-year-olds, who had already received some algebra instruction ($n_1 = 1463$ for one item and $n_2 = 1806$ for another item).

One of the results from the study was that the students who had some algebra instruction sometimes made errors that students who had not had any algebra instruction yet did not make. For example, a greater percentage of students who had received some algebra instruction, thought that when x is not specified, then it means $x = 1$ (e.g., concluding that $10 + x = 11$). In another example, a greater percentage of students who received some algebra instruction, compared to those who had not yet received any algebra instruction, thought that an equilateral triangle with side length x had a perimeter of x^3 rather than $3x$. I interpreted this finding as a case of unproductive backward transfer because students' prior ways of reasoning about algebra notation had changed for the worse and because it appeared the new algebra learning had in some way influenced that change in their prior ways of reasoning.

Macgregor and Stacy's (1997) explanation for why students who had received some algebra instruction made errors in greater percentages was because there was "*interference* [emphasis added] from new learning" (p. 17). Specifically, they attributed the students' misconceptions to an interference effect from instruction about variables with exponents, such as learning that $x^1 = x$ and that $x^0 = 1$.

I did not find the explanation of interference compelling because this explanation implies that our conceptions compete with each other. Furthermore, this explanation is consistent with a *replacement view* of knowledge refinement, namely that learning involves replacing the interfering conceptions with the correct conceptions (Posner, Strike, Hewson, & Gertzog, 1982). However, this explanation goes against the constructivist complex-systems view, espoused by Smith et al. (1993), that learning involves refining and integrating conceptions rather than replacing wrong conceptions that interfere with correct ones.

[2] Although Rebello et al. (2005) also involved negative backward transfer, I did not describe it in this chapter because it focused on backward transfer that occurs within very short time frames (i.e., how doing one problem on an assessment influences doing another problem on the same assessment).

Van Dooren et al.'s (2004) study on student learning of proportional and non-proportional reasoning. The topic of study in Van Dooren et al. (2004) was how students reason about proportional and non-proportional relationships. The research goal was to develop and test experimental lessons that disrupt students' inclination to reason proportionally in non-proportional contexts, such as in the context of using length measures to find areas of two-dimensional shapes and volumes of three-dimensional shapes. This study examined eighth graders from two intact secondary classes in Belgium. The experimental-group class ($n_1 = 18$) participated in 10 special lessons that addressed students' overgeneralization of linearity and took three word-problem tests with proportional and non-proportional items, a pre-test, a post-test, and a delayed retention test. The control-group class ($n_2 = 17$) did not participate in the 10 special lessons but participated in regular lessons instead and took only the pre-test and the delayed retention test.

One result from the study was that scores on the proportional items for the experimental group went down from pre-test to post-test from 83.3% to 52.5%. To illustrate why experimental-group students' scores went down, consider the following statement from an experimental-group student:

> I really do understand now why the area of a square increases 9 times if the sides are tripled in length, since the enlargement of the area goes in two dimensions. But suddenly I start to wonder why this does not hold for the perimeter. The perimeter also increases in two directions, doesn't it? (Van Dooren et al., 2004, p. 496)

I interpreted this finding as a case of unproductive backward transfer because the experimental-group students' ways of reasoning became less correct on the proportional items and the new learning about non-proportional contexts appeared to have influenced that change in their ways of reasoning.

Van Dooren et al.'s (2004) explanation for this finding was that the experimental group "*overgeneralized* [emphasis added] the newly learnt non-proportional strategies to proportional problems they previously solved very well" (p. 497). This explanation aligns in one respect with my view of backward transfer because *generalization* and *transfer* are often referred to interchangeably (e.g., Barnett & Ceci, 2002; Lobato, 2012). However, overgeneralization does not seem like a compelling explanation because it is typically used for transfer in the forward direction (e.g., Hiebert & Wearne, 1985; Zaslavsky, 1997). Therefore, using overgeneralization to explain the Van Dooren et al. findings means hiding the directionality of the generalization (i.e., that new learning generalized back to problems students had previously solved). Second, as with the explanation of interference, overgeneralization in the context of this study would mean that the new knowledge replaced prior knowledge. Using overgeneralization to explain the findings means ignoring the possibility that the new knowledge and prior knowledge interacted in some more complex way than replacement which, as explained earlier, does not align with a complex-systems view of knowledge development.

Lima and Tall's (2008) study on students' problem-solving strategies for linear and quadratic equations. The Lima and Tall (2008) study examined students' solving methods for linear equations. The research goal of this study was to understand students' solving difficulties, particularly for linear equations that have variables on both sides of the equal sign and those that have variables only on one side of the equal sign. One of the constructs used in this study that aligns with backward transfer and that was mentioned earlier is called a *met-after*, which Lima and Tall (2008) defined as follows: "We use the term 'met-after' to denote an experience met at a later time that affects the memories of previous knowledge" (p. 6). In other words, this construct describes an influence in the direction *from* a new experience *to* prior knowledge.

The Lima and Tall study tested Brazilian high school students' ability to solve three linear equations ($N = 68$). Results showed that only 37%, 37%, and 10% of students were successful at correctly solving $5t - 3 = 8$, $3x - 1 = 3 + x$, and $2m = 4m$, respectively. Two additional results from the study that pertained to backward transfer were that, after students learned about quadratic functions, (a) one student inappropriately applied the quadratic formula to solve $5t - 3 = 8$ by assigning a = 5, b = −3, and c = 8 and (b) three other students incorrectly treated $3x - 1 = 3 + x$ as the product of two binomials (i.e., they simplified the equation into the expression $9x + 3x^2 - x - 3$). Lima and Tall (2008) interpreted these findings as "a negative met-after, in which current knowledge is misapplied in solving an earlier problem" (p. 13).

I interpreted these two results as instances of unproductive backward transfer because students' prior ways of reasoning about solving linear equations appeared to have been negatively influenced by the instruction they received about quadratic equations and expressions. Lima and Tall's (2008) explanation for this finding was that "the earlier learning is likely to be fragile to be affected in this way" (p. 14) and that the results reflected "movement of algebraic symbols as a form of functional embodiment that may be performed without meaning" (p. 15). This explanation seems compelling but does not explain how backward transfer could affect conceptual understanding.

Finally, an additional critique of Macgregor and Stacy's, Van Dooren et al.'s, and Lima and Tall's explanations for their backward transfer results is that those explanations best explain unproductive effects (i.e., when new learning muddles, distorts, or disrupts productive aspects of prior ways of reasoning) and do not provide a compelling explanation for backward transfer influences that could be productive (i.e., when new learning enhances, clarifies, or deepens prior ways of reasoning). It would be more unifying if theory about effects like those found in these three studies could account for both unproductive and productive effects.

4.4.2 Theories That Explain Productive Backward Transfer Effects

During my literature review, I identified three studies that reported what I would interpret as productive backward transfer effects: Gentner et al. (2004) addressed learning new negotiation strategies through analogous encoding, Piaget (1968) looked at children's ability to recall visual displays, and Arzi et al. (1985) investigated students' understanding of physical science concepts. Note that, although all three studies come from outside mathematics education, they all address some aspect of cognition.

Gentner et al.'s (2004) study on learning through analogous encoding. The topic of study reported in Gentner et al. (2004) was how analogical encoding facilitates the transfer of learning. *Analogical encoding* is defined as the process of comparing two analogous examples to promote *schema abstraction* over *situation-specific encoding*. In other words, comparing two analogous examples was hypothesized to be a process that helps learners discover a common principle. The research goal of this study was to test the hypothesis that because analogical encoding results in the discovery of a common principle that is not tethered to any one situation, the common principle should subsequently be more readily transferable, forwards and backwards, to other contexts.

This study was situated in the context of a training seminar for full-time professional management consultants who were learning about new negotiation strategies. The participants were divided into an experimental group ($n_1 = 64$) and a control group ($n_2 = 60$). The groups were given two negotiation cases that both illustrated a particular negotiation principle called the *contingent contract principle.* The experimental group, but not the control group, was asked to look for similarities or parallels between the two cases. Both groups were then asked to recall negotiation cases from their own experiences that shared similarities with the cases they had been given. Finally, participants in both groups partnered up with someone from the same group and role-played the negotiation of a new case for which the contingent contract principle applied.

One of the findings from this study was that, compared to the control group, the experimental group recalled more negotiation cases from personal experience (or from a colleague's experiences) to which the new negotiation strategy applied. I interpreted this finding as a case of productive backward transfer because the intervention, in which the experimental group examined two cases for similarities, served as the new learning experience that appeared to influence in a productive way how that group interpreted prior negotiation experiences, which were the prior ways of reasoning. Incidentally, Gentner et al. referred to this effect as *transfer backward*.

Gentner et al. (2004) explained this finding in terms of schema-abstraction. Specifically, they argued that by comparing two partially understood analogous examples, the negotiators in the experimental group developed more abstract schemata for a particular concept and that the changes in their schemata facilitated the reinterpretation of previously encountered, structurally similar experiences. I found this explanation compelling. However, from a constructivist perspective, I would reframe schema-abstraction as *reflective abstraction*, which is "a (more or less conscious) cognitive reconstruction or reorganization of what has been transferred" and which borrows "certain co-ordinations from already constructed structures and to reorganize them in function of new givens" (Piaget as cited in von Glasersfeld, 1995, p. 104). Thus, my characterization of the findings in terms of reflective abstraction is that what the negotiators did was to reorganize their views of negotiations because of new givens encountered during the training seminar.

Piaget's (1968) study on children's ability to recall visual displays. The topic of Piaget's (1968) study was about how children retain their memories of visual displays over different intervals of time. The research goal was to examine the hypothesis that the development of children's operational schemata causes changes in the encoding of their memories.

In one test, Piaget (1968) showed children, ages 3–6, an arrangement of 10 small sticks of differing sizes, ranging from 9 to 15 cm, and arranged in a row from smallest to largest (see Fig. 4.3). The children were asked to look carefully at the arrangement. One week later, they were asked to draw the arrangement without seeing it again. The youngest children in the sample, the 3- and 4-year-olds, typically drew "a certain number of sticks lined up, but all the same length" (Piaget, 1968, p. 4). Six months later, the same children were asked to redraw the arrangement, and "74% of the subjects had a better recollection now than they had after one week" (Piaget, 1968, p. 4). Specifically, those children's memories appeared to have improved during the intervening 6 months because their drawings now showed sticks with organized variations in size, such as half big sticks and half small sticks, or three ordered sizes of sticks and so on. I interpreted Piaget's finding as a case of productive backward transfer because the children's visual and spatial experiences during the intervening 6 months between when they drew what they remembered the first and second time served as the new learning experiences that appeared to enhance their initial memories of the sizes and arrangement of the sticks, which served as their prior ways of reasoning.

To explain this finding, Piaget (1968) conceived of children's initial memories of the sticks as being the result of them having assimilated the visual presentation of the sticks with their current operational schemes. However, during the intervening 6 months, the children's operational schemes developed further. Thus, when they recalled the sticks, 6 months after having first seen them, their further developed operational schemes began to assimilate the original memories differently (i.e., better recollection of the sizes and organization of the sticks). I found this explanation compelling and consistent with a complex-systems view of knowledge development.

Arzi et al.'s (1985) middle school science learning study. The topic of study reported in Arzi et al. (1985) was how middle school science learning is affected by *retroactive facilitation*, which was defined by Arzi et al. as when "subsequent

Fig. 4.3 Arrangement of the sticks shown to children

courses help to consolidate the previously learned subject matter [antecedent learning]" (p. 371). The research goal was to examine how retroactive facilitation changes students' long-term retention of physical sciences learning in real school settings. The study involved science students who, in the seventh grade, all took a chemistry course that examined the three states of matter and the differences between mixtures and compounds (n_1 = 3167). Subsequently, one subgroup (n_2 = 59 of 142 classes) took a follow-up chemistry course on the periodic table in the eighth grade. Importantly, the subject matter from the seventh-grade chemistry course was not covered in the eighth-grade chemistry course. The other subgroup (n_3 = 50 of 142 classes) took a follow-up course on biology and physics in the eighth grade instead of the chemistry course. The biology and physics course did not cover any chemistry concepts. All students took an assessment on the seventh-grade content at the beginning of the eighth grade and again at the beginning of the ninth grade.

Results showed that both groups did better at the beginning of the ninth grade on the seventh-grade assessment then they had at the beginning of eighth grade (i.e., the ratio Grade 9 score/Grade 8 score was greater than 1 for both groups). Furthermore, the subgroup that took the follow-up chemistry course in the eighth grade did statistically better on the seventh-grade content at the beginning of the ninth grade than the group that took the biology and physics course in the eighth grade, even though none of the "facts, concepts, and principles learned in [the seventh-grade science course were]…retaught in [the eighth-grade course] as part of the syllabus" (Arzi et al., 1985, p. 382). I interpreted this finding as a case of productive backward transfer because the new learning in the eighth-grade chemistry course, and to a lesser extent the new learning in the biology and physics course, appeared to enhance students' prior ways of reasoning about the seventh-grade material.

Arzi et al. (1985) called this finding a case of retrospective facilitation and explained the effect using Ausubel's assimilation hypotheses in which "new meanings were incorporated into the students' existing structures of knowledge, via processes termed by Ausubel as progressive differentiation and integrative reconciliation of concepts" (Arzi et al., 1985, p. 385). I found this explanation compelling because it aligns with a complex-systems view of knowledge development.

Summary. As shown, the three studies in this section provided explanations of backward transfer that involved a change to cognitive structures. In Gentner et al. (2004), the cognitive changes were described as schema abstraction; in Piaget (1968), they were characterized as developments in children's operational schemes; and in Arzi et al. (1985), they were explained as differentiations and integrations of structures of knowledge. All three explanations are compelling for explaining the particular productive backward transfer effects with which they are associated. However, the Gentner et al. and Piaget explanations seem less generalizable to all three backward transfer effects (e.g., the Piaget and Arzi et al. studies did not appear to involve schema abstraction). In contrast, the explanation provided by Arzi et al. seems more generalizable (e.g., Piaget's findings of changing memories could be the result of differentiation and integrative reconciliation of visual and spatial concepts). Finally, all three explanations do not seem particularly useful for explaining unproductive backward transfer effects.

4.4.3 Theories That Explain Backward Transfer in Terms of What is Noticed

Finally, I present two articles from the mathematics education literature that provided explanations for backward transfer effects in terms of changes in what is noticed in perceptual or conceptual fields. The first work is a theoretical article by Marton (2006) and involves noticing similarities and differences in contrasting visual displays. The second work is an empirical article by Hohensee (2014) and involves ways of reasoning covariationally about linear and quadratic functions.

Marton's (2006) article on noticing similarities and differences in contrasting visual displays. In this article on noticing similarities and differences, Marton (2006) described *retrospective transfer* as "how the image of an object is affected by experiences following the birth of the image" (p. 520). To explain this idea, Marton cited an example from English philosopher James Martineau about an individual seeing a red ivory ball for the first time. In the example, when the red ball is withdrawn, the individual will retain a mental image of the ball, which Martineau described as "a mental representation of itself, in which all that it simultaneously gave us will indistinguishably co-exist" (Marton, 2006, p. 520). According to Martineau, if the same individual is then shown a white ivory ball, the white ball's contrasting color will bring to the foreground the color in the mental image of the red ball. If, instead of a white ball, the individual is shown an egg after seeing the red ball, its contrasting shape will bring to the foreground the shape of the mental image of the red ball. In other words, the initial image of the red ivory ball can be influenced by the subsequent visual experience of seeing the white ivory ball or the egg, making this an example of retrospective transfer.

I interpret retrospective transfer as a kind of backward transfer because it involves prior ways of reasoning, a new learning experience, and an influence in the direction from the new learning experience back to prior ways of reasoning. Marton's (2006) explanation of retrospective transfer is based on the view that perceptual abilities are hardwired to discern differences. According to Marton, when individuals encounter perceptual objects, they notice some features of objects and do not notice others, or the features form an undifferentiated background. However, when individuals subsequently perceive new objects that serve as a contrast to the initial perceived objects, what makes the initial and new perceived objects different becomes foregrounded in our perception, even if those features were not closely attended to when the initial objects were perceived. Because noticing plays an important role in assimilation, accommodation, and reflective abstraction (Hohensee, 2016), this explanation is compelling and aligns with a constructivist complex-systems view of knowledge development.

Hohensee's (2014) covariational reasoning study. The topic of study reported in Hohensee (2014) was the relationship between quadratic functions instruction and how students reason about linear functions. The research goal was to understand when and in what ways quadratic functions instruction influences students' prior ways of reasoning about linear functions. For this study, a pre-post design was used to examine the reasoning of middle school students who were participating in a summer algebra enrichment program ($N = 7$). The Eight-day enrichment program served as the intervention for the study and focused on quadratic functions. A defining feature of the algebra enrichment program was that covariational reasoning was continuously promoted. Before and after the program, students were interviewed about their ways of reasoning about linear functions. Results from the study showed that most students' level of covariational reasoning on the linear function tasks became more advanced from pre-intervention to post-intervention (e.g., more reasoning with changes in quantities from pre to post) but also that some students became less advanced in their covariational reasoning. I interpreted these results as evidence of productive and unproductive backward transfer.

As indicated earlier, in Hohensee (2016), I offered the process of student noticing as an explanation for the findings. To support this claim, I found evidence that what students came to notice during quadratic functions instruction influenced what they then noticed perceptually and conceptually when reengaging their ways of reasoning about linear functions. In other words, what students noticed during quadratic functions instruction appeared to influence—in some cases productively and in other cases unproductively—the ways they subsequently reasoned with two quantities in linear function contexts. And, as argued above, noticing plays an important role in assimilation, accommodation, and reflective abstraction and thus, this explanation aligns with a constructivist complex-systems view of knowledge development.

Summary. Although Marton's (2006) and Hohensee's (2016) explanations of backward transfer both involve the process of noticing, there are at least five differences between the explanations that suggest that the latter offers a broader explanation than the former. First, Marton's explanation for retrospective transfer was that noticing differences between presentations of perceptual objects is the mechanism, whereas Hohensee's explanation was that noticing more generally, be it of similarities or differences across a new learning experience and a previously encountered context, is the mechanism. Second, Marton's explanation foregrounded noticing that was fairly immediate (i.e., on the timescale of how long it takes to form a visual perception), whereas Hohensee's explanation foregrounded noticing that emerged more gradually (i.e., on the timescale of multiple instructional activities or lessons). Third, Marton's explanation had a strictly psychological basis, whereas Hohensee's explanation had a psychological and social basis. Fourth, Marton's explanation focused mainly on noticing features within a perceptual field, whereas Hohensee's explanation was about noticing features within perceptual and conceptual fields. Finally, Marton's explanation was agnostic to whether retrospective transfer was productive or unproductive, whereas Hohensee's explanation accounted for instances in which backward transfer effects were productive and instances when they were unproductive. Thus, Hohensee's explanation involves a broader view of the influence of noticing.

4.4.4 Summary of Theories That Explain Backward Transfer

Looking across the various theories that explain backward transfer, most of the theories account for either productive or unproductive effects. Interference, overgeneralization, and fragile and meaningless learning were proposed as explanations for unproductive effects. Various changes in cognitive schemata were proposed as explanations for productive backward transfer effects. Only the process of noticing provided an account for both unproductive and productive backward transfer effects. Thus, the theory of noticing could serve as a unifying explanation for backward transfer that subsumes the other hypothesized explanations.

4.5 Aspects of Theory About Backward Transfer in Mathematics Education for Which There is the Most Pressing Need for Development

In this final section, I outline five areas for which I see a need for theory development about backward transfer in mathematics education. The five areas pertain to (a) how the range of backward transfer effects that have been observed are or are not related, (b) the characteristics of ways of reasoning that make them more or less amenable or vulnerable to backward transfer effects, (c) the mechanisms underlying backward transfer effects, (d) the methods for investigating backward transfer effects, and (e) the features of instructional practices and activities that lead to particular backward transfer effects.

4.5.1 Theory That Addresses How the Range of Backward Transfer Effects That Have Been Observed Are or Are Not Related

The first area of need is for theory development that addresses if and how the range of backward transfer effects that have already been observed are related. Thus far, a limited number of backward transfer studies in mathematics and science education have shown a range of backward transfer effects: effects on proportional reasoning (e.g., Van Dooren et al., 2004), effects on covariational reasoning (Hohensee, 2014), effects on action versus process views of functions

(Hohensee, Willoughby, & Gartland, 2020), and effects on the dissociability of science concepts in students' cognitive structures (Arzi et al., 1985). Each finding currently exists in isolation from the other findings, as a singleton, without theory that accounts for how they are or are not related. Theory development about these potential relationships between different types of backward transfer effects is needed and may also help to establish boundaries for the types of backward transfer effects that may be possible.

Several researchers have made initial efforts to develop theory about backward transfer effects that could help explain how different types of backward transfer may be related. For example, Hohensee (2016) hypothesized that one type of backward transfer effect may be when the same ways of reasoning are important for reasoning about two different mathematics topics (e.g., covariational reasoning is important for reasoning about both linear and quadratic functions). Perhaps some of the seemingly different backward transfer effects for different mathematics topics that have been reported are related by this common feature.

A second hypothesis comes from Marton (2006), who proposed that *discerning differences* could be a reason for backward transfer effects (Marton called backward transfer *retrospective transfer*). Marton defined discerning differences as "learning and transferring distinctive features that separate instances from non-instances (as opposed to learning and transferring features that the instances have in common)" (p. 520). In the red ball and white ball example described earlier, an individual's image of the red ball may change after a white ball is visually presented to them. Perhaps some seemingly different backward transfer effects are related because they all involve discerning differences. More theory development about how different backward transfer effects are related would guide researchers about where else to look for backward transfer and how to produce it in other mathematics contexts.

4.5.2 Theory About Characteristics of Ways of Reasoning That Make Them More or Less Amenable or Vulnerable to Backward Transfer Effects

The second area of need for theory development is about the characteristics of ways of reasoning that make them more or less amenable to backward transfer effects. By characteristics of ways of reasoning, I refer to general aspects that could apply to any ways of reasoning, such as strength (fragile vs. solid ways of reasoning), age (new vs. established ways of reasoning), associatedness (isolated vs. well-associated ways of reasoning), comprehensiveness (narrow vs. comprehensive ways of reasoning), abstractness (concrete vs. abstract ways of reasoning), explicitness (implicit vs. explicit ways of reasoning), and so on. Theory development is needed about the amenability or vulnerability of ways of reasoning to backward transfer effects for these and other characteristics of ways of reasoning.

Very little theory currently exists in the literature about what characteristics make ways of reasoning amenable or vulnerable to backward transfer effects. However, Hohensee (2014) hypothesized that ways of reasoning about linear functions that are mostly incorrect or mostly correct may be less amenable to productive backward transfer effects than ways of reasoning about linear functions that are only sometimes correct. Also, Hohensee, Gartland, Melville, and Willoughby (2021) hypothesized that the ways students reason about rates might be less amenable to backward transfer influences than the level of covariational reasoning students reason with. Both hypotheses were motivated by findings from those studies. More theory development about which characteristics of ways of reasoning are amenable or vulnerable to backward transfer effects would help researchers develop more targeted interventions.

4.5.3 Theory About Mechanisms Underlying Backward Transfer Effects

The third area for theory development is about the mechanisms that underlie backward transfer effects. Most studies that have focused on backward transfer, or that have reported what I interpreted as backward transfer effects, have not explicitly examined its mechanisms. For example, when Van Dooren et al. (2004) found that learning about non-proportional relationships unproductively influenced students' understandings of proportional relationships, the mechanisms that led to this effect were not directly investigated. An exception is Hohensee (2016), in which, as explained above, student noticing was explicitly examined as a potential underlying mechanism of backward transfer.

Theory development about the mechanisms underlying backward transfer is of importance to the field because not only would this theory explain how backward transfer occurs, but it may make it more possible to reliably produce productive backward transfer effects and inhibit unproductive effects. Furthermore, theory about the mechanisms of backward transfer would inform other aspects of theory development for backward transfer, including the types of backward transfer effects that are possible and the kinds of instructional practices and activities that lead to backward transfer effects. Furthermore, theory development about mechanisms of backward transfer could, in turn, inform theories of learning and of learning trajectories more generally.

4.5.4 Theory About Methods for Investigating Backward Transfer Effects

The fourth area for theory development about backward transfer pertains to the methods available for investigating backward transfer effects. The previous three points about theory development could indirectly influence researchers to use particular methods to measure backward transfer. For example, as theory about the relationships between backward transfer effects is developed, new methods to investigate related types of effects may emerge as well. However, there is also a need for more direct development of theory about methods for investigating backward transfer. For example, one area of need for theory development that pertains directly to research methods is about conceptual frameworks that connect particular instructional moves, activities, and classroom norms to particular backward transfer effects. Making those connections can be challenging because the new learning experiences and the prior ways of reasoning that are influenced during backward transfer typically occur at different times and places.

An existing conceptual framework that has directly informed my methods for investigating backward transfer is Lobato et al.'s (2012) *focusing framework*. This theoretical framework organizes what students notice during mathematics instruction into four categories: (a) centers of focus, (b) focusing interactions, (c) mathematical tasks, and (d) nature of the mathematical activity. This framework was used as a methodological tool in Hohensee (2016) for thinking about how to connect aspects of instruction to particular backward transfer effects that were measured outside of the instructional context. Whereas the centers of focus and focusing interactions were very helpful aspects of the theory for making connections, the mathematical task and nature of the mathematical activity were somewhat more difficult to use to make connections. This is one area where additional theory development related to methods would be warranted.

4.5.5 Theory About Features of Instructional Practices and Activities That Lead to Particular Backward Transfer Effects

The fifth area for theory development is about the instructional practices and activities that lead to particular backward transfer effects. Typically, the findings of backward transfer have focused more on the effects themselves and less on the instruction that is associated with the effects (e.g., Moore, 2012). An exception is Hohensee (2016), where several instructional practices, such as *quantitative dialogue*, were linked to backward transfer effects. Quantitative dialogue is defined as "verbal communication that directs attention to quantities as measurable attributes of objects or situations" (Lobato et al., 2012, p. 463). Hohensee (2016) showed that when the teacher emphasized quantitative dialogue during instruction about a new mathematical context, students attended more to quantities in previously-encountered mathematics contexts.

Theory development about instructional activities and practices that lead to particular backward transfer effects would help researchers design more effective interventions. This aspect of theory development may also have greater direct relevance for teachers and other practitioners than other aspects of theory on backward transfer.

The five areas of pressing need for theory development about backward transfer described above are broad in scope. This is not unexpected given that backward transfer research is still in its infancy. Furthermore, the broad scope of need for theory development also indicates the significant amount of people power required to bring theory about backward transfer to a comparable level of development as other mathematics education theories. Thus, this chapter could serve as a signal to the field of mathematics education to join in this work.

4.6 Conclusion

In this chapter, I presented a case for theory development about backward transfer in mathematics education from a constructivist complex-systems view of knowledge development. I began by sharing reasons for why theory development is needed in education more broadly and how that applies to backward transfer in the context of mathematics education. I then laid out the current state of theory development and prior research about backward transfer in mathematics education, as well as selectively outside of mathematics education. Finally, I presented five areas that represent a pressing need for theory development about backward transfer in mathematics education.

As stated at the outset, there is a lack of research on backward transfer in mathematics education, which means there is an imbalance in the transfer of learning research in favor of forward transfer, and this is inconsistent with a complex-systems view of knowledge development. I have advocated for correcting the imbalance because the study of backward transfer offers promise for generating new insights into how students learn mathematics and potentially represents untapped potential for improving mathematics learning. My hope is that this chapter will motivate mathematics education researchers to join in the effort to develop theory about backward transfer.

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Chapter 5
Exploration of Students' Mathematical Creativity with Actor-Oriented Transfer to Develop Actor-Oriented Creativity

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There is a rich body of research from various fields including education and psychology on the transfer of learning, with a history of over 100 years (e.g., Bransford & Schwartz, 1999; Detterman, 1993; King, 2017; Lave, 1988; Lobato, 2003; Thorndike, 1903). Commonly defined as the ability to apply knowledge learned in one context to a new context (Mestre, 2003), the transfer of learning plays an important role in many areas of our work in education, for example, in curriculum and program designs at the undergraduate level. To illustrate, note that students are required to take an introductory calculus course as a prerequisite for upper level courses in many science, technology, engineering, and mathematics (STEM) programs, and it is expected that students transfer their calculus knowledge to their respective STEM majors (e.g., Bressoud et al., n.d.; Cui, Rebello, Fletcher, & Bennett, 2006). The focus on transfer goes beyond mere application of knowledge and encompasses the application of processes such as problem solving, reasoning, critical and creative thinking, communication, and so forth. As an example, students in mathematics programs are required to take some form of an introductory proof course with learning objectives that include mathematical reasoning and proof-writing processes that would be used in subsequent, advanced-level mathematics courses. In sum, the transfer of learning is directly related to a goal of most educational programs: providing learning experiences that can be generalized and used by the learner outside the initial learning situation, preferably including their future careers (Bransford, Brown, & Cocking, 1999). As Mayer and Wittrock (1996) put it, "schools are not able to teach students everything they will need to know, but rather must equip students with the ability to transfer" (p. 49).

The development and transfer of various mathematical concepts as well as mathematical processes such as reasoning, problem solving, and proof construction at the undergraduate level are explored in research studies through various theoretical

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frameworks (e.g., Cui et al., 2006; Lockwood, 2011; Weber, 2005). In this chapter, I focus on the process of creative thinking in mathematics at the undergraduate level. It has been recently reported in the *World Economic Forum* that creativity at work is one of the top-three demanded skills, and that it "has jumped from 10th place to third place in just five years" (Schöning & Witcomb, 2017, para. 12). Similarly, Nadjafikhah, Yaftian, and Bakhshalizadeh (2012) claimed that fostering mathematical creativity should be one of the goals of any education system. Hence, it is timely to explore mathematical creativity and its place in mathematics classrooms at the tertiary level.

To date, there are numerous policy and curriculum-standard documents that emphasize the importance of mathematical creativity (e.g., Committee on the Undergraduate Programs in Mathematics [CUPM], 2015; National Council of Supervisors of Mathematics [NCSM], 2012; National Council of Teachers of Mathematics [NCTM], 1980; National Science Board [NSB], 2010; Partnership for 21st Century Skills, 2006). As Askew (2013) pointed out, "calls for creativity within mathematics and science teaching and learning are not new, but having them enshrined in mandated curricula is relatively recent" (p. 169). For example, mathematical creativity was emphasized by the Mathematical Association of America's (MAA) CUPM in its latest guidelines for majors in mathematical sciences (Schumacher & Siegel, 2015). The guidelines state that "a successful major offers a program of courses to gradually and intentionally lead students from basic to advanced levels of critical and analytical thinking, while encouraging creativity and excitement about mathematics" (p. 9). Additionally, Cropley (2015) highlighted these points as, "teaching engineers (and other STEM disciplines) to think creatively is absolutely essential to a society's ability to generate wealth, and as a result provide a stable, safe, healthy and productive environment for its citizens" (p. 140). However, in these moves to include creativity into educational settings, there exists an underlying assumption that this skill would transfer to future situations. For example, Luria, Sriraman, and Kaufman (2017) stated, "Not only can teaching for creativity improve students' understanding of course content, but it also prepares students for the application of learning objectives across domains" (p. 1033).

The purpose of this chapter is to initiate an exploration of this underlying assumption theoretically by examining the ways in which the transfer and the creativity constructs relate to one another, and by offering a possible approach to explore mathematical creativity through one of the contemporary transfer approaches, the actor-oriented transfer (AOT) framework. More specifically, the following question guides my theoretical exploration: In what ways could the construct of transfer aid in the exploration of mathematical creativity at the tertiary level?

This chapter starts with a brief summary of research on mathematical creativity, noting the existence of various definitions, orientations, and perspectives of creativity that impacted the research efforts. In this summary, I extract from mathematics education studies that examined the final products (e.g., proofs, solutions to problems) of students and mathematicians, and also mathematicians' thinking processes, to highlight an existing shift between product and process orientations in creativity research. The chapter continues with a brief summary of the transfer of learning as a research construct under various learning theories. I conclude this section with some research studies implementing the AOT framework as an example of contemporary approaches to transfer. In the Transfer of Mathematical Creativity section, I address the research question with empirical examples illustrating a way to use AOT to explore students' mathematical creativity. In this section, I also offer a way to view these two constructs (transfer and creativity) together through an intersecting relationship. This chapter concludes with a proposal of an emerging research construct, actor-oriented creativity (AOc), as a way to gain more insights into students' mathematical creativity in general.

5.1 Mathematical Creativity

Similar to the construct of transfer of learning, there is a rich history of exploring creativity and mathematical creativity and their roles in educational settings. In his presidential address to the American Psychological Association (APA) in 1950, Guilford asked, "why do we not produce a larger number of creative geniuses than we do, under supposedly enlightened, modern educational practices?" (p. 444). And, he called for explorations of finding creative promise in learners and the ways in which their creativity could be developed. However, Mann (2006) indicated that there are more than 100 definitions of mathematical creativity and claimed that an absence of an agreed-on definition was one reason for the sparse research attempts to study mathematical creativity in mathematics education.

Sriraman (2005) suggested that, in the absence of a precise definition of mathematical creativity in mathematics and mathematics education, "we move away from the specific domain of mathematics to the general literature on creativity in order to construct an appropriate definition" (p. 23). Hence, in this section, and throughout this brief summary of existing studies in mathematics education on mathematical creativity, I intertwine related constructs from domain-general creativity as they influenced research activities in mathematics education.

The exploration of mathematical creativity has been traced back to two psychologists, Claparède and Flournoy, in 1902 (as cited in Borwein, Liljedahl, & Zhai, 2014, and Sriraman, 2009), who focused on the professional mathematicians' process of creativity, rather than conceptualizing the construct of mathematical creativity. Hadamard (1945) extended this earlier study to the mathematicians of his time, focusing on the psychology of mathematical creativity, and discussed the results of his study in light of a psychological four-stage model of the creative process developed by Wallas (as cited in Sriraman, 2009, and Sadler-Smith, 2015).

These four stages of creativity are preparation, incubation, illumination, and verification. The *preparation* stage is the stage when the problem solver thinks about the problem, gathers related information, and offers possible ideas. This conscious stage prepares the mind to work on the problem unconsciously in the second stage. During this *incubation* stage, the problem solver does not consciously work on the problem; the problem is put aside. In the third, *illumination* stage, ideas "suddenly" fit together and the solution appears (also known as an "aha" experience). The process continues in the fourth *verification* stage in which ideas are checked and the solution is examined for correctness and appropriateness.

These proposed stages extend Dewey's (1920) logical sequencing and conscious model of problem solving by including the unconscious stage of incubation, and the between consciousness and unconsciousness stage of illumination. In particular, the illumination stage is believed to reflect an instantaneous "train of association" (Sadler-Smith, 2015, p. 346), suggesting a connection to associationism theory. This theory claims that pairs of thoughts become associated based on a thinker's past experiences (Shanks, 2007). These two stages, incubation, and particularly the illumination stage, are reported to be experienced by many mathematicians and believed to be the "heart" of mathematical creativity. Wallas's four-stage model is still in use to examine the process of creativity. For example, Sriraman (2004) interviewed five research mathematicians and found that these four stages are still applicable to describe modern-day mathematicians' process. His study provided more detail of the stages by considering the roles of personal and social attributes such as imagery, intuition, and interaction with others. Liljedahl (2013), on the other hand, focused on the stage of illumination, through anecdotal reflections of preservice teachers and research mathematicians, and emphasized the inclusion of affective domains in mathematics creativity research. In particular, he claimed that "what sets the phenomenon of illumination apart from other mathematical experiences is the affective component of the experience" (p. 264) and, for students, the unexpected appearance of a solution provided an emotional motivation.

As this four-stage model attempts to explore the process of creative thinking, the main portion of the mechanism of creativity seems to exist in the mind's unconscious work. Guilford (1950) noticed this particular issue and, referring to an analysis of processes of creativity with this four-stage model, stated, "such analysis is very superficial from the psychological point of view" and "tells us almost nothing about the mental operations that actually occur" (p. 451). Noticing that these stages were not testable, he suggested some testable factors such as fluency, flexibility, production of novel ideas, synthesizing and analyzing ability, and evaluation ability (Guilford, 1959). In his *Structure of Intellect* model, Guilford distinguished between three types of thinking, convergent, divergent, and evaluative. *Convergent* thinking refers to providing a single correct answer or a best solution to a problem, whereas *divergent* thinking focuses on the creation of many possible ideas and multiple solutions to an open-ended prompt. *Evaluative* thinking includes judgement about whether an answer is accurate, or a solution approach is consistent, or valid for a problem. Even though Guilford (1967) considered all three forms of thinking as part of the creative process, divergent thinking has received more attention and is commonly used as a way to operationalize creativity in research.

Researchers, mostly using divergent thinking, have focused on the fluency, flexibility, originality, and elaboration components to develop an operational definition of creativity for research studies (e.g., Balka, 1974; Leikin, 2009; Torrance, 1966; Silver, 1997). *Fluency* in general refers to the amount of outputs to a stimulus. Silver (1997) defined it in the problem-solving setting in mathematics as the "number of ideas generated in response to a prompt" (p. 76). *Flexibility* is operationalized as the number of categories of responses to given stimuli. In the problem-solving context, this relates to the number of shifts in approaches, providing multiple approaches to a problem to produce a variety of solutions. This could mean that a student approached a problem and for some reason changed this approach to a new one. *Originality* (or novelty) is described as a unique production or an unusual thinking (Torrance, 1966). *Elaboration* refers to the ability of producing a detailed plan and generalizing ideas (Torrance, 1966). Torrance's (1966) assessment tool (Torrance Tests of Creative Thinking [TTCT]), which leverages these components, is still used in schools and by researchers. For example, Kim (2012) reported that K-12 students' creativity scores, which were examined with TTCT, had decreased from 1990 to 2008, even though their Intelligence Quotients (IQ) and Scholastic Assessment Test (SAT) scores had shown increase since 1966.

The use of these four components demonstrates a shift in perspective from a *process* orientation (i.e., exploration of the nature of the mental mechanisms) to a *product* orientation (i.e., quantification and examination of the outputs that a person provides for tasks). For example, Balka (1974) used the fluency, flexibility, and originality components in his *Creative Mathematical Ability Test* in which participants were given mathematical situations to develop problems. Mathematical creativity of the participants was determined by the responses (outputs) through fluency, which was the number of problems posed by a participant; flexibility was determined as the number of different categories of problems generated by a participant; and originality was determined as the rarity of the response provided by a participant compared to the other participants' responses in the study. Similarly, Leikin (2009) focused on the fluency, flexibility, and originality to create a creativity rubric (using a point system) that evaluated how creative a student was when they produced solutions to certain tasks.

Within the product orientations of creativity, there is also an emphasis on the quality of the end product: Is it original and useful (Runco & Jaeger, 2012)? This framing was adopted by the mathematics education community to develop definitions of mathematical creativity as an ability to produce original, useful, adaptive, unusual, applicable, and so forth proofs and solutions. The word *ability* appears in such definitions to include the person in charge of producing these products; however, the act of judging the created work by an outsider plays an important role in such conceptualizations of creativity. Even though a product orientation provides means to measure mathematical creativity, "a more precise characterization of creativity will require a detailed consideration of the processes used in generating the items leading to" (Ward & Kolomyts, 2010, p. 95) productions that are considered creative.

In fact, with a five-stage model, Sheffield (2009, 2013) focused on processes of creative problem solving at the K-12 level. This model, by offering a nonlinear approach to process, differs from Wallas's four-stage model, which assumes a linear progression between stages. Proposed as a creative problem-solving heuristic, the model includes five stages: investigating, relating, creating, evaluating, and communicating. During the *investigating* stage, a person examines the available information, ideas, and mathematical concepts that are related to this information. Actions such as identifying similarities and differences of ideas, and combining information, are proposed to be part of the *relating* stage. During the *creating* stage, solutions are created or new ideas are identified or new connections are developed. Similar to Wallas's verification stage, Sheffield described the *evaluating* stage as actions that are taken to examine proposed solutions or ideas; however, this stage differs from the verification stage because these evaluative actions can be done throughout the problem-solving process as opposed to appearing just at the end of the process. Ideas, solutions, or approaches are explained (to others) at the *communicating* stage. This is a nonlinear model, meaning that a problem solver could start at any stage. For example, a person could identify similarities and differences between a given task and previous work (relating stage), communicate these observations and then evaluate them, which could lead to the process of investigating. Finally, a person could create solutions after the investigations or enter into the stage of relating again.

In summary of definitions and orientations of mathematical creativity, I refer to Sriraman's framing of this construct (Liljedahl & Sriraman, 2006). He, focusing on experts' work, suggested that mathematical creativity can be defined as the ability to generate original work, which contributes and extends the existing body of knowledge as well as creates new questions or areas for further mathematical explorations. To operationalize this definition in mathematics education research in K-12 classrooms, Sriraman (2005) defined mathematical creativity as "the process that results in unusual (novel) and/or insightful solution(s) to a given problem or analogous problems, and/or the formulation of new questions and/or possibilities that allow an old problem to be regarded from a new angle requiring imagination" (p. 24). This formulation captures two important points for exploration of students' mathematical creativity. First, students' mathematical creativity is viewed from a process orientation as opposed to the product orientation used for experts' mathematical creativity. Second, mathematical creativity in K-12 classrooms may look different than the one employed by mathematicians and hence, student creativity needs to be evaluated accordingly, considering students' prior experiences (e.g., experiences with analogous problems as mentioned by Sriraman, 2005). In other words, students do "have moments of creativity that may, or may not, result in the creation of a product that may, or may not, be either useful or novel" (Liljedahl, 2013, p. 256) to the mathematics community at large. This particular idea relates to an important distinction of mathematical creativity within research: the difference between *absolute* (*extraordinary*) and *relative* creativity, where the former refers to (historical) inventions (discoveries at a global level) and the latter one is defined as "the discoveries by a specific person within a specific reference group, to human imagination that creates something new" (Leikin, 2009, p. 131).

This particular formulation of mathematical creativity (i.e., relativistic and process-oriented) for students aligns with the *mini-c creativity* construct within domain-general creativity. Beghetto and Kaufman (2007) defined mini-c creativity as "the novel and personally meaningful interpretation of experiences, actions and events" (p. 73). They highlighted that the novelty and meaningfulness of interpretations are personal judgements and, as such, they may not be original or meaningful to others. Beghetto and Kaufman (2007) argued that mini-c creativity is an interpretive and transformative process that is part of an individual's learning process. This argument stems from Vygotsky's "argument that creativity (imagination) is one of the basic mechanisms that allows new knowledge to develop" (as cited in Leikin, 2014, p. 61). The construct of mini-c creativity could allow us to explore students' creativity during learning (e.g., in a course) as opposed to at the end of learning (e.g., after completion of a course).

In this chapter, I suggest incorporating mini-c creativity to explore tertiary-level students' mathematical creativity, which would mean taking a relativistic, process orientation in the domain of mathematics. I furthermore propose conducting such explorations from a transfer research lens so that we gain understanding of students' transfer of creative-thinking skills. K-12 students' mathematical creativity through a relativistic, process orientation has been explored (e.g., Sheffield's creative problem-solving heuristic), but without the use of mini-c creativity. Sheffield (2009) discussed the process of mathematical creativity with five nonlinear stages. Given how such stages provide insights into students' mathematical creativity, I question how such stages, namely the processes observed in those stages, transfer to other situations. In the next section, I briefly summarize research on transfer of learning and demonstrate some aspects of the AOT framework within existing studies.

5.2 Transfer and the Actor-Oriented Transfer Framework

Transfer of learning has been traditionally defined as the ability to apply knowledge learned in one context to a new context (Mestre, 2003). Early psychological views of transfer were based on the mental abilities of a person, and these abilities were believed to become stronger by training them in different subject areas. Thus, the training of the basic mental functions was also thought to improve the person's ability to transfer ideas and skills to new situations. After many experiments showing the failure of such claims, Thorndike and colleagues challenged this existing belief and proposed an alternative idea, the *theory of identical elements* (Thorndike, 1903; Woodworth & Thorndike, 1901). Thorndike's work showed that even though learners did well on a test of the specific content they had studied, this content knowledge did not increase their performance in a new situation. Thorndike and colleagues further concluded that transfer from one task to another happened only when two tasks shared identical elements.

Thorndike's studies influenced instructional practices with the inclusion of more skill-repetition activities in the mathematic curriculum. In addition, Thorndike's work influenced the transfer studies conducted later. Many researchers followed a similar research paradigm; an initial learning task was followed by the target task created by researchers who thought that these two tasks shared similar features and examined participants' performance from one task to another as an indication of transfer (e.g., Bassok, 1990; Gick & Holyoak, 1980). Researchers from this paradigm were interested in research questions of the form "Do students transfer?" Most of the studies conducted under this traditional paradigm reported failure of spontaneous transfer from one task to the next.

Judd argued that one of the possible reasons for failure of transfer could be the relationship between two tasks that was declared to be similar by the researchers (as cited in Tuomi-Gröhn & Engeström, 2003). He claimed that the learners might have a different opinion about the sameness and differences of two tasks. For example, in their seminal study, Gick and Holyoak (1980) first gave participants a story about a successful capture of a fortress by dividing the army into small groups (initial learning task). Then, participants were presented with Duncker's (1945) radiation problem (transfer task) in which they were tasked to find the best method to eliminate a tumor by radiation. These two tasks, according to the researchers, shared similar structural features (e.g., fortress = tumor, capture the fortress = eliminate the tumor, whole army into small groups = high intensity into lower intensity of radiation, etc.) but they had different surface features (the contexts). The researchers found that, generally, participants did not successfully or spontaneously (i.e., without a hint) transfer an analogous solution of the initial task to the transfer task where tasks had (structural) similarities. Even though their conclusion claiming that transfer of learning requires the overcoming of surface features (e.g., contexts) is an important one, it would be as valuable to know how these participants approached the transfer task (e.g., their reasoning about solutions, perception of connections between two tasks or prior experiences, etc.).

Bransford et al. (1999) argued that such negative transfer results of initial-task to transfer-task experiment designs were due to the underlying perspectives of transfer. They stated, "evidence of transfer is often difficult to find because we tend to think about it from a perspective that blinds us to its presence" (p. 66). They problematized both the experimental design of initial task to transfer task (also known as sequestered problem solving [SPS]) and the definition of transfer that emphasized direct applications of prior knowledge to a new problem. They claimed such experimental designs with the direct application view of transfer "make people look much 'dumber' (or 'less educated') than is actually the case" (Schwartz, Bransford, & Sears, 2005, p. 6). Lave (1988) also problematized the traditional views of transfer by pointing out that such definitions of transfer (i.e., any derivative application of knowledge from prior learning to a new problem) consist of measures of the proper use of previous learning in the new setting with the assumption that the settings (initial learning and the transfer) and other social and environmental factors do not affect the learner's performance.

Overall, researchers have criticized the traditional definition of transfer and methodologies used during these transfer studies. They suggested that a new framing of transfer should address that learning could not be separated from the environment and it should capture the notion of transfer being an active process. The research questions should be posed in a way that more than binary results are achieved. Also, new methods of inquiry should be in place to gather evidence of transfer (both productive and unproductive), rather than just relying only on SPS types of methods, which could provide insights into mechanisms of transfer. Some contemporary approaches also argued for a need to shift from the researcher's point of view to the learner's point of view during transfer investigations, meaning that

> The researcher does not measure transfer [of mathematical creativity] against a particular cognitive or behavioral target but rather investigates instances in which the students' prior experiences shaped their activity in the transfer situation, even if the result is non-normative or incorrect performance. (Lobato, 2012, p. 235)

Some examples of these contemporary approaches implemented in research studies are transfer by affordances and constraints (Greeno, Smith, & Moore, 1993), preparation for future learning (PFL; Bransford & Schwartz, 1999), and the AOT framework (Lobato, 2006). Because the focus of this particular book is on AOT, I mainly focus on this framework in this chapter.

5.2.1 Actor-Oriented Transfer Framework

The AOT framework views transfer as "the personal construction of relations of similarity between activities, or how 'actors' see situations as similar" (Lobato & Siebert, 2002, p. 89). The main focus of this framework is the learner (actor) and how the learner sees the transfer situation in relation to the initial learning situation. Obtaining evidence for AOT differs from traditional transfer approaches. In traditional approaches, successful or improved performance on transfer task was considered as evidence. Meanwhile, in AOT, regardless of successful or improved performance on transfer tasks, any influence of prior learning experiences is considered as evidence for transfer. Consider an example provided by Lobato (2006, 2012). Students were tasked with finding the slope of wheelchair ramp (transfer task) after they had been introduced to finding the slope of a line (initial learning task). According to the experts (or researchers), these tasks shared the same structural features and could be solved using the similar approach of rise over run, but differed in terms of surface features (contexts).

When Lobato and Siebert (2002) examined one student's reasoning on the transfer task, they observed that this particular student did not "transfer" (in the traditional sense) the slope formula (rise over run) from the initial learning situation. However, when they examined the student's reasoning on the wheelchair task, they noticed that the student's reasoning on the task included identifying the related two quantities (height and length) contributing to steepness and developing a multiplicative relationship between them, all of which were directly linked to finding the slope. Further examination of the student's prior experiences from the teaching interventions revealed that this particular student was most probably using the same reasoning that he had developed during in-class activities. They postulated that the student demonstrated transfer between these two situations by creating his own similarities between these two situations, rather than doing what researchers expected him to do (i.e., use the rise over run formula).

For such inquiries with the AOT perspective, a typical study design needs data in qualitative nature from both transfer and initial learning situations. For example, following a typical AOT design, Karakok (2009, 2019) explored undergraduate physics students' transfer of learning of eigenvalue and eigenvector concepts from various learning experiences to interviews. The researcher conducted three interviews and audio- and video-recorded three different quantum mechanics physics courses over one semester that all participants took. Three interviews were conducted with seven students, all of whom had some course experience with linear algebra topics prior to the study. The first interviews were conducted prior to the quantum mechanics courses. The second interviews were conducted mid-semester, while students were still taking one of the quantum mechanics courses. The last interviews were conducted after students completed all three quantum mechanics courses.

During the first interview, participants' existing conceptualization of eigenvalues and eigenvectors were captured, given that they all had prior experiences of learning these concepts in various courses. The analysis of the first interview data (from the traditional transfer approach) showed lack of transfer of learning of eigenvalues and eigenvectors to the interview tasks because students could not successfully complete tasks on these concepts. With the exception of one student, participants could not describe these concepts and showed limited conception of them. However, as students progressed through three quantum mechanics courses, regardless of their differing initial learning experiences, their conceptions showed changes. When the second interview data were examined, some students still lacked transfer (from the traditional perspective); however, they were able to draw upon their experiences from classroom activities to describe these concepts and demonstrate their thinking with examples that were similar to the in-class activities (see Karakok, 2019, for the case of Gus).

Using students' conceptual changes as the basis, the researcher examined data from all interviews and all quantum mechanics courses with the AOT framework. Analysis suggested that participants were constructing similarities between interview tasks and their experiences in small-group activities and whole-class discussion in quantum mechanics courses. In other words, the AOT analysis provided possible explanations for participants' seemingly idiosyncratic reasoning processes during interview tasks, namely that they represented the production of similarities that students observed to make between interviews and quantum mechanics courses. Using the AOT framework to examine the influences of learning experiences from the courses, facilitated a shift in the focus of analysis from the product (i.e., students' conceptions) and the quality of this product (e.g., successful, complete) to the process of students' reasoning. Particularly, it was observed that one of the course instructor's explicit instructional moves seemed to play a role in students' reasoning and influence their processes during the interviews (Karakok, 2019).

Lockwood (2011) demonstrated another possible way to use the AOT perspective. Rather than calling students' in-class experiences "initial learning," Lockwood explored student-generated connections among a variety of counting problems in the context of combinatorics across multiple interviews. The study identified three types of student-generated connections: (a) elaborated versus unelaborated, (b) conventional versus unconventional, and (c) referent type. Participants of the study were given eight to 18 counting problems to work on in pairs and asked to think aloud and discuss their ideas with each other in these interviews. These video-recorded problem-solving sessions were then analyzed by the researchers to explore student-generated connections and similarities among situations within the problem-solving sessions. The study highlighted that "in addition to mathematical connections that an expert might expect, students make unexpected connections as well" and suggested that "instructors should, therefore, be open to considering student-generated connections as they arise and may want to take advantage of alternative kinds of similarities when planning lessons" (Lockwood, 2011, p. 321).

As demonstrated in this section, exploration of students' thinking with the AOT framework could "detect instances of the generalization of learning experiences" (Lobato, 2012, p. 232) in many forms. Furthermore, the AOT framework could also help explore instances "even when there is a lack of transfer according to traditional definitions" (p. 232). It is my claim that AOT could also help us to "detect" instances of students' mathematical creativity and "distinguish between a person's creativity and his/her prior knowledge and experience" (Zazkis & Holton, 2009, p. 359). More precisely, AOT, with its focus on students' point of view rather than experts', and its focus on the process of construction of similarities and differences rather than end-result production, could help us to answer research questions such as: What is the nature of undergraduate students' mathematical creativity (i.e., what processes are used by students)? How do students "transfer" their mathematical creativity between situations (i.e., how do students construct similarities between their processes among situations)? In the next section, by drawing from existing studies, I provide a theoretical discussion on why and how the construct of transfer could be considered in exploration of students' mathematical creativity.

5.3 Transfer of Mathematical Creativity

When transfer and creativity lines of research are examined, it is notable that they both claim, rightfully, the importance of the abilities to transfer and be creative. In creativity research, loosely speaking, it is expected that a person has knowledge about a domain (e.g., procedural-factual knowledge, technical skills) and direct applications of domain-specific content to expand on or extract from them to develop a creative product. It seems that the person is assumed to have a transfer ability. Transfer studies, on the other hand, have been interested in whether a person could apply learned knowledge and skills in novel situations under various approaches. As I consider these two lines of research, there seems to be an intersecting relationship such that there could be instances of transfer that would not be part of creativity research (or would not "count" as creative), there could be instances of creativity that would not be part of transfer research, and there is the overlapping section where transfer could be creative or creativity could be considered transfer. In this chapter, I situate my arguments within the transfer construct, including the intersection.

As pointed out in the introduction section and the summary of creativity studies, there seems to be an underlying assumption that creativity is a transferrable skill that helps not only the learning of concepts but also applying learned concepts in other situations. Craft (2005), by examining creativity research studies, observed that there is not a consensus to assume that creativity is a transferable skill without a reference to a specific domain. Starko (2017) provided a similar observation: “Most theorist concur that individuals are creative in some subject area and need a base knowledge and skills to succeed” (p. 94). In this chapter, for these reasons, I consider creative-thinking skills within the domain of mathematics and seek to explore students’ mathematical creativity from a transfer perspective. In particular, I argue for examining development of students’ mathematical creativity and how it transfers across different domain-specific situations. With this proposed exploration approach, I attempt to address the points made by Baer and Kaufman (2012). They noted that the relationship between creativity and a specific content domain is part of a larger question on the relationship between the learning of content and thinking skills and that “it [the relationship] is related to questions about the possibilities of transfer of learning and of teaching to promote such transfer” (p. 151).

In the brief summary of transfer research, I discussed various approaches to exploring transfer that included examinations of direct application of prior learning in new settings and explorations of the construction of similarities between different situations. Many of these approaches would equally provide meaningful insights into different aspects of students’ mathematical creativity and transfer of creative thinking in mathematics. In this chapter, I focus on exploration of mathematical creativity from the AOT perspective for several reasons. To explore the phenomenon of students’ mathematical creativity, there is a need for an approach that is “particularly effective at bringing to the fore the experiences and perceptions of individuals from their own perspectives, and, therefore, at challenging structural or normative assumptions” (Lester, 1999, p. 1). The AOT framework provides this opportunity in explorations of students’ mathematical creativity by taking a student’s point of view. In other words, AOT could help understanding students’ “descriptions of what [students] experience and how it is that they experience what they experience” (Patton, 2002, p. 107) in the context of mathematical creativity. AOT, with its view of transfer as “the personal construction of relations of similarity between activities, or how ‘actors’ see situations as similar” (Lobato & Siebert, 2002, p. 89), could provide a lens to gain in-depth understanding of processes of creativity. For example, in the relating stage, Sheffield (2009) stated that students make connections between a given task and their prior knowledge and ideas by identifying similarities and differences. Students’ particular actions in this stage (and also in others) could be explored through the AOT lens and this exploration could help us to detect what instances from prior experiences that learners relate to.

The research efforts in mathematical creativity, as well as domain-general creativity, indicated influences of personal traits, affective domains, and social interactions on development of creativity (e.g., Csikszentmihalyi, 1999; Pitta-Pantazi, Kattou, & Christou, 2018; Sriraman, 2009). With its view of transfer as a distributed phenomenon, AOT also considers these domains. However, in this chapter, the exploration of students' mathematical creativity through AOT is limited to the cognitive domain, and consideration of other domains, in particular the affective domain, will be part of future studies because such inclusion will provide a better picture of students' mathematical creativity and its transfer.

5.3.1 Actor-Oriented Transfer of Mathematical Creativity

As an illustration of the AOT lens, I consider a previous study that on which I collaborated with colleagues to explore benefits of a particular assignment system that implemented the Creativity-in-Progress Rubric (CPR) on Proving (Karakok et al., 2016; Savic, Karakok, Tang, El Turkey, & Naccarato, 2017) in an elective combinatorics course (Omar, Karakok, Savic, & El Turkey, 2019). In this course, in addition to covering course content topics from the area of combinatorics, the instructor (the first author of Omar et al., 2019) aimed to engage all students with challenging tasks, and develop and improve students' technical writing and prose in mathematics. The CPR on Proving was provided to students to guide their thinking process and writing in assignments. This rubric was developed by the other authors of Omar et al. (2019) prior to this study with the aim to explore how to foster students' mathematical creativity in tertiary mathematics courses (see Karakok, Savic, Tang, & El Turkey, 2015; Savic et al., 2017; Tang, El Turkey, Savic, & Karakok, 2015). The instructor in this study asked students to submit a reflection of their work using this rubric as well as their written work answering questions for each assignment. In this assignment system, there were five biweekly homework assignments and two projects. Each homework assignment had several questions for students to practice direct application of the course content (i.e., they were considered as exercises), and one challenging question that was considered nonroutine (i.e., it was considered as a problem as discussed in Schoenfeld, 1985). In contrast, for each project, students had roughly five weeks and the tasks were open in nature. For example, for the first project, students were given a formula and asked to "investigate possible new algebraic proofs or augmentations to proofs in existing literature. Moreover, develop insight on how to approach this problem from a combinatorial perspective" (Omar et al., 2019, p. 87). We examined students' written work for assignments and reflections to understand the benefits of the assignment system using the rubric.

In this empirical exploration, I reexamined one of the students' written work for two homework problems, the course notes, and the interview transcript. My goals here are (a) to illustrate how AOT could aid in exploration of students' mathematical creativity, and (b) to provide empirical examples of AOTs of students' mathematical creativity. The first example is from the student's work on the problem of the second homework assignment:

> Portfolio Problem 2: Let $n \geq 1$ be an integer. Determine the number of walks in the plane with n steps, starting from (0, 0), with steps of type (0, 1), (0, -1), or (1, 0), given the condition that any such walk cannot intersect itself. Find any generalizations if the directions you can move are altered. (Generating functions might help.) (Omar et al., 2019, p. 84)

We examined the student-submitted written work (which was presented in Omar et al., 2019, p. 95) and the instructor's notes from classes prior to the assignment. We observed that this particular student's work demonstrated an understanding of recursive relationships and techniques that were discussed in class and we claimed that this particular understanding helped the student to "discover an explicit expression" (p. 94). To provide an illustrative example, I focused on this particular data piece "by scrutinizing a given activity for any indication of influence from previous activities and by examining how people construe situations as similar" (Lobato & Siebert, 2002, p. 89). In her written work, the student explained her inquiry of the problem. For example, the student stated that the exploration was started with creating visualization of the walks, and then the student used "both symbolic and graphical representations of each of the three possible steps." Prior to presenting the "discover[ed] explicit equation" and the related computer-generated codes, the student wrote, "Because both [another student's name] and I study computer science, our next inclination was to develop several different models for this language." After evaluating their instincts, the student provided a computer program code and a proof of the conjectured recursion.

Before arriving at her invented equation of the recursive formula, the student once again mentioned another course experience: "I remembered a technique of solving recurrence relations called the 'polynomial method' from Discrete [a prerequisite math course]." As observed in this student's written work, the student seemed to be constructing similarities between the given homework problem and her prior experiences in discrete, computer science, and combinatorics courses. This empirical example could be considered as evidence of AOT. Furthermore, the student's provided solution to the problem was considered novel by the course instructor and the researchers (Omar et al., 2019). This brief example only demonstrates a way to use the AOT lens, and thus far, I have not discussed the AOT of mathematical creativity. For this particular exploration, I first need to reiterate an operational definition of students' mathematical creativity.

As discussed in the Mathematical Creativity section, previous studies on mathematical creativity provided different approaches to identifying mathematical creativity. Taking a product orientation, a person's work (e.g., proofs, solutions) could be examined "by fluency (total number of appropriate responses), flexibility (the number of different categories of responses), originality (rarity of responses) and elaboration (amount of detail used in the responses)" (Leikin & Pitta-Pantazi, 2013, p. 160) for the indication of the person's mathematical creativity. It should be noted that the judgement of "appropriate responses," "different categories," "rarity," and "amount of detail," would be made by the researchers of the studies. Studies in creativity have demonstrated shifts toward process orientations given that "researching the creative product may not provide full understanding of the development of creativity, or may not reflect the creativity used to reach that product" (Savic et al., 2017, p. 25). A process-orientation creativity focuses on exploration of actions, behaviors, and stages that take place in the generation of work (e.g., ideas, proofs, solutions). Mostly, pulling from cognitive approaches, a person's thinking or progression of thinking (in the case of stages) is investigated for actions such as examining a given problem (as seen in the preparation stage of Wallas's model and the investigating stage of Sheffield's model), identifying similarities and differences of ideas, combining ideas (as seen in the relating stage of Sheffield's model), developing a solution, verifying a solution, and communicating ideas.

Sriraman (2005) suggested that, when considering students' mathematical creativity, one should consider their processes that result in novel solutions. However, Leikin (2009) emphasized that these novel solutions should be considered with respect to students' prior experiences in mathematics. In sum, I propose an operational definition of students' mathematical creativity that is inspired by these researchers' work, and I also incorporate the mini-c creativity construct from domain-general creativity (Beghetto & Kaufman, 2007): A student's mathematical creativity is a process of engaging with a mathematical situation in which personally meaningful interpretations of experiences, actions, and events are employed; novel connections are made; personal risks are taken; and approaches and ideas are examined for appropriateness in order to propose or produce a solution or proof, or to pose a question.

In my next illustration, I consider this definition for students' mathematical creativity for the purpose of examining AOT of this particular student's mathematical creativity. The analysis started with identification of actions that seemed to be related to the ones in the aforementioned definition of students' mathematical creativity. Once actions were identified, I examined the written work around which these actions seemed to take place. This particular analysis was conducted to gain better understanding of the usages of these actions as they were performed by the student, but also to hypothesize any indication of "influence from previous activities [actions] and by examining how people construe situations as similar" (Lobato & Siebert, 2002, p. 89). The actions that I identified in the student's work for the homework two problem were: visualizing, using symbolic and graphical representations and evaluating representations as a way to interpret the problem, making connections to other courses (e.g., developing and translating a model), conjecturing with observed results (i.e., taking risks), proving for correctness, and posing questions for consideration of another approach. All of these actions suggested an indication of this student's mathematical creativity.

To demonstrate a possible indication of influence from previous activities and actions, I focus our attention to the action of *making connections to other courses*. The student's attempt at relating the given problem to her other courses was previously discussed and was already hypothesized as evidence of AOT. Here, I provide more discussion about this action through the examination of mathematical work when the computer science course was mentioned. The student wrote:

> Because both [another student's name] and I study computer science, our next inclination was to develop several different models for this language. Institutively, it seemed to lend itself to regular expressions, and sure enough I was able to develop a regular grammar for allowed walks. I translated this into a deterministic finite automaton … wrote short programs. … The goal of this was to quickly compute enough values that we could gain insight into the nature of walks as a function of steps.

It seems that the student not only identified a similarity between the given problem and the computer science course content, but also, the student seemed to be influenced by the process of developing models in the computer science course (i.e., previous actions) to develop and translate a model for the given problem. It seems that this instance could be an empirical example suggesting evidence of AOT of the student's mathematical creativity because it seemed that the student was influenced by previous processes (developing models) in a different setting and adopted this process for the given problem.

To provide a clarification of this claim, I focus on the same making connections to other courses action, and this time, I consider the student's written work when the discrete course was mentioned. The student wrote, "Having found a recurrence relation, my next action was to seek a closed form. I remembered a technique of solving recurrence relations called the 'polynomial method' from Discrete … and applied that here." The student, in her work, used the method to find a closed form. This work included procedural steps such setting up equations, solving a quadratic equation, applying initial conditions, subtracting terms, and substituting. Even though the student successfully computed a polynomial model for this given problem, I did not consider computing a closed form in a discrete course (i.e., previous action) as the student's mathematical creativity. In particular, the process of computing by itself does not necessarily tell us much about the students' mathematical creativity in the discrete course. Because the student's written work was limited to procedural computations, it seemed that the student was transferring content knowledge from the discrete course to the homework problem. And, it did not seem appropriate or suitable to hypothesize that this could be a transfer (from the AOT lens) of this student's mathematical creativity from the discrete course to this given problem.

In the first suggested empirical example of AOT of mathematical creativity, I only examined the student's work on one problem to hypothesize "transfer" of an action of developing models from another course (e.g., computer science) to the given problem. Because this analysis had obvious limitations (e.g., computer course information was not available), I examined the written work of this student from another homework problem. In other words, I considered the same student's written work on two problems from two different homework assignments from the combinatorics course: the second homework problem (discussed earlier) and the fourth homework problem. The fourth homework problem (Fig. 5.1) was assigned approximately one month later than the second homework problem. Again, the question was open in nature and it was related to a different content topic (graph theory) than the problem in the second homework.

The actions that I identified in the student's work on this fourth homework problem were: using definitions and theorems from the course (e.g., generating series, generating functions, Burnside's Lemma, Polya's Theorem) as a way to interpret the problem, exploring cases (e.g., examining generating series for special cases of graphs), visualizing, evaluating work to shift to different approaches, making connections (to a different problem), posing questions, and examining work to provide a hypothesis (as the "answer" of the problem). With the definition of students' mathematical creativity, I claim that all of these actions suggest an indication of this

4 Portfolio Problem: Let G be a graph on vertex set $V = \{v_1, v_2, \ldots, v_n\}$. A *k-coloring* of G is a function $c : V \to \{1, 2, \ldots, k\}$ so that any two adjacent vertices in G are assigned different colors. For a given k-coloring c, we define its indicator $\mathrm{ind}(c)$ by

$$\mathrm{ind}(c) = x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$$

where a_i is the number of vertices in V assigned to color i by the coloring c.

Two colorings are said to be *equivalent* if one can be obtained from another by applying an automorphism of G. Let $F(x_1, x_2, \ldots, x_k)$ be the generating series that sums of all the indicators of all inequivalent k-colorings of G. What information about G does F contain? (Start by playing with explicit examples like complete graphs, path graphs, and cycle graphs.)

Fig. 5.1 The fourth homework problem

student's mathematical creativity. For the examination of AOT of this student's mathematical creativity, I examined the identified actions in the fourth homework problem for any indication of influences from the previous identified actions for the second homework problem. The *posing question* action appeared in both lists. On the second problem, it was noted that the student posed a question after providing the "discover[ed] explicit equation" (Omar et al., 2019, p. 94). This question seemed to be initiated from the act of evaluating the "final" work for more generalizations:

> Suddenly, our algorithm for finding possible paths has to have memory. In terms of the strings representing valid walks, our language is no longer regular or even context-free. Therefore, we don't have an obvious path for generalizing our approach in these cases. After hitting this wall, I started to consider how we might use generating functions to approach this problem. Unfortunately, I didn't have time to explore this option very far.

The type of questioning in the form of "how we might use generating functions to approach this problem" did not appear in other parts of the student's written work on this problem. That is to say, the student might have posed questions throughout her inquiry but they were not in the written work. In contrast, the written work of the fourth homework problem had questions and they were not only at the end of the student's written work. For example, relating to another problem (unknown to the researcher), the student wrote in the middle of her work:

> Fortunately, I already had some knowledge about the related problem where we may paint the vertices freely, and so I started wondering: how does the generating function that I have already found change given this restriction? Is there any way I can modify my equation, which takes symmetry into account, to remove the invalid colorings?

The questions that the student posed (wrote) throughout her work seemed to guide the student's continued exploration. The student provided a "final" answer for complete graphs at the end of her written work, and again the student seemed to be wondering about the generalization of her "final" answer and what it could mean for different types of graphs: "I went on to explore cycle graphs, which were closely related to path graphs, as well as completely disconnected graphs and star graphs. Unfortunately, I don't have time to write up my investigations." It seems that the process of posing questions was a way for this student to start another line of investigation or an approach. As it was noted at the end of the second homework problem, the student seemed to continue with this sort of action in the fourth homework problem. In the fourth homework problem, however, the student seemed to pose more questions. I conjecture that this student was influenced from her previous act of posing questions for starting a new line of investigation or an approach (in the second homework problem) to pose (more) questions on the later problem.

This particular observation could be an empirical example suggesting evidence of AOT of the student's mathematical creativity because the process of posing questions in the fourth homework problem seemed to be influenced by the student's previous experience of posing a question in the second homework problem.

With these two empirical examples, I attempted to illustrate (and answer the research question) in a way that the construct of transfer, more specifically AOT, could aid in exploration of students' mathematical creativity at the tertiary level. When transfer and creativity constructs are considered to be in an intersecting relationship, AOT could aid in gaining better understanding of instances in which students transfer their mathematical creativity. The empirical examples presented here raise new possibilities for the field to consider in research efforts focusing on these two constructs (transfer and creativity) together.

5.4 Actor-Oriented Creativity

Previous empirical examples illustrated potentials to examine transfer (using an AOT lens) of students' mathematical creativity from a researcher's perspective, namely that the researcher's observations and interpretations of the indication of influences of prior experiences in a new setting were used to form the hypothesized results. Explorations of AOT of students' mathematical creativity could be extended to better understand the phenomenon of creativity and its transferability by consulting with participants on researchers' observations or asking them to identify their own mathematical creativity in a setting and whether there were any influences of previous settings in their self-identified creative work. Because "novices are likely to demonstrate greater variety in their interpretations of learning environments than experts" (Lobato, 2012, p. 235), it would be beneficial to include students' interpretations of their own mathematical creativity to further this line of research. Kozbelt, Beghetto, and Runco (2010) emphasized the importance of understanding such "subjective" experiences that may not be observable by a researcher:

> The creative experience represents the more subjective forms of creativity, possibly never resulting in a tangible product, never undergoing external evaluation or never traveling beyond an individual's own personal insights and interpretations … Overlooking these subjective creative experiences in favor of objectively evaluated creative products can result in a partial conception of creative phenomena. (p. 23)

To demonstrate what it means to consider students' perspectives, I refer to the same student's data from an interview that was conducted at the end of the course to examine the benefits of the assignment system implemented in the course (Omar et al., 2019). At the interview, the student was asked various questions related to the assignment system and also asked to describe mathematical creativity.

Interviewer: What's your definition of creativity, mathematical creativity?
Student: I definitely think about, just kind of like asking a lot of questions and being, kind of able to have insight to know like which questions are going to be like relevant and are going to be kind of a, like an appropriate level of, like challenge and generality. Um, so kind of which will lead in interesting directions. Um, being able to, like really … bridge different techniques and different … things that we have like learned. So kind of like learning to apply and like generalize concepts.

In this quote, the student included actions, such as asking questions, having an insight to ask relevant questions "which will lead in interesting directions," making connections between existing knowledge and a given situation, and applying and generalizing as part of her view on mathematical creativity. These actions seemed to align with the definition of students' mathematical creativity provided in the previous section. Furthermore, the researcher's observation of "posing question" action that was hypothesized as transferred (from the AOT lens) was part of this student's view of mathematical creativity. The student was also asked if she had any moments of mathematical creativity in the course, to which she responded as follows:

> I definitely … did have um kind of like moments where I kind of felt like being able to … bridge examples, um, and kind of like notice general things. So, I think on … portfolio problems [problems on each homework] I had kind of like little moments of that, and then with the second big project in particular, um, I think just like kind of going through and … coming up with proofs and writing up proofs. And then also even just the process of … revising, um our proofs and definitions, um, I think very kind of creative in that we were like actually developing something and not just in the sense of like 'O I had an epiphany' … but having this process of kind of like going back, more like with say an essay, … reworking it, and making it better.

The student's identification of her own creativity in the homework problems (which were referred to as portfolio problems in class) seemed to align with the observations made by the researcher for both problems discussed in the previous section. The student also mentioned that she felt creative in the last project in the course, which was not examined by the researcher. Limited in its scope, this instance (i.e., the student's identification of another moment in which she felt creative) could be taken as a suggestion of the existence of moments that participants might perceive as mathematically creative, but that might not be considered as such by the others (e.g., researchers, teachers). Inclusion of students' perspectives on mathematical creativity and their perception of their own mathematical creativity could help us detect nonnormative instances of students' mathematical creativity and provide processes that are not reported in earlier studies or included in process-orientation definitions of mathematical creativity.

I propose the term actor-oriented creativity (AOc) to distinguish students' perspectives on creativity and perceptions of their own mathematical creativity from the ones observed or declared by others. I use *actor-oriented* and the lower-case *c* (as in mini-c creativity) to emphasize the students' views and interpretations of their own processes. This emerging construct relates to some of the underlying mechanisms of AOT. For example, in the AOT framework, "when taking an actor's point of view, the researcher does not measure transfer against a particular cognitive or behavioral target" (Lobato, 2012, p. 235). This resonates with the process orientation of AOc and inclusions of students' perspectives of mathematical creativity. To understand AOc's potential, in future studies of creativity, participants could be asked to describe what mathematical creativity means to them, to identify moments of mathematical creativity that they have experienced, and to reflect on how these moments differed from other noncreative moments. Students' responses could be examined for similarities to and differences from the ones observed by the researcher.

5.5 Conclusion

In this theoretical exploration chapter, I argue that there is an intersecting relationship between transfer of learning and creativity constructs. In particular, there could be instances of transfer that would not be part of creativity research (or would not "count" as creative), there could be instances of creativity that would not be part of transfer research, and there is the overlapping section where transfer could be creative or creativity could be considered as a transferrable ability. My exploration in this chapter was centered within the transfer construct, and the AOT framework was gauged to examine the potential of transferability of mathematical creativity, focusing the investigation on the intersection of transfer and creativity constructs.

The research question that guided this theoretical exploration was addressed through illustration of empirical examples. The student's posing-question action as part of the student's mathematical creativity was observed in both problems, and this was presented as an illustrative case for evidence of AOT of the student's mathematical creativity from the second homework problem to the fourth homework problem. Furthermore, this action was mentioned by the student as part of her view of mathematical creativity. The student's identification of her creative moment in another problem was taken as a suggestion of the existence of instances of mathematical creativity that may not be considered by researchers. To consider such instances, I propose an emerging construct of actor-oriented creativity (AOc) that explores students' perspectives of their mathematical creativity in research studies of mathematical creativity, especially when relativistic, process-orientation views are considered. As an emerging construct, AOc could offer a broader range of accounts of mathematical creativity. With its current formulation, this emerging construct could help in exploration within the *other* side of the intersecting relationship of transfer and creativity, where the creativity construct is taken as the main line of research.

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Chapter 6
Transfer as Progressive Re-Mediation of Object-Oriented Activity in School

Joshua Danish, Asmalina Saleh, Andrea Gomoll, Robert Sigley, and Cindy Hmelo-Silver

Transfer is a metaphor intended to describe the application of knowledge that is developed in one context within a second (Day & Goldstone, 2012; Lobato, 2006, 2012). However, researchers have consistently pointed out that transfer is hard to find, particularly when the original and transfer contexts look quite different as when comparing school to work (Bransford, Brown, & Cocking, 2000; Bransford & Schwartz, 1999). The difficulty in identifying episodes of transfer has led researchers to question how it is defined and ultimately to question the metaphor itself, pointing out that it is both vague and problematic (Beach, 1999; Hager & Hodkinson, 2009). As a metaphor, transfer implies that something is moved from one situation to the next. However, there has been quite a lot of debate regarding what exactly moves across contexts, with arguments that it is an individual's knowledge structures that allow movement between contexts or a perception of contexts as similar (Day & Goldstone, 2012; Lobato, 2006, 2012).

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Sociocultural theorists have noted that although individuals move across contexts, there are also aspects of social contexts that must remain similar for the kinds of activity that we typically label as transfer (Hager & Hodkinson, 2009; Lerman, 2000). However, a number of sociocultural theorists have suggested that this kind of explanation remains unsatisfactory because it doesn't articulate how the interaction between individuals and contexts lead to the kinds of performances that individuals are able to successfully engage in within new contexts (Beach, 1999). This led Beach (1999) to suggest that the metaphor of transfer is in fact the problem, and we should instead think about consequential transitions between spaces as individuals learn to adapt their practices in new spaces. In an effort to examine transfer between school and work, Tuomi-Gröhn and Engeström (2003) suggested a further move to focus instead on boundary crossing between activity systems and proposed that we might focus on how new forms of activity develop in this space. Although we agree that it is important to explore the interaction between individuals who move between contexts and the features of those contexts themselves, we agree with Hager and Hodkinson (2009) that shifting metaphors may not go far enough to address the tensions that arise within the transfer metaphor in a manner that allows clear description of the processes that we are in fact interested in. Furthermore, we are interested in designing and developing learning environments, particularly within schools, that can help teachers to support learners in transferring their knowledge across settings in ways that both honor learner agency and help learners to be successful in the kinds of tasks valued by schools.

Therefore, we have been working on an approach that focuses on the *progressive re-mediation of object-oriented activity* (Danish, Saleh, Gomoll, Sigley, & Hmelo-Silver, 2018). This approach is grounded in activity theory and, in particular, the importance of understanding how all of human activity is goal directed, what activity theorists refer to as the object of shared activity (Engeström, 1987; Wertsch, 1981). At the same time, activity theorists also note that all activity is mediated or transformed by the tools that individuals use in their daily activity. Our approach thus focuses on the way that shifts in both mediators and objects of activity change and remain similar across contexts. Thus we use the term re-mediation to refer to intentional changes in mediation. As we will describe below, we also assume that this kind of change is progressive in that learners rarely spontaneously adopt new mediators, or new objects of their activity, but are more likely to incrementally change their mediators, object, or both over time.

Our goal in developing this approach to transfer has been to better understand school-based transfer and to do so in a way that also helps us to better understand transfer at large. In aiming to understand how transfer might work in school, we necessarily want to understand how teachers can support transfer through their actions both when ideas are first introduced and when they are revisited (the transfer context). In the remainder of this chapter, we first lay out our theoretical notion of object-oriented transfer and discuss how it builds upon and expands prior definitions of transfer. We then present a brief summary of a series of interactions where a student, Brandon, works with a researcher and instructor, Amy, to transfer his ideas about how to represent and identify combinatorics across both classroom and clinical interview contexts. Across these interactions, we highlight how Amy worked to support Brandon's orientation to tools as useful for problem solving, making connections about tool use across problems, and articulating his evolving ideas using formal mathematical terminology. This case surfaces a number of instructor actions that support object-oriented transfer. Finally, we propose concrete steps that teachers can take to prepare for and support transfer of mathematical tools and concepts by attending to transfer as an object-oriented phenomenon.

6.1 Theoretical Approach: Object-Oriented Transfer

6.1.1 Activity Theory

Our approach to exploring transfer is grounded in activity theory (Engeström, 1987). Activity theory builds broadly on Vygotsky (1978) and other sociocultural theories of learning (Danish & Gresalfi, 2018). Activity theory explores how human actions and learning are fundamentally shaped by the social context in which they occur and at the same time help to define those social contexts. Activity theory differs from other sociocultural approaches in that it takes collective, object-oriented activity as the unit of analysis for understanding cognition and learning (Wertsch, 1981). Collective activity refers to activities where individuals are working together towards a common end. This focus on collective activity highlights the fact that our experiences are shaped by the ways that we aim to act in coordination with other people. Naturally, individuals have multiple disparate goals, and they are often changing during activity. However, it is our shared goals, or mismatches between them, which define our collective activity and interactions. To help distinguish between the goals that individuals pursue in a given moment and those that a group of people are working towards, the shared goals of the collective group are referred to as the object of activity. From this perspective, to understand cognition and learning, we need to understand how our shared objects and individual goals shape our actions. In the context of transfer, this means further understanding how the presence of both similar and different goals or objects may lead to engagement in similar forms of activity.

To further understand activity, activity theorists also note that all human action is mediated, or transformed by our sociocultural context (Wertsch, 2017). Mediators include the tools that we use, the rules that govern our actions, the community that we are interacting with, and the division of labor through which we all orient towards our shared object of activity (Engeström, 1987; Wertsch, 1981, 2017). It is important to note that from this perspective, tools include both *material* dimensions (e.g., the actual paper on which a student has inscribed a table to represent different combinations of pizza toppings) as well as an *ideal* dimension (e.g., the mathematical concepts embedded within the student's table) that are interconnected (Cole, 1996). The way that a student uses a tool is also shaped by the object of their activity. Put colloquially, students consider: What is it I am trying to accomplish? This, their object or goal, will shape how they take up the tool in action. These actions are then further mediated by the rules, tool, and division of labor. Mediation is also always bidirectional, with the tools that a student has available also shaping their perspective on the object of activity.

By way of an example, consider the aphorism that if you give a child a hammer, everything starts to look like a nail. This suggests that a hammer somehow carries with it the idea that all things should be pounded flat like a nail. However, activity theory challenges us to consider how the tool and object are distinct components of activity. Following this, we might say that if you teach a child that a hammer (a tool) is great for flattening things (an object), the child will link these two concepts in their understanding. Thus, the child is likely to either flatten things when using a hammer (i.e., as a tool) or to consider the relevance of a hammer as they achieve certain objects or goals (e.g., such as flattening nails). We argue that this bidirectional relationship between tools and the object of activity is an important aspect of transfer to consider when teaching. A teacher might choose whether they want children to focus on the tool itself or on the purposes they might pursue with the tool (or, alternatively, the affordances of other tools for that object of flattening a nail), depending on what they want their students to take away from the lesson. At the same time, it is important to explicitly recognize that students' activities always involve both the tools and an object of activity and, thus, both should be chosen intentionally by the teacher. Furthermore, activity theory might ask how other mediators of activity shape a learners' experience with a specific tool. What rules are there that encourage or discourage the use of hammers? What rules shape how hammers are used? For example, children are often taught only to hammer when wearing safety glasses, which might lead them to hesitate when glasses are not present. What about the community and division of labor? How would children coordinate their actions when working with a limited number of hammers or a limited number of nails that need hammering? Activity theory suggests that we need to always consider the role of rules, community, and the division of labor in mediating the use of tools in supporting human learning and activity.

6.1.2 What Transfers?

This analysis suggests that when looking at transfer, it is valuable to consider both the tools that transfer, and the objects that learners are pursuing, as well as the various mediators of learners' activity within both the original and transfer contexts—issues that activity theory can help make visible. From our perspective, traditional transfer accounts focus largely on the "tools" that we see in activity theory. For example, approaches to transfer that are grounded in cognitive theory ask whether learners have a sufficiently abstracted notion of a given tool to apply it within a new context (e.g., Barnett & Ceci, 2002; Goldstone & Sakamoto, 2003). This approach then explores how learning environments provide learners with sufficient opportunities to generalize and abstract their understanding of the tool. Further, this approach then looks to see whether students spontaneously recognize that their tool has value within the new context. However, these approaches typically view a context in the form of a "problem." Social dimensions of the context such as the role of other members of the community, or the rules that might be present and so on, are less visible in this kind of analysis. More importantly, we believe that by conflating a tool with a specific purpose, this approach may obscure how students ultimately need to understand both tool and purpose and their relationship to each other. That is, true understanding of a hammer is tied to how learners recognize that it can be used for both putting nails into wood and pulling them out and that both might support a broad range of activities.

We think it is also important to note that a cognitive approach assumes what we refer to as a *normative account* of transfer. That is, schools and disciplines expect and value certain kinds of transfer, and then scholars often look to see if those happen or not. Thus, whereas creativity researchers are interested in the child who uses a hammer to prop open a door or represent a spaceship, a normative account would only recognize and value the use of the hammer for putting nails into wood or taking them out. We view this as taken for granted, and, thus, one intention of our exploration of transfer as progressively mediated and object oriented is to explicitly note this normative influence and the power differential that allows teachers to indicate which kinds of transfer are valuable and which are to be ignored or discouraged.

As an alternative to cognitive approaches to transfer, actor-oriented approaches to transfer such as the work by Lobato (2012) build on sociocultural theories to identify how transfer is tied to learners' experiences with tools in rich social contexts. An important distinction is that this actor-oriented approach focuses on what learners see as relevant to transfer in a context rather than focusing on a more normative account. That is, scholars using this approach would ask: What does a child notice about hammers in the first place? Or, why does a child view a hammer as being something that is useful for pounding nails? Or, if a child is instead interested in using a hammer to remove nails and other fasteners in new contexts, this approach doesn't view that as a failure to transfer the notion that a hammer is for pounding nails (the normative account) but rather as successful transfer of the role of the hammer in removing nails (also normative but less common) or for emulating a laser gun (not normative but creative and authentic). Then we might ask what it was that led students to view the hammer in that way. Although we agree with the actor-oriented approach's critique of a sole focus on normative accounts of transfer, we believe that there is also value in reconciling teachers' needs to promote specific normative accounts to accomplish their daily tasks. Thus, our goal in building on actor-oriented approaches to transfer within a mediated and object-oriented approach to transfer is to focus on the intersection, connection, and tensions that lie between students' experiences and their own objects. We also want to attend to the kinds of mediators that teachers want their students to be able to use and the objects teachers wish their students to pursue. We believe that focusing on these two dimensions (mediators and objects) can lead us to more deeply recognize the work, and related tensions, that teachers need to engage in when promoting their academic agenda in interaction with students who might not yet share it.

In addition, we believe that prior accounts of transfer have largely focused on how learning environments prepare students for transferring later and then explore if and when students do transfer. That is, we feel that the focus has been on how teachers can support transfer out from a classroom context whereas we would also like to explore how they might support transfer into new classroom contexts because that is often an important building block for school success and later transfer beyond school. We are not the first to explore how transfer can be supported within transfer contexts (Beach, 1999; Tuomi-Gröhn & Engeström, 2003). However, we believe that articulating this movement in terms of the shifting mediators and objects that participants engage with can provide new insights into the process. We now discuss how an object-oriented approach to transfer might build on prior accounts to explore this issue.

6.1.3 Teachers and Intentional Transfer

Actor-oriented approaches to transfer certainly recognize the socially situated nature of learning and thus the impact of multiple social factors on those features of a tool that students are likely to transfer. As Lobato and colleagues (Lobato, 2012; Lobato, Rhodehamel, & Hohensee, 2012) noted, noticing features of a tool and recognizing them as valuable is a socially situated action. Teachers and other learners help define what a student notices as important and relevant and thus what they are likely to transfer into new contexts. In activity theoretic terminology, learning a new tool is mediated by the rules, other tools, community, and division of labor within the activity. Engle, Lam, Meyer, and Nix (2012) further noted how teachers might play a role in framing activities in ways that are more "expansive," helping learners to recognize the potential generality of a tool they are learning about. However, these accounts largely focus on cases where teachers are helping students to transfer out to future activities.

In contrast, we view teachers as continuously aiming to support transfer from one context to another within their own classrooms or from their classroom to the next. This is more modest but equally important. Our goal in exploring object-oriented transfer is to better understand how teachers might intentionally support this kind of transfer in mathematical classrooms and beyond. From this perspective, we first want students to recognize that tools are useful for specific objects and that it is desirable for them to continue using those tools as they encounter similar objects across contexts. That is, we want to think about how learners might transfer both tools and objects, not just one or the other. This suggests that rather than exploring whether students see problems or "contexts" as similar or different, we need to explore whether they see an object of activity as similar, whether they see similarities or differences in the other mediators that support that overlap in contexts, and whether they see those differences as consequential for if and how they use the tool in the new context. In the example of Brandon presented later in this chapter, we are interested in how he transfers the need to identify the possible combinations of elements within a set as well as the value of a specific graphical representation as a tool for supporting efforts to identify those combinations.

An object-oriented approach to transfer also calls our attention to the fact that some aspects of activity are approved by members of the community and some are not. When participants behave in expected and ratified ways, this is made clear in interaction. In contrast, when participants behave in ways that contradict the communal rules, division of labor, or object of activity, this is often made visible to them or leads to a breakdown in activity. In the case of classrooms, the teacher plays a particularly important and salient role in shaping what learners view as appropriate. Similar to Engle et al.'s (2012) notion of expansive framing, we believe that this means we need to attend to how teachers might encourage transfer and how learners experience teachers' different ways of framing transfer. This suggests that it is important to understand both how future transfer is encouraged, if it is, in an original context and how it is then further encouraged in new transfer contexts. Importantly, we ask: How does this encouragement mediate learners' understanding of the tools, object, and other mediators that are part of their transfer experience?

6.1.4 How Do We Look for the Process of Transfer?

Given that our focus is on transfer and not all learning, we think it is important to focus explicitly on the processes through which students recognize tools not just as useful but as potential candidates for transfer and when they recognize objects of activity as related. To understand this within classroom activity, we think it is therefore equally important to attend to how and when teachers support this process, helping students to see tools as relevant for transfer or encouraging or ratifying student belief that particular tools can and should transfer. When teachers do not support this process successfully, it is likely students will resist the effort to help them transfer the target tools. Woven throughout our mediated, object-oriented transfer perspective is our activity theoretic assumption that tools never transfer on their own but rather that tools are used in pursuit of specifics objects. Thus, learners likely view tools and objects as interconnected and learn to pursue objects with tools and vice versa. Taking an object-oriented approach, we work to attend to how and when this connection happens. We see this as similar to the actor-oriented approaches to transfer in that we remain interested in what students' notice and take up, but we also extend those ideas by focusing on how learners' activities are mediated, including the work of a more knowledgeable other (the teacher) in promoting institutionally supported tools and objects for transfer.

Thus, the four key types of moments to look for in a potential transfer situation are:

1. How teachers frame tools and objects as interrelated and whether they indicate through their actions that either or both might be useful in other future contexts.

2. How students see tools and objects as relevant and whether they recognize that this might be worthy of transfer to future contexts.
3. How teachers provide opportunities for students to transfer prior tools into new contexts and then help students view that transfer task as valuable.
4. How students attempt to transfer in new contexts and how they come to view this as successful and valuable and thus worth repeating.

It is important to note that although these moments may appear in sequence, we assume that they are often happening iteratively and in an overlapping manner as learners come to appropriate tools in new ways and recognize their potential for transfer. We now briefly illustrate how this approach can help us explore transfer in a sample data set before returning to a discussion of how teachers might support transfer of mathematical tools and concepts from this perspective.

6.2 Working Towards Transfer with Brandon: An Example

The video data described in this paper come from a larger longitudinal study that was conducted in a small suburban community in the Northeastern United States. The study contained 32 in-class problem-solving sessions (1 hour each) led by researchers. Students explored the development of mathematical ideas in combinatorics and fractions before they were formally introduced to the ideas, algorithms, and formal academic language. Students were guided by facilitators as they worked, with little direct instruction provided to the whole class.

The video highlighted in our analysis comes from one of the early problem-solving sessions and a follow-up interview with one of the participants, Brandon (Maher & Martino, 1998). Our interest in Brandon's work emerged based on the experiences of two of the authors who used this as an example in their own teaching to demonstrate transfer and the use of isomorphisms. As part of a larger project, we used the VMCAnalytic video analysis tool (Maher, Palius, Maher, Hmelo-Silver, & Sigley, 2014) as a space to engage in iterative interaction analysis (Jordan & Henderson, 1995). As a research team, we held data sessions centered on exploring mediated, object-oriented transfer and how it unfolded in Brandon's recurring work with researcher Amy. Participants in these sessions shared initial thoughts about the key events in each video related to our interest in understanding how transfer emerges within activity and then clipped and annotated these events within the VMCAnalytic for the next session. Over the course of several sessions, our research team narrowed our focus to one class session and interview with Brandon. In the classroom segment, two fourth-grade children, Brandon and Colin, were working to solve a combinatorics problem about combinations of pizzas and toppings. The second, and the more extensive focus of our analysis, involved a clinical interview with a researcher, Amy, and the fourth-grade student, Brandon, solving the same problem and then the ensuing interaction when Amy asked Brandon what this task reminded him of.

Elsewhere, we analyzed a sequence of activities that Brandon engaged in, highlighting the various moments where he and the teacher explored the mathematical contexts and how those moments helped make our framing visible (Danish, Saleh, Gomoll, Sigley, & Hmelo-Silver, 2017; Danish et al., 2018). Here, for clarity purposes, we present a streamlined version of this account organized by our four key features to help make visible the role that each might play in what we observed as "transfer." It is important to note that, like many learners, Brandon also had intermediate steps in his activity and benefited from iterative, repeated engagement with these ideas. For clarity's sake, however, we present a streamlined account intended to highlight how an object-oriented approach to transfer can help us think about possibilities for supporting effective transfer. Even in streamlined form, however, we believe the progressive nature of this transition is apparent.

6.2.1 Introducing Generalizable Tools in Math Class

Two events appear to be particularly important in setting the stage for Brandon to transfer his ideas across math classes. First, the teacher asked the students to think about how to combine colored blocks into stacks in different ways. This introduced the idea of listing out combinations as a practice that the classroom could later build on. Then, in our focal sequence, the students were asked to list out combinations of possible pizza toppings. Although the teachers clearly knew these were related activities, they did not initially make that explicit. However, they did support the students in identifying their own productive methods for recording and vetting different combinations of pizza toppings. Here we see that both the process of trying to list all of the possible combinations and the specific representational tool—a grid-like table where the students listed each combination in order (see Fig. 6.1)—are powerful tools that might be repurposed for later math problems. However, Amy didn't simply introduce the table. Rather, Brandon and the other students were able to identify their own process and thus to see how their approach to structuring the information might be useful in addressing the specific problem. Furthermore, Amy helped the students to see the value in their approach by asking

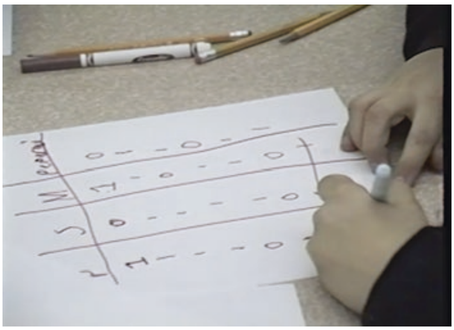

Fig. 6.1 Brandon's pizza combinations. The letters on the top are toppings, and 1 indicates the topping is present and 0 indicates it is absent

clarifying questions about how they went about listing their sequence and why they chose that approach. For example, Amy scaffolded Brandon in indicating how grouping the different topping patterns helped him to identify whether he had already listed a specific combination and thus could help avoid redundancy. Our mediated, object-oriented approach to transfer highlights three key aspects of this part of the process: (a) The structure of the table itself is a powerful "tool" for identifying combinations, (b) the practice of carefully and sequentially constructing the table is also a conceptual tool for identifying combinations in mathematics, and (c) both of these support the object of identifying all possible combinations. However, as we will show below, simply pointing this out is not enough for Brandon (or most students in our experience) to spontaneously transfer these ideas to new contexts. However, this does set the stage for that accomplishment. Importantly, Amy was also attending to Brandon's ideas here so that she could then aim to build on them later.

6.2.2 *Revisiting Useful Math Tools to Explore Their Utility*

Three weeks after the classroom activity where Brandon produced the table of pizza toppings, Amy met with him in a one-on-one interview to discuss his representation. Although on the one hand, this kind of interview is "artificial" in that most classrooms do not have a researcher who engaged in one-on-one meetings with students, we also believe it helps to depict what is possible when a thoughtful educator revisits a students' earlier activity in a discussion with them, something we have seen many successful teachers do within their classrooms. In this interaction, Amy began by asking Brandon to recount what he had done to list his pizza toppings using the table. Brandon began to describe that he had listed all of the different combinations in order starting with one topping (pepperoni) and then combining that one topping with all possible pairs (pepperoni and mushroom, pepperoni and sausage, etc.) and then three toppings and so on. Amy also asked Brandon to indicate how he knows to move on to a new topping (when all combinations have been exhausted) and how he knows to skip some (they were already listed). Brandon didn't have ready answers to these questions and in fact changed his approach part way through the interview when he believed that he had identified an error.

In this way, Amy's prompts were key to Brandon repeating, articulating, and continually refining his tool use (mediating it)—if he had stopped at any point he'd have had an incomplete and potentially incorrect solution. Brandon and Amy both continually noted how this tool is tied to the specific object of identifying the number of combinations in the set, helping to highlight the value of this tool for that specific but also generalizable object. Furthermore, Amy helped him to articulate how his tools (the table and practice of filling it in) are productive for solving the problem of systematically identifying combinations of elements in a set. She effectively highlighted the importance of these aspects of the approach, helping him to notice and articulate them. We believe this also set the stage for Amy to help connect this solution to another context. However, this also highlights how transfer is rarely spontaneous and "correct" in that learners are always refining their understanding and their practices in response to feedback. Being aware of this, a teacher can help the students to build up their repertoire of transferable tools continuously as they engage with them in classroom activities.

6.2.3 Signaling and Ratifying Successful Transfer

At this point, Brandon appeared to see the utility of his approach to identifying combinations of pizza toppings but had not yet discussed its utility in other contexts. Thus, although he had now used this tool more than once, he had not yet demonstrated unaided transfer. We add "unaided" here to highlight that he has in fact transferred his knowledge when we consider the full mediated activity system including Amy, which we view as quite important for teaching contexts. For teachers, unaided transfer is rarely necessary during a curricular unit or even across units. Rather, mediated transfer, where students can perform an action that looks like transfer, is often how teachers support progressive changes in how learners use tools, and one of our goals in articulating the role of mediated object-oriented transfer is to highlight these intermediary steps as a feature of the changing system. In this case, Amy also helped Brandon to recognize that he may have done something unique by asking him if he did, in fact, recognize this as similar to other things they had done in class: "Does this problem with the pizzas remind you of any other problems we've done this year?" Brandon didn't at first think of a similarity, and Amy prompted him, "It could be in the way you've done them." This focused his attention on the approach to a solution rather than just the materials, and he indicated that it reminded him of the "problem with the blocks" where the students had to find all of the combinations of yellow and red Unifix® cubes that might be combined (see Fig. 6.2).

Again, Amy didn't simply stop at Brandon indicating a simple awareness of similarity. Rather, she brought out the Unifix® cubes and asked him first to show how he worked with them and then to try and apply his approach of using the table

Fig. 6.2 Brandon reassembles *Unifix*® cubes to show Amy how he found different combinations

to the problem of different colored cubes. He demonstrated his solution to Amy, and appeared to see the similarity, but was still using different approaches. Thus, by some measures, he had not yet fully "transferred" his knowledge. Fortunately, Amy asked him if he could use the table to also identify the patterns in Unifix® cubes just as he had with the pizza toppings. She also helped scaffold the process by helping him translate between the problem spaces, suggesting he focus on a single color first (yellow blocks) just as he had grouped his pizza toppings, focusing first on pepperoni. Brandon then further unpacked the similarity between the solutions, showing how the grouping helped him identify all of the combinations and reiterating the importance of not repeating patterns that appeared previously but in a different order. In short, Brandon now exhibited the kind of transfer that we are typically interested in promoting!

6.2.4 Who Transferred?

Amy would clearly have liked to see spontaneous transfer but didn't. She didn't, however, give up or treat this as failed transfer. Rather, we see here a messy, realistic process through which she took continuous steps, as did Brandon, to engage in meaningful, mediated, and object-oriented activity. Amy helped mediate Brandon's activity, helping define the object for him, and helped him view both the object and the tools as transferable, and we see nice progress there. A skeptic therefore might point out that Brandon did not in fact transfer knowledge on his own in the way that educators have long hoped for, in the way that we hope will happen out of school, and in the way that is so rarely seen. We agree! But we also think that Brandon has, in collaboration with Amy, achieved exactly the kind of transfer we can and should be promoting in school because it helps him to see the power of the tools he is learning and to connect ideas across class sessions and topics. It is also important to note that, unfortunately, we do not have evidence of whether or not Brandon is later able to engage in this kind of transfer on his own as the class moved on to other topics. It's quite possible that without added opportunity for practice and reflection (Beach, 1999), Brandon would not be able to engage in this activity unaided. Nonetheless, we believe that noticing the continued use of his table as a tool for this broader set of problems was an accomplishment and an important building block in future reflective activity.

Brandon's is not an individual accomplishment, and it does not need to be. Rather, it is a collaborative achievement that builds on the work of both the teacher and the student to see the power of a tool in a new space. On the one hand, Brandon has seen the generalizable and potentially abstracted value of a specific tool for a specific object (Day & Goldstone, 2012); how tables of combinations can help in listing all of the possible combinations without repeating any. At the same time, this was not a solely cognitive effort nor was it solely driven by the student's knowledge. Rather, the student needed help in noticing the key features of the problems and solutions (Lobato et al., 2012) and benefited from assistance in framing the solutions more expansively (Engle, 2006). In this process, Amy also helped Brandon to see not only what he viewed as relevant but what she viewed as relevant. Although it is important to give students agency, we also want to recognize the value of building on ideas that the teacher knows are valued by the discipline and by the school context. That is, it is important and valuable for Amy to help Brandon achieve a normative vision of transfer that will support him within the existing institutional setting.

6.3 What Can Teachers Do With This Knowledge?

Our goal in exploring an object-oriented approach to transfer is twofold. First, we want to better understand how and when learners might perform what analysts view as transfer. Second, we want to build on this knowledge to help teachers reflect on how they might support transfer within their own classrooms, particularly when they are working under the assumption that students will carry certain valuable, normative ideas through their classes. Our analysis above suggests that there are four key elements that teachers might keep in mind both in their original context and their transfer context to help students continuously orient towards overlaps in different objects of activity and the tools—both material and in practice—that might help them pursue those objects throughout their school career. These four elements are summarized in Table 6.1. Note that these elements all build on our core commitments to the notion that (a) transfer is progressive—it is a continuous process of refinement and iteration as learners explore how different tools might transfer—and (b) this process can be continuously supported by other mediators, including the teacher.

First, we think it is important to help students to recognize tools as being useful for specific objects of activity—that is, helping students recognize that they are attempting to achieve certain goals and framing those goals in potentially generalizable terms. In our example above, Brandon only saw the transferability of his object when he recognized the goal of finding all of the combinations of elements in a set rather than viewing it solely as a problem about pizza toppings. Similarly, it is valuable if the student views the tool as supporting this object. Thus, listing a sequence of combinations in a table is a good way to find all combinations of elements in a set rather than just being useful for pizza. And, as we saw, Brandon did not fully appreciate this relationship when these ideas were first introduced but only after they were reexamined in later episodes.

Second, we believe it is helpful for teachers to explicitly call out the fact that a given tool might be useful in new situations. Whereas laboratory experiments benefit from seeing whether students notice this spontaneously, day-to-day classroom activity is better supported when students notice that tools can be reused and are provided guidance in how to use them in their new context. As noted above, we don't believe that it lessens Brandon's accomplishment of transfer because Amy helped him see the overlap across contexts. Rather, it reveals how hard this kind of

Table 6.1 A summary of teaching practices to support object-oriented transfer

Teacher practices for supporting transfer	How this applies in the original context	How this applies in the subsequent contexts
1. Scaffolding awareness of how the tool helps achieve the object.	Help students see the object and recognize it is one with possible future uses (i.e., expansive framing). Also, help students see clearly how the tool is tied to this object and is not just arbitrary.	Help students view new objects as identical or similar to prior objects. Help students to recall the tools that were useful with those prior objects.
2. Scaffolding the realization that a given tool might transfer.	Help students to frame their solutions in generalizable terms and suggest the possibilities for such solutions in future problems.	Help students remember and leverage prior tools and representations in new, different contexts.
3. Providing opportunities and encouragement to attempt transfer.	Curriculum design can include opportunities to apply tools in new contexts as well as opportunities to discuss and reflect on them.	Teachers can validate and encourage opportunities to transfer tools into new contexts. Teachers can promote the kinds of normative language and tools that will be expected while also legitimizing nonacademic language that still bridges tools into new contexts. New activities can be designed to help students fine-tune their tool use so that it is more generalizable.
4. Ratifying students' accomplishments of transfer.	Teachers can create an environment where students see that carrying tools into new contexts is valued by teachers and valuable for their own local objects of activity.	

work really is and how a focus on co-construction of transfer orients us to the value of teachers in promoting transfer with their students.

Third, we should think of sequences of instruction as powerful opportunities for transfer. Although the literature on learning progressions does not always explicitly use the term *transfer*, we see a crucial overlap here in that learning progressions are intended to find the most powerful conceptual tools for students within each discipline and to help students develop and refine those over time. From our perspective, that is a case of continuously supporting object-oriented transfer by identifying the most valuable tools that we would like our students to transfer, introducing them, and then providing opportunities to see them as transferable.

Finally, we think it is important to validate learners' efforts at transfer, helping to motivate them to attempt to transfer and to carry over the tools that are valued by the discipline. It is important to note that activity theory highlights the need for this to be authentic and substantive. That is, we shouldn't simply reward students for attempting transfer that is meaningless to them. Rather, we should help them see the value in transfer on their own so that they are more likely to appreciate the transferred tools and continue to apply them, refining their use over time.

6.4 Conclusions

As a field, we all recognize both the value and the difficulty in supporting transfer of tools and practices from classroom activities to other classroom activities and eventually to the real world beyond. As sociocultural approaches to transfer have long noted, however, transfer does often happen, it is just not always as spontaneous as researchers might hope it to be nor is it always of the same concepts and ideas that educators would like learners to transfer (Lobato, 2006). To understand why this is, sociocultural theorists have noted both that we need to attend to learners and their agency (Lobato, 2012) as well as recognizing that transfer occurs at the interaction between learners and contexts, suggesting that it best be viewed as a transition (Beach, 1999) or boundary-crossing activity (Tuomi-Gröhn & Engeström, 2003).

Our goal has been to build on this tradition by finding theoretically consistent ways to recognize and anticipate how learners engage in the process of moving between these contexts. Our approach builds on this tradition by noting three key elements of transfer: It is (a) progressive, (b) mediated, and (c) object oriented. That is, if all human activity is object oriented, then any account of transfer needs to attend to how learners recognize the object of activity as similar or different across contexts. Similarly, if these activities are also mediated, we need to recognize the role of shifting mediation in transfer contexts. Furthermore, as discussed above, the mediators and object of activity are always interrelated and influence each other. Thus, transfer is really an account of how similar mediators are taken up to pursue a new object, how new mediators are developed to pursue a similar object, or both. Finally, the process of refining our understanding of the potential use of mediators to pursue similar and new objects is a progressive one that is constantly in flux. We believe that explicitly recognizing these elements will support both teachers and researchers in not only promoting transfer but in recognizing when and why it has or has not occurred as anticipated. With this in mind, supporting transfer can be moved from the theoretical realm to the pragmatic, grounded activities of everyday classroom life.

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Part II
Transfer of Mathematics, Science, and Technology Learning in School Settings

Chapter 7
Graphical Shape Thinking and Transfer

Kevin C. Moore

It is widely acknowledged that a learner's currently held cognitive structures afford and constrain her future learning experiences. It is also widely acknowledged that a learner's present learning experiences can shape and modify her previously constructed cognitive structures. Researchers refer to these phenomena in ways dependent on their theoretical framing. Researchers adopting a transfer perspective often appeal to processes of forward transfer and backward transfer to explain these phenomena (Hohensee, 2014; Lobato, 2012). Researchers adopting a Piagetian constructivism lens are disposed to explain these phenomena in terms of assimilation and accommodation (Piaget, 2001; Steffe & Olive, 2010; von Glasersfeld, 1995). Because each of these processes is influential in a learner's mathematical development, researchers have called for more detailed explanations of them in terms of specified mathematical content, concepts, and teaching (Diamond, 2018; diSessa & Wagner, 2005; Ellis, 2007; Hohensee, 2014; Lobato, Rhodehamel, & Hohensee, 2012; Nokes, 2009; Thompson, 2013b).

I address the aforementioned call in the present chapter by discussing forward and backward transfer in the context of students' meanings for graphs. I do so with three related goals. First, I define and elaborate on constructs, which are forms of what is referred to as *graphical shape thinking* (Moore & Thompson, 2015)—that Thompson and I introduced as epistemic subjects to capture students' meanings for graphs.[1] Epistemic subjects (Steffe & Norton, 2014; Thompson, 2013a) are conceptual models that specify categorical differences among students' in-the-moment

[1] Thompson and I initially used *shape thinking* as the stem phrase for the constructs (Moore & Thompson, 2015). We have since updated the stem phrase to *graphical shape thinking* to emphasize our focus on quantitative relationships and their graphs, as opposed to the study of geometric shapes.

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meanings. Second, and relatedly, I situate these meanings in terms of specific transfer processes and their implications for student activity in novel situations. Third, I use a synthesis of a student's actions to assert the role sequential processes of forward and backward transfer can play in students' graphical shape thinking.

I structure the chapter as follows to accomplish these goals. I first provide two vignettes in order to introduce the two graphical shape thinking constructs and motivate a focus on particular aspects of transfer. A concise discussion of the theoretical underpinnings central to this chapter follows the opening vignettes. I subsequently describe the two graphical shape thinking constructs and, using accompanying student data, illustrate them in terms of students' transfer processes. Generalizing from these cases, I introduce a way to frame concept construction in terms of theories of transfer and graphical shape thinking, and I provide a data example to illustrate the productive nature of such a framing. As part of this discussion, I provide suggestions for future research.

7.1 Two Vignettes

The following vignettes are from task-based clinical interviews (Ginsburg, 1997) that occurred as part of investigation into preservice secondary teachers' (PSTs') and undergraduate students' meanings for graphs in the context of noncanonical representations (Moore, Silverman, Paoletti, Liss, & Musgrave, 2019). PST1 is responding to the prompt and graph in Fig. 7.1a. PST2 is responding to the prompt and graph in Fig. 7.2a. Both vignettes are actual excerpts. As the reader engages with the vignettes, consider the particular meanings the PSTs are drawing on in that moment of reasoning, as well as the potential influence of previous learning experiences with respect to each PST's reasoning.

7.1.1 Vignette 1 (PST1): Where the Slopes Were[2]

Vignette 1 concerns the following prompt: *You are working with a student who happens to be graphing y=3x. He provides the following graph* [shown as (a) in Fig. 7.1]. *How might he be thinking about this?*

PST1: Um, like this [*rotating* Fig. 7.1a *90 degrees counterclockwise*—Fig. 7.1b—*laughing*].[3] Like I [*rotating back to* Fig. 7.1a], because if you turn it this way [*rotating to* Fig. 7.1b *again*], then this [*tracing left to right along hori-*

[2] "Int." stands for the interviewer.

[3] Throughout this chapter, as needed, I describe clarifications in participant explanations, gestures, and actions using "[*text*]". Italicized text indicates added information whereas standard text indicates our replacing an ambiguous word or phrase with my interpretation of her intended word or phrase. I also use this convention with a line break to indicate a summary of intermediate discussion.

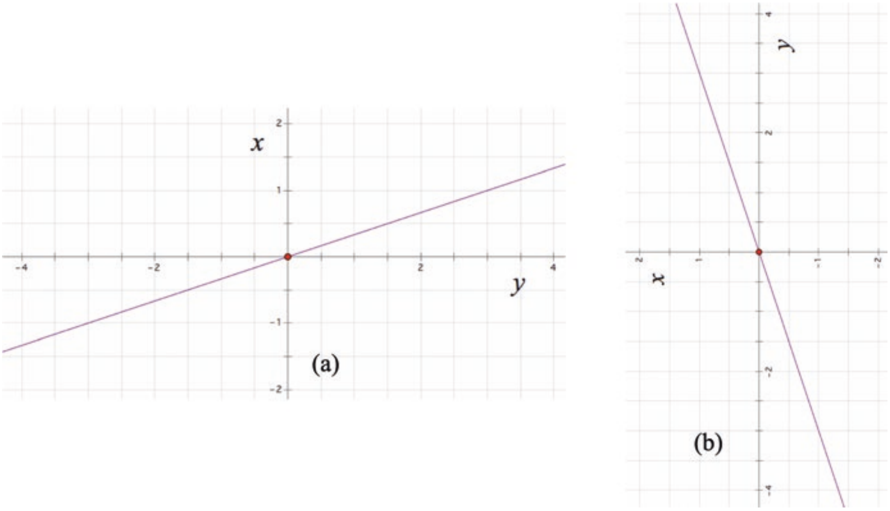


Fig. 7.1 (**a**) A hypothetical student's graph of $y = 3x$. (**b**) PST1's rotated graph

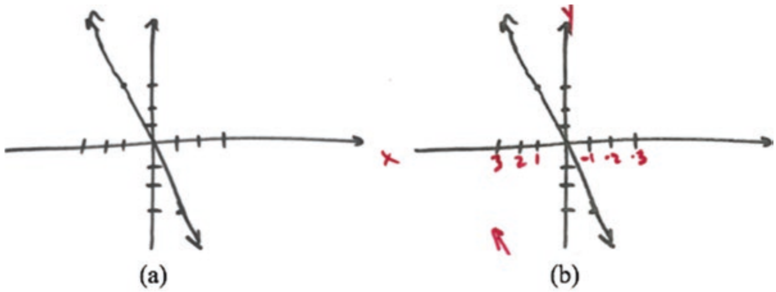


Fig. 7.2 (**a**) A second hypothetical student's graph of $y = 3x$. (**b**) PST2's added markings in red

zontal (x) axis] and then this [*tracing down vertical (y) axis*], it would be still not right though.

Int.: How would you respond to this student if they said, "Well, here's" [*rotating back to* Fig. 7.1a]

PST1: I mean I would tell them that they just labelled, like, well, I guess I would figure out what they were thinking about first because it could have just been something of they don't know which, they don't know that this is the *x*-axis [*pointing to the horizontal axis*] they don't know this is the *y*-axis [*pointing to the vertical axis*] . . . I don't really know if that makes sense. I mean the only way I can think of it is like this [*rotating to* Fig. 7.1b] and it's still wrong because this is negative slope [*laying a marker along the line sloping downward left to right*] . . . [*rotating back to* Fig. 7.1a]. I would just explain to them like the difference between the *x*- and *y*-axis . . . because if they were thinking of it as like sideways [*rotating to* Fig. 7.1b] or whatever [*rotating back to* Fig. 7.1a] it is, or inversely, or whatever, then, show them like the difference between like positive and negative slopes, also.

Because that's something that when I was in middle school we learned kind of like a trick to remember positive [*holds left hand pointing up and to the right*], negative [*holds left hand pointing down and to the right*], and no slope, and zero [*holding left hand horizontally*], like where, that's where the slopes were. And it's stuck with me 'til now, so it's important to know which direction they're going, when it's positive and negative and zero and no slope, too. But in this case positive or negative.

PST1 assimilated Fig. 7.1a essentially as a piece of wire with indexical associations of "slope" based on how it is placed in relation to two other pieces of crossing wires, as evidenced by her describing "where the slopes were," "which direction they're going," and using directional gestures to indicate a line's direction. PST1 understood rotating the paper as changing the line's slope (e.g., "which direction they're going") and she did not perceive any rotation to result in an image of a line associated with $y = 3x$. Furthermore, PST1 explicitly appealed to the previous learning experiences in which she formed these associations, thus anticipating the given task as resolved by producing a line in the "direction" learned during those experiences.

7.1.2 *Vignette 2 (PST2): A Product of How We've Decided to Represent Things*

Vignette 2 concerns the following prompt: *You are working with a student who happens to be graphing y=3x. He provides the following graph* [shown as (a) in Fig. 7.2]. *How might he be thinking about this?*

[PST2 has labeled the axes as shown in Fig. 7.2b; claims that the graph is of y = 3x].

Int.: So, what about a student who says, that says, "That can't be, that can't be right [*pointing at* Fig. 7.2b] because that's sloping downwards left to right. You know, that's going down to the right, so it can't be right. It has to be negative."

PST2: No, um, that sloping downward to the right [*moves hand down and to the right*] is a product of the convention of us labeling our axes with our positives over here [*motioning to her right*] and our negatives over here [*motioning to her left*], so you can look at it and we can trust that [*making hand motion down and to the right*] that's going to be a negative slope as long as everything is within our conventions.

Um, but slope is really just rate of change. And so, what this is telling us, this three [*circles* 3 *in equation*], is that when x, it's like [*writes "rise/run"*]. Is it sad that I still have to use rise over run like this? I feel like this is so bad [*writes* 3/1]. Well, anyways. Okay. So when we're saying that

> when our x, or our y changes on this graph, when our y changes by 3 [*pointing to the* 3 *in* 3/1], our x is changing by one [*pointing to the* 1 *in* 3/1]. So, if we can go up three [*pointing to the dash indicating a value of* $y = 3$ *on the y-axis*] in the positives [*puts plus signs beside* 3 *and* 1], we're still going positive one. But now our positives are over here [*motioning to her left*], so we have to be cognizant of the way our axes were labeled.
> If we were to switch this [*using her hands to indicate changing the orientation of the horizontal values*], it would flip and have that picture or image [*making hand motion up and to the right*] that you're looking for. But that's, again, just a product of how we've decided to represent things.[4]

PST2 assimilated Fig. 7.2a to a system of meanings based in images of coordinating quantities' values as they varied within an unconventional reference system as evidenced by her explicit attention to quantities' magnitudes and values both in her discussion and gestures. After this interaction, PST2 also sketched a graph like Fig. 7.1a and claimed it to be an alternative representation of $y = 3x$. Furthermore, PST2 explicitly raised issues of convention, suggesting that her previous learning experiences directed her attention to arbitrary representational choices when considering the viability of a novel solution. This enabled her to understand each graph (e.g., Fig. 7.1a and the conventional displayed graph for $y = 3x$) in terms of an equivalent relationship between covarying quantities, with perceptual differences between them resulting from different coordinate system conventions.

PST1's actions, which suggest establishing relations based in perceptual cues and figurative properties of shape, are consistent with what Thompson and I (Moore, 2016; Moore & Thompson, 2015) term *static graphical shape thinking*. PST2's actions, which suggest her establishing relations based in covarying quantities and how they are represented within a coordinate system's conventions, are consistent with what Thompson and I term *emergent graphical shape thinking*. The marked differences between the PSTs' meanings and established relations with their previous learning experiences raise several broader questions. Two questions are:

1. In what ways do students' graphical shape thinking influence their construction of relations of similarity between previous and current learning experiences (i.e., transfer)?
2. Relatedly, in what ways do students' attempted construction of relations of similarity between previous and current learning experiences (i.e., transfer) influence their development of graphical shape thinking?

[4] PST2 subsequently discussed "rise over run" as a convention itself and how the graphs are such that the variation in x relative to variation in y is 1/3 and the variation in y relative to variation in x is 3.

7.2 Theoretical Framing—Transfer, Understanding, and Meaning

The two questions raised in the previous section center on transfer, understanding, and meaning. Defined generally, transfer is "the influence of a learner's prior activities on his or her activity in a novel situation" (Lobato, 2008, p. 169). Educational research, particularly in mathematics education, entails numerous perspectives on transfer. Early researchers characterized transfer in ways that reflected an implicit or explicit assumption of there being objectively correct solutions to mathematical problems. Cox (1997) and Lobato (2006) identified that these early approaches to transfer had roots in associationism and behaviorism that can be traced to Thorndike's (1903, 1906) notion of *identical elements*. More recently, researchers have claimed to problematize the relationships between an external environment and the mind (see Anderson, Reder, & Simon, 2000), but Lobato (2006, 2012) and Wagner (2010) argued that these accounts do not problematize these relationships in practice, instead operating "as if situational structure could be directly perceived in the world" (Wagner, 2010, p. 447).

To be more sensitive to nonnormative reasoning or what an expert might deem "incorrect" reasoning, Lobato (2006, 2012; Lobato & Siebert, 2002) introduced the *actor-oriented transfer* (AOT) perspective. The AOT perspective explores transfer as perceived by the learner. It emphasizes a learner's construction of personal relations of similarity between learning experiences and, accordingly, clarifies that claims about the nonnormative (or normative) performances resulting from transfer are from the perspective of the observer; all activity is viable and normative from the perspective of the learner. The AOT perspective also frames transfer in terms of the construction and reconstruction of knowledge. Whereas traditional perspectives have approached transfer as a static application of knowledge, the AOT perspective approaches transfer in terms of active, subjective constructions of similarity. The AOT perspective thus accounts for forms of transfer that promote learning through cognitive reorganization and accommodation (Hohensee, 2014; Lobato, 2012; Lobato & Siebert, 2002). Reflecting this affordance of the AOT perspective, researchers' adoptions of the AOT perspective have yielded explanations of students' (and teachers') meanings and learning in numerous content areas (Diamond, 2018; Ellis, 2007; Hohensee, 2014; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002). Researchers have also used the AOT perspective to identify how particular artifacts, language, and other factors of classroom instruction can explain differences in students' transfer of learning (Lobato et al., 2012).

Because the AOT perspective seeks to explain transfer from the perspective of the learner, it is productive for a researcher to pair the AOT perspective with a framing of meaning that emphasizes its subjective nature; an appropriate framing of students' meanings provides the ground by which a researcher can situate accounts of transfer. In this chapter, I draw on Thompson and Harel's descriptions of meaning and understanding (see Thompson, Carlson, Byerley, & Hatfield, 2014). The distinction between understanding and meaning is rooted in Piaget's characterization of understanding as assimilation to a scheme and of meaning as the space of implications created by a moment of assimilation (Skemp, 1962, 1971; Thompson, 2013b; Thompson & Saldanha, 2003). Thompson and Harel defined *understanding* to refer to a cognitive, in-the-moment state of equilibrium that results from assimilation. Understanding could occur from having assimilated an experience to a stable scheme, or from a functional accommodation specific to that moment and arrived at by an effortful coordination of existing schemes (Steffe, 1991). For instance, a student could perceive two marks on a piece of paper as orthogonal and assimilate the marks as coordinate axes, thereby establishing a state of equilibrium (i.e., an understanding). If the student also perceives an unfamiliar curve within the assimilated coordinate system, he might engage in effortful activity to understand the unfamiliar curve. The student could attempt to relate the unfamiliar curve with the collection of shapes and associated perceptual properties with which he is already familiar through prior learning experiences. Or, the student could attempt to imagine the curve in terms of an emergent trace of covarying values within the respective coordinate system and relate that to previously experienced covariational relationships. Either could result in a state of understanding via assimilating the curve to a meaning.

Meaning in Thompson and Harel's system refers to the space of implications that a moment of understanding brings forth (Thompson et al., 2014). When a person creates an understanding by assimilating an experience (e.g., a perceived word, phrase, diagram, or set of statements) to a scheme, the scheme is that person's meaning in that moment; the person's meaning in that moment consists of an organization of actions, operations, images, and schemes that the person anticipates or enacts (Piaget & Garcia, 1991; Thompson, 2013b; Thompson et al., 2014). Establishing a state of understanding through assimilation to a meaning can occur in many forms. It can be a nearly subconscious, habitual act (e.g., reciting learned multiplication facts), or it can be an effortful progression of reciprocal acts of accommodation and assimilation (e.g., sustained problem solving).

Returning to the notion of transfer, and as I illustrate in this chapter, transfer can occur within either case of establishing a state of understanding through assimilation to a meaning. A researcher can identify different forms of transfer in order to characterize the influence and interplay of a student's meanings constructed during previous learning experiences and their current actions and learning experience. *Forward transfer* and *backward transfer* have emerged as two forms of transfer useful for characterizing such experiences. Hohensee (2014) introduced forward transfer and backward transfer to differentiate between the influence of a learner's prior conceptions and actions on her activity in a novel situation (i.e., forward transfer) and "the influence…new knowledge has on one's ways of reasoning about related mathematical concepts that one has encountered previously" (p. 136; i.e., backward transfer). Stated in terms of meanings, forward transfer is how previously constructed meanings influence the assimilation of a present experience. Backward transfer is how a novel experience and associated meaning influences the learner's previously constructed meanings.

With the constructs of forward and backward transfer formally introduced, I return to the opening vignettes. Recall that PST1's actions suggest her drawing relations of similarity based on previously learned associations between the perceptual direction of a line and "slope." This is a form of forward transfer. She experienced a novel axes orientation in the form of hypothetical student work, and her prior conceptions of "slope" entailed an axes orientation that required that she attempt to modify the graph so that they were relevant. Recall that PST2's actions suggest her drawing relations of similarity based on coordinating quantities' covariation with attention to coordinate conventions. This is also a form of forward transfer, but there are aspects of her actions that suggest backward transfer. Namely, PST2 called attention to the previously learned mnemonic phrase and calculation of "rise over run" being problematic in the context of the unconventional axes orientation. An explanation for her actions is that experiences with graphing in unconventional axes orientations influenced her meaning for the mnemonic phrase and calculation—one typically taught in Grades 6–12 curricula only in the context of conventional axes orientations—so that she came to understand it as subordinate to the concept of forming a multiplicative comparison. Her experiences with graphing covariational relationships in unconventional axes orientations entailed backward transfer with respect to her meaning for "rise over run" so that it could accommodate unconventional axes orientations, and the phrase no longer was absolutely literal relative to the implied physical movements.

As suggested by this interpretation of PST2's actions, transfer can involve accommodations to previously constructed meanings (Hohensee, 2014; Lobato, 2012). Transfer and accommodation can occur in the context of two (or more) concepts or topics. For example, Hohensee (2014) illustrated backward transfer in terms of how students' learning of quadratic relationships can influence their previously constructed meanings for linear relationships. Or, transfer and accommodation can occur in the context of the same concept experienced across many learning experiences, as illustrated by Lobato and Siebert (2002) and Lobato et al. (2012). Because graphical shape thinking primarily refers to meanings for the same concept (e.g., graphing), and after elaborating on each of the graphical shape thinking constructs, I highlight how forward and backward transfer can potentially relate to the development of students' graphical shape thinking.

7.3 Graphical Shape Thinking

Each form of graphical shape thinking represents an epistemic subject that stabilized across a research program initiated by Thompson (1994a, 1994b) and was then extended by Thompson, and other colleagues, and myself. Collectively, we targeted secondary students', undergraduate students', and teachers' meanings for precalculus and calculus ideas, including graphing (Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Moore, 2014, 2016; Moore, Paoletti, & Musgrave, 2013; Moore & Silverman, 2015; Moore, Silverman, Paoletti, & LaForest, 2014; Paoletti

& Moore, 2017; Saldanha & Thompson, 1998; Thompson, 2013b, 2016; Thompson & Carlson, 2017; Thompson, Hatfield, Yoon, Joshua, & Byerley, 2017; Thompson & Silverman, 2007). An epistemic subject is a characteristic of thinking that has stabilized within a researcher's thinking across the second-order models she has created for particular students' mathematical meanings (Steffe & Norton, 2014; Steffe & Thompson, 2000; Steffe, von Glasersfeld, Richards, & Cobb, 1983; Thompson, 2013a). An epistemic subject is a hypothetical way of thinking that proves increasingly viable through a researcher's use in predicting and explaining students' behaviors; it supports a researcher engaging in forward transfer as a mechanism to organize their experiences with future students. The generality of epistemic subjects empowers researchers and educators to interact with students in more productive and targeted ways (Hackenberg, 2014; Thompson, 2000).

Consistent with the AOT perspective, epistemic subjects are nonjudgmental with respect to what an observer might perceive to be correct mathematics; students' meanings are always considered viable from their point of view. Furthermore, characterizing a student's actions as consistent with an epistemic subject is not a statement about the student's capabilities or other potential meanings they hold and transfer. Students can, and do, hold multiple meanings for a concept, each of which are products of students having reasoned about that concept in particular ways. Any claim above regarding PST1, PST2, or an individual below is not a holistic claim about the individual but rather a claim about that individual's actions in that moment.

7.3.1 Static Graphical Shape Thinking

Static graphical shape thinking characterizes actions that involve conceiving a graph as if it is essentially a malleable piece of wire (*graph-as-wire*). Thompson and I (Moore & Thompson, 2015) chose the term *static* to indicate that a student assimilates a displayed graph so that he predicates his actions on perceptual cues and figurative properties of shape, and imagined transformations are with respect to physically manipulating that shape as if it were a wire (e.g., translating, rotating, or bending). Because static graphical shape thinking entails actions based in perceptual cues and figurative properties of shape, an element of thinking statically is that conceived associations are (in that moment of understanding) indexical properties or learned facts of the *shape qua shape*; the associations require further contextualization to entail logico-mathematical operations (cf., emergent graphical shape thinking in which quantitative operations constitute the meaning). To illustrate, PST1's treatment of "slope" suggests that her graph's defining properties were its straightness and its direction, and her graph's direction was associated with learned facts of slope. Her subsequent actions were to rotate the graph-as-wire, inferring that changing the line's direction changed its slope.

In addition to associations like slope, the facts of shape constituting static graphical shape thinking can be equations, names, or analytic rules. For instance, as reported by Zaslavsky, Sela, and Leron (2002), a student could associate a graph that he understands as a line with the analytic form $f(x) = mx + b$ regardless of coordinate system or axes' scales. Another student could assimilate a displayed graph as "curving up" and associate this with "exponential" and the analytic form $f(x) = a \bullet b^x$. In each case, the students' graphs entail indexical associations between shapes (e.g., "line" or "curve up"), function class terminology (e.g., "linear" or "exponential"), and analytic rules (e.g., $f(x) = mx + b$ and $f(x) = a \bullet b^x$); in the moment of assimilation, names and associated analytic rules are little more than memorized facts associated with various graphs-as-wire.

Because of its basis in perceptual cues and figurative properties of shape, static graphical shape thinking enables a learner to establish relations between learning experiences via foregrounding perceptual and figurative aspects of a graph. To illustrate, I first return to Lobato et al.'s (2012) findings. Recall that they identified how particular artifacts, language, and other factors of classroom instruction can explain differences in students' reasoning and transfer. Specifically, the authors likened some students' reasoning about a graph to reasoning about a "piece of spaghetti" (Lobato et al., 2012, p. 452) with a property of visual steepness. They classified such reasoning as focused on physical objects (as opposed to mathematical objects, as described in the next section). True to the AOT perspective, Lobato et al. (2012) illustrated that such reasoning did support students' transfer but that such transfer processes were for the purpose of describing properties of the "piece of spaghetti" (Lobato et al., 2012, p. 452). These students did count squares or boxes on coordinate grids to characterize slope, but they did so ignoring axes labels or scale. Such actions are a hallmark of static graphical shape thinking. Even in cases in which students do identify and reason about numbers, often to handle a perturbation in a novel task, they do so for the purpose of describing perceptual or figurative properties (i.e., how one moves or visual steepness).

As an additional illustration of static graphical shape thinking and forward transfer separate from slope, consider Excerpt 1 and Excerpt 2, which occurred during clinical interviews used to investigate PSTs' meanings for noncanonical displayed graphs (see Moore & Silverman, 2015; Moore et al., 2014; Paoletti, Stevens, Hobson, Moore, & LaForest, 2018). The interview prompt (Fig. 7.3) depicts hypothetical students claiming a Cartesian graph displays the sine function and its inverse simultaneously. We designed the hypothetical students' claim to reflect the understanding that the displayed graph is $[(x, y) \mid -\pi/2 \le x \le \pi/2, y = \sin(x), x = \arcsin(y)]$.

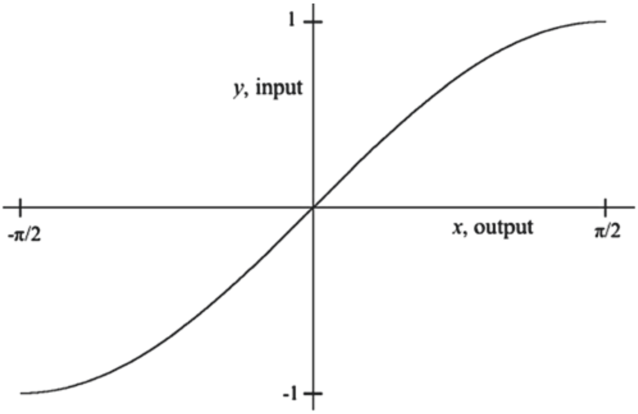
Excerpt 1: Brienne's response to the hypothetical students' statement that Fig. 7.3 represents the inverse sine function.

Brienne: I'm thinking this just kind of looks like the sine graph, like the plain sine graph [*laughs*]. Which is going to be different. So, no…

Excerpt 2: Sansa's response to the hypothetical students' statement that Fig. 7.3 represents the inverse sine function.

Sansa: It looks the same . . . the sine graph . . . I mean he graphed the sine graph . . . um, at pi over 2 sine is 1.

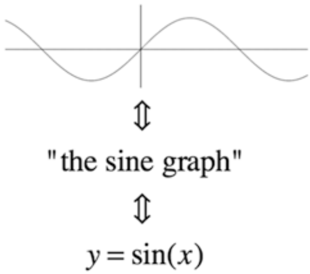
[*The researcher focuses Sansa on the students' statement about and labeling of input and output. She continues by rejecting the student's statement.*]



[The students] claim: "Well, because we are graphing the inverse of the sine function, we just think about x as the output and y as the input." When giving this explanation, they add the following labels to their graph.

Fig. 7.3 Graph and prompt posed to the students

Fig. 7.4 The sine shape qua shape



Sansa: You can't just label it like that. Um, why? Why can't you do that? I don't know. I feel like he's missing the whole concept of a graph . . . Like a sine graph's like a, it's a graph like everyone knows about, you know . . . that's just no. I think they're just missing the concept of graphing [*she continues to reiterate that the student graphed the sine graph and not the arc-sine graph*].

Both Brienne and Sansa's actions indicate their previous learning experiences having resulted in them associating a shape with a name or function (e.g., "sine" or "the sine graph"; Fig. 7.4). For instance, Sansa described her graph as follows: "looks the same . . . the sine graph . . . everyone knows about." I understood her to mean she perceived a learned shape that everyone including mathematics students should recognize as "the sine graph." The students' actions also suggest they had come to associate the name or function uniquely with the recognized shape; the shape could not be given a second name or function, and a different name or function should yield a different graph. This influenced how they perceived the viability of the students' claim, ultimately rejecting the given graphs as potentially representing the inverse sine function. For example, Brienne subsequently added that the "graph of an inverse function" should look different than the graph of the parent function.

7.3.2 *Emergent Graphical Shape Thinking*

Emergent graphical shape thinking characterizes a student's actions that involve conceiving a graph (either perceived or anticipated) simultaneously in terms of what is made (a trace entailing corresponding values) and how it is made (a sustained image of quantities having covaried). Thompson and I (Moore & Thompson, 2015) chose the term *emergent* to indicate that a student assimilates a graph—whether given, recalled, or constructed in the moment—as a trace in progress that is born or derived from images and coordination of covarying quantities. The student conceives the result of this trace to be the emergent correspondence between covarying quantities (Carlson et al., 2002; Frank, 2017; Saldanha & Thompson, 1998; Thompson et al., 2017). I illustrate states consistent with this meaning by showing

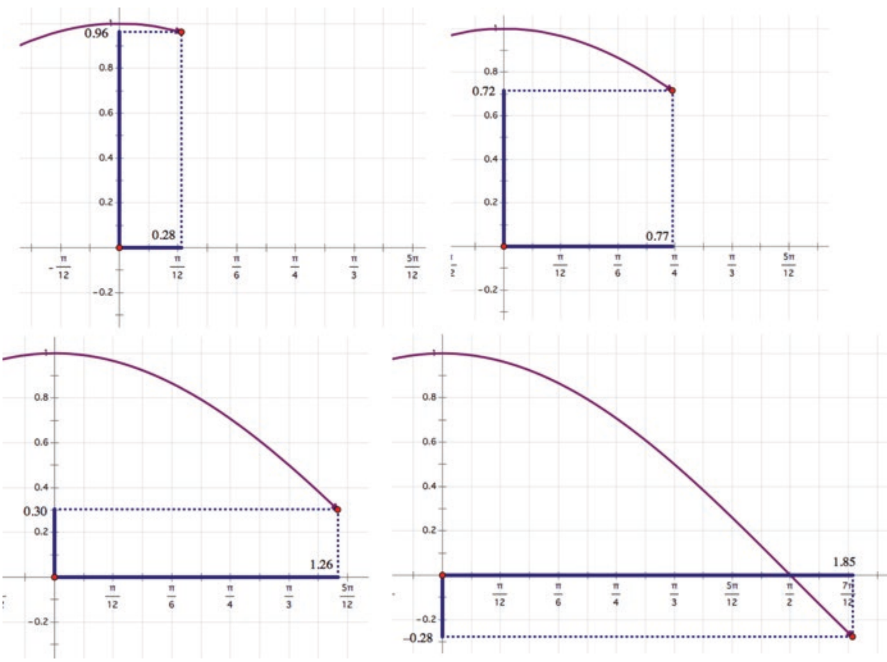


Fig. 7.5 Instantiations of emergent shape thinking

instantiations of an emergent trace of two quantities' magnitudes (Fig. 7.5), but note that static images alone are insufficient to convey emergent shape thinking. Emergent shape thinking is more complex than the displayed instantiations because it entails images of covariation: imagining magnitudes in flow, reasoning about what happens immediately after an instantiation, and reasoning about what happens between instantiations).[5]

An element of thinking emergently is that conceived features or attributes are properties of the covariational and quantitative operations used in assimilation; quantities and their covariation are organic to a student's graph when thinking emergently. Returning to Vignette 2, PST2's treatment of "slope" or rate of change suggests that her graph's defining properties were the covariation that produced it under the constraints of how the quantities were represented within particular axes organizations. Hence, PST2 understood traces in perceptually different orientations as representing equivalent properties of covariation (e.g., no matter the orientation, she conceived a displayed graph such that the rate of change between the two emergent quantities is three).

In addition to mathematical concepts like slope or rate of change, and because of its bases in schemes of covariation, emergent shape thinking associates function class terminology and analytic rules with images of covarying quantities (and the produced correspondence of values). Consistent with PST2, a student thinking emergently could understand a graph as representing a linear relationship of the form $y = mx + b$ via constituting a curve in terms of two values, y and x, covarying at a constant rate with a measure of m (or $1/m$ for changes of x measured relative to changes of y). As another example, a student thinking emergently understands a graph to be "exponential" and of the form $y = a \bullet b^x$ via conceiving a trace such that the rate of change of y with respect to x is proportional to y (Castillo-Garsow, 2010). In the moment of assimilation, images and properties of covariation form the basis of students' associations between their graphs, function class terminology, and analytic rules.

Because of its basis in quantitative and covariational operations, emergent graphical shape thinking enables a learner to establish relations between learning experiences via foregrounding the covariational properties that produce a graph. These properties are consistent with what Lobato et al. (2012) called mathematical objects. As an alternative to foregrounding a graph's "look" as in the case of Sansa and Brienne, consider Shae's focus on covariational properties when responding to the same prompt. Shae was a PST involved in the same series of studies as Sansa and Brienne (Moore, Silverman, et al., 2019; Paoletti, Stevens, Hobson, Moore, & LaForest, 2015). Shae first explained that if x represents angle measure values (in radians) and y represents directed vertical distance measures (in radii), then the sine function denotes x as an input value and y as output value, and the arcsine function

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[5] I direct the reader to other work (Carlson et al., 2002; Castillo-Garsow, Johnson, & Moore, 2013; Confrey & Smith, 1995; Ellis, Özgür, Kulow, Williams, & Amidon, 2015; Johnson, 2012, 2015; Saldanha & Thompson, 1998; Thompson, 1994a; Thompson & Carlson, 2017) for more extensive treatments of the schemes and operations involved in covariational reasoning.

reverses these roles. She further explained that either axis could represent input or output values and therefore understood the graph as being both the sine and arcsine functions. At this point, I was not sure whether Shae conceived her displayed graph covariationally and I presented a canonical Cartesian graph of the inverse sine function (Fig. 7.6). I explained that a second student claimed it to be the graph of the inverse sine function, as opposed to the graph in Fig. 7.3. Shae understood both graphs to represent "the same thing" (Excerpt 3).

Excerpt 3: Shae compares noncanonical and canonical displayed graphs of sine and arcsine.

Shae: Looking at this [Fig. 7.6], I would assume they're meaning sine of negative, sorry, one x equals y [*writing* $\sin^{-1}(x) = y$], where x is their vertical distance and y is their angle measure. So the student, they're both [*pointing at both* Fig. 7.3 *and* Fig. 7.6] representing the same thing just considering their outputs and inputs differently.

Int.: So could you say a little bit more about

Shae: Yeah. So they both kept x the horizontal and y the vertical. But, so here [*referring to* Fig. 7.6] their y's show the angle measure and the x's show the vertical distance. So for the inverse sine their input is vertical distance, output is angle measure. And they're showing the same thing here [*referring to* Fig. 7.3], where their input is the vertical distance, which is their y, and their output is the angle measure, which is their x.

[*Shae uses an input value of 1 to argue that both displayed graphs have the same input and output values relative to her respectively defined input and output axes. The interviewer then asks how she would convince a skeptical student who claims that the graphs look different.*]

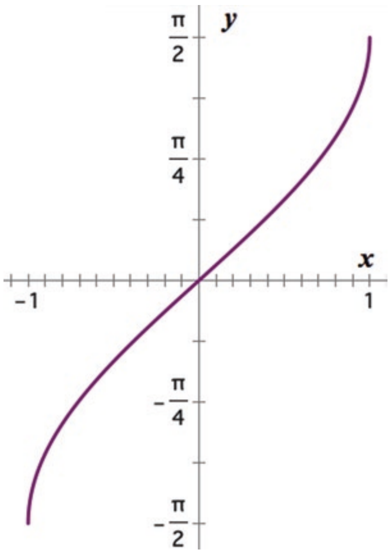

Fig. 7.6 Canonical displayed graph of the arcsine function

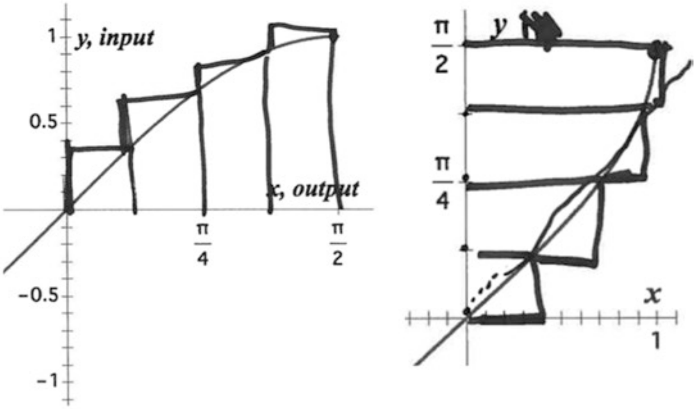

Fig. 7.7 Equivalent conceptions of two displayed graphs

Shae: Oh, you could show the increasing, right. So I mean you could just like disregard the *y* and *x* for a minute, and just look at, like, angle measures. So it's like here [*referring to* Fig. 7.6], with equal changes of angle measures [*denoting equal changes along the vertical axis*] my vertical distance is increasing at a decreasing rate [*tracing curve*]. And then show them here [*referring to* Fig. 7.3] it's doing the exact same thing. With equal changes of angle measures [*denoting equal changes along the horizontal axis*] my vertical distance is increasing at a decreasing rate [*tracing curve*].

Int.: OK.

Shae: So even though the curves, like, this one looks like it's concave up [*referring to* Fig. 7.6 *from* $0 < x < 1$] and this one concave down [*referring to* Fig. 7.3 *from* $0 < x < \pi/2$], it's still showing the same thing. [*Shae denotes equivalent changes on* Fig. 7.3 *and* Fig. 7.6 *as shown in* Fig. 7.7]

Shae's actions indicate her previous learning experiences having resulted in associating a function name with a particular covariational relationship. Furthermore, such a covariational relationship as not constrained to a unique graph or "look," nor was it constrained to a unique function name. Thus, by envisioning each graph to entail some quantity increasing by decreasing amounts as another quantity increases in successive equal amounts, she was able to perceive each graph as mathematically equivalent despite their perceptual differences (e.g., "concave up" versus "concave down"; Fig. 7.7). Mathematical attributes were both properties of Shae's graphs' emergence and the learned function names [$(u, v) \mid -\pi/2 \le u \le \pi/2$, $v = \sin(u)$, $u = \arcsin(v)$], and these learned and reconstructed properties formed the basis for her relating the present task to her previous experiences.

7.4 But What of Development?

Recall that two questions generated by the opening vignettes were:

1. In what ways do students' graphical shape thinking influence their construction of relations of similarity between previous and current learning experiences (i.e., transfer)?
2. Relatedly, in what ways do students' attempted construction of relations of similarity between previous and current learning experiences (i.e., transfer) influence their development of graphical shape thinking?

Regarding the first question, in the case of static graphical shape thinking, indexical associations based on perceptual features form the basis for constructing relations of similarity, supporting students in assimilating those contexts in which figurative aspects of shape prove viable. For instance, when experiencing a novel graph in some coordinate system, students recently completing an instructional sequence emphasizing static graphical shape thinking might anticipate and impose perceptual and figurative features of shape on that novel graph (see Lobato et al., 2012). In the case of emergent graphical shape thinking, the logico-mathematical operations of quantitative and covariational reasoning form the basis for constructing relations of similarity, supporting students in assimilating those contexts in which those covariational and quantitative schemes prove viable. Students recently completing an instructional sequence emphasizing emergent graphical shape thinking might anticipate and impose covariational properties on some novel graph (see Moore et al., 2013).

Whereas the first question is focused on how students' prior learning experiences influence their present experience, the second question opens a focus on explaining the ways in which students' transfer actions can, in turn, result in modifications to those meanings constructed during previous learning experiences. With respect to graphical shape thinking, I contend that sequential processes of forward and backward transfer occasion reciprocal acts of assimilation and accommodation that

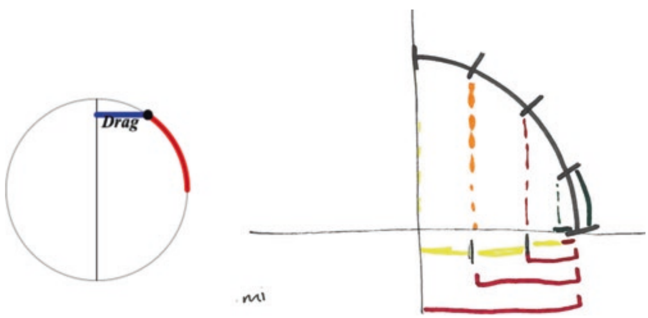

Fig. 7.8 An example of partitioning activity to show horizontal segments decreasing by increasing amounts for successive equal variations in arc (Stevens & Moore, 2017, p. 712)

provide the basis for constructing abstracted meanings rooted in emergent shape thinking.

To illustrate, I draw on a synthesis of a student's actions when prompted to construct a graph representing how two quantities vary together in the context of circular motion (see Stevens & Moore, 2017, for a more detailed account of the student's actions). What follows occurred after a group session in which the student, Lydia, and two other students engaged in partitioning activities (e.g., Fig. 7.8) with a diagram of a circle to identify and reason about variations in horizontal or vertical distance from the vertical or horizontal diameter, respectively, for equal variations in arc length (i.e. the sine and cosine relationships).

After the group session, we engaged Lydia in an individual session. We asked her to return to the circular motion context to gain insights into how her experiences during the group session might have influenced her reasoning. She first constructed variations in horizontal distance for equal changes in arc length. She appropriately concluded that the horizontal distance decreased by an increasing magnitude for an equal change in arc length as the point rotated from the start to the 12 o'clock position (consistent with Fig. 7.8). Her actions and claims were consistent with the group conclusions from the previous session.

We then asked Lydia to create a graph representing this relationship (i.e., the normative Cartesian graph for the cosine relationship), again attempting to gain insights into how the group sessions influenced her thinking as well as how she drew relations of similarity between circle and graphical contexts. Lydia immediately drew a curve that perpetually resembled the normative Cartesian graph for the sine relationship (Fig. 7.9, bottom, with only the axes and curve).

What occurred next was an interaction in which Lydia attempted to engage in compatible physical actions with the circle context and her drawn curve while

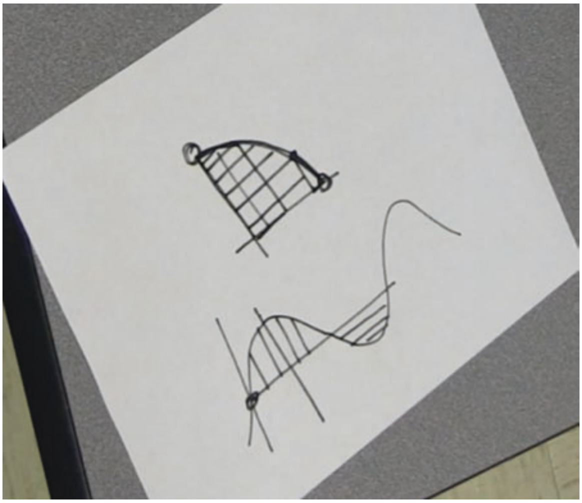

Fig. 7.9 Lydia's drawn graph and circle context (Stevens & Moore, 2017, p. 713)

simultaneously constructing the same covariational properties. Namely, she attempted to:

- Partition along an arc (the circle in the circle context and the curve in the Cartesian context)
- Draw horizontal segments
- Draw vertical segments
- Identify segments that were increasing by decreasing amounts for equal, successive variations in arc length, a property consistent with the sine relationship
- Identify segments that were decreasing by increasing amounts for equal, successive variations in arc length, a property consistent with the cosine relationship
- Draw and identify equal changes along the horizontal Cartesian axis, which was an action done repeatedly in the group session and class in which she was enrolled

Attempting to construct and identify all of these in both the graphical and circle contexts perturbed Lydia. After several different attempts and a sustained period of time, she explained, "I like see the relationship, and I can explain it to a point, and then I get like—I confuse myself with the amount of information I know about a circle that I was just given to me by a teacher, and then what I've like discovered here [*referring to the teaching sessions*]."

Recall that the group session included a focus on both the sine and cosine relationships. An explanation for Lydia's perturbation is that she expected all of the actions and properties of both to be relevant to both the circle and her graph. Thus, in her attempt to relate her current activity to that in the group sessions, she conflated particular figurative and perceptual features of her and her classmates' actions (i.e., static graphical shape thinking) and those quantitative and covariational conclusions those actions and their results indicated (i.e., emergent graphical shape thinking). This left her unable to relate the present experience, the group session outcomes, and her previous instructional experiences to her satisfaction.[6]

I interpreted Lydia's actions to indicate both elements of static graphical shape thinking and emergent shape thinking, and her conflating these elements constrained her ability to relate the group session to her present experience. I thus decided to engage Lydia in another sustained round of interactions so that she could further reflect on her activity and those actions she attempted to transfer from the group session. I also drew her attention to identifying the quantities of the circle context, illustrating several particular values of those quantities in the circle context, identifying how those values related to her graph, and repeating this process (see Fig. 7.10

[6] It is important to note that a traditional transfer perspective would frame Lydia as not transferring her knowledge from the group sessions because of her not successfully completing the problem in ways aligning with researcher intentions. The AOT perspective, however, allows for a much more nuanced and productive account of Lydia's transfer actions because of its sensitivity to transfer from her viewpoint. Lydia was transferring actions from the group session, and far too many to establish a personal state of understanding.

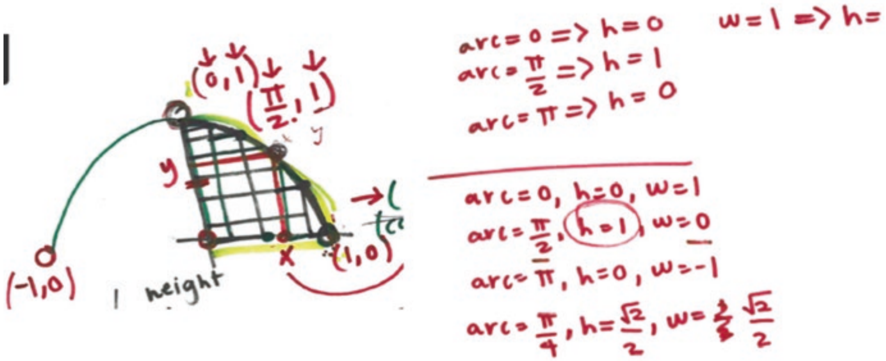

Fig. 7.10 Lydia's annotated diagram of identifying quantities and values (Stevens & Moore, 2017, p. 714)

for her work with the diagram). It was during this process that Lydia had a realization (Excerpt 4).

Excerpt 4: Lydia has a realization (Stevens & Moore, 2017, p. 714, with "{ }" denoting modifications added for clarifying purposes).

Lydia: Because this is my – This is *x* – um, *x-y* plane, then here I'm saying at this point [*the origin*], my width is 0, my arc length is 0, and my height is 0.
Int.: Width is 0, my arc length is 0 and my height is 0.
Lydia: Wait, but then I said {*referring to the situation*} at arc length 0, and [*laughs*] height is 0, then my width should be 1.
Int.: And your width should be 1, right? What about at pi-halves? What should we have?
Lydia: Then I should have a height of 1 [*pointing to curve for an abscissa value of pi/2*].
Int.: Okay.
Lydia: And then my width should be 0 {*focus remaining on her graph*}. So this graph does not do anything with the *x-y* plane.

[*Lydia summarizes this claim and then the researcher asks Lydia to consider an arc length of pi radians.*]

Lydia: Then my arc length on the *x*-axis [*motions across horizontal axis*] should be pi. My height should be 1 – or 0, and then my *x*-value should be negative 1. So this [*referring to her drawn graph*] just doesn't have – then this doesn't relate to the *x*, the width [*referring to width from the situation*], just this graph. So my whole circle talks about width and height and arc, but then this graph itself only talks about arc and height. [*speaking emphatically*] Done it. [*laughs*]

{Lydia then reasons emergently about her graph.}

The beginning of this interaction continues to illustrate the influence of the group session on Lydia's activity. Namely, Lydia continued her attempt to incorporate each of the three relevant quantities and their corresponding contextual segment orientations into her drawn graph. In this case, however, she exhibits a more explicit focus on quantities' values so that figurative actions were subordinate to quantitative operations. In doing so, Lydia had a realization about the outcome from the group session; she came to conceive her drawn graph as an emergent trace of two particular quantities—arc length and height—in a way compatible with the circle context. Notably, Lydia indicated that this was a pivotal moment for her (e.g., "Done it"), and her developing emergent graphical shape thinking as a way to relate a context and graph became a meaning she transferred forward for the remainder of the study (Stevens & Moore, 2017).

Reflecting on Lydia's progression, I underscore that her initial actions in the circle context were stable and such that we interpreted her to have reasoned quantitatively and covariationally. It was in the act of transferring those actions to her recollection of the drawn graph from the group session (i.e., forward transfer) that she was perturbed. It was then through several processes of reconstructing and relating her actions in the present contexts and from the group sessions that she was able to identify and isolate those actions critical to her (and her group's) activity and those that were merely a product of the representational system. More broadly, Lydia's actions highlight the potential affordances of sequential processes of forward and backward transfer in the context of representational activity and graphical shape thinking. Namely, when a student experiences the opportunity to construct and represent a particular relationship in multiple and varied ways across multiple learning experiences, they are afforded the opportunity to identify and differentiate between those (physical and mental) actions associated with emergent graphical shape thinking so that only vestiges of figurative activity remain. Both Thompson (1994b) and Lobato and Bowers (2000) identified that such an opportunity is the underlying foundation to a productive view of multiple representations.

Before closing, I note that when speaking of constructing and representing a relationship in multiple and varied ways, I am referring to a multitude of contexts that permit anticipating and enacting quantitative operations on available figurative material. For instance, event phenomenon and coordinate systems (e.g., a Ferris wheel ride, a bottle filling with water, and the polar coordinate system) permit quantitative operations on figurative material associated with quantities (e.g., a traversed arc length, a segment representing the height of water, and a directed angle measure and radial distance). If event phenomenon, multiple coordinate-system orientations, and multiple coordinate systems are used in tandem, it provides students a plethora of opportunities to differentiate quantitative operations from figurative forms of action (Moore, Stevens, Paoletti, Hobson, & Liang, 2019). In contrast, tables, formulas and written phrases—each a representation—do not entail figurative material that permit quantitative operations. I do not downplay the important use of tables, formulas, and phrases, but rather highlight the difference in their use as compared to that of event phenomenon and graphs as it relates to affording students the opportunity to simultaneously engage in and differentiate between quantitative operations and figurative forms of action.

7.5 Moving Forward

Von Glasersfeld (1982) defined a *concept* as "any structure that has been abstracted from the process of experiential construction as recurrently usable" (p. 194). The term *abstraction* has a long history in mathematics education, and the term accordingly is met with far too many different interpretations and perspectives to describe and synthesize here (e.g., Dubinsky, 1991; Piaget, 2001; Sfard, 1992; Simon et al., 2010; von Glasersfeld, 1991; Wagner, 2010). As a concise and simplistic definition for operational purposes, abstraction is the process of becoming consciously aware of and differentiating between one's actions (physical and mental) that are critical to some conceived concept and those that are not (Moore, Stevens, et al., 2019; Piaget, 2001). As Wagner (2010) explained, abstraction is not a decontextualizing process that results in constructing something devoid of context, but rather, an abstracted concept becomes more sensitive to both the similarities and differences among perceived contextual instantiations of the concept.

Lydia's actions illustrate such a process of abstraction in her differentiating between those actions and operations that are quantitative and covariational in nature and those that are a product of representational conventions and figurative aspects of a context perceived as entailing that relationship (Moore, Stevens, et al., 2019). In doing so, Lydia eventually constructed a meaning for graphing—emergent graphical shape thinking—that consisted of a covariational structure she could describe as if it is independent of the specific figurative material associated with a context. She could also transfer this way of thinking to assimilate novel contexts or situations permitting the operations constituting her way of thinking. It is in this way that her thinking became abstract, that she constructed a concept; she constructed a structure so that its mathematical properties and actions were anticipated independent of any particular instantiation of them, thus not being tied to any particular two quantities and associated context.

Lydia represents only one case, and it remains to be seen how students' learning can be supported through sequential processes of forward and backward transfer in the context of repeated and varied opportunities to construct and represent covariational relationships. Much is left to understand about the initial and ongoing development of graphical shape thinking, especially in the context of students who are experiencing graphing for their first time. The forms of graphical shape thinking do not currently represent developmental stages, nor are the graphical shape thinking constructs as predictive and explanatory as those in areas like units coordination (Steffe & Olive, 2010). To make a claim of developmental stages requires research focused on students' persistence in using them as ways of thinking and evidence that their current schemes impede their thinking at a higher level, and thus research along those lines is a necessary and important next step of research. Furthermore, a current limitation of shape thinking and its forms is that they stem from working primarily with secondary students, undergraduate students, and postgraduate students (i.e., teachers). Detailed insights regarding the initial development of students' meanings for graphs as related to graphical shape thinking are thus needed, especially during students' formative years of constructing displayed graphs. Importantly, there is evidence suggesting that emergent graphical shape thinking is a readily accessible meaning for middle-grades and secondary students (Ellis, 2011; Ellis et al., 2015; Johnson, 2012, 2015).

One productive future line of inquiry will be investigating students' meanings for graphs in the context of some topic, such as students' derivative or rate of change meanings. Researchers taking a topical focus will contribute nuanced descriptions of the schemes and operations that comprise the forms of shape thinking and are specific to those topics (e.g., conceiving a displayed graph as relating multiplicative and additive structures, Ellis et al., 2015). Additionally, researchers that take a topical approach can gain insights into the extent that the forms of shape thinking enable productive transfer as it relates to learning those topics. A complementary line of inquiry to a topical focus will be investigating students' meanings across multiple contexts and topics. Researchers who consider shape thinking and its forms across multiple contexts and topics will have opportunities to make generalizations with respect to students' meanings and transfer.

Another productive future line of inquiry will be characterizing relationships between students' graphical shape thinking, backward transfer, and their learning. At the prospective and practicing teacher level, there is evidence suggesting their meanings not only foreground static graphical shape thinking (Thompson et al., 2017), but that their meanings can conflict with reasoning emergently (Moore, Stevens, et al., 2019). Moore, Stevens, et al. (2019) specifically illustrated that prospective teachers can produce graphs emergently that differ from those produced statically, especially under noncanonical coordinate orientations. In such cases, the prospective teachers experienced a perturbation. Although not the focus of the authors' study, their findings suggest the potential for backward transfer. When perturbed as a consequence of reasoning emergently, the prospective teachers showed evidence of reflecting on and beginning to analyze their previously constructed meanings, which had been consistent with reasoning statically. These initial acts of perturbation and reflection can be the genesis of backward transfer (Hohensee, 2014; Lobato & Siebert, 2002), and future researchers should explore the affordances of these situations in promoting productive backward transfer.

In closing, I make an instructional and curricular comment for both educators and researchers. Lobato et al. (2012) convincingly illustrated how numerous classroom factors can influence students' propensity to construct meanings consistent with emergent or static graphical shape thinking. Complicating the matter, researchers have provided results working with teachers and students that suggest emergent graphical shape thinking is not currently a widely held learning goal in classrooms (Carlson et al., 2002; Thompson, 2013b; Thompson et al., 2017). Thus, it will take concerted and intentional efforts, both inside and outside the classroom, if emergent graphical shape thinking is to become a targeted learning goal of mathematics educators. Specific to curricular materials, I view typical K-16 textbooks and curricula to be nearly devoid of intentional or sustained efforts to engender and support emergent graphical shape thinking. At best, textbooks and curricular narratives sustain a focus on displayed graphs as consisting of coordinate pairs and states of values, which is not equivalent to a focus on covariation, magnitudes, or a displayed graph's emergence and is especially problematic when combined with examples like those above that treat displayed graphs statically (Carlson et al., 2002; Frank & Thompson, 2019; Thompson & Carlson, 2017; Thompson et al., 2017). Based on this observation, I find an important area of work to be the design of curriculum and instructional experiences that target students' emergent shape thinking. More specifically, I perceive a need for instructional activities and interactions in which it is productive for students to differentiate between mathematical properties necessary to all graphs of a relationship and those properties that are a consequence of the conventions of a coordinate system.

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Chapter 8
Using an Actor-Oriented Perspective to Explore an Undergraduate Student's Repeated Reference to a Particular Counting Problem

Elise Lockwood and Zackery Reed

In this chapter, we use the lens of actor-oriented transfer (AOT, Lobato, 2003, 2012, 2014; Lobato, Rhodehamel, & Hohensee, 2012; Lobato & Siebert, 2002) to examine instances in which one student, Carson, referred back to a problem to develop his reasoning about important combinatorial ideas. Carson was a participant in a teaching experiment in which a small group of four undergraduate students solved counting problems and engaged in generalizing activity. In this teaching-experiment study, we had students solve a set of problems, categorize those problems, and eventually use those categories to describe problem types and articulate general formulas. Carson focused on one particular problem that had been meaningful in his initial problem solving (the Horse Race problem, which states: "There are 10 horses in a race. In how many different ways can the horses finish in first, second, and third place?"), and he referred to that problem frequently throughout the sessions and returned to it in several different settings. Given the fact that counting problems can be difficult for students to solve correctly (e.g., Annin & Lai, 2010; Batanero, Navarro-Pelayo, & Godino, 1997; Lockwood & Gibson, 2016), we think it is worthwhile to examine ways in which this student effectively drew upon this particular problem in his combinatorial problem solving.

We not only want to demonstrate that Carson referred to problems, but we also want to uncover what cognitively afforded his forming and leveraging of these connections. In doing so, we seek to answer the following research questions: What cognitive mechanisms facilitated a student's repeated connections to a particular counting problem? In what ways did these student-generated connections affect the

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student's work in subsequent combinatorial situations? By carefully examining ways in which Carson thought about and used this particular problem, and by studying how he compared and contrasted it with other problems and situations, we can gain insight into how students might establish and use prototypical problems. Such insights can inform ways in which students may make connections or differentiate between situations and problems they encounter. In addition, we adopt an AOT perspective, and we highlight affordances that such a perspective affords us.

8.1 Literature Review

8.1.1 Connections to Problems and Problem Types in Combinatorics

Some previous studies have examined students' connections between particular problems and problem types. Maher, Powell, and Uptegrove (2011) offered examples in which students drew on particular problems repeatedly over time, sometimes referring to problems by name. For example, in describing students' work on a problem involving counting pizzas with certain combinations of toppings, Muter and Uptegrove (2011) reported that one student said, "Everything we ever do is like the tower problem" (p. 107). These authors described instances in which students made connections between representations as a way to reason about why problems might be similar. For example, Muter and Uptegrove reported that students could identify relationships between towers made from blue and red cubes, pizzas with certain toppings, and binary sequences. Tarlow (2011) also reported on connections students made among problems involving pizzas and towers and settings involving binomial coefficients. In these studies, these pizza or towers problems represented specific problems, but they came to be emblematic of types of problems that were isomorphic to other problem settings. The contexts of pizzas and towers were typically leveraged to help students make connections between binomial coefficients and Pascal's triangle, ultimately to make sense of binomial identities. Such work demonstrates that a particular problem (whether it is a specific problem or a problem type and whether the problem implicitly or explicitly represents a broader type) is something to which students can refer back. Here, the researchers did not attempt to account for what cognitive mechanisms were facilitating students to make these kinds of connections. We build on this literature base by providing a specific example of how a student used a particular problem in several subsequent counting situations and by offering insights into the cognitive mechanisms that contributed to this transfer. Our example also emerges in a different context than Maher et al.'s (2011) work, as their connections occurred over the course of a longitudinal study where students had repeated, sustained exposure to combinatorial concepts, spanning many years. The connections we describe in this chapter occurred with an undergraduate student over the course of a concentrated but relatively short period of time.

We also situate this work within Lockwood's (2011) investigation of student-generated connections in combinatorics. Lockwood used AOT (discussed in detail in the Theoretical Perspectives section) to examine students' combinatorial thinking and activity. She found instances in which students made unexpected connections back to previous counting situations (the connections were unexpected in the sense that students connected problems that would not traditionally be considered to be isomorphic). Using the lens of AOT, Lockwood categorized student-generated connections as being *elaborated* versus *unelaborated* (describing the extent to which a student expounded upon the connection they made) or *conventional* versus *unconventional* (describing the extent to which a connection aligned with conventions or expectations of the broader mathematical community). She also characterized referent types (the types of objects to which students refer when making connections) as involving *particular problems*, *problem types*, or *techniques/strategies*. A reference to a particular problem means that a student is connecting to a single instance of a particular problem, whereas a problem type indicates a broader class of problem (which may be referred to by a specific name, such as "hot dog problems"). When students refer back to techniques/strategies, they may refer to a particular approach that they are familiar with that is applicable in the current situation. Lockwood (2011) illustrated cases of elaborated, conventional and elaborated, unconventional instances of AOT, both of which involved students' referring to particular problems. These cases suggested ways in which the AOT perspective could be used in a combinatorial setting.

In this chapter, we build on Lockwood's (2011) characterization in a couple of ways. By incorporating a Piagetian cognitive perspective (discussed in the Mechanisms for Student Reasoning section), we gain some insight into cognitive mechanisms that may underlie such student-generated connections. We suggest that, in the case of our student, we can offer a nuanced analysis into the referent type he identified that offers more detail than Lockwood's categories. Thus, we broadly use the categorization of AOT presented in Lockwood (2011), but we also modify and adapt some of the categories to align with our cognitive perspective toward the data. Indeed, we view this cognitive perspective as a way to account for the mechanisms by which students' prior learning experiences may influence their reasoning on novel problems.

8.1.2 Mathematical Discussion

We briefly provide a mathematical discussion of the Horse Race problem to contextualize further discussions. The Horse Race problem states: "There are 10 horses in a race. In how many different ways can the horses finish in first, second, and third place?" We emphasize two ways to solve this problem, each of which highlights an important combinatorial principle (we note that there are additional ways to solve the problem that we do not discuss here). First, one solution is to use the multiplication principle[1] and recognize that we have 10 options for which horse finishes first, and then, for any of those options, we have 9 options for which horse finishes second, and we have 8 options for which horse finishes third. Using the multiplication principle yields 10·9·8 = 720 possibilities. This is a correct, common answer (and is in fact how the students in our study solved the problem initially). We can observe that this answer, which counts the number of 3-permutations from a 10-element set, is also commonly expressed as $\frac{10!}{7!}$ (more generally, $\frac{n!}{(n-r)!}$). We note that, on the one hand, this expression is simply an efficient way to write that product because the 7! in the denominator cancels out 7! in the numerator, leaving only 10·9·8, which is the product we desire. However, there is also a way to view that product $\frac{10!}{7!}$ in terms of equivalence. Specifically, we can argue that for any of the 10! arrangements of the 10 horses, for a given way the first three positions finish, there will be 7! arrangements of the last 7 positions (for the horses who did not finish in the top 3). However, each of those 7! arrangements should only contribute to one outcome that we care about because we only want to count unique ways the first three elements can be arranged. For example, if the horses are labeled A, B, C, D, E, F, G, H, I, and J, and we say A, B, and C finished first (in that order), we would get 7! permutations of the letters D through J. We contend that this is a useful way to reason about this expression because it emphasizes that we are dividing by sizes of equivalence classes. This is a way of thinking about the problem that we think is valuable for students, both because it orients students to think about sets of outcomes (Lockwood, 2013, 2014) and because such thinking arises in a variety of combinatorial settings (see Lockwood & Reed, 2020, for additional insight into an equivalence way of thinking in combinatorics). Indeed, if well understood, this way of thinking can be a valuable resource in approaching counting problems.[2] This problem has the potential to reinforce equivalence as a useful way of thinking about counting, and it provides a combinatorial (and not simply numerical) justification of the formula. In the teaching experiment, then, we asked students to consider this alternative perspective on their answer to the problem.

[1] Broadly, the multiplication principle (sometimes referred to as the Fundamental Principle of Counting; e.g., Richmond & Richmond, 2009) is the idea that if a problem can be broken down into successive stages, and if the number of options at each stage is independent of the choice of options in any previous stages, then we can multiply the number of options at each stage to find the number of outcomes of the problem. We prefer Tucker's (2002) statement of the multiplication principle. See Lockwood, Reed, and Caughman (2017) and Lockwood and Purdy (2019a, 2019b) for additional discussions of the multiplication principle.

[2] We highlight the importance of this equivalence way of thinking in combinatorics, but reasoning about equivalence is a vital aspect of mathematical reasoning that has widespread applications in a variety of domains, such as abstract algebra and in reasoning about equivalence classes more broadly. Thus, there is perhaps an additional motivation to foster reasoning about equivalence that extends beyond just the combinatorial context.

8.2 Theoretical Perspectives

8.2.1 Actor-Oriented Transfer

In this chapter, we adopt an actor-oriented view of transfer. This perspective, which we outline in this section, is manifested mainly through our methodology and our attention to student-generated connections (and, in particular, one student's repeated references to the particular Horse Race problem). Lobato (2003) introduced the term AOT, which she described as a shift from "an *observer's* (expert's) viewpoint to an *actor's* (learner's) viewpoint by seeking to understand the processes by which individuals generate their own similarities between problems" (p. 18, emphasis in original). Lobato and Siebert (2002) described AOT as "the personal construction of relations of similarity between activities, or how 'actors' see situations as similar" (p. 89). Lobato (2012) distinguished between a more traditional view of transfer and an actor-oriented perspective on transfer:

> From a mainstream cognitive perspective, transfer is characterized as "how knowledge acquired from one task or situation can be applied to a different one" (Nokes, 2009, p. 2). From the AOT perspective, transfer is defined as the generalization of learning, which also can be understood as the influence of a learner's prior activities on her activity in novel situations (Lobato, 2008, p. 233).

Lobato (2012) went on to explore five dimensions across which the mainstream cognitive perspective and the actor-oriented perspective of transfer differ: "(a) the nature of knowing and representing, (b) point of view, (c) what transfers, (d) methods, and (e) goals" (p. 234). We do not detail each of these dimensions, but we highlight that there are different kinds of evidence one looks for when studying these respective views of transfer. In the traditional transfer perspective, evidence of transfer is illuminated through "paired tasks that are similar from the researcher's point of view" (Lobato & Siebert, 2002, p. 89). In AOT, however, evidence of transfer is revealed "by scrutinizing a given activity for any indication of influence from previous activities and by examining how people construe situations as similar" (Lobato & Siebert, 2002, p. 89). Indeed, methods for uncovering instances of AOT consist of closely examining students as they work and seeing what connections students make to previous situations.

It is important for us to emphasize that, in this chapter, the AOT perspective is evident in our focus on connections that *students* (and not experts) initiated among situations. That is, rather than seeing if students could apply prior knowledge, we looked to examine ways in which students (and in our case, a particular student) made connections among problems. This view is reflected in our methodology, as we are qualitatively examining a student's language and activity to better understand the connections they are making. We want to clarify that sometimes these connections are expected (or, as Lockwood, 2011, would say, "conventional") in the sense that they align with connections the mathematical community might approve. So, some of the episodes described in this chapter still represent examples of AOT even though the student was making conventional connections. Further, our use of

AOT allows us to focus on particular aspects of Carson's work that traditional transfer might not prioritize or acknowledge. Specifically, we can focus less on Carson's performance only (and whether or not he correctly solved a novel problem), but rather, we can examine his explanations, ways of reasoning, and personal relations of similarities. We view this as an affordance of the AOT perspective that gives us richer insights into Carson's thinking and activity.

Finally, we also point out that the AOT perspective views transfer as involving both psychological and social aspects, whereas a traditional transfer perspective focuses on a purely cognitive perspective (e.g., Lobato, 2012, 2014). Our study involves interaction between a small group of four students, and this design allows us to examine social (and not just psychological) factors that might occasion transfer. This is another way in which our use of an AOT perspective affects our design and data analysis, and, as we share our results, we will point to times in which social engagement seemed to contribute to instances of transfer.

8.2.2 Mechanisms for Student Reasoning

We make use of constructs from Piaget's genetic epistemology (Piaget, 1971; von Glasersfeld, 1995) to support our use of AOT in analyzing Carson's work. We identify aspects of Carson's learning and understanding of the Horse Race problem that influenced his effective leveraging of the problem to solve subsequent counting problems. This radical constructivist approach considers knowing and learning to be inextricably linked to mental activity in the form of applying cognitive structures called *schemes* (Piaget, 1971; Thompson, Carlson, Byerley, & Hatfield, 2014). Succinctly put, schemes are "organizations of mental activity that express themselves in behavior, from which we, as observers, discern meanings and ways of thinking" (Thompson et al., 2014, p. 10). Hypothesizing aspects of a thinker's mental structures by analyzing their utterances and observed mathematical activity allows us to discuss how students develop and make changes to their mathematical knowledge over time as they engage in specific tasks. We define knowing as the "conferring of meaning" to an object or concept in reference to previously constructed schemes (Jonckheere, Mandelbrot, & Piaget, 1958, p. 59), and we use the phrase "assimilate to a scheme" (Thompson, 2013, p. 60) to describe this. Assimilation is accompanied by the interrelated process of accommodation, which is the mechanism through which learning occurs as a thinker alters her scheme due to unexpected aspects of an experiential reality inconsistent with structures to which she assimilates (von Glasersfeld, 1995, p. 66). In this way, knowing and learning rely on the assimilation and accommodation of mental activity being organized into flexible and malleable cognitive structures.

Subsumed within the mechanism of accommodation is the construct of abstraction, which accounts for certain changes to a thinker's schemes characterized by the borrowing and repurposing of operations to higher levels of cognitive activity (Piaget, 1977, 2001). This specifically occurs through "projection (as if by a reflector) onto a higher level of what has been drawn from the lower level" and "'reflection' involving a mental act of reconstruction and of reorganization on this higher level of that which has been thus transferred from the lower one," a process we specifically identify as *reflective abstraction* (Piaget, 1977, p. 303).

Our aim is to leverage these constructs from Piaget's genetic epistemology to complement our use of the AOT perspective with a fine-grained analysis of the cognitive elements informing Carson's engagements in transfer. We aim to unpack relevant aspects of Carson's developing schemes for counting specific combinatorial scenarios, and we seek to demonstrate coherence between Carson's assimilatory mechanism and instances of transfer as being unified in the carrying out of specific mental acts. Specifically, we will demonstrate that many of Carson's engagements in transfer were occasioned by his repeated assimilation to a counting scheme that coordinated the structure of outcomes as being inherently similar to his coordinations of the Horse Race outcomes. We will also demonstrate the abstractions through which Carson transformed his knowledge structures to facilitate such assimilations.

A key aspect of such analysis is attention to Carson's conveyed mental acts (we will use the term *operations*) and the combinatorial objects on which Carson envisioned carrying out such acts. We feel that this engagement with the cognitive subtleties of Carson's instances of transfer provide a consistent narrative supporting the utility of the AOT perspective in mathematics education research, which we will demonstrate and discuss below.

8.3 Methods

8.3.1 Data Collection

The data presented in this chapter are part of a broader study in which we were examining the role of generalization in the context of combinatorics. We have reported on some details of the study and on Carson's initial work on the Horse Race problem in particular, elsewhere (Lockwood & Reed, 2018), but we elaborate additional data and adopt different perspectives than we reported previously. For the broader study, we conducted a small-group teaching experiment that consisted of nine 90-minute sessions with four undergraduate students, during which the students solved a variety of counting problems. We recruited the four students from vector calculus courses based on selection interviews and scheduling availability. We sought novice counters who were willing to engage with the material and communicate their ideas. By novice counters, we mean students who were not familiar with basic counting formulas and who would not simply try to recall such formulas during the study; through selection interviews we determined that this was the case for each of the students. The students sat at a table facing each other and wrote their individual work on sheets of paper. We videotaped and audiotaped each session.

A teaching experiment (Steffe & Thompson, 2000) includes teaching episodes that consist of several main elements: “a teaching agent, one or more students, a witness of the teaching episodes, and a method of recording what transpires during the episode” (p. 273). Steffe and Thompson (2000) say that a main purpose for implementing this teaching experiment methodology is to allow for “researchers to experience, firsthand, students’ mathematical learning and reasoning” (p. 267). In teaching experiments, researchers can examine students’ reasoning over time and see how they think about and learn particular content or mathematical ideas. We have reported about the data collection elsewhere (Lockwood & Reed, 2018; Reed & Lockwood, 2021), and we do not include too many details here because we are most interested in one of the student’s work. Broadly, throughout the teaching experiment, the students engaged in a number of different kinds of activities. In the first three sessions, they solved a variety of counting problems and then categorized and characterized those problems to articulate general formulas for some key problem types, including permutations and combinations (see Reed & Lockwood, 2021, for more details of their categorization). In subsequent sessions they solved more counting problems and worked on proofs of binomial identities. The Horse Race problem, described above, is the main problem we emphasize as we analyze Carson’s connections to other problems. We also describe several other problems Carson solved, but we will outline details of those problems in the results.

We focus on the work of one particular student over the course of several sessions of a teaching experiment. Carson was a student enrolled in vector calculus. His selection interview revealed that he was an insightful and resourceful problem solver who could reason through problems he had not seen before. During the teaching experiment, he was a strong student who often provided insights for the group and could explain and justify his reasoning. We focus on Carson in this chapter because as we reviewed the teaching experiment data as a whole, we realized that he returned several times to the Horse Race problem, connecting back to it in multiple ways. We were interested in examining this phenomenon more closely, and so we sought to trace Carson’s references to (and uses of) the Horse Race problem throughout the experiment. Again, we note that we examined Carson’s use of the Horse Race problem from an AOT perspective, and doing so affected where we looked for evidence of transfer. We looked not at his performance on subsequent, similar (to us) tasks, but rather, we examined Carson’s own explanations, ways of reasoning, and personal relations of similarity, as seen in his language and activity throughout the rest of the experiment.

8.3.2 Data Analysis

The interviews were all transcribed, and we made enhanced transcripts in which we inserted relevant images and descriptions into the transcript. For the data analysis for this paper, we first searched the transcripts for any mention of the Horse Race problem. This involved searching for words such as *horse*, *race*, *racing*, *ranking*, *podium*, etc. This yielded episodes in Sessions 1, 2, 3, 7, and 9 (out of 9 total sessions). We then reviewed the transcripts of those episodes and identified instances in which Carson referred to the Horse Race problem (either by name or by making some explicit connection to racing, rankings, or podiums, which we infer as being inherently connected to that problem). Then, for each episode, we analyzed in what broad context Carson referred to the problem, looking from Carson's point of view at what explanations, justifications, and connections he himself made. In doing this, we considered Lockwood's (2011) framework for characterizing AOT, and we identified the kinds of connections Carson made in each case: elaborated or unelaborated, conventional or unconventional, and referent type. Then, we described specifically how Carson used the problem and the nature of the connection he made with the given context. Lobato (2006) says the following about data analysis:

> In contrast, evidence for transfer from an actor-oriented perspective is found by scrutinizing a given activity for any indication of influence from previous activities and by examining how people appear to construe situations as similar using ethnographic methods, rather than relying upon statistical measures based on improved performance (p. 436).

Following Lobato (2006), we scrutinized what Carson was doing in these episodes and what connections he seemed to make. These methods of analysis were fundamentally informed by our use of an AOT perspective, and we used different methods for observing instances of transfer than we would have had we adopted a more traditional perspective.

8.4 Results

In this section, we explore some of the variety of ways in which Carson leveraged the Horse Race problem over the course of the teaching experiment. We first present Carson's thinking and activity on relevant tasks prior to his work on the Horse Race problem to establish some of his initial ways of operating. Then, we describe Carson's initial work on the problem, and we feel this is particularly noteworthy because it gives potential reasons for why the problem became so important for him. Then, we discuss several episodes in which he referred back to the Horse Race problem, and we demonstrate several ways in which he used the Horse Race problem that suggest transfer (and AOT in particular). Throughout this section, we attempt to make connections with Carson's thinking and frame our analysis of his work in terms of constructs described in the section on mechanisms for student reasoning, and we also highlight ways that the AOT perspective allows for extensions of such analysis. We also note that although we ultimately want to focus on Carson's work, we occasionally include some of the other students' comments because these at times help to contextualize Carson's comments and activity.

8.4.1 Carson's Work Prior to the Horse Race Problem

Carson first demonstrated proficiency in working with multiplicative structures during the selection interview, where he successfully solved five problems. Carson's first engagement in transfer involved identifying a structural similarity between the following two counting problems: "How many ways are there to arrange five numbers in a row?" (the answer is 5!) and "How many ways are there to flip a coin, draw a card from a standard 52-card deck, and roll a 6-sided die?" (the answer is 2·52·6). He justified this claim of similarity by appealing to the construction of independent events in both processes that would imply a solution via successive multiplication.[3] Carson described the independent events in arranging numbers, which demonstrates aspects of his developing scheme:

Carson: Right, so like the 1 has 5 possible outcomes. And where the 1 ends up, it does kind of decide where the 2 can end up, because it takes away one of the 5 possible outcomes for the 2. Which means there's going to be 4 possible outcomes, but which one of those 4 it ends up on is kind of independent of where the 1 ends up right? So it's not gonna, where 1 ends up isn't going to change the number of outcomes for 2, so they are independent events. I guess in reality they are not independent but for the math they are.

We interpret that he was coordinating a new scheme involving multiplication and independent events in a sequence that he could (and would) apply in subsequent counting sessions. Specifically, the scheme that Carson developed through his initial engagement with these counting problems involved coordinating the construction of an outcome through a sequence of independent choices, resulting in a sequence of products that enumerated the outcome set. We refer to this perceived outcome structure generally as a *multiplicative structure*[4] involving the coordination of combinatorial objects that can be enumerated multiplicatively. We refer to the outcome of such a multiplicative structure as a multiplicative outcome.

Carson assimilated to this multiplicative structure while solving the first problem given in Session 1 of the teaching experiment, involving arrangements. This first problem, the Line problem, asks "How many ways are there to arrange 5 children in a line for recess?" He described arranging five kids in a line by having five choices for the position of the first kid, then four choices for the next kid, and so on, resulting in a multiplicative outcome (Fig. 8.1). Carson's language suggests implementation of this same multiplicative structure from his selection interview. Specifically, this was the first instance in the teaching experiment of him assimilating to his new (and developing) scheme for coordinating multiplicative combinatorial objects.

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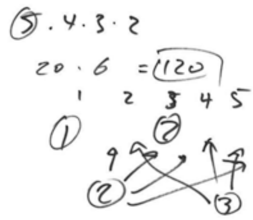
[3] Although it is possible that Carson was recalling some basic combinatorial structures from high school algebra, Carson's problem solving in each task throughout the selection interview appeared to be novel. We thus infer that Carson was constructing new schemes through each activity in the selection interview.

[4] Our use of the word *structure* rather than *scheme* is to convey the instability of his understanding in his initial work, which, through repeated use, would eventually stabilize to a scheme.

Fig. 8.1 Carson's work on arranging five people in a line



Another illuminating assimilation occurred when he solved the MATH problem ("In how many ways can we rearrange the letters in the word MATH?"), also during Session 1 of the teaching experiment.

Carson: Well, each letter is a kid in line.
Interviewer: Okay, say more about that.
Carson: So, it's the same problem. It's what we just did.
Aaron: You mean 4! in this case.
Interviewer: Okay, and what makes you make that association?
Carson: Well, if you have 4 kids and their names are Matt, Alice, Theo and Hanson, then you number them 1, 2, 3, and 4.

In this instance, not only did Carson assimilate to the same multiplicative structure, but he justified his perceived similarity by making direct identification of the outcomes of the MATH problem to kids in a line from the previous Line problem. In particular, Carson spontaneously suggested that he could associate the objects being arranged in the MATH problem (M, A, T, and H) with objects arranged in the Line problem (the names of specific kids). His mental activity in this instance can be modeled as constructing an isomorphism between the outcomes. Carson explicitly identified the letters as individual children, naming the children "Matt," "Alice," etc. This naming served the purpose of making a correspondence as if defining a bijective function from the set of 4 kids to the set of first letters in their names. We use the term isomorphism to convey that his identification communicated a perceived combinatorial structure on top of his explicit bijection. We argue that this itself is an instance of transfer, where the referent (what is being transferred) is the set of outcomes, where Carson was formulating a connection from one set of outcomes (arrangements of M, A, T, and H) to another set of outcomes (arrangements of kids). Here, Carson's transfer served as a means of explicating what he perceived to be a similar structure between the two situations, thus implying a course of action that would generate a solution to this new problem. Thus, Carson's engagement in transfer in this way was occasioned both by his cognitive associations and, we would argue, also by the social component of his interactions with the interviewer and other participants.

Carson additionally described a process of arranging the M, then holding it constant and arranging the A, and so on, again conveying the sequence of "independent events" that he described when arranging 5 numbers (from the selection interview).

Carson: So, we locked in the first letter, so we have 4 options what the first letter can be and then for each option for the first letter we have 3 options for what the second letter can be, so that's 3 sets there. And then, for each second letter there's gonna be 2 more options, that means we have 3 sets of 2. And then, for that third letter if you were to group those together there's–oh, I'm sorry once the first 2 letters are constant there's only one more option.

Carson's comments were made in the context of discussing why 4·3·2·1 made sense as a solution and involved him organizing a list of outcomes, thus demonstrating that he reasoned according to the multiplicative structure in this new MATH problem. This supports that his developing scheme involved completing an independent sequence of events in a multiplicative manner.

Thus, in summarizing Carson's work prior to the Horse Race problem, we note that he paid attention to the multiplicative enumerations of counting problems, which was in line with envisioning a sequential counting process broken down into independent events. Beginning with n-permutations, Carson's assimilatory mechanism involved recognizing individual outcomes as being the results of ordered sequences of events. Further, we have an initial instance of Carson leveraging an isomorphism construction as a means of explicating a perceived multiplicative structure, a component of his engagements in transfer throughout the teaching experiment. So, in approaching the Horse Race problem, Carson (a) had experience with reasoning about multiplication in counting as involving independent events and (b) could construct isomorphic relationships between the set of outcomes in one counting context with another. This initial analysis of Carson's thinking provides the context in which future abstractions shaped Carson's engagements in transfer, and it also demonstrates both the cognitive and social aspects of Carson's activity (which is an affordance of the AOT perspective). We now share Carson's initial experience with solving the Horse Race problem, which will set the stage for subsequent discussions of his work.

8.4.2 Carson's Initial Experience with the Horse Race Problem

We discuss the episode of Carson's initial work on the Horse Race problem in some detail because we think it is important to see how this problem developed for him. In particular, we view this as an instance of abstraction through which Carson constructed a new scheme that incorporated an equivalence way of thinking, which is something he would later leverage as he constructed additional isomorphisms when engaging in transfer. This account suggests that Carson's solving of the Horse Race problem was formative, and that the resulting abstraction provided him a means of productively engaging in transfer in future episodes.

We gave this problem to the students as part of an initial set of 14 problems in the first two sessions of the teaching experiment. In Session 1, the interviewer posed the problem for the students and had them think about it individually.

Interviewer: Great, love it. Now, I'll have you work on this one, maybe a little think time first. So, I've got 10 horses in a race, how many different ways can the horses finish first, second and third place?

Interviewer: Okay, so tell me what did you guys get.

Anne-Marie: So, I did it the same way as before, I know we have 3 places and then first place you have 10 horses that can be in first place, and so if you look at second place you only 9 now, because 1 horse already has first place. So, you have 9 horses left to choose from for that second place, and once you choose one for second place now you only have 8 left to put in third place. So, you have 10 times 9 times 8 to get your total amount.

Interviewer: Okay, great. And, you said something.

Aaron: Yeah, the way that she did it is right. I thought that there would be 10 different unique, like the same horse could – how do I explain this, 10 horses could be in the same place, but it's 10 times 9 times 8, because you are already using up one of the combinations on previous.

Interviewer: Great, like, yeah – go ahead.

Carson: If you think about it in the real world, like anything can happen mid race, like a meteor could come and hit the horses and then the different one wins than you thought was going to. But, after the first one finishes there's still 9 horses on the race track, so there's 9 different ways that the next horse could cross the line and then after that second-place horse has finished, there's still 8 horses on the race track, so any of those 8 horses could take third. So, 10 times 9 times 8 (Fig. 8.2).

Initially, then, all three of the students seemed to have used the same approach, and they arrived at the correct product of 10·9·8, arguing that the number of options decreases to 9 after the first horse finishes, and then to 8 after the second horse finishes. The students had all arrived at the same answer and seemed confident. We note that Carson said that anything could happen in the race to change the order of the racing horses (including a meteor strike), but as soon as that first horse crossed the finish there were then 9 horses left that could be arranged. This highlights

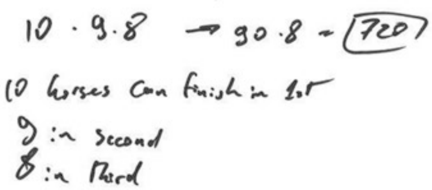


Fig. 8.2 Carson's initial written work on the Horse Race problem

Carson's assimilation to his multiplicative scheme involving the imagining of a sequence of independent events that would be enumerated multiplicatively, which he had established earlier in the interviews.

We also wanted to see if and how the students could reason about the solution to the problem $\frac{10!}{7!}$ not just numerically but combinatorially (as discussed above in the Mathematical Discussion section). In light of this, the interviewer asked the students how their initial answer of 10·9·8 would relate to a solution of $\frac{10!}{7!}$. One of the students, Aaron, noticed that it would be the same number because cancellation of terms in 10! and 7! would occur, and Carson agreed. Thus, the students had seen that they could numerically simplify the expressions to yield the same value, but again, the interviewer wanted to see if they could make sense of this quotient combinatorially and in terms of equivalence. The interviewer asked the students to explain why $\frac{10!}{7!}$ might make sense "aside from the fact that it's numerically equivalent to 10 times 9 times 8." The students worked for several minutes, and in the excerpt below, we see that Aaron noted that he could not explain the quotient combinatorially. Ultimately, Carson was the only student who seemed able to make sense of the situation, and he explained his thinking below.

Interviewer: What are you thinking about so far, it's a hard question?

Aaron: Yeah, I can't see it right now.

Carson: So, the way I'm thinking about it, is that <u>we know kind of the method to get the number of ways that 10 horses can finish a race, and that's 10!, and that something we did, like kids in line. So, it's the same problem there</u>. So, there's 10! total outcomes, and then we know for any given first 3 there's going to be 7!, *because that's saying we know the first 3 horses have finished, how can the last 7 horses finish*, so that's going to be 7!. But all we care about is how many given first 3 s there are. So, if we divide the total number of outcomes by the number of potential of outcomes for the last 7 horses that will give us the potential number of outcomes for the first 3. If that makes sense?

We suggest that the underlined portion represents an instance of transfer, with Carson referring back to a particular problem (the Line problem) and relating it to arranging 10 horses in a race. He then built from that to make a case about why division by 7! might make sense. Carson's explanation of the solution was that the 10! represented arranging all 10 horses, and that for any given first 3 there were 7! ways the remaining 7 horses can finish. Then, division by those 7! ways of arranging the last 7 horses gave the number of outcomes for how the first 3 could finish.

We argue that this discussion contributed to Carson's deeper understanding of this phenomenon and that justifying this idea provided a context for Carson's understanding of permutations in which he could ground and develop important ideas. In particular, the prompt to explain the $\frac{10!}{7!}$ expression combinatorially facilitated a significant moment of accommodation for him. He had to abstract key operational aspects of his previous scheme to accommodate this new mathematical phenomenon. We contend that Carson envisioned carrying out a specified sequence of events (counting ways that 7 horses could finish a race) within the context of a larger counting process (counting ways that 10 horses could finish a race), and he coordinated the multiplicative structures of the specific desired sequence and the more general counting process. Notice that Carson first described arranging 10 horses, and then he described arranging the total 7 horses after the first three finished. This is similar to his sequencing of individual events, but he was now conceiving of these as together comprising a broader two-stage process. We infer that rather than counting the number of potential outcomes of the next event, he imagined a new independent counting process to occur after the first three horses finished (the italicized portion of the above quotation). As we will further discuss below, Carson's accommodation entailed a reflective abstraction of key operations from his multiplicative scheme to construct a new scheme that coordinated new operations accounting for equivalent outcomes.

The interviewer recognized that Carson's answer was correct and saw that Carson seemed to understand the problem, but she wanted to emphasize the point for Aaron and Anne-Marie, neither of whom seemed totally convinced. They then proceeded to have a bit more discussion; in this case, the interviewer sought to help the students reason about equivalence. We view the next exchange as an intervention, in which the interviewer tried to help students make sense of equivalence in counting (and to develop an equivalence way of thinking). By taking the time to explore this idea of equivalence more deeply on this problem, we note that this discussion provided additional opportunity for Carson to continue to develop his thinking and to explain his thinking to his fellow students. He repeatedly used the Horse Race problem and the idea of a podium as a means to explain the phenomenon, and this episode also emphasizes the way that social interaction can influence transfer, something that an AOT perspective affords (and something for which a more traditional perspective would not explicitly account). We contend that, in this case, the social interaction of explaining to other students about equivalence seemed to play a part in facilitating Carson's explicit use of the Horse Race problem as a way to articulate important aspects of equivalence upon which he would later draw.

The interviewer then asked why division instead of subtraction made sense to Carson, seeing if he could provide an explanation to his fellow students. Here, the interviewer wanted to give Carson the opportunity to explain rather than for her simply to tell the students why division made sense.

Interviewer: That's fair. So, you agree that it's 7! for any one of these, but why are we dividing? Okay, that's a great question. Why do you think you're dividing instead of subtracting?

Carson: So, for any arrangement of the first 3, so for any 1, 2, 3, we have 7! options for the rest of them. And, what we're being asked for is to find how many arrangements in the first 3 there are. <u>So, that 7! is only gonna be worth one of the things we are looking for, which is</u>

> the podium finishers. So, if we divide the total number of the combinations for this last bit we can get, while holding these 3 constant, by 7! we're gonna get 1. Because 7! over 7! is 1. But, the total arrangement is 10!, and if we divide that by 7! we're gonna get just the number of arrangements there are for the first 3.

The underlined portion again highlights the equivalence way of thinking Carson was bringing to this problem, and it suggests that he realized that for any arrangement of the first three horses, there were 7! equivalent ways to arrange the remaining horses who do not finish. His language again reflected a new (and still stabilizing) scheme of coordinating two multiplicative counting processes with the collection of outcomes that isolated a desired subsequence of events. His language of "holding these 3 constant" reflected operations that he had used in his previous multiplicative scheme, which were being coordinated with the ways the first three and the last seven horses could finish the race.

While continuing this discussion, Aaron talked about seeing groups of objects and dividing by the size of the groups (he was referring to groups generally and not necessarily to horses or to a particular context). Carson picked up on this and again tried to explain the problem, connecting the idea of groups to the "podium" of the three horses that finished the race (see Fig. 8.3). In this instance, Carson was referring back to their arrangements of kids in a line and pointing to Aaron's written work on the problem.

Carson: Well, that's sort of what you're doing, doing this too. So, like this is a group. Or [pauses] this here is a group, or this is the podium [points to the first six numbers], those first 6 places are the podium, these [the arrangements of 7 and 8] are the number of combinations for the end for the tail is what she was calling it. Or this is the podium [points to the first 5], the first 5 places at the podium and that's the number of combinations for the end [the arrangements of 6, 7, and 8]. So, if I wanna find the total number of ways the podium can finish, it would be the entire thing divided by the size of the group.

Fig. 8.3 Carson describes equivalent arrangements of groups

Both of Carson's above descriptions of why division made sense further illuminates a component of his new scheme, specifically the use of a representative element of the equivalence class of 10 arranged horses with the first three (or five or six as he mentioned above) horses fixed. This is consistent with his sequential construction of a desired outcome, where Carson's abstracted scheme leveraged a particular outcome as a representative of a class of outcomes, and this was in line with an equivalence way of thinking. We suggest that this was a reflective abstraction of his scheme for creating independent events in which he made a sequence of choices for placing objects in a number of positions, or "slots." The projected operation was the sequencing of events resulting in a multiplicative operation, which he then reflected to incorporate the equivalence structure into his new scheme by arranging the remaining outcomes. We argue that, in this case, the Horse Race problem was very salient for Carson, and we will see that his new scheme incorporated the specific Horse Race context as language through which he could engage in transfer while articulating certain ideas to his peers. Again, this demonstrates the interaction between the cognitive and social elements that the AOT perspective affords, in that Carson's abstracted constructs provided him with specific means of communicating to his peers the mathematical structures he would later encounter in his combinatorial activity throughout the teaching experiment.

Moreover, he (twice) explained this reasoning to his fellow students, and he did so by using either the problem context directly (mentioning a horse race) or specific features of the problem (such as a podium or a race). We feel that these factors contributed to the problem achieving particular importance for him, and, as we will see, Carson continued to use the Horse Race problem itself (or features of the problem) as referents in a number of additional contexts. Cognitively, we view the following instances of transfer in the remainder of this chapter as primarily deriving from Carson assimilating to this newly abstracted scheme that attended to the multiplicative structure of the outcomes he was counting. In this way, our use of Piaget's constructs enable us to present a unified account of the cognitive elements behind Carson's effective engagements in transfer, which typically involved use of some isomorphism.

8.4.3 Carson's Use of the Horse Race Problem as a Referent and His Formulation of Isomorphism

We now offer an example of how Carson made connections to the Horse Race problem on subsequent occasions in the interview. This came about in a couple of different ways and contexts. Here, we focus on how he made explicit connections to the Horse Race problem when solving new problems, and his work suggests to us that his uses of transfer were supported by assimilation to his newly abstracted scheme for permutation problems that involved envisioning a specific sequence of independent events that resulted in an equivalence structure. We would characterize this as a conventional, elaborated connection (in the sense of Lockwood, 2011) because

Carson made connections to problems that we might conventionally consider to be isomorphic. We also emphasize the role that the set of outcomes played in Carson's development and uses of isomorphism. We highlight two episodes in which Carson made connections to new problems.

We first emphasize a situation at the end of Session 1, when the interviewer gave them the Book problem: "You have 7 books and you want to arrange 4 books on a shelf, where their order on the shelf matters. How many ways are there to do this?" The students all answered the problem correctly, and they seemed able to make sense of it as being similar to the Horse Race problem. Aaron related it to the previous Horse Race problem, saying, "It's the same type of problem," and Anne-Marie said that she was "looking at what we were just talking about." They both explained that the answer would be $\frac{7!}{3!}$, although they did not explicitly make a connection to elements of the Horse Race problem and why it was similar to the current problem.[5] When Carson explained his work, we see that in explaining his answer of $\frac{7!}{3!}$ for the Book problem, he explicitly made a connection to the previous Horse Race problem. Even more, though, we emphasize that, in doing so, he actually changed the language of the given Book problem to match salient language in the Horse Race problem, notably, talking about books racing.

Carson: Yes, kind of similar to the horse problem, you can say they're all in a race, you want see how many ways the first 4 books could finish in the race. So, the equivalent, you could think about you got a bunch of books and a bucket, and you're reaching in and grabbing one randomly to put on the first spot on the shelf. So, you have 7 options for that first one and then there's already a book on the shelf, so there's only 6 left in the bucket and you're going to grab one randomly and put it on the second spot, so 6 ways that could come out for the second spot and onwards to the fourth spot. So, it's just 7 options times 6 options, times 5 times 4. I think about it in the grouping again and do 7! divided by 3!

It is interesting to note that Carson made a clear connection to the objects in the current Book problem (books) to objects in the Horse Race problem (horses), and he could think of the books as competing in a race.

In this episode, we again argue that Carson engaged in transfer by creating an isomorphism. His description of the books finishing a race is an instance in which he used the outcomes as referents by articulating a bijection between outcomes that preserves a perceived combinatorial structure. Further, we see evidence of Carson's assimilatory mechanisms at play. First, he gave a clear description of sequencing the placing of books that was similar to the way he described the horses being sequenced

[5] We note that Aaron and Anne-Marie were similarly making connections to prior problems, and these utterances represent instances of transfer. However, we continue to focus primarily on Carson's work.

in a race. Carson conveyed both the borrowed multiplicative operations as well as his new attention to equivalence in his last two sentences, where his appeal to "groupings" was reminiscent of the groups of podiums he had previously constructed in the Horse Race problem. Importantly, Carson was not just noting that the solution activity was the same, but even more, he seemed to recognize that the combinatorial structure was isomorphic.

Carson again engaged in a similar moment of transfer when solving the Cats and Collars problem (which states: "You have a red, a blue, a yellow, and a purple collar to put on seven cats, where no cat will get more than one collar. In how many ways can you give the four collars to seven cats?"). The students worked on this problem during Session 2. This problem was slightly more challenging for the students, and it took them some time to make progress on the problem and arrive at a solution. After several minutes of reasoning about the problem, the students correctly agreed that the answer should be $\frac{7!}{3!}$. They argued that, in this case, they had four slots or positions that represented the collars, and then they considered choices for which cats could go into each collar. There is a noteworthy feature of this problem, which is that it is perhaps not clear how to encode the problem and whether it makes sense to think about giving the cats to the collars or giving the collars to the cats. In the following exchange, we highlight a productive discussion that all of the students had about this problem, where they were trying to explain why having the spots represent the collars would make sense.

Josh: Yeah, I recognize that it was 7! over 3! and I actually drew seven spaces at first for the seven cats but I found that this was clearer for saying that their collars were the spaces.

Aaron: Yeah, I did that, too.

Josh: Because it represents this more clearly, I think. […] You can actually see where this can come into play here because you have seven spaces and three of the spaces can't be filled. So, you can say that this is 7! over 3! just like this.

The interviewer then asked for some clarification (seen in the excerpt below), asking the students how they were thinking about distributing cats to collars (and not vice versa). We note that here, the interviewer was genuinely trying to think about their idea and to get a sense of whether their approach made sense.

Interviewer: Yeah, so are you thinking of those A, B, C, Ds as being fixed, then, and you're arranging the spots around those A, B, C, Ds?

Carson: <u>Yeah, so that's how I thought about it is you have four spots and – It's a race, right, and the cats that get first, second, third, and fourth get collars. Right? So, seven cats can get the first collar, onward to the fourth collar and then, the last three cats, it doesn't really matter what place they come in in that race because they're not going to get collars. Right? But there are 3! ways that those last three cats could finish so you need to divide the total number of outcomes for the race by those 3! for the last place.</u>

Interviewer: Okay. And you feel like that's reflected in this as much as in that?
Aaron: Yes.
Carson: Right. Well, it's a little like you see the empty spaces that are going to get arranged when you represent the cats with the spaces rather than the collars.

In the underlined portion, we see that Carson again connected the Cats and Collars problem to a racing context. He explicitly says, "it's a race, right?" and later talks about ways cats "could finish." As with the Book problem, Carson drew an explicit isomorphism to identify the outcomes as having the same assimilated combinatorial structure as the Horse Race problem (i.e., a multiplicative structure that could result in equivalence classes). One aspect of AOT is that it allows researchers to look for how a prior learning experience influences reasoning on novel problems. In this case, we see evidence of such influence, where Carson's sophisticated reasoning about how a certain counting process generated outcomes in the Horse race problem also applied to the Cats and Collars problem. The language of "cats finishing" suggests the influence of the racing context on his current situation.

Again, Carson was not just using the Horse Race problem as a general referent, but instead, he was paying explicit attention to the outcomes as a means of describing the same assimilated structure. This attention to outcomes is not only productive, but it also demonstrates the increasing stability of Carson's assimilated scheme. Further, it shows the specifics of Carson's assimilatory mechanism underlying his engagement in transfer of yet another situation. Moreover, we contend that Carson's engagements in transfer had a reflexive effect in that his use of isomorphic language was socially motivated (to explain to the interviewer or his peers), and this repeated articulation of the multiplicative structure he was perceiving further stabilized his scheme associated with the Horse Race problem. Indeed, this episode, and the lengthy exchange among all four students, highlights that AOT allows for considering the influence of social interaction on the emergence of transfer. The elaborated connections between the Cats and Collars problem and the Horse Race problem were articulated during Carson's discussions with his peers. As an example, we would consider Aaron and Anne-Marie's initial comments on the book problem as unelaborated connections from which Carson then built to engage in isomorphic reasoning.

8.4.4 Carson Leveraged a Connection to the Horse Race Problem When Developing and Justifying a General Counting Formula

In Session 3, the students engaged in another kind of activity in the teaching experiment. We had asked them to categorize problems they had solved, and the goal was for them then to use these categories to come up with and justify general formulas for each problem type. The categorization is described in more detail elsewhere

(Reed & Lockwood, 2021), but here we point out that the Horse Race problem again came up for Carson in this context. Specifically, Carson still used the Horse Race problem to describe his understanding of general permutation processes, thus leveraging his scheme to analyze more general phenomena than just solving another new permutation problem.

First, we highlight a brief reference he made to the Horse Race problem as the students were categorizing 14 problems that they had initially solved. They had categorized all of the problems involving either permutations or combinations together, and they were in the process of separating those problems into two groups. First, he put slips of paper with the Horse Race problem and the Restaurant problem (which states, "Corvallis has 25 restaurants, and you want to rank your top 5. How many different rankings can you make?") together, and said "these two are the exact same problem." Then he also added the slip of paper with the Cats and Collars problem. In the following exchange, Carson associated three colors with three places.

Carson: So, these two are the exact same problem [the Horse Race problem and the Restaurant problem]. This [the Cats and Collars problem] is the exact same problem too because you could say red's first place, blue's second place, yellow's third place, right?

The interviewer then explicitly asked him what he meant by problems being the "exact same problem," and his response again demonstrates his use of isomorphism to elaborate the specific structures to which he was assimilating. Notice that, as before, Carson both referenced the context of a race, implying isomorphic racers in the race, and referenced the "podium" and dividing by the racers that did not make the podium, which more explicitly suggests assimilation of the equivalence structure within the set of the outcomes.

Interviewer: Okay, great. When you say something like, "these are the exact same problem" can you say what you mean by that and also how you know that they're – like how are they the same to you? What do you mean?

Carson: So, essentially all of them are asking for a ranking of a given set of objects and asking how many arrangements there are for a given number of places, right? So, <u>the cats are racing to get the collars</u> [points to the paper with the Cats and Collars problem] you could say <u>or the restaurants are racing to get the top five rankings in the town</u> [points to the paper with the Restaurant problem] or the <u>horses are racing in a race</u> [points to the paper with the Horse Race problem]. Then <u>each of the rankings or the collars are a ranking in the race</u>. Yeah, then you can <u>just divide by the duplicates for leftover ones, the ones that didn't make the podium finish or whatever amount of finishes there are or whatever podium they're asking for</u>.

The students eventually categorized the problems into four groups, – arrangements with repetition, n-permutations (arranging n distinct objects), r-permutations (arranging r from n distinct objects), and r-combinations (selecting r from n distinct objects). After the students had categorized problems into groups, they were asked to articulate and describe what that problem type was counting. Specifically, the interviewer asked, "What is the problem type that each of these four categories represents? And I'd like you to actually write down as a group what you want that to say." The students worked to characterize problem types, and they first articulated what we would refer to as *n*-permutations. They described those problems as the number of ways to arrange a set of objects, and they also wrote down the general formula that there are $x!$ ways to arrange x objects (they had used the variable x). They then moved to the next set of problems, which were permutations of some subset (of size r) of the n objects (so, r-permutations).

Interviewer:	Okay, nice. How about the next?
Carson:	So, same as above, except only asking for unique number of places.
Josh:	Well, we have more unique objects than unique places.
Carson:	Right. I mean thinking about the method for solving this, it's the factorial from above, right? <u>So, we have 10 horses in a race</u>. <u>How many ways can the horses finish, but then how many of those have a unique podium, right? So, how many times are the first, second, and third place different?</u>
Aaron:	So, you're not really looking at 4 through 10 in that case?
Carson:	Right.
Aaron:	So, it'd be over 6! in that case.
Carson:	Right.
Aaron:	So, just getting rid of all the arbitrary combinations that you're not looking for.
Carson:	Right.
Anne-Marie:	Yeah.
Carson:	So, you could say we're arranging the horses in a random way and then selecting three of them, right?
Anne-Marie:	Mm-hmm.
Carson:	So, how many ways could that selection come out?
Josh:	Well, not necessarily an arrangement, you're just selecting three horses from a certain number of things.
Carson:	Right.

The students were discussing this problem type, and they decided that they wanted to count arrangements of some number of elements (but not necessarily all of them). Carson connected this problem type to the Horse Race problem, and he used the context of the Horse Race problem as a prototypical problem of this type. The students were then able to come up with the general formula (Fig. 8.4) of $\frac{a!}{(a-b)!}$, where they noted that "*a* is the total amount and *b* is how many you're choosing."

The interviewer asked why they had an $(a - b)!$ in the denominator, and, as Carson explained the formula, he again referred to the podium arrangements. The language in the following excerpt again demonstrates that the equivalence structure was a primary aspect of his scheme. We emphasize in the underlined portions below

$$\frac{a!}{(a-b)!}$$

Fig. 8.4 The students' general formula for *r*-permutations (in their case, *b*-permutations from *a* distinct objects)

that Carson was using the context of the Horse Race problem (and referring to the podium, for example) to talk about a general process and not the specific example and instance of the Horse Race problem.

Interviewer: Okay, nice and why is it *a* minus *b* factorial in the denominator?

[…]

Carson: Yeah, so I guess the way that I think about this is that *a* is your total number of arrangements for the entire thing and then you want to divide by the number of ways that the places you're not selecting can be arranged, right?

Anne-Marie: Mm-hmm.

Carson: So, if you're selecting first, second, and third, then you have fourth through 10th and those can be arranged in 10 minus 3 factorial ways, right?

Interviewer: Mm-hmm.

Carson: So, we can just divide by that number of arrangements (begins motioning slots with hands) for the backend to get just one for the frontend because that's what we're asking for is how many ways can that podium be arranged.

Thus, in this section, we see the Horse Race problem continuing to be an aspect of Carson's work, even when engaging in an activity of developing a general formula for the combinatorial operation of combinations. This demonstrates that Carson's scheme extended beyond just solving additional permutation problems, but he drew on the Horse Race problem in a different kind of combinatorial activity.

8.4.5 Carson Draws on the Racing Context to Justify the Formula for Combinations

During Session 7, the students were trying to justify a formula for combinations ($\frac{n!}{(n-r)!r!}$), and for several minutes they were going back and forth to try to explain the formula. They talked for a couple of minutes, trying to work through an example and make sense of what might be happening. Ultimately, the notion of "redundancies" was important for them, which we take to mean duplicate outcomes. Notably, as they were working through this, Carson ultimately used a context of a race and a podium to explain his reasoning about the formula.

Aaron: So this is finding the arrangements of these two slots within these six.
Interviewer: Exactly.
Carson: That's what the whole thing is doing.
Interviewer: No, and then this is getting rid of—
Aaron: Redundancies.
Interviewer: Yes.
Carson: So if you think about this like a race, so all the numbers are in a race. It's asking how many of them can finish in the first two places. That's this many. That's 6! over 4!, right? So that's how many different ways you can get the podium, but then we really only care about the different ways that the podium can be. Who are the different people that can be on the podium? Not the different arrangements of people in first and second place. So that's the 2! redundancies there, because there's two ways that that podium can get arranged.
Interviewer: Mm-hmm, and so I just wanted to reinforce this piece is really just letting you focus on the number of things you care about.
Aaron: So you don't care about order. You just care about number of things.
Carson: Yeah, and this is the term that dictates order, because that's dividing out the redundancies of the—
Interviewer: Yeah, and that term is making it so you don't care about the rest of it. It just lets you focus on two things.
Carson: 6! tells you how many arrangements. Well, 4! tells you how many arrangements of the backend, so that leaves you with how many in the backend there are.

Here again we point out that this race context that began with the Horse Race problem was an important part of him being able to reason about and explain and communicate his thinking on problems, even when justifying a general counting process that is not strictly a permutation. In this instance, Carson again made use of an isomorphism to set up the combinatorial objects as being structurally similar to the Horse Race problem, but he then made a change by altering the conditions of the race. Specifically, Carson leveraged an isomorphism to discuss a race with two finishers on the podium. We infer that this represents an instance of transfer in which he attended to the set of outcomes, and he then altered what mattered about the podium to answer the specific situation they were characterizing. This demonstrates that Carson leveraged the Horse Race problem as a means of identifying and communicating a perceived combinatorial structure on which he could operate when solving specific problems. This is significant because it highlights that as Carson's scheme stabilized, the Horse Race Problem became a context for mathematical exploration rather than just a template on which could fit an existing structure. Thus, Carson's engagements in transfer were more nuanced than purely engagement in assimilation, though we primarily have access to Carson's engagements in transfer through his assimilations.

In some sense, Carson's connection to the formula for combinations was a bit more unconventional because the Horse Race problem is a permutation and not a combination problem. However, we see that he was drawing on the salient equivalence that had been so meaningful for him in the Horse Race problem, and he used that in reasoning about the formula for combinations.

8.4.6 Carson Reflects on the Role of the Horse Race Problem Throughout the Teaching Experiment

To conclude our results, we highlight a reflective comment to demonstrate that Carson was himself aware of how regularly he referred to the Horse Race problem. These conversations took place during Session 9, the final session of the teaching experiment. He did not seem to have a particular reason for why he returned to it so often, but we offer some suggestions about why he brought up the problem repeatedly. We saw this in the final session, when we asked the students to reflect on the overall teaching experiment and their work in general. The interviewer had asked about the extent to which the students reflected on prior work, asking the following:

Interviewer: So, maybe one question just broadly is to what extent do you feel that you were reflecting back and looking back on prior work as you were solving given problems? I mean, you did it a lot, so, it seemed like you were either referencing the other day's work or particular problems. And so, can you talk at all about how that was for you?

Carson's response indicated that he clearly referred back to the Horse Race problem. At one point, he reflected that he brought "the horse races to every problem," so he was aware that he regularly made connections to that particular problem. Further, Carson said the following upon reflection.

Carson: For whatever reason, the Horse Race problem is the one that's in my head forever. And it must have just been where it clicked in the interview because that's kind of what I refer to. If somebody says how many ways can a horse finish in the podium, how many ways can the podium be organized, things like that. And that's kind of where I keep going back to. And I don't know why that is.

Even though Carson said, "And I don't know why that is," we infer that he tended to explicate perceived combinatorial structure by making an isomorphism between the outcomes in a particular combinatorial event sequence that reflected various structures (e.g., multiplicative and equivalence structures). Although Carson initially attended to the multiplicative structure of counting problems, his work on the Horse Race problem entailed a significant abstraction of his multiplicative operations that he was able to coordinate with an equivalence structure.

Further, in the situations that followed this abstraction, Carson had opportunities to reinforce his reasoning about that problem—both in solving similar problems and by being asked to justify his reasoning to his fellow students. It seemed that in addition to being the setting in which he made sense of the problem, he also used the problem as a setting to communicate and explain fundamental ideas about that problem. This proved to be successful in that when he explained and communicated in the context of that problem, people seemed to understand, or at least agree with, what he said. These moments of transfer served to demonstrate the subtleties of his assimilatory mechanism and abstracted operations, but they also served a key role in Carson's contributions to the group's mathematical activity. This successful communication perhaps also stabilized his scheme for the equivalence combinatorial structure that he had abstracted when working on the Horse Race problem. That is, his successful communication of the assimilated structures in subsequent problems likely reinforced the horses and podium arrangements as meaningful outcomes for him to leverage. We emphasize again that being able to account for the influence of Carson's social interaction on his development of connections is a benefit of adopting a perspective of AOT rather than traditional transfer. As we have noted, a feature of AOT is that it allows for both psychological and social perspectives, and Carson's communication and explanation (which were necessitated by the social environment in which he was situated) at times provided motivation to make connections to the Horse Race problem.

8.5 Discussion and Conclusions

In this chapter, we reported on one student's repeated reference to a particular problem in a teaching experiment that focused on combinatorial problem solving. We demonstrated several contexts in which Carson referred back to this problem, often constructing and leveraging isomorphism to do so. Carson's engagement in transfer across the teaching experiment involved assimilation to a rich combinatorial scheme. We attribute much of Carson's success to his attention to the outcomes described in a counting problem and his careful coordination of specific counting processes to be carried out to generate a set of outcomes (this is in line with Lockwood, 2013, 2014). By unpacking the cognitive mechanisms of Carson's learning (and subsequent application) of the Horse Race problem, we have provided a means of characterizing the salient and unifying aspects of Carson's engagement in transfer throughout the teaching experiment. We have also demonstrated a cyclic interplay between the social interactions that occasioned Carson's transfer and his subsequent assimilations and minor accommodations that effectively stabilized his multiplicative and equivalence scheme. Thus, we see both the social and cognitive components of the AOT perspective providing explanatory mechanisms for Carson's transfer activity. We now offer a couple of points of discussion about Carson's case.

We have used Lockwood's (2011) ways of characterizing student-generated connections among counting problems, highlighting especially cases in which Carson made elaborated, conventional connections to a particular problem. Carson's conventional connections between problems often involved creating an isomorphism (even implicitly) between components of the problems (this involved language like having cats or restaurants racing, as we have discussed). We find this kind of transfer particularly useful for students' learning within combinatorics, given that the nature of combinatorial outcomes can be an important underlying structure in counting problems. Indeed, spontaneously mapping sets of outcomes to each other as a means of communicating underlying mathematical structure is a powerful tool in advanced mathematics, and we see elements of such activity in Carson's instances of transfer. There are, however, many instances of AOT that might involve activities other than creating an isomorphism. Indeed, this is a benefit of AOT that we did not explicitly mention in this chapter (because our data did not highlight it) but elsewhere (e.g., Lockwood, 2011), researchers have shown that students may relate situations even when a formal isomorphism may not be established. Further, although the connections were conventional in some sense (they would be considered normatively correct, for example), the frequency with which he repeatedly focused on the Horse Race problem was at times surprising. He brought it up on many occasions that we might not have expected, which underscores how important he found his initial experience with that particular problem. Specifically, his reference to the Horse Race problem in reasoning about the formula for combinations was particularly illuminating and important.

In addition, even though he regularly referred to the Horse Race problem (a particular problem), we see that this particular problem often implicitly served as a broader problem type for Carson; it became emblematic of problems that had certain structural characteristics, namely, a multiplicative event sequencing and notions of representative equivalent outcomes. Thus, we see perhaps some blurring of two referent types Lockwood had previous articulated, namely particular problems and problem types. Related to this, although we have not framed our work in terms of prototypical problems, we note that our findings could shed some light on the nature of prototypical problems and how students might think about and use such problems. As noted previously, Maher et al. (2011) described students making repeated references to a "pizza problem" or "towers problem" in similar ways that Carson referred to the Horse Race problem. We have built on prior work by Maher et al. by attempting to account for cognitive mechanisms that facilitated Carson to make such connections. In addition, although Maher et al.'s work occurred over many years, we show how robust connections to certain problems and problem types occurred within just a handful of sessions in a concentrated study. Thus, we have demonstrated that students can establish reliable and meaningful prototypical problem types that they can use in a variety of settings even over relatively short periods of time.

We also suggest that our study sheds some light on ways in which both the cognitive and social dynamics of the AOT perspective can be engaged and support each other. Our radical constructivist lens provided explanatory mechanisms for Carson's engagements in transfer, specifically for the purpose of unifying what might initially seem like disparate engagements in transfer. For Carson, these were all parts of what became a robust scheme for counting. We demonstrated some sophisticated ways that social interaction could contribute to and support transfer, both for individuals and for a group of multiple students. On several occasions, Carson's connections to the Horse Race problem emerged in the midst of social interaction, whether that was during a group discussion or whether he was communicating or explaining an idea to the rest of the group. Again, this is something that an actor-oriented (as opposed to a traditional) view of transfer affords. Moreover, we see these social interactions as occasioning Carson's engagement with the structural aspects of his abstracted scheme, thus facilitating a stabilization of certain aspects of his scheme. Though the social dynamic of Carson's interactions was not itself the focus of our study, we posit that adopting both the social and cognitive analyses in accordance with the AOT perspective has provided a richer account of the ways Carson's understanding developed. Other researchers could draw more direct relationships between social and psychological aspects of transfer, and studying the nature of transfer as stemming from social interaction is fascinating and could benefit from additional investigation.

To conclude, we discuss some potential implications and takeaways of our study. In particular, we believe that Carson's case demonstrates some sophisticated and compelling ways of reasoning about equivalence in combinatorics, and there is much to learn from Carson about how students might productively develop and articulate such reasoning. Specifically, Carson seemed to have a deep understanding of sets of outcomes (Lockwood, 2014). He could reason about certain outcomes being equivalent to each other, and he realized he could use division to account for this. This way of thinking was reinforced across multiple similar situations, and Carson could notice relevant structural similarity in outcomes (and the way in which outcomes were generated) that supported his conceptual development. Although we are of course not surprised to see students' impressive and inspiring work, we suggest that the AOT perspective can put researchers in situations to recognize and appreciate the excellent work that their students do. This point has been made elsewhere (e.g., Lobato, 2014; Lobato & Siebert, 2002; Lockwood, 2011), but our specific empirical example again demonstrates the rich insights about students' thinking and activity that can result from adopting an AOT perspective. This suggests that it would be worthwhile for teachers and researchers to continue to view students' mathematical activity from the students' point of view.

As a final takeaway, we also highlight the role that justification played in Carson's work, both as he established a robust initial understanding of equivalence in the Horse Race problem, and as he subsequently drew on that problem in additional situations. By having to justify and explain his reasoning to the interviewer and to his peers throughout the teaching experiment, Carson reinforced fundamental ideas that were meaningful for him. Thus, a potential practical takeaway is to give students opportunities to formulate justifications and explanations and to articulate those regularly. Such justifications could reinforce important mathematical ways of thinking that could become important and foundational for students, creating initial situations and contexts to which they can subsequently refer.

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Chapter 9
Promoting Transfer Between Mathematics and Biology by Expanding the Domain

Claus Michelsen

At the beginning of the twenty-first century, Niss (1999) identified and presented five major findings in the scientific discipline of mathematics education. The findings were the results of thorough theoretical or empirical analyses and offered solid insights of considerable significance to our understanding of the processes and outcomes of mathematics teaching and learning. Among the findings was the key role of domain specificity:

> For a student engaged in learning mathematics, the specific nature, content and range of a mathematical concept that he or she is acquiring or building up are, to a large part, determined by the set of specific domains in which that concept has been concretely exemplified and embedded for that particular student. (Niss, 1999, p. 15)

Thus, if a mathematical concept is introduced for the student in a narrow mathematical domain, the student may construct and see it as a formal object with arbitrary rules without any connection to an extra-mathematical context. Without a direct reference, the finding of the key role of domain specificity addresses the problem of transfer between a mathematical and an extra-mathematical context. This raise concerns in education, where a growing number of disciplines contain elements from mathematics. Mathematics plays a crucial role in science, which relies on widespread mathematization. Many science phenomena and their patterns of interaction are best described in the language of mathematics, which then becomes a bridge between the students' verbal language and the scientific meaning that they seek to express (Osborne, 2002). However, although mathematics and science might be considered as intertwined disciplines, this description is not straightforward as far as the students are concerned. Confusion may be caused, for example, when mathematics and science teachers use different terminology or approaches when explaining ideas. Lappalainen and Rosqvist (2015) used a study of students' experiences of

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applying knowledge across disciplines to highlight that the connections between the topic at hand and past experiences are an important and all too often neglected part of teachers' day-to-day work. Furthermore, they noticed that the connections made to previous knowledge that the students relied on in a given situation might often differ from the connections that their teacher would make. Consequently, the teacher's hints at association might hinder the student's problem-solving process. This chapter addresses transfer between mathematics and biology in an interdisciplinary learning setting. This issue was investigated in the context of upper secondary Danish education and the starting point was that students' development of mathematical models of biological phenomena can easily be adopted and adapted for use in both mathematics and biology classrooms and establish intercontextuality between mathematics and biology. Two progressive perspectives on transfer, expansive framing and actor oriented, were applied to examine the transfer that occurs between the school disciplines of mathematics and biology.

9.1 Theoretical Framework

Looking at the challenges of educational theory and practice in the twenty-first century, Schoenfeld (1999) identified six arenas in which significant progress needs to be made. One arena was transfer, which was labelled as ubiquitous. The importance of transfer was explicated by emphasizing that we could not survive if we were not able to adapt what we know to circumstances that differ from the circumstances in which we learned it. We "are making connections all the time. The issue is to figure out which ones they make, on what basis – and how and why those connections are sometimes productive" (Schoenfeld, 1999, p. 7). Transfer is one of the classic problems in education and, according to Lester and Lambdin (2004), transfer should be the goal of education because the educational system prepares students for a world outside the classroom and for solving future problems that we might not even imagine today. And, one might add that to prepare the students for this, they should, in their schooldays, experience transfer, for example between a mathematics class and a biology class.

9.1.1 Transfer from the Students' Perspective

Transfer is a general entity, but there are different definitions and different research approaches. In this chapter, I adopt the contemporary perspectives on transfer characterized by looking at transfer from the students' perspective and consider transfer "as an activation of associations between tools in the source (learning) and target (transfer) contexts" (Rebello et al., 2005, p. 246), and consider that transfer occurs "when learning to participate in an activity in one situation [i.e., learning context] … influence[s] (positively or negatively) one's ability to participate in another activity in a different situation" (Greeno, Moore, & Smith, 1993, p. 100). The aim of my study was to examine the influence of a learner's prior activities on his or her activity in a novel situation within the actor-oriented transfer perspective "to understand the interpretative nature of the connections that people construct between learning and transfer situations, as well as the socially situated processes that give rise to those connections" (Lobato, 2012, p. 239). This approach to transfer is productive because the focus is not on the transfer of learning when it does not happen, but on identifying how students make connections between learning that took place in previous contexts and learning taking place in new, currently encountered contexts.

9.1.2 Expansive Framing and Intercontextuality

Prior knowledge influences the comprehension of any new context, as Dewey (1938) pointed out with the principle of continuity of experience that "means that every experience both takes up something from those which have gone before and modifies in some way the quality of those which come after" (p. 35). Clearly this has instructional implications, a fact which is underscored by using the term *expansive framing* (Engle, Lam, Meyer, & Nix, 2012) for the framing of learning and transfer contexts as opportunities for students to actively contribute to larger conversations that extend across times, places, people, and activities. By engaging in this kind of temporal framing, teachers make it clear to students that they are not just getting current tasks done but are preparing for future learning while regularly drawing on past learning. Transfer "is more likely to occur to the extent that learning, and transfer contexts have been framed to create what is called intercontextuality between them" (Engle, 2006, p. 456). Intercontextuality occurs when two or more contexts become linked with one another. When this occurs between learning and transfer contexts, the content established during learning is considered relevant to the transfer context. Thus, when the teacher is helping students to see and experience relationships between different contexts, the more likely—all other things being equal— students will transfer content between them. Engle, Nguyen, and Mendelson (2011) investigated the degree to which high school biology students transferring knowledge from a learning session about the cardiovascular system to a session about the respiratory system depended on framing conditions and concluded that students in the expansive condition were generally more likely to transfer facts, conceptual principles, and a learning strategy from one system to another. Engle et al. (2012) proposed the idea that transfer can be promoted by the instructional practice of framing learning contexts in an expansive manner and that expansive framing may: (a) foster an expectation that students will continue to use what they learn later, which may affect the learning process in ways that can promote transfer; (b) create links between learning and transfer contexts so that prior learning is viewed as relevant during potential transfer contexts; (c) encourage learners to draw on their prior knowledge during learning, which may involve them in transferring additional examples and making generalizations; (d) make learners accountable for intelligently reporting on the specific content that they have authored; and (e) promote authorship as a general practice in which students learn that their role is to generate their own solutions to new problems and adapt their existing knowledge in transfer contexts. Thus, the proposal offers five not mutually exclusive and eventually even complementary explanations for how transfer can be promoted by expansive framing.

Expansive framing and intercontextuality between learning and transfer contexts are aligned with the idea of expanding the domain suggested by Michelsen (2006) to transcend the problem of domain specificity. Expanding the domain highlights that interdisciplinary activities between mathematics and science offer a great variety of domain relationships and context settings that can be linked and serve as a basis for students to see and experience connections between situations almost all the time, guided by the aspects they find personally salient. When enough connections are made, the degree of intercontextuality can become so strong that a larger encompassing context is formed that seamlessly incorporates the contexts of mathematics and science. This raises the question: What kind of classroom can one envisage being filled with opportunities for the students to make connections? This chapter investigates this question using an interdisciplinary mathematics–biology teaching sequence in Danish upper secondary education by means of the following research questions:

1. Which elements of intercontextuality are experienced by the students in the interdisciplinary mathematics–biology teaching sequence?
2. To what extent did the students participating in the interdisciplinary mathematics–biology teaching sequence experience (a) expectation for transfer, (b) continued relevance, (c) positioning as authors of their own ideas, (d) accountability to the content, and (e) participation in practices of generating new knowledge and engaging in adaptive problem solving?

9.2 Intercontextuality and Interdisciplinary Mathematical Modelling

The Danish upper secondary school system is organized in what are known as study packages, where an important feature of a package is that the participating disciplines form a coherent program, which is ensured by closer interaction between the disciplines. This calls for interdisciplinary teaching across the traditional boundaries between the disciplines, both at the level of discipline content and at the level of pedagogy. Indeed, the very aim of regular upper secondary education is "to prepare the students for further education, hereunder to acquire … knowledge and competencies through the education's combination of disciplinary breadth and depth and through the interplay of the disciplines" (Ministry of Children and Education, 2017; author's translation). Some of the study packages include mathematics and biology as core disciplines. To fulfil the objective of coherence in the study packages, interdisciplinary teaching across mathematics and biology is demanded.

9.2.1 Mathematical Modelling in an Interdisciplinary Context

Interdisciplinary learning and teaching involving mathematics education has become of considerable interest to some mathematics educators (e.g., Sriraman & Freiman, 2011), and interdisciplinary teaching in mathematics and science offers students the opportunity to experience a coherent curriculum. An educational integration of science and mathematics was suggested by Berlin and White (1998) as a promising path towards improved student understanding, performance, and attitudes related to mathematics and science. A historical analysis spanning from 1901 to 2001 and related to integrated science and mathematics analysis documented strong philosophical support for the integration of science and mathematics education to improve students' understanding of the two disciplines. It was emphasized that although each of the human enterprises of mathematics and science has a character and history of its own, each of the disciplines depends on and reinforces the other (Berlin & Lee, 2005). One can go a step further and point to the widespread use of models and modelling in the scientific enterprise and in mathematics and science education as well. Modelling is accepted as an important issue in mathematics and science education at all levels, and there is extensive literature recognizing the importance of models and modelling, both in mathematics education and in science education (e.g., Gilbert & Justi, 2016; Kaiser & Sriraman, 2006; Stillman, Blum, & Biembengut, 2015). Mathematical models and modelling were also suggested as tools to transcend the obstacles preventing the integration of mathematics, physics, and engineering into the biology curriculum and vice versa (Chiel, McManus, & Shaw, 2010), and Jungck (2011) made the point that instead of focusing on how to overcome the challenges of implementing mathematics into biology, the focus should be on the development of individual biological models that can be easily adopted and adapted for use in both mathematics and biology classrooms. Across different versions of the modelling cycle, the mathematization of an extra-mathematical domain is the starting point, which entails the connection of an extra-mathematical setting to a mathematical one and vice versa when the constructed model is validated by confronting the model output with the known reality of the extra-mathematical setting (Niss, 2010; Niss & Højgaard, 2011). The reference to the modelling of an extra-mathematical situation emphasizes that mathematical modelling is not only a matter for mathematics. This interdisciplinary aspect highlights that mathematical modelling has the potential to support the students in making forward and backward links between mathematics and extra-mathematical situations.

I argue that interdisciplinary mathematical modelling activities connect settings, promote student authorship, and accommodate the five explanations for how transfer can be promoted by expansive framing. Mathematical modelling activities emphasize social interactions between students and the students' development of conceptual tools that include explicit descriptive or explanatory systems that function as models to reveal important aspects about how students are interpreting the modelling situation (Lesh & Doerr, 2003). In this way, the student's explanations are revoiced and credited with authorship. Galbraith (2015) made the point that it is essential that modelling is about real-world problems with the purpose of applying the students' knowledge proficiently to problems located in personal, work, or civic contexts or in other discipline areas. This makes students become publicly recognized as the authors of particular transferable content and adopts the practice of authoring knowledge.

9.3 The "Laboratory for Mathematics Teaching" Project

In the period between 2014 and 2018, the mathematics and science education research group at the University of Southern Denmark participated in the project called "Laboratory for Mathematics Teaching." Interdisciplinary teaching sequences in mathematics and biology in upper secondary education were designed and implemented within the project teaching sequences via modelling. The model-eliciting framework proposed by Lesh and Doerr (2003) was applied as a didactic tool for interdisciplinary teaching in mathematics and biology. The framework confronts the students with the need to develop a mathematical model to make sense of a meaningful situation. The starting point is a model-eliciting activity designed to elicit the students' initial ideas about a problem situation. This is followed by more structurally related model-exploration activities and model-application activities. In our approach, the problem situation to be modelled by mathematics is an experimental situation from biology.

9.3.1 Design-Based Research and Exploration of How Students Interpret Transfer Situations

The mathematics and biology teaching sequences were developed according to a design-based research approach (Design-Based Research Collective, 2003; Lesh & Sriraman, 2005) by translating research in interdisciplinary teaching at the University of Southern Denmark into interdisciplinary modelling activities based on the model-eliciting activity approach in assembled project teams of scientists, as well as in-service and pre-service upper secondary school teachers. The approach was focused on developing greater clarity and coherence when working with mathematical ideas, language, and procedures in mathematics and biology lessons to support students to transfer their mathematical skills and understanding effectively to their biology learning.

For the purposes of this chapter and inspired by the point made by Lobato (2012) that design-based research opens a space to explore how students interpret transfer situations, the socially situated nature of transfer processes in classrooms, and how contextual sensitivity can play a productive role in the transfer of learning, it was decided in one of the teaching sequences to focus specifically on transfer in an interdisciplinary mathematics–biology context with a focus on modelling activities. The teaching sequence was implemented in two upper secondary schools in the Region of Southern Denmark and involved 45 Grade 10 students, three teachers with mathematics or biology as their discipline, and one researcher (the author of this chapter). All students were enrolled in a study package program with mathematics as one of the core disciplines. According to their teachers, the students' prior experiences with interdisciplinary mathematics–biology teaching was limited. The population of students was representative of a typical upper secondary school in a small middle-class town with a close to equal gender distribution.

9.3.2 The Interdisciplinary Mathematics–Biology Teaching Sequence

The two interdisciplinary mathematics–biology teaching sequences were planned through three workshops of two hours each with the participating teachers and the researcher at each of the two schools. The general theme of the workshops related to the ways in which the teachers could support the students' experience of intercontextuality in an interdisciplinary mathematics–biology setting. The researcher introduced the teachers to the model-eliciting framework which led to a general agreement among the teachers that an experimental situation in biology should act as a starting point for the students' modelling activities. To prepare the students for this, the teachers planned to introduce the students to linear and exponential regression before implementing the interdisciplinary teaching sequence. The teachers were very keen on developing relevant, meaningful activities that could engage the students emotionally, and they decided to design a two-week interdisciplinary mathematics–biology teaching sequence consisting of four modules, each 90 minutes long, with the human body as the overall teaching theme. In one of the schools, the theme of the sequence was the muscles of the human body and in the other the theme was consumption and elimination of alcohol in the human body. Finally, to give the students immediate feedback, it was decided that at the end of the sequence the students should communicate their activities, experiences, and findings through a written report or a poster.

After a short introduction to the theme and exchange of ideas among teachers and students about potential investigations within the theme, the students were distributed in groups of between three and six and given the task of planning and implementing experiments, collecting data from the experiments, and analyzing the data with mathematical tools. The students were informed by their teachers that the teaching sequence would be structured to provide them with opportunities in the groups to investigate self-chosen problems within a theme (muscle strength or consumption of alcohol), to search for possible solutions, to make observations, and to collect data and to develop a mathematical solution to explain the problems. The teachers encouraged the students to draw on their prior knowledge about regression during their work.

At the school that was working with the muscles of the human body, the students focused on measuring the volume of the biceps and thigh muscle and relating this to weight, jump, and sprint ability and the maximum weight lifted by the arm. The students made different hypotheses about relationships between the measures and entities and checked the hypotheses using regression. Figure 9.1 below and the excerpt that follows come from a student report in which the students investigated the relationship between the circumference of the biceps and the maximum weight lifted in boys:

> Here, we again tried to find a relationship, this time between the circumference of the biceps and the maximum weight in a bend. Here, we get a good r^2 as evidence for an exponential relationship between the two variables. The explanation rate is also above 0.8, which is good.

In the second school, the students worked with consumption and elimination of alcohol in the human body. The school's principal gave eight students permission to consume four standard units of alcohol. The eight students acted as test persons to be called in for experiments by the other students. Most of the groups decided to use a breathalyzer to measure the blood alcohol concentration of the test persons at different times. Some groups decided to compare the reaction time and the distance travelled of the test subjects and some of the sober students. The former was investigated either using an online reaction-time tester or by a simple experiment with a dropped ruler to be caught by the person, whereas, in the latter, the test subjects had to walk in a straight line and the distance walked was measured. Figure 9.2 below and the excerpt that follows come from a student report in which the students investigate the relationship between the blood alcohol concentration and number of consumed standard units of alcohol:

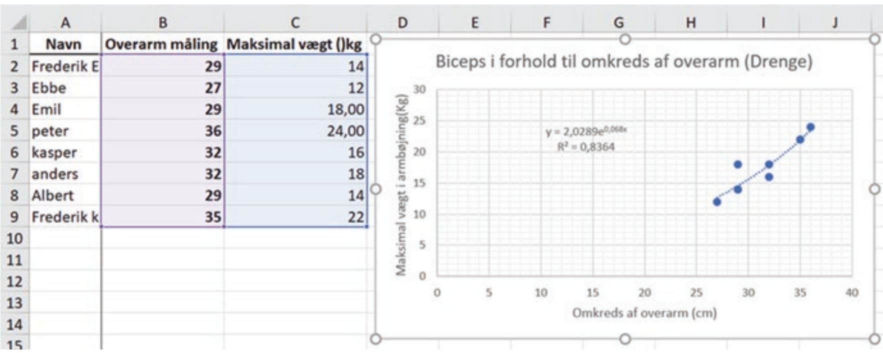

Fig. 9.1 From a student report: Investigating the relationship between the circumference of the biceps and the maximum weight lifted in boys

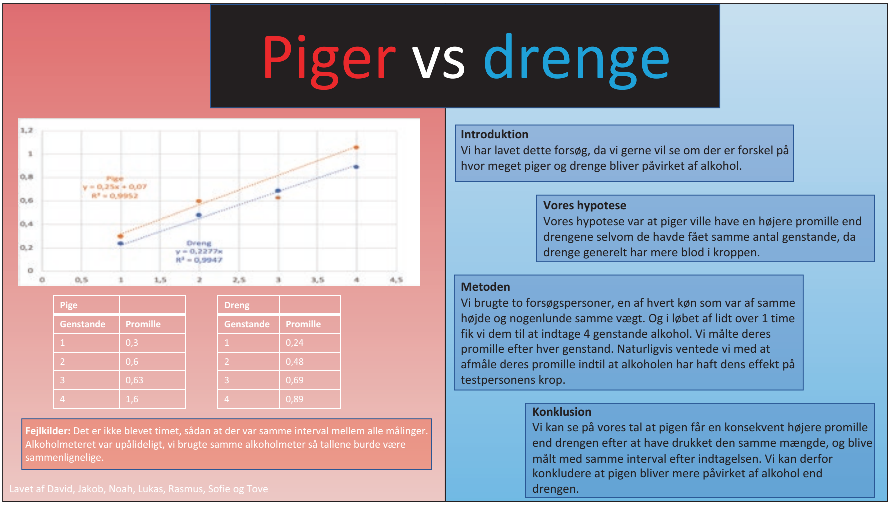

Fig. 9.2 From a student report: Investigating the relationship between the blood alcohol concentration and number of consumed standard units of alcohol for a girl (Pige) and a boy (Dreng)

> From the graph [Fig. 9.2] we can see that the girl has a consistently higher blood alcohol concentration than the boy after drinking the same amount. Therefore, we conclude that the girl is more affected by the intake of alcohol than the boy.

9.4 Method

The primary data for this chapter came from records of the classroom work, the students' reports and posters, and two focus group interviews with students. One month after the completion of the sequence, the researcher conducted three 25-minute focus group interviews with three randomly selected groups of three students at each of the schools. Two months later, the researcher interviewed each of the groups for 25 minutes. The focus group interview approach was chosen to obtain general information about the students' experience with, interest in, and expectations of a future application of the mathematic–biology teaching sequence. With reference to the description by Dufresne, Mestre, Thaden-Koch, Gerace, and Leonard (2005) of transfer as a complex dynamic and ongoing process, I also wanted to discover what the students had to say about the phenomena addressed in the sequence with a specific focus on the knowledge elements that the students brought to the fore. To provide an authentic setting for the students, the interviews were conducted at the schools during a mathematics lesson. Furthermore, the students were interviewed in the groups in which they had worked during the instructional sequence, which, among other things, made it possible to take their reports or posters into account.

The interviews were conducted as semi-structured interviews with an interview guide. The students explained how they themselves experienced intercontextuality. The first interview focused on the students' experiences from the interdisciplinary teaching sequence, whereas the second was centered around a question put to the students about what kinds of experiments, data sampling, and analyses they could imagine were they to investigate vital capacity, which is the maximum amount of air a person can expel from the lungs after a maximum inhalation. This entailed an incorporation of instructional elements, and the second interview might be categorized as a kind of teaching interview based on the premise that students construct their responses to interview questions dynamically and often do things on the spot (Rebello et al., 2005).

The interviews were videotaped and then transcribed. Together with field notes from the classroom activities and the students' reports and posters, the transcripts of the interviews with students comprise the primary data for this article. The data were analyzed by the researcher as a deductive category application (Mayring, 2000) with a focus on the actor-oriented perspective on transfer and the five potential explanations for how expansive framing may promote transfer.

9.5 Findings

Based on the analyses of the data, I claim that the students saw greater relevance in the content and experienced greater freedom and sense and that this led to a greater exchange of ideas between the students. The findings from the study are presented in two parts. In the first part, the students' perception of intercontextuality is investigated from the actor-oriented perspective. The focus is on how the students interpreted the interdisciplinary teaching sequence and its situational structure, how they interacted with prior learning experiences and tasks, and the ways in which they generalized their learning experiences. In the second part, I use the five potential explanations for how expansive framing may promote transfer as an analytical tool to shed light on the students' experiences with the teaching sequence and to consider how the context of the students' learning activities was framed to promote intercontextuality.

9.5.1 The Students' Perception of Intercontextuality

The students considered the teaching sequence as a new, unfamiliar, interesting, and challenging situation, but at the same time they referred to learned knowledge in prior mathematics and biology classes and recognized that knowledge that is relevant in one setting is also relevant in other settings:

> We measured our own muscle strength and muscle size and that was different from the normal lessons. Normally we only get tasks from the book. We used the thing with plots, which we have previously worked with in mathematics and biology as well, for example with Excel. We felt quite well prepared for the tasks.

The students were aware of the interdisciplinary aspects of the sequence, and a group of students wrote in the introduction to their report:

> In this project we used tools from mathematics and biology for tests and measurements of different parts of our body and muscles. The measurements have given us insight into domains relating to our body and made us aware of views on areas relating to elements of our body, where these views have made us aware of how to use both mathematics and biology in our data.

During the first interview, the students were shown Fig. 9.3 below and asked to reflect about possible links to their experiences from the teaching sequence.

The interview excerpts below show that the students experience was such that the teaching sequence connected curriculum units across time and trained them to make connections across topics: "Data processing is clearly both mathematics and biology, while the experiments aimed at measuring muscle strength are biology and regression is mathematics. We are used to having disciplines, and it has been exciting to connect disciplines. It is challenging."

However, the students' perception of the connections showed great variation. Some students considered data regression mainly as a biological activity whereas others saw it as a mathematical activity.

The teachers expansively framed their interactions with students by temporally connecting to prior and future interactions in which students could use what they were learning. The students were aware of the focus on learning important new information to be used in future settings:

> It became very real when Nicole had to breathe into the breathalyzer. And we became aware of all the uncertainties in the experiments, whereas there are never any uncertainties in the textbook. And it's not just mathematics, but also social sciences and biology, probably mostly biology. We expect to use data processing and regression in physics and other disciplines. But it is also about training in the native language, as we should communicate our results.

This excerpt also shows that the students responded to the framing with a broad interpretation of intercontextuality by including several disciplines and by detecting a difference between a textbook and a lab setting in terms of the presence or absence of uncertainties.

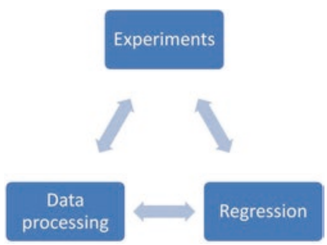

Fig. 9.3 Handout sheet for the interview: Possible links in the students' experiences from the teaching sequence

During the researcher's informal observations of the teaching sessions, a high volume of activity was observed in terms of exchanges of ideas between the students in the groups and between the groups and the teachers. This highlights the importance of sociocultural factors, and this perspective was also addressed by the students in the interviews:

> We started in upper secondary school six months ago, and it's a lot about social relationships among the students. In our class the social relationships are good, and it made sense in the project when we worked together to design the experiments and make measurements.

A recurring theme in the interviews was the students' expectations for transfer of mathematics. All the students expected to use mathematics in the future, but, as the two excerpts below show, the nature of their expectations varied widely, including everyday life, research, and jobs intertwined with references to the teaching sequence: "Mathematics can for example be used when you are shopping, if you need to take out a loan and something like that, or if you want to be a banking advisor, and regression can be used to predict" and "Mathematics is difficult, but it is used for many things, such as measuring muscle strength, building something, something with money, and medicine when mixing something."

Some students also questioned whether all mathematics topics can be used outside mathematics: "We haven't yet used the polynomial of second degree, and we do not know if we ever will use it…. Results that can be related to everyday life are better than just calculating the discriminant."

The second group interviews were conducted with the aim of investigating how the students interpreted a context centered on investigating vital capacity. Across the six groups, I found evidence that most students generalized their learning experiences, by indicating that mathematics can be applied to handle data collected in a biological context. When asked about what kinds of experiments, data sampling, and analyses they could imagine were they to investigate vital capacity, the students immediately referred to health issues:

> The height has something to do with the size of your body. If your body is big, then your lungs are also big. Could it have something to do with genes; we have twins in the class. It could also have something to do with lifestyle. We have both smokers and nonsmokers in the class, so one could investigate whether there is an effect on vital capacity.

For most of the students the response to the question about how to investigate vital capacity was to calculate the average vital capacity for each of the two sexes to find the expected difference: "We chose five girls and five boys of different heights and weights, measured their vital capacity and then calculated the average for the two sexes. Initially, we are satisfied that there is a difference."

The relationships constructed by the students were motivated by an expectation of a gender difference, and then they searched for a simple mathematical tool, such as averaging, to compare girls and boys. Only in rare cases did the students give the short and precise response from the teacher's point of view, namely that you should measure the vital capacity and height of the students in the class and then check the relationship using a regression analysis. However, when challenged by the interviewer, the students suggested collecting data and making graphs to investigate the difference:

> We could make some graphs. It must have something to do with boys and girls. Perhaps we could do something about vital capacity as a function of height. Yes, measure how tall you are, then measure your vital capacity and look for differences.

To sum up, I observed that the students temporally connected to prior and future interactions in which they could use what they were learning. When the students drew on and used their prior knowledge from mathematics and biology classes, they constructed a new setting with connections to former settings. In this process, the students began to see lab and classroom settings as being interconnected and established a connection between an experimental setting in biology and a more theoretical setting in mathematics where the students worked with their laptops.

Although the methods used to investigate vital capacity are in many ways isomorphic to investigating alcohol consumption and muscle strength, the students made sense of the situation by establishing a context based on several former learning experiences—for example, calculating the average, making graphs, identifying variables—and everyday life experiences—for example, sports, twins, and smokers in their class. There were also implicit references to the interdisciplinary teaching sequence—for example, gender issues, the experimental method, and mathematics as a tool to handle and compare the collected data.

9.5.2 The Students' Experiences with Expansive Framing

The first explanation for how transfer can be promoted with expansive framing is the students' expectation for transfer. The students had expected to transfer what they had learned in the teaching sequence. Most of the groups pointed to the use of regression in mathematics and biology and the communication of their results through posters and reports: "We learned how to process data and how to find answers: We will use it in reports, and we can make graphs in both mathematics and biology" and "We are sure that we will use regression in the future. We have already used it in biology."

The second explanation concerns the students' experience of continued relevance. Some of the groups addressed the interdisciplinary aspect of the teaching sequence and their expectations of future interdisciplinary work at school: "The mathematics–biology collaboration probably shows us a little bit more about what we can face in the future, namely that we must look at more than one discipline at the same time."

The relevance of the teaching sequence for the students' future studies and jobs was also an issue for some of the students: "I want to study to become a physiotherapist, and I think I am going to use some of the methods of collecting and studying data that I have learned here."

Positioning the students as authors of their own ideas is the third explanation. The freedom to work with self-chosen problems within a theme focusing on the human body was highlighted by the students as a challenge as well an opportunity to connect to everyday life experiences:

> The project was unstructured, we were not quite sure of what to do, but it was cool. It was great to work with the project, and we had the freedom to choose experiments ourselves. And alcohol is part of our everyday lives and we know about alcohol. For example, how much we consume in an hour and how weight matters.

This freedom positioned the students as the sole authors and made them, rather than the textbook or the teacher, accountable for the content. It also involved the students in explaining their ideas to and exchanging their understanding with each other and, as the excerpt below highlights, extending their perspective to include reflections about connections between formulas and measurements:

> Usually we use formulas handed out by the teacher, and that's just the way it is. Building a formula by ourselves is much more productive. The group work required us to make some compromises. But it was also good to exchange ideas. And then we got to know each other. We discussed which experiments we should do. We chose experiments where we could both make graphs and tables, and we used formulas. It was interesting to see how formulas differ from reality. We had a formula for the blood alcohol concentration, and then we measured it with a breathalyzer.

The experience of authorship was highlighted by the students with reference to their collaboration and the resulting products in the form of reports or posters:

> We had some disagreements about the results from our experiments and which results to present on the poster. It was nice that we worked together on something which resulted in a product instead of just ending up in the bin.

The fourth explanation is that the product may help the students to identify with their knowledge and become accountable for sharing the content. A nod to the regularization of author content in such a way that the students eventually assumed authorship as a standard practice is found in the students' description of their experiences from the teaching sequence:

> We had to plan and make the measurements ourselves, and we gained a much better understanding. We knew many of the formulas from the mathematics and biology classes, but it was different anyway. After all, we had to prepare the tasks ourselves. Normally the tasks just appear in the textbook. We made measurements, diagrams, and we even found the best correlation.

However, although the students viewed what was learned in the teaching sequence as having continued relevance, there is no indication that the students expected authorship to become a general practice for generating new knowledge and engaging in adaptive problem-solving as stated in the fifth explanation.

Seeking evidence for the students' experiences of expansive framing, I noticed that the students experienced learning in preparation for transfer, connecting curriculum units from different disciplines across time, and taking on the role of someone who authors knowledge to generate reasonable responses to a problem.

9.6 Discussion

In my study, I found indications of students responding to and making sense of interdisciplinary settings by establishing a context based on several former learning strategies and experiences. They expected to continue to use what they had learned, created links between different settings, drew on prior knowledge, generated their own problems, authored content in reports, and expected to generate their own solutions to new problems. In addition to connecting settings, the promotion of students' authorship stands out as the main feature of expansive framing. The opportunity to explore a self-chosen problem through modelling activities gave the students a sense of ownership and positioned them as authors of their own ideas. This helped the students to engage in a context about vital capacity in which they could use the knowledge that they had come to identify with and construct a new setting in which they used their knowledge. The responses from the students participating in the study emphasize the point made by Lobato (2012) that transfer involves some experiences of similarities or sameness across situations.

The experiences from the study show that the students need opportunities to apply their knowledge across a broad set of contexts. In the actor-oriented transfer perspective, transfer is a distributed phenomenon across individual cognition, social interactions, material resources, and normed practices (Lobato, 2012). Thus, the more relationships students see between the learning and transfer context, the more likely it is that transfer will occur.

In this study, I conducted interviews with students in which they were asked to explain how they themselves addressed transfer situations, with the rationale of examining whether students involved in interdisciplinary model-eliciting activities experienced the five explanations for how expansive framing promotes transfer. I do not claim that the students experienced the five explanations. There were indications of expectation for future transfer; that the connecting of settings made prior content continuously relevant; and that the students became positioned as authors who shared their knowledge, making them more likely to contribute what they know more generally. I did not observe an emerging practice that would allow students to author content regularly. This requires continuity and regularity such that the students eventually assume authorship as a standard practice, and this was not achieved in a study based on a short intervention in a traditional monodisciplinary teaching culture. However, the study was conducted within the design-based research approach that is inherently a cyclic process of development and research in which the theoretical ideas of the designer feed the development of products that are tested in real classroom settings, leading to theoretical and empirical products and local instructional theories. Therefore, the question is which challenges should be addressed in the next iteration? With intercontextuality as the starting point, the productive role of contextual sensitivity in transfer (Lobato, 2012) should be considered. For students enrolled in a mathematics–biology study package, model-eliciting activities involving both disciplines offer a wide range of settings where the students can engage more frequently with contexts in which they can use the knowledge they have come to identify with, sometimes even helping to construct new settings in which they can use their knowledge. The emphasis should be on creating links back to prior learning contexts, which may encourage students to make use of transfer opportunities by *transferring in* additional examples and generalizations related to what they are learning about (Engle et al., 2012; Lobato, 2012). In my study, I observed weak indications of students' generalizing activity (e.g., that a situation in biology can be modelled with mathematics to obtain new biological information). The ideal is that the students generalize their experiences with identifying and structuring an extra-mathematical situation (e.g., in biology), mathematizing the situation to be modelled, analyzing and tackling the model, interpreting the results, validating the model, communicating about the model, and monitoring the modelling activity. To reach the ideal, structuring should be addressed as the active "process that occurs through an interaction of contextual affordances, personal goals, and prior learning experiences" (Lobato, 2012, p. 243).

9.7 Concluding Remarks

Making connections and transferring ideas to a new context are difficult processes that many students cannot accomplish on their own. In mathematics education, there is a broad acceptance that modelling provides opportunities for students to experience and discuss the role of mathematics and the nature of their models as they study systems taken from extra-mathematical reality. However, despite this acceptance, the potential for mathematical modelling to act as a "glue" between mathematical and extra-mathematical settings is only sparsely addressed in mathematics education. Issues of interdisciplinary teaching have been underestimated in mathematical modelling and the same goes for context in transfer research. Modelling makes sense of complex situations, and the purpose of the resulting models is to provide meaningful ways for students to construct, explain, describe, explain, manipulate, or predict patterns and regularities associated with complex situations. The significance of authenticity empowers students to use their mathematical knowledge proficiently to identify and solve problems located in personal, work, or civic contexts or other discipline areas and should be recognized as the starting point for positioning the students as active participants in a learning context where they serve as authors of their own ideas and respondents to the ideas of others. My point is that interdisciplinary model-eliciting activities offer learning environments where students are recognized as authors of ideas which are integrated into class discussions and other activities. Therefore, I consider these kinds of learning environments to be examples of expansive framing which encourage the students to regularly use what they already know with an expectation of continued use. I do not argue that interdisciplinary model-eliciting activities per se lead to transfer. My point is that expansive framing should be regularly paired with activities where the students critically evaluate the knowledge they have transferred in for its relevance and validity.

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Chapter 10
Transfer of Learning as Boundary Crossing Between Cultural-Historical Activity Systems

Vanessa Sena Tomaz and Maria Manuela David

The issue of transfer is recurrently present in discussions about learning (Detterman, 1993; Engeström, 2015; Greeno, Moore, & Smith, 1993; Lave, 1988). There are countless transfer conceptions, associated with as many learning conceptions, forms of cognition and of knowledge, liable to various criticisms according to the theoretical orientation adopted. Activity and situated-learning theorists (Beach, 2003; Greeno, 1997; Lave, 1988; Tuomi-Gröhn & Engeström, 2003; van Oers, 1998) have criticized theories of transfer that treat learning as a process of internalizing portable knowledge in the head of the individual. According to Kagawa and Moro (2009), activity theory formulates learning as changes in the holistic and invisible relationship among the individual, artifacts, and other people and expands the concept of transfer into the interrelations among collective activity systems. Tuomi-Gröhm (2003, p. 202), in particular, discussed a Finnish internship program for nurses, wherein school and workplace collaboratively interacted and changed, and proposed a view of transfer as an increase in collaboration between the two *activity systems* (Engeström, 1987), school and workplace, to develop new theoretical concepts and to apply them to solve everyday practical problems.

Following a similar view, although in a different context for transfer, Tomaz (2007) structured some school practices as activity systems (Engeström, 1987) to capture the complexity of the relations between three school disciplines. Articulating her analysis with the ideas of Greeno et al. (1993), who consider that the transfer process is influenced not only by the cognitive abilities of the person, but also by the cultural aspects through the interactions between people occurring historically in the environment, Tomaz (2007) elaborated a perspective on transfer which she considered more adequate to describe the students' learning in the school disciplines. According to this perspective, transfer of learning can be seen "as a social and historical practice, in permanent transformation, that may occur by a

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recontextualization process of the attunements and restrictions of actions within or between activity systems, in an environment" (Tomaz, 2007, p. 197).

In this chapter, we expand the investigation of transfer of learning made by Tomaz (2007), and deepen it theoretically, considering as a starting point, that transfer of learning is not only a cognitive ability; it is, above all, a practice structured across social-historical and cultural activity systems. In this, practices are seen as a set of interrelated processes of production of knowledge in particular settings, and learning is seen as a recontextualization process of new ideas, experiences, and procedures emerging from the practices themselves.

Because the discussion involves school activities that make connections with the students' daily life, the theoretical and methodological deepenings made in this chapter are guided by the following central issue: How do artifacts that are common to different school disciplines, when used and adapted by the students to deal with the contradictions that evolve in the activity system formed by these disciplines, facilitate the boundary crossing between activities configured within the school disciplines and reveal transfer of learning in this context?

To answer this question, we sought support in the theory of expansive learning (Engeström, 1987) as an alternative learning theory that enlarges the focus of analysis to multiple interacting systems, capable of capturing the improvement of innovation and change in collective processes. Accordingly, we adopt a perspective of learning transfer based in the *boundary crossing* theoretical concept (Engeström, Engeström, & Kärkkäinen, 1995) characterized by these authors as a "horizontal expertise where practitioners must move across boundaries to seek and give help, to find information and tools wherever they happen to be available" (p. 332).

We consider that this perspective on transfer of learning is different from the more traditional ones because it puts the emphasis on the ideas of expansion of the activity and of transformation of knowledge. It allows a more comprehensive approach to the process of transfer, capturing the complexity of and the interrelations among the processes involved, for example, by focusing on the subjects, individually and collectively; on the artifacts; on the power relations; and on the rules that compose an activity system, as explained in the next section. In this chapter, we focus initially on certain artifacts and their role in facilitating students' boundary crossing between three activities, configured within the disciplines of mathematics, Portuguese, and geography, to further exemplify how the activities reveal transfer of learning according to the perspective we adopt.

10.1 Expansive Learning as Boundary Crossing

In recent works devoted to analyzing classroom activities, we used activity theory perspectives (Engeström, 1987) to highlight the role that visual representations play in structuring mathematics activity and to discuss how school mathematical activity is modified when students' everyday situations are brought into the classroom (David & Tomaz, 2015; Tomaz & David, 2015). In this chapter, we adopt a similar perspective, so as to develop and refine an alternative perspective of transfer of learning across three elementary school disciplines (mathematics, Portuguese and geography).

The activity concept, according to Leont'ev (1978), represents a specific form of social existence that includes crucial changes of social reality. It emerges from a necessity, which drives motives towards a related object. To satisfy motives, actions are needed. These, in turn, are accomplished in accordance with the conditions that determine the operations related to each action.

According to Engeström's (2015) perspective, an activity is always understood as a collective phenomenon in a community, and individuals can only perform actions inside a larger system of at least two interactive collective activities,[1] which should be taken as the unit of analysis. Every activity system is characterized by its object that can be expressed as concern, motivation, effort, and meaning. To represent a collective activity system, Engeström (1987) offered a model composed of triangles and, in the nodes of this model, he placed the following components: subject, object, mediating artifacts, community, division of labor, and rules. *Subject* consists of an individual or group engaged in a common purpose whose agency is the focus of the analysis; *object* is the "problem space" towards which the activity is directed and which is molded and transformed into outcomes; *mediating artifacts* are instruments, tools, and signs; *community* refers to people who share the same object; *division of labor* is the division of tasks, power, and status between the members of the community; and *rules* "refer to the explicit or implicit regulations, norms, conventions and standards that constrain actions and interactions within the activity system" (Engeström & Sannino, 2010, p. 6). This activity system is one of the conceptual tools available to explain networks of interacting activity systems, dialogue, and multiple perspectives and voices.

Engeström's (1987) expansive learning theory is based on the cultural-historical activity theory, and involves the multidimensional treatment of the learner as an individual and as a community. Engeström and Sannino (2010) explained that the core idea of expansive learning is qualitatively different from the ideas related to the metaphors of acquisition and participation; it relies on the metaphor of expansion, in which learners learn something that is not yet there.

The expansive learning theory was based on the ideas of Russian cultural-historical school representatives such as Vygotsky (1978), Leont'ev (1978), and Davydov (1990) as well as on the work of Bateson (1993) and Bakhtin (1981). According to the theory of expansive learning, contradictions are the necessary but not sufficient engine of expansive learning in an activity system, and the process of expansive learning should be understood as the construction and resolution of successively evolving contradictions (Engeström & Sannino, 2010).

According to this theory, learning manifests itself primarily as changes in the object of the collective activity, which contrasts with the traditional perspective,

[1] Because, according to Engeström's perspective, every activity is formed by at least two individuals, from now on, for the sake of clarity and brevity of the text, when we write *activity* we are always considering it as an activity system.

wherein learning manifests itself as changes in the subject. Thus, contradictions are a key concept in this theory because they may produce changes in the object, expansive transformations of the activity, and new cultural patterns of the collective activity. Furthermore, an interesting stream of studies on expansive learning, mentioned below, takes into account the horizontal dimension of learning, that is, across activity systems. Engeström and Sannino (2010) emphasized studies in which the expansive learning is manifested in different ways. Among those, the studies on boundary crossing will be of a special interest for the purpose of this chapter. According to Engeström, Engeström, and Kärkkäinen (1995, p. 333), boundary crossing entails stepping into unfamiliar domains and is a creative endeavor, which requires new conceptual resources.

Moreover, we agree with a broader metaphor of the boundary-crossing approach because it is oriented towards both the personal and the collective (Engeström & Sannino, 2010). Akkerman and Bakker (2011) defined boundaries as "sociocultural differences that give rise to discontinuities in action and interaction" (p. 139). According to these authors' perspective, boundary crossing consists of the efforts made by individuals or groups at boundaries to establish or restore continuity in action or interaction across practices.

Consistent with the boundary-crossing process, artifacts have an important role, and they may form interlinked combinations that may be called *instrumentalities* (Engeström, 2007). Artifacts are not merely standard devices that could be transmitted to the subjects in a readymade form. To the contrary, "the design and implementation of *instrumentalities* is obviously a stepwise process that includes fitting together new and old tools and procedures as well as putting into novel uses or 'domesticating' packaged technologies" (Engeström, 2007, p. 33). These are evolving toolkits needed, created and used to deal with tensions and to support agentive actions (Engeström, 2015). Star, as cited by Engeström (1990), also used the notion of boundary objects, that is, "objects that are both plastic enough to adapt to local needs and constraints of the several parties employing them, yet robust enough to maintain a common identity across sites" (p. 190). According to Paavola and Miettinen (2018), boundary objects are usually interpreted as stable and concrete thing-like artifacts and allow interoperability and communication in the activity systems. For example, maps, models, forms, knowledge repositories, and graphic representations can perform the role of boundary objects and play an important role in the expansion of the object of activity and the crossing of boundaries, but they are not the object of the activity (Engeström, as cited by Vetoshkina, 2018, p. 118).

It is important to emphasize that boundary objects should not be confounded with the object of the activity; instead, we consider that they compose the *instrumentality* of the activity system as artifacts that may boost transfer of learning between activity systems. Therefore, like Tuomi-Gröhn and Engeström, as cited by Kagawa and Moro (2009), we understand that "transfer of learning takes place through interaction between collective activity systems" (p. 185).

Artemeva (2007), reviewing a collection of studies on transfer, suggested that researchers who adopt the framework of activity theory and expansive learning do not necessarily interpret transfer as the effect of a prior task on the subsequent task of the same level of complexity, but rather see it as a continual learning from one changing situation to another, a more complex one, and from one activity system to another. Tuomi-Gröhn, Engeström, and Young (2003) argued that "what is transferred is not packages of knowledge and skills that remain intact; instead, the very process of such transfer involves active interpreting, modifying and reconstructing the skills and knowledge to be transferred" (p. 4).

The prevalent studies from this perspective of transfer concern mainly vocational education and work. Engeström and Sannino (2010) brought forward some studies in which learning is manifested as boundary crossing. One of these is in the field of vocational teacher education, developed by Hasu and Engeström (2000), who observed that bridging the gap between the developers and the users may require new types of software tools.

Another study, by Tuomi-Gröhn et al. (2003), discussed a Finnish internship program for nurses, wherein school and workplace collaboratively interact and change. Bakker and Akkerman (2014) used the boundary-crossing approach with vocational students during work experience when they crossed the boundaries between school and work. They discussed how vocational students were supported to integrate the statistical knowledge, learned mainly at school, with work-related knowledge they developed mainly during internships. Similarly, Morselli (2017, p. 288) also discussed the role of enterprise education in vocational education, based on the studies on expansive learning that take into account the horizontal dimension of learning, that is, across organizations. The author argued that it is at the boundary where the need for dialogue and negotiation of meaning may generate new ideas, given that the movement from school to work is bidirectional, meaning that students bring their expertise when moving from school to work and vice versa.

The studies mentioned here on boundary crossing revealed several learning processes triggered at some practices' and/or activities' boundaries. In a similar way, in our study, we take boundary crossing as one form of manifestation of expansive learning to analyze productive ways to interact between activity systems within the school domain. Furthermore, because there is a very close relationship between transfer and learning in the activity-theory approach, we tend to agree with Säljö, as cited by Artemeva (2007), when he said that "there may be no need for a separate concept of transfer because it cannot be distinguished from the concept of learning" (p. 363). In what follows, we present some examples of boundary crossing, from our data, to show how they can reveal expansive learning and transfer of learning.

10.2 Data and Method of Analysis

10.2.1 Origin of the Study

We take a set of data collected on a sequence of classes analyzed in a larger study (Tomaz, 2007). This study involved three teachers from a public school in Brazil who taught mathematics, Portuguese, and geography. The data for the larger study were collected by one of the authors (Vanessa Tomaz) through participant observations in four classrooms (two seventh-grade and two eighth-grade classrooms, each with 35 students) for a period of 6 months. Data were also collected through interviews with students and teachers. We produced field notes and transcriptions of audio and video recordings for all empirical data in Portuguese. For the purpose of this chapter, a selected set of data was translated to English.

The three participants in this study were a mathematics teacher, Telma, a Portuguese teacher, Rosângela, and a geography teacher, Noêmia, who decided to collectively develop an interdisciplinary approach to the Water theme, involving the students of two seventh-grade classes (Class 1 and Class 2). The two groups of students were very similar in many respects. Both had mixed socioeconomic backgrounds, and the students' ages varied from 13 to 15 years. There was also no significant disparity in the students' performance on the mathematics tests or in the level of participation of the students observed in both classes. The teachers had vast experience (over 23 years for each teacher) teaching at the middle school level. In the interviews, the students stated that they were good teachers, and during our observations of their classes, we noticed that they usually succeeded in establishing open and amicable relationships with the students. For the most part, students participated actively in their lessons.

10.2.2 Method of Analysis

Aligned with the methods used in other works (David & Tomaz, 2015; Tomaz, 2007; Tomaz & David, 2008, 2015), the methodological approach used in this study was also grounded in ethnography as a logic of inquiry in education (Green, Dixon, & Zaharlick, 2003). According to this logic, the focus is on the process, and there are no strict protocols previously defined for the observations and interviews. Ethnography assumes that it is not possible to avoid a certain degree of subjectivity in the data collection and in the analysis, which is essentially interpretive. However, it is possible for the researcher to reach the required scientific rigor by carefully describing all the research procedures and by contrasting his or her interpretations with the other subjects' perspectives, for example, through interviews and discussion of the video records.

10.3 Water Theme as an Activity System

The study of the Water theme was proposed by the teachers of mathematics, Portuguese, and geography with the objective of promoting learning to make the students expand the meanings of the content studied in the school disciplines, especially with respect to the use of school knowledge in and out of school situations. Each teacher tackled the theme by proposing increasingly complex tasks, which required further knowledge beyond that required for their discipline. The proposals of the mathematics and Portuguese teachers were more focused on awareness actions and on the application of the disciplinary content. In turn, the geography teacher requested that students create elaborate proposals to solve the problem of water shortage in the world. The mathematics teacher initially proposed a study of the students' water bill, with the mathematical goals of applying the *cross-multiply and solve for x* strategy (Post, Behr, & Lesh, 1988), as well as *percentages*, content already being covered in her classes. As the work progressed, she proposed other problems of application of these content areas not directly related to the water bill but, instead, with other texts that also addressed the Water theme.

The Portuguese teacher discussed the theme and produced, with the students, different types and genres of texts about the Water theme. The geography teacher only started the discussion of the Water theme after it had been completed in the other disciplines. She introduced the role of the supranational organizations (i.e., the United Nations) in the current world conflicts and proposed a seminar involving the students as representatives of the different countries to debate the shortage of water in the world and to create proposals to solve this problem. From the moment that the theme was introduced in the three disciplines, each teacher followed their own planning, developing and proposing of activities on water within their planned content at different times and without formally meeting to discuss the classroom work. The communication between the teachers about the progress of work was made by the students themselves who would comment on or use in one class what they had done in another.

Thus, when the students were asked to perform tasks more closely related to disciplinary content addressing the Water theme, produce texts, make calculations, or even submit environmentally and economically viable proposals to avoid shortage of water in the world, the study of the Water theme became a great challenge for them.

Behind this challenge, there was a persuasive and persistent *contradiction*, namely that students had to be aware of excess water consumption and to create alternatives to avoid shortages, using the society in which they lived as a reference, and at the same time having to align awareness actions and creation of proposals to the assessment tasks according to the school content taught in each discipline. This illustrates how this primary contradiction, sometimes referred to as *use value* versus *exchange value*, manifests itself in education, and in this specific case. This contradiction is also manifested in the teachers' activity as they oscillated between two opposing forces: discussing the theme by focusing on the awareness actions or on the creation of consumption alternatives versus connecting the discussion of the theme with the disciplinary content to show its application in the students' lives. Furthermore, Engeström (2015) argued that contradictions in an activity system may exist on different levels (within the components of an activity system, between the components of an activity system, and between activity systems), and later, we show some examples of these from our data.

According to our interpretation, in the study of the Water theme, there was a network of activities in which subjects interacted and brought together dialogues and multiple perspectives and voices, which allowed it to expand into an activity system. This activity system constitutes the *unit of analysis* to be considered in this chapter. We consider that, in this unit of analysis, from now on referred to as the *Water activity system,* there was integration of ideas, tools, languages, rules, and concepts from the different disciplines involved.

According to Leont'ev (1978), the main characteristic that distinguishes one activity from another is its object because the object gives the activity a specific direction. In our case, the actions of the subjects (i.e., the students and teachers), when they participate in the tasks organized by the disciplines of the specific areas, are oriented towards the shared object, namely water and its shortage. Although students and teachers may be mobilized by different motives towards the shared object, as we will show, in the development of the activity, they redirect their motives, resulting in them being blurred at times.

The primary contradiction referred to earlier, inherent in the activities' objects in the educational field, radiates to all the components of the Water activity system, bestowing dynamism to this system and making it impossible to delimit it to a specific disciplinary field. This contradiction projects itself onto the activities developed within each discipline, as well as onto the boundaries of the activities that compose the Water activity system and even outside of it.

In the Water activity system—our unit of analysis—it is possible to identify a series of actions related to the same theme within the disciplines of mathematics, Portuguese, and geography. Zooming into this system, the set of actions related to each discipline can each be characterized as an activity subsystem (Engeström, 1987), forming a constellation of interconnected activities, A_1, A_2, and A_3, configured respectively in the mathematics, Portuguese, and geography disciplines.

Figure 10.1 represents a general scheme for the Water activity system, formed by a constellation of these three interconnected activities that share the same object (i.e., water and its shortage).

Given the complexity of the Water activity system, for the purpose of the present analysis, in Fig. 10.2 we specify all of its components, following the triangular model (Engeström, 1987). We also outline the interconnected three subsystems without losing their connection with the Water activity system: A_1 focuses on mathematics problems involving water (object: to solve mathematics problems on water); A_2 focuses on texts on water that raise youth awareness (object: to create a text on water to raise youth awareness); and A_3 focuses on projects to solve the issue of water worldwide (object: to create "scientific" proposals to solve the water

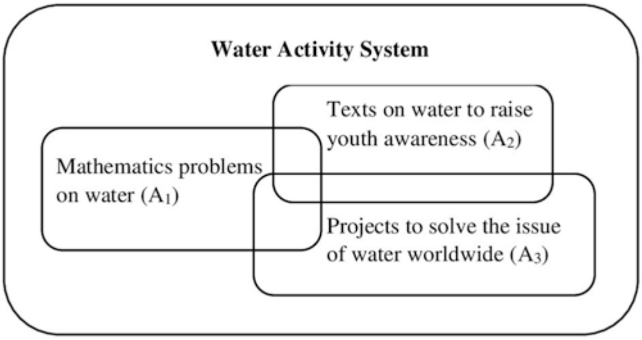

Fig. 10.1 The activity A_1 connects with the other two activities A_2 and A_3

shortage). In Fig. 10.2, the contradictions are highlighted by the two-headed lightning-shaped arrows.

All components of each system are summarized in Table 10.1. In the next sections, the discussion of some of these components will be presented to elicit the role of the artifacts in crossing boundaries between the three systems, and the expansions of the Water activity system as a whole, as a result of changes perceived in the objects.

10.3.1 Mathematics Problems on Water Activity System (A_1)

When Telma taught the topic of proportionality, she introduced the cross-multiply and solve for *x* strategy, which she called *rule of three*, to solve typical school problems involving proportions. Afterwards, to discuss the Water theme, she proposed a problem related to the students' water consumption bill. The students were supposed to find in their bills the necessary data to answer the following questions: How many days of consumption? What is the average daily consumption of the family? What is the average consumption per person? What is the average consumption per person per day? What will you do to save water in your home?

Besides the work with the water bill, the teacher proposed problems created from texts about water whose resolution demanded the use of the rule of three, percentages, and other numeric notions. One of the problems was created from the following small text retrieved from a booklet from *Campanha da Fraternidade*[2] (Conferência Nacional dos Bispos do Brasil [CNBB], 2003):

[2] *Campanha da Fraternidade*, or the Brotherhood Campaign, is an initiative of the Brazilian Catholic Church, which proposes an annual theme to be debated.

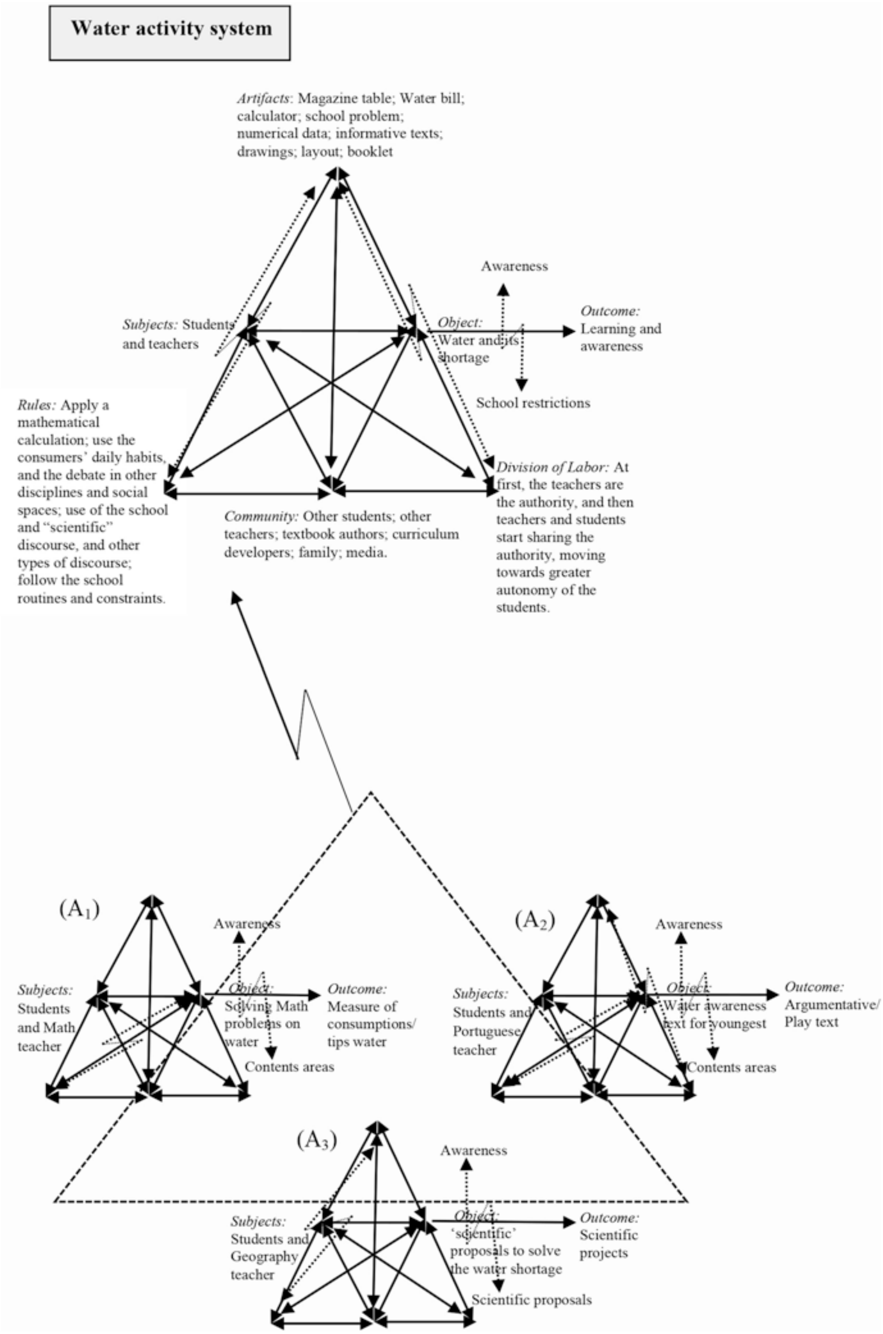

Fig. 10.2 The constellation of three interconnected activities composing the Water activity system

Table 10.1 Constellation of activity subsystems and their components

	Mathematics Problems on Water Activity System (A_1)	Texts on Water to Raise Youth Awareness Activity System (A_2)	Projects to Solve the Issue of Water Worldwide Activity System (A_3)
Object	**O_1** – Solving mathematics problems on water	**O_2** – Creation of a text on water to raise youth awareness	**O_3** – “Scientific” proposals to solve the water shortage
Subjects	**S_1** – Students and mathematics teacher	**S_2** – Students and Portuguese teacher	**S_2** – Students and geography teacher
Mediating Artifacts (Tools)	**T_1** – Water bill, magazine table, booklet, rule of three, questions done by math teacher; school problems, etc.	**T_2** – Magazine table, booklet, numerical data, drawings and graffiti, etc.	**T_3** – Layouts, maps, lab material, magazines table, booklet, other books and magazines, texts, ICT[a], drawings, etc.
Community	**C_1** – Other students; teachers of other subjects; mathematicians; textbook authors; curriculum developers, researcher, and COPASA[b]; parents; media; religious and civic organizations	**C_1** – Young people, teachers, school staff, parents, and other community members; teachers of other content areas; religious and civic organizations; researcher	**C_2** – Other students, teachers of other content areas, other school professionals, researchers, United Nations, international communities
Division of Labor	**D_1** – Starts with mathematics teacher and students sharing the authority and moves towards greater autonomy for the students	**D_1** – Starts with Portuguese teacher and students sharing the authority and moves towards greater autonomy for the students	**D_2** – Starts with geography teacher and students sharing the authority and moves towards autonomy for the students: They gain empowerment to act and evaluate their colleagues. The geography teacher only follows what they had previously agreed on
Rules	**R_1** – Use the water bill to extract the data; apply a mathematical calculation (cross-multiply and solve for *x*, simple division); compare your averages with the one shown by COPASA; determine the number of water consumers at home; use the averages previously found, the consumers’ daily habits, and the debate in other disciplines and social spaces	**R_1** – Assemble arguments from other classes of different disciplines; write following the standard language; organize and systematize information from different sources; use of the numerical data; use of visual representations; follow the school routines and constraints	**R_2** – Search for information about the physical, economic, political, and social aspects of each continent and include these aspects in an oral presentation; present at least five projects to solve the problem of lack of water in your continent; assume that, starting in 2025, the water could be rationed and that, after 2050, there could be a lack of water in the world; the proposals should have the format of “scientific” projects; they should be handed to the jury 3 days before oral presentation, etc.

Note. Adapted from Tomaz and David (2015)
[a]ICT – Information and Communications Technology
[b]COPASA – Companhia de Saneamento de Minas Gerais (Sanitation Company of Minas Gerais)

Table 10.2 Water consumption data from Pinho and Barros (2004)[a]

Activity	Time (min.)	Water (tap running; L)	Water (tap off; L)
Brushing teeth	5	12	1
Shaving	10	24	4
Washing up	15	117	20
Watering the plants	10	186	96
Washing the car	30	560	40

[a]*Note*. Translated by the authors from the original in Portuguese

> From a consumption perspective, 20% of the Brazilian population (35 million) do not have access to drinking water . . . 80% of the excess sewage is dumped into rivers. Around 105 million Brazilians live under a state of insecurity regarding the water they use. (p. 8)

The problem based on the text proposed by the mathematics teacher was the following: "According to the text what is the rate of Brazilian population represented by people with water insecurity?" (proposed in Class 1 and Class 2 on March 30, 2004).

The second problem situation was suggested by a table published in the magazine *IstoÉ* (Pinho & Barros, 2004), which indicated the consumption of water used to brush teeth, do the laundry, and so on (see Table 10.2). The teacher created problems involving percentages and rule of three.

The teacher showed her students this table, which was taken from the magazine, and asked them the following: "How much water would a person use if he kept the tap running while brushing his teeth? And how much water would he use if he turned the tap off?" (mathematics class on March 30, 2004, in Classes 1 and 2).

In the context of these problems, the development of the activity uses the codes, the textual genres, and the symbols typical of the mathematics discipline. Even though it is not a mathematical language in the format used by professional mathematicians, a school language of the content area is being constructed and defines the participation code of the students in their practices.

The school tasks involved in the mathematic problems on water are structured as an activity system because they have a *motive* to solve mathematics problems on water (the object), expressed through the actions that operate within the conditions established by the environment. In this case, according to our classroom observation and perceptions from the interview with the teacher, the activity motivation for the teacher was to solve problems on water so as to continue her awareness work on the water issue. Simultaneously, this activity allows the teacher to continue to propose possible applications of the rule of three and percentages in problems that are closer to the students' reality. This motive was expressed in her actions during reading, calculating, and registering the school-math methods to solve the problems. The actions operated in the use of artifacts that enabled the accomplishment of the activity. These artifacts, which can be represented by the table in the magazine, the booklet, the cross-multiply and solve for *x* procedure, the school problems, the water bill, and so on are different textual genres and work as resources to solve the problems.

The students also applied the cross-multiply and solve for x strategy, but their motives were mainly focused on the awareness and dissemination of water-saving tips.

In this context, it is possible to notice a manifestation of the primary contradiction in education, between object and rules, wherein the students have to deal with a situation that involves water consumption as a school problem and fulfill their student role, versus in a real situation, raising questions that could result in different ways to reduce the water consumption that go beyond the school situation (for example, leave the tap just a bit open, use just a glass of water instead of opening the tap). In this way, we understand that the mathematical problems on water configure themselves as an activity system (A_1), summarized in Table 10.1.

As explained next, the problems on water, solved by the students in the domain of Activity A_1, were later used as artifacts in the Portuguese class to create the texts that raise awareness among the students (A_2), as well as in the geography schoolwork to create proposals to solve the water issue in the world (A_3).

10.3.2 Texts on Water to Raise Youth Awareness Activity System (A_2)

During the discussion about the Water theme, several textual genres (informative, narrative, and argumentative) on the theme were produced in the classes of mathematics, Portuguese, and geography, keeping, in each case, the specificities of each content area. Particularly in A_2, the creation of these texts boosted the discussion of the Water theme, aiming to raise youth awareness of the importance of saving water. Each produced text is marked by a language situationally constructed, within and among the content areas, as shown in the examples below (see Fig. 10.3a, b).

The texts to raise awareness among the youth were proposed by the Portuguese teacher, whereas the students solved problems on water consumption in the mathematics classes. Her guidance was that the students should use the discussions on the other content areas to write their arguments, using scientific data. The following are guidelines written by the Portuguese teacher on the blackboard (April 1, 2004):

- Text on water
- Use the scientific data you have researched on water
- Produce a text to raise the awareness of young people on their importance to lead a change on water use
- Text structure:
  - Produce a text to young people
- Objective:
  - Raise their awareness on the influence they can have to change the posture and the habits of their families on the rational consumption of water.

Fig. 10.3 (**a**) Poster for the play by Class 2; (**b**) poster for the play by Class 2 (Tomaz, 2007)

– Argue on the risk of lacking water in the world.
– Comment the works done in the school.
– Present some measures to save water through rational use.

The teacher intentionally proposed a text to allow students to make connections to other texts and content areas, as can also be confirmed in the following excerpt from the interview:

V: What objective did you have . . . what did you want when you were proposing the water activities with the students?
Rosângela: What I really wanted was that, through reading, they had a pathway to research and noticed that this text dialogues with the others. (collective interview with the teachers on June 29, 2004)

The students of Class 1 wrote informative and argumentative texts aiming to raise readers' awareness of the water issue, which were read and discussed in class; the students of Class 2 wrote individual texts and collectively assembled a theatrical play that they presented in the closing of the work on water. In both types of texts, they used the information presented in the mathematical problems. The texts clearly presented more than a systematization of the discussion, information, and knowledge acquired by the students through the study of the Water theme. The objective of the two types of texts, dissertation and play, was directed toward youth awareness.

The products of the objective are exemplified below with excerpts from the texts of two students, Joaquim and Cássia, both from Class 1. In the first excerpt, Joaquim's text stated the following:

> In my school (and I believe in yours as well) they are working with this in Mathematics, Portuguese, Religious Studies, and other content areas. My Mathematics teacher, Telma, showed us a very interesting table with the help of Vanessa (UFMG). I will show you. (The text also included an image of the table students were shown in Table 10.2)

In a second excerpt from Joaquim, the text stated:

> There was a time we brought some texts on water. I brought a text called "Manifest of water" that informs about the water, the water of the planet, and the work of project "*Manuelzão*". In it we have some information: "The Earth surface is covered by ¾ of water,

> 97% are oceans, 2.7% are polar glaciers, that, when melt become salted water. Thus, 0.3% are river waters.

Cássia also referred to the data presented on the mathematics problems, as shown in the following excerpt:

> On a consumption level, 20% of Brazilian population (35 million) have no access to drinkable water. I think that 100% of the Brazilian population should have access to drinkable water, not only 80%. In many content areas at school I did some works on how to save water and one of them showed a table like this.

This student proceeds with her text, reproducing the same table of water consumption from an article titled "Água Enxuta" in the Brazilian newsmagazine *IstoÉ* (Pinho & Barros, 2004).

In Class 2, even though the theatrical play is a different kind of text from the one used by the students of Class 1, the numerical data used for the play were the same as used in Class 1. These numerical data, taken from the booklet from *Campanha da Fraternidade* and the *IstoÉ* magazine table, strengthened students' arguments with evidence of percentages of river water, ocean water, and potable water on Earth, and this evidence was added to the posters created to represent the play scenarios.

We consider that the production of texts to raise youth awareness by the students and Portuguese teacher (subjects) organizes itself as an activity system, Texts on Water to Raise Youth Awareness (A_2), whose components are depicted in Table 10.1.

In A_2, the subjects (students and Portuguese teacher) guide their actions in the creation of a text on water to raise youth awareness (object). These actions are mediated by different artifacts (numerical data, magazine table, drawings, booklet) and according to rules (i.e., assemble arguments from other classes of different disciplines, write following the standard language, organize and systematize information from different sources, use of the numerical data, use of visual representations, and follow the school routines and constraints) so that the results of the activity system are texts, both argumentative text and text in the form of a play.

As seen in the examples presented from Class 1, the guidelines written by the Portuguese teacher on the blackboard played a fundamental role in the systematization and organization of the information of the students' texts to target a specific segment of the public, whereas the play produced by Class 2 reflects the engagement of the class in a collective action to raise youth awareness and their option of using the visual appeal as a tool to convince other youth.

As the teacher asserted, the discussion on water in the Portuguese classes had the following motives:

Rosângela: I focused on the importance [of the theme] which is evident to all of us . . . even considering the lack of it [water] . . . also to encourage the student to search information . . . when he wants to know . . . to learn something more . . . that he could such different sources, supports . . . in different materials. . . . (Individual interview with the Portuguese teacher on June 20, 2004)

In Class 1, the students interacted to discuss their own texts and assembled their ideas in individual text productions for use later in their Portuguese class. In Class 2, to produce the play, the students needed to interact in a way that allowed the group to reach the common goal, the theatrical play. In this effort, we can see that the written text was insufficient for the purposes of the task at hand, leading to other types of texts, such as the posters created by the students.

Although in both cases the students tried to produce their texts following the teacher's rules, at times they made choices, individual or collective text, argumentative text or a theatrical play, and the added new rules, such as the use of visual representations. We perceived that these changes introduced by the students promoted a less hierarchical and more horizontal division of labor in the classroom because, in both classes, they did not restrict themselves to just following the guidelines of the teacher. Especially in Class 2, the option for the play, by combining different languages, required a more complex work organization and power negotiations among colleagues and with the teacher.

We perceive that the creation of an argumentative text and the play were motivated by a contradiction manifested in the current educational activity between object and rules: producing texts to raise youth awareness using a language that best mobilizes the young person—for instance, a play or a leaflet with numerical data—versus producing texts to show the mastery of linguistic tools attuned to the content covered in the discipline.

Another contradiction arises in A_2 between artifacts and division of labor. When the students of Class 2 opted to produce a collective play, there was the need for other textual supports to favor the visualization of arguments capable of convincing young people of the problem. The students used the magazine table and numerical data from the Mathematics Problems on Water activity (A_1), which worked as boundary objects for activity A_2. These artifacts allowed for the visualization of the arguments needed, as well as for the general rules for text production established by the teacher and the appeal for awareness. That is, to face this contradiction, students crossed content-area boundaries, gained power of action, and expanded their participation in the Water activity system by changing the use of the magazine table and numerical data. These were not used as a source of data to calculate but as a reference to build the text arguments.

It should be noted that some artifacts used by the students in this activity system, A_2, which were also artifacts in A_1, were used differently in the two activity systems. For example, in A_2 (Portuguese), the table represented on the posters was paired with other images, highlighted for making comparisons with the information in the photos (Fig. 10.3a, b),but also complemented by them. In A_1 (mathematics), however, the table worked as a data base for the students to solve the proposed problem. In the A_2 texts, the same table was used to give numeric arguments used to convince youth through different textual supports.

10.3.3 Projects to Solve the Issue of Water Worldwide Activity System (A_3)

The idea for the geography work also originated in the initial teachers' meeting, when it was suggested that a simulation of the United Nations Security Council could be conducted to discuss the water problem in the world.

To develop this work, some of the students in each class were divided into groups representing different continents. There was a special group, called by the students the "jury group", that played the role of the countries' members of the United Nations Security Council. The development of the work was registered in a notebook under the responsibility of the general secretary of the jury group. This group was also in charge of establishing and legitimizing the rules to be followed by the other classmates and of evaluating their work. The jury group was solely composed of students from Class 2 because, according to the teacher, they were the ones who first showed interest in participating in this group.

The guidance given by the teacher was that each group should show research on the physical, economic, political, and social aspects of a specific continent and, considering that the people who would watch the final work presentation may not know much about the continent the group was supposed to represent, the students should give them a general view of this continent, including its water status. Paired with this description, the students should publicly present at least five solution projects for the water shortage problem on the continent. All proposals were to assume that, starting in 2025, the water could be rationed and that, after 2050, there could be a lack of water in the world, according to the information from the booklet from *Campanha da Fraternidade* (CNBB, 2003):

> 'In 2050 when we will have 3 billion people more, we will need 80% more water for human use; and we do not know from where this will come.' This scenario is dramatic, as it clearly endangers the survival of human race and of a great part of beings . . . . (p. 5)

The teacher's guidelines worked more as a set of references for students than as actual rules dictating the work execution.

From the beginning, the students were engaged with the work and, at a certain point, the jury group decided to establish some guidelines to elaborate the projects that would be presented by their classmates, as well as the oral presentation the groups would have to do. They also determined that the projects should have the format of *scientific* projects with concrete solutions to the water shortage problem, rather than just raising awareness of a future problem. They determined, for instance, that each group should present five proposals to solve the water issue on the continent, to be handed to the jury three days before the oral presentation. In this presentation, all group members should explain their projects without using their notes, and they should be capable of discussing their projects with the jury members. Thus, during the work, the jury group members played, in the classroom, the roles of advisors and evaluators of their classmates' work. They checked with the groups the progress of their work; they studied and analyzed, in advance, their classmates' projects and offered criticisms and suggestions on the ideas presented, leading to modifications of the presented projects. All these initiatives were countersigned by the geography teacher, who followed the students' movements without directly interfering. According to Engeström and Cole (1997), we can say that the jury group promoted a situated interventionism making use of selective discoordinations, which become "a tool for revealing and traversing zones of proximal development at both individual and collective levels" (p. 308).

Faced with the demand and the belief that "scientific" projects would be able to solve, definitely, the lack of water in the world, the students, using different resources (layouts, maps, lab material, magazines table, booklet, other books and magazines, texts, ICTs, etc.), presented the following projects: construction of dams, excavation to find water tables, searches for new technologies, implementation of environmental policies including taxes, desalinization of sea water, *faggara* (underwater aqua-ducts), rainwater harvesting, siphoning, and water reuse. All the groups' proposals, except for one from a group in Class 1, were considered by the jury group as pertinent to the continent represented by the group and adequately justified according to physical, economical, and social characteristics.

Thus, in the geography classes, the students were guided by the jury group to create *science-based projects* able to solve the serious problem of lack of water, and the discussions of awareness, held by the students in the other content areas, were relegated to a secondary level of importance. This change of focus was stimulated by the teacher, despite not being a requirement in her demand for scientific projects, as will be seen later. Therefore, together, the students' and the geography teacher's actions were driven towards the object, "scientific" proposals to solve the water shortage. As in the mathematics and Portuguese classes, the set of actions observed in the geography classes can, during their turn, also be seen as an activity system, and that we labelled Projects to Solve the Issue of Water Worldwide (A_3; Fig. 10.2), involving the students and the geography teacher (i.e., the subjects) in the creation of solution proposals to solve the water problem in the world, mediated by artifacts (layouts, maps, lab material, magazines table, booklet, other books and magazines, texts, ICTs, drawings, etc.). To follow the rules established for this activity system (Table 10.1), the subjects' actions should, during their turn, consider a broader community, involving other students, teachers of other content areas, other school professionals, researchers, the United Nations, international communities, and so on.

The role of the jury group deserves special attention because, as the activity unfolds, its members alternate their roles and the division of labor in the activity system. Sometimes they assumed the responsibility of leading the work of their colleagues, acting as if they were teachers. However, while the other students were preparing to represent "their" countries and continents, the jury members also deeply researched them, acting more like the other students. Moreover, as they interacted with several groups, they learned about several research projects and about their colleagues' proposals, gathering arguments to conclude on the viability of different "scientific" projects and gaining empowerment to act, again, more like teachers in the evaluation of their colleagues' projects. The following are excerpts from a group interview with the jury:

José: As they were saying things, I tried to see in my house if there was any proof for it . . . then I tried to find other projects about it . . . you know?
V: Ah . . . so they also looked for those things there . . .
José: They searched their way . . . I don't know how . . . they did it but not very scientifically . . . they did things for themselves . . . they established their projects . . .
V: But things like . . . desalination . . .
Soraia: Desalination they took from one of my projects . . .
V: What type of things did they do by themselves?
José: To melt ice caps . . .
Alan: Then we would ask "how will you melt the ice caps?" then they would say "ah, I don't know," we will take it and send . . .
José: Melting a bit and putting the water in a ship . . . remove it with a water truck . . .
Alan: Things they would make up . . . .
Geraldo: It seems they didn't make any research!
Alan: Then Gerson reproved two . . . artificial rain and this one about melting . . . (Group interview with the jury, June 6, 2004)

The coordination work implemented by the jury group succeeded in making sure that almost all students shared the same purpose of gathering projects and arguments to show the viability of their proposals to solve the water issue worldwide, but with different strategies. In the specific case of the jury group, the cohesion of the members around the "scientific" character requirement for the projects was clear. The effort to make their classmates' research turn out as scientific possibilities was reinforced when they eliminated beforehand the proposals that they thought did not attend to this requirement. With this effort, they shared behaviors, languages, habits, values, and tools used by the members of the community, embodying an activity system with a group of students in a strong position of authority.

In A_3, the poster layouts used to present the solutions for the water problem, served as the artifacts that could better represent the students' projects in a scientific format, as demanded for by the jury (see Fig. 10.4).

When the jury group assumed leadership and guided the discussion towards the solutions, and no longer towards the awareness of the problem of water, the students had to change the direction of their actions to fit the new rules of the activity. However, a group from Class 1 kept the focus on awareness, and this was not well evaluated by the jury (see Fig. 10.5).

Later on, during an interview with the jury group, they explained why the work of this group of Class 1 (i.e., Fig. 10.5) was not well evaluated by them:

Alan: Do not take a long shower . . . .
Soraia: Everyone already knows that . . . .
Alan: It is worthless . . .
Soraia: It is worthless . . . I know that since I was born . . . but it is worthless nobody does that . . .

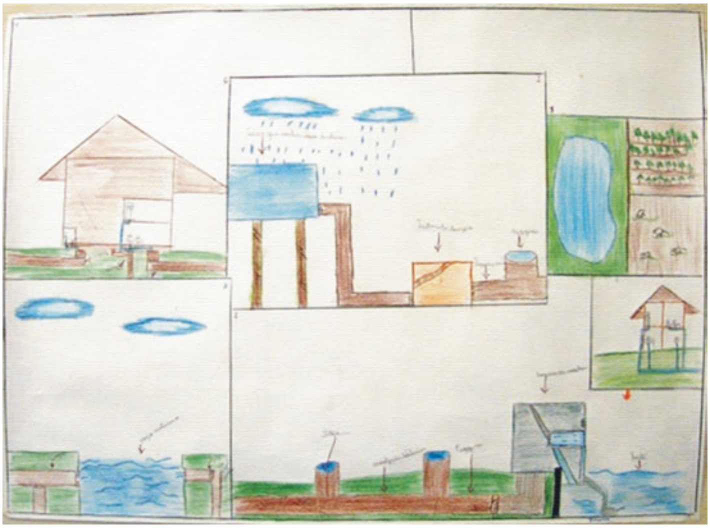

Fig. 10.4 Layout of the project of water treatment in Africa from Sebastião, presented in Class 2 of the geography class (Tomaz, 2007)

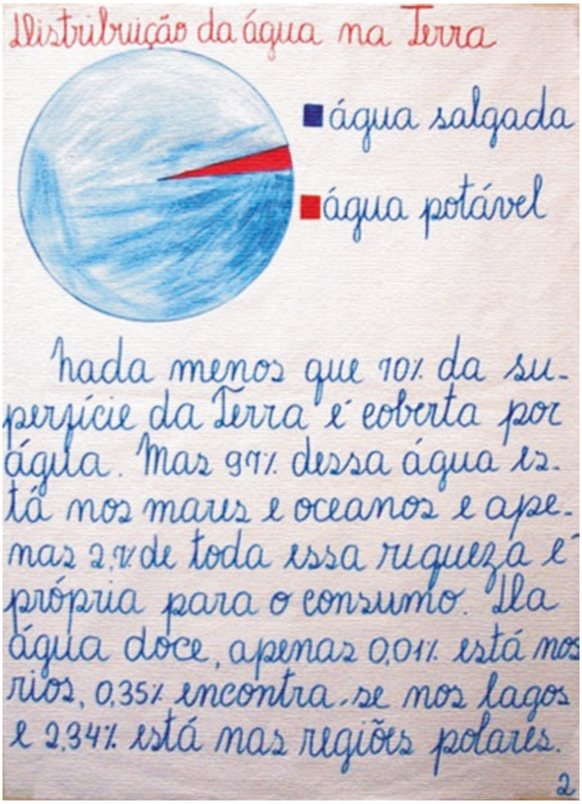

Fig. 10.5 Poster presented in Class 1 of the geography class (Tomaz, 2007)

José: That is what they did . . . save water . . . preserve water . . . treat water . . . are things . . . everything that people knows . . . then after I . . . they didn't propose anything . . . we let them present . . . in the end I asked them " in places with no water in Europe, what will you do?" . . .

Alan: Because everyone already knew . . . and the things nobody knows were worthless until now . . . so much that their work was about Europe but they were saying things about Brazil . . . (Group interview with the jury group, July 6, 2004)

In this excerpt, the jury argued that they had changed the focus of actions, which were now guided towards the creation of scientific projects with a broader geographical reach (the world), but the group of Class 1 was only saying things that "everyone already knows" and "saying things about Brazil."

The requirement that the students should present scientific projects to solve the lack of water in the world, challenged students to convince the jury group that their projects were really viable and efficient, but did not eliminate the need to raise awareness of the water issue. This idea was reinforced by the teacher during an interview after the end of this work:

Noêmia: They are proposals that everyone knows . . . for example . . . I'll tell someone that he should wash his car with a bucket instead of a hose . . . everyone knows . . . but washes it with a hose . . . I mean, it is not because the person doesn't know that . . . that they can save water this way . . . but he has no interest . . . he doesn't have this awareness . . . so to work on [people's] awareness is much worse than to use something that doesn't depend on the will of people. (Individual interview with the geography teacher on June 15, 2004)

Behind this challenge, there was a persuasive and persistent contradiction manifesting itself between artifacts and rules, which can be restated as follows: Maintain the discourse of awareness to reduce water consumption, even though the people know what they should do, versus completely abandon the awareness actions and instead try to create "scientific" projects able to produce more water, which could lead the classmates to relax in their actions to reduce consumption.

This contradiction becomes more evident in activity A_3 when the students are requested to represent their solutions in the form of scientific projects, which is a new rule in this activity. Given this rule, the students' actions oriented towards the activity object—"scientific" proposals to solve the water shortage—and demanded mediating artifacts that effectively communicate how science driven by technology could support projects on reuse of water or exploration of new sources, where there is no space for doubts and daydreams but, instead, a call for what is concrete, quantified, and classified, and for hypotheses verified through the scientific method (Rosa, 2012). From this perspective, the eagerness for scientific solutions aligns with people's welfare needs (i.e., the needs to consume and make use of water without restrictions) because globalization reconfigured the use of the space to transcend the geographical and to incorporate cultural, ethical, and behavioral aspects of the people. Thus, for the scientific projects to be really effective, they must break with the local reality and aim for a global reach, by being presented as a remedial alternative for people's lack of awareness of the water problem and by providing an immediate result for the global society.

The rejection of a group proposal by the jury can be seen as a way to reinforce the new object of the activity and sheds light on the discoordination of actions in this group during this activity. However, the jury group worked as a crucial boundary-crossing change agent, carrying, translating, and helping to implement new ideas between the educational institution and the out-of-school society (Engeström & Sannino, 2010, p. 13).

The geography teacher's belief that new demands were established by the jury group reinforces our perception of the manifestation of the contradiction between artifacts and rules in A_3:

Noêmia: I think they looked for more elaborate proposals . . . I think this is because they would have to discuss with the jury . . . so if they stuck to something too simple . . . "what will they evaluate? You know?" . . . if the judges will evaluate the work . . . by the discussion they would have to present something more . . . so that is when they researched . . . and sent those complicated proposals . . . they did it . . . (Individual interview with the geography teacher, June 15, 2004)

In activity A_3, once again, the artifacts mobilized in activity A_1 (the magazine table and the booklet from *Campanha da Fraternidade*) are mobilized to compose the scientific projects, the outcome of A_3, which were mainly focused on people's awareness.

10.4 Transfer as an Outcome of Boundary-Crossing Processes

The analysis of activity systems A_1, A_2, and A_3 revealed some mediating artifacts common to all three subsystems: magazine table, booklet, numerical data, drawings, and texts. However, these artifacts were used in different ways in each subsystem.

When we *zoom out* and consider the Water activity system itself, we can imagine how these common artifacts act as boundary objects in this system, supporting the subjects' actions—mainly those of the students—which are targeted towards the discussion of the Water theme. Although each subsystem has its own object, when we focus on the Water activity system, we realize that its object is shared by the three subsystems that compose the Water activity system. Boundary crossing between A_1, A_2, and A_3 takes place because relations are established between them, ensuring the continuity of the actions directed towards the object, water and its shortage, and avoiding the system's disintegration.

These relations are facilitated by the multiple voices that hybridize the system, which is characteristic of a horizontal movement of learning and development within and between activity systems, taking the dialogue as a search for shared meanings (Engeström, 2003). The reorchestration of these multiple voices is facilitated when they are seen in their historical context because, as Engeström (2015) argued, "An activity system is by definition a multi-voiced formation" (p. xxiv).

Thus, in the unfolding of the Water activity system, we perceived at various times changes in the students' flow of actions, without losing sight of the dialogue between them. This horizontal aspect of learning in the activity system aligns with the idea of boundary crossing as a powerful lens for analyses of sideways interactions between different actors and activity systems (Tuomi-Gröhn & Engeström, as cited by Engeström, 2015, p. xxiv). Next, we discuss one of such flows of actions within the Water activity system, highlighting the role of some boundary objects.

Initially, in A_1, the focus of the subjects' actions was on the awareness of the excessive consumption of water in the family when they made calculations of some water consumption averages using the rule of three and the numerical data from their own water bill, the table from the magazine, and data from the booklet from *Campanha da Fraternidade*. In A_1, numerical data, the water bill, the magazine table, the rule of three, and the booklet were some (but not all) artifacts used by the students to support their actions. In A_2, the actions were also directed towards awareness, but now to that of the youth, and the students created different textual genres (informative, argumentative, and a theater play). For this, they used several artifacts, among which were numerical data, the magazine table, the booklet, drawings, magazine snippets, and posters. They used numerical data from various sources in their drawings to inform water consumption measures, included in their folders and posters the percentages of water presented in the booklet, and used the magazine table in the theater scenario. All those artifacts were used in A_2 to support the arguments for increasing youth awareness. Finally, as the focus and the geographical coverage of A_3 changed, the students proposed scientific projects to solve the serious problem of water in the world. They also used drawings, in which they included numerical data about water consumption, the magazine table to show how much water is spent on everyday activities and how the water treatment process works, and layouts to visually represent the flow of their projects.

Actually, within activity A_1, whose object was "solving mathematics problems on water," two artifacts (the *Campanha da Fraternidade* booklet and the magazine table) took a central role because all the problems referred to them and the numerical data required to make the calculations were to be extracted from them.

As shown before, in the texts and drawings produced by the students in A_2, the artifacts of activity A_1 (booklet and table) favored boundary crossing between activities A_1 and A_2 because they were used in both of them. However, in A_2, the texts were produced to discuss, search for alternatives for, and raise awareness of the water issue among youth. Now, we have an activity in the discipline of Portuguese that interacts with the activity of mathematical problem-solving on water, in the discipline of mathematics, through several common artifacts. However, in A_2, the role of the booklet and of the table was less relevant. For example, the table appeared just as an illustration on a poster and an excerpt of the booklet was cited in some student work.

In contrast, in the activity proposed to solve the water shortage in the world (A_3), not all groups produced "scientific" projects, which was one of the rules of the activity. One group produced folders appealing to reduce water consumption, which was denied by the jury group, showing a lack of coordination of actions in this group towards the object of the activity A_3. To create the posters for the geography class, the students of this particular group also wrote texts using information from the booklet, the magazine table, and the numerical data from the mathematical problems on water (A_1), which were also used in the awareness texts during the Portuguese classes (A_2). But, according to the jury, the approach used by this group was just going back to the awareness proposals and not going further to the solutions projects. They discarded it, together with the artifacts they had used, for not being attuned to the object of A_3.

Despite this, it is possible to say that all students remained engaged in the same activity, the Water activity system, and that the artifacts, the *Campanha da Fraternidade* booklet and the magazine table, crossed the boundaries between activities A_1, A_2, and A_3. The relevance of these artifacts, according to the objects of the three activities, diminished to the point that, in A_3, they were considered inappropriate, although still present in the classroom discussions. Therefore, we argue that these two artifacts acted as boundary objects in the Water activity system. Of course, they were not the only ones to act as such in the Water activity system; there were others, such as the numerical data and drawings, that could also be subjected to the same analysis.

There were also *new artifacts*, introduced in A_3, such as the projects' layouts, maps, lab material, and other books and magazines, which can be seen as innovations in the Water activity system associated with an expansion of the object of this activity, by incorporating new solutions for the water problem beyond people's awareness.

Therefore, we consider that the set of common artifacts—magazine table, booklet, numerical data, drawings, texts—was sufficiently robust and plastic to operationalize, together with other artifacts, the actions of the subjects in the activity systems A_1, A_2, and A_3. These characteristics guarantee the communication between the three activities, keeping the assemblage of the Water activity system, even when there were innovations in its development, exemplified by the activity system A_3. So, we can say that these artifacts acted as boundary objects (Star, as cited by Engeström, 1990, p. 190), composing a necessary toolbox, created and used to deal with the contradiction manifested in the Water activity system and to support greater power of action of the subjects (i.e., acting as an *instrumentality* [Engeström, 2007] of this activity).

As mentioned before, to analyze the Water activity system, composed of three subsystems (Fig. 10.2) and its ruptures and internal contradictions, we used the expansive learning theory (Engeström, 1987), which moves the scope of learning from the individual to the relations between multiple activity systems. According to this theory, transfer of learning takes place through interaction between collective activity systems in which boundary crossing works as a tool for promoting transfer of learning. In the case analyzed in this chapter, the three subsystems were configured within three different school disciplines. Therefore, to reach their common object, related to the awareness of and solutions for the water shortage worldwide, the students needed devices and artifacts to connect the content areas, even if they were from different domains. And, as we could see, some artifacts (especially the magazine table and booklet) performed as boundary objects (Engeström, 1990) and facilitated boundary crossing between activities A_1, A_2, and A_3. The presence of boundary objects contributed to the non-disintegration of the Water activity system.

The analysis of the activity system composed of A_1, A_2, and A_3 allowed us to understand the interactions between the three activities connected to the Water theme without losing the focus on any one of them. This system composed of school activities, in turn, interacts with real-world issues, connecting the school with society through its community.

The perception of the connections and boundary crossing between the three activity systems (A_1, A_2, and A_3) led us to understand how the primary contradiction in education was manifesting in the object of the Water activity system. The boundary crossing among the activities within the Water activity system, boosted by the manifestations of the contradictions among its components, influences a series of actions mediated by artifacts that act as boundary objects among activities. Figure 10.6 shows how A_1, A_2, and A_3 are connected with each other through the artifacts.

The proposal to discuss a theme (water) as a way to approach school content and raise students' awareness of a social problem (water shortage worldwide) faced successive contradictions inherent to the educational activity because, during its course, the awareness actions were put to examination. This triggered successive boundary crossings between A_1, A_2, and A_3, leading to an expansion of the Water activity system object. Supported by Engeström and Sannino (2010), we can say that these boundary crossings boosted a new level of understanding about the water problem and its shortage, therefore leading to expansive learning in the Water activity system.

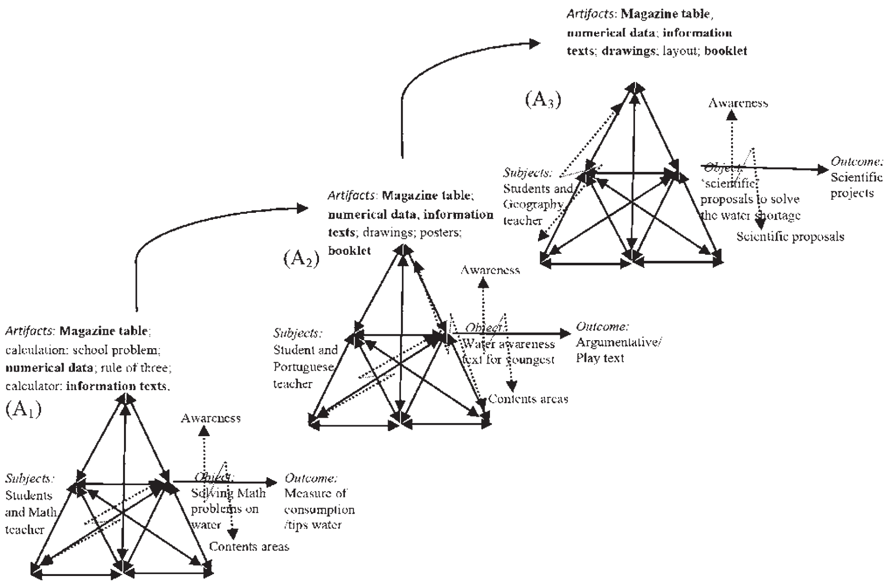

Fig. 10.6 Boundary crossing between activity systems

In this chapter, our perspective on transfer of learning was grounded in the expansive learning theory (Engeström, 1987) and the analysis performed on the Water activity system showed that successive boundary crossings between activities boosted expansive learning in this system. On the other hand, Tuomi-Gröhn and Engeström (2003) associated the idea of transfer with "an outcome of boundary crossing in an expansive learning process" (p. 185). Therefore, we agree with Säljö, as cited by Artemeva (2007), that there is such a close relationship between expansive learning and transfer of learning that they cannot be differentiated from each other. Consequently, it is possible to say that we have also identified transfer of learning in our data because transfer is not being considered as a simple transportation of knowledge from one activity to another, but as the formation of a new pattern of activity oriented to the object, namely water and its shortage.

10.5 Final Remarks

In this chapter, we analyzed an activity system called Water, composed of three other systems, developed in different content areas (mathematics, Portuguese, and geography) and related to the same object: water and its shortage. We showed how some artifacts used and adapted by the students to deal with the contradictions that evolve in the activity system acted as boundary objects and facilitated the boundary crossing between activities configured within the school disciplines.

The historical development of the activities that compose the Water activity system was not linear nor orderly; it was established by internal contradictions within each activity and between activities. In the configuration of this activity system, some artifacts worked as boundary objects favoring boundary crossing between activities, in turn leading to expansive transformations of the system, that is, to expansive learning on the Water theme. The students showed a broader and deeper understanding of the water issue, as highlighted by the student Tereza in the following statement: "This study we did was really worthwhile . . . many things I didn't know and now I do . . . learning . . . so it worked out to raise my awareness . . . because I had no idea it was so serious . . . ."

We conclude then that the perspective of expansive learning, adopted in this chapter, provides valuable theoretical and methodological tools to analyze transfer of learning within a school context. Because we make a strict association between transfer of learning and expansive learning, our conception of transfer carries characteristic aspects from expansive learning, which put the primacy on the following: communities as learners, not on individual learners; transformation and creation of culture, not on transmission of knowledge; and process of horizontal movements and exchange between different cultures' contexts, not on a process of a vertical improvement along some uniform scale of competences (Engeström, 2015). This implies that transfer of learning is a developmental process that includes construction or transformation of knowledge, identities, and skills rather than the application or use of skills that have been acquired elsewhere.

In conclusion, although the analysis was performed within a system of school activities, this perspective on transfer of learning helped us to show how it is possible to generate a kind of learning that penetrates and grasps pressing issues that humankind is facing today and will face tomorrow, such as a new understanding about water and its shortage worldwide.

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Chapter 11
Teaching and Assessing for Transfer from Block-to-Text Programming in Middle School Computer Science

Shuchi Grover

11.1 Introduction

There is a growing movement worldwide to teach computer science (CS) and programming (or coding) to all students as part of their K–12 education. There is broad consensus that teaching CS and coding should be aimed at preparing students for a future that will require these skills in all Science, Technology, Engineering, and Mathematics (STEM) careers as well as non-STEM careers (White House, 2016). Programming is central to most introductory CS curricula. It is seen as a key literacy for the twenty-first century and an essential means to develop computational thinking (CT) skills (Grover & Pea, 2013, 2018), a means for self-expression (Resnick, 2012), and a tool for social participation (Kafai, 2016) as well as computational action for the socially inclined "do-er" (Tissenbaum, Sheldon, & Abelson, 2019). Often, introductory programming experiences are situated in easy-to-use and engaging block-based programming environments including Scratch, Alice, Snap!, and MIT App Inventor, among others. These environments are designed to lower the barrier to learning programming through affordances that eliminate troublesome issues of syntax. However, programming in postsecondary learning settings and professional contexts involves text-based programming languages such as Python, Java, Javascript, C, C++, and Scheme, among others (Chan, 2019; Guo, 2014). Clearly, early experiences in block-based environments should ease the path for future learning in text-based environments in college and beyond. It has been observed that the transfer from blocks to text does not happen easily or automatically (Weintrop & Wilensky, 2017). Few studies in the realm of computing education have focused on the transfer of learning from visual block-based environments

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to text-based ones, and none have been conducted in middle school settings. The research and ideas presented in this chapter point to strategies that can be effectively adopted for mediating transfer and deeper conceptual learning in middle school as well as other levels of K–12 education. The broader strategies presented can also be applicable for closely aligned fields like mathematics and science.

This chapter describes research that examines design of introductory block-based programming experiences in Scratch aimed at preparing middle school students for future text-based programming. The designs were informed by new and old approaches to fostering and assessing transfer. The curriculum worked to help students build an appreciation of the varied careers in computing as well as a conceptual understanding of core algorithmic concepts to aid successful transfer to future programming contexts. The "expansively framed" (Engle, Lam, Meyer, & Nix, 2012) curriculum was designed to foster conceptual understanding and transfer to text-based programming and also included the use of multiple examples, analogous and multiple representations of computational solutions, and cognitive apprenticeship through modeling and worked examples. A unique "dynamic assessment" was designed to assess for transfer to text-based programming. This chapter details curricular design features and presents empirical research conducted in middle school classrooms. Findings from the two iterations of this design research should inform future inquiry in this space.

11.2 Research Framework

11.2.1 Approaches to Tackling the Thorny Issue of Transfer

Transfer of learning, or the application of something that has been learned in one context to a future context, is a key goal of formal learning and education. Without it, what students learn as part of formal school experiences would have little effect on the rest of their lives (Engle et al., 2012). The preponderance of studies in education literature, however, suggests that appropriate transfer takes effort on the part of the curriculum (Pea, 1987). The seminal literature on how people learn (Brown & Cocking, 2000) points to several critical features of teaching and learning that affect people's ability to transfer and suggests ways to facilitate transfer. Although there is no single prescribed strategy for fostering learning for transfer, suggested instructional strategies aim to help students *assemble new mental platforms for subsequent learning*. A focus on building conceptual understanding is a way to achieve this. Additionally, Pea (1987) argued, "Successful studies for teaching thinking skills for transfer have been explicit in describing for learners the need for and purpose of these new learning activities (e.g. Bereiter & Scardamalia, 1986; Palincsar & Brown, 1984; Pressley et al., 1984; Schoenfeld, 1985)" (p. 50). This suggests that students should appreciate the what and why of the knowledge they are acquiring as important. In a similar vein, Dweck and Elliot (1983) asserted that orienting learners to the value of what they are learning leads to additional effort to learn. To create an educational culture that encourages transfer-enhancing learning and thinking processes through bridging instruction, pedagogical moves must convey knowledge and skills in functional contexts, synergistic curriculum design (Bransford, Sherwood, & Hasselbring, 1985), and the provision of multiple examples of knowledge transfer (Pea, 1987). The bottom line is that ***any curriculum that aims for deeper learning and transfer needs to intentionally incorporate strategies to facilitate transfer***. Or, quite simply, transfer needs to be designed and assessed for.

11.2.2 Preparation for Future Learning

With a view to preparing learners for "lifelong learning," Bransford and Schwartz (1999) called for broadening old conceptions of transfer by including an emphasis on learners' *preparation for future learning* (PFL), where the focus shifts to assessments of a learner's abilities to learn from new resources. They critiqued traditional tests of transfer for predominantly testing direct application of one's previous learning to a new setting or problem with no opportunities for learners to demonstrate their abilities *to learn* to solve new problems. The PFL perspective suggests that assessments of people's abilities can be improved by involving assessments that provide opportunities for new learning. Such "dynamic assessments" (Campione & Brown, 1990; Feuerstein, Rand, & Hoffman, 1981; Schwartz, Bransford, & Sears, 2005; Schwartz & Martin, 2004) measure how well students apply learned skills to new learning. They also argued that using approaches for fostering flexible learning for transfer aided PFL.

In a salient demonstration of the promise of the PFL approach, Schwartz and Martin (2004) worked with 15 classes of ninth-grade students studying descriptive statistics and compared the value of asking students to invent statistical methods (a strategy to promote more flexible learning for transfer) with that of students simply practicing methods they were shown. Students in the two conditions performed no differently on regular assessments testing students' learning on statistical problems. But on problems in specially designed dynamic assessments that required application of new learning that was embedded in a worked example (which constituted a new learning resource in the transfer test), students who were in the invention condition did much better than the control group.

Other approaches for teaching STEM subjects with a view to promoting successful transfer include Dede (2009), who describes the design of an immersive multiuser virtual environment that provides authentic scenarios for scientific inquiry and for collaboratively identifying problems through observation and inference, forming and testing hypotheses, and deducing evidence-based conclusions about underlying causes. He suggested that the "potential advantage of immersive interfaces for situated learning is that their simulation of real-world problems and contexts means that students must attain only near-transfer to achieve preparation for future learning" (Dede, 2009, p.67). Our approach to promoting PFL draws on the late Randi Engle's work on expansive framing as well as older ideas of helping learners see deeper structures in learning content.

11.2.3 Expansive Framing (For Socially Framed Context-Based Transfer)

Engle et al. (2012) suggested that learning and transfer contexts can be socially framed in different ways and that such framing influences the learner's ability to transfer what they learn (Engle, 2006; Engle et al., 2012; Engle, Nguyen, & Mendelson, 2011; Engle, Roberts, Nguyen, & Yee, 2008). Framing is a person's sense of what kind of activity they are engaged in. It is their rarely spoken but ever-present answer to the question "What is going on here?" (Goffman, 1974). A person's framing involves a set of expectations and contextual assumptions of how to proceed in a given context (Tannen, 1993). Engle et al. (2012) draw on prior literature (Goffman, 1974; Goodwin & Duranti, 1992; Tannen, 1993) to articulate *framing* as "the meta-communicative act of characterizing what is happening in a given context and how different people are participating in it" (p. 217).

Engle et al. (2012) proposed the idea of expansive framing by building on the socializing approach to teaching for transfer from Pea (1987) and other extant literature that recognized that transfer can be promoted by creating an *expectation for future use* .When there is such an expectation, students see that what they are learning will maintain relevance over time (e.g., Bereiter, 1995; Brown, 1989). *Expansive framing* is a strategy for mediating transfer that explicitly fosters an expectation that students will continue to use later what they are learning. Engle et al. hypothesized that framing curricular content as having the potential for transforming students' everyday experiences initiates a series of processes of cognitive encoding of the learning that eventually lead to greater transfer. Simply put, students who expect they will need to continue using what they have learned may prepare for such future use through better mental representations that they can draw upon in later transfer contexts.

Engle et al. (2012) contrasted expansive framing with *bounded framing*, where the learning is treated as a one-time event in the classroom relevant only for that lesson or unit, as learning that students are unlikely to ever use again. Expansive framing, on the other hand, is treated as a discussion of an issue that students will be actively engaging with throughout their lives. They shared data from multiple studies including an example of a high school biology teacher who constantly framed the learning in an expansive way by encouraging students to see linkages between the biology learning to contexts outside of school and also other subjects (such as chemistry). Students in that classroom were shown to score well on researcher-designed transfer tests as well as on end-of-year standardized tests. Engle et al. (2012) believed that the teacher's expansive framing resulted in many of his students developing "an *expectation for future transfer* that was equivalent with, or perhaps even stronger than, the degree to which they noticed the teacher emphasizing future usefulness" (p. 222).

Furthermore, new learning contexts must make connections to the earlier contexts from which learners are expected to transfer in knowledge to the new context. Pea (1987) suggested that by creating links back to prior learning contexts, students are encouraged to make use of transfer opportunities by using their relevant learned knowledge. Gagne (1985) offered a similar suggestion—that learning transfer is a circumstance influenced by the number of common cues between learning and transfer situations. By making such explicit connections, learners are encouraged to draw on their prior knowledge during learning in the new context, which leads to them transferring in prior examples and abstract (general) principles. Schwartz, Chase, and Bransford (2012) described this as helping learners recognize "the old in the new."

11.2.4 Multiple Examples and Analogous Representations for Content-Based Transfer

Early transfer research focused mostly on the content that learners should ideally transfer (Engle et al., 2012). For example, Gick and Holyoak (1983) believed that "the induction of a general schema from concrete analogs will facilitate analogical transfer" (p. 1) and that learners are more likely to apply what they have learned from one analogous problem to another if they form a generalization or "schema" such that it can be applied to a new problem.

Research on these *content-based generalizations* suggests that multiple examples of acquisition and application of new knowledge are important for transfer. When a topic is taught in multiple contexts and includes multiple (well-chosen) examples that demonstrate the range of application of the concept being taught, then learners are more likely to abstract the general principles and relevant features of concepts and develop a flexible representation of knowledge (Brown & Cocking, 2000; Gick & Holyoak, 1983; Hakel & Halpern, 2005; Pea, 1987).

Another strategy for helping learners see deeper structure of conceptual ideas is through *comparing cases and analogous representations* (Bransford & Schwartz, 1999; Gentner, Loewenstein, & Thompson, 2003) through which learners develop a more general problem-solving schema that primarily captures common structure rather than the surface elements. Consequently, I believe that analogous representations that are presented as part of an expansive framing should be more easily retrievable when the learner encounters a new case (or a computer program in our context) with the same structure. Schwartz, Chase, Oppezzo, and Chin (2011) reported on the success of this strategy in the context of physics learning. Such a pedagogical design also exploits the method of perceptual learning and pattern recognition in problem solving where attention is drawn to relevant features that are critical to solving different problems and problem types (Bransford, Franks, Vye, & Sherwood, 1989; Polya, 1957).

Wagner (2010) argued that the notion of extracting deeper structure in a learning situation is rooted in the old information-processing view of learning and transfer (Anderson & Lebiere, 1998) that is at odds with the Piagetian constructivist view of learning (Piaget, 1937/1954). However, the situational perspective in mathematical problem solving in which Wagner grounded his argument is quite distinct from the context of this work—the context of programming is common to both the learning and transfer contexts, with representational encoding being the only difference.

11.3 Teaching Programming: Why Blocks First and Transfer from Blocks to Text

Learning to program is hard. A large body of literature points to the difficulties that novice learners face on their path to acquiring an understanding of coding (e.g., du Boulay, 1986; Soloway & Spohrer, 1988). These past two decades have seen a plethora of visual programming environments emerge that ease learning of programming for young children through features and affordances that not only remove the issue of syntax errors but also lower the threshold to creating working programs through the use of visual, drag-drop graphical tools. As a result, these "block-based programming environments"—such as Scratch (Resnick et al., 2009), Alice (Cooper, Dann, & Pausch, 2000), App Inventor (Wolber, Abelson, Spertus, & Looney, 2011), and Snap*!* (formerly Build Your Own Blocks; Harvey & Mönig, 2010), to name just a few—have become popular vehicles for introducing novice learners in K–12 classrooms to CT and programming. Several recent studies have also shown that students learn programming better and are more engaged when they start with block-based rather than text-based programming (e.g., Price & Barnes, 2015; Wagner, Gray, Corley, & Wolber, 2013; Weintrop, Killen, Munzar, & Franke, 2019). This gain on engagement meshes well with a growing recognition of the critical need to encourage groups such as females and people of color (who have traditionally been marginalized in CS and the technology industry) to learn coding and computing, and has strengthened the case for the "blocks-first" approach to teaching programming in K–12.

However, the use of block-based programming is largely restricted to primary and secondary age learners, with virtually no use of block-based programming environments in postsecondary education or in industry. Preparing students for transfer from blocks to text is thus a key concern that many researchers and educators would like to see addressed (e.g., Armoni, Meerbaum-Salant, & Ben-Ari, 2015; Bagge, 2019; Weintrop & Wilensky, 2017). Only a few studies in the realm of computing education have focused on the transfer of learning from visual block-based environments to text-based ones. Dann, Cosgrove, Slater, Culyba, and Cooper (2012) demonstrated how to mediate transfer of learning from a specially designed version of the Alice block-based environment to the text-based Java environment. They argued that by using the exact same example in both Alice and Java, their students succeeded in achieving better learning results. Wagner et al. (2013) adopted a similar approach—they introduced learners to programming mobile apps in App Inventor (AI) and then used the AI Java Bridge to have students code the same examples in Java:

> When transitioning to the Java Bridge, the students were challenged, but because we repeated the same applications (HelloPurr and PaintPot), they were able to create a mental

> map between the Java syntax and what they previously created with the blocks language. The majority of the students appeared to understand how to use Java in creating an app. (p. 624)

Touretzky, Marghitu, Ludi, Bernstein, and Ni (2013) also leveraged the idea of analogous representations described above—children between the ages of 11 and 17 in a 1-week summer camp transitioned from Kodu to Alice to Robotics NXT-G programming environments in a structured way. By scaffolding instruction to help children see analogies between formalisms and computing constructs in each, they aimed to foster deeper conceptual understanding. For example, their strategies attempted to help children appreciate that "WHEN/DO in Kodu, If/Then in Alice, and SWITCH blocks in NXT-G all function as conditional expressions, even though they look different" (p. 610). My work drew on these prior studies to design a Scratch-based introductory programming curriculum for middle school students that also builds the foundation for future learning of text-based programming and assesses for PFL of blocks to text programming.

11.4 Designing for Mediating and Assessing Transfer in Introductory Programming

This section describes the design and development of an expansively framed curriculum—*Foundations for Advancing Computational Thinking* (FACT)—that aimed for transfer from block- to text-based programming. Because introductory experiences in FACT were set in the context of a block-based programming language (Scratch), the research effort was guided by the question, "How can early experiences in block-based programming be designed so that learners can transfer their learning successfully to future text-based programming?"

Prior publications detail the designed features of the curriculum for deeper learning of introductory programming with a focus on algorithmic and computational thinking (Grover, Pea, & Cooper, 2015) as well as formative and summative assessments of learning of foundational concepts (Grover, 2017). Here, I focus on the designs for transfer and PFL.

The 7-week FACT curriculum (Table 11.1) included topics that focused largely on algorithmic problem solving in the context of programming in addition to broader notions of computing as a discipline.

A unique feature of FACT (compared to other middle school CS curricula) was its attention to targeting deeper conceptual understanding of computational problem solving and solutions through pedagogies such as scaffolding (Pea, 2004) as well as modeling cognitive apprenticeship (Collins, Brown, & Newman, 1988). It involved working through examples (Renkl & Atkinson, 2003) and thinking aloud, à la *live coding* (Rubin, 2013), to model solutions of computational problems in a manner that revealed the underlying structure of the problem and the process of composing the solution in pseudocode or in Scratch. Code reading and tracing were also mod-

Table 11.1 FACT curriculum unit-level breakdown

Unit	Targeted topics in the Unit
Unit 1	Computing is everywhere!/what is CS?
Unit 2	What are algorithms and programs? Computational solution as a precise sequence of instructions
Unit 3	Iterative/repetitive flow of control in a program: loops and iteration
Unit 4	Representation of information (data and variables)
Unit 5	Boolean logic and advanced loops
Unit 6	Selective flow of control in a program: conditional thinking Final project (student's own choice; could be done individually or in pairs)

eled throughout, and were part of formative assessment. Often students were expected to think about and discuss programming scenarios or problems before the solution was modeled. Academic language and computing vocabulary were used during this scaffolding process. These strategies in and of themselves set the learner up for deeper conceptual learning and better transfer; however, the curriculum and assessments also consciously attended to mediating transfer and preparing learners for success in future text-based programming contexts through the use of expansive framing and analogous representations. In addition, students' ability to transfer was also assessed through a special PFL assessment designed for this purpose. Design features for each of these are described in the following sections.

11.4.1 Expansive Framing

FACT's introductory expansively framed experiences with CS for middle school were seen as bridges to high school curricula such as Exploring CS (Goode, Chapman, & Margolis, 2012) and AP CS Principles (College Board, 2017) as well as future work-related computing experiences in text-based programming environments.

Expansive framing of computing—the role computing plays and will continue to play (no matter what career students choose)—served the twin purposes of improving students' perceptions of computing as well as generating interest and excitement and, thus, motivating the learning, priming, and preparing of students for future use (transfer) of what they were learning in future contexts and settings.

The design of the first unit, titled *Computing is Everywhere!,* played a key part in expansively framing the curriculum for transfer. Specifically, it was designed to generate motivation and excitement about CS among students—its prospects for their futures and also for remainder of the FACT course that the students were learning. Commencing the course with a motivating experience that underscores future relevance aimed to socially frame the learning context to influence students' mental encoding and propensity to transfer what they learn (Engle et al., 2012). This unit also served to remedy misconceptions of computing as a discipline among young students (as suggested in a large body of past literature, e.g., Carter, 2006; Mitchell, Purchase, & Hamer, 2009; Yardi & Bruckman, 2007) and educate them about the true nature of computer science.

Computing is Everywhere! showcased example uses of computing for varied purposes and in diverse fields. FACT worked to build awareness of the many uses of computing through "showing" rather than "telling" using a corpus of engaging and interesting videos—an idea that was inspired by the power of stories and narratives (Bower & Clark, 1969; Graesser, Olde, & Klettke, 2002; Haberlandt & Graesser, 1985). These video narratives fostered an expectation that students will likely find their learning of programming and CT relevant for personal passion projects (such as crafts) and for their future (no matter what field they choose). Additionally, the example videos aimed for students to develop an appreciation of CS as a protean discipline. For this purpose, publicly available videos were curated from YouTube. For example *MIT Computer Program Reveals Invisible Motion in Video, Untangling the Hairy Physics of Rapunzel,* and *IBM's Watson Supercomputer Destroys Humans in Jeopardy,* among others, exemplified innovations in computing in engaging ways and also demonstrated the use of computing in contexts that were believed to be novel for the average middle school student. These and other videos selected for this purpose were added to a publicly available playlist (http://bit.ly/CS-rocks) titled *Computing is Everywhere* as shown in Fig. 11.1.

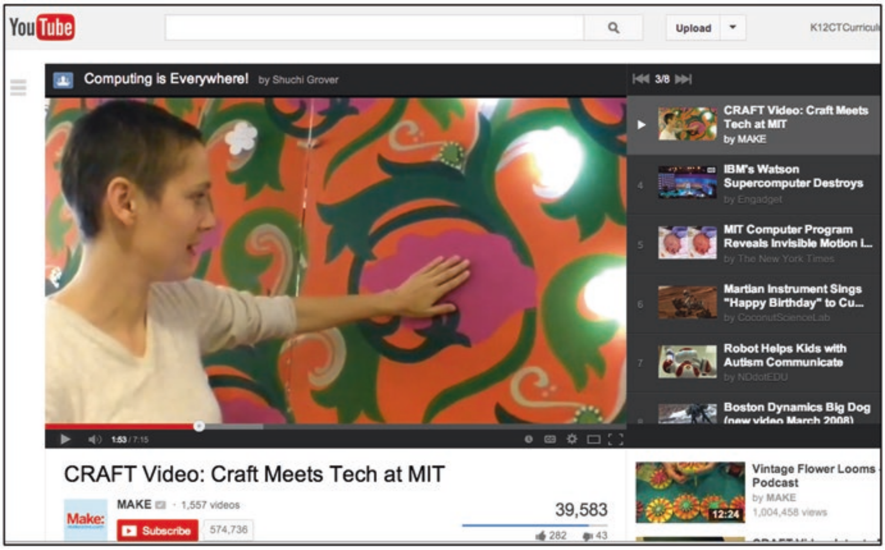

Fig. 11.1 Publicly available playlist of videos called *Computing is Everywhere!*

In addition to this playlist of publicly available videos, the author recorded and produced a corpus of videos titled *Vignettes of People and Computing* that comprises short 1- to 3-minute videos of everyday people representing diverse genders and racial/ethnic groups drawn from diverse fields describing how they use computing in their work—from art to music to education research. This set of videos was created in an effort to mimic the idea of bringing in speakers into the classroom who can lend a personal connection to the ideas being discussed and also serve as role models for students (Dasgupta & Stout, 2014). This video repository is also publicly available (http://goo.gl/oatj4H). For the curricular intervention, all these brief videos were shared with learners via the online learning platform used by the school, and were accompanied by prompts for anchored discussions (Guzdial & Turns, 2000) right below the video to solicit students' reactions.

Over the course of the subsequent units of FACT, the transferability of the deeper ideas of CT and algorithmic thinking was repeatedly highlighted to reinforce the initial message of *Computing is Everywhere!* For example, in the early introduction of the unit on what programming is, images of code in various block- and text-based languages were used to underscore the ideas that algorithmic solutions can be programmed in various languages and that the same algorithm can be coded or represented in various languages.

11.4.2 Analogous Representations

Successful PFL in an introductory programming curriculum demands that students develop not only coding skills but also computational and algorithmic thinking skills. Successful transfer thus requires an understanding of the underlying structures of programs beyond the syntax and surface features of the environment in which children are initially learning programming. Such understanding encompasses more expansive frames in which similarities in deep structures across programming environments are anticipated, recognized, and productively used. Unlike earlier research that had attempted this by employing different programming languages to help students abstract deeper features of constructs, FACT was distinct in that these ideas were applied while using a single programming environment (Scratch) by employing the strategies described below.

The FACT approach to mediating transfer relied on the use of expansive framing and analogous representations of algorithmic solutions to help learners see computational constructs in forms more expansive than the shackles of a specific syntactical structure. It is contended that guiding students to draw analogies between different formalisms can foster deep and abstract understanding of fundamental concepts and structures of algorithms. To this end, in the process of modeling the construction of algorithmic solutions in FACT, English text and pseudocode were frequently used to describe algorithms so that students would see programming concepts at play in an algorithmic solution that is represented in ways distinct from Scratch. This was based on a belief that such use of pseudocode would set learners up for better abstraction than merely learning to program in Scratch (or any specific programming language being used). Formative assessments also included exercises and quizzes involving short programming snippets in pseudocode and plain English, in addition to Scratch.

Throughout the course, pseudocode was used not only to describe and deliberately lay out the steps involved in organizing the algorithmic solution to accomplish a goal, which has its own benefits (Mayer, 1989), but also to introduce students to analogical terms and representations of algorithmic solutions distinct from the Scratch environment. The use of pseudocode thus bolstered familiarity with textual representations of programs and also with analogous terms and description of iterative (looping) and conditional structures that are different from Scratch. For instance, Scratch has only "REPEAT" and "REPEAT UNTIL" blocks for iteration. However, using terms like "WHILE" or "FOR" in pseudocode aims to help students recognize that different computational vocabulary can be used to describe the same idea of repetition of steps (even though there are subtle differences in the ways in which these constructs operate in different programming languages).

The image on the left in Fig. 11.2 shows an example of an algorithmic solution presented to learners in pseudocode. The solution for calculating the average of a set of numbers shown in Fig. 11.2 represents the final step of a problem decomposition exercise that involved breaking down the problem and composing the various subparts that this problem was first broken into. Additionally, introductory programming concepts were expansively framed at frequent points in the course by highlighting their relevance in text-based programming languages such as Java and Python that students are likely to use in the future. The point would be driven home through assertions such as, "Even though a loop in other languages like Java or Python will be expressed with terms like *While* or *For*, they help to accomplish the same things in an algorithmic process like the *Repeat Until* loop does in your

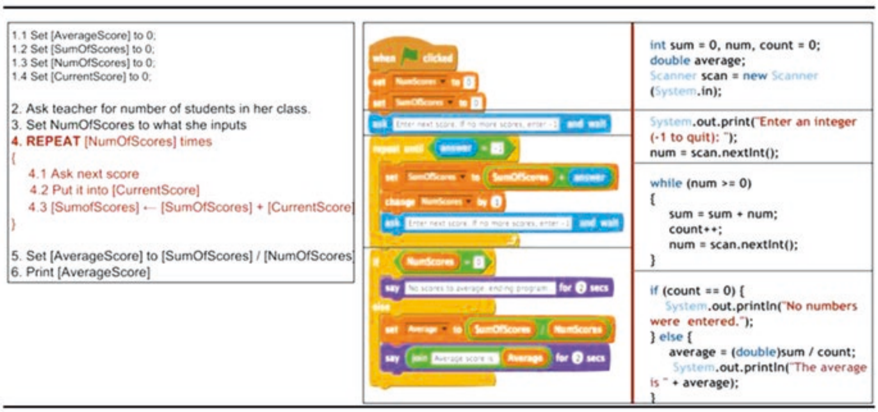

Fig. 11.2 (Left) Presentation slide of algorithm to calculate the class average represented in pseudocode. (Right) Analogous representations of the same algorithm in Scratch and Java

Scratch program to help you compute the average test score for a class." The image on the right in Fig. 11.2 provides an example of how a FACT activity required learners to observe the analogous representation of the same algorithm in Scratch and Java (a text-based language students use in advanced high school courses). This was shown to students after they had coded the pseudocode version of the "Find the Average" algorithm (the image on the left in Fig. 11.2) in Scratch.

11.4.3 PFL Assessment

The concern for transfer of CT and programming experiences was examined using PFL assessments as summative measures that assessed students' ability to transfer their learning from the Scratch context and readiness to work with text-based programming environments.

Based on the design of "dynamic" PFL assessments in prior research (such as Schwartz & Martin, 2004), these assessments aimed to assess how well students learned from a new resource and applied new learning to read and comprehend code presented in a text-based language. The problems were preceded by "new learning" in the form of syntax details for fictitious (Pascal-like or Java-like) text-based languages for constructs such as output to the screen, loops, conditionals, and variable declaration and assignment that students had encountered in the context of Scratch in FACT. To help learners make connections back to the past learning context (Pea, 1987) and see "the old in the new" (Schwartz et al., 2012), references were made to the equivalent constructs in Scratch, for example: "PRINT displays things specified in parenthesis to the computer screen one line at a time (like SAY in Scratch)" (see Fig. 11.3).

Two different types of syntax were explained, followed by questions that involved programs coded in the new syntax. For example, the explanation shown in Fig. 11.3 was the new (Pascal-like) syntax description that preceded the first couple of questions. Then, a new (Java-like) syntax was explained, and the remainder questions were based on snippets of code written in that language (Fig. 11.4).

11.5 Methods

In this section, I describe the classroom research conducted to examine the success of the FACT learning experience among middle school students. Grover et al. (2015) provides details of the classroom research and the results of students learning of algorithmic and computational thinking. Here I focus on the research and analysis of only those data from empirical research that were captured to examine transfer and PFL. The section first describes results from preliminary explorations (that aided the design phase) and then findings related to PFL from two iterations (called Study 1 and Study 2) of design-based research (Wang & Hannafin, 2005) in middle school classrooms using FACT.

'<–' (left arrow) is used to assign values to variables. *For example:* n <– 5 assigns the value 5 to the variable n
If there are blocks of compound statements (or steps), then the **BEGIN**..**END** construct is used to delimit (or hold together) those statement blocks (like the yellow blocks for REPEAT and IF blocks in Scratch).
FOR and **WHILE** are loop constructs like REPEAT & REPEAT UNTIL in Scratch **WHILE** (*some condition is true*)
BEGIN
... (Execute some commands) .....
END
PRINT displays things specified in parenthesis to the computer screen one line at a time (like SAY in Scratch). *Commas are used inside the PRINT command like JOIN in Scratch to combine a text message with a variable*
===
Question #1: *When the code below is executed, what is displayed on the computer screen?*

```
PRINT("before loop starts");
num <– 0;
WHILE (num < 6) DO
BEGIN
        num <– num + 1;
        PRINT("Loop counter number ", num);
END
PRINT("after loop ends");
```

Fig. 11.3 Sample PFL question following new syntax specification

11.5.1 Preliminary Empirical Explorations and Findings

To better guide the research effort, the design of curriculum elements and PFL assessments was preceded by preliminary explorations. These were aimed at examining the conjecture that programming courses offered at the middle school level in a block-based programming environment like Scratch often focus on giving learners introductory experiences with the programming environment; preparing for transfer to text programming are likely not part of the typical learning experience.

The PFL assessment was administered to students in a seventh-grade classroom of 24 students studying introductory programming in a local school district. On two of the five questions, the class scored an average of 25% (only about one third of the class even attempted an answer). On the remaining three questions, only three or fewer students gave a correct answer (or something close to a correct answer). Most students simply responded, “I don’t know” or left the question blank.

This suggested a lack of ability to transfer learning of algorithmic flow of control to the point of not even attempting questions that (most likely) ***appeared*** totally unfamiliar to them. To test these questions with a different cohort of students, the same classroom was revisited in a different semester around 8 weeks into the term, and the same test was administered. The results were very similar to the earlier pilot.

Clearly, these explorations bore out initial hypotheses that students in a typical introductory programming classroom (at the time of these studies) did not understand the deeper structures of algorithmic control flows in text-based code snippets even after weeks of working on similar problems in block-based Scratch-like

[Section of new syntax presented preceding Questions 3-7]

(1) int is a way of defining an integer (numeric) variable

(2) = is used to assign values to variables.
For example: n = 5 assigns the value 5 to the variable n□

(3) == checks for equality (same as checking if something *is equal to* something else) and returns true or false.
For example (i == 100) returns true if the value of i is 100 and false otherwise□

…

(8) **for** and **while** are loop constructs (*like REPEAT & REPEAT UNTIL in Scratch)*

```
for (initialize a variable; boolean condition (loop while it's true); update the variable)
{
   .. (Execute some commands) ...
}
```

…

(11) ++ changes (increments) the value of a variable by 1 *(like the '**Change .. by** .. 'block in Scratch)'*
So for a variable n, n++ it is equivalent to saying n = n+1
For example: If n = 5 then executing n++ changes the value of n to 6

==

Q6. Which of the options must be true after the while loop terminates-

```
while (Counter < FinalNum) && (CurrentNum != NextNum)
{
     Counter++;
}
```

❍ Counter >= FinalNum
❍ Counter < FinalNum
❍ (Counter < FinalNum) && (CurrentNum != NextNum)
❍ (Counter >= FinalNum) || (CurrentNum != NextNum)
(Counter >= FinalNum) || (CurrentNum == NextNum)

Fig. 11.4 Sample PFL question following new Java-like specification

programming. These findings strengthened the case for actively using strategies to mediate for transfer and to prepare students for future learning through a curriculum such as FACT.

11.5.2 Study 1 and Study 2

Two studies were conducted in a middle school classroom in the western United States with two different cohorts of students. According to school data provided online by the California Department of Education at the time of the study (2013), about 8.6% of the student population was socioeconomically disadvantaged, 12.6% were English Language Learners, and 13.6% were students with disabilities. The racial/ethnic composition of the school was 43.2% Asian, 37.8% White, 10%

Hispanic or Latino, 2% African American, and 7% are students of other ethnicities.

The studies were conducted over two 10-week periods in the spring and fall quarters of the same year. These 10 weeks included the time before and after the 7-week-long FACT intervention that included visits to the classroom for Institutional Review Board (IRB) permissions and pre–post tests as well as post-course activities such as final project completion, presentations, and student interviews. FACT was taught as part of the semester-long *Computers* elective that met four times a week for 55 minutes each. The elective was offered to Grades 7 and 8, and approximately one fifth of the class comprised students who had been placed in the class by the school counselors. In both studies, these students happened to be English Language Learners (ELLs) or students with other learning difficulties who could not be accommodated in other elective classrooms. Unfortunately, because this was an elective class, these students did not get the same para-specialist supports they received in core-subject classes.

The only differences related to PFL between Study 1 and Study 2 (in the iterative design research) were in the PFL assessment—the wording of two questions was refined to improve clarity. Because all five questions in the original PFL assessment involved loops (and variables), two additional PFL assessment questions were added for Study 2—one involved variables and Boolean expressions and the second involved conditional statements.

11.5.3 Data Measures and Analyses

Besides the PFL measures, the following student data were collected: (a) pretest and posttest of programming knowledge; (b) technology fluency and prior experience with programming (survey); (c) interest in and attitudes towards CS (survey); (d) reactions to *Computing is Everywhere!* videos; (e) academic preparation (as measured here by math and English levels in state tests); (f) learning issues (English language issues especially); (g) demographics (age and gender); and (h) student experience surveys and interviews (post-FACT).

Mixed methods analyses In both studies, student reactions to the videos in *Computing is Everywhere!* were coded based on the same set of codes:

1. Positive valence (e.g., awesome, amazing, cool, exciting),
2. Real-world connections of CS,
3. Curiosity to learn more about something addressed in the video,
4. Seeing CS in a way they had not thought of before,
5. Commenting on the use of CS for creative expression, and
6. Personal connection with the ideas or story in the video on a personal level—to their own lives or in the lives of the people in the video.

The open responses to the PFL questions were graded by the main researcher according to the detailed rubric. Each question was graded out of 3 points. In Study 1, a second doctoral researcher re-checked the grading and used this set as training for grading the PFL test for Study 2. The mean and standard deviation for the PFL test were 63.27 and 28.86, respectively. In Study 2, the two graders graded the PFL test. The inter-rater reliability (Cohen's Kappa) for the grading was 82.1%.

To understand the factors that predicted students' PFL scores, data from the two cohorts were combined (for greater statistical power). This was justifiable because there was no significant difference between the two cohorts on any of the measures captured—pretest, posttest, PFL test, or prior experience survey measures. Regression analyses were done to find which variables among the data gathered were significant in predicting student performance on the PFL assessment. (Three students were dropped from the analyses due to incomplete data.) Univariate regressions were used as the first step in determining which variables should be dropped in the stepwise multivariate regressions. If a variable could not significantly predict the dependent variable in a regression all by itself, it was not included in the multivariate regression analysis. Age was one such variable.

11.6 Results

In both studies, on the post-test, students scored the highest on questions related to serial execution followed by conditionals and then loops. The average scores between the two studies were not different statistically. In Study 2, these scores were 91%, 85%, and 77% respectively for an overall mean of 81.6%.

11.6.1 Reactions to Computing is Everywhere!

Qualitative coding and analysis of student responses to the videos gave a glimpse into how the different videos influenced students' thinking (Fig. 11.5). Student responses included comments like "I think that was a really cool opportunity that she got to work with Disney and I would take it if I got the chance to work with Disney" and

> I liked the mural they made on the wall. In the future everybody will have their rooms painted however they want and you can turn off the lights by slapping the wall. It would be useful to not have to get up to turn on the fan or lights. You just slap the wall.

Suggestions of a new appreciation of the importance of, and wanting to learn more about, CS and coding were made in some comments, for example: "Some of the most interesting computer science videos I've seen I think. And I now really want to learn more about coding and what I can do with it" and

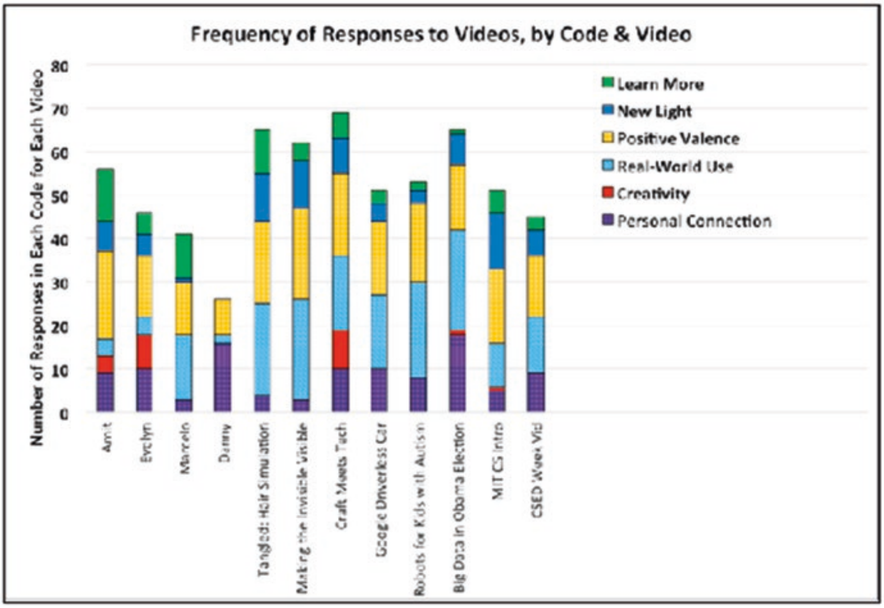

Fig. 11.5 Frequency of responses to individual videos by coded categories

> I liked the second video with the music because that's something that I can relate to. I love music so I like things that have to do with music. When I thought of computer science before the video I think of computers and only about computers. But now I see that computer science goes so much further than that. Now I can't think of anything that doesn't use computer science. It's really cool.

11.6.2 Performance on the PFL Assessment

The performance of students on the PFL test in Study 2 was quite similar to that in Study 1, and there was no statistical difference in the scores across the two cohorts ($t = -0.22$, $p = .82$). The mean and standard deviation were $M = 65.07$ and $SD = 26.47$, respectively. The class average scores (out of 3) on each of the seven questions in Study 2 are shown in Table 11.2.

Students seemed to struggle the most on the question involving the FOR loop. Prior research suggests that FOR loops are the most problematic looping construct for older novice programmers too because of the complex "behind-the-scenes" action involved with incrementing the loop variable (Robins, Rountree, & Rountree, 2003). It is therefore not surprising that less than 50% of the students tackled that question correctly. Questions 4 and 5 were based on the same snippet of code that was very similar to Question 3 but had the added complexity of checking for divisibility by 2 and then incrementing one or the other counter variable. As in Question

Table 11.2 Question-wise breakdown of PFL test scores (Study 2)

Question	Average score by question (out of 3)	
Question 1	WHILE loop, with variable increments	1.65 (55%)
Question 2	FOR loop, with variable increments	1.32 (44%)
Question 3	IF statement within WHILE, counter variables	1.96 (65%)
Question 4	IF statement within WHILE, counter variables for mod operation	1.68 (56%)
Question 5	Number of iterations of WHILE loop	2.32 (77%)
Question 6	Boolean condition for WHILE loop	1.79 (60%)
Question 7	Multiple conditionals in sequence	2.54 (85%)

3, the answers were coded with one of four scores: 3, 2, 1 or 0. The wording of Question 5 ("How many numbers will be processed by the program below?") also seemed to have caused some confusion because some students counted the number of variables that the code was using rather than the count of numbers processed by the loop. Of the students who understood the question as intended, the majority answered correctly, and a few answered 99 instead of 100, which is a common "off-by-1" loop error (Soloway, Bonar, & Ehrlich, 1983).

11.6.3 Factors Predicting PFL Test Scores

The mean PFL test score for the combined sample was $M = 64.3$ ($SD = 27.4$). According to the regression analyses (Table 11.3), both the pretest and posttest significantly predicted PFL test performance, with the posttest, not surprisingly, being the most predictive ($\beta = 0.58$, $p < .01$). Prior experience factors did not significantly predict performance on the PFL test and neither did math academic preparation. However, being an ELL student was negatively correlated with the PFL score. This was perhaps due to the fact that the test was very text heavy as seen in Figs. 11.3 and 11.4.

The high degree of correlation between the posttest and PFL test prompted further regression analyses of the PFL test and the posttest as broken down by CS constructs to investigate which construct (Serial Execution, Conditionals, or Loops) was the best predictor of performance on the PFL test. Not surprisingly, the regression analyses suggested that the PFL test was most closely correlated to the Loops concept (the only significant predictor among the three question categories). This was as expected because many of the PFL test questions were based on code snippets that involved loops (with variables). Students who were able to master loops also did well on the PFL test. However, Loops was the topic that students in Study 1 and 2 seemed to have the most difficulty with—the lowest scores on the posttest were on the questions involving loops (with variables).

It would therefore appear that the PFL test was much more difficult when compared to the posttest. This explains the relatively lower score of 65% on the PFL test

Table 11.3 Regressions predicting PFL and posttest score for the combined student sample

	PFL test score	
Variable	Beta	*SE*
Posttest score	0.58***	0.16
Pretest score	0.40**	0.19
Math standardized test	−0.13	3.69
English language learner	−0.27**	7.79
Prior experience 1: coder	−0.10	2.40
Prior experience 2: creator	−0.08	1.64
Prior experience 3: consumer	−0.02	1.73
Constant	–	14.31
N	49.00	
Adjusted R-squared	0.68	

*$p < .05$. **$p < .01$. ***$p < .001$

compared to an average of approximately 80% on the posttest. It would not be unreasonable to assume that PFL test scores would likely have been much higher had it been more comparable to the posttest in terms of topics associated with the different questions.

11.7 Discussion

Based on the results described above, it appears that the *Computing is Everywhere!* unit succeeded in providing an expansive framing of computing and programming that helped learners appreciate the relevance of what they were learning. It helped foster an expectation that learners would continue to find future use for what they were learning in the introductory CS and programming curriculum due to the relevance of computing in every field. This expectation was reinforced through highlighting the deeper structures of programming and algorithmic thinking—that students learned through modeling, cognitive apprenticeship, and analogous representation—in future coding experiences in text-based environments.

The performance on the PFL test suggested that a majority of learners were successfully able to transfer most of the basic ideas of algorithmic control flow to comprehend code completely foreign to them. It is worth noting that:

1. In most cases, students were able to correctly get a sense for the program flow and recognized the concept of looping or conditional execution in the code even though their responses were not always completely accurate.
2. Consistently right or consistently (completely) wrong responses mapped closely to performance on the Scratch test. This is to be expected because skills mastery in the original context is essential for transfer (Kurland, Pea, Clement, & Mawby, 1986).

3. Unfortunately, most of the questions on the PFL test involved loops with variables—a topic that is traditionally known to be difficult. Repeated execution has been known for a long time to be a thorn in the side of novice programmers (Parsons & Haden, 2007; Pea, 1986).
4. The nature of many of the errors committed on the PFL test were similar to those committed on the Scratch test, suggesting weak initial learning of some concepts.
5. Problems such as the "off-by-1" looping error or issues with the FOR construct are common even among older novice programmers at the undergraduate level as well (du Boulay, 1986; Soloway & Ehrlich, 1984).
6. The qualitative analysis of the PFL responses suggests that several students provided descriptions of the code rather than answering the pinpointed question, such as the final value of a variable. Those explanations were suggestive of understanding of the algorithmic flow of control.

Perhaps the most telling takeaway on the strategies for mediating transfer and PFL such as expansive framing and analogous representations used in FACT was what one seventh-grade student had to say about FACT and the PFL questions in his post-FACT interview:

> It was definitely interesting, especially the ones that related to other programming languages; I could definitely see that it's really important to not only get people in this course exposed to Scratch but also have the knowledge and basis and realize that this is not just programming in Scratch but these skills can all be applied to any programming language, so I especially like those questions, and they were also the ones that were probably most informative because they showed what some of the different syntaxes were in different programming languages.

That said, these results suggest that such a PFL test would benefit from some revision. It has been established that the PFL test was difficult compared to the posttest because most questions involved loops with variables. Not only was it testing learners on new learning and programming in a context completely alien to most students, it also focused mainly on loops and variables, which were the hardest concepts for learners to grasp as evidenced by the breakdown of posttest performance by constructs (Grover et al., 2015). Additionally, performance on the PFL test clearly depended on learners' ability to read and work with long sections of textual "new learning" on which the PFL questions were based.

11.7.1 Limitations

The study would have benefited from a better experimental design with a control condition where student learning in a different but comparable setting could be tested using the same set of assessments. This was achieved in some sense through the preliminary explorations where students learning Scratch in another school in the same district were administered a preliminary version of the PFL test as a validation of the hypothesis that middle school computing experiences don't usually teach to promote block-to-text transfer, a finding that has since been reproduced in other research (e.g., Weintrop & WIilensky, 2017). However, this was not a control group in a true sense.

11.8 Conclusion and Implications

The goal of all formal learning is to equip learners with the skills and abilities for the near and not-so-near future. However, few curricula in K–12 computing education pay due heed to explicitly teaching for transfer. In an effort to answer the question, *"How can early experiences in block-based programming be designed so that learners can transfer their learning successfully to future text-based programming?"* this article describes approaches to consciously teaching and assessing for transfer in introductory CS curricula in ways that leverage the promising and powerful ideas of "expansive framing" and "preparation for future learning" in introductory CS teaching and learning at the middle school level. This design research describes the promise of creating environments that strive for deeper learning to prepare learners for future computing experiences. This research also provides middle school teachers with curricular and pedagogical ideas to promote a deeper engagement with CT concepts even as they use block-based environments like Scratch that are friendly to younger programmers. Implications from this research point to the need for introductory CS curricula to (a) teach programming to strive for acknowledging deeper structures of programs rather than only the (shallow) syntax of a programming environment and (b) give learners a sense for the commonality of programming structures in various programming environments (through the use of pseudocode or examples from multiple programming environments to express the same solution).

The dynamic PFL assessment that assesses transfer of learning from a block-based programming context to a text-based one is also new to education research. The related text-based assessments, which can be used as PFL assessments after teaching middle school students introductory CS using environments like Scratch and Alice, are a unique contribution of this research. To conclusively establish the merits of this curricular approach, comparative investigations would be needed with students who are learning block-based programming in other types of curricula. Other approaches for mediating transfer should also be investigated. For example, working with different programming languages in the course of the same intervention could be another strategy worthy of investigation . Which languages should be used? In what sequence? What would be appropriate transition points? What would bridging strategies look like? These are just some of the promising questions that merit future investigation. Additionally, new dual-modality languages and environments, such as Pencil Code and Code.org's AppLab, have emerged. These tools have both block-based and text-based interfaces allowing learners to move back and forth between the two, and are thus promising for promoting transfer with appropriately-designed curricular experiences.

Perhaps the most salient implications of this research concern the foundational premise of this effort—that students as young as 12 and 13 years of age can and should be taught computational and algorithmic thinking in a way that will successfully transfer from the novice-friendly block-based programming to the more mature text-based programming. Furthermore, making learners aware of CS and its diverse applications serves the dual purpose of connecting to identity and engagement while also underscoring the relevance of what learners are learning to the expectation of usefulness for the future. This is ever more important given that middle school years are crucial for identity building as well as cognitive development for analytical work required by STEM disciplines. The key lies in teaching with a view to deeper conceptual understanding so that learners master the core disciplinary ideas in ways that attend to transfer of that learning to productive future efforts. Expansively framing the curriculum to consciously create bridges to future learning contexts and assessing for transfer through specially designed PFL assessments are unique contributions that this research makes to the fields of STEM and computing education research.

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Part III
Transfer that Transcends School Settings

Chapter 12
Tenets of Ethnographic Accounts of Cross-Setting Learning in Relation to Interpretive Accounts of Transfer

Kara Jackson

The observation that humans regularly make their way across settings in which values may be differentially privileged, and in which social relations and the norms of communication and practices may vary, has been a persistent object of inquiry for philosophers, anthropologists, psychologists, cognitive scientists, and educational researchers alike (Beach, 1999). Specific to education, as has long been observed (e.g., Dewey, 1916), understanding how it is that people forge connections and experience contradictions across settings is especially important, given that much of public discourse regarding the purposes of education centers on preparing students to use what they learn in classrooms in settings outside the classroom.

My personal interest in exploring cross-setting learning is rooted in practice. As a mathematics teacher, I wrestled with issues of whether what I was teaching in a classroom would be of use in youth's present and future lives outside of educational institutions. I also wrestled with reconciling genuine commitments to building on youth's existing experiences in classroom instruction while also being cognizant of what both I and youth were being held accountable for by the educational system. And, I observed differences in the ease with which it appeared that some youth, as compared to others, made their way in and across home and school settings. These dilemmas sparked my interest in understanding how scholars theorized how people experience continuity, and discontinuity, specific to (mathematics) learning and education. I was especially interested in the implications of this scholarship for the design of learning settings, particularly for youth from communities that historically have not been supported well in math classrooms and schools. Questions like these led me as a graduate student to conduct an ethnographic study of two 10-year-old youth and their families' pursuit of mathematical understanding and identities across home, school, and community settings, with a focus on how, when, and why people engaged in mathematical activity, and to what effect.

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I share this to contextualize my interest and intent in writing this chapter. In the study mentioned above, I took, broadly speaking, a sociocultural perspective on learning and built on a then nascent set of ethnographic studies taking a cross-setting perspective on learning to theorize youth's participation and identity work across settings. I did not initially conceive of the study in relation to the literature on transfer. However, as I worked to theorize the work the youth and others were doing to establish or maintain productive relations across settings in relation to mathematics, it became apparent that the literature on transfer, especially what I will refer to as interpretive accounts of transfer, was relevant in many regards. By interpretive accounts of transfer, I mean accounts that place emphasis on the meaning that learners generate in context (e.g., Engle, Lam, Meyer, & Nix, 2012; Greeno, 2006; Hohensee, 2014; Lobato, 2006, 2012; Pea, 1987). In such accounts, "knowing and representing arise as a product of interpretive engagement with the experiential world, through an interaction of prior learning experiences, task and artifactual affordances, discursive interplay with others, and personal goals" (Lobato, 2012, p. 234).

The purposes and methodologies guiding ethnographic, cross-setting accounts of learning are not the same as those guiding interpretive accounts of transfer. In cross-setting accounts of learning, researchers deliberately and expansively trace, as best they can, the varied pathways that people take in relation to focal content, which is often broadly construed as including concepts, tools, and practices. In accounts of transfer, the scope of what is of interest (both the content and the focal settings) is narrower and more precisely defined from the outset. However, both bodies of scholarship aim to understand how it is that people generate meaning that is consequential for future action in events that are, or are intended to be, linked.

In what follows, I first describe the purposes of ethnographic accounts of cross-setting learning in general terms. I then elaborate on four tenets of such accounts of learning that I view as useful to consider in relation to interpretive accounts of transfer. The tenets are as follows: (a) people's participation varies in relation to context; (b) people and settings change over time, and in relation to one another; (c) learning is not complete in any one event; and (d) power relations matter in understanding what pursuits are possible, when, by whom, and with whom. Throughout, I ground my discussion in the study I conducted of youth pursuing mathematics across settings. My hope is to initiate productive conversation regarding overlap and tensions between ethnographic accounts of cross-setting learning and interpretive accounts of transfer, while fully recognizing that the two bodies of scholarship are not aimed at understanding synonymous processes. In the body of the chapter, I give primacy to considering what tenets of ethnographic accounts of learning across settings might suggest for interpretive accounts of transfer. In the conclusion, I consider some tensions in doing so, as well as the potential benefit of working across these two approaches to make sense of cross-setting learning.

12.1 Ethnographic Accounts of Learning Across Settings

In the last couple of decades, it has become increasingly common to carry out ethnographic studies of people learning content and forging interests and identities across settings (see, for example, Barron, 2010; Barron & Bell, 2015; Penuel, DiGiacomo, Van Horne, & Kirshner, 2016; Tuomi-Gröhn & Engeström, 2003; Vossoughi & Gutiérrez, 2014). Ethnographic studies, in general, aim to understand a particular phenomenon (e.g., participating in classroom mathematics) from the members' points of view; to do so, researchers attend especially to the ways in which members interpret, or give meaning, to their participation in particular events and practices in context (Hammersley & Atkinson, 2007). Ethnographic accounts of learning across settings typically reflect and build on assumptions and findings associated with sociocultural accounts of learning.

Sociocultural accounts posit that learning is evidenced by a change in participation in the practices associated with a community. Moreover, a change in participation reflects both a change in what one knows (e.g., understandings of content) and does (e.g., how one participates, how one uses a tool) as well as in who one is, which entails both how a person views and positions herself and how she is viewed and positioned by others within the community (Lave & Wenger, 1991; Packer & Goicoecha, 2000). From this perspective, "Knowledge—perhaps better called *knowing*—is not an invariant property of an individual, something that he or she has in any situation. Instead, knowing is a property that is relative to situations, an ability to interact with things and other people in various ways" (Greeno, Moore, & Smith, 1993, p. 99).

Learning is conceptualized as occurring along a trajectory of participation, whereby a person participates in events that are linked (via people's participation, but also possibly through tools and other resources available); and it is across linked events that shifts in a person's participation as well as social positioning occurs (Lave & Wenger, 1991; Wortham, 2006). Emphasis is placed on characterizing the quality of shifts in participation (i.e., learning) that occur, and in explaining why and when such shifts occur or not in relation to key features of the focal context, specific to the various events. Scholars typically attend to features like the nature of social relations, discourses, ideologies, and the tools available and routines for using the tools to explain why particular people become more central in a given community—and why others either remain in or take up more peripheral positions (e.g., Brown, Collins, & Duguid, 1989).

Until more recently, empirical accounts of mathematics learning reflecting a sociocultural perspective tended to focus on individuals' shifting participation in a particular setting, like a classroom (e.g., Boaler, 2000; Gresalfi, 2009) or in an out-of-school setting (e.g., Nasir, 2000; Stevens, Mertl, Levias, & McCarthy, 2006). However, increasingly (not specific to mathematics learning), there has been attention to people's participation in linked events *across* settings, with a particular focus on how their participation in one setting has bearing on their participation in an alternative one. In particular, studies have focused on understanding the relationships between participating in out-of-school and in-school activity, often with the intent of designing for increased continuity between the two, especially for groups of students from communities that have been historically marginalized through schooling (for reviews of this literature, see Barron & Bell, 2015; Bronkhurst & Akkerman, 2016). Some scholars have argued for the importance of attending to cross-setting participation, in part, in terms of its ecological validity (Barron, 2006; Stevens, Wineburg, Herrenkohl, & Bell, 2005; Vossoughi & Gutiérrez, 2014). Tracing participation across settings represents a "truer" understanding of how it is that people actually experience the world and deepen their knowledge, senses of themselves and others, and so forth. Accounts that are more ecologically valid support better design of learning environments.

One strand of ethnographic accounts of cross-setting learning has focused on how youth develop and grow their interests and engagement across settings, which, in turn, "advance[s] their learning in particular domains" (Barron, 2010, p. 114). For example, Barron traced the development and nurturing of youth's interests in technology (e.g., web development, computing) across "networks of support" (p. 114); youth pursue their interests, often with encouragement from family members, and in pursuing their interests, they interact with others with similar interests who span a number of settings. Barron showed that youth's interest and their sustained engagement in activities through which they continue to deepen their knowledge and skills specific to computing is clearly not confined to a particular setting.

Another strand has focused on the development of "hybrid" practices and settings that emerge when youth are explicitly encouraged to share and leverage resources that are typically associated with participation in one setting in a new setting. For example, Gutiérrez, Higgs, Lizárraga, and Rivero (2019) describe how an after-school Science, Technology, Engineering, Arts and Mathematics program was deliberately designed for youth to relate in different ways to one another and to adults than is typical in school settings. Youth were positioned as the "principal designers" in digital activities in the after-school program (e.g., designing a video game, creating a video about an interest), and they were encouraged to use their interests to inspire the focus of their design. Ethnographic study of the youth's resulting activity indicated that they "leveraged [digital] tools and practices from their after-school program and everyday gaming to new media practices in their households" (pp. 69–70). As they did so, family relations and typical home practices shifted. Moreover, the youth's engagement and learning specific to their design work was sustained across home and after-school settings.

12.2 Four Tenets of Ethnographic Accounts of Cross-Setting Learning: Considerations for Interpretive Accounts of Transfer

Although the focus and specific frameworks guiding ethnographic accounts of cross-setting learning vary somewhat, depending on theoretical commitments and the focal content being explored, they share a set of tenets. In what follows, I elaborate on each tenet, make observations regarding how interpretive accounts of transfer may or may not reflect the tenets, and raise questions regarding the potential value in considering the tenets in transfer research. To ground the discussion of each tenet, I provide examples from a 14-month ethnographic study I conducted of two 10-year old African American youth (Nikki Martin and Timothy Smith[1]) and their families engaging in mathematical activity across home, school, and community contexts. I provide a bit of background here, before moving to the tenets, to contextualize the examples on which I draw. (For more details, including about methodology, see Jackson, 2009, 2011.)

Nikki, Timothy, and their families lived in a low-income neighborhood in a large city in the northeastern United States. I had worked with them as a mathematics specialist for a number of years prior to the start of the study. I initially began the study in the spring of their fourth-grade year at their neighborhood school. However, Nikki and Timothy moved from their neighborhood school to a charter school, Johnson Middle School, across the city at the start of fifth grade. I used ethnographic methods (e.g., participant observation, interviews, document collection) to trace and analyze how the youth and their families experienced and made sense of their participation in and across home and classroom settings, specific to mathematics. Over the 14-month period, observations included approximately 60 hours in each of Nikki's and Timothy's home; about 18 hours in their fourth-grade math classrooms in the spring; and about 130 hours, usually two consecutive days a week, over the course of an academic year in their fifth-grade mathematics class. In addition, I conducted eight interviews with Nikki and five interviews with Timothy, three interviews with each of Nikki's and Timothy's mothers, three interviews with their fifth-grade mathematics teacher (Ms. Ridley), and two interviews with a special-education teacher who worked with Timothy.

12.2.1 Tenet One: People's Participation Varies in Relation to Context

The unit of analysis for most ethnographic, cross-setting accounts of learning concerns the person in context (Lave, 2012). The assumption that people's participation in any given activity varies, or is in relation to, aspects of the context is perhaps trivial. However, as Engle et al. (2012) noted, issues of context are "underemphasized in most transfer research. When context is addressed, it is primarily treated as a 'physical reality'" (p. 216); attention is given to who was present, the tools available, and where the test of transfer occurred. However, interpretive accounts of transfer center person in relation to context; they attend to the interactions through which participants generate meaning specific to the focal content (Pea, 1987). For example, the actor-oriented theory emphasizes how "social interactions, material resources, and normed practices" shape the learner's (i.e., actor's) understanding of

[1] All names are pseudonyms.

a given situation, from the perspective of the learner, not an outside observer (Lobato, 2012, p. 241). As another example, Engle et al. (2012) emphasized the importance of how focal activity is framed, and in relation to how learners are positioned. They convincingly show the importance of expansive framing, or of encouraging youth to draw upon their past activity to engage in the current activity, and in suggesting the value of the current activity for future activity, and thus, of positioning youth as having authority with respect to their learning (Greeno, 2006).

Social interactions, social positioning, the use of material resources, and norms of established practices are central in making sense of person in context in ethnographic, cross-setting accounts of learning. In addition, cross-setting accounts suggest the value in broadening even more so what counts as relevant context in making sense of a person's participation in a given activity. Of course, what is relevant context in any given event is an open, empirical question. However, ethnographic studies indicate the value in attending to broader discourses that circulate, often across settings, regarding the particular youth, the content they are learning, and so forth (Gresalfi, Taylor, Hand, & Greeno, 2009). For example, although mathematics may be constructed as a sense-making activity in a given classroom, youth have often been subject to narrower constructions of mathematics in prior classrooms. How context shapes a youth's interpretation of a given task is not restricted to, or bounded by, the dominant discourses of a given setting.

Moreover, ethnographic, cross-setting accounts illustrate that "persons are not 'the same' in different situations: Their identities are partial and plural" (Lave, 2012, p. 162). Further, histories matter for how it is that youth interact with particular people in a particular setting, and these histories are not solely individual. Histories are informed by long-standing relations between, for example, the community in which a student is located and educational institutions. And, as discussed earlier, ethnographic accounts also suggest that people's participation is shaped by their own interests, that is, what they find compelling to engage in. These interests may emerge, deepen, or fade in a given event relative to the context.

The value in expanding what counts as context was especially clear in making sense of Nikki and Timothy's participation in their fifth-grade mathematics classroom. The charter school was part of a network that explicitly aimed to support economically disadvantaged youth of color to attend college. The school leaders and the overwhelming majority of the teachers, including Ms. Ridley, were white. As I have documented elsewhere (Jackson, 2009, 2018), deficit discourses circulated throughout the school regarding the incoming fifth graders' supposed inabilities to "do school" and lack of moral character, and these discourses shaped policies and practices, especially regarding discipline and the completion of homework. Moreover, these discourses impacted how mathematics was constructed as a subject matter and how it was taught. The charter-school network's fifth-grade mathematics curriculum as intended and enacted was primarily aimed at supporting students to memorize procedures for solving routine sets of problems. Speed and accuracy were valued over making sense of central ideas and problem-solving processes. Of course, it is not unusual in the United States for mathematics classroom instruction to prioritize learning procedures without understanding. However, as I showed elsewhere (Jackson, 2009), the decision to prioritize memorizing procedures went hand in hand with the construction of the youth as unprepared to engage in making sense of ideas. More generally, understanding these broader discourses was essential to making sense of how it was that students like Nikki and Timothy were positioned in the classroom, and their subsequent learning of mathematics.

Of course, individuals differ in how they interact and are positioned in any given setting. As it turned out, in Ms. Ridley's classroom, Nikki was regularly recognized as a top student by her teacher and other students. She was quick to participate in math class, and she was generally accurate and satisfied with providing answers to routine problems. Timothy was routinely positioned as an unsuccessful student and found it challenging to participate in the mainstream math classroom. There was evidence that he had some cognitive delays, and especially in a classroom where speed was valued, he stood out in relation to many of his peers; he struggled to respond quickly to questions, and it took him longer than the teacher expected for him to complete written work.

As McDermott (1993) observed, assets and deficits only become visible and "real" in relation to what is prioritized or valued in a given setting. To this point, for about 30 minutes at the start of several lessons in a given week from October to January, Timothy participated in a pull-out mathematics class with a few of his peers, led by Ms. Sanchez, a teacher trained in special education. In Ms. Sanchez's class, speed was no longer valued, and Timothy eagerly participated and completed his work and was positioned competently with respect to mathematics (Jackson, 2009).

The contrast between Nikki and Timothy within the mainstream class, as well as the contrast between Timothy in Ms. Ridley's and Ms. Sanchez's class, illustrates that people's participation must be understood with respect to what is valued and made visible (or not) in a given setting. Moreover, to make sense of the guiding norms of participation and social interactions, it is likely important to consider broader discourses that shape activity and people's positioning (in this case, discourses about youth and mathematics), as well as the histories that individuals bring from alternative settings.

Adopting a broader view of what might count as relevant context in studies of transfer is likely of value in investigating and explaining desired as well as undesired outcomes, including variation in outcomes in a given setting. As Pea (1987) observed, characterizing, or judging, transfer necessarily involves cultural values and assumptions on the part of both the learner and the observer. For example, deciding whether to use a particular set of skills to accomplish a task is, in part, dependent on whether one views those skills as relevant or of value in the context in which the task is situated. And, judgments about what is valuable cannot be taken for granted, nor are they necessarily consistent across learners in a given situation.

There are certainly limits in what it is feasible to attend to, contextually, in a given transfer situation. However, the findings of sociocultural accounts of learning mathematics, more broadly, suggest the importance of at least attending to focal learners' patterns of participation in relation to what it means to know and do mathematics and who is positioned as capable in mathematics (Gresalfi et al., 2009).

Attention to these issues reflects Greeno's (2006) observation that transfer rests, in part, on individuals' agency to decide to act in a particular way across events. As Greeno suggests, students who are positioned as having mathematical authority and are provided opportunities to develop deep understandings of mathematics and ways of reasoning in one setting are more likely to participate with agency (e.g., apply prior understandings to solve a new problem) in a new setting. However, if students are positioned primarily as receivers of knowledge in a prior setting and mathematics is constructed as a set of procedures to mimic, students are less likely to risk drawing on their prior understandings when presented with an unfamiliar problem. Attending, then, to how students are positioned with respect to what it means to learn and do mathematics across settings may, at a minimum, allow for a researcher to explain variation in terms of students' ways of reasoning within a given setting and between settings, or at least raise questions about variation that are worthy of future attention.

12.2.2 Tenet Two: People and Settings Change Over Time and in Relation to One Another

A second, related tenet of sociocultural accounts of learning is that both people and settings change, and in relation to one another. Within any given setting, people are transformed as they participate in practices over time. And, the practices themselves, as well as the settings in which they are located, are transformed as people participate in them. Moreover, people's changed participation in a given activity is, oftentimes, informed by their activity in an alternative setting. In addition, part of what changes as learners move across settings is the meaning of what it is they were seeking to make sense of at an earlier point in time. Dreier (2008) writes:

> As learners move into different contexts from the context in which they first addressed a learning issue, the meaning of this issue for them may change, and other contexts may reveal other aspects of the learning issue and other opportunities for learning about it. (p. 3)

Tracing Timothy and his family's work in response to a strict homework policy, as enforced by the school, illustrates this tenet well. Johnson Middle School teachers generally assigned homework in each subject area, and it was required that parents sign off on their students' completed homework each evening. Students were punished if they did not bring completed, signed homework to school the following day. Prior to attending Johnson Middle School, Timothy's parents had checked when he arrived home what his homework was for the evening, and they checked again that he completed his homework each evening. However, at Johnson, the amount of homework was substantially more as compared to his fourth-grade class at his neighborhood school, and, especially given the consequences of incomplete homework and the speed at which Timothy worked, his parents closely monitored his homework time. As soon as he came home, he shared what he needed to do for homework with a parent and then sat at the dining room table until it was complete.

One of his parents, typically his mom given that his dad often worked in the evenings, checked his work over the course of the evening.

If Timothy had difficulty solving a mathematics problem, his parents worked with him; they, too, were often unsure of how to solve a given problem but regularly turned to Timothy's notes from class and resources on the Internet, as well as asked Timothy's older sibling for assistance. If they continued to struggle in solving the problem, either Timothy, or sometimes one of his parents, would call Ms. Ridley. Johnson Middle School had a policy whereby teachers were required to answer their cell phones until early evening in relation to homework questions.

As substantiated in Jackson (2011), over the course of the year, Ms. Ridley shifted her interpretation of Timothy's participation patterns in the classroom, and in large part in response to her evolving interactions with Mr. and Mrs. Smith and Timothy, primarily regarding homework. At the start of the year, in an interview, Ms. Ridley initially described Timothy's apparent struggles in class as due to a cognitive disability. However, over time, as Ms. Ridley engaged in homework phone calls and meetings with the Smiths, she suggested that his supposed lack of "speed" and "effort" was because his parents had "coddled" him, as evidenced by their involvement in his homework, and thus, he lacked intrinsic motivation to participate in math class. In addition, Ms. Ridley's evolving interpretation of his family's participation in Timothy's schooling affected how she socially positioned Timothy in the classroom. At the beginning of the year, she encouraged his participation; however, nearer to the end, she occasionally chastised him for what she perceived was a lack of effort on his part.

Stepping back, as evidenced in this example, it was the *change* in Timothy's relative positioning in the classroom with respect to changes in how his family was perceived by his teacher that were central to making sense of what he was learning when and why. Attention to the changing relations between person and context highlights that the shifting context impacted Timothy's learning, and vice versa. Further, the related changes in classroom setting and person were not restricted to events that happened in the classroom.

Vossoughi and Gutiérrez (2014) suggested that the assumption that persons and setting change in relation to one another "challenges traditional notions of 'transfer,'" which privilege attention to the reproduction of particular practices across events (p. 610). This tenet of cross-setting accounts of learning raises the question of the value in investigating how a "transfer context" might, more broadly, shift as a result of the various meanings people come to make in a given setting. Are relations between people changed in the focal setting as learners deepen their understanding of particular ideas? If so, in what ways?

Engle et al.'s (2012) work on expansive framing is an example of interpretive transfer research that, at least implicitly, is suggesting the value of attending to changing relations between settings and people. They write, "In an expansive framing of roles, learners are positioned as active participants in a learning context where they serve as authors of their own ideas and respondents to the ideas of others" (p. 218). Presumably, as learners are increasingly positioned as such, this leads to changes in the social relations in the transfer context, and potentially the norms of participation, more generally, in the focal setting. It could be useful to know how expansive framing might, in turn, support deeper learning of focal content as well as changes in how the learners identify with respect to the content and one another.

12.2.3 Tenet Three: Learning Is Not Complete in Any One Event

A third tenet of ethnographic accounts of cross-setting learning is that learning is not complete in any event. As Dreier (2008) writes, cross-setting accounts indicate that "persons' pursuits of learning in relation to a particular learning issue are rarely finished in one learning situation and context" (p. 87). Instead, much learning, or people's changing participation in practices, is "open-ended" (p. 89). Moreover, "this open-endedness makes it complicated for persons to define the status of their prior learning and their reliance on it" (p. 89). The transfer literature often, either explicitly or implicitly, suggests that learning (whether it be understanding a procedure or a concept or, for example, developing a process for handling symbolic representations) is complete in the first event. In fact, the common nomenclature of the "learning context" and the "transfer context" reflect (whether intentionally or not) this assumption. It suggests that learning has a beginning and an end.

Yet, cross-setting accounts of learning repeatedly show how what it is that people pursue ebbs and flows (and sometimes goes dormant) in relation to their personal interests, as well as how particular activities are organized; and, as such, what they know and who they are becoming develops over time, and often across diverse settings. I illustrate this again in relation to homework, but this time from an event that happened in Timothy's fourth-grade classroom.

As described in detail in Jackson (2011), in the spring of fourth grade, Ms. Jones's class was focused on adding and subtracting fractions with unlike denominators. The homework provided a new context with which to reason about fractions: shaded sectors on an analog clock. For example, in one problem, a clock face had a sector from 12:00 to 4:00 shaded in dark blue (20 minutes out of 60 minutes, or 4 hours out of 12 hours), and another sector from 4:00 to 6:00 shaded in light blue (10 minutes out of 60 minutes, or 2 hours out of 12 hours), and students were expected to use what they knew about time to add them together (e.g., $1/3 + 1/6 = 1/2$). Timothy puzzled over the problem, as did his mother and sister. His mom then suggested that Timothy ignore the context of time and instead estimate the fractional amount shaded for a combined sector of the circle (e.g., dark blue plus light blue). Timothy did so and made reasonable estimates. His estimates for 15 minutes (1/4), 30 minutes (1/2), 45 minutes (3/4), and 20 minutes (1/3) were exact. His estimates for 5 minutes, 10 minutes, and 40 minutes were not wholly accurate, but were reasonable.

It turned out that most students expressed that they were confused by the clock context and therefore had not completed the homework. In response, Ms. Jones went through each of the problems in a procedural manner, and, rather than attend to Ms. Jones's explanation, Timothy began to read a book. When I asked Ms. Jones about Timothy's participation and understanding of fraction addition and subtraction a couple of days later, she told me that his answers were "close enough" and that he likely didn't listen to her explanation because he was "like a special education student" and that the activity was too difficult.

In this case, some learning about fraction addition and subtraction was initiated in Ms. Jones's classroom. While working on homework, Timothy puzzled through identifying areas of sectors, clearly drawing on some past learning about how to reason about a part in relation to a whole. Although he had not yet demonstrated evidence of full understanding of adding or subtracting fractional areas, he accurately combined two sectors to make a larger part. Moreover, the learning he appeared to be doing at home had the potential to be deepened and extended the following day; however, it was not, because it was assumed that Timothy was not capable of deepening his learning regarding this topic, at least at that moment.

This is a brief illustration of a quick sequence of potential learning events characterized by initial promise, which was, at least in the immediate moment, not capitalized upon. In this sequence, it becomes evident that, for Timothy, learning—that is, deepening his understanding, in this case, of a mathematical concept—could not be mapped to one particular event. Further, his opportunities to deepen his understanding (in both his home, where his mother supported him to continue working on the problem, and in the classroom) were bound up with conceptions of Timothy as a person. And, of course, he would revisit the concept of fractions, and of addition and subtraction of fractions, in his fifth-grade classroom. What comes to endure over these events—specific understandings of concepts, like a fraction, as well as conceptions of mathematics more generally, as well as how Timothy is positioned and positions himself in relation to mathematics—is of particular interest from a cross-setting perspective. And, from this perspective, it assumed that both understandings of content and of personhood—that is, how one is positioned and positions themselves—remain open to revision, at least to some degree.

The assumption that learning is never complete in an event, and is indeed open to revision and refinement, is at odds with an assumption that undergirds most accounts of transfer. Hohensee's (2014) recent scholarship on *backward transfer*, or "the influence that constructing and subsequently generalizing new knowledge has on one's ways of reasoning about related mathematical concepts that one has encountered previously" (p. 136), is a notable exception. In an earlier study of middle-grades students' understanding of *quadratic* functions, Hohensee noticed unproductive shifts in some students' understanding of *linear* functions. This prompted him to revise and teach a unit on quadratic functions in which he investigated how to simultaneously support students' deepening understandings of quadratic functions and support productive shifts in students' prior ways of reasoning about linear functions. On the basis of his work, he convincingly argued for the importance of considering both *forward* and *backward* transfer, both in accounting for and in designing for learning.

From a cross-setting perspective on learning, Hohensee's (2014) study illustrates well the tenet that learning is not complete in any one event. Students' prior ways of reasoning about linear functions were revised in the context of making sense of quadratic functions, in both productive and unproductive ways. As such, his work suggests the value in investigating the potential for new events (what is often called the "transfer context") to provide opportunities to deepen, question, or revise prior ways of reasoning, rather than as opportunities to confirm what are assumed to be prior ways of reasoning (based on what was demonstrated in a "learning context"). Further, as Hohensee's (2014) study illustrates, there is value in transfer research in attending to understandings about mathematical ideas that are different from the focal target of instruction.

12.2.4 Tenet Four: Power Relations Matter in Understanding What Pursuits Are Possible Where, When, by Whom, and with Whom

A fourth tenet of ethnographic accounts of cross-setting learning is that it is crucial to attend to power relations in making sense of people's participation within a setting, as well as how a person's participation in one setting bears on their participation in an alternative setting (Esmonde, 2017; Vossoughi & Gutiérrez, 2014). Moving across settings in pursuit of particular interests, ideas, and understandings can be empowering. As Penuel et al. (2016) wrote, "By moving across settings of social practice, people are able to pursue diverse concerns and become aware of new possibilities for action and arrangements for participation in practice (Dreier, 2008)" (p. 32). However, not all opportunities are equally available to all people at all times. Patterns of opportunity within a setting are often shaped by broader forces that operate to maintain particular power relations at the expense of groups or individuals (Martin, Anderson, & Shah, 2017). These patterns often become especially evident in studies of youth from communities that have been historically marginalized in and through educational systems (Bronkhurst & Akkerman, 2016).

The importance of attending to power relations was very apparent in making sense of Nikki's and Timothy's trajectories of participation with respect to school mathematics. As I indicated above, they were positioned in drastically different ways within the same fifth-grade math classroom. However, both were subject to an impoverished construction of mathematics, which I have suggested was coupled with deficit-oriented discourses of children of color from low-income backgrounds (Jackson, 2009).

And, even though Nikki in many ways was quickly positioned as a successful student in the classroom, she struggled at times within the classroom to maintain full participation given gendered relations. As I describe in detail elsewhere (Jackson, 2009), discourses regarding boys and girls in relation to mathematics shaped classroom interactions. There was a daily practice in the classroom called

"Math Royalty" in which students competed with one another to accurately complete five problems as quickly as possible. The winner was publicly recognized as the Math Queen or King, and Ms. Ridley tracked the number of Queens versus Kings, with the intention of encouraging female students to be seen as equally as capable of doing well in math as their male peers. Nikki was consistently identified as a Queen, and male students increasingly refused to acknowledge her. At one point in February, she deliberately lost Math Royalty so as to avoid the gaze of her male peers. In many ways, she still maintained her positioning as a successful student; however, what she learned and who she was positioned as were clearly shaped by political relations in the classroom.

As another example of the importance of attending to power, as illustrated above, the political relations between the home and Johnson Middle school clearly shaped Timothy's opportunities to learn mathematics, and his evolving positioning. Ms. Ridley's changing interpretation of Timothy's participation in the math classroom was in relation to her interactions with his family, that were likely shaped by broader discourses regarding the role of parents—especially parents of color with minimal formal education—in children's schooling.

Through each of these examples, it becomes clear that what happens in any one event is dependent on discourses, ideologies, and so forth that have developed across time, as well as on individuals' histories of participation, and that these relations are political (Vossoughi & Gutiérrez, 2014). This tenet reflects a growing understanding of the significance of the sociopolitical in the field of mathematics education more broadly (Gutiérrez, 2013) and poses important questions for interpretive studies of transfer. Relevant questions include: What role do political relations play in the extent to which transfer is observed? How might political relations explain variation in transfer, even within the same event, in the same setting? How might political relations that circulate outside of the focal event (e.g., within the school, between the school and community) shape the forms of participation and positioning that are observed?

Attending to the political relations seems especially important when working to iterate and improve upon prior instructional sequences that do not result in desired outcomes. Based on close study of Nikki's and Timothy's experiences, I conjecture that targeting content issues in an instructional sequence absent raising questions about the discourses that circulated in Johnson Middle School regarding what youth of color were capable of, mathematically, would have done little to alter enduring patterns—both relational and academic learning—in the classroom. What might it mean to design for transfer in which an explicit goal would be to interrupt enduring, problematic patterns about what it means to know mathematics and in relation to who is positioned as capable of engaging in academically rigorous mathematics? As an example, consider Engle et al.'s (2012) work on expansive framing. What would it look like to infuse attention to the sociopolitical into the expansive framing that teachers do to support students to view themselves as having the authority and motivation to apply previous ways of reasoning in a new context? As another example, consider Nathan and Alibali's (2021) argument in this volume for the importance of attending to the social relations between teachers and learners across events in an embodied theory of transfer. They explicitly call for attention to how "dimensions of social relationships, such as warmth, respect, and power" may shape whether and how learners notice and respond to opportunities to bring to bear initial learnings in novel contexts (Nathan & Alibali, 2021, p. 54). What might it mean to consider those dimensions of relationships in both organizing for transfer (e.g., supporting teachers to design for cohesion) and in making sense of learners' resulting experiences?

12.3 Conclusion

In this chapter, I identified four tenets of ethnographic accounts of cross-setting learning: (a) people's participation varies in relation to the context; (b) people and settings change over time and in relation to one another; (c) learning is not complete in any one event; and (d) power relations matter in understanding what pursuits are possible, when, by whom, and with whom. Each of these, as I indicated above, may provide those engaged in interpretive accounts of transfer with questions or ideas worth considering, in service of both designing to support learning and in broadening how we explain when more desirable forms of learning happen and when they do not.

Of course, there are tensions in broadening what one attends to in the study of transfer. One potential tension regards the precision with which one can identify differences in students' ways of reasoning with respect to targeted understandings. Clearly, attention to detailed differences in students' ways of reasoning is a strength and foundation of interpretive accounts of transfer. Analysts may worry that increasing attention to other aspects of context (e.g., ideologies regarding youth and mathematics) will result in decreasing attention to students' ways of reasoning.

A second, related tension regards acting on what is learned from studies of transfer that broaden their scope. Another strength of interpretive accounts of transfer is that they often inform tight instructional sequences that can then be trialed and investigated in new contexts. As one expands the scope of investigation, it is presumably more challenging to design a subsequent instructional sequence that takes into consideration, for example, broader discourses about mathematics and who is capable of engaging in mathematics alongside the development of core mathematical ideas. And, targeting discourses may be a longer-term activity, as compared to targeting students' understanding of some content. Making principled decisions about what to target when, and how, in an instructional sequence is already difficult, intricate work; one can imagine the additional learning demands for instructional designers when choosing to broaden the scope of what to target in a sequence.

A third tension regards establishing standards of evidence to both guide and evaluate analyses that bring together a fine-grained analysis of students' reasoning with a focus on, for example, political relations. Doing so would entail forging new methodologies. It would also require figuring out how to communicate the work to different audiences, which would presumably require some perseverance.

More generally, both investigating extant learning and designing for subsequent learning would require prolonged collaborations between researchers with varied expertise. Such collaborations could enrich the ethnographic study of cross-setting learning just as it might enrich studies of transfer. Ethnographic accounts of cross-setting learning tend to foreground learning as "becoming"—that is, becoming a person who is socially recognizable as capable and competent in a specific set of practices, who identifies with and is judged by that evolving community to be a valued member. Interpretive accounts of transfer tend to foreground how a person develops (and can be supported to develop) sophisticated understandings of and ways of reasoning about core mathematical ideas. Working across these perspectives could go some way to understanding how to design and implement instruction that supports people to pursue their interests and develop desired identities and deep understandings of meaningful ideas across the varied settings of their lives.

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Chapter 13
Transformative Experience: A Motivational Perspective on Transfer

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Transfer research has primarily focused on the capacity to transfer and, increasingly, the social and cultural affordances of and constraints to transfer (e.g., Greeno, 1997; Haskell, 2001; Nokes-Malach & Richey, 2015). Less research has addressed the motivation to transfer: that is, the choice to apply learning in a transfer context when such application is not required. A helpful model for considering the role of motivation in transfer is the detect–elect–connect model proposed by Perkins and Salomon (2012). According to this model, transfer involves detecting a potential application of prior knowledge, electing to pursue this application, and working out the appropriate connections. Motivation is inherent to this model and Perkins and Salomon (2012) argued that motivational aspects have received less attention in the transfer research. For example, considerable research explains the kind of cognitive structures needed to detect correspondences and work out the connections (e.g., Chi & VanLehn, 2012; Spiro, Collins, Thota, & Feltovich, 2003). Less research addresses the motives that may drive a transfer episode.

The issue of motivation becomes particularly important when we are concerned with transfer of learning from a school setting to everyday life. Such transfer is often implicit in our conceptions of the purpose of education. For example, Dewey (1938) advocated for an education in which students take the ideas they learn in school and use them to enrich their everyday lives outside of school. When we are concerned with transfer of learning to everyday experience, the detect phase requires more than a cognitive capacity to recognize a correspondence between prior knowledge and a task that is set in front of the student by a teacher or researcher. It does require this, but, in many cases, it also requires a motivational desire to seek out a correspondence between school knowledge and life outside of school. Further, the elect phase is no longer driven by a need to meet the demands of the teacher or

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researcher. It now needs to be driven by a more autonomous form of motivation: a personal choice to pursue a connection.

Consider a high school student who has recently finished studying ecology in science class. The ecology unit examined the relationships between organisms and their environments as well as organisms and their physical surroundings. Suppose a story comes up on the students' Facebook newsfeed about the possibility of releasing wolves into certain regions of Colorado. Lacking a motivation to transfer, the student may scroll past without thinking twice. But possessing a motivation to transfer, the student will be primed to *detect* a connection to the ecology unit. Such motivation to transfer would further drive the student to *elect* to read the story, work out *connections* to the ecology unit, and fully engage in transfer of learning. When we neglect a motivational perspective on transfer, we run the risk of developing the capacity to transfer without the drive to transfer. Knowledge learned in such a way that it could be applied may still be inert because the student does not choose to apply it.

We believe our work on transformative experience (TE) theory uniquely positions us to contribute to a motivational perspective on transfer. TE theory seeks to define, understand, and foster experiences in which students use in-school learning to see and experience the world differently in their everyday lives outside of school and find value in doing so (Pugh, 2011). In this chapter, we clarify how this conception of experience relates to the detect–elect–connect framework and intersects with other transfer perspectives touching on motivation. Further, we draw on TE theory to identify principles related to fostering a motivation to transfer. We then consider the meaning and application of these principles in the domain of math education. Most of the work on TE theory has been in the domain of science education, so our extensions to math education are speculative and in anticipation of future research.

Before discussing TE theory and its relation to motivated transfer, we briefly define the conceptions of transfer and motivation we are drawing on. Conceptions of transfer are diverse, ranging from more traditional conceptions defining transfer as the successful application of knowledge or skill in novel or more complex situations, to alternative conceptions defining transfer as, for instance, preparation for future learning (Bransford & Schwartz, 1999) or the influence of participation in one activity setting on participation in another activity setting (Greeno, 1997). We adopt a traditional conception of transfer and, as illustrated in the ecology example above, we have a particular interest in the transfer of in-school learning to everyday experience. According to the taxonomy proposed by Barnett and Ceci (2002), such transfer represents far transfer in terms of the physical and functional context and, consequently, is difficult to achieve.

Motivation is also conceptualized in many diverse ways. We are particularly interested in autonomous forms of motivation: that is, purposeful and self-initiated forms of action (Ryan & Deci, 2017; Wentzel & Brophy, 2014). Although in-school to everyday-experience transfer situations may involve some form of extrinsic motivation (e.g., an individual is motivated to apply basic math skills during a monetary exchange to avoid being cheated), we are especially concerned with situations lacking a strong extrinsic component. The ecology example above represents such a situation in that the student is not being rewarded, punished, or pressured to read the story and make connections to ecology. The motivation to transfer referenced in this example is autonomous motivation. Finally, our interest in motivation to transfer is more focused than the broader issue of the relation between motivation and transfer. There are many ways by which motivation patterns during learning can influence later transfer success (for a review, see Pugh & Bergin, 2006). For example, students' goal orientation while learning can influence their ability to transfer their learning to a new learning situation (Belenky & Nokes-Malach, 2012). However, we are interested in motivation driving the transfer process itself.

13.1 TE Theory and Motivated Transfer

TE theory begins with the construct of a transformative experience, which Pugh (2011) defined as "a learning episode in which a student acts on the subject matter by using it in everyday experience to more fully perceive some aspect of the world and finds meaning in doing so" (p. 111). As an example of a TE, Pugh (2004) described a middle school student (Ed) who applied Newton's Laws many times in his everyday experience. For instance, Ed saw his niece slide across the kitchen floor and crash into a door. He applied his physics lens and concluded that his niece was a great example of an object in motion continuing in motion until being acted upon by another object. Through such application, Ed came to perceive the world differently. He commented, "I can look at, like, when two cars crash into each other, I can look at that in a different way, and when I watch a movie I can look at that in a different way. Now I'm going to see things that I'm used to seeing in a different way" (Pugh, 2004, p. 189). This expansion of perception was meaningful to Ed and it led him to value Newton's Laws because they "made me think about stuff that I'm not used to thinking about in that way" (Pugh, 2004, p. 187). This conception of TE is derived from Dewey's work on aesthetic experience and ideas. Below we give a brief summary of Dewey's work and its connection to TE (for a more detailed explanation, see Pugh, 2011). Then, we provide a more elaborated definition of TE and, finally, clarify the connection between TE and the detect-elect-connect model of transfer.

13.1.1 Dewey and TE

Much of Dewey's (1934/1980) work on aesthetics revolves around the construct of "an" experience. Central to "an" experience is a transformation of our relationship with the world. Dewey believed authentic engagement in the arts epitomized such transformation as explained by Jackson (1998):

> Our interactions with art objects epitomize what it means to undergo an experience. . . . The arts . . . expand our horizons. They contribute meaning and value to future experience. They modify our ways of perceiving the world, thus leaving us and the world itself irrevocably changed. (p. 33)

"An" experience is transformative in that it involves an expansion of one's perception of the world. Moreover, value and meaning are transformed as one comes to perceive successive layers. What once was taken for granted is now appreciated. Ultimately, one's relationship with the world is transformed and the individual comes to be in the world differently.

Content ideas can have a similar transformative effect. Dewey (1933/1986) distinguished between *concepts*, which are established meanings, and *ideas*, which are possibilities. Building from Dewey, Prawat (e.g., 1996, 1998) argued that big *ideas* awaken anticipation and such anticipation leads individuals to act on the world in new ways (i.e., to try out the ideas). The worth of an idea is then determined by the difference it makes in everyday experience. Prawat (1998) explained,

> Children, like adults, can get excited about big ideas. The anticipations or expectations that accompany the development of new ideas generate sufficient interest to carry individuals into the more effortful process of reflectively testing whether the idea has cash value. (p. 219)

Cash value refers to the significance the idea has in everyday experience (i.e., the degree to which it opens up new experiences). Thus, engaged learning is centered around a process of anticipation leading to action and a meaningful transformation of everyday experience.

Pugh and colleagues (Pugh, 2004, 2011; Wong et al., 2001) have taken this meaning of engaged learning as a basis for defining TE. TE represents engagement with an idea resulting in the type of transformation of perception and value characteristic of "an" experience.

13.1.2 Further Defining TE

Pugh (2002, 2011) further defined TE in terms of three characteristics: motivated use, expansion of perception, and experiential value. These characteristics represent an attempt to translate Dewey's constructs of "an" experience and ideas (e.g., anticipated action, transformation of perception) into a researchable construct. These three characteristics all need to be present at some level for a TE to transpire.

Motivated use refers to applying school content in everyday life outside of the classroom when such application is not required. In the example presented earlier, Ed applied Newton's Laws a number of times in his everyday life simply because it occurred to him to do so. Motivated use is intertwined with the aspect of non-constrained transfer (Pugh & Bergin, 2005). In addition, the construct is in line with conceptions of interest as a willingness to reengage (Hidi & Renninger, 2006;

Renninger & Su, 2012), particularly when such reengagement occurs in out-of-school settings (e.g., school-prompted interest; Bergin, 1992).

Expansion of perception refers to seeing the world through the lens of the content. Through this process, the student begins to re-see successively deeper layers of meaning (Girod, Rau, & Schepige, 2003). For example, Ed came to see events of motion (a car crash, a niece and door crash) in novel ways, ways he had not thought of before. The notion of expansion of perception overlaps with other cognitive perceptual models (e.g., schema development, conceptual change, role of perception in transfer), but the unique difference is the role it plays for expanding and enriching future experiences.

Experiential value refers to valuing content for the way it expands perception and experience. When students use content to perceive deeper layers of meaning, they often develop greater interest in and appreciation for the world and the content itself. This notion is similar to Eisner's (1991) contention that value and meaning grow as one becomes a connoisseur and is able to perceive successive layers of an object. For example, Ed developed greater interest in events of motion and increased appreciation for Newton's Laws as he used them to perceive the world.

13.1.3 TE in Relation to the Detect-Elect-Connect Framework

The notion of transfer is implicit in the definition of TE, particularly as it relates to the detect and elect phases of transfer. In this section, we discuss the relation between TE and the detect-elect-connect framework. Further, we clarify the role of motivation in each phase.

TE and the detect phase To undergo a TE, students must first detect a connection between content learned in school and events, objects, or issues encountered in everyday life outside of school (i.e., motivated use). Such detection may be of either the high-road or low-road form of transfer (Salomon & Perkins, 1989). High-road transfer involves deliberate thought, reflection, and mindful application of knowledge in new settings. Low-road transfer is more automatic. Many cases of TE involve deliberate, mindful application of learning (i.e., high-road transfer). For example, an elementary student learning about erosion commented, "At recess I look around on the blacktop for weeds and bugs and stuff that might be causing erosion" (Girod, Twyman, & Wojcikiewicz, 2010, p. 818). This student was actively and consciously detecting opportunities to transfer her erosion knowledge. Other accounts of TE suggest detection can also be of the low-road form in the sense that seeing the world a particular way is an almost automatic, uncontrollable reaction. For example, in a study by Pugh, Bergstrom, Heddy, and Krob (2017), one middle school student talked about seeing the everyday weather she experienced in terms of science ideas as an almost unbidden process: "I can't really get it out of my head. Like, I can't help it, when I see something that involves weather, or the heat transfer…It's stuck in my head and I can't get it out" (p. 387).

In the context of TE, both high-road and low-road detection of a transfer opportunity are motivationally driven. The role of motivation is more obvious in the high-road example, but it is also pertinent to the low-road example and, in both cases, associated with anticipation. The construct of anticipation is central to TE theory. Anticipation is seen as the force that instigates action and moves an experience forward (Dewey, 1938, 1934/1980). Commenting on Dewey's theory of aesthetic and educative experience, Wong (2007) explained,

> Anticipation is what transforms an ordinary occurrence into an event saturated with significance and moving forward with dramatic energy. Whether the learner is engaged in reading a story, watching a film, or conducting scientific inquiry, anticipation is what moves us to the edge of our seat so that we may see better and be better prepared for what we might see. (p. 208)

Thus, we propose that the student's act of looking for erosion at recess was instigated by anticipation developed in class. Such anticipation may have been along the lines of "Could I find erosion? Is it really happening here? What might I see?" This type of high-road, deliberate seeking out of opportunities to transfer may relate to Haskell's (2001) idea of the spirit of transfer, or a tendency to seek out opportunities for deep learning and transfer. Haskell's view of the spirit of transfer was more dispositional or trait based; however, it may be that some individuals utilize anticipation related to transfer differently than others.

We believe anticipation played a similar activating role for the student who said she couldn't help but think about weather in terms of science ideas that were "stuck" in her head. However, this example requires further explanation. Many forms of low-road transfer are not driven by motivation. For example, virtually all of us apply formal units of time (e.g., minutes, hours, days) whenever we solve problems involving time. We don't need to do so intentionally because these formal units have come to mediate our thinking of time (Vygotsky, 1978). Further, it is hard to see a role for anticipation in this example. Does anyone excitedly look forward to thinking about time in terms of formal units? Many other examples are similar such as the transfer of reading skills to read new texts or the application of basic math in routine situations. The example of the girl automatically thinking about science ideas when viewing weather events differs in two important ways. First, most of the other students in her class did not automatically perceive weather events in terms of science ideas even though they learned the same ideas. Thus, there was something unique about this student's relationship with the content. Second, this student expressed a deep fascination and interest in the weather ideas she was learning (Pugh et al., 2017). We believe it likely this fascination created an anticipation that triggered her science thinking whenever she saw weather events. Thus, although the detection was described as more automatic by the student, it was still anticipated and potentially triggered by that anticipation.

TE and the elect phase To undergo TEs, students must also elect to pursue the connection detected. Expansion of perception requires a sustained effort to see the world differently. If the students in the examples above ceased their engagement upon detecting a transfer opportunity, then their experience would be forestalled at mere recognition. Perception requires "an act of reconstructive doing" (Dewey, 1934/1980, p. 53): that is, an active re-seeing of a familiar object, event, or issue. This perspective parallels Engle's (2006) emphasis on the role of agency in transfer: "Transfer involves not just knowing but doing, and that doing inherently involves an exercise of human agency. Thus, if transfer is going to happen, I argue, it is necessary that learners choose to use what they have learned" (p. 455). This act of agency denotes a clear role of motivation and seems to apply to both the detect and elect phases. In the elect phase, it is choosing to follow through on an application opportunity once it has been detected. Research from the workplace literature shows that *motivation to transfer* (i.e., desire to use knowledge and skills acquired through training) influences actual application of training (Holton, 1996; Noe & Schmitt, 1986). For example, Axtell, Maitlis, and Yearta (1997) found motivation to transfer to be a significant predictor of transfer of interpersonal skills a full year after training, even when controlling for other important predictors such as self-efficacy, perceived autonomy, management support, and transfer success at 1 month.

Within the context of TE, motivation to transfer and agency take the form of a desire and choice to apply learning in everyday experience. After a detection is made, it is choosing to deliberately use content as a lens to see and understand the world in new ways. What is it that propels an individual to carry through such an expansion of perception? Again, we propose that anticipation plays a central role. Not only is anticipation an instigator of actions, but it is the force that carries an experience through to completion (Dewey, 1934/1980; Wong, 2007).

Thus, the constructs of detect and elect are inherent to the meaning of TE and anticipation is proposed as a driver in the transfer process. However, further research is needed on the role of anticipation within TE. Overall, this motivational approach to understanding the detect and elect phases of transfer would be a nice compliment to emerging research on how social factors influence these phases by establishing norms about appropriate contexts for application (e.g. Engle, 2006; Lobato, 2012; Pea, 1987).

TE and the connect phase Motivation to transfer plays a less salient role in the connect phase of transfer. Consequently, the construct of TE yields less insight into this phase of transfer. Nevertheless, there are motivation issues worth discussing, particularly with respect to how motivation during the initial learning phase can influence students' abilities to cognitively work out the connections in the connect phase.

During the learning phase, motivation can manifest as a focus on learning with the intent to transfer what is learned. Sternberg and Frensch (1993) referred to this learning orientation as a *mental set* for transfer. Such intent can influence the way that individuals study and process information, which in turn influences their transfer success. For example, a student may learn statistics with the intent to apply it to his involvement in a fantasy football league. This intent likely influences his learning of statistics and success at being able to apply it in contexts outside the classroom. Intentionality in this phase denotes a motivational approach to learning in that it is purposeful and goal driven. Students who undergo TEs may be more likely to adopt a mental set for transfer. That is, the orientation toward making learning transformative may be initiated in the original learning phase. Students may anticipate applying learning in everyday experience and engage learning strategies that help them learn content at the level of application.

Undergoing TEs may also support future connect phase efforts through the development of flexible knowledge structures. That is, students who undergo TEs, by definition, practice the application of learning in everyday contexts and such application in multiple contexts is key to developing the type of flexible knowledge structures needed for transfer (e.g., Haskell, 2001; Spiro et al., 2003). Indeed, existing research confirms that students who undergo TEs perform better on traditional transfer tasks, which target students' ability to work out connections (Pugh, Linnenbrink-Garcia, Koskey, Stewart, & Manzey, 2010a, 2010b). However, more research is needed to understand the relationship between TEs with particular content and skill in working out connections when transferring such content.

13.2 Teaching for TE

In past research, we found that science learning is not transformative for most students (e.g., Pugh et al., 2010a). Consequently, we and our colleagues have focused on identifying and developing instructional strategies effective at fostering TEs, resulting in the Teaching for Transformative Experiences in Science (TTES) instructional model (Girod et al., 2003, 2010; Heddy & Sinatra, 2013; Pugh, 2002; 2020; Pugh et al., 2010b; Pugh & Girod, 2007). Pugh et al. (2017) observed that sixth-grade earth science students receiving TTES instruction reported a higher degree of TE on a survey measure and in interviews than students receiving a combination of direct instruction, discussion-based, and activity-based science instruction. The TTES model yields insight into how motivation to transfer may be fostered via TEs. In this section, we describe three key strategies of the TTES model and consider how they may be adapted to math education. These three strategies include (a) framing content as ideas, (b) modeling TE, and (c) scaffolding re-seeing.

13.2.1 Framing the Content as Ideas

Framing has been identified as a method for establishing particular student orientations toward learning (Engle, Nguyen, & Mendelson, 2011; van de Sande & Greeno, 2010; Watanabe, 1993). Such framing is typically accomplished through metacommunicative signals about context, learning purpose, participation norms, and so on (Engle et al., 2011). In the TTES model, framing is used to establish a perspective of learning as engagement with ideas, in the Deweyan (1933/1986) sense of the term: that is, framing the content as possibilities that need to be acted upon and tried out in everyday experience. Central to such framing is an emphasis on creating idea-based anticipation, which is anticipation in connection to a content idea. As explained previously, anticipation is what differentiates an idea from a concept and is the driving force in a TE. Hence, fostering idea-based anticipation is a focus of the TTES model. There are various ways in which such anticipation may be fostered. One approach is to artistically craft topic introductions (Pugh, 2002; 2020; Pugh et al., 2017). Just as a novelist may craft text to create anticipation, so the teacher may craft activities or dialogue to create anticipation about the potential of the ideas to add value to one's experiencing of the world. Walter Lewin, a legendary physics teacher at the Massachusetts Institute of Technology, fostered anticipation in connection with physics ideas by saying things like,

> All of you have looked at rainbows, but very few of you have ever seen one. Seeing is different than looking. Today we are going to see a rainbow. Your life will never be the same. Because of your knowledge, you will be able to see way more than just the beauty of the bows that everyone else can see. (Rimer, 2007, p. 2)

Such framing creates anticipation about learning the content and then acting on it (e.g., "What am I going to learn and how is this learning going to change my life? Will I really see rainbows differently? How will I see way more beauty than everyone else?").

A related approach to framing concepts as ideas is to emphasize the experiential value of the content: that is, discuss and highlight the value the content has in everyday experience. Many times, students don't develop anticipation about acting on content because they don't appreciate the value of doing so. Thus, teachers can support TEs by helping students perceive the value that comes from applying particular content in everyday experience. For example, a middle school teacher shared an experience of being in a tornado and talked to students about how fascinating it is to be able to understand weather from a science perspective. He gave examples of seeing weather events from a science perspective and added, "That's what's going to be so cool, is that we're going to be able to do that" (Pugh et al., 2017, p. 642).

Metaphors can also be used to frame content as ideas and foster idea-based anticipation. Metaphors denote possible relationships between something unknown (i.e., the content to be learned) and something known (i.e., something from prior experience). Compelling metaphors do this in a way that generates anticipation about the unknown thing (Pugh & Girod, 2007). For example, as reported by Girod and Wong (2002), a teacher used the metaphor "Every rock is a story waiting to be read" as a way of presenting geology content to fourth graders as an idea that generated anticipation and action. The simple metaphor transforms geology from a set of established concepts that need to be learned to an idea, a possibility: "Really? Rocks have stories? I can learn to read the story of a rock? What if…?"

This approach to framing content aligns with Engle's (2006) work on expansive framing. According to Engle (2006), learning environments differ greatly in the degree to which they encourage students to learn with the goal of using what they are learning. The degree to which future application is conveyed in the classroom then plays a significant role in how likely students are to learn in preparation for transfer. Engle and colleagues (Engle, 2006; Engle, Lam, Meyer, & Nix, 2012) found that framing learning broadly in terms of contexts of application, instead of narrowly (e.g., "This is important to know for the test"), supported transfer through the establishment of intercontextuality, or the linking of one context to one another. Intercontextuality facilitates transfer because new information is viewed as relevant to both the learning and the transfer contexts. Further, expansive framing establishes norms for where and when application of particular content is appropriate. Such norms encourage students to *detect* application opportunities and *elect* to pursue such application. Framing content as ideas (in the Deweyan sense of the term) is one particular approach to expansive framing that emphasizes not only establishing intercontextuality but motivating actions through the generation of anticipation.

Application to math education It may be fair to say that many students are inclined to learn math concepts instead of engage with math ideas. That is, they gain mathematical understanding but not the anticipation that leads them to actively try out such understanding in everyday experience. For instance, a student may learn how to construct an algebraic equation but not see this as an exciting way of making sense of and acting upon the world. Hence, the student is not motivated to seek out transfer opportunities in everyday life. Deliberately focusing on creating anticipation through artistically crafted introductions, experiential value statements, and metaphors may help math students engage with the content as ideas. What might this look like?

"You already have two strikes against you: Your name and your complexion. Because of these two strikes, there are some people in this world who will assume that you know less than you do. *Math* is the great equalizer." This quote is spoken by a math teacher in the movie *Stand and Deliver* to his Latinx students. Such a statement creates anticipation and emphasizes the potential of math to empower students and enrich their experience. It represents an artistic crafting of content. Using statements like this, teachers can frame the purpose of math as personal empowerment.

However, framing is needed at a more content-specific and immediate level too. That is, math content needs to be framed as ideas that have the potential to enrich students' current everyday experience. This may seem problematic given that math content is generally perceived as established meanings (i.e., concepts) as opposed to possibilities (i.e., ideas). However, the teacher can present math principles as possible ways of seeing and making sense of the world—possibilities that need to be tried and judged by the students themselves. For example, an elementary teacher may tell students that the world is made of shapes and learning to recognize shapes and understand their properties will give them a glimpse into the secret structure of the world. A secondary teacher introducing how to graph relationships may tell students,

> The ability to graph relationships is a window into a whole new world. Everything in our world exists in relation to something else. These relationships are what define our world. We all understand these relationships to some degree. But math, *math* allows us to visualize and understand these relationships concretely. If you are willing to "do the math," you will

> see and understand things that no one else sees and understands. You can actually do this and I will give you the tools.

These are just a few examples. The key is that teachers frame math content in ways which create anticipation about acting on the content in everyday experience. We fear that math learning is often non-transformative because, for the most part, none of us bother to "do the math." Perhaps if we saw value in doing so, if we saw how math can make the world a far more interesting and profound place, if we anticipated this enrichment, we would be more willing to do so.

13.2.2 Modeling TE

Modeling is an important strategy for developing cognitive skills and establishing particular values and norms within a community of practice (e.g., Akerson, Cullen, & Hanson, 2009; Brophy, 2008; Palincsar & Brown, 1984). Within the TTES model, modeling is used to help students acquire such skills as re-seeing everyday objects through the lens of science content. It is also used to convey the value of the content and establish norms for engaging with the content in everyday experience. Enthusiastic modeling can also foster idea-based anticipation.

Modeling is done by sharing personal experiences, expressing a passion for the content, and generally showing students what it means to live the content (Pugh, 2020; Pugh & Girod, 2007). For example, a high school biology teacher modeled his proclivity to see the world through the lens of science by displaying a bag of potatoes. The teacher explained that he saw this bag at the grocery store and noticed the potatoes were named after a man who did work on natural selection. He commented, "Science is everywhere and now the grocery store is haunting me" (Pugh et al., 2010b, p. 289). This teacher shared other examples of thinking about natural selection in everyday life, such as reading a newspaper article about bacteria developing resistance to drugs through natural selection and analyzing the survival characteristics of animals in a series of Mountain Dew "Be Nocturnal" commercials. He shared these examples energetically and added, "I'm pumped up about it; so was Darwin" (Pugh et al., 2010b, p. 290).

Application to math education Just like science teachers, math teachers can model their own TEs with the content. That is, they can share their examples of using math to re-see and find new meaning in aspects of their everyday experience. Inspiration can be drawn from popular books. For example, in the *Freakonomics* series (e.g., Levitt & Dubner, 2009), Steven Levitt showed what it is like to see everything from cheating by schoolteachers to the relative dangers of drunk driving versus drunk walking through the lens of math and economics. Likewise, Jordan Ellenberg (2015) illustrated how he sees everything from obesity in America to hidden messages in sacred texts through a mathematical lens. Both of these individuals model what it means to truly live and breathe math.

We do not expect teachers to be Levitt or Ellenberg, but they can make a conscious effort to see the world through a math lens and share these experiences with their students. For example, a statistics teacher known to one of the authors used data from a climbing gym app to draw interesting conclusions about variability in route grades[1] at her local gym and probabilities for levels of discrepancy between original route-setter grades and consensus-climber grades. She shared this with her class as an example of how statistics brings a deeper understanding to everything and can sooth your ego when you are struggling on a moderately graded route. Likewise, a high school math teacher known to one of the authors shared with his students how he sees real estate through the lens of math. He took an in-depth look at the real estate market in different states across the United States, used math to determine which ones could yield the best returns with the least amount of capital, and made investments that provided a relatively quick return on investment. This teacher modeled the power of math to make something more interesting and bring about positive outcomes.

Modeling allows students to see firsthand what it means to perceive the world through the lens of math. In doing so, it can help set a norm that math is worth doing just for the fun of it. When students see the value that others get out of applying a mathematical lens to the world, they may be more likely to do so themselves.

13.2.3 Scaffolding Re-seeing

Even when students develop anticipation and want to seek out opportunities to apply content in everyday experience, they still need support in doing so. One problem is that students simply aren't aware of the many opportunities they have to apply content in everyday life. Thus, teachers can support students' TEs by helping them identify re-seeing opportunities (i.e., opportunities to see objects, events, or issues through the lens of curricular content) (Pugh, 2020). For example, a middle school teacher fostered TE by, in part, helping his students identify opportunities to re-see the world through the lens of air pressure (Pugh et al., 2017). He first demonstrated how everyday food packages (bag of chip, yogurt cup) are puffed up at high altitude due to air pressure (the school is around 5000 feet in elevation). Then he led the students in a discussion of other objects and events that could be re-seen. Together they identified a range of re-seeing opportunities including wind blowing through the doors of the school, ears popping when driving down from the mountains, and burping. The teacher encouraged the students to look for examples of air pressure in their everyday lives and later gave them opportunities to share their re-seeing experiences.

In addition to identifying re-seeing opportunities, teachers can scaffold re-seeing by helping students do the cognitive work of perceiving the world through the lens

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[1] Climbing routes are given a grade indicative of how hard the route is (e.g., 5.11a).

of content at a deep level (Pugh, 2020). For example, when the teacher in the prior example allowed his students to share examples of re-seeing the world in terms of air pressure, the students often made surface-level connections (Pugh et al., 2017). To support deep re-seeing, the teacher and researchers developed case studies out of the experiences shared. The students then studied their own experiences under the support of the teacher and in collaboration with a set of resources. They referenced this approach as experientially anchored instruction because the instruction was anchored in the students' own everyday experience. Students who engaged in experientially anchored instruction displayed deeper learning and a higher degree of TE than students who engaged in similar inquiry experience with weather case studies from a workbook.

Application to math education Despite the fact that math is everywhere, students need help identifying opportunities for re-seeing the world through the lens of particular math ideas. Teachers can help their students identify meaningful aspects of their (the students') world that can be re-seen. For example, many students are into social media. Although many people see social media platforms such as Facebook, Instagram, and Twitter as distractions for students, it is possible for teachers to use these platforms to explain mathematical concepts in terms to which students can relate. In 2018, Instagram changed the way posts are viewed so that the chances of receiving a high number of likes or even views decreased. The new algorithm uses "engagement" to decide which posts will be seen the most. This means posts with the most views, saves, comments, shares, and likes will be seen by more people than posts that receive less engagement. Although this may seem irrelevant to most, this algorithm change presents a unique opportunity to show students how mathematics can be used to enhance their social media presence. Providing students with examples relevant to their lives is a great way to help them re-see the concepts they are being taught. If students are shown that what they learn about in math class exists in something as prevalent as social media, it is likely they will re-see these concepts daily as they use their own social media accounts as well as become more motivated to seek out other areas where similar concepts may apply.

In a study reported by Kaplan, Sinai, and Flum (2014), an eighth-grade math teacher helped students identify aspects of their everyday lives that could be re-seen through the lens of "functions." An example included extracurricular activity time as a function of required homework time. Students were encouraged to identify functions in their lives, graph them, and post them in the classroom for other students to see. Students then reflected on their experiences with functions in writing journals as part of an identity exploration activity. For some students, these activities were transformative. For example, one student explained that these activities not only helped her transfer math to her everyday experience, they also helped her understand the math:

> I didn't really understand this topic well and what are those *X* and *Y* axes, and I even got annoyed. But then, I sat by myself and started thinking about functions that connect to me, and then "the coin dropped"; I suddenly understood what these functions are…in one

second I succeeded to understand this material in math, and it was amazing. (Kaplan et al., 2014, p. 274)

Asking students to document objects in their everyday environment that relate to math might also encourage re-seeing opportunities. In a study by Meier, Hannula, and Toivanen (2018), in-service teachers were asked to use photography to capture objects that could be used to teach math. Examples of objects and the mathematical concepts they connected to included car tires (representing geometry, perspective, symmetry), windows (representing patterns, symmetry, geometric shapes, calculation of area and volume), and a football or soccer pitch (representing geometry, calculating volume, calculation of percentages, statistics). Using the lens of photography allowed the participants to re-see everyday objects through their understanding of math.

Teachers can also employ experientially anchored instruction to scaffold re-seeing. A good example can be found in culturally relevant pedagogy approaches to math instruction. For example, Tate (1995) reported on a middle school mathematics teacher who shifted the curriculum to focus on solving community issues connected to students' everyday experience. The students decided that drugs, alcohol, and violent crime were all major issues in the community, and many students had experienced these problems firsthand. The students determined that one of the main catalysts for these problems was the number of liquor stores near the school. They used math to figure out how they could close or move liquor stores close to the school. For example, they developed a revised city tax system using mathematical principles discussed in class to counter the current system which gave tax incentives to liquor stores within a certain distance of the school. They also used geometric concepts to measure the distances of the liquor stores from the school and identified stores in violation of city code. The students presented their findings to the city council and helped pass new laws. Although not confirmed in this study, we believe such experiences of anchoring math learning in students' current experience will encourage students to continue to transfer math to their everyday experience as found in our studies in the domain of science education (e.g., Pugh et al., 2017).

13.3 Conclusion

Transfer of learning research has primarily addressed the question, "What are the factors influencing whether students can apply their learning to novel problems and in novel contexts, including real-world contexts?" Less research has addressed the question, "What are the factors influencing whether students *choose* to apply their learning to novel problems and in novel contexts?" The former question is sufficient for many transfer situations involving math. However, if we are interested in math education enriching students' experience of the world beyond necessary practical applications, then the second question becomes important; that is, motivation to transfer arises as a critical consideration. We believe TE theory brings insight to the issue of motivation to transfer and pedagogical methods effective at fostering motivation to transfer. In this chapter, we discussed the connection between TE theory and motivation to transfer. We contextualized this discussion in terms of the detect-elect-connect transfer framework proposed by Perkins and Salomon (2012). Because TE theory has been developed primarily in the domain of science education, the implications for math education proposed in this chapter are speculative. Future research is needed exploring the meaning of TE in math and assessing the effectiveness of the teaching for TE strategies in the domain of math education.

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Chapter 14
Transfer, Learning, and Innovation: Perspectives Informed by Occupational Practices

Stephen Billett

The process of thinking and acting, often referred to as transfer in the educational literature, has long been positioned as a key issue for educational programs and methods and modes of instruction (Prawat, 1989; Raizen, 1991); that is, it needs to be addressed educationally. Yet, researchers have long emphasized that, ultimately, it is a problem best understood through considerations of individuals' knowledge, its utilization, and further development (Pea, 1987; Prawat, 1989; Royer, 1979). The aim in this paper is to position what is referred to as transfer as being shaped by and central to both individuals' learning and development (i.e., change in what individuals know, can do, and value) and societal progress (i.e., changes to societal practices, norms, and forms) as others have done (Lobato, 2012; Volet, 2013). What is commonly labelled *transfer* is not qualitatively distinct from what is referred to as learning, adaptation, problem-solving, and being innovative (Billett, 2013). However, none of these processes of thinking and acting can be understood without accounting for the actual or imagined circumstances in which the person act and the relations between them (i.e., sociopersonal). So, more than transfer being understood through the activities and interactions—including those intending to promote it—afforded by physical and social settings (e.g., schools), it is fundamentally a process premised on and mediated by individuals' thinking and acting, which is, by degree, person dependent. These premises and their consequences are discussed here in relation to adapting what has been learned from one physical, social, or imagined circumstance (e.g., educational setting) to others (e.g., workplaces) and in ways not fully accommodated in the educational, cognitive, or sociocultural literatures. These premises draw on two related lines of development: personal and social. Here, these lines are illustrated by accounts of individuals engaging in and adapting their occupational knowledge when engaging in their paid work. During that

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engagement and when facing problems or novel requirements or requests, workers engage in the process labelled as transfer. That is, they adapt what they know, can do, and value to the workplace tasks they confront. However, this process of adaptation is personally mediated in ways not fully accommodated by explanations privileging either mechanistic cognitive processes or universalist social suggestions. Instead, individuals' thinking and acting (i.e., experiencing) is shaped by sociopersonal factors that include brute facts (e.g., maturation) and social suggestions, albeit mediated by what individuals know, can do, and value. Occupations are social practices manifested situationally and culturally, and they continually evolve to meet changing social and cultural needs. Adults' occupations are often central to who they are and how they exercise their sense of self. Hence, they provide a context for understanding how these processes of thinking and acting play out in terms of personal (i.e., learning) and societal continuity (i.e., occupational change), with the mediation of individuals being a central quality.

Through this discussion, the aim here is to position and explain transfer as being premised on interdependencies between personal and situational factors. These interdependencies comprise those among persons' personal epistemologies (i.e., what they know, can do, and value; Billett, 2009) and the social and physical environments in which they act. Hence, both the mediation of the social suggestions and individuals' epistemologies are central to this explanatory account. Importantly, rather than being an educational problem, what is referred to as transfer needs to be responded to as an issue for individuals' learning and development.

The aim here is also to elaborate these processes so that they can be understood—not as an educational phenomenon or problem but in a way far more broadly cast. Earlier, Royer (1979) advised against taking such a perspective (e.g., transfer as a form of general behavior), stating that the body of theory was too nascent. Now, 40 years on, it is perhaps timely to take up this challenge.

14.1 Transfer as a Problem for Education

Transfer of what is learned in and through educational institutions and educators is a central concern for them given that it is crucial to their purposes and continuity. The key role of these hybrid institutions and those who teach in them is to assist students to learn knowledge that can be utilized elsewhere beyond the circumstances of their learning (e.g., "schools"). Hence, concerns about a lack of transfer from educational programs is a crucial problem for these institutions and educators because it questions their worth, purpose, and continuity. From the educational perspective—its institutions, provisions, and teachers—developing transferable outcomes is fundamental to their central purpose and rationale. Securing learning outcomes whose applicability is not restricted to the circumstances of their acquisition (i.e., school-like activities) but extends to social settings and circumstances beyond where they were learned is important. Such adaptable outcomes are taken to be educative rather than reproductive. The enduring criticism is, however, that the learning acquired through "schooling" often fails to be adaptable to other circumstances (Raizen, 1991). Those concerns are levelled at school-learned knowledge, such as mathematics, that would seem to be inherently transferable or adaptable to a range of circumstances as well as at domain-specific procedures, such as those generated in vocational and professional education. This criticism raises unhelpful questions about the worth and privileged status of education and teaching.

The salience of this criticism is that it confounds expectations associated with the massive societal investment in education, education institutions, and those who teach in them. Clearly, there are very many adaptable learning outcomes arising for students from experiences in educational institutions and through teaching. This is evident in measures of literacy and numeracy that distinguish countries with effective schooling and tertiary education systems. Such outcomes are often overlooked in criticisms of the efficacy of educational provisions, particularly when such achievement is measured in terms of international comparisons (e.g., Programme for International Student Assessment [PISA] or Programme of International Assessment of Adult Competence [PIAAC]). However, expectations that what is being taught and learnt in educational institutions can be comprehensively and confidently adaptable to other circumstances and nonschool-like tasks are unreasonable and unrealistic. Not the least here is that these institutions and their activities are often quite hybrid and abstracted from the activities and social settings where what has been learned will need to be applied (Brown, Collins, & Duguid, 1989). Understanding how situational factors (i.e., social and physical) influence what is being learned is shaped by the kinds of activities and interactions we engage in because they are held to structure cognition (Rogoff & Lave, 1984) and this complicates the process of transferring or adapting that knowledge elsewhere. Hence, activities and interactions provided by and privileging the goals of educational institutions may generate legacies (i.e., students' learning) that are not directly applicable to situations that have other kinds of goals and processes. Consequently, expectations about wholesale adaptability of what has been learned to other social and physical circumstances need to be restrained (Pea, 1987).

Concerns about the limits of adapting what has been learned in and through educational programs are not restricted to what occurs in school classrooms and schooling for young people or in disciplines such as mathematics (Schwartz, Bransford, & Sears, 2005). There is also a concern to understand about how transferable or adaptable learning might arise for young people's engagement in activities outside of schooling and also for adults' learning and development across the lifespan, including for their work life (Gruber & Harteis, 2018). An analysis through PIAAC of adults' learning across working life in all kinds of occupations indicates the need to adapt to changing circumstances and to apply what workers know, can do, and value to changing workplace and occupational requirements. However, much if not most of this kind of learning occurs outside of educational provisions and even outside the close guidance of more expert or experienced workers (Organisation for Economic Co-operation and Development [OECD], 2013). So, there are good reasons and important educational imperatives to elaborate how the process referred to as transfer arises and can be promoted for educative purposes more broadly beyond schooling.

Indeed, viewing transfer as an artefact within educational processes and outcomes restricts understandings about what it comprises and how the “problem of transfer” might be addressed. If conceptions of, and responses to, transfer are limited to the educational provision and what occurs in educational institutions, then the important considerations of individuals’ contributions and their learning and development may be overlooked. As is now understood, whether referring to core or generic competences (Green, 1998), problem-solving strategies (Voss, Tyler, & Yengo, 1983) or the adaptation of what individuals know, can do, and value from one setting to another (Ericsson & Lehmann, 1996), there is a need to account for both situational and personal factors. For instance, the efficacy of procedures that appear to be widely applicable, such as mathematical calculations, has long been understood as being shaped by situational factors (Bishop, 1991; Carraher, Carraher, & Schliemann, 1985; Lave, 1988), as are occupational-specific capacities (Billett, 2001b). So, more than a concern about what happens in educational institutions, more broadly, considerations of the circumstance in which what has been learned is to be applied needs to be accommodated. Thus, situational factors play a role here. However, beyond these factors are those brought by individuals. For instance, it was found that, when asked to respond to the same set of nonroutine problems, hairdressers in five different hairdressing salons responded in ways that reflected the practices of those settings, but their responses (i.e., solutions) also emphasized person-dependent preferences in their construction of goals and preferred hairdressing procedures (Billett, 2003).

It follows that the process that is the focus of this book—transfer—is held as a significant problem for the educational project (i.e. the provision of organised experiences to achieve particular societal purposes as usually exercised through education practices and institutions). However, to understand this problem and how it might be addressed requires an explanation of what it constitutes. Following others (Lobato, 2012; Volet, 2013), it is proposed that transfer cannot be understood without accounting for (a) the individuals’ personal epistemologies, (b) situational requirements and factors, and (c) the relations and interdependence among them (Billett, 2013). These premises and their consequences are illuminated and discussed here in relation to the social and cultural practices of paid work—occupations—and their enactment in work settings. This case is advanced, overall, by elaborating consonances across processes referred to as learning, problem solving, adaptability, and innovation. This is achieved by conflating a set of concepts that speak to the issues associated with what is referred to as transfer. In elaborating what constitutes these processes, consideration is given to situational factors and personal mediation. Finally, some bases for elaborating factors that shape and influence what is referred to as transfer are discussed.

14.2 Conflating Transfer with Cognate Concepts and Processes

As foreshadowed, what is referred to as transfer is not a specific and hybrid process. Instead, concepts labelled problem solving, adaptability, innovation, and learning are all cognate processes that have both social and individual dimensions and lines of development: a sociopersonal process (Billett, 2013). However, to understand changes in both cognition and culture necessarily involves duality between the activities and interactions that constitute a social practice and its suggestions, on the one hand, and how individuals come to engage with those suggestions, on the other—that is, accounting for individuals' cognitive experience (Valsiner, 2000; i.e., their capacities and ways of knowing the world) and what is afforded them by social institutions and practices. Hence, it is not only schooling activities per se that are key factors in the prospects for the transfer of knowledge to circumstances outside of them: Situations and individuals mediate those prospects. That mediation is premised, on the one hand, on how the circumstances of learning are generative of the capacities to adapt to other circumstances and, on the other hand, personal factors associated with the construction and organization of individuals' knowledge (i.e., their personal epistemologies), which include their intentionalities, interests, and the direction and focus of the deployment of their knowledge.

The process referred to as transfer, albeit advanced and championed under a range of labels, is a central and common basis for both individuals' learning and development and societal progress. It is, therefore, more than an artefact of education as it is often portrayed. For instance, although he opens the prospect for it to be cast far more broadly, Royer (1979), in his seminal paper, noted that transfer is commonly seen as the extent to which learning of an instructional event contributes to or detracts from subsequent problem solving or the learning of subsequent instructional events. Likely, such a definition focused on instructional events is quite commonly accepted. Yet, in doing so, this definition positions transfer as being a product of, and to be judged in terms of, educational experiences (i.e., instructional events). However, it is more than that, being fundamental to individual learning and societal development.

The point here is that considerations of transfer or adaptability about their efficacy need to be aligned with the kinds of problems, goals, and imperatives that arise in specific physical and social settings and through the interdependency with the person who is acting in them. Take the following three vignettes that are drawn from studies of learning in and through work.

> The client returned to the hairdressing salon in tears. She had left several hours earlier after a transformational haircut that turned her long dark hair into a short, sharp and a peroxided blonde style. Her boyfriend had not appreciated the change. He said, she looked like a "bikie's moll." Peter, her hairdresser, had to respond to his visibly distressed client while also completing the work on his current client. Having spoken to the distressed client, and getting her a warm drink, he discreetly told his apprentice to reschedule his next appointment. Over the next few hours, he softened the hairstyle and re-introduced some color into the haircut, being careful to avoid complications arising from coloring hair that had been

recently treated with chemicals. He also gave the client a lot of attention and affirmation and by the end of the afternoon, she was smiling, and they both agreed that her boyfriend was a "dick head."

Chan is an engineer who works in a precision engineering company that produces items used for surgical work and implants. He was working on an instrument that provided visual examinations of manufactured components to ensure they met specifications of size and finish. The existing optical instrument, produced by a German company, was very expensive and not wholly suited to this task. He was comparing this instrument with one produced in China that was far less expensive but also was not wholly suited to the task. Through the process of disassembling both instruments and through engagement with a co-worker with experience in the production of optical instruments, he constructed a hybrid device that was suited to the tasks required by his company. Later, the company came to realize that the instrument he had produced might well become something they could manufacture and sell to other engineering firms.

The aged care facility was experiencing a high level of elderly residents with bedsores. This led to other complications, including infections in the most fragile of residents. Rosa was one of the few nurse-qualified members of staff, the majority of whom were minimally trained aged care workers. She developed a program of rotating patients and ensuring their posture was changed, and used other events in the day (e.g., meal and shower times), and a stock phrase that could be remembered by all of the staff to remind them to move residents to different physical positions at those times and also encouraged all the residents to exercise greater mobility, with the assistance of other aged care workers and on-site physiotherapists. Innovation to practice was supported by providing a phrase and the timing to ensure that residents were not immobile for extended periods.

These vignettes refer to processes, as noted, variously labelled as learning, problem solving, adaptability, innovation or, in the educational literature, transfer. The hairdresser is faced with a novel situation (i.e., problem) that he needs to address for the sake of a specific client but also for his business's reputation. Hence, he brings to bear considerations of customer satisfaction and procedural skills of hairdressing to the problem, working in constraints of a cut with short hair (i.e., transfer). The engineer is responding to a specific problem (i.e., quality and price of instrumentation), opening up the possibility of new products for the company (i.e., innovation) by drawing upon what he knows and can do and then extending it to a new sphere of application (i.e., transfer). Similarly, the nurse must adapt healthcare practices to an aged-care facility and with staff who are not fully healthcare trained. She generates a solution to the problem by adapting procedures used in one setting to this particular healthcare situation (i.e., transfer). Along the way, it is evident that each of these three workers has learned through the process of adapting what they know to address the problem and to generate novel solutions.

Here, we could add other terms from other kinds of literature, such as securing equilibrium or reconciliation between a goal state and what is being enacted and monitored, to emphasize that the process referred to as transfer is not hybrid. What is common to the process described in those vignettes and the various labels given to it is are persons engaging what they know, can do, and value in construing a goal to be achieved and then constructing a solution to achieve it (Voss, 1987). These processes necessarily exercise considerations of the domain-specific forms of knowledge associated with the task (e.g., hairdressing, electronic engineering, nursing) that constitute the "problem" (e.g., occupational requirements) and the kinds of solutions likely to be acceptable, appropriate, or viable in that situation (e.g., a hairdressing salon in which each hairdresser has their own clients, a small precision engineering workplace, an aged-care facility with few qualified nursing staff) and acceptable within the context of their enactment (e.g., a satisfied customer, enhanced profitability, improved patient care and safety; Billett, 2001b; Brown et al., 1989).

What is perhaps unusual in presenting these vignettes is that a process often associated with schools and schooling (i.e., transfer) is described in terms of what occurs within workplaces when addressing an individual's need to adapt that individual's knowledge to a novel circumstance (i.e., innovation). However, each vignette describes processes that are person particular (Voss et al., 1983), shaped by what each of these workers knows, can do, and values (i.e., cognitive experience and experiencing). Each also emphasizes the importance of individual mediation rather than educational interventions to secure transfer. That is, rather than the instructional events that are used to promote transfer in educational settings, here they were largely mediated by these individuals' personal epistemologies. Lobato (2012) also made this point about individuals being the actors who mediate the transfer process. Moreover, she also referred to accounts within cognitive science that support, albeit implicitly, the sense making and actions of individuals that are central to the transfer process and its efficacy. As noted, these cognitive experiences—personally shaped means of constructing experience from earlier (i.e., premediate – prior to the immediate) experiences (Valsiner, 2000)—arise through individuals' personal histories. These are formed progressively and iteratively across their life histories or ontogenies (Billett, 1998; Scribner, 1985) through particular kinds and combinations of experiences (i.e., personal processes of experiencing) and how the individuals have engaged in these tasks, such as the hairdressers mentioned earlier (Billett, 2003). This includes the processes through which they identify and enact problem-solving strategies to new circumstances and tasks, discount the worth of alternatives, and decide on goals and preferred courses of action to achieve outcomes. So, these individuals' responses to the nonroutine problem solving were shaped by the legacies of earlier experiences and learning that shape how they engage with tasks they encounter in the immediacy of their engagement in tasks.

So, individuals' application of what they know, can do, and value (i.e., their conceptual, procedural, and dispositional knowledge) and changes arising from that enactment come to the fore here. Those legacies also define what comprises "near transfer," or generating new understandings, abilities, or values, versus what comprises "far transfer," including the development of innovation and the kind of transitions they must negotiate. Because these processes bring together the two lines of development—social and personal—it is necessary to understand the situational contributions to cognition and how individuals come to mediate what they experience and their responses to the social suggestions to which they are subject. Consequently, the explanatory power of universal maxims, such as Gagné's (1965) "lateral" and "vertical" transfer, Royer's (1979) near and far transfer and, more recently, Beach's (1999) consequential transitions, needs to be appraised in how they are able to accommodate both lines of development.

14.2.1 Situatedness of Cognition and Transfer

Given our understandings about how cognition is both shaped by and indexed to specific circumstances and events (Barsalou, 2003, 2009; Greeno, 1989) and also the bases by which problems emerge, need to be resolved, and are judged as effective or otherwise (Billett, 2001b), it is necessary to consider the situational consequences for the process of transfer. As proposed earlier, educational institutions are hybrid physical and social institutions designed for the purpose of achieving particular kinds of learning outcomes: the intended curricula, or the achievement of the school's goals, to use an early definition (Tyler, 1949). Their physical and social environments afford specific kinds of activities and interactions that are the product of those institutions and are directed towards achieving intentional educational outcomes. The outcomes of the curriculum being enacted (i.e., the intended curriculum) and then construed by students (i.e., the experienced curriculum) are a product of what is afforded and mediated by those experiences and how students come to engage with and mediate them. In their own terms, these outcomes are highly successful because much of the judgement about what has been learned is often made through appraisals of activities germane to those institutions (e.g., assessment tasks). Perhaps most students are judged to have succeeded, by degree, through educational programs as they meet these institutional requirements. They pass.

However, those circumstances and the kind of activities and interactions that are engaged within hybrid educational settings are often quite abstracted from those that the students or graduates may encounter elsewhere, such as in workplaces. What passes as calculations, computations, and means of communicating, let alone specific occupational procedural skills that have been developed, may be unhelpful in those situations (Bishop, 1991). Indeed, drawing on a review of anthropological literature, Lave (1977) proposed that there is little evidence to suggest that school-learned knowledge is any more inherently transferable than that arising from activities in other kinds of settings. In some ways, this is not surprising, if activity structures cognition. Therefore, cognition in the form of what we know, can do, and value is not something uniformly applied across circumstance and setting (Lave, 1991): As a universal maxim, it is subject to personal mediation. Indeed, Lave (1988) stated that:

> Learning transfer is meant to explain how it is possible for there to be some general economy of knowledge, so that humans are not chained to the particularities of literal existence. The vision of social existence implied by the notion of transfer . . . treats life's situations as so many unconnected lily pads. This view reduces the organisation of everyday practice to the question of how it is possible (for the frog) to hop from one lily pad to the next and still bring knowledge to bear on the fly, so to speak. (p. 79)

The critique of universalist maxims about transfer is not restricted to the failure to accommodate the exigencies of literal existence (e.g., what mathematics constitutes in social settings as different as schools and workplaces). It is also applicable to sociocultural accounts of transfer that claim to address failings of other universalist approaches. For instance, Beach's (1999) categories of consequential transitions were offered as universal sociocultural phenomena. However, these maxims do not accommodate the personal experiencing and mediation of such transitions. They fail to account for the situation that what for one individual might be a lateral transition and for another might be a collateral transition. The scope for the personal mediation is evident in the examples Beach provided. When illustrating lateral transition from an excerpt from the travels of Marco Polo, Beach referred to what this experienced traveler encounters when reaching the city of Tamara. He used a range of understandings to describe and account for what Marco Polo experiences in viewing this city. However, other travelers arriving in this city and who might be less or differently experienced (e.g., from the countryside) may lack Marco Polo's premediately derived concepts or sufficiently developed language to respond as he does. In this excerpt, reference is made to the land outside of the city as being "empty to the horizon" (Beach, 1999, p. 115). However, perhaps the person who lives on the land outside of the city perceives this land as being far from empty. For them, it may be full of life and meaning that is unavailable to Marco Polo. Elaborating such a distinction, Higgins (2005) referred to the personal nature of mediation in Dewey's example of a small room with little more than a telescope in it. He stated that "to a brute realist the room seems relatively barren constricted; but to the astronomer who lives there, it opens up onto the entire universe" (Higgins, 2005, p. 453).

Also, in the example of collateral transition, Beach (1999) referred to Nepalese shopkeepers learning to become literate and numerate. The claim is that this collateral transition is such that those engaging with that learning will be shaped by their common interests and concerns associated with shopkeeping. However, universalism is unlikely among these learners if the findings of small-business owners engaging in learning about the administration of a goods and service tax are any indication (Billett, 2001a). Rather than having common goals, focuses, and intentionalities, diversity amongst these were identified in that study. Likely, at least one of these shopkeepers has a different set of intentions for becoming literate and numerate than the others. That goal and basis for transitions might be about moving away from the family business or the retail sector or about attending university or emigrating from Nepal. So, in these instances, the individual's transfer would not be from classroom to shopkeeping but to a different form of practice: for example, to do something other than be a shopkeeper. Beach suggested that "social institutions called schools are about elevation of young people" (Beach, 1999, p. 117) when this may not be the case. For some students, including myself many years ago, school is not a source of elevation but rather something to engage with as peripherally as possible and to be away from as soon as possible to avoid bullying, boredom, or enforced and unwelcomed order. So, how students view schooling and engage with its efforts to develop transferable knowledge is not shaped by beliefs about elevation. In referring to encompassing transition, again, the individual mediation of those transitions is ignored. Yet whereas experienced machinists likely encountered an encompassing transition in shifting from manually to computer-controlled lathes (Beach, 1999), younger or trainee machinists who do not possess or rely on those manual skills may have welcomed this new technology because it fits with their understandings in ways that for more senior machinists it simply would not. So, for these less experienced machinists, programming a Computer Numerically Controlled lathe might be a lateral transition associated with their knowledge about computers, not an encompassing one. The point here, again, is that the categories of transitions Beach proposed can never be universal as proposed because, ultimately, they are construed and constructed by individuals. It is they who engage in, and whose engagements determine, whether they are best categorized as lateral, collateral, or encompassing rather than being prespecified. So, this categorization would be person dependent and relative, not universal.

Consequently, explanations of transfer need to account for (a) what actually constitutes the task being undertaken, (b) how to achieve the required goal, and (c) the acceptability of that solution to the particular situation, with all of these being construed and mediated by the actor. So, what passes as transfer is ultimately labelled through individuals' construing and constructing knowledge and its subsequent utilization. The degree by which that utilization is broadly applicable seems to be at the heart of the transfer problem for educational institutions and teachers. However, beyond what educational programs intend and implement are a range of factors that are largely outside of the scope of schools and schooling. There are the situational factors and how they might differ across physical and social circumstances. Not the least here is the difference between the kinds of activities and interactions that are afforded by educational institutions and those accessed in other kinds of social and physical settings. However, sitting above these situational considerations is the process of meaning making—that is, how individuals come to construe and construct and engage in problem solving, adaptation, or transference. Hence, there is a need to acknowledge and account for the contributions of both individuals' mediation of what they experience and the circumstances in which learning arises.

14.2.2 Individuals' Mediation

As indicated, regardless of whether referring to problem solving, adaptability, innovation, or transfer, individuals' mediation of these processes is not negotiable. That is, how they construe what they experience and respond to it are essential elements of these processes. Consequently, a consideration of what, for individuals, are either routine or nonroutine problem solving and near or far transfer and the kind of transitions they are engaging in are all mediated personally. The emphasis on the personal is, however, far from novel although often underplayed (Lobato, 2012; Volet, 2013). What others have indicated and what now needs to be considered more comprehensively is how individuals mediate these processes, and this needs to be positioned more centrally than is often stated within the educational literature, where often individuals are viewed as being members of cohorts (classes of students) rather than individuals who mediate experiences in different ways. This was foreshadowed earlier by Pea (1987), who proposed an interpretive perspective of transfer based on a socioculturally defined model of appropriate transfer rather than one that is objectively defined:

> Elements perceived by the thinker as common between the current and prior situation are not given in the nature of things but read in terms of the thinker's culturally-influenced categorisation of problem types. (p. 639)

Consistent with this conception, Lobato (2012) offered the perspective of actor-oriented transfer, in which it is necessary to consider the individual as the key mediator in this process. She advanced this perspective to reprivilege the active engagement by individuals (i.e., actors) in this process. So, recognizing and emphasizing how individuals mediate the processes referred to above (i.e., learning, adaptability, transfer, and innovation) has long existed and is now becoming more widely acknowledged. All too often in the accounts of transfer within the educational literature, individuals are subsumed into cohorts of students who have specific qualities and attributes (e.g., Schwartz et al., 2005) rather than viewed with a focus on the person-particular basis for, and mediation of, transfer. Indeed, much of the cognitive literature, particularly that on expertise, emphasizes the important role of individual schemas that arise through repertoires of experiences that are a key factor in experts' ability to respond to nonroutine problems (i.e., far transfer), including in domains of mathematical knowledge (Sweller, 1989). Part of this capacity to respond arises from the automatization and practice that occurs through repeated engagement in domain-specific activities: that is, particular experiences leading to specific kinds of outcomes.

However, this development arises in different ways and from different kinds of experiences for individuals: They are person specific by degree. Not the least here is that individuals develop their own domains of knowledge through the particular repertoire of personal experiences they encounter across their life courses (Billett, Harteis, & Gruber, 2018). That construction of individuals' domains of knowledge is a product of individuals' ontogeny that is likely to be person dependent, if not unique. It arises from their personally unique set of experiences and constructions. That process of construction that leads to individuals developing domains of knowledge, albeit related to performing mathematical calculations or hairdressers cutting clients' hair, is, in some ways, person dependent. That is, based on what they know, can do, and value (i.e., their personal epistemologies), how these epistemologies shape the construal and construction of knowledge and themselves is further extended or enriched. However, this process progresses in ways which inherently are individually specific given the particular and myriad kinds of experiences they encounter across their lives (Gergen, 1994). In this way, the ongoing intrapsychological construction of individuals' knowledge domain is not the replication of a textbook or some canonical set of concepts, procedures, and values captured in a syllabus or other document. Instead, it is what individuals generate through reconciling what they experience and how this relates to what they know, can do, and value. Gergen (1994) captured this process as follows:

> As people move through life . . . we are continuously confronted with some degree of novelty – new contexts and new challenges. Yet our actions in each passing moment will necessarily represent some simulacrum of the past; we borrow, we formulate, and patch together various pieces of preceding relationships in order to achieve local coordination of the moment. Meaning at the moment is always a rough reconstitution of the past, a ripping of words from familiar contexts and their precarious insertion into the emerging realisation of the present. (pp. 269-270)

Yet again, there is nothing particularly new here. Much earlier, Hoffding (1892) claimed that failure of transfer occurs most frequently because the learner fails to recognize that the new situation is similar to one encountered previously, thereby emphasizing person dependence and also how they have constructed their knowledge. These propositions were followed by those of Baldwin (1894), who emphasized the reconstruction of experiences by which each individual makes sense of experiences and did so selectively, and of Janet (1930), who viewed intrapsychological processes as shaping beliefs and reacting to and regulating actions in response to what is experienced. Hence, regardless of whether referring to instructional events or experiences in workplaces and the need to respond to problems or generate innovations, these processes rely upon contributions of the social world but also individuals' mediation of them.

Ultimately, therefore, it is individuals who engage in these processes and select possible options for action, implement and then evaluate their efficacy, and make changes and adapt to new circumstances, as Voss et al. (1983) have more contemporaneously proposed. This emphasis on the person seems to be important because so much of the existing literature on transfer emphasizes either universal and obstructed cognitive processes or situational factors that, when addressed separately, fail to account adequately for the person mediating what they experience. Central here is how individuals direct their capacities, interest, and intentionalities in responding to those experiences. Engaging in demanding cognitive processes, such as nonroutine problem solving, adaptation, far transfer, and innovation, cannot occur unless individuals invest effort, intention, and energy in those activities. If, for instance, an individual does not see the worth of attempting to adapt, transfer, learn, or innovate, it is likely that the personal and situational outcomes will be very different than for the individual who sees such activities as being important and worthy of investing effort and their capacities. Regardless of whether referring to students in high school, technical colleges, or universities or workers engaged in their occupational tasks, parents responding to the challenges of parenthood, or activities within communities, it is individuals who take what they have learned (i.e., know, can do, and value) and attempt to use that to respond to emerging problems, challenges, and circumstances. The person engaging in the thinking and acting is, therefore, central to all these processes, not the least because these processes are person dependent.

As discussed, although transfer (or adaptation or problem solving) is seen as being a problem for educational institutions, it is also something that is required and routinely exercised in adults' working life, as indicated in the PIAAC data (OECD, 2013), and it is also central to remaking cultural practices, albeit mathematics or nursing. Considering these processes outside of educational settings provides a means by which the processes can be understood and by which the importance of situational and personal factors in this process can be recognized.

14.3 Transfer Situated in Occupational Practice

The discussions above hold that transfer and processes supporting it are not something restricted to educational programs and teaching. Certainly, it is evident and well illustrated in accounts of individuals engaging in their paid work as they confront novel tasks and generate innovations as part of their everyday work activities. When addressing problems or novel requirements or requests, workers engage in these processes as they adapt what they know (i.e., conceptual knowledge), can do (i.e., procedural knowledge), and value (i.e., dispositional knowledge) in enacting workplace tasks. It would seem that this capacity is required by all classes and kinds of paid work and globally. Data from PIAAC (OECD, 2013), conducted across over 30 countries with large samples of working-age adults, demonstrate that the majority of workers report engaging weekly in both routine and nonroutine problem solving in their work, with the latter being associated with the generation of innovations. That is, they are routinely engaging in what would be categorized as both near and far transfer as set out above. Moreover, these workers report that they more frequently learn through their own engagement with work tasks than through being supported by more experienced or expert coworkers (OECD, 2013): Their participation in processes that within the educational literature are referred to as transfer are more likely to be generated through individuals' mediation of these activities than through being guided by others (more experienced workers). This suggests that transfer—far from being a hybrid process and reliant upon educational provisions and the direct guidance of a more informed partner (e.g., teacher, expert) and processes of adapting what individuals know, can do, and value to other circumstances—is part of everyday thinking and acting in activities such as work (Billett, 2013). It is also through these activities that the co-occurrence of learning and innovation arises (Billett, Yang, et al., 2018). Occupations are social practices that arise historically and culturally but whose specific performance requirements are manifested situationally and also need to be adapted (i.e., remade and transformed) to meet changing social and cultural needs. Work and workplaces provide a physical and social context through which to understand what is referred to in the educational literature as transfer, shifting it from being viewed as a problem to be addressed solely within education to a process that is central to personal and societal continuity.

Instead, the personally premediated process of shaping how these individuals construe the task and select goals and procedures to do so is enacted by individuals (Lobato, 2012). This orientation is often overlooked in the educational literature that seemingly prefers to view learners (i.e., students) as discrete cohorts. This person-dependent process includes whether, for each individual, what they encounter is a routine or nonroutine problem-solving activity (Groen & Patel, 1988); what constitutes the "problem space" (Glaser & Bassok, 1989) and how that is generated and addressed; whether accommodation or assimilation is required (Piaget, 1971); whether it comprises far or near transfer (Royer, Mestre, & Dufresne, 2005); and whether it constitutes the remaking or transformation of situated practice. Whatever kinds of labels are used, they all refer to an interdependence between two lines of development—social and personal—as the person acts to bring to bear what they know, can do, and value in responding to a specific task.

All of this goes to the heart not only of individual development, cognition, and learning but also of societal continuity through innovation: that is, the remaking of societal practices such as occupations and sometimes their transformation. Human society exists and is perpetuated by the generation of responses to the problems that need to be addressed contemporaneously (Donald, 1991), and these are typically experienced and responded to by individuals engaged in their everyday work or community lives (Epstein, 2005). In responding to tasks that are generated by particular circumstances or settings, it is likely that individuals also shape what constitutes solutions to them. Individuals engaging in their work provide instantiations and illustrations of these processes. When responding to work tasks and problems, processes of transfer, learning, and innovation arise with consequences for the occupational practice and the circumstances of its enactment (i.e., workplaces; OECD, 2013). So, although tsunamis of change brought about by new technologies can be profound, it is at the local level that these changes cause the remaking and transformation of societal practices, such as those at work (Billett, Yang, et al., 2018). However, these processes are person dependent; they cannot be viewed as being in any way uniform or standard. Importantly, sociopersonal processes emphasize the duality between the person and the social world.

14.4 Transfer, Learning, and Innovation

The key points arising from the discussion here are fourfold. These are, firstly, to view transfer as something explained by encompassing considerations of human cognition and addressed in ways not constrained by educational discourse and practices. Secondly, it is to be considered outside of particular kinds of physical and social settings (schools and schooling). Thirdly, rather than the practices of addressing individuals as cohorts of learners, there is a need to engage them more individually. Fourthly, there is a need for a sociopersonal account of learning to arise.

Firstly, what is referred to as transfer is not a special or hybrid process. Indeed, there is no unique process called transfer given that it seemingly comprises a label to describe the processes and outcomes of construal and construction of school-learned knowledge (Billett, 2013). Instead, it is just one among a number of labels for what humans commonly engage in and have needed to do to advance themselves across their lives (i.e., ontogenetic development) and how they have responded to both familiar and unfamiliar challenges across human history (Donald, 1991). Even within the field of educational inquiry, concepts such as accommodation, appropriation, adaptation, problem solving, learning, and innovation are at least analogous to what is referred to as transfer. So, transfer needs to be explained by encompassing considerations of human cognition and addressed in ways not constrained by educational discourse and practices.

Secondly, the process referred to as transfer is typically portrayed in the educational literature as a problem (Royer, 1979; Schwartz et al., 2005), as one privileged by particular social circumstances and settings (i.e., schools and schooling), and as something to be promoted through specific educational strategies (e.g., instruction, teaching). All of these considerations position what is referred to as transfer as being an institutional fact (Searle, 1995): something arising through and shaped by social institutions. However, this process is instead a personal fact because it is shaped by how individuals experience, mediate, and respond to what is experienced (Lobato, 2012; Volet, 2013). Importantly, however it is labelled, this process is far from being an institutional fact alone and one that is restricted from what occurs in schools and in the learning and use of mathematics, for instance. Instead, it is in large part a personal fact: shaped by personal histories, agency, capacities, and dispositions.

Thirdly, and following from above, although much of the educational literature focuses on the qualities of the schooling experience provided and variability in cohorts of students (e.g., those with different levels of school achievement, gender, year level; Beach, 1999; Hatano & Greeno, 1999; Schwartz et al., 2005), this fails to acknowledge that it is something undertaken and mediated by individuals and in personally particular ways. That is, instead of referring to learners in terms of cohort qualities (i.e., academic high achievers, low achievers, second language learners, those with disabilities), there is a need to understand how these learners come to engage what they know and adapt it to new circumstances. As noted, Lobato (2012) has captured the many implicit references to this within cognitive accounts of transfer, but, when translated to discussions about educational institutions, the emphasis on individuals' thinking and acting becomes lost. Therefore, to understand fully the process of transfer, how it is enacted, and how it might be enhanced requires a consideration of individuals' construal and construction from what they experience and then how this contributes to their immediate and ongoing development.

Fourthly, to understand the factors shaping this process requires accounting for, as well as going beyond, earlier conceptions of those factors (e.g., identical elements [Thorndike & Woodworth, 1901], purely cognitive accounts of adaptability—transferring from one situation to another [Stevenson, 1991], and also accounts that emphasize socially generated factors [e.g., Pea, 1987]) to include personally dependent or shaped factors. Beyond adding considerations of ontogeny to account for personal experiences and processes of experiencing, and positioning these within the sociogenesis of learning, it is also necessary to consider brute factors (Searle, 1995; i.e., those of nature). This includes fatigue and maturation that impact human thinking and acting, including how and what individuals direct their intentionalities towards (Malle, Moses, & Baldwin, 2001). Certainly, the process often described as transfer and, in particular, far transfer requires effortful and focused thinking and acting on the part of the individual (Royer et al., 2005): consequently, more than what is afforded by intentional educational experiences or workplace activities or the brute facts of how individuals can engage with them. Human cognitive functions are not the mere exercise of algorithms or heuristics nor determined by social suggestions. Instead, they are shaped by their dispositions and energies and how particular experiences fit within those that they are currently negotiating and even the interest to do so. Hence, the demands, timing, and effort required for this kind of thinking and acting may not always coincide with individuals' efforts and interest to conduct such an exploration.

The case being made here has been founded on premises that arise from understanding how people learn through everyday practices rather than from those whose purpose is to intentionally promote learning. The first is the importance of considering not only the social suggestion and individuals' engagement with that suggestion but also the relational interaction between the two. The second is to understand that people's processes of engaging in what is referred to as transfer, adaptability, problem solving, and learning is, in many ways, person dependent. However, in raising this issue, there is often a distinction made between the individual and the social. Indeed, in some forums, referring to the individual is seen as being anathema. Yet, ontogenetically, there is nothing more than social than the personal.

14.5 Transfer as a Problem of Learning

This chapter has aimed to contribute to the growing body of literature that seeks to broaden and extend explanation of what constitutes transfer and how it might best be promoted (Lobato, Rhodehamel, & Hohensee, 2012; Schwartz et al., 2005; Volet, 2013). Along with these perspectives, this chapter proposes that, rather than being an educational problem, what is referred to as transfer needs to be primarily understood through a consideration of individuals' learning and development, albeit one that is socially shaped but personally mediated. It is held that cognitive and sociocultural explanations, in different ways, tend to universalize this process to an unhelpful extent, which excludes or minimizes, rather than accentuates, the role of personal mediation. Cognitive theories helpfully describe knowledge structures and forms that underpin transference of knowledge (e.g., Schwartz et al., 2005) with abstracted domains of activities (e.g., occupations). However, they fail to acknowledge fully the often socially situated geneses of problems and what constitutes their resolution (Billett, 2001b). Also, individuals' knowledge structures are developed by individuals in personally particular ways arising through their moment-by-moment experiences (i.e., microgeneses) across their life course (i.e., ontogeneses). No two people will ever have the same sets of experiences or mediate them in the same way. So, for instance, just as what constitutes routine or nonroutine problem solving is person dependent, the same applies to what is referred to as near or far transfer (Royer, 1979). That is, what is a novel problem-solving task for one person may be routine for another and, equally, far transfer for one individual might be near transfer for another and vice versa. Similarly, although sociocultural accounts helpfully refer to the contributions and suggestions of kinds of social settings and practices, these suggestions cannot be taken as universals because how those suggestions are perceived is personally mediated. For instance, Beach (1999) referred to a set of transitions as unambiguous: categorizing transitions into a number of universal kinds. However, how individuals construe and construct these events may see the "same" experience be a lateral transition for one person but a collateral transition for another and so on. So, labelling events as transitions cannot occur without accounting for what these transitions mean to, and are construed, constructed and enacted by individuals themselves.

Therefore, what is held to be transfer requires accounting for and emphasizing what is suggested and mediated by factors beyond the person (i.e., interpsychologically or intermentally) that comprise not only social contributions but also those of the brute world (i.e., nature) on the one hand. Then, on the other, it needs to account for how individuals mediate what is experienced (i.e., intrapsychologically or intramentally) based on their personal epistemologies (Billett, 2009). These epistemologies arise from earlier or premediate (Valsiner, 1998) experiences and, subsequently, contribute to what is construed and constructed through the immediacy of experiences in the social and physical world. However, such considerations are not just about the transfer of learning. Instead, as foreshadowed, the process labelled as transfer in the educational literature is at least analogous to what is referred to in other literature as learning, adaptability, problem solving, and being innovative. All these processes commonly comprise generating change to what individuals know, can do, or value through the process of experiencing. Hence, whatever label is applied, personal mediation is a central explanatory concept, as is personal epistemologies. So, in seeking to address the problem of transfer in and from educational provisions, the starting point is to view it from the perspective of the personal.

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Chapter 15
Studying Apprentice Students' Transferring Process: The Case of a Functional Relation

Chrissavgi Triantafillou and Despina Potari

The issue of transfer has received extensive discussion during the last 20 years in mathematics education. This discussion has been concerned mainly with what happens when people perform differently with the same (through the eyes of a mathematician) tasks in different contexts (Lerman, 1999). Interest in transfer in the context of mathematics education started primarily when researchers provided empirical evidence showing that children in Brazil could solve mathematics problems in their everyday activity in selling and buying things but they could not solve similar problems in the school context (Nunes, Schliemann, & Carraher, 1993). The Realistic Mathematics Education approach has used the construct of horizontal mathematization to describe, to some extent, the transfer process from the realistic context to the mathematical one (van den Heuvel-Panhuizen, 2003). In the context of mathematical literacy, mainly, the question is whether what the students do at school can be used to interpret phenomena in their society in a critical way (Skovsmose, 2014). Numerous studies have explored students' attempts to transfer their academic or school knowledge to the workplace domain (e.g., Williams, Wake, & Boreham, 2001). The purpose of these studies was to challenge students' mathematical knowledge when they face authentic workplace situations.

The transfer issue brought up different ways of theorizing cognitive aspects of the relation between a person and the societal activities she is involved with. Particularly, different conceptualizations of transfer have been proposed to explain the difficulties that students face to connect their school with their out-of-school knowledge. The cognitive approaches view transfer of learning as the application and use of past learning to new situations, focusing mostly on the individual who is involved in these situations (Haskell, 2001). In this approach, the notion of transfer was realized as a unidirectional process (usually from school to work) and was oriented toward how individuals are performing on specific tasks without taking into

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account the situational and contextual aspects of their practice. The situational approaches to transfer (e.g., Lave, 1988) view learning between different domains of knowledge as a dynamic process distributed across the knower, the environment in which knowing occurs, and the activity through which the learner is participating when learning or knowing occurs (Barab & Plucker, 2002). Hence, current views of transfer place emphasis on the interactional relations between the individuals and the collective (social, cultural) aspects of the different activities they participate in life.

In this direction, new theoretical constructs such as the developmental transfer and the boundary-crossing approach (Engeström, 2001; Tuomi-Gröhn & Engeström, 2003; Wenger, McDermott, & Snyder, 2002) have been developed. Engeström, as reported in Konkola, Tuomi-Gröhn, Lambert, and Ludvigsen (2007, p. 222), proposed the model of *developmental transfer* as a form of collaboration between two different activity systems. The developmental transfer shifts the emphasis from the individual transfer of knowledge to the collaborative efforts of different activity systems (e.g., school and workplace) to create new knowledge and practices. This theoretical construct is based on activity theory and expansive learning and the transferring process in this approach is multidirectional and multifaceted, involving transitions from school to workplace and vice versa. Furthermore, it is because of its dynamic nature that this transfer is named as developmental transfer. In the boundary-crossing approach, researchers assume that different activities are separated by boundaries, and that individuals participating in two or more activities then have to "cross" these boundaries (Tuomi-Gröhn & Engeström, 2003). From this perspective, "Transfer is not based on the transition of knowledge only, but on collaboratively creating new theoretical concepts and solutions to problems that lack ready-made answers" (Tuomi-Gröhn, 2007, p. 42). This approach allows researchers to observe the possibility of transferability of learning from one activity system to another through the process of boundary crossing, where boundaries are viewed as sociocultural differences that give rise to discontinuities between different activity systems.

Another issue that is still open for discussion among researchers is the description of the learning that takes place during the transferring process. Many studies connect the transfer of mathematical knowledge between different activities with the idea of generalization providing evidence on how individual understandings are generalized beyond the specificity of their originating contexts (Beach, 2003; Lobato, Ellis, & Muñoz, 2003).

In the present study, we adopt the developmental transfer approach to analyze three cases of apprentice students' transferring process while they try to relate authentic workplace phenomena and representations with their academic mathematical and scientific knowledge. The workplace was a telecommunication organization, and the three students, as apprentices in this organization, legitimately participated in two different activity systems: their academic institution and this specific workplace. Our focus is on students' transferring process and on how aspects of the environment that was developed in their internship facilitated or constrained this process. These aspects could include workplace practices in which the students participated, authentic workplace tasks and representations selected by the researchers and tackled by the students, workplace norms and conventions, and existing academic and workplace goals and motives related to mathematics, as well as the support provided by mentors from both activity systems. Our study in Triantafillou and Potari (2014) referred to the same setting, focusing on how the same students made sense of authentic representations related to the place value concept. Here, our focus is on a workplace phenomenon described by a functional relation between three variables (i.e., length, diameter, and resistance) and illustrated through four different workplace representations (WRs). In this chapter, we elaborate on the nature of the shared object between work and academic mathematical activities that the students formed and we also scrutinize how this shared object developed in relation to aspects of their internship. We use constructs from Radford's objectification theory (2003, 2008) to understand the nature of the shared object while students were engaging with the different WRs. We further elaborate on our conceptualization of transfer in the following section.

15.1 Theoretical Framework

We view transfer through the theoretical lens of the cultural–historical activity theory (CHAT; Leont'ev, 1978; Vygotsky, 1978) and we use Engeström's idea of developmental transfer as our methodological and analytical tool. Developmental transfer implies that the basic unit of analysis of learning is a collective, object-oriented activity system like a school or a workplace (Engeström, 1987). Engeström (2001) identified three characteristics of developmental transfer: (a) Learning is a process in which different activity systems interact in implementing a shared developmental project; (b) during the learning process, the object of work is understood and reconstructed in a new way; and (c) the new concepts are implemented as tools or models of new activities. Developmental transfer has been indicated by changes in the object of the work of the entire activity system, and in the conscious creation of knowledge and practices. For example, Tuomi-Gröhn (2003) discussed a Finnish internship program for nurses, in which school and work collaboratively interacted and changed.

Many researchers argue that internship periods foster collaboration between schools and workplaces (e.g., Akkerman & Bakker, 2012; Tuomi-Gröhn, 2007). Tuomi-Gröhn (2007) illustrated how this works in a developmental project aiming to build collaboration between a daycare and a school organization. Under the perspective of developmental transfer, Konkola et al. (2007) illustrated how the different activity systems interact and a potentially shared object emerges (see Fig. 15.1). This shared object arises from the work activity and its developmental needs (Tuomi-Gröhn, 2007).

In our case, the subjects were students from an academic engineering institution who were doing their 6-month internship in different sectors of a telecommunication organization. We consider the academic and the workplace as the different

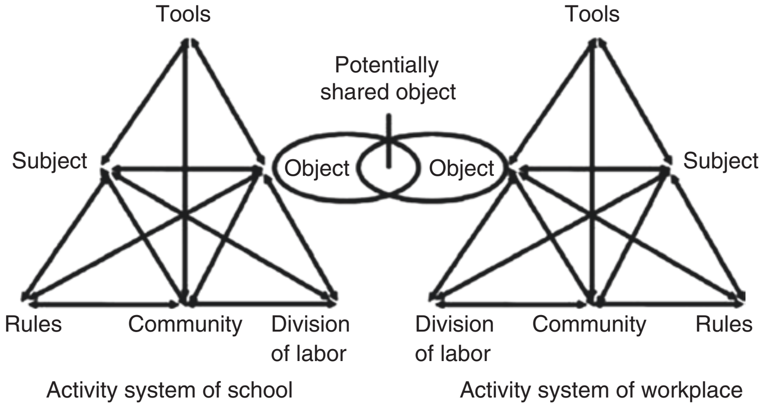

Fig. 15.1 School and workplace as interacting activity systems. (Konkola et al., 2007, p. 216)

activity systems. These systems are, by their nature, diverse in their cultural, historically developed ways of learning and knowing. For example, academic institutions are explicit and mostly rely on abstract and explicit ways of learning and knowing practices, whereas workplace practices mostly rely on applied and practical ways of learning and knowing (e.g., Pozzi, Noss, & Hoyles, 1998; Williams & Wake, 2007). Akkerman and Bakker (2012) referred to the above different types of epistemologies as differences in school and work "epistemic cultures" (p. 156). These differences in the objects of the activity systems bring tensions and contradictions that the students must overcome to develop the potential shared object.

In this study, we focus on a specific task (named here as the TASK) that was identified in the ethnographic phase of our study (Triantafillou & Potari, 2010). This TASK concerned the selection of different types of diameters of copper wires in subscribers' network for certain distances. The arguments behind this selection were (a) mathematical (i.e., the functional relation between three quantities of the copper wire, the length, the cross section, and the resistance of the wires); (b) practical (i.e., the lowering of the cost); and (c) scientific (i.e., taking into consideration the upper limit of the resistance [1000 Ω] to have an appropriate signal quality). The mathematical aspect of this TASK was based on a formula well known to all participants from their school and academic studies. Moreover, the above functional relation was represented in the specific workplace through a number of different WRs. These representations included a table, professionals' explanations (i.e. metaphors and practical examples), an elaborated formula, and a graph. Some of the above WRs are cultural products produced by experts in this organization (i.e., the table, the elaborated formula, and the graph) or developed by the practitioners in this organization while trying to communicate to an outsider (the researcher or the students) the mathematical and the scientific aspects of the TASK. Moreover, these WRs were related to practitioners' everyday workplace activity (Triantafillou & Potari, 2010).

Because one of the central ideas of developmental transfer is to give students the appropriate tools to deal effectively with new situations (Tuomi-Gröhn, 2007), we decided to use the WRs to analyze students' responses while trying to link their work with their academic mathematical and scientific experiences. We consider these representations—they had been provided by the researcher, the first author, who was also a lecturer in their academic institution—as the tools that mediated students' transferring activity. Some of these WRs are closer to those the students had encountered at school and university (i.e., the algebraic formula and the graph) whereas others have more of a black box nature that constrains these connections (i.e., the table and professionals' explanations). Apprentice students, in their attempts to interpret the different WRs and make sense of the TASK, created their own images of the shared object between their academic and workplace experiences. According to Leont'ev (1978), the mental image of the object of an activity is considered to be the product of the subject's "detection" of the object's properties (p. 4). Our focus in this study was on the nature of the shared object and on the way it was formed and developed during students' engagement with the four WRs.

To understand students' images of the shared object while engaging with the different WRs, we use Radford's objectification theory (2003, 2008). Radford (2003) defined objectification as a creative process of noticing something and is "linked to the individuals' mediated and reflexive efforts aimed at the attainment of the goal of their activity" (p. 41). The reflexive nature refers to the relationship between individual consciousness and a culturally constructed reality, whereas the mediated nature refers to the means that orient thinking and allow individuals to become aware of and understand the cultural reality. Mediated means in our study are the four WRs as well as students' semiotic means used while trying to objectify the phenomenon under consideration. These semiotic means were scientific notions relevant or irrelevant to the TASK, sensual perceptions, and linguistic tools, as well as forms of reasoning and action, either from their academic or their workplace activities. Furthermore, with respect to the transfer issue, because we view knowledge transformation as connected with the idea of generalization, we perceive different layers of generalization in students' transferring process. Specifically, to analyze the levels of students' transferring process, we adapted the three layers of generalization that are described by Radford (2003): factual, contextual, and symbolic. These layers are used as our analytic tool to describe the nature of the shared object during students' transferring process. Radford (2003) defined the *factual type of generalization* as abstractions of actions undertaken on objects bound to the concrete level. The semiotic means are related to the words that show noticing of the features of the pattern, and to gestures that indicate rhythm and movement or drawing. In our study, we considered that students were at the *factual level* of generalization when they used incomplete and sensual ways of reasoning very close to the actual scene where our intervention unfolded but which were mostly irrelevant to the specific TASK. For example, they argued that the covering of wires with shields leads to the increase of the diameter of the wire. Radford (2003) realized *contextual generalizations* as students' abstraction from specific figures presented to them. The semiotic means at this level are the generic and locative terms that are used as new mathematical objects emerge. These mathematical objects are contextually situated objects having spatial temporal characteristics. In this study, we considered that students were at the *contextual level* of generalization when they used arguments that were mostly bound to the particular WRs. For example, they described the algebraic relations between the three variables in the formula ($L = R \cdot 45 \cdot d^2$) without understanding how these relations affect the technician's choice. The shared object at this level is considered to be contextual because their understanding is context bound and partly shared between the two activity systems. According to Radford (2003), what characterizes the *symbolic generalizations* is the use of algebraic symbols to symbolize the variables and their relation. In our study, we conceptualized the *symbolic level* of generalization when students developed a new understanding of the mathematical object. This means that they made links between the different WRs and developed control of how to use them (e.g., which aspects of one representation were useful and which to abandon), used elements from both their academic and workplace experiences, and used the mathematical object as a tool to explain the TASK. We consider the above new understanding of the mathematical object as a potentially shared understanding between the two activity systems.

Going beyond the nature of the shared object and looking at the way that the shared object is created in relation to the two activity systems, the developmental transfer approach offers us a way to address the way that tools as mediators, the two communities, the rules, and students' division of labor during their internship influenced the creation of the shared object. We also consider motivational and emotional aspects that frame the above activity. We see emotions and motivation linked to the subject's actions and goals as driving forces of the formation of the mathematical object in the way that Roth (2007) extended Engeström's third-generation of activity.[1] Transfer in this way is seen through the creation of this shared object that has meaning in both activity systems and is developed through the students' actions and goals being in dialectical relation to their emotions, motives, and identities. The actions and goals are also developed through the mediation of tools, rules of the communities, and division of labor of each activity system and the emerging tensions and contradictions between the objects and the elements of the two activity systems. Because we traced students' efforts to master and cultivate the shared object between the two activity systems, we addressed the following research questions:

1. What are students' images of the shared object?
2. How do these images develop?

[1] Vygotsky and Leont'ev were responsible for the first and second generation of activity theory, respectively.

15.2 Methodology

15.2.1 The Context of the Study

The study presented in this chapter is part of a research project that aimed to explore the issue of transfer of mathematical knowledge between the academic and the workplace domains. The entire project was divided into two phases: the ethnographic and the interventional. In the ethnographic phase, through the analysis of a variety of activities performed by different groups of practitioners in a telecommunication organization, a number of mathematical concepts such as the place value and the linear function notions emerged (Triantafillou & Potari, 2010). The interventional phase was based on the idea that apprenticeship programs are a very common environment for observing students' reciprocal influences of school or college and workplace practices (e.g., Akkerman & Bakker, 2012). Particularly in this phase, we studied how five engineering students from an engineering educational academic institute, who were doing their compulsory 6-month practicum in the organization, tried to link their workplace activities with their academic knowledge. In the interventional phase of the project, through semi-structured interviews with the participants, the researchers studied how the students tried to link their academic and workplace knowledge of the above mathematical notions (i.e., the place value concept and the notion of linear function). In the academic institute, there are no systematic structures to relate the workplace to the academic courses, especially in the area of mathematics. Mathematical courses are organized according to the presentation of the mathematical content without modelling activities or any connection to practical situations.

An extended presentation on how the five apprentice students developed their school-based knowledge on place value to incorporate aspects of this notion in the new context can be seen in our previous work (Triantafillou & Potari, 2014). In this chapter, we report on the transferring process of three illustrative cases focusing on the notion of the functional relation that is hidden behind the specific TASK.

15.2.2 The TASK

The TASK was about the selection of the proper diameter of copper wire in the local network for certain distances. The practitioners knew that when the length of the wire increases, this results in an increase in its electrical resistance, and the only way to control this effect is to use wires with larger diameters (larger cross-sectional areas) for long distances. The mathematical dimension of this task was the variation of the electrical resistance (R) of a copper wire in terms of its length (L) and its cross-sectional area (s). In particular, because the electrical resistance (R) of the copper wire varies proportionally in relation to the length (L) of the wire and in inverse proportion to the cross-sectional area (s) of the wire, it forces the practitioners to change the diameter of the wire to control its resistance. At the same time, practitioners knew the upper limit of the resistance of a copper wire required for a good signal quality for the subscriber (about 1000 Ω).

Hence, the TASK was about the proper selection of the copper wire type. The argument for this selection was related to the mathematical object described above while taking into consideration the signal quality and trying to lower the cost. The above algebraic relation is mentioned in many physics school and academic textbooks. All the apprentices knew this relation as the electrical resistance formula, $R = \rho \cdot L/s$, where R represents the resistance in Ω, ρ represents the resistivity of the material that the wire is made of, L represents its length in meters, and s represents the cross-sectional area of the wire in m^2.

The above mathematical object was expressed in the particular workplace through a range of representations. The WRs were presented to the participants in a sequential order (table, professionals' explanations, elaborated formula, and graph) and were the basis of four tasks that were designed by the researchers to analyze participants' transferring process. We present below the four different WRs and the questions addressed to the participants in each case.

15.2.3 The Four Authentic WRs

The table In the beginning of the research activity, students were given the table representation (Fig. 15.2). It came from a technical book used in this organization that gives information about the diameter of the copper wires, expressed in millimeters, that are used in a local subscriber line and the corresponding distance in kilometers between the subscriber and the organization building or, in other words, the length of the wire. The symbol Φ represents the image of a circle with a diameter. The table represents the workplace phenomenon in a nontransparent way to a person who is not from the workplace given that the resistance, which in school science is the central variable, is hidden. Moreover, the table provides information about the need to change the diameter when working with wires of certain lengths but does

Fig. 15.2 The table representation

Distance L(Km)	**Wire diameters (mm)**
Up to 3	Φ 0.4
From 3 to 6	Φ 0.6
From 6 to 9	Φ 0.8
From 9 to 10	Φ 0.9

not explain the necessity for doing so. The choice of presenting only the distance and the diameter in the table indicates a tendency that exists in the workplace to give immediate answers for a specific problem, that is, for how to choose the appropriate wire for a certain distance. Comparing also the characteristics of this workplace representation to characteristics of conventional tables used in mathematics for representing a function, we see different units of length measurement and a combination of verbal (words *up to*, *from*), arithmetic, and figural symbols. Moreover, the function of the diameter in terms of the distance is a step function, a function that is not often met in school mathematics. In this case, there is a transformation of a well-known academic formula to a functional inscription embedded in the workplace expertise.

For all the above reasons, we chose to address the following questions to the participants: "How do you interpret this representation? How do you explain the relation between the quantities mentioned (i.e., distance and diameter?) Do you link it with your academic or workplace experiences?"

Professionals' explanations Many practitioners used the following metaphor when they were asked by the researcher to explain the table information (Fig. 15.2): "The flow of charge through wires is like the water in a canal; when you want to send water through a great distance, you must use a canal with a larger diameter to avoid the losses." Here the water canal is the copper wire, the length and the diameter of the canal is the length, and the diameter of the wire and the losses is the resistance. Although this metaphor describes the phenomenon, this can still be non-transparent for someone who does not understand this relation. The students were asked to make connections between the metaphor and the function represented in the table, for example: "Do you connect in some way the metaphor with the table representation? What can be the 'losses' mentioned in the metaphor in the case of a copper wire?" In one of the three illustrative cases, the technician provided practical examples to explain to the student the mathematical object hidden in the table representation.

The elaborated formula Practitioners in this organization were using an elaborated formula ($L = R \cdot 45 \cdot d^2$), where L stands for the length of the wire, R for its resistance, and d for its diameter, to make measurements in the underground networks. Even though this formula is material specific, it is very close to the conventional school-type formula because it represents the three main quantities, namely the resistance, R, the length, L, and the diameter, d, of the wire. However, it is solved in terms of the length of the wire for convenience because practitioners replace the known quantities, R and d, to find the distance (L) between the point of measurement and the unknown, faulty point in the wire. At the same time, the number 45 represents the constant terms, namely the combination of the resistivity of the wire and characteristics of the area of the circle. The students were asked to describe the elaborated formula, relate this formula to their previous knowledge, and also compare it with the conventional one. Typical questions addressed to the participants were: "Can you describe this formula? Have you ever seen this formula before?"

The graph The graph came from a technical book from the specific organization (see Fig. 15.3). It is a typical graphical representation of two quantities (resistance versus length) and represents this relation for three wire diameters (i.e., 0.4 mm, 0.6 mm, and 0.8 mm). At the same time, it explains the need to change the diameter at certain distances to keep the resistance from exceeding the value of 1000 Ω (this is the maximum value of resistance on the *y*-axis). Hence, this representation is the only one that explains in detail the TASK.

The students were asked to interpret the information provided in the graph and compare the three different drawings and relate them to the initial table and the TASK in general. The typical questions addressed to the students were as follows*:* "Can you interpret the information presented in the graph? Can you relate the graph with the table information?"

We consider this application of the above linear function, through the use of the different representations, to be a "novel situation" for apprentice students because it was something they had never encountered or been trained to navigate. The novelty of this situation is the authenticity of the setting (the workplace phenomenon and the four WRs that describe aspects of this phenomenon) and its different orientation compared to students' academic and school experiences.

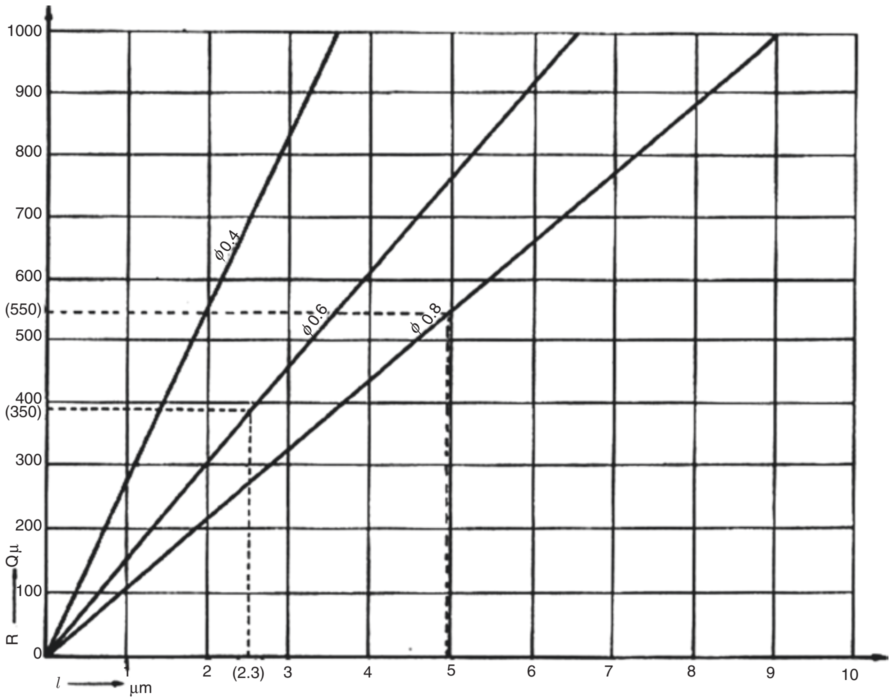

Fig. 15.3 The graph representation

15.2.4 The Participants

The apprentices The apprentices were five male undergraduate students of an engineering educational institute in their last year of their studies. Three were from the Department of Electronics (St3, St4 and St5), one from the Department of Electrology (St2), and one from the Department of Informatics (St1). All of them had successfully completed the courses required for their degree and they were doing their 6-month internship in various departments of the organization. During their internship, they mostly assisted a technician on some fieldwork assignments.

St1 (23 years old) was studying informatics. Even though mathematics was one of his favorite subjects, he did not usually attend the academic lectures because he could not see any links between mathematics and the field of informatics. The object of his work activity was to report faults appearing in the central network (between Athens and the local area). He was not allowed to work outside the main building. Hence, due to apprentice restrictions, he did not have the chance to see technicians performing the specific workplace task we focus on in this study. St2 (22 years old) was studying electrical engineering. He had a positive attitude towards mathematics and, as he mentioned, he had not come across many difficulties in mathematics throughout his school and academic studies. In all our discussions, he was in favor of empirical ways of learning. "I believe that seeing and working on something helps me to know it better," he told us, and this was the reason he enjoyed learning during his internship more than during his academic studies. During much of his internship, St2 was accompanying the technicians in their fieldwork. St3 (22 years old) and St4 (23 years old) were both studying electronic engineering. Although they both expressed positive attitudes towards mathematics in their early school years, they admitted they had subsequently lost interest. St3's internship was in the Athens central building of the organization. He was enthusiastic about his internship activities and was looking for chances to take on responsibilities. St4's internship was in a small Greek town and he was accompanying professionals in their fieldwork. St5 (21 years old) was studying electronics as well. He expressed negative feelings towards mathematics, and, as he claimed, he faced a lot of difficulties during his school and academic studies. In addition, he found it difficult to relate mathematical processes to engineering concepts in his academic studies. St5's internship took place in the central station of this telecommunication organization, located in a small Greek town.

The practitioners The mentors at the workplace were practitioners with whom the students collaborated. They contributed to this process by offering artifacts and explanations that could act as boundary objects in this process.

The researchers The first author was teaching mathematics in the students' institution. Both authors designed this intervention to investigate the issue of transfer of mathematical knowledge between mathematics in the academic and workplace domains. Moreover, the first author acted as a boundary person from the university to the workplace, by bringing authentic tasks from the workplace where she had seen connections to mathematics and also by encouraging students to reflect on mathematics they had met at school or at college. The researcher, the mentors, and the students created a partnership to bring changes to the way that internships had been implemented for several years, and to facilitate the creation and the development of a shared object between academic and workplace mathematics.

15.2.5 The Process of Data Generation

The whole process lasted 8 months given that not all students started their internship at the same time. The data came from semi-structured individual interviews and ethnographic observations in the first half of students' internships and structured interviews during the second half of their internships (Table 15.1).

Concerning the ethnographic observation phase, the first author followed the students in a fieldwork activity and discussed with them the object and the goal of the specific activity. Field notes were kept during this part of the study. Each individual interview lasted approximately 1 h and was audio recorded. Only in the case of St3 was there no ethnographic observation. This was due to the strict rules at the particular place at which he was completing his internship. Initially, the questions focused on students' school and academic experiences and their dispositions and attitudes towards the subject of mathematics. Subsequently, the questions focused on their work experiences and on connections they made between these and their schooling. Finally, our aim was to design three interventions that we had identified in the first part of our research project (Triantafillou & Potari, 2010). In this chapter, we present our results concerning one of these interventions related to the linear function concept. This intervention took place during the second half of students' internship program.

Table 15.1 Research tools—duration of the whole project

	No. of interviews	Ethnographic observation (duration)
St1	6	3 (3 h)
St2	5	5 (1.5 h)
St3	5	–
St4	4	1 (2 h)
St5	5	2 (2.5 h)
Total	25	11 (9 h)

15.2.6 Data Analysis

Initially, the analysis of the data was based on grounded theory techniques such as open coding and the constant comparative method (Charmaz, 2006). Then, the emerging categories were conceptualized and characterized in relation to our theoretical framework.

Concerning the first research question about the shared object that the students formed, we coded participants' statements in terms of three dimensions that reflect the semiotic means of Radford's objectification theory (2008): (a) students' meanings of the mathematical object (the functional relation), (b) the origin of students' meanings in relation to the two activity systems (academic or workplace), and (c) the students' reasoning while making sense of the TASK. The analysis was conducted for each student and for every representation with respect to the above dimensions. Then, synthesis of the above dimensions was made to identify the image of the shared object each student developed while facing the different WRs. This image was characterized using the three layers of generalization (i.e., factual, contextual, and symbolic). Thus, three cases were identified in relation to the shared object that the students formed while facing all the WRs. In Case 1 (St1) and Case 2 (St2 and St4), the students developed the potentially-shared object and reached the symbolic level of generalization, whereas in Case 3 (St3 and St5), the students remained at the contextual level, not managing to develop this object.

To address the second research question about how the shared object was developed, we analyzed the data in relation to the elements of the extended mediational triangles of the two activity systems for each student (tools, community, rules, division of labor, and the object or motive). The tools included the WRs, the knowledge resources and terms students themselves brought from their academic and their internship experiences, or the information provided by the researcher and the professionals. The workplace community included experts and practitioners whereas the academic community included the researcher and teachers. The rules of the two communities and students' division of labor were identified mainly from the ethnographic phase of this study. The object or motive included students' emotional reactions while trying to overcome tensions in their attempts to link the objects of the two activity systems. In this chapter, we present our analyses of St1, St2, and St3 as illustrative cases of different forms of the shared object and of the manner in which this process was developed.

15.3 Results

We present the three illustrative cases of different forms of the shared object and of the ways the different forms that developed emerged from the analysis. Case 1 refers to St1, who formed a shared object at the symbolic level through his persistence to overcome the tensions he met; Case 2 refers to St2, who also managed to form a shared object at the symbolic level, where the technician's support contributed to helping St2 overcome the tensions he met; and Case 3 refers to St3, who remained at the contextual level.

In the first part, we present students' images of the shared object individually while they were facing each one of the four WRs. We use data extracts and students' statements to illustrate semiotic elements of students' images. In the second part, we try to identify how the shared object developed for each case with respect to the elements of the academic and workplace activity systems.

15.3.1 Students' Images of the Shared Object

Case 1 (St1) In the table task, the researcher presented the table representation to St1 and asked him if previously he had had the chance to hear about the TASK as it was presented in the table. He responded positively by relating the TASK with a discussion he had heard among technicians in his internship (Line 1.1). This discussion was about the limitations that specific diameters of copper wires have for high-speed data transmission. His further reasoning was naïve, given that St1 related the telecommunication data amount with the volume characteristics of copper wires (Line 1.3).

1.1. *St1:* Yes, I did. There was a discussion about the doubling of speeds from twenty-four [1024 *kb*/s] to two forty-eight [2048 *kb*/s] and because organization's entire network is working mostly with zero point six [St1 refers to 0.6 mm diameter copper wires], it [the network] could not afford such high speeds.
1.2. *R:* Does it sound reasonable to you?
1.3. *St1:* Maybe it is because there is a bigger data amount. I am not so satisfied with my response though […] it has to do with the resistances and with some magnetic fields that are created inside the cable, this is what I remember … but I'm not so sure, maybe I make a mistake. We could ask a technician [to verify his response].

The semiotic means that St1 used while facing the table representation came from the workplace context (i.e., recalling technicians' discussions [Line 1.1] or sensual perceptions ["there is a bigger data amount"]), and from his academic experiences ("it has to do with the resistances and with some magnetic fields"). St1 made incomplete arguments by using raw information from both activity systems, and even though he started to objectify the role of the hidden variable (i.e., the resistance), he was not able to describe a complete argument about the TASK.

On the one hand, when facing professionals' explanations in the form of the metaphor, St1 seemed to interrogate this workplace practice ("It is a little weird logic"). This was perhaps because his academic experiences conflicted with this way of reasoning. On the other hand, the metaphor seemed to strengthen his initial idea that there is a relation among the three quantities *R*, *l*, and *S*, as indicated by his positive statement, "Yes, it has to do with a relation" (Line 1.4).

1.4. *St1:* Here I see the losses as an error in signal transmission and what is the reason for the errors of the signal transmission? [He asks himself and he responds:] maybe because of the length of the wire… maybe it has to do with the magnetic fields but also with the resistance, as it comes to my mind, but I do not know if it is true. It reminds me of school physics. […] Yes, it has to do with a relation …it is difficult to remember; it has been years now.

The metaphor helped St1 to verify that the hidden variable in the table was the resistance but he expressed many hesitations about his capability to objectify the mathematical object presented in a metaphorical way. He was relating the task with the subject of school physics but his argument about the workplace task was still incomplete.

Hence, St1's images of the shared object, at this point, were factual and fragmental because they were mostly based on scattered information from the workplace and his academic and school experiences. Even though he objectified the central role of the resistance, he could not make a complete argument about the TASK.

The elaborated formula ($L = R \bullet 45 \bullet d^2$) verified St1's initial idea about the role of the resistance and provided him the tools to change his mode of describing the relation from general ways ("when one increases and the other increases as well") to describing the mathematical structure of this relation (Line 1.6).

1.5. *R:* What do you see in this relationship, can you describe it to me?

1.6. *St1:* I see that as the length increases the resistance increases as well. The parameters are proportional [explains] if *d* is known and *R* gets the value 1 let's say, *L* will be forty-five times *d* squared; if it gets the value two will be two for forty-five *d* squared.

Then, the graphical representation was given to him and he was asked to interpret the information presented in it. St1 referred to the slope notion and he argued as follows: "The bigger the diameter the lower is the slope." At the same time, it was hard to select where to focus on the graphical data ("For the same distance . . . for different distance"). Subsequently, he took the same value of resistance ("550 Ω") and found the length of the copper wire for different values of diameter ("when is 0.4 mm the distance is 2 km when is 0.8 mm it is 5 km").

In the formula task, St1 recognized and described the mathematical structure presented in this representation (Line 1.6) by relating the resistance with the other two quantities. The origin of these explanations were his mathematical and academic experiences. In the beginning of the graph task, St1 used mathematical notions (slope); he identified covariance between different variables (slope and diameter); and he was feeling confused about where to focus. We consider his level of generalization to be contextual because the new object was partly shared between the two activity systems and, at this point, his argument about the TASK was still incomplete.

After the graph task, St1 proposed to solve the formula for different values of *L* with the following statements: "if I take *L1* and *L2*," and "$L1 = R \times 45 \times 0.16$ and

$L2 = R \times 45 \times 0.64$ and in this way we can see which line corresponds to each diameter." Then, the researcher explained that the maximum resistance of a wire is 1000 Ω and St1 responded as follows by providing a complete argument about the workplace task:

1.7. *St1:* I take the limits [he refers to the upper limits of the resistance when the copper wire is of a certain diameter], on the 6 km [we use wires with diameters] 0.6 [refers to the table representation], on the 0.8 [wires with diameters 0.8 mm] it is better; of course if they tell me that they can use all three types of diameters then yes! In the informatics we do the same on similar matters we take the smaller one which satisfies the restrictions and budget we have.

In the last part of the intervention, we see that St1 took initiative to relate the function to the graph and used this relation to explain the table. Also, he realized the practical consequences of the use of different types of diameters (Line 1.7). The shared object at this point had a symbolic character because St1 seemed to develop control of how to use his math-related and work-related knowledge, by realizing which aspects of the representations he should focus on, identifying the practical consequences of the technicians' selections, and relating aspects of this task with his own academic and professional experiences.

Case 2 (St2) When the researcher asked St2 to explain the table representation initially, he tried to explain the TASK by himself ("I know from my studies in electrical engineering [it is due to] the power transmission or it is due to heat losses"). At this point, he realized the insufficiency of his reasoning, so he referred to a technician who was present in this discussion. The discussion started as follows:

2.1. *St2:* [he refers to his workplace supervisor] Mr. [...] we have a table with distances and cross sections in the urban network and as the distance increases the cross section of the cable increases as well.

In the beginning, St2 related the table information with his academic studies "in electrical engineering" and used notions such as "power transmission" and "heat losses." He did not identify the hidden variable as St1 did, and his argument was false and irrelevant to the situation. He described the relation presented in the table to the technician in a general way ("as [one] increases [the other] increases as well") but instead of referring to diameters of the copper wires, he referred to "cross sections."

In Lines 2.2 and 2.3, we present the discussion between the technician (T) and St2.

2.2. *T.:* [he corrects St2] It is not that the cross section increases… we must increase the cross section in order to have less voltage drop. We have losses … if we use 0.4 from here to [he refers to a suburb 8 km from the local organization station] it is not possible the signal to arrive… we must use 0.8.

2.3. *St2:* Why do we have a voltage drop?

The technician extended St2's general presentation of the proportional relation between cross sections and distances (Line 2.1) by emphasizing the increase in the cross section as a necessity in the workplace context ("we must increase…"; Line

2.2). In the same line, he reasoned about this necessity ("it is not possible the signal to arrive [at certain distances]." Afterwards, the technician referred to the notion of "voltage drop" and he explained to St2 that voltage drop "is due to the resistance … all the copper wires do not have the same resistance." Furthermore, he presented the inverse proportional relation between resistance and diameter as follows: "When the copper wire has diameter 0.4 the resistance is high but when we take wires with larger diameter the resistance becomes smaller … so when the distance increases … what do we do? We increase the diameter." In addition, the technician provided examples of how they managed to balance the increase of the distance when a subscriber moved from near the station to another place further away from the main station as follows:

2.4. *T:* I do not know if you heard yesterday that I said something when the [names the particular subscriber] moved from here to [names the new place] I said we would need two couples of wires, remember what I said? [then he explains why the use of a pair of wires was necessary] when we use one pair of wires [the signal] will go to no more than 5 km with 0.6 [0.6 mm diameter] if I use a pair of wires instead of a single one the signal will go up to 10 km or maybe 8 or 9 km.

The above excerpt indicates that St2 was present in an instance of the particular workplace task this study focuses on [Line 2.4]. During this instance, St2 did not have the chance to make connections with his academic knowledge. The technician's reasoning through the use of practical examples was helpful to St2 because it revealed the role of the hidden variable in the table representation (i.e., the resistance). However, when the researcher asked him to explain the specific workplace task, St2 argued as follows: "When we increase the distance, we use more amount of copper and this results in an increase of the cable's resistance." We consider this argument to be incomplete because his reference to the functional relation is only partially complete.

When the researcher presented the algebraic relation in a symbolic form ($L = R \bullet 45 \bullet d^2$), St2 was ready to make an argument about how the relations between the three variables affect each other: "The longer the length will be, the larger the cross section is [he corrects] the cross section must be so as to lower the resistance." Subsequently, St2 identified the graphical relation between R and L to be a line and he argued as follows: "If we increase the length it is impossible the resistance not to increase as well." In the graph task, St2 was asked to reason about the changes that he observed in the different lines, and he responded, "There is a different rate of change between the resistance and the length of the wire," and continued by reasoning about the difference in the three types of wires: "The more the cross section increases the less the resistance increases, while the length also increases, more smoothly …the increase between resistance and length is smoother in large cross sections than in lower ones."

St2's images of the shared object at this point were based on relating the three main quantities R, L, and S ("the longer the length [L] will be the larger the cross section [S] must be so as to lower the resistance [R]") in a flexible way. He used a comparative phrase ("so as to") to indicate how the increase in the cross section of the wire balances the increase in the resistance due to the increase in the length. In the graph task, to describe the different slopes of the three lines, St2 used a mix of academic terminology ("different rate of change") and embodied expressions ("smoother increase"), and his argument was almost complete. However, we did not consider his argument about the workplace task to be fully complete because he did not yet know the workplace restrictions about the maximum resistance of 1000 Ω. This information was essential for understanding the reasons for changing wire diameter beyond certain distances.

St2 referred to graphical details such as the values that the resistance takes for different types of diameters and across a range of distances, and he argued that "if we go to 5 kms and we use 0.4 then we will go beyond the 1000 Ω so we use wires with diameter 0.6." Even though some of his expressions referred to perceptual aspects of the graph ("smooth changes"), we can see that his explanations went beyond the particular data. St2 connected the graph with the table as follows:

> Until 3 kms [he refers to values on the table], if we use copper wires with diameter 0.4 ... we are up to here [pointing to the maximum value of the resistance]. If we go to 5 kms and we use 0.4 then we will go beyond the 1000 Ω so we use wires with diameter 0.6.

Then, the researcher asked him why not always use wires with diameter 0.8 mm, and he responded: "This could be better, but it would cost a lot …they save money in this way, otherwise a big amount of money could go there."

The shared object at this point had symbolic characteristics because St2 used the graphical details to explain the table representation (i.e., the limitations of using specific diameters of copper wires for certain distances). Moreover, he argued that the use of different types of diameters instead of the larger one, which is "better," was to lower the cost expenses ("they save money in this way").

Case 3 (St3) St3 reasoned about the distance–diameter relation presented in the table by using his immediate work-related experience: "When we are nearby, we leave the wires without any protector. As the distances increase, we see that they put more protectors, plastics, etc. so they become thicker." To further reason about why this really happens, he used the notion of signal quality. This notion was central in his studies in electronics and his workplace experiences in the Central Telecommunication Centre in Athens.

3.1. *St3:* It sounds reasonable to me. When you have a copper wire to cover a short distance the small diameter is enough because you do not have many demands from the cable … that is when it has to travel a distance of one hundred say kilometers it needs ... let's say 100% quality after five hundred kilometers drops a little it becomes 95%.

St3's images of the shared object were completely fragmental and bound to concrete objects from his work-related and academic experiences ("signal quality"). He did not identify the role of the hidden variable and his argument related to the TASK was incorrect at this point.

In the professional explanation task, St3 encountered the metaphor. St3 related the metaphor to the TASK as follows: "I can see that the water canal as something analog to the wire" and he identified the losses that the metaphor referred to as "noise distortions." Thus, the metaphor did not add to St3's understanding. Furthermore, he moved from false arguments ("relating length of the wire and signal quality") to naive ones ("protectors in wires result in increasing the diameter").

St3's image of the shared object at this point was based on fragmental perceptions from his work-related experiences ("we see that they put more protectors, plastics") and his academic experiences ("noise distortions" and "signal quality"). The latter notions were central to his academic engineering studies but irrelevant to the specific functional relation.

In the formula ($L = R \bullet 45 \bullet d^2$) task, St3 identified algebraic relations between quantities presented in this sign (i.e., the *R*–*L* proportional relation and the *R*–*d* inverse proportional relation). He identified the first directly from the formula and the latter by solving the equation for *R*: "*R* is equal to *L* over 45 times *d* squared." Moreover, he eventually related the elaborated formula and the well-known academic one by referring to the role that the parameter ρ plays: "Yes, it is the resistivity …it is an intrinsic property of the wire. It depends on the material the wire is made."

In the graph task, St3 recalled the "slope notion" and he used it to relate the slope with the inverse proportional relation of *R*–*d*: "Since the qualities are inversely proportional the bigger cross-sectional area the lower the slope is." Then, the researcher asked him to relate the graph with the table representation, but St3 responded, "Both are correlated with resistance of the copper wire."

The formula representation helped St3 to shift from his fragmental and factual understanding of the TASK to a contextual understanding when he identified algebraic relations between the different quantities presented in the formula (*R*–*L* proportional relation and the *R*–*d* inverse proportional relation) but not in a connected way (i.e., how the one relation affects the other). In this explanation, we can identify typical ways of treating algebraic relations in academic mathematical tasks where there is no need to take into consideration how the one affects the other. Finally, even though St3 recalled mathematical notions from his academic studies (e.g., the slope, the diagonal line), he could not manage to use them meaningfully to provide a complete argument about the workplace task.

We concluded that St3 remained at a contextual understanding because, on the one hand, his mathematical knowledge seemed to be fragmented and situated and, on the other, his working experience was mostly irrelevant to the particular TASK. Furthermore, he did not manage to use the graph data as tools for reasoning about the workplace task presented in the table.

In Table 15.2, we summarize students' images of the shared object while facing the four WRs across the three cases. This shared object was characterized as factual, contextual, and symbolic.

The shared object all students developed while facing the table and professionals' explanations was factual. Their semiotic means were coming from both their academic fields (e.g., heat losses for St2 or signal quality for St3) and their internship experiences. These means consisted mostly of irrelevant scientific notions and sensual perceptions (e.g., using protectors to increase the diameter of the wire). The meaning of the mathematical object was partial (only St1 identified the role of the hidden variable) and their arguments about the TASK were mostly false or incomplete.

The shared object all students developed while facing the elaborated formula and the graph was contextual. Even though they used their academic experiences to describe the mathematical object, these explanations were not sufficient to explain the TASK. This was mostly because they did not know the scientific aspect of the maximum resistance that preserves the signal quality. When the researcher informed them about this scientific aspect, St1 (Case 1) and St2 (Case 2) linked aspects of the different WRs and used the mathematical object as a tool to provide a complete argument about the TASK. Furthermore, St1 related the practical argument behind the TASK (the lowering of cost) with his own professional experiences. Cases 1 and 2 are considered to be the *developmental transfer cases* because they had developed the potentially shared object between the two activity systems whereas Case 3 was considered to be the *non-developmental transfer case*.

Table 15.2 The shared object across cases for all WRs

<table>
<tr><th>Shared object (WRs)</th><th>Case 1</th><th>Case 2</th><th>Case 3</th></tr>
<tr><td rowspan="5">Factual (table & professionals' explanations)</td><td>Recalling technicians' discussions about aspects of the TASK</td><td>Using technician's explanations and technical terminology.</td><td>–</td></tr>
<tr><td>Identifying and verifying the hidden variable (R).</td><td>The hidden variable (R) is presented by the technician.</td><td>–</td></tr>
<tr><td colspan="3">Recalling scientific notions from their academic fields mostly irrelevant to the mathematical object; developing sensual perceptions from their internship.</td></tr>
<tr><td colspan="3">Making general description of the mathematical object in relation to two quantities (L, s).</td></tr>
<tr><td colspan="3">Making false or incomplete arguments about the TASK.</td></tr>
<tr><td rowspan="5">Contextual (formula & graph)</td><td>Academic-based description of the mathematical object.</td><td>Using the balance metaphor to describe algebraic relations in the mathematical object.</td><td>Academic-based description of the mathematical object.</td></tr>
<tr><td>–</td><td>–</td><td>Relating the mathematical object with his academic studies.</td></tr>
<tr><td colspan="3">Identifying and describing algebraic relations between three quantities.</td></tr>
<tr><td colspan="3">Recalling scientific notions relevant to the mathematical object (e.g., the slope).</td></tr>
<tr><td colspan="3">Making incomplete arguments about the TASK.</td></tr>
<tr><td rowspan="2">Symbolic</td><td colspan="2">Linking all WRs; arguing about the scientific and practical dimensions of the TASK.</td><td>–</td></tr>
<tr><td colspan="2">Using the mathematical object as a tool in reasoning about the TASK.</td><td>–</td></tr>
</table>

15.3.2 The Process of the Development of the Shared Object

Students' objects or motives concerning the TASK Aiming to build relations between the objects of the two activity systems (academic and workplace) caused difficulties and tensions for all students. However, St1 and St2 managed to overcome them and develop a shared object whereas St3 did not succeed. St1 showed interest in understanding the TASK and this was present in all his activity. In St1's case, we identified a number of emotional reactions during his activity. These were, for example, his initial reluctance to proceed with the task or his self-evaluation of his ideas and suggestions. In the latter case, he was articulating the possibility of other, more appropriate interpretations that might exist besides his own ("maybe I make a mistake… we could ask a technician"). Even though the TASK was not so close to his informatics studies, he persisted and managed to overcome his hesitations and his difficulties (e.g., where to focus on the graphical display). Furthermore, we can see changes in his statements of his involvement (i.e., from negative "I do not know" and "I do not remember" to positive ways of acting: "I'll solve the formula," "I'll take the limits"). St2 was motivated to be engaged with the TASK, which was relevant to his professional field. When he realized his initial difficulties in understanding the TASK, he searched for help from a technician. This brought him some tensions due to the different ways of explaining mathematical relations in his academic studies and the ones presented by the technician. He managed to overcome these tensions eventually while facing the formula and the graph, which were closer to his academic experiences. St3 did not make substantial effort to understand and argue about the TASK, maybe because it was beyond his professional field (electronics).

Mediated tools and boundary objects The table representation was nontransparent for all cases. Students' first interpretations of the table were fragmental (something they heard, they saw, something it reminded them of). The professionals' explanations (metaphor or practical examples) seemed not to help the students to make significant shifts in the shared object they had developed in the table task because these explanations were bounded in the unfamiliar-for-them workplace context. When all students faced the formula and the graph, which were closer to their academic experiences, they changed their mode of mathematical denotation (referring to and describing algebraic relations). For example, the formula provided St2 mathematical tools that were absent in the technician's reasoning (i.e., algebraic relations presented in a symbolic form). Moreover, this particular mathematical object is present only in physics texts in students' academic studies, with an emphasis on its scientific characteristics, whereas it is unusual for it to become a tool in students' academic mathematical practices.

The communities, rules, and students' division of labor The researcher (as a member of the academic mathematics community) facilitated the interactions between the different activity systems by probing all students for reasoning and clarifications, provided information about the limits of the resistance, and stimulated students' reflections. The technician (as a member of the workplace community) supported Case 2 to develop understanding by providing examples of the usage of the functional relation in his everyday work. This support was absent in Cases 1 and 3. The technician's support was crucial because he revealed the role of the hidden variable, he provided instances of this TASK in action, and he provided professional explanations. Concerning the rules in the two communities, in the academic context, science and mathematics are taught separately whereas in the workplace context, the TASK incorporates aspects from both fields. Finally, the fact that St3 did not have access relevant to the TASK fieldwork activities maybe acted as a limiting factor in his transferring process.

15.4 Concluding Remarks

In this chapter, we presented three illustrative cases of students' transferring process while they were trying to objectify the role that a functional relation from their academic studies played in their reasoning about an authentic TASK. This TASK concerned the selection of the proper copper wire in a local subscriber network. The above functional relation was represented by four different WRs (a table, professionals' explanations, an elaborated formula, and a graph). By conceptualizing transfer as a sociocultural, meaning-making process of objectification, we used the theory of developmental transfer by Engeström (2001) and Radford's (2008) objectification theory as our analytic and theoretical tools. Transfer in this way is conceptualized as the development of a shared object between the academic and the workplace activity systems. The above theoretical constructs helped us to zoom in on the nature of the shared object across the WRs and on how this shared object was developed. The three illustrative cases were: Case 1, which refers to St1, who managed to revisit the functional relation through many tensions and with much persistence; Case 2, which refers to St2, who also managed to revisit the functional relation but with the technician's support; and Case 3, which refers to St3, who made several connections with his academic studies but who could not manage to overcome the contextual restrictions that the WRs as well his division of labor in his internship were imposing on him.

We identified shifts in participants' meanings of the functional relation in their attempts to interpret the four WRs. Students' first interpretations of the table were factual and fragmental (something they heard or saw in their fieldwork or something they remembered from their academic studies). The table had a form of a cultural inscription that was satisfying particular workplace collective and communal purposes and was thus nontransparent for the newcomers in this community (Williams & Wake, 2007). The professionals' explanations did not help participants to make shifts in the shared object they had developed in the table task. The formula, on the one hand, helped all participants to change their mode of mathematical denotation (referring to and describing algebraic relations) whereas, on the other hand, they relied mostly on their academic-oriented ways of treating algebraic relations. Thus, students' understanding of the formula task was mostly contextual, originating from their academic and school experiences. The same happened when students faced the graph. Afterwards, Case 1 (St1) and Case 2 (St2) managed to link the different WRs and to reason by taking into consideration the requirements of this specific context and thus develop a symbolic understanding of the underlying relation. However, St3 (Case 3) was not ready to overcome contextual restrictions that this inscription was imposing on him. Hence, the type of WRs seemed to influence students' transferring process in certain ways.

St1 (Case 1) was the only participant in this study who managed, without any help from the workplace community, to develop his understanding of the functional relation as required in this context. A factor that contributed in this direction was his persistence to overcome many challenges (hesitations, drawbacks, tensions) he met throughout this activity. This result is in line with other relevant findings about the role of affective factors in transfer (Evans, 1999). Another factor that promoted participants' attempts to relate their academic to the work-related knowledge was the role of the members of the academic community (the researcher, for all cases) and the workplace community (the technician for Case 2) who acted as mentors by providing the necessary scaffolding that participants needed to complete the tasks. In contrast, the embeddedness of certain representations in workplace expertise (the table, the graph) and certain ways of reasoning (through metaphors and practical examples) acted as constraints in participants' transferring process. Constraints were also present in certain academic behaviors and practices, such as knowledge compartmentalization (Mandl, Gruber, & Renkl, 1993), as well as procedural ways of treating algebraic relations (Smith, 2011). In addition, participants had to overcome the differences in the learning processes between their academic and internship experiences and to identify their role as apprentices in the workplace setting. Akkerman and Bakker (2012) argued that apprentices are at the periphery of both activity systems (being a student but away from their academic duties and being a professional and novice at the same time in their fieldwork).

The results of this study suggest that personal and communal dimensions are in a dialectic relationship when apprentice students try to link academic and workplace experiences. For example, the affective dimension was related to tensions that an apprentice might meet due to his division of labor in the work community whereas the transfer development presupposes students' familiarity with the workplace tools and rules or the type of reasoning used in this setting. In addition, mentors from both activity systems might affect students' transferring activity and authentic representations could act as boundary objects between the above systems.

Many researchers have identified internship programs as valuable learning and working experiences for apprentice students to relate their school and work-based learning processes (e.g., Akkerman & Bakker, 2011). In the Greek educational system, these programs do not give any attention to making apprentices' experience more meaningful to them. This study suggests that the use of authentic tasks as well as having mentors from the university and the workplace may help to facilitate apprentices' learning. This could be a help for both workplace and academic institutions. Of course, the rules of teaching and learning mathematics at the university need to change to include more opportunities for students to reason with and develop their conceptual understanding rather than just solving a packet of exercises for their final exams.

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Part IV
Transfer Research that Informs Teaching and Research

Chapter 16
Teachers' Beliefs About How to Support Students' Transfer of Learning

Jaime Marie Diamond

Transfer, or the idea that one's learning influences their engagement in novel situations, provides the foundation on which many teachers interact with students. As a mathematics teacher educator, I often find myself thinking about how to structure class activities so that the preservice teachers in my classroom are prepared for their future interactions with children: What finite set of activities can I present them with so that they are prepared to interact with all (or even most) children? When I teach mathematical content, the question is similar: What finite set of activities can I present students with (and how can I structure the interactions around those activities) so that students are enabled to productively engage in future problem-solving situations? In other words, what can I do now to positively influence students' future interactions? As it turns out, educators in educational systems around the world aim to foster the development of students who can use their classroom learning in ways that allow for their successful engagement in new situations (e.g., Australia's Ministerial Council on Education, 2008; England's Department of Education, n.d.; Hong Kong's Curriculum Development Council, 2017; Singapore's Ministry of Education, 2017; The United States' National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

Not only do educators seek to support the development of students who can productively transfer their learning, but items appearing on both national and international assessments are explicitly designed to test students' transfer, which is often conceived in terms of the *successful application* of concepts and procedures. For example, the National Assessment of Educational Progress reports the results of a student's mathematical achievement at one of three levels: Basic, Proficient, or Advanced. The three levels are distinguished by the degree to which students show "evidence of understanding the mathematical concepts and procedures" comprising particular content areas as well as the degree to which students can "*apply*

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[emphasis added] mathematical concepts and procedures" in both routine and non-routine situations (National Assessment Governing Board, 2017, pp. 71–72). In this way, the ability to productively and successfully make use of one's learning while engaging in increasingly novel situations is taken as an indicator of one's mathematical sophistication.

Similarly, the 2015 Trends in International Mathematics and Science Study (TIMSS) distinguished between and assessed both what students *know* (i.e., "the facts, concepts, and procedures students need to know," Mullis & Martin, 2013, p. 24) and were able to *apply* (i.e., "the ability of students to apply knowledge and conceptual understanding to solve problems," p. 24) across a range of mathematical content areas. Students in many countries including the United States performed significantly better on tests of the former than of the latter. Interestingly, there were several countries (e.g., Sweden and New Zealand) where the opposite was true—students performed significantly better on tests of applying than on tests of knowing.

These results raise many questions, including: What are teachers doing to support their students' transfer of learning? Alternately, what do teachers believe they should be doing to support students' transfer? Moreover, what is the relationship between teachers' beliefs about and approaches to transfer and those of the people who wrote the aforementioned assessments?

16.1 Teachers' Beliefs About Transfer

16.1.1 Prior Work

Interestingly, research examining teachers' beliefs about and instructional supports for transfer is only just beginning to emerge. A curious reader may wonder whether teachers even think about their students' transfer of learning (i.e., how the learning that emerges and develops in the classroom influences students' engagement in future novel situations). In 2013, I reported the results of a pilot study that I conducted to determine just that (Diamond, 2013). In the pilot study, I interviewed seven teachers and posed a range of questions including more general questions like "What are your goals as a math teacher?" Analysis of the interview data showed that teachers do indeed think about transfer, for example, (a) when explaining their desire to support the development of students who "know how to transfer their knowledge and problem solve in the real world;" and (b) when discussing their frustration that so many students fail to make use of their learning in novel situations (e.g., on a quiz or a test) despite their best efforts to prepare them for those situations. Hohensee (2016) reported similar findings. In particular, he found that all of his teacher participants were explicitly aware of when their students failed or succeeded in making use of their prior knowledge in a novel context.

More recently, I reported the results of a study that provided the field with rich images of how teachers think about students' transfer of learning (Diamond, 2019).

In particular, I described and illustrated six distinct beliefs about transfer (i.e., what it is and how it occurs) that the mathematics teachers in my study provided evidence of holding. Of the six beliefs, two were focused on students' content knowledge, two were focused on students' disposition (i.e., their problem-solving approaches), and two were focused on students' affect. Whereas transfer researchers have tended to maintain a sole focus on the role that content knowledge plays in students' transfer of learning, I found that teachers do not. Instead, all but one of the teachers provided evidence of holding multiple beliefs about transfer wherein they focused on at least two of the following: content knowledge, disposition, and affect.

16.1.2 This Chapter

The purpose of the present chapter is to shed light on how the teachers referenced in the previous paragraph believed students' transfer of learning should be supported. The guiding research question was: What are teachers' beliefs about how to support students' transfer of learning? Eight practicing teachers engaged in two clinical interviews involving instructional tasks related to slope. Analyses of these data revealed 12 beliefs about how to instructionally support transfer, providing insight into teachers' images of what it means to "teach for transfer."

To examine these teacher beliefs, I drew from Philipp (2007) who defines beliefs as:

> Psychologically held understandings, premises, or propositions about the world that are thought to be true…. Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual. (p. 259)

In other words, when developing instruments to generate data regarding how the teachers believed students' transfer of learning should be supported, I followed Philipp and others in assuming there exists a complex relationship among teachers' beliefs, perceptions, and actions.

As the definition suggests, beliefs shape one's perception of and interactions with the world (Pajares, 1992). They serve as filters bringing forth certain aspects of a situation while allowing others to fade into the background; in this way, beliefs affect what one notices and therefore shape one's interpretation of a situation (Grant, Hiebert, & Wearne, 1998; Mason, 2002). For instance, a teacher may believe that to support students in productively transferring their learning, one should support the development of conceptually meaningful understandings of various topics; this teacher may therefore attend to the nature of a student's explanation over the correctness of his or her final answer. Moreover, beliefs position one to act in a certain way (Cooney, Shealy, & Arvold, 1998; Di Martino & Zan, 2011; Forgasz & Leder, 2008; Rokeach, 1968). That is, given a particular situation, beliefs may draw one toward one action over another. For example, the teacher above may be moved to integrate particular tasks and activities into his or her lessons over others or to probe students for further justification after answers have been given. Such assumptions informed the design and sequencing of the instruments I used during data collection and helped to illuminate the ways in which the teachers in this study believed students' transfer of learning should be supported.

16.2 Methods

16.2.1 Participants

Eight practicing teachers[1] from the southwestern United States participated in this study. I recruited both middle school and secondary teachers who provided evidence, during pre-recruitment observations, of considering their students' transfer of learning. Such evidence included verbal statements and enacted practices focused on how students' current learning might inform their engagement in novel situations (e.g., asking students, "Where might you use this in *your* life?"). I also recruited teachers on the basis of the mathematics courses they taught (e.g., Pre-Algebra, Algebra 1, or Algebra 2) and whether they had recent opportunities to teach about slope and linear functions; participants were recruited based on whether they viewed slope as part of their curriculum. I chose a specific mathematical topic to provide participants with a context in which to ground their conversations about transfer. Slope was chosen due to its prominence and importance in mathematics curricula. Incidentally, the amount of formal teaching experience varied across the eight participants from 0 to 33 years as did their educational backgrounds. See Diamond (2019) for more information about the participants as well as the participant-selection process.

16.2.2 Data Collection

Overview of the research design Each participant engaged in two 2-hour semi-structured clinical interviews (Clement, 2000; Ginsburg, 1997). Prior to the first interview, I asked each teacher to reflect on a time they taught about slope and linear functions, review their teaching materials, and select an item (e.g., a lesson plan, test, class activity) they believed showed an instance wherein they attempted to support students in "generaliz[ing] their learning to new tasks and/or activities" (Diamond, 2013, p. 320). Participants then discussed these items during the first interview.

At the end of the first interview and after acknowledging that they may have many different goals when creating lessons, I asked each participant if they believed

[1] Pseudonyms were used for all participants in the study.

they could create or adapt an existing lesson plan on slope for the explicit purpose of supporting students in successfully generalizing their learning beyond the specifics of the lesson. All participants responded in the affirmative. They then discussed these lesson plans during the second interview.[2]

I video- and audio-recorded all interviews with a video camera and a table microphone. I aimed the video camera to capture the teachers' verbal reports, gestures, and written inscriptions. All written work and materials were collected.

Instruments I implemented four sets of interview questions over the course of the two interviews. The participants' responses were consistent across the sets of questions. I, therefore, focus here on the analysis of data emerging during the question sets involving each teacher's Teaching Item and Lesson Plan because these were the sets that most explicitly drew from the participants' actual teaching practices. In contrast, another set of interview questions included *hypothetical* teachers' classroom activities. It should be noted that the term "transfer" was not used during the interviews. Rather, when appropriate, a general description was used to orient participants to the phenomenon of interest. An example from the Interview 1 protocol follows:

> Tell me a little bit about the artifact/item you chose and how you think it shows that you were thinking about helping students to make future use of their learning. (Alternate/additional wording: …helping students to generalize their learning to new situations).

The Teaching Item Activity During Interview 1, I asked the participants to engage in discussion surrounding their teaching items. To gain access to a spectrum of beliefs (i.e., from more explicitly to more tacitly held beliefs) about supporting students' transfer of learning, I began by providing teachers with opportunities to espouse their beliefs. For example, I asked the teachers: "What in particular were you doing to help your students so that they would be enabled to successfully engage with new tasks, activities, or situations?" In addition, I designed items to generate data from which I could infer beliefs. These inferred-beliefs items included showing the teachers mathematical tasks and asking them whether they believed students would be able to productively engage with those tasks as a consequence of having engaged in activity involving the teachers' teaching items. The mathematical tasks that I showed to teachers varied in terms of the (a) real-world contexts in which they were set; (b) mathematical terms used (e.g., whereas "slope" appeared in one task, "rate" or "ratio" appeared in others); (c) representations shown (e.g., whereas more standard representations like a table or a graph appeared in some tasks, non-standard representations like a drawing of a hill appeared in others); and (d) quantities (i.e., measurable attributes of an object or event; Smith & Thompson, 2008) and quantitative relationships described (e.g., whereas most tasks involved linearly related quantities like a burning candle's height over time, one task involved

[2] Time between interviews: 8 weeks (mean); 4.5 weeks (median); 4 weeks (mode); 21 weeks (range).

a quadratic motion situation). I then probed the teachers' responses, for example, asking how they would go about preparing students for their future engagement with tasks the teachers identified as being potentially too difficult.

The Lesson Plan Activity During Interview 2, I asked the participants to engage in discussion surrounding their lesson plans. As part of the Lesson Plan Activity, I asked the teachers to include a new task or activity that they believed students would be able to productively engage with as a consequence of having engaged in their lessons. Because I told the teachers that they may be asked to subsequently implement their lesson plans in their classrooms, this activity was designed to generate data regarding the instructional supports the teachers might actually enact during their lessons and the teachers' beliefs about how those supports would serve their students' transfer of learning. As part of the larger study, some of the teachers were indeed observed as they implemented their lessons; the results of the analyses of those data are not presented here.

16.2.3 Data Analysis

Data analysis involved a multistep qualitative process wherein I examined teachers' responses to interview items to identify and characterize their beliefs about how to support students' transfer of learning. In the first step, I transcribed all interviews and identified those episodes in which the teachers appeared to address students' transfer of learning. I then reduced the data set to these episodes to prevent data overload and subsequently summarized these episodes to establish an account of what the teachers did and said prior to any interpretive coding (Miles & Huberman, 1994).

In the next step, I coded the data using what Miles and Huberman (1994) considered "partway between a priori and inductive coding" (p. 61). To capture themes about teachers' beliefs regarding how to support students' transfer of learning, I used a priori codes contained within the transfer literature. For example, some of the teacher participants expressed the belief that multiple examples should be used when aiming to support transfer—a belief found in the literature (e.g., Gentner, Loewenstein, & Thompson, 2003; Markman & Gentner, 2000; Singley & Anderson, 1989). I induced other categories of teachers' beliefs using *open coding* from grounded theory (Strauss, 1987). Throughout the analysis, I sought disconfirming evidence using the *constant comparative method* of grounded theory (Strauss & Corbin, 1990).

A curious reader may wonder about the relationship between the 12 categories of instructional supports presented below and whether, perhaps, any categories might be combined. For example, the following three instructional supports may sound similar: *choose tasks and pose problems that support students' quantitative reasoning*, *encourage students to infuse a real-world context into a given (decontextualized) problem*, and *make use of real-world situations* (see Appendix for an organized table of the 12 categories of teachers' beliefs regarding instructional supports for students' transfer of learning). However, the teachers' beliefs about how each move would mediate transfer—via the development of a mathematically valid interpretation of slope, a visualization and sense-making approach to problem solving, or a belief that mathematics is relevant and useful outside of the classroom—is what informed the collapsing or separating of categories. See Diamond (2019) for further elaboration of methods used during data analysis.

16.3 Results and Discussions

In this section, I illustrate the participants' 12 beliefs about how to support students' transfer of learning. These instructional supports correspond with the particular ways in which the teachers believed transfer occurs (see Appendix). This section is therefore organized by the teachers' beliefs about transfer (i.e., what it is and how it occurs).[3] Relations among the teachers' and researchers' instructional supports for transfer are also discussed.

16.3.1 Mathematical Content

Seven teachers provided evidence suggesting they believed that students' transfer of learning involves the role of mathematical content. In particular, these teachers believed students would be able to productively transfer their learning to novel situations if students develop and make use of particular types of mathematical knowledge. In what follows, I describe each of these beliefs about transfer and then present the teachers' corresponding beliefs about how to instructionally support transfer.

Procedures Three teachers believed students transfer their learning to a novel situation when the novel situation prompts the use of a learned *procedure*, or predetermined set of steps. For example, the "rise over run" procedure for finding slope in a linear context involves isolating two coordinate pairs, finding the "rise" between the pairs (i.e., the difference in y values), finding the "run" between the pairs (i.e., the difference in x values), and placing the "rise" over the "run." These teachers did not provide evidence of considering the conceptual underpinnings of such procedures.

Corresponding instructional supports There were two instructional moves these teachers believed would support students' transfer of a learned procedure: (a) tell students the desired procedure, and (b) use multiple examples. I illustrate each pedagogical support for transfer in turn.

[3] For further elaboration and exemplification of the teachers' beliefs about transfer, see Diamond (2019).

Tell students the desired procedure. All three teachers provided evidence suggesting they believed students would be supported in productively transferring their learning if they were shown or told the procedure that should be used to solve particular types of problems. For example, Blake explained that the goal of his teaching item (a collection of six activities) was to introduce students to the "rise over run" procedure for finding the slope of a graphically represented line (see Fig. 16.1 for an example of two of Blake's activities):

> I start with two points on a graph and I'll define, I will give them an equation, you know, the y_2 minus y_1, but, I really want them to be able to find the slope of a line by identifying two points and counting rise and run.

In other words, when asked about how his teaching item illustrated the way in which he worked to support students' transfer of learning, Blake highlighted his demonstration of the procedure he ultimately wanted his students to use to find slope.

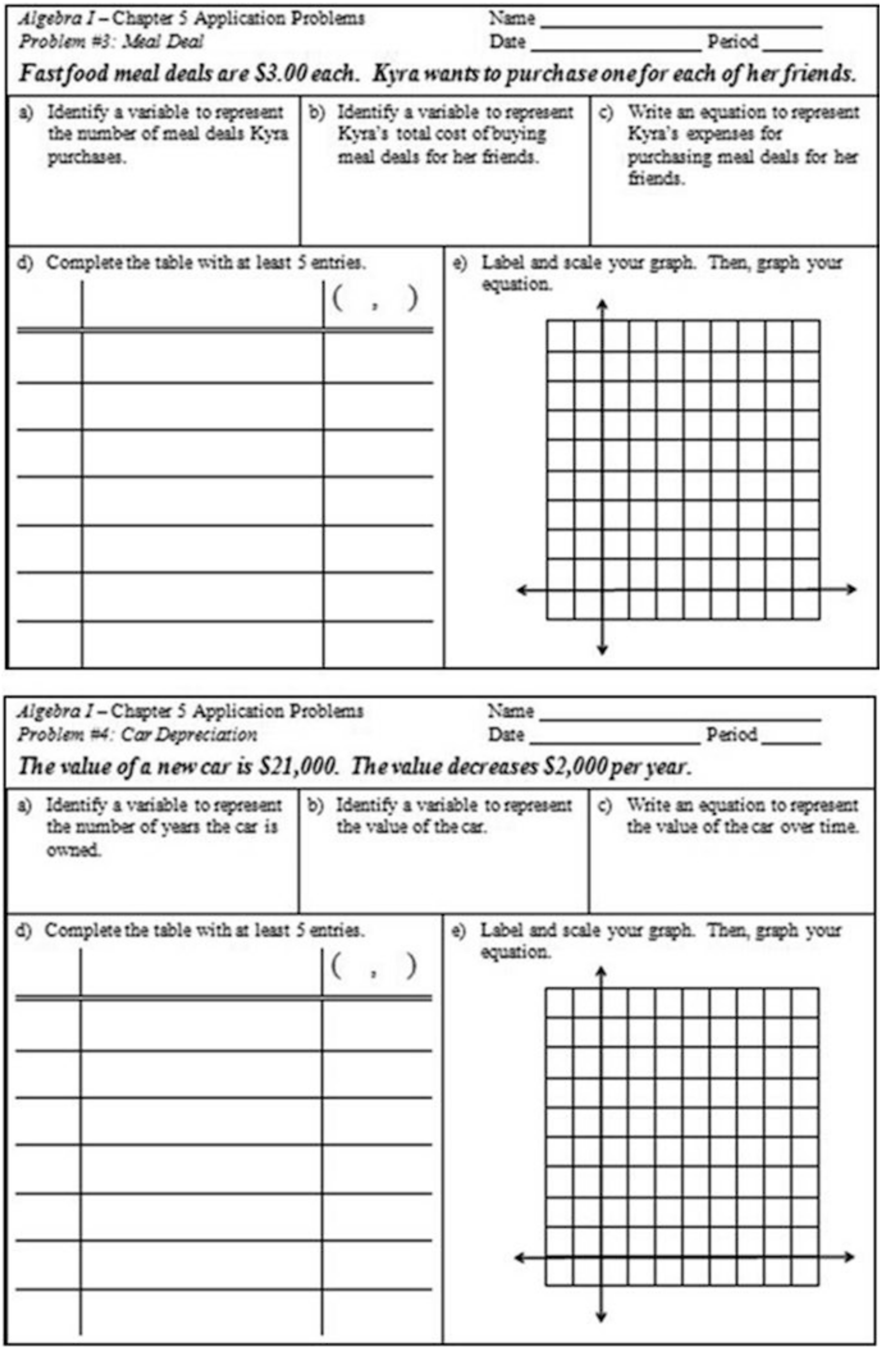
When shown mathematical tasks and asked whether he believed his students would be able to productively engage with them after having engaged with his teaching item, Blake made predictions based on whether he believed his students would be prompted to make use of the learned procedure. More specifically, Blake explained that his students would be better supported in productively transferring their learning to The Burning Candle Task than to The Water Pump Task because The Burning Candle Task contained a graphical representation; thus, his students would be prompted to employ the learned procedure (i.e., they would be prompted to choose two points on the line, count the corresponding rise and run, and place the rise over the run).

In contrast, Blake believed his students would experience difficulty during their engagement in The Water Pump Task, which contained a tabular representation including data for the amount of water in a pool at four different times, because the task did not contain a graphical representation or a table *to be* filled in. As he explained:

> Kids would look at [The Water Pump Task] and think, "Well, I have never done this before" and stop there…. They would go, "This doesn't look the same. I don't understand what you want me to do." Because it doesn't give them a table to fill in. The table *is* filled in and then the table looks different that the table[s] before [See Fig. 16.1] and [The Water Pump Task] doesn't give them a graph.

In other words, he believed his students would be unsure of how to proceed when confronted with a situation that "looks different." Similarly, when explaining the way in which their teaching items supported students' transfer, both Anne and Donna emphasized the act of telling their students precisely what to do when engaging in particular types of problems involving slope. As Anne said, students "are not going to make the connection unless it's explicit."

Use multiple examples. Two teachers (Donna and Blake) provided evidence that they believed students would be supported in productively transferring their learning if multiple examples were used during initial learning. Donna believed that the use of multiple examples would support her students in knowing *when* to use a learned procedure and Blake believed that the use of multiple examples would



Algebra I – Chapter 5 Application Problems
Problem #3: Meal Deal
Name ____________________
Date ______________ Period _____

Fast food meal deals are $3.00 each. Kyra wants to purchase one for each of her friends.

a) Identify a variable to represent the number of meal deals Kyra purchases.
b) Identify a variable to represent Kyra's total cost of buying meal deals for her friends.
c) Write an equation to represent Kyra's expenses for purchasing meal deals for her friends.
d) Complete the table with at least 5 entries.

		(,)

e) Label and scale your graph. Then, graph your equation.

Algebra I – Chapter 5 Application Problems
Problem #4: Car Depreciation
Name ____________________
Date ______________ Period _____

The value of a new car is $21,000. The value decreases $2,000 per year.

a) Identify a variable to represent the number of years the car is owned.
b) Identify a variable to represent the value of the car.
c) Write an equation to represent the value of the car over time.
d) Complete the table with at least 5 entries.

		(,)

e) Label and scale your graph. Then, graph your equation.

Fig. 16.1 Two of Blake's six activities support students in *seeing* the procedure. For instance, Donna's teaching item consisted of a lesson on arithmetic sequences and their connection to slope. The lesson began with multiple examples of arithmetic sequences and the query to her students, "What do they all have in common?" Specifically, Donna said:

> I just give them "55, 49, 43," [and tell them to] continue the pattern. And most kids will go, "Oh, it's going down by 6." So then we do a couple of those. It can be decimals. It can be fractions. It can be—and I want it to be added or subtracted. It doesn't matter. And we do a couple of those and I say, "OK, what do they all have in common? They all have in common that this difference, whether it be subtracting 6 from each term or adding 5 to each term, *that* [the difference] is constant."

In this example, when explaining how her teaching item illustrated her attempts to teach for transfer, Donna foregrounded her request that students look across multiple examples that varied in terms of number type (e.g., whole numbers, decimals, or fractions) but that all shared a common feature (i.e., a constant difference in consecutive terms). Donna explained that after students identify the constant difference as the feature shared by all of the sequences, students either spontaneously or via a sequence of guided prompts link arithmetic sequences to slope, at which point Donna demonstrates the procedure for finding slope, $slope = \frac{y_2 - y_1}{x_2 - x_1}$. In this way, Donna described how she uses multiple examples to illustrate to students *when* it is appropriate to make use of "slope" while engaging with sequences, namely when there is a constant difference between successive terms of a sequence.[4]

Blake seemed to make use of multiple examples for a different purpose. Rather than helping students to determine whether a particular situation necessitated their use of a demonstrated procedure, Blake believed multiple examples helped students *see* the procedure itself. The six activities comprising Blake's teaching item varied in terms of context but did not vary in terms of formatting or wording (see Fig. 16.1). As Blake explained, "this [pointed across his activities] is trying to create, I am trying to show them a pattern … then they would start seeing that pattern." Here, Blake seemed to express his belief that by making use of multiple examples he was helping his students learn to recognize a particular "pattern" for finding slope, namely the "rise over run" procedure he had shown them.

Meanings Four teachers believed transfer is related to the meanings students develop during mathematical activity. With respect to the topic of slope, these teachers believed that students would productively transfer their learning if students developed mathematically valid interpretations such as *slope (in a linear context) is a ratio describing of the multiplicative relationship between two quantities*. When making predictions about students' transfer, these teachers thought about the meanings students might have developed rather than whether a task might cue students' use of a predetermined procedure.

[4] The independent variable, with respect to Donna's arithmetic sequences, is taken to be the ordinal placement of a term in the sequence; thus, the "run" or the "$x_2 - x_1$" is 1, which means that the slope in this case, calculationally, is the same as the constant difference between consecutive terms of an arithmetic sequence.

Corresponding instructional supports There were three instructional moves these teachers believed would support students' transfer of mathematically valid interpretations of slope: (a) choose tasks and pose questions that support students' quantitative reasoning, (b) use a curriculum that progresses from contextualized to decontextualized situations, and (c) provide students with opportunities to explain their reasoning. Each instructional move is discussed in turn.

Choose tasks and pose questions that support students' quantitative reasoning. All four teachers provided evidence suggesting they believed students would be supported in productively transferring their learning if the tasks and questions they posed provided students with opportunities to reason quantitatively (i.e., with measurable attributes). When discussing how their teaching items illustrated their attempts to support students' transfer of their slope learning, these teachers described the ways in which students would be supported in conceiving of slope in terms of *two relevant quantities* (e.g., the amount of a particular supply and the amount of people in a family) as well as the *multiplicative relationship* between those quantities (e.g., we need 12 times as much supply as there are number of people in our family).

For example, Patrick explained that to support students' transfer, he tends to avoid using more "traditional," decontextualized problems and instead selects tasks set in real-world contexts and poses accompanying questions to get students thinking about the measurable attributes of objects in those situations. In his words:

> Instead of collecting data in a traditional sense where it's *x* and *y* tables and it's we're graphing *x* and *y*, we are graphing *distance* versus *time* or we are graphing *how much coffee they have* as *days pass* and it's going down; they are using up supplies. Or *how much water do they have in storage* and why does the graph go up and then go down?

Patrick believed that by posing questions about the quantities in real-world situations (e.g., "How much water do they have in storage?"), students would begin to focus on measurable attributes such as *the amount of water*. Once his students are able to identify relevant quantities in various contexts, Patrick explained that he supports them in conceiving of the relationships that exist between those quantities by asking questions like "How much coffee are they using every day?" and "How many miles have they traveled each day?" As a result, he believed students would begin to conceive of slope in terms of "how much change is happening."

Patrick described how he planned to be even more explicit about supporting students in conceiving of the numerical value of slope in terms of the quantities in a specific situation. For instance, when students produced 12 as the slope of the function, he planned to ask, "What does the 12 mean?" explaining that he wanted to give his students "time to think about 'well, 12 what? It's 12 what? And why is it *12* of something?'" Moreover, Patrick reported that he would ask his students *where* they see the slope of 12 in their graphs, as a way to help students conceive of and coordinate both quantities in the ratio (12 pounds of beans for each 1 person). In this way, these teachers consistently discussed posing tasks and asking questions that supported students in developing a quantitative meaning for slope as a way to support students in transferring their slope learning.

Use a curriculum that progresses from contextualized to decontextualized situations. Two teachers (Patrick and Kay) provided evidence that they believed students would be supported in productively transferring their learning if they made use of curricula that progressed from contextualized problem situations to decontextualized tasks and symbolic formulas. Returning to Patrick, he believed that such a progression would be particularly helpful in supporting students to make use of the meanings they developed while engaging in contextualized problem situations later, when they are confronted with decontextualized tasks. That is, students would be supported in productively transferring their understanding of slope because they could leverage their meaningful interpretations to make sense of decontextualized problem situations (as "transfer" tasks). Patrick explained how such a progression could be powerful for students:

> The first problems they do are all conceptual. So, [for example] they do one and it says "You need four inches of shoe lace for every child" and they're graphing number of children and amount of shoe lace … So those are the variables and then we go into some traditional examples where it's purely *x* and *y* and it's an algebraic rule; it's just the typical Cartesian coordinate grid. But they still have to identify where you start at and what is the change? So they realize "Oh, it's the same; it's either in a word problem, a context, or it's not." So, that's what I hope they would be able to do later, is still be able to, we call that [slope], but "Oh, no, it's the same thing."

In other words, by beginning a unit with a problem which supports students in thinking about slope in terms of quantities like "the amount of shoelace" and "the amount of children" and, in particular, as the unit ratio 4 inches of shoelace for every 1 child, students are first supported in developing a mathematically valid interpretation of slope as a ratio of two quantities. Students can then make use of such ways of reasoning when presented with "traditional examples where it's purely *x* and *y*," eventually realizing that both kinds of situations involve slope, or a ratio of two quantities. In this way, these teachers believed a progression from contextualized to decontextualized problem situations supported students in recognizing both types of problem situations as involving the interpretation of slope developed during initial learning activities.

Provide students with opportunities to explain their reasoning. Two teachers (Richard and Emma) believed that allowing students to explain (verbally or in writing) their interpretation of a particular problem or mathematical topic would support transfer. As Richard stated, "If they never can explain it [i.e., slope] in something that they do know, I think they are going to have a really hard time later on trying to take it to something that is new to them."

Similarly, Emma expressed her belief that "having students explain their answers" ("right or wrong") supports them in being able to make future use of their learning. She believed such explanations would help make students' ways of reasoning more explicit: "I think it's nice to really talk through what went through their mind [be]cause it may make [their thinking] more obvious for them." Once ways of reasoning are made more obvious to students, Emma believed those ways of reasoning would be available for future use. For instance, Emma believed that if students are supported in reasoning about and verbally explaining the meaning of a particular slope value, then they should be able to make use of those same ways of reasoning to explain the meaning of additional slope values.

Discussion of instructional supports for beliefs involving mathematical content Researchers operating from information-processing views of transfer have also suggested versions of the first two instructional supports discussed above (i.e., *tell students the desired procedure* and *use multiple examples*; e.g., Anderson, Kline, & Beasley, 1979; Brown, Kane, & Echols, 1986; Carbonell, 1983; Gentner et al., 2003; Gick & Holyoak, 1980, 1983; Singley & Anderson, 1989). However, there are significant differences in the ways in which researchers and teachers conceive of these pedagogical supports. For example, researchers tend to create systematic variation in the superficial surface details of the multiple examples they use during instruction and maintain the underlying structure, or solution strategy, to encourage students' encoding of the common strategy (e.g., Catrambone & Holyoak, 1989; Reeves & Weisberg, 1990, 1994). In contrast, Blake believed that sameness in the surface details across multiple examples (in terms of formatting and wording) was essential for transfer and instead sometimes varied the structure (in this case the solution method—see the two examples shown in Fig. 16.1, where solving Part E of each problem requires a different procedure).

As discussed in Diamond (2019), the teacher belief about students' *meaning* mediating their transfer of learning has points of contact with Lobato's (2006, 2012) actor-oriented transfer perspective. However, the teachers in this study significantly contributed to the pedagogical actions previously reported in the literature as being related to instances of productive transfer from an actor-oriented perspective (Lobato, Rhodehamel, & Hohensee, 2012). Specifically, these teachers articulated the importance of quantitative reasoning to students' transfer of learning and discussed this idea using a variety of curricular materials that practicing teachers currently use (e.g., The Overland Trail unit of the reform-oriented and National Science Foundation-funded Interactive Mathematics Program curriculum). Similarly, the teachers elaborated potentially important roles for the sequencing of activities and students' explanations in students' transfer of learning.

16.3.2 Disposition

Three teachers provided evidence that they believed the transfer of learning involves students' dispositions. In other words, these teachers believed that students would be able to productively transfer their learning to novel situations if they develop and make use of particular dispositions. Here, *disposition* is used to refer to a student's approach to problem solving. Whereas the previous section involved teaching actions to support students' transfer of mathematical knowledge, this section involves teaching actions to support students' transfer of a learned problem-solving approach. Here, it is the learned disposition itself (rather than the learned procedure or meaning) that is believed to transfer across situations.

A visualization and sense-making disposition Two teachers believed students' transfer of learning is mediated by a visualization and sense-making approach to problem solving wherein students imagine themselves in a problem situation in an effort to reason about what is going on in that situation.

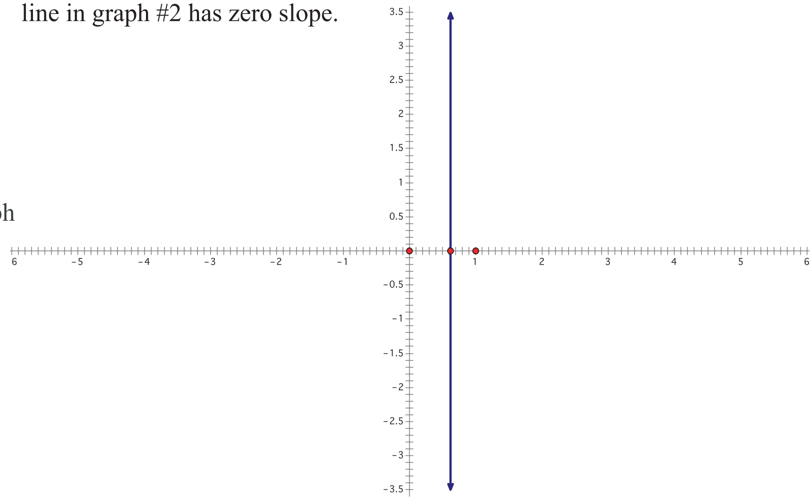
Corresponding instructional supports There were two instructional moves that these teachers believed would support students' transfer of a visualization and sense-making approach to problem solving : (a) encourage students to infuse a real-world context into a given problem, and (b) avoid problem statements that tell students how to proceed.

Encourage students to infuse a real-world context into a given problem. Emma believed that to support students' transfer of a visualization and sense-making disposition, she should encourage students to apply real-world situations to problems that are not already set within such contexts. Emma believed that leveraging real-world situations would enable students to visualize themselves in problems and thus support students in making use of their learned sense-making dispositions. To illustrate, note that Emma's teaching item (shown in Fig. 16.2) is devoid of a real-world context. She discussed her hope that students would apply such a context to the task to solve it (note the cue at the bottom of the task: "You may use a real world example to create your argument."). She went on to explain that if students did not infuse the situation with a real-world scenario, she would remind them to do so and then help them to apply a distance versus time scenario to both graphical representations to make sense of them. Emma believed that the tendency to apply a real-world context to decontextualized situations would "be something students could use [in the future], like if they are grappling with something, put it in a real-world context to explain it more." In this way, Emma believed students who learn to approach decontextualized problems by asking themselves "What if I just attach some sort of real-world situation to this? How would that help me? Would that reveal more about this slope?" will be supported in making use of their visualization and sense-making dispositions and, as a consequence, be supported in solving novel problems.

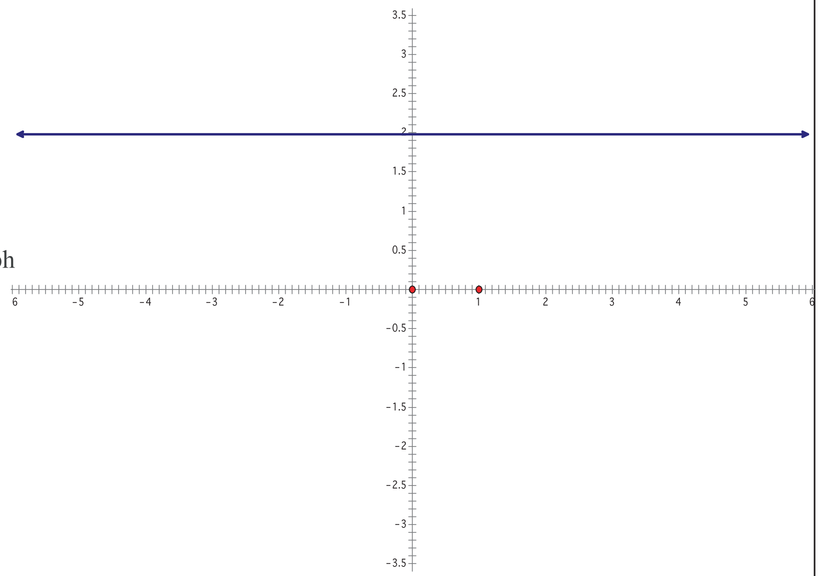
Avoid problem statements that tell students how to proceed. Kay believed that to support students' transfer of learning, she should provide students with open-ended problems that avoid step-by-step instructions regarding how to solve them (e.g., "Graph the points and then calculate the slope of the resulting line"). She believed that open-ended problems would facilitate students' use of their sense-making dispositions because such problems provide students space in which to reason about how to proceed. As Kay explained, problems that explicitly tell students what to do to may actually "create a roadblock" because they call for specific steps students may not know how to carry out. Moreover, Kay believed such problems might actually prevent students from thinking and reasoning. As she noted, "If you look at a lot of textbooks … a lot of the problems are broken down into steps. 'Step 1: Find slope. Step 2: Graph it. Step 3-,' you know, OK great, where is the thinking?" In other words, Kay believed problems that include "steps" deny opportunities to make

Task: Explain why the line in graph #1 has an undefined slope and why the line in graph #2 has zero slope.

Graph #1



Graph #2



Your explanation can range from the formulas or definition of slope, or you may use a real world example to create your argument. If you think of some other argument to form, great! I want your argument to be creative and convincing!

Fig. 16.2 Emma's teaching item

use of a sense-making approach to problem solving because they directly tell students *what to do* rather than allowing them to think about *what they could do*.

A group-brainstorming disposition Donna believed that students are more likely to productively transfer their learning when they develop a group-brainstorming approach to problem solving that involves working in a collaborative setting wherein multiple people contribute ideas about how to solve novel problems.

Corresponding instructional support There was one instructional move Donna believed would support students' transfer of a group-brainstorming disposition: model the disposition.

Model the disposition. Donna believed that to support students' transfer of learning, she should model or enact the disposition she wants her students to develop and use. To help students develop a more productive or effective group-brainstorming disposition, Donna described how she enacts its various components, for example, demonstrating to students how to share ideas in a group setting. For instance, when confronted with a novel problem-solving situation, Donna explained that she tells students what the problem reminds her of, what she notices about the problem, and how she comes up with ideas regarding how to solve the problem:

> We do a lot of teacher modeling – "This is how I'm thinking; look I saw this; oh, this reminds me of this," you know, so I do a lot of modeling of how I just kind of reach out and grab these things…. So, really once you get that going, then they can start problem solving.

In this way, Donna believed that if students are exposed to productive aspects of a disposition, they will be supported in reenacting those aspects, thus taking them on as part of their own dispositions. Note that Donna believed students would "*start* problem solving" once they develop the disposition.

Discussion of instructional supports for beliefs involving students' dispositions Like the teachers in this study, Bereiter (1995) focused on the development of dispositions that are necessary or useful. According to Bereiter, previous approaches to students' transfer of learning have emphasized the transfer of facts, strategies, or principles and have ignored or backgrounded the transfer of dispositions. Bereiter argued that when teaching for transfer, disposition is of primary concern since the success or failure of a lesson depends on whether or not students are supported in making future use of a learned disposition. Bereiter (1995) used *disposition* to refer to "some way of approaching things" and the *transfer of disposition* to refer to the idea that students carry over a particular way of approaching things into novel situations (p. 23).

In this way, Bereiter suggested teachers reflect on and identify the kinds of dispositions that should be fostered. This is what some of the teachers in this study appeared to do. Specifically, they identified two dispositions they believed would be particularly helpful to students' engagement with novel problems: a visualization and sense-making disposition and a group-brainstorming disposition. In addition, Bereiter suggested teachers create situations in which those desirable forms of thinking and approaching problems take place. Again, this is precisely what some of the teachers in this study did, yielding the three teaching actions discussed above, two of which were believed to support students' development and enactment of the former disposition and one of which was believed to support the latter.

16.3.3 Affect

Seven teachers provided evidence suggesting they believed students' affect plays an important role in their transfer of learning. More specifically, six teachers believed *students' condidence in their own abilities to engage in mathematical activity* mediates their transfer of learning and three teachers believed *students' beliefs that mathematics is relevant and useful outside of the mathematics classroom* mediates their transfer of learning. This follows McLeod (1992) who conceived of such elements as components of the affective domain.

Confidence Six teachers believed that students would productively transfer their learning to novel situations when those students develop confidence in their abilities to engage in mathematical activity. Here, *confidence* refers to the way in which a student views his or her "competence in mathematics" (McLeod, 1992, p. 583) or the "belief that one can learn to do that which is expected of one" (Broekmann, 1998, p. 18). In other words, these teachers believed that students who view themselves as competent and capable doers of mathematics will productively transfer their learning, whereas students who view themselves as incompetent and incapable doers of mathematics will be unsuccessful when confronted with a novel problem-solving situation.

Corresponding instructional supports There were several ways in which these teachers believed they could support students' transfer of learning via the development of students' confidence. I present the two most prevalent instructional moves here: (a) monitor the language used in the classroom, and (b) support students' independence.

Monitor the language used in the classroom. When explaining how they instructionally support students' transfer of learning, Donna and Blake focused on the nature of the language used in the classroom. In particular, these teachers described attending to the ways in which people in the classroom verbalize their mathematical experiences. As Donna explained, she and the students in her classroom are "very very careful of words." Donna provided the following example:

> In my classes, we don't say, "It's easy." I don't let them say that. We say, "I understand that" or "I don't understand that." And if someone says, "Oh, that's so easy," I'm like, "Excuse me?" and then they're all like, "You can't say that."

In this excerpt, Donna distinguished between *a problem* being an "easy" problem and the relative ease or difficulty *an individual* experiences while engaging with that problem. In the former, the relative ease or difficulty of a problem is an inherent characteristic of the problem itself and in the latter, the relative ease or difficulty of a problem is inherent to the student. It seemed Donna aimed to support students in reflecting on and articulating the latter rather than the former. For if one student in the classroom categorizes a problem as "easy," then other students in the classroom may feel stupid for not understanding the "easy" problem. However, if a student says, "*I* understand this," then the other students may be less affected by the statement because it does not imply that another student should also be able to understand it. In another example, Donna explained that she does not perpetuate the view that courses like calculus are "hard" but rather tells her students, "*Everybody* can take calculus." In this way, Donna described wanting to support her students in viewing themselves as competent and thinking "Oh, I *can* do that" regardless of the topic or content area and using language as a vehicle to support the development of such views. Similarly, Blake made use of personal anecdotes to let students know that the relative ease or difficulty they experience does not dictate *who they are* but is simply a part of *what it means to participate* in mathematical activity.

Support students' independence. When explaining how they instructionally support students' transfer of learning, Sam and Emma focused on supporting the development of students who could answer their own questions and solve their own problems. To illustrate, consider the two-part activity Sam brought to the first interview. Sam explained that the first part, which consisted of six questions, was to be completed as a class, whereas the second part, which consisted of Questions 7–12, was to be completed by students in small groups. In each of the first six questions, students were provided with (a) two coordinate points (x_1, y_1) and (x_2, y_2), wherein x_1, x_2, y_1, and y_2 are known values; (b) space in which to plug those values into the slope formula; and (c) a Cartesian plane on which to graph the associated line. However, in Questions 7–12, students were provided with a different set of information. For instance, in Question 7, (a) y_1 is an unknown, (b) the slope is given, and (c) a line has already been graphed.

When asked how he thought the first part of the activity supported students in productively engaging with the second part of the activity, Sam did not focus on relationships between mathematical details of the two parts of the activity, but rather on providing students with opportunities to figure out the second part on their own:

> Kids would be like "What do we do about Number 7?" and I'm like "I don't know; figure it out, you know this number [pointed to the missing y-value in Question 7] could be anything," and I just walk off, you know, and then I come back 5 minutes later and they're like "Yeah, we figured it out," like I didn't have to teach anything, you know, and just try to like change the problem in such a way where they have to like take some process that they thought they understood pretty well here [pointed to the first part of the activity] into some new situation.

In this excerpt, Sam expressed his belief that allowing students the time and opportunity to think through and answer their own questions (e.g., by walking away after hearing a student's question rather than directly answering the question) supported them in making use of their learning. Sam went on to explain that he hoped such a move would support students in feeling "confident" about their ability to engage in mathematical activity and, in particular, their ability to productively engage with novel mathematical tasks. Similarly, Emma believed that allowing students to develop their own problem-solving strategies builds confidence "because now they see sort of the steps it takes to get to the solution and they did it themselves;" thus, when presented with a novel problem, students will have had experience solving such problems, believe that they can solve such problems, and therefore be enabled to solve them.

Belief that mathematics is relevant and useful Three teachers believed that students would productively transfer their learning to novel situations when those students develop the belief that mathematics is relevant and useful outside of the classroom.

Corresponding instructional supports There were two ways in which these teachers believed they could support students' transfer of learning via students' beliefs about mathematics: (a) make use of real-world situations, and (b) ask students to identify real-world situations in which particular mathematical topics or ideas are at play.

Make use of real-world situations. All three teachers spoke about making use of real-world situations as a way to support students in viewing mathematics as useful and relevant outside of the classroom. For instance, Patrick believed that students' transfer of learning is supported when students are asked to engage with situations that spark their curiosity and that students find "interesting" and "relevant." In particular, when explaining how he developed his lesson plan so that it supported students in making future use of their learning, Patrick said that he was not inclined to choose traditional "practice" problems or to use worksheets, but rather to choose problems that would support students in thinking about *why they would want to* learn about slope and how it could be useful in their lives outside of the classroom:

> I wanted it to be interesting. I wanted it to be something they would be curious to figure out. And kind of relevant … My inclination wasn't to say "Well, let's do a worksheet" or "let's do more practice." It was "Alright, so when would you ever come across slope; when would you ever want to even have a thought about something that's steep or there's changing amounts?" So, that's when I thought about cars, working so much, making money, so much per week.

This excerpt highlights Patrick's belief that particular kinds of problems support students in viewing mathematics as relevant, namely problems that students find interesting and that involve specific mathematical ideas. Similarly, when discussing how they supported students' transfer of learning, Richard and Blake stressed the importance of making students aware of how mathematics can be seen and used in their lives (e.g., when reasoning about football scores or the costs involved with various purchases).

Ask students to identify real-world situations in which particular mathematical topics or ideas are at play. Patrick and Richard believed that to support students in seeing mathematics as relevant and useful and, as a consequence, support students in transferring their learning, they should ask students to come up with real-world examples that involve the mathematical topics discussed in class. When explaining how the goal of supporting students' transfer of learning shaped his lesson plan on slope, Patrick said it led him to ask questions like: "Are there any other instances where you hear something that is stated like this rate of pounds of beans per person; what else have you ever heard or seen that is so much of an amount per something?" Patrick reasoned that when students share that they have heard "miles per hour" or seen a commercial advertising a car's mileage per gallon, they often realize, "Wow, there is [sic] a lot of rates that I hear or I know about, but I never think about them as it's *this* much *per* something." Similarly, Richard said that he believed asking students to identify jobs wherein employees make regular use of mathematics and to write reports responding to the question "How do you use mathematics in your life?" helped students view mathematics as relevant and useful in the real world and, as a result, supported transfer. In this way, Patrick and Blake believed that supporting students in conceiving of the connections between classroom mathematics and real-world situations would help them to see mathematics as relevant and useful and consequently support students in making productive use of their classroom learning in the real world.

Discussion of instructional supports for beliefs involving students' affect The idea that one's confidence mediates their transfer of learning is largely absent from the transfer literature. That said, the teaching actions discussed by the teachers in this study resonate with ideas that have been articulated by transfer researchers. For example, Engle (2006) placed emphasis on the situations in which students' learning is embedded and more specifically on the ways in which teachers *frame* those situations. Engle focused on the degree to which various teaching actions bind students' activities to the classroom and suggested that teachers support students' transfer of learning by framing learning situations expansively (i.e., as relevant in and related to situations occurring across spans of time, groups of people, physical locations and topic areas; Engle, 2006; Engle, Nguyen, & Mendelson, 2010). That way, students may be more likely to choose to make use of their classroom learning while, for instance, shopping at the grocery store. The teachers in this study also believed that framing matters; however, their focus was on the framing of the student and, in particular, the student's confidence. They therefore suggested teachers pay attention to the language used to communicate the degree of difficulty students experience while engaging in mathematical activity, suggesting that teachers avoid and even prohibit language attributing that difficulty to specific mathematical tasks and topics.

The notion that one's beliefs about mathematics mediates transfer has points of contact with both Engle's (2006) and Pugh's (2011) approaches to transfer. As noted above, Engle argued that framing learning situations as relevant in and related to other situations supports transfer. Pugh also focused on the transfer of classroom learning to out-of-school situations, examining whether and how students' classroom learning reappears in their everyday experiences. Moreover, Pugh offered teaching actions that are very similar to those suggested by the teachers in this study but did so for a different purpose. While the teachers in this study suggested teaching actions for the purpose of fostering the belief that mathematics is relevant and useful outside of the classroom, Pugh and colleagues suggested teaching actions they believe motivate students to make use of subject matter while outside of the classroom (specifically when said use is not required) to see new aspects of the world and to find value in doing so; like the teachers in this study, they suggested (a) emphasizing the real-world importance of classroom topics and (b) supporting students in reseeing their worlds (Pugh, Linnenbrink-Garcia, Koskey, Stewart, & Manzey, 2010).

16.4 Conclusion

My goal in writing this chapter is to illuminate the ways in which teachers believe students' transfer of learning should be supported. The teachers in this study often exhibited evidence of holding more than one belief about what constitutes transfer and how it should be supported instructionally, suggesting that, in practice, students' transfer may be best supported by multiple approaches. For example, Patrick believed that transfer involves the development of mathematically valid interpretations of slope and therefore believed he should (a) choose tasks and pose questions that support students' quantitative reasoning and (b) use a curriculum that progresses from contextualized to decontextualized situations. He also believed that students' transfer necessitates their belief that mathematics is relevant and useful; he thus believed he should (c) make use of real-world situations and (d) ask students to identify real-world situations in which particular mathematical topics or ideas are at play. It would be interesting to examine whether and how (he believes) these four instructional supports interact to support transfer and what they ultimately support the transfer of. How might various teachers enact these supports and how would the students in those teachers' classrooms engage in novel problem situations? Might future examinations be able to identify more and less productive supports for transfer?

Just like the conceptualization of what it means to be mathematically proficient benefited from conversations with teachers and, in fact, resulted in the addition of a fifth strand of mathematical proficiency (Jeremy Kilpatrick, personal communication, September 5, 2017), so too might conceptualizations of transfer benefit. Whereas transfer research in mathematics education has tended to focus on the nature of students' mathematical knowledge, the mathematics teachers in this study focused on students' affect and problem-solving approaches in addition to their mathematical knowledge. Their beliefs about how to instructionally support students' transfer of learning therefore targeted all three of these aspects of the student. My hope is that the results presented here inspire others to move their transfer investigations into actual classrooms and to examine the phenomenon through the eyes of teachers so that we are all better positioned to support students' productive and successful engagement with novel situations.

Appendix

The 12 Categories of Teachers' Beliefs Regarding Instructional Supports for Students' Transfer of Learning organized by their corresponding beliefs about how transfer occurs

Mathematical Content

Students productively transfer their learning to a novel situation when the novel situation prompts the use of a learned *procedure.* (3)

Instructional Supports:
- Tell students the desired procedure
- Use multiple examples

Students productively transfer their learning to a novel situation when they develop mathematically valid *meanings* for topics like slope. (4)

Instructional Supports:
- Choose tasks and pose questions that support students' quantitative reasoning
- Use a curriculum that progresses from contextualized to decontextualized situations
- Provide students with opportunities to explain their reasoning

(7 teachers)

Disposition

Students productively transfer their learning to a novel situation when they develop and make use of a *visualization and sense-making disposition*. (2)

Instructional Supports:
- Encourage students to infuse a real-world context into a given problem
- Avoid problem statements that tell students how to proceed

Students productively transfer their learning to a novel situation when they develop and make use of a *group-brainstorming disposition*. (1)

Instructional Support:
- Model the disposition

(3 teachers)

Affect

Students productively transfer their learning to a novel situation when they develop *confidence in their abilities to engage in mathematical activity*. (6)

Instructional Supports:
- Monitor the language used in the classroom
- Support students' independence

Students productively transfer their learning to a novel situation when they develop the *belief that mathematics is relevant and useful outside of the mathematics classroom*. (3)

Instructional Supports:
- Make use of real-world situations
- Ask students to identify real-world situations in which particular mathematical topics or ideas are at play

(7 teachers)

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Chapter 17
Transfer of Mathematical Knowledge for Teaching as Elicited Through Scripted Role-Play

Ami Mamolo

Teacher educators are no strangers to the dictum "this is the way I was taught." We hear it as rationale for solving strategies, as motivation for lesson planning, as push-back against alternative pedagogies, and as stinging critique from those who grew to understand that school math was "not for them." Explicitly and implicitly, prospective teachers grapple with inconsistencies between their generalizations of school learning from their own experiences as students and the ideas and experiences to which they are exposed during their teacher education program. Petrarca (2016) has conceptualized the experiences of teacher education as *the (Un)Making of the teacher* in acknowledgment of the well-documented concerns with the resilience of misconceptions of teaching that prospective teachers often hold (Darling-Hammond & Baratz-Snowden, 2007). In *(un)making* a teacher, experiences of disturbance can help bring to the surface some of these conceptions and offer an opportunity to confront and learn from them. In considering prospective teachers' responses to such disturbances through a lens of actor-oriented transfer (AOT), researchers can gain insight into the habits and expectations that have been generalized from K-12 school experiences and that can tacitly impact a prospective teacher's approach with students, sometimes in direct conflict with their professional development education and expressed intentions. To that end, this chapter explores an extension of AOT to the case of mathematical knowledge for teaching, with a particular focus on how experiences of contingency can help shed light on preconceived notions of mathematics and mathematics learning that may conflict with prospective teachers' intended pedagogical approaches.

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17.1 Mathematical Knowledge for Teaching

The construct of *mathematical knowledge for teaching* (MKT) has been widely discussed in education research, with attention focused on what knowledge is required in teaching, for teaching, and of teachers (e.g., Adler & Ball, 2009; Davis & Simmt, 2006; Hill, Ball, & Schilling, 2008). Its roots go back to Shulman's (1986) distinction between subject matter knowledge (SMK) and pedagogical content knowledge (PCK), which were further refined by Ball, Thames, and Phelps (2008). This refinement, and the subsequent research it stimulated, have highlighted the common, specialized, structural, and connected ways of understanding mathematics, as well as the practices, values, and sensibilities necessary to the discipline and its teaching (e.g., Ball et al., 2008; Ball & Bass, 2009; Zazkis & Mamolo, 2011). Although the focus was on distinguishing facets of teacher knowledge, Ball et al. (2008) acknowledged their interconnected relationships. They suggested that "teachers who do not themselves know a subject well are not likely to have the [pedagogical content] knowledge they need to help students learn this content" (Ball et al., 2008, p. 404). In other words, how mathematics is understood and experienced by teachers can influence their understanding of, and expectations for, students and student learning (Mamolo & Pali, 2014).

MKT has been linked to teachers' abilities to choose appropriate examples and problems (e.g., Rowland, Thwaites, & Huckstep, 2003); to plan lessons (e.g., Wasserman & Stockton, 2013); to recognize mathematical significance in different contexts and student work (e.g., Zazkis & Mamolo, 2011, 2018); and to articulate in the moment what needs to be done, why, and how (Mason, 1998). Mason (1998) framed this knowledge in terms of awarenesses of, or ways of "being" with (Mason & Davis, 2013), mathematics that enables the understanding and articulation of relevant pedagogical practices and decisions. Such awarenesses can develop through shifts of attention that help a learner progress from a student (who can articulate what to do), to a master (who can articulate what to do and why), and then to a teacher (who can articulate what to do, why to do it, and how). Each level of awareness encompasses and extends previous levels, highlighting how the mathematical knowledge required in and for teaching differs qualitatively from mathematical knowledge required of students. A point of particular interest in a teacher's awareness of mathematics is in the articulation of mathematical practices and sensibilities that convey "how" a learner might approach the subject, in other words, in their ways of being mathematical that can engender mathematical ways of being in others.

Rowland and Zazkis (2013) suggested that "one's stance regarding the mathematical knowledge needed (or essential) for teaching depends on one's perception of teaching itself," and that teaching "involves attending to students' questions, anticipating some difficulties and dealing with unexpected ones, taking advantage of opportunities, making connections, and extending students' horizons beyond the immediate tasks" (p. 138). Teachers are required to respond in the moment to situations for which they may not have prior experience or awareness; they must "figure out what is right practice in the situation" and "be prepared for the unpredictable"

(Lampert & Ball, 1999, p. 39). This aligns with Rowland and Zazkis's notion of teaching as response to contingencies, where such contingencies can offer moments of *disturbance* (Mason, 2002) that invite individuals to revisit, rethink, and refine previous understanding, and thus broaden their disciplinary awareness. In this perspective, teaching moments are also learning moments, and research supports the idea that mathematics teachers learn through their teaching experiences (e.g., Leikin & Rota, 2006). Simulating such experiences in teacher preparation programs through the use of performed or scripted role-playing can also offer valuable learning opportunities to advance disciplinary knowledge and help prospective teachers prepare for future interactions (e.g., Lajoie & Maheux, 2013; Lawson, McDonough, & Bodle, 2010; Mamolo, 2017; Zazkis & Zazkis, 2014).

17.2 Scripts and AOT

As an instructional tool, scripted role-playing (or scripting) can be useful in co-developing mathematical understanding and pedagogical awareness (Mamolo, 2017), and in shedding light on personal understandings, biases, and perceptions of student difficulties (e.g., Koichu & Zazkis, 2013; Zazkis & Zazkis, 2014). Role-playing offers a venue through which teachers can enhance their abilities to articulate mathematical ideas and related pedagogical practices; such abilities in articulation are considered crucial for the practice of teaching (Mason, 1998). As a methodological tool, scripting has shed light on prevalent conceptions, common errors, and mathematical reactions to unexpected questions. Extending this work, I use scripts as a lens through which to gain insight into the disciplinary knowledge, practices, and sensibilities evoked by the setting of the scripting task, and to tease out inconsistencies in participants' approaches to, and intentions for, teaching. Specifically, my interest is in the knowledge and practices drawn upon when prospective teachers are presented with an unanticipated mathematical idea or question to which they must respond, as influenced by prior learning in, and before, their teacher education program. As such, the AOT perspective, wherein transfer is defined as "the generalization of learning" and "the influence of a learner's prior activities on her activity in novel situations" (Lobato, 2012, p. 233), is well suited for this study.

The typical approach to AOT includes using the same set of transfer tasks both before and after the design experiment (e.g., Lobato, 2008). However, "the primary distinguishing feature of the actor-oriented approach is the effort to relinquish normative notions of what counts as transfer and immerse oneself in the learner's world instead" (Lobato, 2008, p. 174). With this in mind, when engaging in script-writing, the original "task" can be considered as the experience of learning mathematics as a K-12 student, the design experiment as the experiences of learning to teach mathematics in a teacher education program, and the transfer task as the script (see Fig. 17.1 for the general scripting prompt). In this conceptualization, I look at the influence of participants' prior experiences interacting with teachers as learners on

> (A) You are given the beginning of an interaction between a teacher and a student and your task is to extend this imaginary interaction in the form of a dialogue between a teacher and a student (or several students). You may also wish to explain the setting, that is, the circumstances in which the particular interaction takes place.
> (B) You are also asked to explain your choice of action, that is, why did you choose a particular example/approach, what student difficulties do you foresee, why do you find a particular explanation appropriate, etc.

Fig. 17.1 The scripting prompts

their novel activity of interacting with learners as teachers. One of the benefits of scripting is that it can be done individually, and as such, each prospective teacher must imagine playing both the teacher and student characters. Participants must therefore draw on prior experiences of school learning in general, and of learning (school) mathematics in particular. Evoked through the scripting process are aspects of mathematics teaching that include the mathematics content or practices that participants see as relevant or connected to the topic in question, the pedagogical approaches and attitudes towards mathematics that are seen as relevant to the setting and that are projected onto the teacher-characters, and the attitudes and abilities expected of the pupil-characters in the context of learning mathematics. The goal is not to pinpoint specifics of where from a generalization might have developed, but rather to bring to the surface generalizations of learning that are inconsistent with participants' expressed intentions for learning, as informed by the experiences and expectations of a teacher education program. Thus, the possibilities for transfer relate to various facets of *mathematical being*, extending the scope of applicability of the AOT lens.

Lobato (2012) observed that "transfer is supported through the incremental growth and organization of smaller elements of knowledge, which are *highly sensitive to context* and are only gradually refined to extend to a widening circle of situations" (p. 243, emphasis in original). The pedagogical and content knowledge elicited in the context of lesson planning during, say, a methods course might be "too contextualized" to readily transfer to moments of teaching in the classroom. In setting the scene for their scripts, and in voicing student and teacher characters, participants reveal evidence of generalizations of their learning experiences as students and student-teachers. For instance, in a well-planned lesson plan, a high-achieving prospective teacher advocated strongly in favor of student-centered approaches and designed a lesson plan that expertly aligned with his intentions (Mamolo, 2017). When that same student-teacher responded to a scripting task that centered on an unexpected question, his script seemed to contradict his intentions. It relied heavily on a teacher's explanation, with few opportunities to listen to, or for, student ideas. A breakdown of his script highlighted this. The teacher-character uttered over 450 words compared to the student-characters' combined utterance of 77 words. Of the teacher-characters' 27 lines of dialogue, only five included questions and each of those questions invited the student-characters to repeat an utterance already made by the teacher-character in the script. And, one of the two total times a student-character raised a question, it went unanswered by the teacher-character. Incidentally, the script ends with the student-characters praising the teacher-character as “the best Math Teacher ever” (Mamolo, 2017, p. 240). Although there may be many ways to account for this incongruency, viewing it through the lens of AOT provides an interesting interpretation that can help teacher educators unearth and address aspects of becoming a teacher that might need to be “unmade.”

17.3 Methods

17.3.1 Setting and Participant

This case study reports on the data collected from a prospective middle school teacher, Kumi, who was enrolled in a mandatory mathematics content course that took place in the second semester of a four-semester teacher education program. The course emphasized diverse ways of reasoning with and about mathematics, and included a variety of task-based activities aimed at inviting prospective teachers to reflect on, critique, and reconstruct their perspectives on mathematics and mathematics learning. An important focus of the course was on effective mathematical communication, including verbal, pictorial, and technology-enhanced modes of communicating with students. As part of the course, students completed weekly reflections on their experiences, growth, concerns, and struggles with mathematics, both in general and with particular reference to problems and themes from Mason, Burton, and Stacey’s (1982) *Thinking Mathematically*. Recurring themes addressed in the written reflections included the “healthy state” of “being STUCK!” (p. 56), posing questions to advance mathematical thinking and problem solving, the “hidden assumptions” (p. 101) people might have about mathematics and mathematical thinking, and the importance of “mulling” over problems (p. 97). These themes were emphasized during class discussions and explorations in addition to homework activities and course assignments. Typically, in-class problems were solved using a combination of visual and geometric imagining and conjecturing, physical 3-D constructions and modeling, interactive and dynamic software explorations (such as with *Geometer’s Sketchpad* © and *Gapminder* ©), and computational and algebraic resolutions.

Kumi was chosen for this study because she was a keen student who was open to critically reflecting on her prior experiences, current thinking, and future ambitions for her students. She was forthcoming, bright, and thoughtful in her contributions in class, but struggled with secondary level mathematics and had failed her Grade 11 mathematics class when she was a secondary student. Kumi was a generally motivated student, who was about 10 years older than the average student in her cohort. Her aim was to become an English and history teacher; she held a graduate degree in the humanities and taught college-level courses. In Kumi’s jurisdiction, middle school teachers are certified to teach all subject matter, including mathematics, regardless of their teachable focus. Thus, although Kumi hoped to teach English and history, she nevertheless was required to take a mathematics course, as were all prospective teachers in her program. In the first written reflection for the course, students were asked to introduce themselves and share a bit about their past experiences. Kumi described herself as "pretty educated," commenting, "it's not often that I'm set back on my heels when it comes to school." She considered herself "extremely comfortable with math through Grade 8" and "pretty good with numbers" but claimed that "the joy of doing anything beyond what I would call simple math is gone." The highest level of math that she had taken was Grade 11, which she failed, and she felt that she "should've failed Grade 10 as well," if not for "a friendly teacher."

17.3.2 Data Collection and Analysis

Data for this research included participants' responses to a scripting prompt depicted in Fig. 17.1, as well as field notes from in-class discussions and written reflections. The focus of this chapter is on Kumi's response to, and reflections on, a scripting prompt concerning an unfamiliar equation for solving the area of an equilateral triangle (see Fig. 17.2). Students were given several weeks to work on their scripts; there were a few script options to choose from and, for the purposes of this chapter, I focus on the option depicted in Fig. 17.2. Students were encouraged to use multiple representations in their responses, including digital ones, and could use any resource they found applicable for completing the script. The aim was to give prospective teachers a task which could shed light on what they transferred from their teacher education programs to novel teaching situations. The scripting task was a new kind of activity for the prospective teachers, and as such, served as a transfer task for which the AOT perspective is well suited. Of interest in analyzing Kumi's script response were prior influences (i.e., from K-12 schooling, teacher education)

You're working with students to find the area of an equilateral triangle with side lengths 6.
To do so, students used the following learned procedure:
Find the height, determine the base length, and substitute into the formula $A = \frac{1}{2}bh$.

T:	What have you done there?
S1:	I multiplied $\frac{\sqrt{3}}{4}(6)^2$ and got $9\sqrt{3}$ when I simplified everything.
S2:	Does that always work?
S1:	Yes, my teacher taught it to me last year.
S2:	I used the formula and got a different answer… 15.5884572681.
T:	…

Fig. 17.2 Scripting task: Area of equilateral triangle

that might have shaped how she approached and interacted with her hypothetical students. Rather than assessing the quality of Kumi's response to the scripting activity, all of the data (script, written reflections, field notes) were analyzed with an eye toward where the influences might have come from.

17.4 Results and Analysis

17.4.1 Kumi's Intentions for Her Future Students

Kumi's reflections on her approaches to teaching and her experiences learning mathematics revealed some of her intentions for her future students and the learning environment she hoped to cultivate. For example, she was asked to opine on the quote: "Being STUCK! is a healthy state, because you can learn from it" (Mason et al., 1982, p. 56). She wrote:

> I suppose the simple answer is that getting stuck forces you to really work through a problem. Getting it on the first try doesn't actually result in any learning; it simply shows you what you know. In order for real learning to occur, you need to get yourself to a place that is somewhat 'uncomfortable'. This is important because it develops a sense of grit, one that teaches them [students] two things: not to give up and to search for other ways to solve a problem… and letting students know that you too can get stuck on things makes you much more relatable.

Kumi's characterization of "real learning" as requiring some discomfort needing to be worked through for the benefit of developing grit and solving strategies aligns with ideas introduced during her teacher education program, such as the role of disequilibration in learning. Her comments suggest her intentions to provide "real learning" opportunities for students in a relatable way were influenced by her teacher education program, and her response further suggests that she was not averse to students seeing her as someone who can also be stuck on problems. Kumi also offered her ideas about the value of mulling over and posing questions:

> Mulling a problem over can go a long way to getting UNSTUCK. Mulling is something I did frequently in university, particularly when writing my thesis. I think that this strategy is greatly undervalued and underutilized in all aspects of life, not just math. Sitting back to reflect on something can be incredibly useful and can produce some inspiring results.

In this excerpt, Kumi's intentions for her future students' learning seem to include generalizations from her own deep learning as an undergraduate student. From her written responses and from field notes collected from her contributions to class discussions, it was clear that Kumi valued deep thinking, meaningful learning, and the role good problems and questions had to play in fostering these things. For example, she wrote:

> Personally, I think that questions have incredible value, and in the context of a classroom they likely trump the value of answers. By questioning things we don't know – or even by being posed questions for which we do not have answers – we are forced to think and rationalize problems, which is far more conducive to learning than simply receiving answers. A classroom that values questions over answers is one that truly values learning.

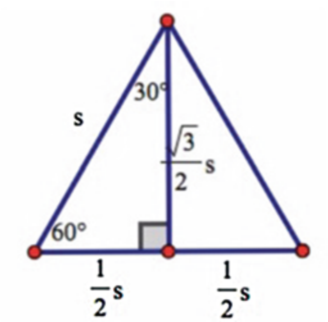
17.4.2 Inconsistencies Between Intentions and Interactions

The scripting task Kumi chose includes an exchange between two students, where Student 1 (S1) applies the equation $A = \frac{\sqrt{3}}{4}s^2$ to determine the area of an equilateral triangle with side length of 6 units. It was an approach with which Kumi was not familiar, and as such, it served as an experience of contingency to which she had to respond. This special case can be derived from the general formula by some straightforward algebra, as illustrated in Fig. 17.3.

In her script, Kumi sidestepped the unfamiliar approach right away, writing:

T: Alright, settle down. We'll work this out together. S1, do you understand the formula that you used?
S1: No, not really,
T: OK, that's a good place to start. Knowing formulas is a great way to help us solve equations, but if we don't understand their meaning it can be difficult to understand how and why they are meant to work in the first place. Does that make sense?
S1: Ya, I guess so.
T: Now, I'm not saying this is wrong, but I think there's a bit of confusion here, you said you found a solution using the [prescribed] formula; what was it?

In this beginning segment from her script, Kumi expertly pivots away from the unfamiliar toward more comfortable footing. She offers helpful advice about using formulae before turning her attention exclusively toward the numeric response given by S2 and engaging students in a step-by-step determination of the triangle's area using $A = \frac{1}{2}bh$. Notably, there is nothing in the script prompt that suggests either of the students has difficulty computing the area of a triangle with the prescribed formula. However, Kumi's approach was typical of how prospective teachers addressed a script prompt that took them outside of their comfort zones: ignore or avoid the original question, impose an expected misunderstanding on students, and address that misunderstanding (usually in a step-by-step explanation where the



$A = \frac{1}{2}bh$ where $b = s$.

Then, $h^2 = s^2 - \left(\frac{1}{2}s\right)^2 = \frac{3}{4}s^2$, and

$h = \frac{\sqrt{3}}{2}s$. Thus, $A = \frac{1}{2}s \cdot \frac{\sqrt{3}}{2}s = \frac{\sqrt{3}}{4}s^2$

Fig. 17.3 Area of equilateral triangle teacher does much of the heavy lifting). Notably, this approach stood in contrast to her intentions to foster "real learning" and her recognition of the value of mulling over a problem we do not understand. Rather, she seemed to generalize strategies experienced prior to her teacher education program during which friendly step-by-step explanations were seen as helpful, and even necessary, for passing her courses.

There are further inconsistencies between Kumi's script dialogue and her proclamations about teaching and learning. For example, although she believed that questioning has "incredible value … which is far more conducive to learning than simply receiving answers," none of the students in her script posed a single question. Indeed, all of the questions posed came from the teacher-character, and each question was rather bite-sized, eliciting immediate and typically correct responses, as exemplified in the following exchange:

T: Now, what is the Pythagorean Theorem?
S2: It's $a^2 + b^2 = c^2$
T: Good, now what about with the measurements we know?
S2: That would be $3^2 + b^2 = 6^2$
T: And simplified?
S2: It would be $9 + b^2 = 36$
T: Excellent! Now how would we work the equation so we can solve for b^2?
S4: We would have to isolate b^2
T: Spot on!

Across Kumi's script, student suggestions that aligned with her intended approach were praised as "excellent," "perfect," or "spot on." This is in contrast to her responses to S1, the student who was deemed in need of a better understanding. Indeed, aside from telling S1 they are "not wrong" but possibly "confused," very little subsequent attention is paid to S1 or his ideas. For example, a student other than S1 answered every question Kumi posed seeking explanations or validations. The only subsequent time S1 contributed to the dialogue at all was to offer "measure it" as an approach to determining the height of the triangle. This was acknowledged by the teacher but was quickly dismissed: "We certainly could… But what if we didn't have a ruler? Is there a way to figure it out mathematically?" Kumi's response to S1's suggestion highlights the influence her K-12 student experience had on her developing mathematical knowledge for teaching, generalizing from those experiences as to what counts as "mathematical." Even in this context of shape, measurement, and area, "mathematically" is understood to mean "algebraically." This is in direct conflict with key objectives of the course, which emphasized the value of physical construction, spatial visual reasoning, and modeling as ways of supporting, explicating, and extending algebraic expressions.

Further, if "simply receiving answers" is less conducive to learning than "questioning the things we don't know," then there is another inconsistency in how S1 was addressed. Kumi's expectation was that S1 lacked understanding of how to compute the area of a triangle, and yet S1 was constantly put in a position of "receiving answers" and was never invited to question anything or mull anything over. Indeed, despite Kumi's proclamation that "for real learning to occur, you need to get yourself to a place that is somewhat ‘uncomfortable,’” she orchestrated the exchange in such a way that there were no opportunities for “real learning to occur.” At the end of her script, Kumi readdresses S1. Her words are telling. She has her students determine and round off the decimal representation of S1’s radical answer and then ends her script with the following:

> Now, we can see that S1’s approach worked here, and they ended up with the same answer when everything was simplified all the way. It is, however, a more complicated formula, and we aren’t certain that it will work for every triangle that we encounter. So, what I’d like you all to do is look at the triangles that I’ve posted on the board and try using the formula $A = \frac{1}{2}bh$ to find their areas. Then we’ll reconvene and see what we came up with.

It is difficult to reconcile this statement with any of Kumi’s proclamations about teaching and learning that would have been influenced by her teacher education program. The baseless characterization of S1’s approach as “more complicated” is surprising, especially considering the care Kumi took earlier in the script to avoid opining on it or on the knowledge of S1’s “previous teacher” (e.g., “I’m not saying this is wrong, but …”). There are many ways Kumi could have ended her script that acknowledged S1’s approach, still avoided actually dealing with it, and nevertheless offered advice that was consistent with her professed values for teaching and learning. That aside, what is really notable is the assumption that an unknown formula must necessarily be “more complicated” (in this case, it is not). This can be interpreted as a generalization made from Kumi’s experiences as a pupil for whom “the joy of doing anything beyond simple math is gone.” Kumi struggled with secondary school mathematics and attributed any successes to the kindness of her teachers. This influenced her approaches to dealing with a struggling student and overshadowed her conscious efforts to value student contributions and promote questioning and moments of struggle for meaningful learning.

17.4.3 Seeking Familiarity in the Unfamiliar

Kumi chose to engage with a script on equilateral triangles because she had “taught triangles a bit during [her] first practicum and frankly, most of the others [script options] confused [her] entirely.” Kumi described her experiences with the scripting tasks as “a bit odd,” stating: “I typically hate ‘not knowing’, and I’ve found myself in that position an awful lot in completing this script.” She wrote:

> This was difficult in the respect that I had no clue what was happening with S1’s solution, and try as I might, I couldn’t figure it out… not having any idea how he got there limited my options when it came to how I approached the problem of ‘teaching’ this…I felt it necessary to tip-toe around it.

She noted that “some might say ‘fake it ‘til you make it’, but I would suggest that’s asinine… students deserve far better.” The novelty of the scripting task and its content seemed to have placed Kumi in a position that aligned more closely with her typical experiences in secondary school mathematics than her typical experiences as prospective teacher. Kumi's struggles with secondary mathematics had left her "not knowing" on many occasions, and formula manipulations and balancing equations "really ended things for [her] in math." Given that transfer of knowledge is considered "highly sensitive to context" (Lobato, 2012, p. 243), it is perhaps not surprising that her experiences as a pupil had an influence on her approaches as a teacher in a context for which she felt unsure of the mathematics.

The fact that Kumi openly admitted not understanding S1's approach to determining the area of an equilateral triangle helped rationalize her avoidance of this approach. "Right or wrong," she wrote, "if I was confronted with this in a class of my own, I would default to what I know… keeping things moving and addressing something I'm unsure of later." Yet this is in conflict with her desire to connect with her students; whereas "letting students know that you too can get stuck … makes you much more relatable," the teacher-character was consistently and solely positioned as an authority and assessor, a generalization influenced by her typical experiences as a student in K-12 mathematics. Nearly every utterance made by the teacher-character included some value statement ("Excellent!" "Fantastic!" "Perfect!"), and the conversation was kept firmly within the boundaries of Kumi's comfort. Her desire for "keeping things moving" and her approach of asking bite-sized, easily answered questions, are typical of instructional approaches that have been adopted to facilitate learning for struggling students, to which Kumi would have been well exposed in her K-12 education and which served to influence her script response. Such approaches have been shown to compound student difficulties (Watson, 2006), an idea Kumi was exposed to in her teacher education program. They also conflict with Kumi's recognition of the importance of taking the time to mull things over, a strategy that she considered "greatly undervalued and underutilized in all aspects of life, not just math."

One of the expectations about becoming a teacher that Kumi may have generalized is the sense that teacher education programs ought to focus on introducing pedagogical approaches for known curricular content. This is evidenced in the fact that the only document that Kumi drew upon in her original script was the curriculum document for her jurisdiction and via her admission that her "experience with this assignment – and this course as a whole – has admittedly, been a bit odd." The course (and assignment) required students to engage with familiar content in new-for-them ways, so as to broaden their horizons and expectations for student learning, abilities, and achievement. The approach in this course aligns with Rowland and Zazkis's (2013) perspective that, in teaching, "mathematical knowledge beyond the immediate curricular prescription is beneficial and demonstrably essential" (p. 138).

The curriculum document for Kumi's jurisdiction makes no explicit mention of the formula $A = \frac{\sqrt{3}}{4}s^2$ and it was interesting that, despite admitting that she had "no idea what it meant," she did not seek any external sources to help make headway. She stated:

> I tried to work it out several times and kept getting 'stuck', which was quite frustrating. I ended up taking an alternative route – working strictly with the Pythagorean Theorem –

> largely because I feel that's where I would have gone if I was confronted with this in a class of my own: I would default to what I know and then figure out the problem when I wasn't on the spot.

Kumi was not technically "on the spot" for this script—she had several weeks to work on it and could have called on any resource for assistance. A Google search of "area of equilateral triangle" yields the special case formula immediately, and the first image in the image search includes a derivation of the special case from the general case. A search of "different ways to find the area of a triangle" yields similar results. Kumi repeatedly commented that addressing S1's approach in a "follow up lesson" would give her the "time to become confident with the theory behind it," and it is interesting to note that she held this expectation despite "being completely confused" during script writing for which she had several weeks to work.

17.5 Discussion

Typically, the lens of AOT has been applied to study students' mathematical knowledge and its development. This chapter illustrates how AOT can also be helpful in interpreting the development and application of prospective teachers' MKT, highlighting a new context in which to consider the transfer of learning. The specific focus on competing influences that shaped a prospective teacher's responses to an experience of contingency makes a new contribution to work on AOT, which typically has not been examined from the perspective of competing influences. This work is significant for the field of teacher education because it furthers understanding of salient influences that can overshadow approaches introduced in professional development programs, and it provides a new methodological approach for shedding light on these experiences and their influences.

Knowledge of mathematics-for-teaching includes an interplay of content and pedagogical awarenesses, skills, and routines accrued from a variety of influences, including teacher education programs and K-12 schooling. The issue of the role of school knowledge about mathematics subject matter, teaching, and learning for pre-service teacher development has been central in mathematics education research. Teachers' first instructional experiences with mathematics teaching occur during their years as pupils of mathematics and influence the generalizations of teaching and learning that are elicited in their novel experiences as teachers. Early generalizations of mathematics teaching and learning can remain tacit even when these generalizations conflict with expressed intentions and beliefs. Experiences in teacher education programs offer contexts that explicitly draw to the surface pedagogical approaches that prospective teachers are expected to grapple with and eventually adopt. Prospective teachers who are quite skilled at applying these approaches in prescribed scenarios, such as in lesson planning, can nevertheless experiences moments of disturbance when an unexpected situation "sets them back on their heels." In these experiences of contingency, tacit generalizations of mathematics-for-teaching (or mathematics-in-teaching) can surface and influence pedagogical approaches.

A scripted role-play breaks away from typical contexts for learning MKT, and can highlight how prospective teachers envision their future exchanges with pupils, providing a window into what may be viewed as relations of similarity across situations. That is, when a prospective teacher imagines standing in front of a room of students and engaging with them in conversation, what is evoked are generalizations of learning from both experiences as a prospective teacher and experiences as a K-12 student. In Kumi's case, when she was in control of the conversation—for instance, via considered responses to class discussions or reflection prompts—she could act purposefully, generalizing her ideas about positive support for students and the role of questioning and mulling in promoting learning. When Kumi was outside of her comfort zone, experiencing the uncertainty involved in negotiating a moment of contingency, she seemed to default to ideas and expectations that conflicted with her expressed intentions and with the educational experiences to which she was exposed in her teacher education program. Instead, she seemed to generalize situations and attitudes she had experienced as a pupil, drawing from memory how "friendly teachers" supported her through secondary school.

Lobato (2008) highlighted the research question "How does the environment structure the production of similarity?" (p. 173). In a real classroom environment, the experience of contingency is unavoidable; for a responsive educator, it might even be common. Grappling with moments of real contingency in teacher education programs—in situations without, say, the safety net of a prescribed lesson plan or an associate teacher steering the way forward—may be valuable learning opportunities to draw to the forefront generalizations of learning that stemmed from (at times) detrimental experiences as students. Drawing such generalizations to the forefront can in turn help teacher educators rethink how the learning environment in teacher education programs can be structured so as to lend itself to being more readily transferable to real classroom situations. Engaging prospective teachers in enacted or scripted role-play offers a venue through which to elicit contingency, both in terms of new-for-them mathematical ideas, as well as with respect to new-for-them pedagogical approaches for promoting mathematics learning in K-12 schooling. This is important because, "if preconceptions about teaching are not addressed, prospective teachers can unconsciously cling to ineffective practices and fail to learn more-beneficial approaches" (Darling-Hammond & Baratz-Snowden, 2007, p. 117).

This chapter makes a new contribution to advancing the field of research in mathematics teacher education by introducing AOT to the study of competing influences that can inform the development and enactment of teachers' mathematical knowledge for teaching. It extends research into the uses of scripted role-playing for investigating and fostering MKT and offers a new way to examine AOT. Looking forward, there is great potential for this approach to help inform teacher educators' considerations when structuring learning experiences for their students. Prospective teachers are exposed to new (for them) ways of being with mathematics via their experiences in professional development programs that adopt technologically enhanced pedagogies, culturally responsive or diverse pedagogies, context-based or inquiry-based pedagogies, and so forth. For researchers interested in how these competing influences might inform prospective teachers' decisions in future classroom interactions, AOT may provide a useful lens through which to interpret whether the abstractions and generalizations being made during professional development programs are recognized as applicable to K-12 classroom situations and, if so, when and when not. Further, this research sets the stage for extensions of AOT to the context of in-service teacher education. In this conceptualization, the original "task" relates to the experiences had when learning mathematics via interacting with teachers as learners, the design experiment as the experiences of learning mathematics in a teacher education program, and the transfer task as the novel experiences of learning mathematics via interacting with learners as teachers.

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Chapter 18
Reflections on the Idea of "Context" in the Transfer of Research Knowledge

Jeff Evans

Recent discussions of "transfer" in education have distinguished (explicitly or implicitly) among different types of transfer. *Learning transfer* is of course a major concern of educators and the focus of most of the contributions to this book. Sometimes, the original context and the "destination" context are mentioned—for example, for *learning transfer* "from school to work."

Other types of transfer are generally considered to be more specialized. *Knowledge transfer* normally refers to the "application" of research knowledge and has come to the fore in discussions of the missions of higher education institutions, certainly in the United Kingdom. *Technology transfer* (Bozeman, 2000; Gilbert, 2006) and *policy transfer* (e.g., Ozga, 2012) refer to the reapplication of techniques or policies produced or formulated elsewhere. In this chapter, I focus on a central aspect of transfer—namely, the transfer of knowledge and research findings from the original context of research to their application in a different context.

In my experience as a reviewer of articles, in mathematics education and in educational studies in general, researchers often pay insufficient attention to the ways they review and consider the application of the work of others and to the ways they locate the findings of their own work. There is an understandable tendency for many researchers to view their own research context as in some sense "natural" and, thus, to report their work as if the scope of applicability of their findings does not need to be assessed. This tendency is heightened in mathematics education research, since some researchers and teachers seem to view knowledge in that field as neutral, "context-free," and timeless. Thus, a typical methods section in a research report might describe the setting of the study as follows: "[*n*] children were recruited from two neighborhood schools, in [town *t*], both non-fee-paying. The sample included [*m*] boys and [*n* - *m*] girls, with mean age [*k* years, l months]." (This is based on the context described in a recent article in a high-quality journal; it is not acknowledged

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here because the intention is to highlight a widely shared problem or dilemma and not to criticize particular authors.) The challenge of providing an appropriate description of the setting is certainly greater for comparative studies, especially those with an international dimension.

I will discuss what I argue are key aspects of the *context of research* and present some ideas on the way the context of research might be more informatively acknowledged and described in research reports. I also aim to draw out some implications for ways of strengthening the applicability or "transferability" of research findings from one context to another. Note that this appears to fly in the face of advice to educators for most of the last century, which has been that they should seek the most general formulation of their findings to assure transferability. In the literature relating to *learning transfer*, this has been strongly argued against by researchers like Lave (1988), Walkerdine (1988), and contributors to Tuomi-Gröhn and Engeström (2003). At the same time, this view certainly leads to much frustration for teachers when students appear to "fail to transfer" their learning.

18.1 Describing the Context of Educational Research

My theoretical position is broadly sociocultural, with an emphasis on discursive approaches and a concern with policy issues and the broader social setting (e.g., Evans, 2000; Evans, Tsatsaroni, & Czarnecka, 2014). Recent studies have begun to offer a social, economic, and political examination of the context of the production of mathematics education research, especially as it has developed over the last 25 years or so. This has meant a broadening of the study of the determinants (and correlates) of performance, affect, and other aspects of learning, away from a focus on the individual and their immediate constellation of face-to-face relationships (including, for students, the classroom interactions)—which I call here the "micro" aspects of the learning context—to a concern with more "macro" aspects.

Some of the discussion focuses on the "international" context of mathematics education research (e.g., Atweh et al., 2008). For example, Cao, Forgasz, and Bishop (2008), in reviewing the vast range of international studies in mathematics education, pointed to the (neglected) influence of cultural factors on research activities:

> Within the literature reviewed, little has been found of studies with a focus on designing and administering a survey, and how cultural factors can influence researchers' activities and research results, even though international studies evidently take place in different cultural settings. (p. 304)

Brown and Clarke (2013) pointed to national-level concerns when they argued that "schooling is increasingly shaped and judged by its perceived capacity to deliver success in terms of international success linked to economic agenda" and "research increasingly finds its terms of reference set according to measuring delivery in these terms" (p. 459); this results in "national means [averages] of performance [being] given priority over the inequalities they conceal" (p. 460). Tsatsaroni and Evans

(2014) reinforced this and examined further the dominance of international performance surveys like the Programme for International Student Assessment (PISA) and the Programme for the International Assessment of Adult Competencies (PIAAC; Organisation for Economic Co-operation and Development [OECD], 2016).

This context of production of any research is also located in time as well as space; thus, we need to take account of historical changes in these broad contexts, too—in terms of agenda setting, purposes, value orientations, practices, and so on. For example, Clements, Keitel, Bishop, Kilpatrick, and Leung (2013) argued that there has been "a historical progression towards the achievement of mathematics for all: from schooling for all, to arithmetic for all, to basic mathematics for all; to secondary mathematics for all; to mathematical modeling for all; and to quantitative literacy for all" (p. 7).

It is worthwhile to separate the macro aspects of the context from the "meso" aspects. The macro level includes societal, cultural, national, international and "political economy" aspects, close to what Jean Lave (1988) earlier called the "constitutive order" (pp. 177–180). The meso level can be considered to include characteristics of schools, education authorities, and communities.

Concerning the meso level, educational research communities have already developed good ideas about what might be important aspects of the setting concerning teaching, learning, and other outcomes in schools and other institutions. Thus, they can offer some broad suggestions as to the range of meso characteristics that can affect educational outcomes—and that therefore should be considered in reviewing or reporting research findings. This is important because it can help other researchers to understand a reported set of findings and perhaps to apply them to their own concerns, locales, and time periods.

18.2 The Description of Contexts of Research in Practice

Nevertheless, when we look at many of the research reports submitted to, and indeed published in, good-quality journals in mathematics education (and elsewhere), we still often see a number of ways that the context of research is insufficiently reported. First, findings from previous research are often reported as if they were *timeless*. For example, research on secondary schools in the late 1980s or early 1990s is reported and reviewed as if it still holds without question. This tacit claim is made despite the fact that schooling in most countries has clearly changed greatly in a number of ways, including through an increasing hold of neoliberal government policies and the development and intensification of international testing (Lingard, Martino, & Rezai-Rashti, 2013; Rizvi & Lingard, 2010).

Second, findings from previous research are often reported as if they were *independent of the macro context* in which they were produced. Some authors may believe that much of the research reported in the "top international journals" is in fact produced in a narrow range of countries which do not vary much in terms of the national context they provide for education and for educational research. But this assumed homogeneity can be shown to be illusory; Ernest (2008) and Atweh and Keitel (2008) discussed the dominance of "Anglo-European" views of mathematics and mathematics education.

We also need to consider what might be the meso differences in contexts—in institutions, teams of teachers, and pupil catchments—in different times and places. One way is to draw on research areas where these aspects of the context have been extensively researched—for example, in studies on school effectiveness. Again, many of these studies have been done in relatively few countries which appear to resemble each other a great deal. Nonetheless, even within one of these countries, we would need to consider in advance precisely which set of characteristics of schools, teacher teams, and student catchments might be relevant aspects of the research context, given the focus of the study. Thus, for example, in a study of the effectiveness of methods of teaching geometry in secondary school, in country *x*, it might be relevant to take account of differences in students' backgrounds in terms of whether they had attended urban or rural schools. In contrast, in a study of the development and evaluation of a new method of teaching statistical modeling at a university, it might not be considered (theoretically) necessary to take account of those particular differences in students' backgrounds.

This discussion also shows how, despite the analytical value of separating macro, meso, and micro aspects of the context, the three levels interpenetrate in actual contexts.

18.3 Towards a Method for Describing Relevant Aspects of the Context of Research Studies

The question is how to describe the relevant aspects of the context of a study when we review it for possible relevance to our enquiries—or when we are reporting on our own study. I begin by scrutinizing two studies of educational differences done at different times and in different countries. I first chose a country that was "anthropologically unfamiliar" to me and a study that aimed to take account of relevant differences among schools to describe and account for differences in performance: namely, the Indian National Achievement Survey (NAS; National Council for Educational Research and Training, 2012). This is a national survey of the performance of pupils in Class V (mostly aged 10 and 11 years old) in India which aims to control for a wide range of background characteristics in understanding and estimating differences in performance found, for example, between states, using multiple regression. The list of school, teacher, and student characteristics used provides a convenient "long list" of characteristics that might be considered for (theoretical) relevance, and taken account of, in other studies (National Council for Educational Research and Training, 2012, pp. xxvi–xxix).

Hutchison (2013) discussed whether a particular Indian government program, the Sarva Shiksa Abhiyan, was making progress towards equality of opportunity, or equity of treatment in primary education, in line with the Millennium Development

Goals. He drew on the results of National Council for Educational Research and Training (2012) and produced a short list (summarized here) of the key aspects of the school factors—for the purposes of his study—as:

- *Physical resources* (including level of resourcing and student-teacher ratio)
- *Qualities of teaching staff* (including qualifications, experience, and whether the teacher gives regular homework)
- *School atmosphere and ethos* (including unpleasant experiences of students, problem behavior, and difficulty in recruiting teachers)
- *Home-school interaction* (including student absenteeism, homework given and checked every day)

Comparison of this short list with the longer list in National Council for Educational Research and Training (2012) shows that Hutchison chose to focus on a subset of meso characteristics of the educational context—that is, those characteristics of schools, education authorities, and communities that he considers most relevant to the specific problem that he was addressing.

The other example I take is from research done in the United Kingdom in the 1970s (Rutter, Maughan, Mortimore, & Ouston, 1979). This research studied 12 schools of the then Inner London Education Authority (ILEA) as “social institutions” and concluded that various aspects of the “processes” of the school had effects on children’s performance, attendance, and behavior over and above other factors measured. They labeled a key part of these processes “school ethos,” a set of values which they claimed were generally accepted by teachers and students. This research created a stir within educational and wider circles and led to a number of critical summaries and commentaries (e.g., Radical Statistics Education Group, 1982; Tizard et al., 1980).

Young (1980) summarized the factors studied by Rutter et al. (1979) as *school processes*, *pupil intake factors*, *ecological influences*, and *physical and administrative features*. He also pointed to the kinds of “factors” that the Rutter et al. model appeared not to regard as important: “questions of power, conflict, boundary maintenance, identity protection, classification (of pupils, teachers, and knowledge), as well as the ways in which the social relations of the wider society might mediate processes within the school” (Young, 1980, p. 31). We can see many of these additional factors as meso in our terms, though classification and the social relations of the wider society clearly relate to the macro level too.

Focusing on the meso aspects of the two original pieces of research, and comparing Hutchison’s and Young’s summaries of key characteristics, we can discern some overlap despite the different timing and national origin of the two studies. Thus, Rutter et al.’s and Young’s physical and administrative features relate closely to the National Council for Educational Research and Training’s and Hutchison’s physical resources. But Rutter et al.’s ecological influences (e.g., geographical area, balance, and intake) overlap only slightly with home-school interaction (e.g., absenteeism, giving and checking of homework), so these two categories can be seen more usefully as supplementing each other. Some of Rutter et al.’s (1979) school processes (pp. 217–225) relate to what Hutchison calls school atmosphere and ethos, and the remainder relate to qualities of teaching staff. And any summary of relevant aspects of the research context must include pupil intake factors in aggregate for an analysis focused on the meso level. (In contrast, studies focused on the micro level would relate individual intake factors to individual performance.)

Thus, as a starting point, I propose a list of aspects of the meso context of educational research for researchers wanting to think about characterizing the context of an educational study; the discussion above suggests that it include at least the following features:

- Aggregate pupil intake factors
- Qualities of teaching staff
- Physical and administrative features of the institution
- Ecological influences
- Home-school interaction
- School atmosphere and ethos

The important facets within each of these overall categories will vary from setting to setting. A fuller idea of what they might include in particular settings is given in the descriptions of school and other meso characteristics in the two studies. And what are considered as important features will vary depending on the context to which one aims to transfer the knowledge.

18.4 Maintaining Breadth and Depth in the Idea of "Context"

In discussing the context above, I also emphasized its macro aspects—the historical context and the international and national political and policy dimensions. We can see that the work of National Council for Educational Research and Training (2012) is inspired by today's international student achievement surveys, the Trends in International Mathematics and Science Study (TIMSS; from the International Association for the Evaluation of Educational Achievement [IEA]) and PISA (from the OECD). These surveys are conducted within the dominant overall perspective of policies of quality assurance, effective educational systems, and performativity. Despite the National Council's emphasis on context, they appear to assume that knowledge is a neutral, technical matter, and the related checklists are tools for improvement. But in the contemporary global context, such instruments can be seen as "knowledge-based regulatory tools" aiming to promote the transnational governance of education, even if they are used somewhat differently in different contexts (Carvalho, 2012); see also Ozga (2012) on *governing through data*.

The Rutter et al. (1979) study differed in terms of the issues prevailing in educational research at its time: It was not so much aimed at "effectiveness" and performativity. Rather, there were attempts to appreciate the importance of school processes in the reproduction or reduction of educational inequalities: For one thing, the Rutter et al. research was commissioned by the ILEA, a broadly progressive body (until it was abolished by the Thatcher government in the United Kingdom in the mid-1980s). And this research was committed to looking into the "black box" of schooling processes at a time when there was a competing theoretical interest of investigating the extent to which schools are mechanisms of social reproduction (e.g., Bowles & Gintis, 1976; Willis, 1979). Thus, the Rutter et al. research was done at a time when there was perhaps a more "balanced" relation between the policy and the research fields. There was more autonomy in the research field and less intrusion of the policy field through tightly specified conditions on funding, research assessment exercises, and other forms of regulation of what now counts as "relevant" educational research.

18.5 Discussion

The question of what makes up the "context" of research—and how to communicate it to readers of research journals—is indeed challenging because of the difficulty in deciding where to draw its limits. Here I have taken an "empirical" approach in revisiting two significant and critically reviewed studies with the aim of specifying, in general, particular aspects of the context that should be focused on when considering educational research findings. Thus, the list of potentially relevant aspects of the meso context presented above must be considered as suggestive only. For example, the list would have to be changed somewhat for research situated in institutions of higher education (cf. Evans, 2000, pp. 1–4, 16–17). Nevertheless, in this chapter, I am proposing that authors of educational research reports consider various aspects of the research context from the list above and that they make explicit what they consider to be relevant aspects of the context in the description and presentation of findings from their own work.

An alternative approach is to develop a set of key aspects of the context on the basis of theoretical principles. One possible framework is suggested by the mention of "classification" in Young's (1980) critique of Rutter et al.'s (1979) work cited above. From an early focus on *classification* (the separation and stratification of groups of pupils, teachers, areas of study, and forms of knowledge) and on *framing* (the means of regulation of the pedagogic relationship), Basil Bernstein developed a fuller theory of the pedagogic relationship, with contributions from colleagues and students (e.g., Bernstein, 2000). It is likely that a fruitful way ahead will involve a combination of "empirical" and "theoretical" approaches.

One of the central concepts in the Bernstein approach is *recontextualisation*, the transfer, translation, transformation, or "travel" of knowledge or policies originating in one context to another. Here, our examples also raise the question of the transfer of knowledge, policies, practices, and tools from the West to other educational systems, such as those of India—in the context of what has become a global reform movement towards a certain type of educational improvement. Surveys such as the Indian NAS constitute knowledge-based regulatory tools (Carvalho, 2012), which mold policy and practice in particular directions (e.g., Rizvi & Lingard, 2010). This discussion shows that we also need to think about the changing forms and role of knowledge and policy when thinking about the issue of knowledge transfer (Ozga, 2012).

Despite the fact that the two studies considered here could be seen as "quantitative," I would argue that the process of situating research findings recommended here is an essential component of all effective research strategies. Certainly, many "qualitative" researchers have made similar contributions concerning the need to describe "fully" the context, especially in the case of single-institution "case studies" (e.g., Yin, 2003). Further, researchers familiar with qualitative methods may recognize in my concerns some overlap with what ethnographic researchers sometimes call a "reflexive account," based on the view that researchers are part of the world that they study (e.g., Hammersley & Atkinson, 1983, pp. 14–23). Indeed, the usual reflexive account may well feed into the approach that I am advocating—through its concern with what I call aspects of the micro and the meso contexts, which are basically the relationships that the particular researcher (or research team) has with the other social actors and institutional gatekeepers in the research setting. Ethnographers have provided an indispensable contribution through their description of the micro and some aspects of the meso contexts. Nevertheless, in addition, the aim here is to urge greater attention than is currently given to wider aspects of the meso level and to the macro context, too.

So far, we have discussed the levels of the context as if they are reasonably "objective." Lave (1988) considered the differences between *arena* and *setting*—respectively objective and subjectively experienced contexts—in her study of the mathematics deployed by adult shoppers (pp. 145–169). More recently, the importance of the subjectively (or partially) experienced context has been argued to be relevant to the perception of the "numerate environment" (Evans, Yasukawa, Mallows, & Kubascikova, 2022) by a typical adult engaged in a range of numeracy and mathematical practices. Recognizing the role of subjective perceptions of particular (groups of) actors is important because their perceptions of the context of a particular project will condition their attitudes towards its findings and recommendations: Are they sound, important, worthy of support? Thus, they will be crucial, when we come to consider the response to research of particular factions or philosophies of teachers in the "enactment" of policy, at the school level (Ball, Maguire, Braun, & Hoskins, 2011) or, in the possibilities of "scaling-up" changes recommended by research, at the school-system level (e.g., Cobb, Jackson, Henrick, & Smith, 2018).

18.6 Conclusion

From among the important issues relating to transfer, I have focused in this chapter on the problem of how to communicate the important aspects of the context of research to readers of research reports. I have argued that, given that educational research in its execution and reporting has to take account of the context in one way or another, it is advisable to try to make explicit the principles of description so that the issues of power and control in the research process can be more effectively dealt with.

Further, a description that is as clear and full as possible—and that takes account of varying subjective perceptions—will enhance the possibilities of making judgments about the likely take-up of particular recommendations for change by particular groups of practitioners and institutions.

Why are these ideas important and urgent at this time? First, with the advent of a more globalized set of systems of education around the world (Rizvi & Lingard, 2010), there is a "generalizing" trend that threatens to override local norms, values, and meanings (Evans, Wedege, & Yasukawa, 2013); this trend is strengthened by the strong emphasis given nowadays to the results of international studies like PISA, TIMSS, and PIAAC (e.g., Evans, 2019; OECD, 2016; Tsatsaroni & Evans, 2014). There is an additional risk that, despite substantial differences between educational settings, studies done in certain national contexts, usually the "Anglo-European" ones (Atweh & Keitel, 2008), will be considered "the norm" and therefore as not needing careful description.

Second, with trends towards greater accountability and performativity, plus a greater need for researchers to attract a measure of attention to their work amidst the cacophony of competing claims, there is pressure on researchers and journals to present the "message" of the research in a way that is as concise and straightforward as possible. This leaves little room for specifics of the context, subtlety of interpretation, or pointing to "limitations" of the findings.

Seen from another perspective, the views expressed here emphasize the multi-level nature of the context of research or of any other human endeavor. Above, I showed how this can be seen to have flowed from the concerns of qualitative researchers with micro and meso levels especially. Another approach to more quantitative educational research has come to the fore in the last 30 or 40 years; this research aims to use mathematical modelling not only at the individual-case level (pupil or system user) but also to model one or more "higher" levels at the classroom, school, and perhaps system level. The principles of this approach have been set down (http://www.bristol.ac.uk/cmm/learning/multilevel-models/) and are being used increasingly in effective educational research reporting. This is another example of the view that the levels of the context of research need to be taken account of in the production as well as the reporting of research knowledge.

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