Packing and Covering Triangles in Dense Random Graphs

Zhongzheng Tang[1] and Zhuo Diao[2(✉)]

[1] School of Sciences, Beijing University of Posts and Telecommunications, Beijing 100876, China
tangzhongzheng@amss.ac.cn

[2] School of Statistics and Mathematics, Central University of Finance and Economics, Beijing 100081, China
diaozhuo@amss.ac.cn

Abstract. Given a simple graph $G = (V, E)$, a subset of E is called a triangle cover if it intersects each triangle of G. Let $\nu_t(G)$ and $\tau_t(G)$ denote the maximum number of pairwise edge-disjoint triangles in G and the minimum cardinality of a triangle cover of G, respectively. Tuza [25] conjectured in 1981 that $\tau_t(G)/\nu_t(G) \leq 2$ holds for every graph G. In this paper, we consider Tuza's Conjecture on dense random graphs. We prove that under $\mathcal{G}(n, p)$ model with $p = \Omega(1)$, for any $0 < \epsilon < 1$, $\tau_t(G) \leq 1.5(1 + \epsilon)\nu_t(G)$ holds with high probability, and under $\mathcal{G}(n, m)$ model with $m = \Omega(n^2)$, for any $0 < \epsilon < 1$, $\tau_t(G) \leq 1.5(1+\epsilon)\nu_t(G)$ holds with high probability. In some sense, on dense random graphs, these conclusions verify Tuza's Conjecture.

Keywords: Triangle cover · Triangle packing · Random graph · $\mathcal{G}(n,p)$ model · $\mathcal{G}(n,m)$ model.

1 Introduction

Graphs considered in this paper are undirected, finite and may have multiple edges. Given a graph $G = (V, E)$ with vertex set $V(G) = V$ and edge set $E(G) = E$, for convenience, we often identify a triangle in G with its edge set. A subset of E is called a *triangle cover* if it intersects each triangle of G. Let $\tau_t(G)$ denote the minimum cardinality of a triangle cover of G, referred to as the *triangle covering number* of G. A set of pairwise edge-disjoint triangles in G is called a *triangle packing* of G. Let $\nu_t(G)$ denote the maximum cardinality of a triangle packing of G, referred to as the *triangle packing number* of G. It is clear that $1 \leq \tau_t(G)/\nu_t(G) \leq 3$ holds for every graph G. Our research is motivated by the following conjecture raised by Tuza [25] in 1981, and its weighted generalization by Chapuy et al. [7] in 2014.

This research is supported part by National Natural Science Foundation of China under Grant No. 11901605, and by the disciplinary funding of Central University of Finance and Economics.

W. Wu and Z. Zhang (Eds.): COCOA 2020, LNCS 12577, pp. 426–439, 2020.
https://doi.org/10.1007/978-3-030-64843-5_29

Conjecture 1. **(Tuza's Conjecture** [25]**).** $\tau_t(G)/\nu_t(G) \leq 2$ holds for every simple graph G.

To the best of our knowledge, the conjecture is still unsolved in general. If it is true, then the upper bound 2 is sharp as shown by K_4 and K_5 – the complete graphs of orders 4 and 5.

Related Work. The only known universal upper bound smaller than 3 was given by Haxell [14], who shown that $\tau_t(G)/\nu_t(G) \leq 66/23 = 2.8695...$ holds for all simple graphs G. Haxell's proof [14] implies a polynomial-time algorithm for finding a triangle cover of cardinality at most 66/23 times that of a maximal triangle packing. Other results on Tuza's conjecture concern with special classes of graphs.

Tuza [26] proved his conjecture holds for planar simple graphs, K_5-free chordal simple graphs and simple graphs with n vertices and at least $7n^2/16$ edges. The proof for planar graphs [26] gives an elegant polynomial-time algorithm for finding a triangle cover in planar simple graphs with cardinality at most twice that of a maximal triangle packing. The validity of Tuza's conjecture on the class of planar graphs was later generalized by Krivelevich [18] to the class of simple graphs without $K_{3,3}$-subdivision. Haxell and Kohayakawa [15] showed that $\tau_t(G)/\nu_t(G) \leq 2-\epsilon$ for tripartite simple graphs G, where $\epsilon > 0.044$. Haxell, Kostochka and Thomasse [13] proved that every K_4-free planar simple graph G satisfies $\tau_t(G)/\nu_t(G) \leq 1.5$.

Regarding the tightness of the conjectured upper bound 2, Tuza [26] noticed that there exists infinitely many simple graphs G attaining the conjectured upper bound $\tau_t(G)/\nu_t(G) = 2$. Cui et al. [11] characterized planar simple graphs G satisfying $\tau_t(G)/\nu_t(G) = 2$; these graphs are edge-disjoint unions of K_4's plus possibly some vertices and edges that are not in triangles. Baron and Kahn [2] proved that Tuza's conjecture is asymptotically tight for dense simple graphs.

Fractional and weighted variants of Conjecture 1 were studied in literature. Krivelevich [18] proved two fractional versions of the conjecture: $\tau_t(G) \leq 2\nu_t^*(G)$ and $\tau_t^*(G) \leq 2\nu_t(G)$, where $\tau_t^*(G)$ and $\nu_t^*(G)$ are the values of an optimal fractional triangle cover and an optimal fractional triangle packing of simple graph G, respectively. [16] proved if G is a graph with n vertices, then $\nu_t^*(G) - \nu_t(G) = o(n^2)$.

We can regard the classic random graph models $\mathcal{G}(n,p)$ and $\mathcal{G}(n,m)$ as special graph classes, and we can also consider the probabilistic properties between $\tau_t(G)$ and $\nu_t(G)$. Bennett et al. [3] showed that $\tau_t(G) \leq 2\nu_t(G)$ holds with high probability in $\mathcal{G}(n,m)$ model where $m \leq 0.2403n^{1.5}$ or $m \geq 2.1243n^{1.5}$. Relevant studies in random graph models were discussed in [1,19,24]. Other extensions related to Conjecture 1 can be found in [4–6,8–10,12,17,20–23].

Our Contributions. We consider Tuza's conjecture on random graph, under two probability models $\mathcal{G}(n,p)$ and $\mathcal{G}(n,m)$.

– Given $0 \leq p \leq 1$, under $\mathcal{G}(n,p)$ model, $\mathbf{Pr}(\{v_i, v_j\} \in G) = p$ for all v_i, v_j with these probabilities mutually independent. Our main theorem is following: If

$G \in \mathcal{G}(n,p)$ and $p = \Omega(1)$, then for any $0 < \epsilon < 1$, it holds that

$$\mathbf{Pr}\left[\tau_t(G) \leq 1.5(1+\epsilon)\nu_t(G)\right] = 1 - o(1).$$

– Given $0 \leq m \leq n(n-1)/2$, under $\mathcal{G}(n,m)$ model, let G be defined by randomly picking m edges from all v_i, v_j pairs. Our main theorem is following: If $G \in \mathcal{G}(n,m)$ and $m = \Omega(n^2)$, then for any $0 < \epsilon < 1$, it holds that

$$\mathbf{Pr}\left[\tau_t(G) \leq 1.5(1+\epsilon)\nu_t(G)\right] = 1 - o(1).$$

The main content of the article is organized as follows: In Sect. 2, the theorem in $\mathcal{G}(n,p)$ random graph model is proved; In Sect. 3, the theorem in $\mathcal{G}(n,m)$ random graph model is proved; In Sect. 4, the conclusions are summarized and some future works are proposed. The appendix provides a list of mathematical symbols and classical theorems.

2 $\mathcal{G}(n,p)$ Random Graph Model

In this section, we discuss the probability properties of graphs in $\mathcal{G}(n,p)$. Given $0 \leq p \leq 1$, under $\mathcal{G}(n,p)$ model, $\mathbf{Pr}(\{v_i, v_j\} \in G) = p$ for all v_i, v_j with these probabilities mutually independent. Theorem 1 is our main result: If $G \in \mathcal{G}(n,p)$ and $p = \Omega(1)$, then for any $0 < \epsilon < 1$, it holds that

$$\mathbf{Pr}\left[\tau_t(G) \leq 1.5(1+\epsilon)\nu_t(G)\right] = 1 - o(1).$$

The primary idea behind the theorem is as follows:

– First, in Lemma 2, Lemma 3, we prove that $\tau_t(G) \leq (1+\epsilon)\frac{n(n-1)}{4}p$ holds with high probability by combining the Chernoff's bounds technique;
– Second, in Lemma 4, Lemma 6, we prove that $\nu_t(G) \geq (1-\epsilon)\frac{n(n-1)}{6}p$ holds with high probability through combining the Chernoff's bounds technique and the relationship between $\nu^*(G)$ and $\nu_t(G)$[16].
– By using the previous two properties, Theorem 1 holds.

The following simple property will be used frequently in our discussions.

Lemma 1. *Let $A(n)$ and $B(n)$ be two events related to parameter n. If $\mathbf{Pr}[A(n)] = 1 - o(1)$, then $\mathbf{Pr}[B(n)] \geq \mathbf{Pr}[B(n)|A(n)] - o(1)$ where $o(1) \to 0$ as $n \to \infty$.*

Proof. This can be seen from the fact that $\mathbf{Pr}[A] \cdot \mathbf{Pr}[B] = \mathbf{Pr}[B] - o(1) \geq \mathbf{Pr}[A \cap B] - o(1)$ and $o(1)/\mathbf{Pr}[A] = o(1)$.

Denote the edge number of graph G as m. Let $b(G)$ be the maximum number of edges of sub-bipartite in G. There are four basic properties of graph parameters. The first three holds in every graph, while the last one shows the boundary condition of triangle-free in $\mathcal{G}(n,p)$.

Lemma 2.

(i) $b(G) \geq m/2$ *for every graph* G.
(ii) $\tau_t(G) \leq m/2$ *for every graph* G.
(iii) $\nu_t(G) \leq m/3$ *for every graph* G.
(iv) If $G \in \mathcal{G}(n,p)$ *and* $p = o(1/n)$, *then* G *is triangle-free with high probability.*

Proof. Suppose $b(G) < m/2$ and the corresponding sub-bipartite is $B = (V_1, V_2)$. Thus, there exists one vertex, without loss of generality, $u \in V_1$ satisfies that $d_B(u) < d_G(u)/2$. We can move vertex u from V_1 to V_2, and let $\widetilde{B} = (\widetilde{V}_1, \widetilde{V}_2)$ where $\widetilde{V}_1 = V_1 \backslash \{u\}, \widetilde{V}_2 = V_2 \cup \{u\}$. We have $|E(\widetilde{B})| > |E(B)| = b(G)$, which contradicts with the definition of $b(G)$. Therefore, statement (i) holds.
Statement (ii) follows from the definition of $b(G)$ and the result of statement (i).
Statement (iii) is trivial.
Applying Union Bound Inequality, Statement (iv) is due to

$$\mathbf{Pr}[G \text{ contains at least a triangle}] \leq \binom{n}{3} \cdot p^3 = o(1)$$

In view of Lemma 2(iv), we consider henceforth $\mathcal{G}(n,p)$ with $p = \Omega(1/n)$. Under this condition, we give the following upper bounds for $\tau_t(G)$ and $\nu_t(G)$ with high probability.

Lemma 3. *If* $G \in \mathcal{G}(n,p)$ *and* $p = \Omega(1/n)$, *for any* $0 < \epsilon < 1$, *it holds that*

$$\mathbf{Pr}\left[\tau_t(G) \leq (1+\epsilon)\frac{n(n-1)}{4}p\right] = 1 - o(1). \quad (1)$$

$$\mathbf{Pr}\left[\nu_t(G) \leq (1+\epsilon)\frac{n(n-1)}{6}p\right] = 1 - o(1). \quad (2)$$

Proof. For each edge e in complete graph K_n, Let X_e be the random variable defined by: $X_e = 1$ if $e \in E(G)$ and $X_e = 0$ otherwise. Then X_e, $e \in K_n$, are independent 0–1 variables, $\mathbf{E}[X_e] = p$, $m = \sum_{e \in K_n} X_e$ and $\mathbf{E}[m] = n(n-1)p/2 = \Omega(n)$. By Chernoff's Inequality, for any $0 < \epsilon < 1$ we have

$$\mathbf{Pr}[m \geq (1+\epsilon)\mathbf{E}[m]] \leq \exp\left(-\frac{\epsilon^2 \mathbf{E}[m]}{3}\right) = o(1).$$

Thus, it follows from Lemma 2(ii) and (iii) that

$$\begin{aligned}
&\mathbf{Pr}\left[\tau_t(G) \leq (1+\epsilon)\frac{n(n-1)}{4}p\right] \\
= &\mathbf{Pr}\left[2\tau_t(G) \leq (1+\epsilon)\frac{n(n-1)}{2}p\right] \\
\geq &\mathbf{Pr}\left[m \leq (1+\epsilon)\frac{n(n-1)}{2}p\right] \\
= &\mathbf{Pr}\left[m \leq (1+\epsilon)\mathbf{E}(m)\right] \\
= &1 - o(1)
\end{aligned}$$

Similarly,

$$\begin{aligned} &\mathbf{Pr}\left[\nu_t(G) \le (1+\epsilon)\frac{n(n-1)}{6}p\right] \\ =\ &\mathbf{Pr}\left[3\nu_t(G) \le (1+\epsilon)\frac{n(n-1)}{2}p\right] \\ \ge\ &\mathbf{Pr}\left[m \le (1+\epsilon)\frac{n(n-1)}{2}p\right] \\ =\ &1 - o(1) \end{aligned}$$

proving the lemma.

Along a different line, we consider the probability result of the lower bounds of the fractional triangle packing $\nu_t^*(G)$ as follows:

Lemma 4. *If $G \in \mathcal{G}(n,p)$ and $p = \Omega(1)$, then for any $0 < \epsilon < 1$, it holds that*

$$\mathbf{Pr}\left[\nu_t^*(G) \ge (1-\epsilon)\frac{n(n-1)p}{6}\right] = 1 - o(1).$$

Proof. Consider an arbitrary edge $uv \in K_n$. For each $w \in V(G) \setminus \{u,v\}$. Let X_w be the random variable defined by: $X_w = 1$ if $uw, vw \in E(G)$ and $X_w = 0$ otherwise. Assuming $uv \in E(G)$, let T_{uv} denote the number of triangles of G that contain uv. Notice that X_w, $w \in V(G) \setminus \{u,v\}$, are independent 0–1 variables, $\mathbf{E}[X_w] = p^2$, $T_{uv} = \sum_{w \in V(G)\setminus\{u,v\}} X_w$, and $\mathbf{E}[T_{uv}] = (n-2)p^2$. By Chernoff's Inequality, we have

$$\mathbf{Pr}\left[T_{uv} \ge \left(1+\frac{\epsilon}{2}\right)(n-2)p^2\right] \le \exp\left(-\frac{\epsilon^2(n-2)p^2}{12}\right),$$

and by using Union Bound Inequality

$$\mathbf{Pr}\left[T_e \ge \left(1+\frac{\epsilon}{2}\right)(n-2)p^2 \text{ for some } e \in E(G)\right] \le n^2 \cdot \exp\left(-\frac{\epsilon^2(n-2)p^2}{12}\right) = o(1).$$

Now taking every triangle of G with an amount of $\dfrac{1}{(1+\frac{\epsilon}{2})(n-2)p^2}$, we obtain a feasible fractional triangle packing of G with high probability, giving

$$\begin{aligned} &\mathbf{Pr}\left[\nu_t^*(G) \ge \sum_{T \in \mathcal{T}(G)} \frac{1}{(1+\frac{\epsilon}{2})(n-2)p^2}\right] \\ =\ &\mathbf{Pr}\left[\nu_t^*(G) \ge \frac{\mathcal{T}(G)}{(1+\frac{\epsilon}{2})(n-2)p^2}\right] \\ =\ &1 - o(1) \end{aligned} \qquad (3)$$

For each triangle $T \in K_n$, let X_T be the random variable defined by: $X_T = 1$ if $T \subseteq G$ and $X_T = 0$ otherwise. Then

$$\mathbf{E}[X_T] = \mathbf{Pr}[X_T = 1] = p^3 \text{ and } \mathbf{Var}[X_T] = p^3(1-p^3).$$

For any two distinct triangles T_1, T_2 in K_n, we have

$$\mathbf{Cov}[X_{T_1}, X_{T_2}] = \mathbf{E}[X_{T_1}X_{T_2}] - \mathbf{E}[X_{T_1}]\mathbf{E}[X_{T_2}] = \begin{cases} p^5 - p^6, & \text{if } E(T_1) \cap E(T_2) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Denote $\mathcal{T}(G) = \sum_{T \in \mathscr{T}(K_n)} X_T$. Combining $p = \Omega(1)$, we can compute

$$\mathbf{E}[\mathcal{T}(G)] = \binom{n}{3} p^3 = \Theta(n^3).$$

$$\mathbf{Var}[\mathcal{T}(G)] = \binom{n}{3} p^3(1-p^3) + 2\binom{n}{2}\binom{n-2}{2}(p^5 - p^6) = \Theta(n^4).$$

Thus, Chebyshev's Inequality gives

$$\begin{aligned} &\mathbf{Pr}\left[\mathcal{T}(G) \leq \left(1 - \frac{\epsilon}{2}\right)\mathbf{E}[\mathcal{T}(G)]\right] \\ \leq\ &\mathbf{Pr}\left[|\mathcal{T}(G) - \mathbf{E}[\mathcal{T}(G)]| \geq \frac{\epsilon}{2}\mathbf{E}[\mathcal{T}(G)]\right] \\ \leq\ &\frac{4\mathbf{Var}[\mathcal{T}(G)]}{\epsilon^2(\mathbf{E}[\mathcal{T}(G)])^2} \\ =\ &o(1) \end{aligned} \tag{4}$$

Then, since $\frac{1-\epsilon/2}{1+\epsilon/2} > 1 - \epsilon$ when $0 < \epsilon < 1$, we obtain

$$\begin{aligned} &\mathbf{Pr}\left[\nu_t^*(G) \geq (1-\epsilon)\frac{n(n-1)}{6}p\right] \\ &\geq \mathbf{Pr}\left[\nu_t^*(G) \geq \frac{1-\epsilon/2}{1+\epsilon/2} \cdot \frac{n(n-1)}{6}p\right] \\ &\geq \mathbf{Pr}\left[\nu_t^*(G) \geq \frac{1-\epsilon/2}{1+\epsilon/2} \cdot \frac{n(n-1)}{6}p \;\middle|\; \nu_t^*(G) \geq \frac{\mathcal{T}(G)}{(1+\epsilon/2)(n-2)p^2}\right] - o(1) \\ &\geq \mathbf{Pr}\left[\frac{\mathcal{T}(G)}{(1+\epsilon/2)(n-2)p^2} \geq \frac{1-\epsilon/2}{1+\epsilon/2} \cdot \frac{n(n-1)}{6}p\right] - o(1) \\ &= \mathbf{Pr}\left[\mathcal{T}(G) \geq (1-\epsilon/2)\,\mathbf{E}[\mathcal{T}(G)]\right] - o(1) \\ &= 1 - o(1), \end{aligned}$$

where the second inequality is implied by Lemma 1 and (3), and the last equality is implied by (4). The lemma is established.

We take advantage of the following result in [16] to bridge the relationship of $\nu_t^*(G)$ and $\nu_t(G)$. This result shows that the gap between these two parameters is very small when graph G is dense.

Lemma 5. **([16]).** *If G is a graph with n vertices, then $\nu_t^*(G) - \nu_t(G) = o(n^2)$.*

Combining the above lemma, we derive naturally the lower bound of $\nu_t(G)$ with high probability.

Lemma 6. *If $G \in \mathcal{G}(n,p)$ and $p = \Omega(1)$, then for any $0 < \epsilon < 1$, it holds that*

$$\mathbf{Pr}\left[\nu_t(G) \geq (1-\epsilon) \cdot \frac{n(n-1)p}{6}\right] = 1 - o(1).$$

Proof. Using Lemma 5, when n is sufficiently large we have

$$\begin{aligned}
&\mathbf{Pr}\left[\nu_t(G) \geq (1-\epsilon) \cdot \frac{n(n-1)p}{6}\right] \\
&= \mathbf{Pr}\left[\nu_t^*(G) \geq (1-\epsilon) \cdot \frac{n(n-1)p}{6} + o(n^2)\right] \\
&\geq \mathbf{Pr}\left[\nu_t^*(G) \geq (1-\epsilon) \cdot \frac{n(n-1)p}{6} + \frac{\epsilon}{2} \cdot \frac{n(n-1)p}{6}\right] \\
&= \mathbf{Pr}\left[\nu_t^*(G) \geq \left(1 - \frac{\epsilon}{2}\right) \frac{n(n-1)p}{6}\right].
\end{aligned}$$

The result follows from Lemma 4.

Now we are ready to prove one of the two main theorems:

Theorem 1. *If $G \in \mathcal{G}(n,p)$ and $p = \Omega(1)$, then for any $0 < \epsilon < 1$, it holds that*

$$\mathbf{Pr}\left[\tau_t(G) \leq 1.5(1+\epsilon)\nu_t(G)\right] = 1 - o(1).$$

Proof. Let A denote the event that

$$\tau_t(G) \leq \left(1 + \frac{\epsilon}{3}\right) \frac{n(n-1)}{4} p \quad \text{and} \quad \nu_t(G) \geq \left(1 - \frac{\epsilon}{3}\right) \frac{n(n-1)p}{6}.$$

Combining Lemmas 3 and 6 we have $\mathbf{Pr}[A] = 1 - o(1)$. Note that $1 + \epsilon > \dfrac{1+\epsilon/3}{1-\epsilon/3}$. Therefore, recalling Lemma 1, we deduce that

$$\begin{aligned}
&\mathbf{Pr}\left[\tau_t(G) \leq 1.5(1+\epsilon)\nu_t(G)\right] \\
&\geq \mathbf{Pr}\left[\tau_t(G) \leq 1.5 \cdot \frac{1+\epsilon/3}{1-\epsilon/3}\nu_t(G)\right] \\
&\geq \mathbf{Pr}\left[\tau_t(G) \leq 1.5 \cdot \frac{1+\epsilon/3}{1-\epsilon/3}\nu_t(G) \;\middle|\; A\right] - o(1) \\
&= 1 - o(1),
\end{aligned}$$

which establishes the theorem.

Remark 1. In $\mathcal{G}(n,p)$, $p = \Omega(1)$ implies $\mathbf{E}[m] = \binom{n}{2}p = n(n-1)p/2 = \Omega(n^2)$, thus our main theorem is a result in dense random graphs.

3 $\mathcal{G}(n, m)$ Random Graph Model

In this section, we discuss the probability properties of graphs in $\mathcal{G}(n, m)$. Given $0 \le m \le n(n-1)/2$, under $\mathcal{G}(n, m)$ model, let G be defined by randomly picking m edges from all v_i, v_j pairs. Theorem 2 is our main result: If $G \in \mathcal{G}(n, m)$ and $m = \Omega(n^2)$, then for any $0 < \epsilon < 1$, it holds that

$$\mathbf{Pr}\left[\tau_t(G) \le 1.5(1+\epsilon)\nu_t(G)\right] = 1 - o(1).$$

The primary idea behind the theorem is as follows:

- First, in Lemma 2, $\tau_t(G) \le m/2$ holds;
- Second, in Lemma 7, Lemma 8, we prove that $\nu_t(G) \ge (1-\epsilon)m/3$ holds with high probability through combining the Chernoff's bounds technique and the relationship between $\nu^*(G)$ and $\nu_t(G)$ [16];
- By using the previous two properties, Theorem 2 holds.

For easy of presentation, we use N to denote $\binom{n}{2}$.

Now we give the high probability result of the lower bound of $\nu_t^*(G)$ in $\mathcal{G}(n, m)$ model:

Lemma 7. *If $G \in \mathcal{G}(n, m)$ and $m = \Omega(n^2)$, then for any $0 < \epsilon < 1$, it holds that*

$$\mathbf{Pr}[\nu_t^*(G) \ge (1-\epsilon)m/3] = 1 - o(1).$$

Proof. Consider an arbitrary edge $uv \in K_n$. For each $w \in V(G) \setminus \{u, v\}$. Let X_w be the random variable defined by; $X_w = 1$ if $uw, vw \in E(G)$ and $X_w = 0$ otherwise. Assuming $uv \in E(G)$, let T_{uv} denote the number of triangles of G that contain uv. Then we have

$$\mathbf{E}[X_w] = \frac{m(m-1)}{N(N-1)},$$

$$\mathbf{Var}[X_w] = \frac{m(m-1)}{N(N-1)}(1 - \frac{m(m-1)}{N(N-1)})$$

$$\mathbf{Cov}[X_w, X_{w'}] = \frac{m(m-1)(m-2)(m-3)}{N(N-1)(N-2)(N-3)} - (\frac{m(m-1)}{N(N-1)})^2 \le 0$$

where $w, w' \in V(G) \setminus \{u, v\}$. It follows from $T_{uv} = \sum_{w \in V(G)\setminus\{u,v\}} X_w$ that

$$\mathbf{E}[T_{uv}] = (n-2)\frac{m(m-1)}{N(N-1)} = \Theta(n).$$

Using Chernoff's Inequality, we derive

$$\mathbf{Pr}\left[T_{uv} \ge \left(1 + \frac{\epsilon}{2}\right)\frac{(n-2)m(m-1)}{N(N-1)}\right] \le \exp\left(-\frac{\epsilon^2 \mathbf{E}[T_{uv}]}{12}\right) \le \exp\left(-\frac{\epsilon^2 \Theta(n)}{12}\right);$$

$$\mathbf{Pr}\left[T_e \ge \left(1 + \frac{\epsilon}{2}\right)\frac{(n-2)m(m-1)}{N(N-1)}\ \exists\, e \in E(G)\right] \le n^2 \exp\left(-\frac{\epsilon^2 \Theta(n)}{12}\right) = o(1).$$

So taking every triangle of G with an amount of $\left[(1+\frac{\epsilon}{2})\cdot\frac{(n-2)m(m-1)}{N(N-1)}\right]^{-1}$ makes a feasible fractional packing of G with high probability. Thus

$$\begin{aligned}
&\mathbf{Pr}\left[\nu_t^*(G)\geq\sum_{\forall T}\frac{1}{(1+\frac{\epsilon}{2})\cdot\frac{(n-2)m(m-1)}{N(N-1)}}\right]\\
=\;&\mathbf{Pr}\left[\nu_t^*(G)\geq\frac{\mathcal{T}(G)}{(1+\frac{\epsilon}{2})\cdot\frac{(n-2)m(m-1)}{N(N-1)}}\right]\\
=\;&1-o(1).
\end{aligned}\tag{5}$$

For each triangle $T\in K_n$, let X_T be the random variable defined by: $X_T=1$ if $T\subseteq G$ and $X_T=0$ otherwise. Then

$$\mathbf{E}[X_T]=\frac{m(m-1)(m-2)}{N(N-1)(N-2)}.$$

$$\mathbf{Var}[X_T]=\frac{m(m-1)(m-2)}{N(N-1)(N-2)}\left(1-\frac{m(m-1)(m-2)}{N(N-1)(N-2)}\right).$$

For any two distinct triangles T_1,T_2 in K_n, we have

$$\begin{aligned}
&\mathbf{Cov}(X_{T_1},X_{T_2})\\
&=\mathbf{E}[X_{T_1}X_{T_2}]-\mathbf{E}[X_{T_1}]\cdot\mathbf{E}[X_{T_2}]\\
&=\begin{cases}\frac{m(m-1)(m-2)(m-3)(m-4)}{N(N-1)(N-2)(N-3)(N-4)}-\left(\frac{m(m-1)(m-2)}{N(N-1)(N-2)}\right)^2, & \\ & \text{if } E(T_1)\cap E(T_2)\neq\emptyset;\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

Notice that

$$\mathbf{E}[\mathcal{T}(G)]=\binom{n}{3}\frac{m(m-1)(m-2)}{N(N-1)(N-2)}=\Theta(n^3)$$

$$\begin{aligned}
&\mathbf{Var}[\mathcal{T}(G)]=\binom{n}{3}\frac{m(m-1)(m-2)}{N(N-1)(N-2)}\left(1-\frac{m(m-1)(m-2)}{N(N-1)(N-2)}\right)+\\
&2\binom{n}{2}\binom{n-2}{2}\left(\frac{m(m-1)(m-2)(m-3)(m-4)}{N(N-1)(N-2)(N-3)(N-4)}-\left(\frac{m(m-1)(m-2)}{N(N-1)(N-2)}\right)^2\right)\\
&=\Theta(n^4).
\end{aligned}$$

By Chebyshev's Inequality, we have:

$$\begin{aligned}
&\mathbf{Pr}\left[\mathcal{T}(G)\leq\left(1-\frac{\epsilon}{4}\right)\mathbf{E}[\mathcal{T}(G)]\right]\\
\leq\;&\mathbf{Pr}\left[|\mathcal{T}(G)-\mathbf{E}[\mathcal{T}(G)]|\geq\frac{\epsilon}{4}\mathbf{E}[\mathcal{T}(G)]\right]\\
\leq\;&\frac{16\mathbf{Var}[\mathcal{T}(G)]}{\epsilon^2(\mathbf{E}[\mathcal{T}(G)])^2}=o(1).
\end{aligned}\tag{6}$$

Since $\frac{1-\epsilon/2}{1+\epsilon/2} > 1-\epsilon$, we deduce from (5) and Lemma 1 that

$$\begin{aligned}
&\mathbf{Pr}\left[\nu_t^*(G) \geq (1-\epsilon)\frac{m}{3}\right] \\
\geq\ &\mathbf{Pr}\left[\nu_t^*(G) \geq \frac{1-\epsilon/2}{1+\epsilon/2}\cdot\frac{m}{3}\right] \\
\geq\ &\mathbf{Pr}\left[\nu_t^*(G) \geq \frac{1-\epsilon/2}{1+\epsilon/2}\cdot\frac{m}{3} \;\middle|\; \nu_t^*(G) \geq \frac{\mathcal{T}(G)}{(1+\frac{\epsilon}{2})\frac{(n-2)m(m-1)}{N(N-1)}}\right] - o(1) \\
\geq\ &\mathbf{Pr}\left[\frac{\mathcal{T}(G)}{(1+\frac{\epsilon}{2})\frac{(n-2)m(m-1)}{N(N-1)}} \geq \frac{1-\epsilon/2}{1+\epsilon/2}\cdot\frac{m}{3}\right] - o(1) \\
=\ &\mathbf{Pr}\left[\mathcal{T}(G) \geq \left(1-\frac{\epsilon}{2}\right)\binom{n}{3}\frac{m^2(m-1)}{N^2(N-1)}\right] - o(1)
\end{aligned}$$

As $(1+\frac{\epsilon}{4})\frac{m-2}{N-2} > \frac{m}{N}$ holds for sufficiently large n, we have

$$\begin{aligned}
&\mathbf{Pr}\left[\nu_t^*(G) \geq (1-\epsilon)\frac{m}{3}\right] \\
\geq\ &\mathbf{Pr}\left[\mathcal{T}(G) \geq \left(1-\frac{\epsilon}{2}\right)\left(1+\frac{\epsilon}{4}\right)\binom{n}{3}\frac{m(m-1)(m-2)}{N(N-1)(N-2)}\right] - o(1) \\
\geq\ &\mathbf{Pr}\left[\mathcal{T}(G) \geq \left(1-\frac{\epsilon}{4}\right)\mathbf{E}[\mathcal{T}(G)]\right] - o(1) \\
=\ &1-o(1),
\end{aligned}$$

where the second inequality is implied by $(1-\epsilon/2)(1+\epsilon/4) \leq 1-\epsilon/4$, and the last equality is guaranteed by (6). This complete the proof of the lemma.

Similar to the the proof of Lemma 6, the combination of Lemma 5 and Lemma 7 gives the following Lemma 8.

Lemma 8. *If $G \in \mathcal{G}(n,m)$ and $m = \Omega(n^2)$, then for any $0 < \epsilon < 1$, it holds that*

$$\mathbf{Pr}[\nu_t(G) \geq (1-\epsilon)m/3] = 1-o(1).$$

Proof. Using Lemma 5, when n is sufficiently large we have

$$\begin{aligned}
&\mathbf{Pr}\left[\nu_t(G) \geq (1-\epsilon)m/3\right] \\
=\ &\mathbf{Pr}\left[\nu_t^*(G) \geq (1-\epsilon)m/3 + o(n^2)\right] \\
\geq\ &\mathbf{Pr}\left[\nu_t^*(G) \geq (1-\epsilon)m/3 + \frac{\epsilon}{2}\cdot m/3\right] \\
=\ &\mathbf{Pr}\left[\nu_t^*(G) \geq \left(1-\frac{\epsilon}{2}\right)m/3\right].
\end{aligned}$$

The result follows from Lemma 7.

Now, we are ready to prove the main theorem in $\mathcal{G}(n, m)$ as follows:

Theorem 2. *If* $G \in \mathcal{G}(n, m)$ *and* $m = \Omega(n^2)$*, then for any* $0 < \epsilon < 1$*, it holds that*

$$\mathbf{Pr}\left[\tau_t(G) \leq 1.5(1+\epsilon)\nu_t(G)\right] = 1 - o(1).$$

Proof. Let A denote the event that

$$\tau_t(G) \leq \frac{m}{2} \text{ and } \nu_t(G) \geq (1 - \frac{\epsilon}{2})\frac{m}{3}.$$

It follows from Lemmas 2(ii) and 8 that $\mathbf{Pr}[A] = 1 - o(1)$. Since $1+\epsilon > (1-\epsilon/2)^{-1}$, we deduce from Lemma 1 that

$$\begin{aligned}
&\mathbf{Pr}[\tau_t(G) \leq 1.5(1+\epsilon)\nu_t(G)] \\
&\geq \mathbf{Pr}\left[(1-\epsilon/2)\cdot\tau_t(G) \leq 1.5\nu_t(G)\right] \\
&\geq \mathbf{Pr}\left[(1-\epsilon/2)\cdot\tau_t(G) \leq 1.5\nu_t(G) \mid A\right] - o(1) \\
&= 1 - o(1)
\end{aligned}$$

verifying the theorem.

Remark 2. In $\mathcal{G}(n, m)$, the condition $m = \Omega(n^2)$ implies that our main theorem is a result in dense random graphs.

4 Conclusion and Future Work

We consider Tuza's conjecture on random graphs, under two probability models $\mathcal{G}(n, p)$ and $\mathcal{G}(n, m)$. Two results are following:

- If $G \in \mathcal{G}(n, p)$ and $p = \Omega(1)$, then for any $0 < \epsilon < 1$, it holds that

$$\mathbf{Pr}\left[\tau_t(G) \leq 1.5(1+\epsilon)\nu_t(G)\right] = 1 - o(1).$$

- If $G \in \mathcal{G}(n, m)$ and $m = \Omega(n^2)$, then for any $0 < \epsilon < 1$, it holds that

$$\mathbf{Pr}\left[\tau_t(G) \leq 1.5(1+\epsilon)\nu_t(G)\right] = 1 - o(1).$$

In some sense, on dense random graph, these two inequalities verify Tuza's conjecture.

Future work: In dense random graphs, these two results nearly imply $\tau_t(G) \leq 1.5\nu_t(G)$ holds with high probability. It is interesting to consider the same problem in sparse random graphs.

Acknowledgement. The authors are very indebted to Professor Xujin Chen and Professor Xiaodong Hu for their invaluable suggestions and comments.

Appendix: A List of Mathematical Symbols

$\mathcal{G}(n,p)$	Given $0 \le p \le 1$, $\mathbf{Pr}(\{v_i, v_j\} \in G) = p$ for all v_i, v_j
	With these probabilities mutually independent
$\mathcal{G}(n,m)$	Given $0 \le m \le n(n-1)/2$, let G be defined by
	Randomly picking m edges from all v_i, v_j pairs
$\tau_t(G)$	The minimum cardinality of a triangle cover in G
$\nu_t(G)$	The maximum cardinality of a triangle packing in G
$\tau_t^*(G)$	The minimum cardinality of a fractional triangle cover in G
$\nu_t^*(G)$	The maximum cardinality of a fractional triangle packing in G
$b(G)$	The maximum number of edges of sub-bipartite in G
$\delta(G)$	The minimum degree of graph G
$f(n) = O(g(n))$	$\exists\ c > 0, n_0 \in \mathbb{N}_+, \forall n \ge n_0, 0 \le f(n) \le cg(n)$
$f(n) = \Omega(g(n))$	$\exists\ c > 0, n_0 \in \mathbb{N}_+, \forall n \ge n_0, 0 \le cg(n) \le f(n)$
$f(n) = \Theta(g(n))$	$\exists\ c_1 > 0, c_2 > 0, n_0 \in \mathbb{N}_+, \forall n \ge n_0, 0 \le c_1 g(n) \le f(n) \le c_2 g(n)$
$f(n) = o(g(n))$	$\forall\ c > 0, \exists\ n_0 \in \mathbb{N}_+, \forall n \ge n_0, 0 \le f(n) < cg(n)$
$f(n) = \omega(g(n))$	$\forall\ c > 0, \exists\ n_0 \in \mathbb{N}_+, \forall n \ge n_0, 0 \le cg(n) < f(n)$

Union Bound Inequality:
For any finite or countably infinite sequence of events $E_1, E_2, \ldots$, then

$$\mathbf{Pr}\left[\bigcup_{i\ge 1} E_i\right] \le \sum_{i\ge 1} \mathbf{Pr}(E_i).$$

Chernoff's Inequalities:
Let $X_1, X_2, \ldots, X_n$ be mutually independent 0–1 random variables with $\mathbf{Pr}[X_i = 1] = p_i$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbf{E}[X]$. For $0 < \epsilon \le 1$, then the following bounds hold:

$$\mathbf{Pr}[X \ge (1+\epsilon)\mu] \le e^{-\epsilon^2\mu/3}, \quad \mathbf{Pr}[X \le (1-\epsilon)\mu] \le e^{-\epsilon^2\mu/2}.$$

Chebyshev's Inequality:
For any $a > 0$,

$$\mathbf{Pr}[|X - \mathbf{E}[X]| \ge a] \le \frac{\mathbf{Var}[X]}{a^2}.$$

References

1. Baron, J.D.: Two problems on cycles in random graphs. Ph.D. thesis, Rutgers University-Graduate School-New Brunswick (2016)

2. Baron, J.D., Kahn, J.: Tuza's conjecture is asymptotically tight for dense graphs. Comb. Probab. Comput. **25**(5), 645–667 (2016)
3. Bennett, P., Dudek, A., Zerbib, S.: Large triangle packings and Tuza's conjecture in sparse random graphs. Comb. Probab. Comput. **29**(5), 757–779 (2020)
4. Botler, F., Fernandes, C., Gutiérrez, J.: On Tuza's conjecture for triangulations and graphs with small treewidth. Electron. Notes Theor. Comput. Sci. **346**, 171–183 (2019)
5. Botler, F., Fernandes, C.G., Gutiérrez, J.: On Tuza's conjecture for graphs with treewidth at most 6. In: Anais do III Encontro de Teoria da Computação. SBC (2018)
6. Chalermsook, P., Khuller, S., Sukprasert, P., Uniyal, S.: Multi-transversals for triangles and the Tuza's conjecture. In: Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms, pp. 1955–1974. SIAM (2020)
7. Chapuy, G., DeVos, M., McDonald, J., Mohar, B., Scheide, D.: Packing triangles in weighted graphs. SIAM Journal on Discrete Mathematics **28**(1), 226–239 (2014)
8. Chen, X., Diao, Z., Hu, X., Tang, Z.: Sufficient conditions for Tuza's conjecture on packing and covering triangles. In: Mäkinen, V., Puglisi, S.J., Salmela, L. (eds.) IWOCA 2016. LNCS, vol. 9843, pp. 266–277. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-44543-4_21
9. Chen, X., Diao, Z., Hu, X., Tang, Z.: Total dual integrality of triangle covering. In: Chan, T.-H.H., Li, M., Wang, L. (eds.) COCOA 2016. LNCS, vol. 10043, pp. 128–143. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-48749-6_10
10. Chen, X., Diao, Z., Hu, X., Tang, Z.: Covering triangles in edge-weighted graphs. Theory Comput. Syst. **62**(6), 1525–1552 (2018)
11. Cui, Q., Haxell, P., Ma, W.: Packing and covering triangles in planar graphs. Graphs and Combinatorics **25**(6), 817–824 (2009)
12. Erdös, P., Gallai, T., Tuza, Z.: Covering and independence in triangle structures. Discret. Math. **150**(1–3), 89–101 (1996)
13. Haxell, P., Kostochka, A., Thomassé, S.: Packing and covering triangles inK_4-free planar graphs. Graphs and Combinatorics **28**(5), 653–662 (2012)
14. Haxell, P.E.: Packing and covering triangles in graphs. Discret. Math. **195**(1), 251–254 (1999)
15. Haxell, P.E., Kohayakawa, Y.: Packing and covering triangles in tripartite graphs. Graphs and Combinatorics **14**(1), 1–10 (1998)
16. Haxell, P.E., Rödl, V.: Integer and fractional packings in dense graphs. Combinatorica **21**(1), 13–38 (2001)
17. Hosseinzadeh, H., Soltankhah, N.: Relations between some packing and covering parameters of graphs. In: The 46th Annual Iranian Mathematics Conference, p. 715 (2015)
18. Krivelevich, M.: On a conjecture of Tuza about packing and covering of triangles. Discret. Math. **142**(1), 281–286 (1995)
19. Krivelevich, M.: Triangle factors in random graphs. Comb. Probab. Comput. **6**(3), 337–347 (1997)
20. Lakshmanan, A., Bujtás, C., Tuza, Z.: Induced cycles in triangle graphs. Discret. Appl. Math. **209**, 264–275 (2016)
21. Munaro, A.: Triangle packings and transversals of some K_4-freegraphs. Graphs and Combinatorics **34**(4), 647–668 (2018)
22. Puleo, G.J.: Tuza's conjecture for graphs with maximum average degree less than 7. Eur. J. Comb. **49**, 134–152 (2015)
23. Puleo, G.J.: Maximal k-edge-colorable subgraphs, Vizing's Theorem, and Tuza's Conjecture. Discret. Math. **340**(7), 1573–1580 (2017)

24. Ruciński, A.: Matching and covering the vertices of a random graph by copies of a given graph. Discret. Math. **105**(1–3), 185–197 (1992)
25. Tuza, Z.: Conjecture. In: Finite and Infinite Sets, Proc. Colloq. Math. Soc. Janos Bolyai, p. 888 (1981)
26. Tuza, Z.: A conjecture on triangles of graphs. Graphs and Combinatorics **6**(4), 373–380 (1990)