# Chapter 6
# A Brief Overview of the Evolution of the Scientific Theory of Gearing

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## 6.1 Introduction

Gears are the means by which power is transferred from source to application. Gears and gear transmissions are extensively used in the nowadays industry. Gearing and geared transmissions drive the machines of modern industry. Gears move the wheels and propellers that transport us over the sea, on the land, and in the air. Transmission and transformation of rotation from an input shaft to an output shaft is the main purpose of gearing of all kinds. A sizeable section of industry and commerce in today's world depends on gearing for its economy, production, and livelihood. No doubt gearing of all kinds will be extensively used in the future.

It should be realized here that there are two different considerations when the state of the art of gearing is discussed. Gear design, and gear manufacture, that is based on common sense of smart handicrafts and on accumulated practical experience is one of the two. An engineering approach that is based on scientific accomplishments in the theory of gearing is the other one.

Taking into account the incompleteness and the inconsistency of the nowadays knowledge in the theory of gearing, an in-depth investigation into the gear kinematics and the gear geometry has been undertaken in the recent years by the author.

This chapter of the book is written in the following manner. At the beginning, a brief overview of the pre-*Eulerian* period of the gear art is done. Then, the fundamental accomplishments in the "*scientific*" theory of gearing are identified, and a name of the corresponding key contributor (s) is associated (where possible) with each of the accomplishments. As the overall number of the "*fundamental*" accomplishments in the scientific theory of gearing is limited, and it is not large, the overall number of the fundamental contributors to the "*scientific*" theory of gearing is also limited. Irrespective of many other researchers (not mentioned in this section of the book) that have also contributed a lot to the field of gearing, they cannot be regarded as the "*fundamental contributors*" to the "*scientific*" theory of gearing.

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Most of the results of the research that has been carried out are discussed in the monograph [1]. The fundamentals of the proposed scientific theory of gearing are based on the key accomplishments in the gear kinematics, and the gear geometry. With that said, it makes sense to begin the discussion with a brief overview of the evolution of the scientific theory of gearing. This will help us to identify what is already done in the field, where we are now, and what has to be done in the future.

Such an analysis has to be carried out in order to credit every accomplishment in the scientific theory of gearing with the name of the key contributor. In the meantime, not all the achievements in the field of gearing can be attributed with a right name of a gear researcher. A gear researcher who has contributed significantly to the theory of gearing deserves to be credited with a corresponding scientific result.

Several achievements in the theory of gearing cannot be credited to right persons. For example, the names of the key contributors for (a) the condition of contact of the tooth flanks and (maybe this accomplishment is NOT exactly from gearing but from another area of the theory of machines and mechanisms) (b) equal base pitches are not known. A few more achievements in this regard can be mentioned. It is desirable to get the appropriate names identified.

The aforementioned, as well as other accomplishments, are vital to the scientific theory of gearing. It is desirable to know who was the first to come up with these meaningful results, as well as in what way these results have been achieved.

The main goal of the book chapter titled "A Brief Overview of the Evolution of the Scientific Theory of Gearing" is to briefly outline "all" the known fundamental accomplishments in the scientific theory of gearing and to credit right gear researchers with the corresponding scientific achievements in the field, that is, an effort is undertaken aiming to associate each of the fundamental accomplishments in the scientific theory of gearing with the name of the corresponding gear researcher who contributed a particular accomplishment. In order to mention all the key researchers in the field and to miss none of them, the following approach is adopted below.

First, all (with no exclusions) the fundamental accomplishments to the scientific theory of gearing are listed in a chronological order.

Second, a name of a key gear researcher is associated with each of the accomplishment, that is, the name of a researcher who was the first either to discover or to contribute the most to a particular accomplishment in the scientific theory of gearing. For example, Leonhard Euler is credited with the application of the involute of a circle for the gear tooth profile, as he was the first to prove that the involute tooth profile fits the best the needs the gear tooth geometry, regardless of the involute of a circle is known[1] long before Euler has made his discovery in 1760.

In addition to that, a few huge mistakes committed in the past by the gear researchers when investigating gears are also mentioned in order to better understand the theory and to properly value those scientists who contributed much to the

[1]The involute of a circle was first proposed by *Philippe de la Hire* in 1696, and it was later in the eighteenth century when *Leonhard Euler* proposed the involute curve as a viable tooth profile.

scientific theory of gearing. Mistakes of this type can be referred to as the "key mistakes" in the theory of gearing. Following the adopted approach, it is helpful to separate the names of the principal contributors to the scientific theory of gearing from those who contributed less to the subject, and, moreover, from those who committed mistakes that significantly affected the evolution of the theory of gearing.

The results of the research carried out by the author, and a few papers earlier published by the author [2–4], are extensively used in this section of the book. Other sources are extensively used as well [5, 6], and others. The consideration begins with ancient gear designs that are created only due to common sense and ends with the modern scientific theory of gearing.

Tons of sources where investigated prior to making a possible representation of the principal accomplishments in the scientific theory of gearing in a chronological order. A limited number of the sources were selected for a more detailed analysis. These sources are summarized in [1]. The reported analysis is based mostly on the results of the research listed in [1].

The scientific theory of gearing is the foundation of gear design, production, inspection, and application of gears. Below in this chapter of the book, the purpose, principal features, and evolution of the gear theory are discussed.

Prior to beginning the discussion on the scientific theory of gearing, it makes sense to clarify the meaning of the term "scientific theory." As it is understood from the text below, a "scientific theory" has to be based on a few postulates (on a limited input information), which the entire theory can be derived from. The fewer the total number of the postulates, the more powerful scientific theory can be derived, and vice versa. In the case of the scientific theory of gearing, the minimum required input information is limited to:

(a) A disposition of an input shaft relative to an output shaft.
(b) Input rotation.
(c) Input torque.
(d) Output rotation (or output torque).

The entire analytical description of gearing, the kinematics and the geometry of gearing, the elements of gear dynamics, and so forth, can be derived from the clauses (a) through (d).

All the results in a scientific theory of gearing are interconnected with one another. An additional information on gearing (if any) can also be incorporated into the analysis. With that said, it is clear now that all the earlier developed so-called theories of gearing (proposed by T. Olivier (1842) [7], Ch. Gochman (1886) [8], F. Litvin (2004), D. Dooner (2012), as well as numerous others) cannot be referred to as "scientific theories of gearing," as they represent just a collection of scientific/engineering results that are independent from one another, are not interconnected with one another, and do not form a self-consistent theory.

The art and science of gearing have their roots before the Common Era. Yet many engineers and researchers continue to delve into the areas where improvements are necessary, seeking to quantify, establish, and codify methods to make gears meet the ever-widening needs of advancing technology.

Despite gears and geared transmissions are investigated for a long while, the nowadays knowledge of the gear theory is poor and is completely insufficient. Moreover, the author is doubtful that all the principal accomplishments in the theory of gearing are known to all the gear researchers, those who are actively involved in the research in the field.

Motivation: The necessity of "cleaning" of the gear science from incorrect, wrong, and loosely statements, as well as from other inconsistencies, is the main reason to write this section of the book.

The bottom line is as follows: All the principal accomplishments in the theory of gearing have to be identified, and a gear researcher, who has contributed significantly to the theory of gearing, deserves to be credited with the corresponding scientific result.

Those who don't know their own history have no chance for success in the future.

### *6.1.1 Main Periods in the Evolution of the Theory of Gearing*

Gears are used to transmit and transform a rotation from an input shaft to an output shaft. Depending on a particular application, gearing has to meet certain additional requirements, namely, high accuracy of transmission of rotation and high power density[2], which have to be ensured by gears.

The development and investigation of gearing with a constant angular velocity ratio[3] (i.e., gearing for which the equality $\omega_p/\omega_g = \text{const}$ is valid) is one of the main goals of the scientific theory of gearing [1]. Gearing with a constant angular velocity ratio (or gearing with a pre-specified function of the angular velocity ratio) is commonly called "geometrically accurate gearing," or "ideal gearing," or just "perfect gearing," for simplicity. More generally, the design of gearing with a prescribed function of the angular velocity ratio (i.e., (a) non-circular gearing with a constant center-distance, (b) non-circular gearing with a variable center-distance, (c) gearing with a variable shaft angle, and (d) gearing with a variable center-distance, and a variable shaft angle simultaneously) is also covered by the scientific theory of gearing.

Many efforts were undertaken in the past by hundreds of researchers aiming for the development of the "theory of gearing." However, not many of them have really contributed to the theory.

Below in this chapter of the book, the evolution of gearing from the earliest times to the present day is concisely discussed with the emphasis on the "theory of gearing." The consideration is mainly focused on the kinematics of gear pairs, the

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[2]The term "*power density*" is commonly used as an equivalent to the term "*power-to-weight ratio*" (this concept deserves to be investigated more carefully).

[3]In a more general sense, that is, when non-circular gears are taken into account, use of "*perfect gearing*" makes possible an exact function of the pre-specified angular velocity ratio.

geometry of the interacting tooth surfaces, as well as on some other kinematic and geometric aspect of gears and gearing pairs.[4] The following rule is adopted in this section of the book: first, all principal contributions to the scientific theory of gearing are identified. Second, the names of the scientists who contributed these accomplishments are named. Following this rule, not many gear researchers are credited with the fundamental contributions to the field of gearing – only those who were the first to obtain a fundamental result of the research in the field.

As it was earlier (circa 2012) proposed by the author [1, 2, 9, 10], the evolution of the theory of gearing falls into three periods, namely:

(a) Pre-Eulerian period of the gear art.

(b) The time of the fundamental contribution by L. Euler.

(c) Post-Eulerian period of the theory of gearing.

The principal accomplishments in the scientific theory of gearing are considered below in a chronological order in alignment with the just mentioned three periods of the evolution. It is believed that all (or, at least, almost all) principal accomplishments are covered in the paper.

#### 6.1.1.1 Pre-Eulerian Period of Evolution of the Gear Art

The earliest account of gears comes from ancient Chinese and Greek literature. Because of force-multiplying properties of gears, early engineers used them for hoisting heavy loads such as building materials. The mechanical advantage of gears was also used for ship anchor hoists and catapult pre-tensioning.

The earliest written descriptions of gears are said to have been made by Aristotle [5] in the fourth century B.C. It has been pointed out that the passage attributed by some to Aristotle, in “Mechanical Problems of Aristotle” (ca. 280 B.C.), was actually from the writings of his school. In the passage in question, there was no mention of gear teeth on the parallel wheels, and they may just as well have been smooth wheels in frictional contact. Therefore, the attribution of gearing to Aristotle is most likely incorrect. The real beginning of gearing was “probably” with Archimedes, who in about 250 B.C. invented the endless screw turning a toothed wheel, which was used in engines of war. Archimedes also used gears to simulate astronomical ratios. The Archimedean were early forms of the wagon mileage indicator (odometer) and the surveying instrument. These devices were “probably” based on “thought” experiments of Heron of Alexandria (ca. 60 A.D.), who wrote on theoretical mechanics and the basic elements of mechanisms.

In the Ancient times, transmission and transformation of a rotation were the only purpose of gearing. The quality of rotation of the output shaft, that is, smoothness of its rotation, was out of importance in the earliest designs of gears. Therefore, it was a common practice to build pin gears made up of wood, the gear tooth profile

[4] The evolution of the geared mechanisms is out of the scope of this chapter of the book.

geometry was not considered at all, and pin gears successfully met all the requirements of that time.

Judging from the history books is one thing. Finding hard evidence of actual gears is another. The biggest problem in finding archeological evidence of gears is that early gear materials were not built to last. Gears manufactured at this time were probably made of bronze. When bronze tools and mechanical pieces broke, they were simply melted down and refashioned into something else.

The oldest surviving relic containing gears is the Antikythera mechanism[5], named for the Greek island near which the mechanism was discovered in a sunken ship in 1900. The mechanism is not only the earliest relic of gearing[6] but is also an extremely complex arrangement of epicyclic differential gearing. The mechanism is identified as a calendrical Sun and Moon computing mechanism and is dated to about 87 B.C.

The Antikythera mechanism (see Fig. 6.1) is the oldest[7] known artifact consisting gears. Here, an image of the original Antikythera mechanism is shown in Fig. 6.1,a.

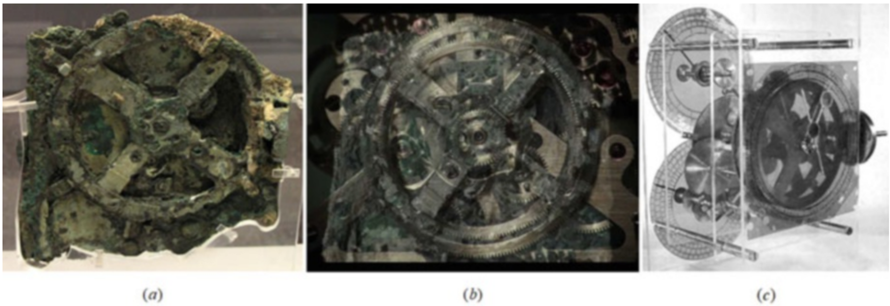


**Fig. 6.1** The *Antikythera mechanism* (100 BC to 205 BC)

[5]The artifact was recovered in 1900–1901 from the Antikythera shipwreck off the Greek island of Antikythera. Its significance and complexity were not understood until decades later. Believed to have been designed and constructed by Greek scientists, the instrument has been dated either between 150 and 100 BC, or, according to a more recent view, at 205 BC. This precious example of antique genius complexity grade was so high that artefacts of a similar complexity and workmanship did not reappear for a millennium and a half, when mechanical astronomical clocks were built in Europe.

[6]The *South-pointing chariot* (invented in the fifth century BC) is another known device that contains gears. Unfortunately, only numerous nowadays designed ***reconstructions*** (not ***replicas***) of the *South-pointing chariot* are known, and no original artifact remained.

[7]The earliest known reference to a gear was around 50 A.D.; *Hero of Alexandria*, through the "*Book of Song*," suggests that the South-pointing chariot may have employed differential gears as early as the reign of the *Zhou* Dynasty (1045–256 BC) of China (Radzevich, S.P, *Dudley's Handbook of Practical Gear Design and Manufacture*, 3nd ed., Boca Raton, FL: *CRC* Press, 2016, 629 pages.). However, no artifact of the South-pointing chariot is discovered so far. Only nowadays designed ***reconstructions/simulations*** are known. Therefore, in the meantime, the South-pointing chariot cannot be considered as a relic of a mechanism with gears.

**Fig. 6.2** Old-style gears made out of wood

Fig. 6.1,b illustrates the Antikythera mechanism overlapped with the image of its replica. A 3D image of a similar overlap is depicted in Fig. 6.1,c. The device has more than 30 gears, although some scientists suggested as many as 72 gears, with teeth formed through equilateral triangles.

Commonly gears of early design were made out of wood with cylindrical pegs and were often lubricated with animal fat grease. An example of old-style gears made of wood is depicted in Fig. 6.2.

Gears were used in wind and water wheel machinery for decreasing or increasing the provided rotational speed for application to pumps and other powered machines. An early gear arrangement is used to power textile machinery. The rotational speed of a water or horse-drawn wheel was typically too slow to use, so a set of wooden gears was needed to be used to increase the speed to a usable level.

In gearing of old designs (see Figs. 6.1 and 6.2, and others), no special care was taken of the geometry of the interacting tooth surfaces of the gears. Practical men were able by various empirical means to get gears adequate for their needs, at least until the early nineteenth century, when the mathematician's work was translated into practical language. Purely empirical solutions for the form of gear teeth can only be accounted for by the fact that gears operated at "low speeds" and under "small loads." No theory of gearing was necessary to design old-style gears as the rotations were low, and there were no constraints on power density of the gearing. Common sense was the only tool used by the smart ancient craftsmen when designing and manufacturing the gears.

The art of gearing was carried through the European Dark Ages, appearing in Islamic instruments such as the geared astrolabes that were used to calculate the positions of the celestial bodies. Perhaps the art was relearned by the clock- and instrument-making artisans of fourteenth-century Europe, or perhaps some crystallizing ideas and mechanisms were imported from the East after the crusades of the eleventh through the thirteenth centuries.

It appears that an English abbot of St. Alban's monastery, born Richard of Wallingford in 1330 A.D., reinvented the epicyclic gearing concept. He applied it to an astronomical clock that he began to build and that was completed after his death.

A mechanical clock of a slightly later period was conceived by Giovanni de Dondi (1348–1364). Diagrams of this clock, which did not use differential gearing, appear in the sketchbooks of Leonardo da Vinci, who designed geared mechanisms himself [11].

Numerous famous names have indicated their interest to gears and gear drives. Leonardo da Vinci (1452–1519), Albrecht Dürer (1471–1528), Robert Hooke[8] (1635–1703), and numerous others can be mentioned in this regard.

Numerous designs of gearing are discussed in the famous book by Leonardo da Vinci [11]. In 1967, two of Leonardo da Vinci's manuscripts, lost in the National Library in Madrid since 1830, were rediscovered [11]. One of the manuscripts, written between 1493 and 1497 and known as "Codex Madrid I" [11], contains 382 pages with some 1600 sketches. Included among this display of Leonardo's artistic skill and engineering ability are his studies of gearing. Among these are tooth profile designs and gearing arrangements that were centuries ahead of their "invention."

Albrecht Dürer[9] is credited with discovering the epicycloidal shape (ca. 1525).

For a long while, the most accurate gears were produced by clockmakers and instrument makers. Questions of exact tooth form, pressure angle, and strength did not enter into the designs of the clockmakers and instrument makers.

Contemporary gears for the uniform transmission of power and rotation are based in much on the application of mathematical curves discovered by scientists in the sixteenth and seventeenth centuries, in the design of teeth flanks. In the period 1450 to 1750 the mathematics of gear tooth profiles and theories of geared mechanisms became established.

Mathematicians turned their attention to gear tooth profile only in the seventeenth century. Girard Desargues (1591–1661), Philippe de La Hire (1640–1718) [12], and Charles Camus (1699–1768) [13] are the names of the most prominent contributors to the gear science in the pre-Eulerian period of evolution of the gear theory.

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[8]In 1666, *R. Hooke* demonstrated for *The Royal Society* a model of gearing that he has invented earlier. Later on the gearing of this kind *Hooke* described in his 1674 book *Lectiones Cutlerianae*. The gearing of this particular kind is nowadays known as *White*'s *gearing*. May be this is somehow associated with Mr. *Christopher White* of London who manufactured a microscope for *R. Hooke*.

[9]***Albrecht*** **Dürer** (May 21, 1471–6 April 6, 1528), a German painter, printmaker, mathematician, engraver, and theorist

Desargues, de La Hire, and Camus have summarized the main accomplishments in the field of gearing in the pre-Eulerian period of time. The results of the research obtained by these scientists are very close to the origin of the scientific theory of gearing.

In particular, the first record on the use of cyclic curves as the tooth profile of a gear is related to Gerard Desargues[10]. Desargues' work on gearing is known mostly from the records made by his student Philippe de la Hire[11]. A treaty on epicycloids and their usage in mechanics is discussed in his book [12]. Philippe de la Hire was the first to describe the use of epicycloids for gears that ensured (as he loosely meant) a uniform transmission of rotation. De la Hire considered the involute as the best among exterior cycloids, since he recognized that it is the special case in which the generating circle's radius is infinite. He also noted that the involute tooth gives the teeth of the corresponding rack as having straight sides. It took 150 years before this principle found practical application.

The first major contribution to the geometry of gear wheels came from France. Charles Etienne Louis Camus[12] is the first mathematician to work the theory of gear teeth into a systematic and general theory of mechanism.

Camus was close to discover the "conjugate action law" for the case of parallel-axes gearing [13]. He showed that in order to get as output a uniform angular velocity, it is necessary to shape the two teeth so that they can be generated like epicycloids by rolling one and the same curve on two different circles.

Charles Camus was the first [13] who formulated the condition that, in his opinion, has to be fulfilled for a pair of gears to be able transmitting a rotary motion smoothly. According to Camus, this condition is formulated as follows:

If, in a uniform rotation, power is to be transmitted via a pair of teeth, then the normal to the teeth flanks at the contact point (within the path of contact) must pass through the pitch point.

Another formulation of that same condition by other researchers is represented in the form:

If an auxiliary curve is rolling on the pitch circles of circular gears, any point attached to this curve traces conjugate profiles.

This sounds similar to the fundamental theorem of gearing known nowadays (see below in this chapter of the book). Camus' principle of gearing is illustrated with 1733 schematic (see Fig. 6.3). In this schematic, the path of contact, namely, a curved line segment[13] *KBC*, and the line of action at different angular configurations

[10]***Girard* Desargues** (February 21, 1591–September 1661) was a French mathematician and engineer.

[11]***Philippe de la* Hire** (March 18, 1640–April 21, 1718) was a French physicist, astronomer, mathematician, and engineer.

[12]***Charles Étienne Louis* Camus** (August 25, 1699–February 2, May 4, 1768) was a French mathematician and mechanician

[13]The line of action, *KBC*, cannot be a curve, as a force acts only along a straight line, that is, along a straight line of action, and not along a curve.

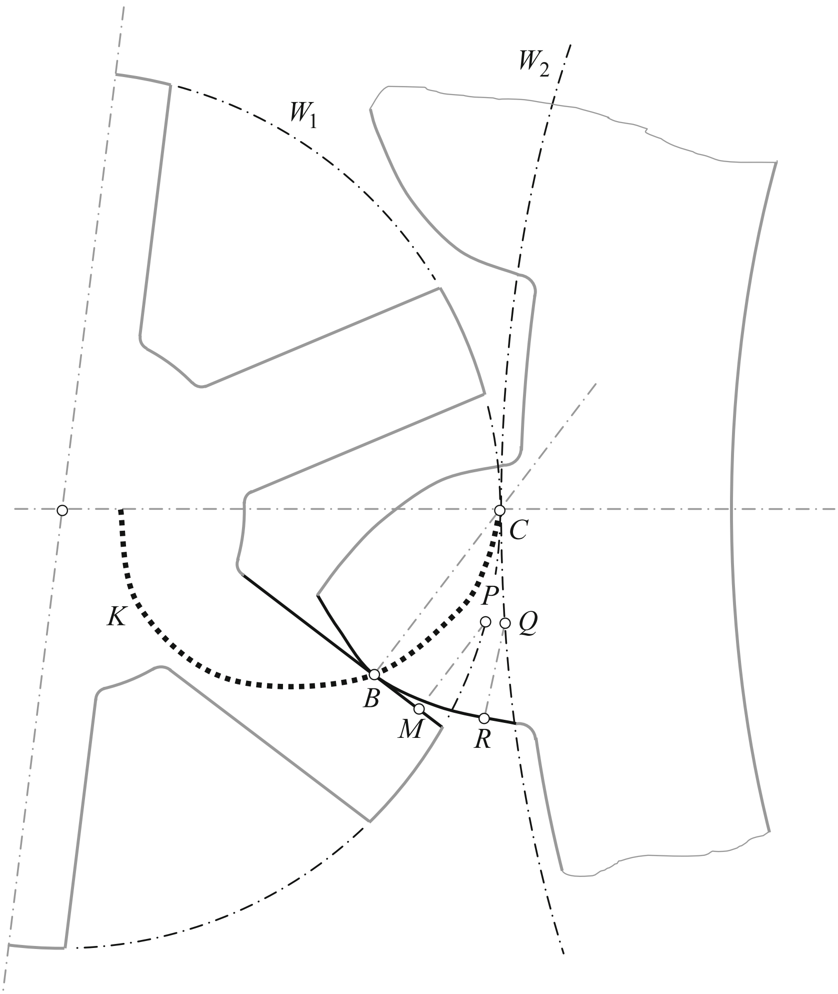



**Fig. 6.3** Illustration of *Camus*' gearing principle (1733)

of the mating gears, that is, *BC*, *MP*, and *RQ*, do not align to one another.[14] The schematic (see Fig. 6.3) reveals that Camus did not realize the difference between the "path of contact" and the "line of action." Camus loosely assumed that the line of

[14]It is instructive to note here that the schematic shown in Fig. 6.3 is a kind of mistake because of the following reasons. First, the path of contact is an envelope to consecutive positions of the instant line of action. Therefore, it is not permissible that the line of action, *BC*, intersects the path of contact, *KBC*. The path of contact must be in tangency with the line of action, *BC*. Second, when numerous instant lines of contact are through the pitch point *C*, then no enveloping curve (i.e., no path of contact) can be constructed. A few more reasons for infeasibility of gearing shown in Fig. 6.3 are to be mentioned.

action, *LA*, in geometrically accurate gearing can be shaped in a form of an arbitrary planar curve (as illustrated in Fig. 6.3), which is incorrect.

Despite Camus was close to discover the fundamental theorem of gearing, he did not succeed doing that. As it is shown below in this chapter of the book, for gears that operate on parallel shafts, the only feasible case is when both the "path of contact" and the "line of action" are straight lines that align with one another, as it is observed in involute gearing.

Therefore, it is incorrect to grant Camus with the discovery of the "conjugate action law."

Camus repeated much of de La Hire's work, although he added many important elements of his own. He gives a detailed analysis of the teeth desirable for the combination of a spur and lantern gear.

Camus did, however, correct de La Hire in that he recognized the fact of sliding of even the epicycloidal teeth one upon the other and said that this phenomenon is one of the principal sources of friction and wear in gearing.

The action of engaged teeth relative to the line of centers is discussed, and he points out that the action is best when engagement takes place after the working face of the driving tooth has passed the line of centers, that is, during the receding action.

Camus goes on to consider the problem of the minimum number of teeth and that of the proper form for the ends of the teeth. He deals with true bevel gears and uses the rolling-cone principle for their analysis. But he considers only the case of interaction of a crown and a bevel gear.

Camus does not consider the involute tooth at all. Although he analyzes trains of gears, he says nothing of the form of teeth required in a series of three or more gears. This can probably be accounted for by the fact that he had only clockwork in mind. The mills of this era seldom had trains of more than two gears engaged.

Clearly, Camus had the basis for a theory of mechanism of gear teeth, but it was not systematically and completely worked out, as in R. Willis [14].

Despite the mathematicians began investigating some curves aiming for their application for the purpose of gearing, no foundations in the theory of gearing has been made at that time.

The condition of contact of the interacting tooth flanks is the only contribution to the scientific theory of gearing attained in the pre-Eulerian period of the gear art. The initially proposed for mechanics of machines in a more general sense, the condition of contact of two machine elements can also be implemented with respect to gears. This condition is schematically illustrated in Fig. 6.4.

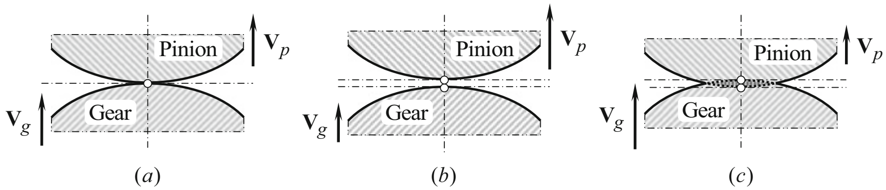



**Fig. 6.4** Condition of contact of a gear and a mating pinion tooth flank: (**a**) perfect contact, $\mathbf{V}_g = \mathbf{V}_p$, (**b**) separation of the tooth flanks occurs, $\mathbf{V}_g < \mathbf{V}_p$, and (**c**) interference of the tooth flanks occurs, $\mathbf{V}_g > \mathbf{V}_p$

In order to fulfill the condition of contact, the contact point, both of a gear and a mating pinion's tooth flanks, has to travel with an equal velocity along the common perpendicular, that is, the equality $\mathbf{V}_g \equiv \mathbf{V}_p$ has to be valid (see Fig. 6.4,a). In this scenario, the relative velocity is zero ($\mathbf{V}_{rel} = 0$). In a case $\mathbf{V}_g < \mathbf{V}_p$, the relative velocity is of a positive value ($\mathbf{V}_{rel} > 0$), and the components 1 and 2 separate from one another (see Fig. 6.4,b). Inversely, in a case $\mathbf{V}_g > \mathbf{V}_p$, the relative velocity is of a negative value in this scenario ($\mathbf{V}_{rel} < 0$), and the components 1 and 2 interfere into one another (Fig. 6.4,c). None of these two scenarios are permissible in geometrically accurate gearing.

The condition of contact is important to the theory of gearing. Unfortunately, it is not known who should be credited with this important accomplishment (it is likely this is because the condition of contact has been discovered for a more general case of mechanical engineering, and not for the purposes of gears).

Accomplishments in the field of gearing and gear art in the pre-Eulerian period of evolution of the gear art are briefly summarized immediately below:

- Various primitive designs of wooden gearing were developed with the purpose to transmit a rotary motion between two shafts.
- Gears that operate (a) on parallel shafts, namely, parallel-axes gearing; (b) on intersected shafts, namely, intersected-axes gearing; and (c) on crossed shaft, namely, crossed-axes gearing, are already known in pre-Eulerian period of evolution of the gear art.
- All the early designs of gearing operate at low rotations and transmit a low torque.
- No tooth flank geometry was taken into account, first of all because of the absence of necessity of doing that: Low-power-density wooden gears that operate at low rotations met the current customer requirements of that time.
- It became clear that the performance of a gear pair depends on a specific tooth profile of the mating gears, namely, teeth wear in gearing depends on the actual shape of a gear and a mating pinion tooth.
- Mathematicians indicated an interest to a special tooth profile of a gear and a mating pinion that allows the lowest tooth flank wear.
- Epicycloid is investigated as a potential candidate that can be used to shape the gear teeth, and epicycloidal tooth flank geometry was proposed for gearing that operates on parallel shafts.
- It was realized at that time that in gearing with epicycloidal geometry of the gear teeth, a rotation cannot be transmitted smoothly – with a constant angular velocity ratio.
- Involute of a circle was known at that time. However, there was no understanding that this curve best meets the needs of gearing.

The gearing that was common in the pre-Eulerian period of evolution of the gear art are far from being referred to as "perfect gearing" as they are not capable of transmitting a rotary motion smoothly. Geometrically inaccurate gears (those feature variable angular velocity ratio) are still in use even in the nowadays industry in cases when the rotations are low, and the transmitted power is low as well.

#### 6.1.1.2 The Time when the Fundamental Contribution by L. Euler Has Been Done – The Origin of the Scientific Theory of Gearing

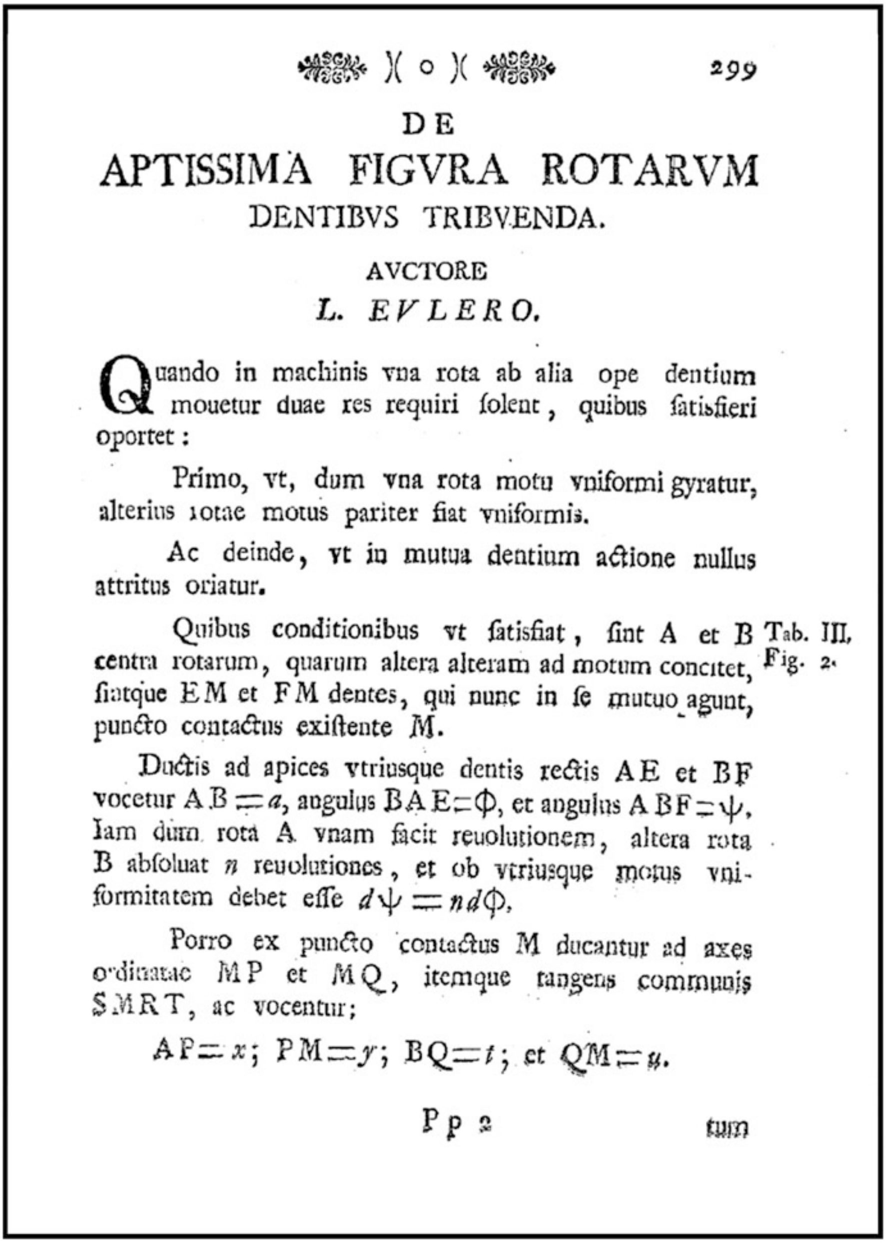
The interest of mathematicians (at the beginning of Desargues, de La Hire, and Camus, and later on of Euler) seems to have come from a desire to increase efficiency and reduce wear in mills of various types where, although the speeds were low, the load was not substantial. Indirectly, these problems were associated with the quality of the transmitted rotation, that is, with the smoothness of rotation of the output shaft. It could happen that the problem of design of geometrically-accurate gears can be traced back to this period of time.

It was Leonhard Euler[15] (1707–1783), a famous scientist (born in Switzerland), whom the origin of the scientific theory of gearing can be traced back to. In the first half of the 1750s, L. Euler (Fig. 6.5) has proved that involute of a circle is the best planar curve that fits to shape a gear tooth profile in geometrically accurate parallel-axes gearing [15]. The main contribution by L. Euler to the scientific theory of gearing is outlined in his two papers [15, 16].

In Euler's first paper (see Fig. 6.6) on gears [15] (written in the first half of the 1750s), he proved that in gearing, the tooth profile sliding is inevitable. As for the shape of the teeth, Euler in this paper did not succeed in going beyond what was already done by Camus. However, Euler's second paper (see Fig. 6.7) [16] (written presumably 10 years later) is very original.

**Fig. 6.5** Leonhard Euler (1707–1783)

[15] ***Leonhard*** **Euler** (April 15, 1707–September 18, 1783) was a pioneering Swiss mathematician and physicist



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# APTISSIMA FIGVRA ROTARVM DENTIBVS TRIBVENDA.

AVCTORE

*L. EVLERO.*

Quando in machinis vna rota ab alia ope dentium mouetur duae res requiri ſolent, quibus ſatisfieri oportet:

Primo, vt, dum vna rota motu vniformi gyratur, alterius rotae motus pariter fiat vniformis.

Ac deinde, vt in mutua dentium actione nullus attritus oriatur.

Quibus conditionibus vt ſatisfiat, ſint A et B centra rotarum, quarum altera alteram ad motum concitet, ſiatque EM et FM dentes, qui nunc in ſe mutuo agunt, puncto contactus exiſtente *M*. Tab. III. Fig. 2.

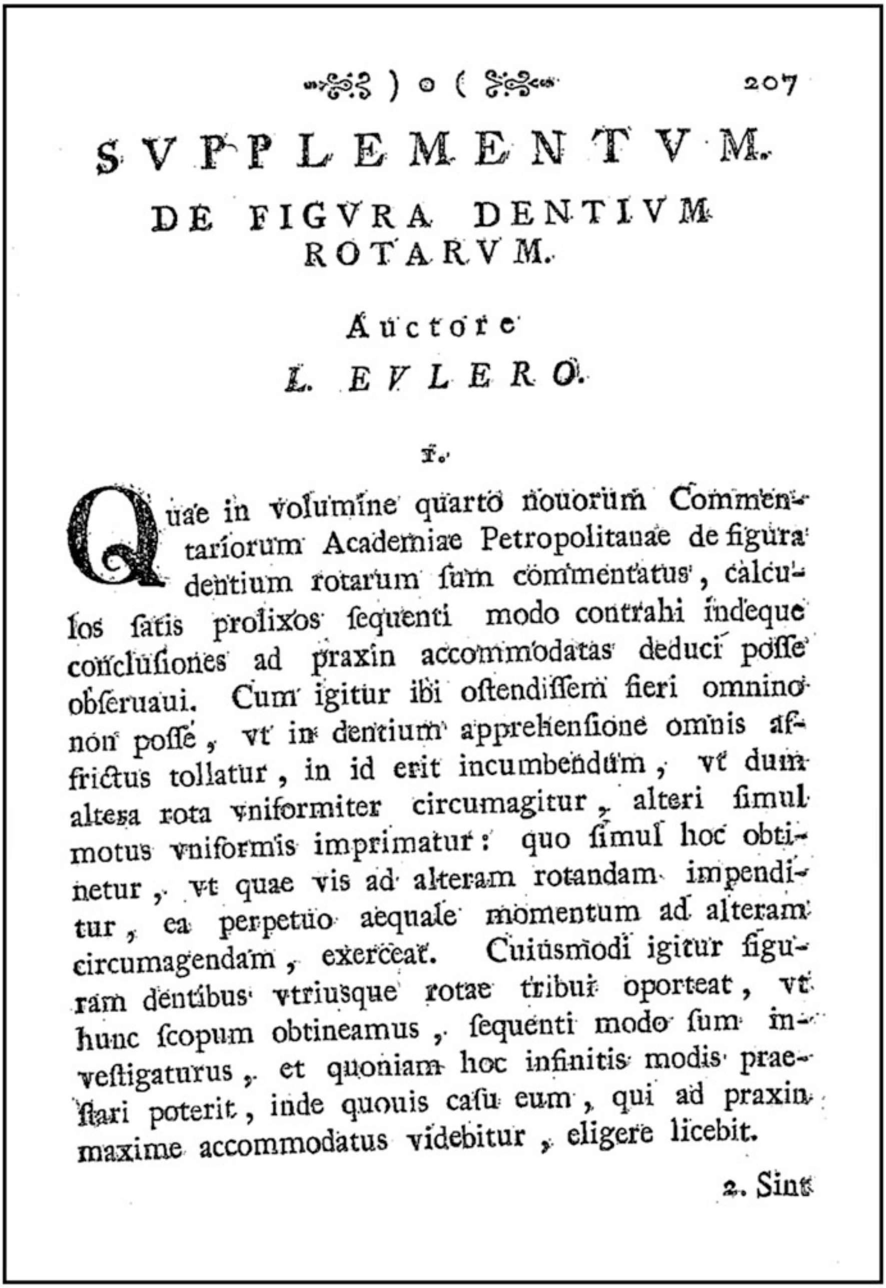
Ductis ad apices vtriusque dentis rectis AE et BF vocetur $AB = a$, angulus $BAE = \Phi$, et angulus $ABF = \psi$. Iam dum rota A vnam facit reuolutionem, altera rota B abſoluat $n$ reuolutiones, et ob vtriusque motus vniformitatem debet eſſe $d\psi = nd\Phi$.

Porro ex puncto contactus M ducantur ad axes ordinatae MP et MQ, itemque tangens communis SMRT, ac vocentur;

$$AP = x;\ PM = y;\ BQ = t;\ \text{et}\ QM = u.$$

Pp 2 tum

**Fig. 6.6** Title page of the paper by Euler, L. (1754-55), "De Aptissima Figure Rotarum Dentibus Tribuenda" ("On Finding the Best Shape for Gear Teeth"), in: *Academiae Scientiarum Imperiales Petropolitae, Novi Commentarii*, t. V, pp. 299-316



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SVPPLEMENTVM.

DE FIGVRA DENTIVM ROTARVM.

Auctore

L. EVLERO.

1.

Quae in volumine quarto nouorum Commentariorum Academiae Petropolitanae de figura dentium rotarum ſum commentatus, calculos ſatis prolixos ſequenti modo contrahi indeque concluſiones ad praxin accommodatas deduci poſſe obſeruaui. Cum igitur ibi oſtendiſſem fieri omnino non poſſe, vt in dentium apprehenſione omnis affrictus tollatur, in id erit incumbendum, vt dum altera rota vniformiter circumagitur, alteri ſimul motus vniformis imprimatur: quo ſimul hoc obtinetur, vt quae vis ad alteram rotandam impenditur, ea perpetuo aequale momentum ad alteram circumagendam, exerceat. Cuiusmodi igitur figuram dentibus vtriusque rotae tribui oporteat, vt hunc ſcopum obtineamus, ſequenti modo ſum inveſtigaturus, et quoniam hoc infinitis modis praeſtari poterit, inde quouis caſu eum, qui ad praxin maxime accommodatus videbitur, eligere licebit.

2. Sint

**Fig. 6.7** Title page of the paper by Euler, L., "Supplementum. De figura dentium rotarum". In: Novi Comm. Acad. Sc. Petropol, 1767. (Originally published in *Novi Commentarii academiae scientiarum Petropolitanae* 11, 1767, pp. 207-231)

### Involute Tooth Profile for Parallel-Axes Gearing

It is proven in these papers [15, 16] that in parallel-axes gearing the involute of a circle[16] is the only planar curve that minimizes friction between mating gear teeth, and, therefore, involute tooth profile is the only kind of tooth profile that should be used to design conjugate gear pairs. In these papers [15, 16], Euler also shows the grasp and precision of his great mathematical mind. He specifically states the conditions:

- Uniform rotary motion of both gears.
- In the mutual action of the teeth "nullus atritus oriatur," no interference between the mating teeth flanks (however, a gap between the mating teeth flanks, that is, equality of base pitches of the mating gears, is not considered yet).

The following conclusion can be drawn from the discussion above:

Conclusion 1: The scientific theory of gearing originated from Euler's two famous papers on the geometry of the ideal shape of gear teeth [15, 16] in parallel-axes gearing.

Conclusion 2: The scientific theory of gearing is tightly connected with the application of an involute of a circle to shape gear teeth in parallel-axes gearing. Despite the fact that involute of a circle was known to mathematicians long before the time, when L. Euler has carried out his research on the shape gear teeth, it was L. Euler who proposed to use involute of a circle in the design of geometrically accurate (i.e., perfect) parallel-axes gearing.

By now, the kinematics and geometry of parallel-axes involute gearing are investigated so extensively that there is no need to discuss here gearing of this particular design more in detail, as it is trivial and is outlined in textbooks on machine and mechanisms theory.

### The Euler-Savary Formula

Euler's papers on gear wheels [15, 16] are part of a development that started essentially with the investigation of the ordinary cycloid: the curve described by a point on the circumference of a circle when this circle rolls without slipping on a straight line [17]. In this paper [16] Euler derived a formula that is equivalent to the Euler-Savary formula[17] in the nowadays interpretation. It is remarkable that although Euler was merely studying a very specific subject, gear wheels, the Euler-Savary[18] formula belongs from a modern point of view to planar theoretical

[16]Invention of the involute tooth profile, which best fits the practical needs of the industry, is commonly credited to *Leonhard Euler* (1707–1783).

[17]The consequences from the *Euler-Savary formula* (the *involute tooth profile*, and the *conjugate action law*) are important to the theory of gearing, while the formula itself is of less importance.

[18]***Felix* Savary** (October 4, 1797–July, 15, 1841) was a French mathematician and mechanician

kinematics and has general validity. Euler in this context also discovered the so-called involute gearing, nowadays the most popular form of gearing.

When a planar motion is considered at a particular instant of time, a modern kinematician thinks of a particular position of a moving axode in relation to a stationary axode. The point where the two curves touch each other is the "instantaneous center of rotation" (or the "pitch point," in other terminology). Clearly, an arbitrary point $P$ of the moving plane describes a curve in the stationary plane. At a particular moment under consideration, point $P$ coincides with a particular point of the curve that it traces. The tangent to the curve in this particular point can be constructed easily by means of the pitch point. How about the center of curvature at this particular point? Nineteenth-century kinematicians have extensively studied the relation between the points of the moving plane and the corresponding centers of curvature of their trajectories in the stationary plane. This particular relation has many properties.

Refer to Fig. 6.8,a, where an arbitrary configuration of two interacting planar curves is shown. The stationary axode is $\pi_1$. The moving axode is $\pi_2$. The point $O$ at which the two axodes touch each other is the instantaneous center of rotation or pitch point at the moment that we are considering. The $k_2$ is a curve in the moving plane. The $k_1$ is the envelope in the stationary plane of the set of positions in the stationary plane of $k_2$. In the position under consideration, the curves, $k_1$ and $k_2$, touch each other at the point $C$. The points $N_1$, $N_2$, $M_1$, and $M_2$ are, respectively, the centers of curvature of the planar curves $k_1$, $k_2$, $\pi_1$, and $\pi_2$, corresponding to the points $C$ and $O$. Let $\theta$ be the angle between the common tangent to the axodes and the common perpendicular at $C$ to the curves $k_1$ and $k_2$. Then the following expression is valid:

$$\left(\frac{1}{ON_1} - \frac{1}{ON_2}\right) \cdot \sin\theta = \frac{1}{OM_1} - \frac{1}{OM_2} \tag{6.1}$$

This is the Euler-Savary formula or theorem. The variables $ON_1$, $ON_2$, $OM_1$, and $OM_2$ correspond to the directed line segments and are signed values. The pitch point, $O$, is the origin of a Cartesian coordinate system with $pr$ as positive $x-$axis and $pn$ as positive $y-$axis. Similarly, $O$ is also the origin of a Cartesian coordinate system $O\,\xi\,\eta$ with directed line segment $OC$ defining the positive direction of the $\xi-$axis. The two systems have the same orientation. As for the signs of the variables in the Euler-Savary formula, $ON_i$ is positive if moving from $O$ to $N_i$ is a move in the direction of the $\xi-$axis. $OM_i$ is positive if moving from $O$ to $M_i$ is a move in the direction of the $x-$axis.

A modern proof of the Euler-Savary formula was given in 1970 by G.R. Veldkamp [18].

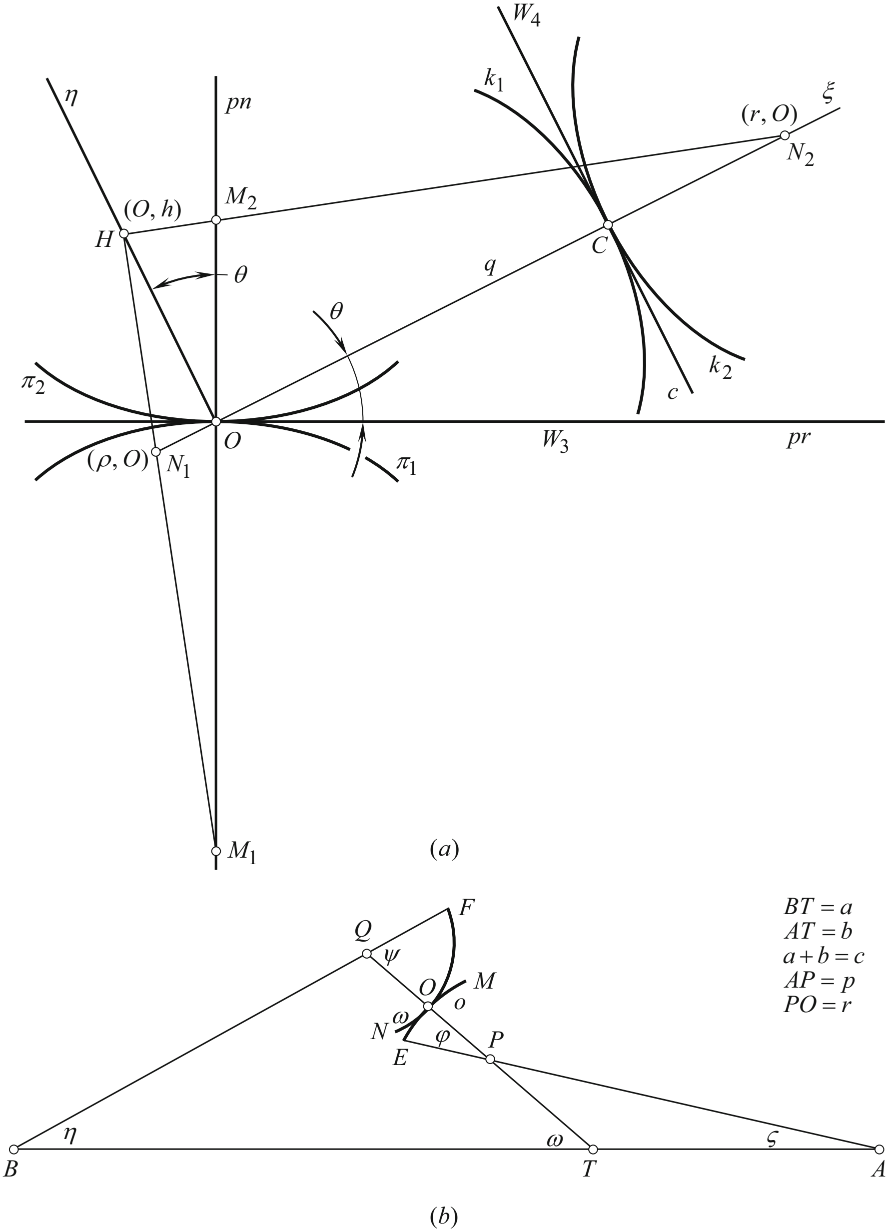


**Fig. 6.8** The schematics used by *L. Euler* for the derivation of the involute gear tooth profile. (Adapted from: Euler, L. "Supplementum. De Figura Dentium Rotarum". *Novi Commentarii adacemiae Petropolitanae* 11, 1767, pp. 207-231. (E330, *Opera omnia*, 17, pages 196-219). [29])

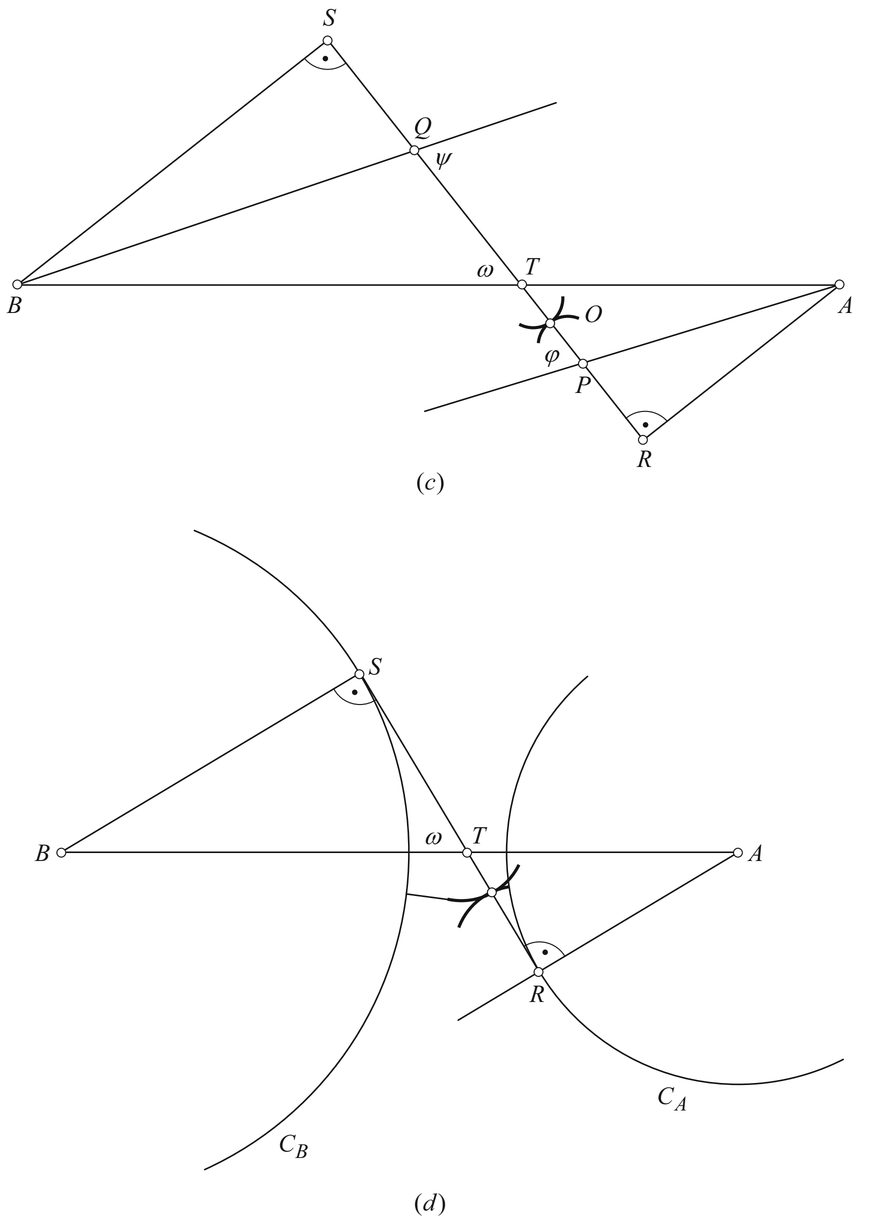


**Fig. 6.8** (continued)

## Leonhard Euler and the Euler-Savary Formula

In the first paper on gears [15] (written in the first half of the 1850s), and in the second paper [16], written presumably 10 years later, Euler didn't investigate general planar motion at a particular instant of time. Instead, he investigated the form of the teeth of gear wheels. The general validity of the formula that he discovered is an accidental spin-off of his research. This arises because, in general, just as for first and second order properties, a planar motion at a particular instant can be represented by a circle rolling without slipping on another circle. This is exactly what we are dealing with when we have planar circular gear wheels fulfilling the Euler's condition of a constant velocity ratio.

Euler started with Fig. 6.8,b. The points *A* and *B* are the centers of the two wheels. The curves, *EOM* and *FON*, are the two profiles of the teeth of the wheels. *O* is the point where the two profiles touch and the line perpendicular to the tangent in *O* cuts *AB* in the point *T*. When the gear wheels rotate, a moment $M_A$ about the center *A* yields a moment $M_B$ about the center *B*. It is easy to see that at the instant of time under consideration, the ratio of these two moments equals *BT*/*AT*. Euler argues that the condition of a constant velocity ratio implies that the ratio of these two moments must be constant, which leads to a kinematic result: The common normal in the point where the profiles touch each other intersects *AB* in a stationary point *T*. From a modern point of view, point *T* is the pole of the motion of the two gear wheels with respect to each other. The axodes of the two gear wheels are two circles, one of which is centered at *A* and another one is centered at *B*. The two circles touch one another at point, *T*. Euler has proved a kinematic result by means of a dynamic argument.

After making clear that the point *T* is stationary, Euler determined several relations between the parameters depicted in Fig. 6.8,b, and differentiated them. He basically considered a slight change in the position of the two profiles with respect to each other, using the fact that the common normal intersects the center-distance, *AB*, always at the stationary point *T*. After some calculations are executed, this reveals that the ratio of the angular velocities, $d\eta/d\xi$, is equal to the ratio *TA*/*TB*. For an arbitrary configuration of the profiles, Euler then derives the following expression that enabled him, in principle, to calculate the actual value of the radius of curvature $\rho'$ of the profile *NOM* out of the parameters of profile *EOM*:

$$\rho' = c \cdot \cos\omega - r - p \cdot \cos\varphi - \frac{b^2 \cdot \cos\omega \cdot d(p\sin\varphi)}{c \cdot d(p \cdot \sin\varphi) - a^2 \cdot d\varphi\cos\omega} \tag{6.2}$$

The explanation for the parameters in Eq. (6.2) can be found in Fig. 6.8,b.

In a particular configuration, one can assume, without loss of generality, that the profile *EOM* is a circle and that the center of curvature of the profile *NOM* coincides with *Q*. Then, the equalities, $dp = 0$ and $\rho' = OQ$, are valid. Moreover, if the foot points of the perpendiculars, from, respectively, *A* and *B*, on the line *PQ* (see Fig. 6.8,c), are designated as *R* and *S*, then it can be shown that Eq. (6.2) implies:

$$RT \cdot SQ \cdot TP + ST \cdot RT \cdot TQ = 0 \quad \text{or} \quad \frac{RT \cdot TP}{RP} + \frac{ST \cdot TQ}{SQ} = 0 \tag{6.3}$$

This is Euler's version of the Euler-Savary formula[19]. It can be shown that Eq. (6.3) is equivalent to Eq. (6.1).

The Euler-Savary formula has an amazing interpretation. It turns out that when $p$ coincides with $R$, then $Q$ coincides with $S$. And naturally Euler considered the possibility that this is the case during entire motion. The profiles then are involutes of the circles $C_B$ and $C_A$ (see Fig. 6.8,d). At this moment involute gearing has been discovered [17].

In Fig. 6.9, a schematic of the equivalent three bar mechanism that is used nowadays for the derivation of Euler-Savary formula is shown. In particular, a formula for the calculation of an actual value of the radius of curvature, $\rho_p$, is derived

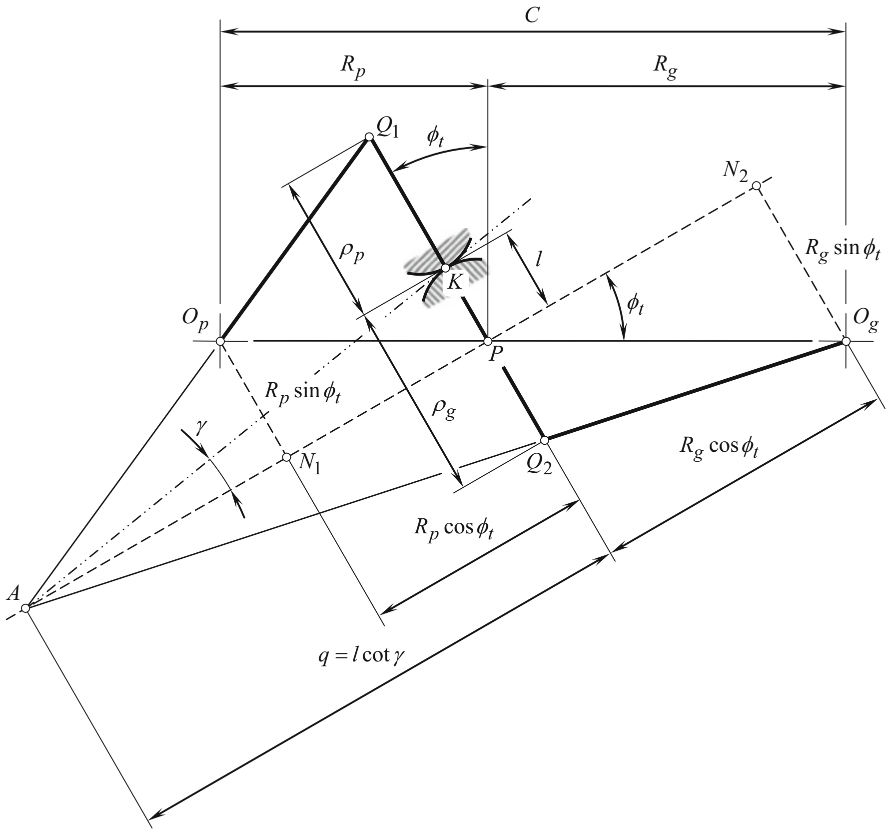


**Fig. 6.9** Schematic of the equivalent three bar mechanism that is used nowadays for the derivation of *Euler-Savary formula*

[19] ***Félix* Savary** was the first to derive the *Euler-Savary formula* in its modern form. *Savary*'s proof can be found in: *Leçons et cours autographiés, Notes sur les machines*, par le professeur *F. Savary*, Ecole Polytechnique, 1835–1836 (unpublished lecture notes; available in the Bibliothéque Nationale in Paris).

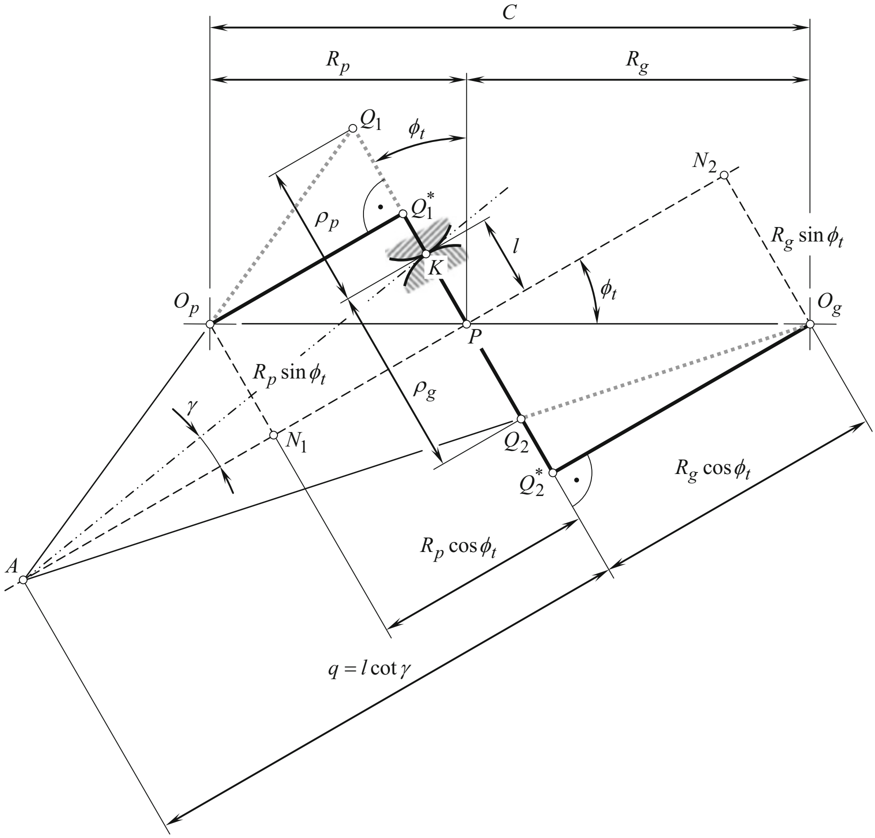


**Fig. 6.10** Schematic of the equivalent three bar mechanism that is used nowadays for the derivation of an involute gear tooth profile

based on the similarity of the triangles $\triangle AQ_1P$ and $\triangle AO_1N_1$. A formula for the calculation of an actual value of the radius of curvature, $\rho_g$, is derived based on the similarity of the triangles $\triangle AO_2N_2$ and $\triangle AQ_2P$.

A reduced case of the equivalent three bar mechanism that is used nowadays for the derivation of an involute gear tooth profile is depicted in Fig. 6.10. The joints are located at points $Q_1^*$ and $Q_2^*$, instead of points $Q_1$ and $Q_2$ correspondingly.

The proposed parallel-axes involute gearing with zero axes misalignment/displacement by L. Euler deserves to be referred to as "Eulerian gearing," or simply as "$E_u$−gearing":

**Definition 6.1.** Eulerian gearing (or just $E_u$−gearing, for simplicity) is a kind of parallel-axes involute gearing that features zero axes misalignment/displacement.

Such a terminology can be used for science purposes; similar to that are the terms "Newtonian fluid," "absolutely rigid body," and "absolutely black body" that can be easily found in the public domain.

Despite the invention of the involute tooth profile is of critical importance, at the time of L. Euler the difference between the line of action and the path of contact in a gear pair has been not understood in detail. This is, in much, because in the case of parallel-axes gearing, both the lines, namely, the line of action, $LA$, and the path of contact, $P_c$, are straight line segments that align to one another. Later on, this inconsistence in interpretation of involute gearing became the root cause of many mistakes when gearing of other designs were proposed and investigated. The main reason for that is as follows.

For gearing that operates on parallel shafts, involute tooth profile is the only tooth geometry under which the tooth flanks (a) are enveloping to one another and (b) are conjugate to each other (or, in other words, they are "reversibly enveloping profile," or just "$R_e$−profile," for simplicity [19]). Epicycloidal tooth flanks of the mating gears are enveloping to each other, but they are not conjugate to one another – they are not a type of "$R_e$−surfaces."

The principle of common tangent has been detailed by Euler. He specifically points out the need for the proper design of gear teeth to avoid friction and wear and indicates this application for clocks. Most clockmakers, however, ignored this, if they ever heard of it. Euler's treatment of gear teeth was very general and was carried out by the application of principles of analytic geometry using both differential calculus and integral calculus. He set up mathematical expressions for gears to move without friction between their teeth (actually for a minimum value of friction). Then, he set up expressions for gears to move with uniform motion.

L. Euler and F. Savary together have devised an analytical method for determining the curvature centers of gear teeth flanks.

The importance of the "conjugate action law" worked out by L. Euler (gears designed according to this law have a steady speed ratio) became correctly realized much later.[20]

For over a century the invention of involute tooth profile was not used in practice. The industrial revolution in Britain in the eighteenth century saw an explosion in the use of metal gearing. A science of gear design and manufacture rapidly developed through the nineteenth century. The invention and the beginning of application of steam and gas turbines that operate at high rotations and produce lots of power immediately turns the attention of engineers to involute gearing.

It should be stressed here that the concept of the "gear/pinion base pitch" (linear base pitch), as well as the concept of the "operating base pitch" (linear operating base pitch) in a parallel-axes gear pair was not known to L. Euler.

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[20] There is no evidence on whether or not *Euler* stressed [15, 16], on the difference between the line of action, $LA$, and the path of contact, $P_c$.

### *6.1.2 Accomplishments in the Theory of Gearing in the Time of the Fundamental contribution by L. Euler Are Briefly Summarized Immediately below*

- It is proven by L. Euler in the mid of the eighteenth century that involute tooth profile meets the best the requirements of parallel-axes gearing.
- The fundamental theorem of parallel-axes gearing was known due to Euler and Savary.
- There is no evidence that a difference between the line of action, *LA*, and the path of contact, $P_c$, was realized by Euler and Savary, as in cases of parallel-axes gearing these two lines align to each another.
- No significant accomplishments at that time are done in the area of intersected-axes, as well as crossed-axes gearing.

The invention of involute tooth profile for parallel-axes gearing is one of the cornerstone accomplishments in the scientific theory of gearing. This achievement is referred to as the beginning of the scientific theory of gearing.

#### 6.1.2.1 Post-Eulerian Period of Evolution of the Theory of Gearing

Since the time when L. Euler carried out his research on involute gearing, scientific theory of gearing got a significant impulse. Numerous contributions to the scientific theory of gearing have been done in the post-Eulerian period of evolution of the gear theory [1, 2, 9, 10]. Principal accomplishments in the scientific theory of gearing are outlined below in a chronological order [1].

Robert Willis and the Fundamental Theorem of Parallel-Axes Gearing

In the nineteenth century, a profound investigation of mechanisms in general sense has been undertaken by Robert Willis[21]. In his 1841 book [14] titled “Principles of Mechanisms,” R. Willis compiled the lectures for his students and knowledge about gears which could be used in practice. In the book, gearing was discussed by the author to the best extent possible in his time.

Despite the “fundamental theorem of parallel-axes gearing” was known to L. Euler, and to F. Savary, this theorem got an extensive recognition in Europe due to publication of the famous book by Robert Willis [14]. Because of this the “fundamental theorem of parallel-axes gearing” is often referred to as the “Willis’ theorem.” The latter is incorrect.

[21]***Reverend Robert* Willis** (February 27, 1800–February 28, 1875), an English academic, was a professor at Cambridge.

The "fundamental theorem of gearing" is known now mostly due to the 1841 book by R. Willis [14]:

Fundamental theorem of parallel-axes gearing (according to R. Willis): The angular velocities of the two pieces are to each other inversely as the segments into which the "line of action" divides the line of centers, or inversely as the perpendiculars from centers of motion upon the line of action.

Nowadays, the "fundamental theorem of parallel-axes gearing" is commonly referred to as "Camus-Euler-Savary fundamental theorem of gearing" (or as "*CES*−theorem of gearing," for simplicity).

As already stressed in this chapter of the book, in parallel-axes gearing, the line of action, *LA*, and the path of contact, $P_a$, represent two different straight lines that align with one another. The "fundamental theorem of parallel-axes gearing" gives an insight to make a difference between these two lines, *LA* and $P_a$. Unfortunately, in the meantime, this difference is not realized by most of the gear researchers.

Generalizing "*CES*−theorem of gearing" to a case of spatial gearing, one can come up with a conclusion (~2008, Prof. S.P. Radzevich), according to which:

Conclusion 3: Two smooth regular surfaces that travel in relation to one another are called "conjugate surfaces" if and only if the surfaces contact each other along a line, and a common perpendicular through every point of the line of contact intersects the axis of instant rotation of the surfaces.

And further:

Conclusion 4: Two (spatial) curves within two smooth regular surfaces that travel in relation to one another are called "conjugate curves" if and only if the curves are always in (point) contact, and a common perpendicular through contact point intersects the axis of instant rotation of the surfaces.

### A Mistake Committed (1842) by Theodore Olivier

The necessity of the theory of gearing for the needs of gear practitioners is realized for a long while. It is likely the 1842 book by Th. Olivier [7] is the first book ever titled as Theory of Gearing. This book by Th. Olivier is followed by the 1852 book by E. Sang [20], then by the 1886 master thesis by H.I. Gochman [8], as well as by numerous other books on the topic, published later on.

The research in the field of theory of gearing has been significantly affected by Theodore Olivier[22]. As early as 1842 a monograph by Th. Olivier on the theory of gearing [7] was published. This monograph is the first monograph ever to be titled Geometric Theory of Gearing ("Théorie Géométrique des Engrenages"). Therefore, it is incorrect to claim that F. Litvin is the author of the "first book on the theory of gearing," as some gear experts loosely do.

It is a right point to mention here that in the research undertaken by Th. Olivier, graphical methods developed in descriptive geometry are extensively used.

[22] ***Théodore*** **Olivier** (January 21, 1793–August 5, 1853), a French mathematician and engineer

In his 1842 book [7], Th. Olivier proposed two principles for generating tooth flanks of the gear teeth. These principles are commonly referred to as the first and the second "Olivier's principles of generating of enveloping surfaces." Later on, both these principles got an extensive usage by gear scientists. Instead, he considered the tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, just the enveloping surfaces (i.e., insufficient).

In a general case of gear meshing, both the principles proposed by Th. Olivier are incorrect, as the condition of conjugacy of the interacting tooth flanks of a gear, $\mathcal{G}$, and a mating pinion, $\mathcal{P}$, is not taken into account (the condition of conjugacy is just ignored).

The violation of the condition of conjugacy of the tooth flanks is a huge mistake committed by Th. Olivier.

Both Olivier's principles are valid just in reduced cases, when the traveling surfaces allow for sliding over themselves, and the sliding occurs in the direction of the enveloping motion. In these reduced cases, the principles by Th. Olivier are getting useless. Therefore, there is no reason in applying "Oliver's principles" for the purpose of generation of conjugate tooth flanks in a gear pair.

Due to the mistake committed by T. Olivier, no geometrically accurate gears can be designed, and only approximate gears can be designed instead. There is no chance to anticipate any significant improvements if gears are designed following "Oliver's principles." Therefore, Th. Olivier cannot be considered as a contributor to the scientific theory of gearing as his accomplishments are a kind of mistake that has negatively affected further development of the gear science.

Later on, that same mistake was committed (1886) by Ch. Gochman [8], the Russian researcher of gears and gearing. This mistake is also observed in all the books by F. Litvin (1914–2017), V.A. Shishkov [21], G.I. Shevel'ova [22], as well as in many other books by authors who adopted Olivier's approach.[23]

## Miscellaneous Improvements to the Gear Art

The proposed curved tooth configuration by A.C. Semple[24] in the first half of the nineteenth century (1848) captured the interest of many mechanical engineers and inventors.

The second ever known monograph on the theory of gearing has been published in 1852 by E. Sang [20]. This book, titled A New General Theory of the Teeth of Wheels, is nothing more rather than a compilation of the known achievements in the field of gearing. No contribution to the theory of gearing has been done by E. Sang [20].

[23]It is likely Dr. *Fraifeld* [23] is among those most affected (influenced) with the two "*Olivier principles*." Generating (hobbing) of gears for "*Novikov gearing*" is another example where ignorance of the condition of conjugacy resulted in insufficient accuracy of the machined gears.

[24]US Patent No. 5.647, *Rack and Pinion*, *Amzi C. Semple*, June 27, 1848

Among the experts in the field of gearing in that period of time, the name of Thomas Tredgold[25] should be mentioned as well. As a gear person he is mostly known for his proposed approximation of bevel gears (i.e., of intersected-axes gears) by appropriate cylindrical gears (i.e., by parallel-axes gears). The proposed approximation, that is, the so-called Tradgold approximation, significantly simplifies the calculation of bevel gearing in engineering practice.

### The Research Carried out by Chaim Gochman

In 1886 a new effort to evolve the theory of gearing was undertaken by Chaim Gochman[26]. In his master's thesis, he converted the results earlier obtained by Th. Olivier (who used graphical methods for solving problems in the field of gearing) into that same results obtained by means of the methods developed in analytical geometry [8]. As it is claimed on page 7 in the research by Ch. Gochman [8], no new scientific results are contributed by Gochman to those already obtained by Olivier [7]. The interested reader is referred to [4] for details on this research.

In his master's thesis [8], Ch. Gochman loosely considered the tooth flanks of a gear and a mating pinion only as surfaces enveloping to one another. The requirement of conjugacy of the mating tooth flanks was ignored, which is a huge mistake. The fulfillment of the condition of contact is sufficient only in the cases when "no" rolling motion is observed. Otherwise, this condition needs to be complemented with (a) the condition of conjugacy and (b) the equality of a gear base pitch and its mating pinion base pitch to the operating base pitches of a gear pair [1].

It must be clearly realized that the terms "conjugate surfaces" and "enveloping surfaces" are not equivalent to one another: all conjugate surfaces are enveloping to each other, but NOT vice versa, that is, not all enveloping surfaces are conjugate to one another. In detail, the committed mistake is discussed by Professor S. Radzevich [4].

The bottom line of this discussion is as follows: there is no chance to develop a scientific theory of gearing based only on the condition of contact of a gear and a mating pinion's tooth flanks, and ignoring:

(a) The condition of conjugacy of the interacting tooth flanks.
(b) Equality of base pitches of a gear and a mating pinion to operating base pitch of the gear pair, and so forth.

The direction of evolution of the gear theory that strictly follows the Olivier-Gochman approach represents the dead end in the evolution of the theory of gearing.

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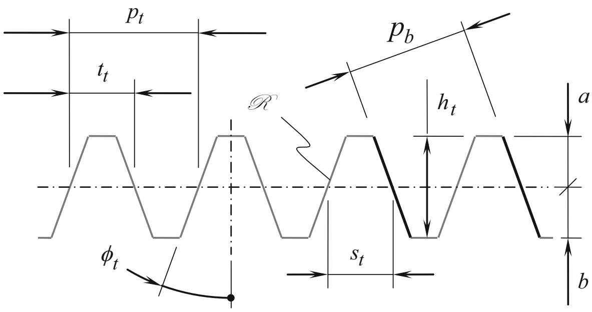
[25] ***Thomas*** **Tredgold** (August 22, 1788–January 28, 1829), an English engineer and author

[26] ***Chaim I.*** **Gochman** (1851–1916), a Russian mechanician (Novorossiysk University, Odessa, now in Ukraine)

**Fig. 6.11** Base pitch, $p_b$, in a basic rack, $\mathcal{R}$




## Equality of Base Pitches in Geometrically Accurate Parallel-Axes Gearing

The interaction of tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, of a gear and a mating pinion to a certain extent can be construed as that in a cam mechanism, especially in cases when just one pair of teeth is engaged in mesh. It is common in gearing that two or even more pairs of teeth are engaged in mesh at that same time. In order to make multiple engagements possible, base pitches in interacting tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, have to be equal to one another; therefore, fulfillment of an equality $p_{b.\ g} = p_{b.\ p}$ is a must in geometrically accurate parallel-axes gearing. Here, $p_{b.\ g}$ and $p_{b.\ p}$ are base pitches (see Fig. 6.11) of a gear and a mating pinion, correspondingly. Only involute gears feature base pitch, and only involute gear pairs are capable of transmitting a uniform rotary motion smoothly from a driving shaft to a driven shaft. No other gear tooth profiles are capable of ensuring that. Gears with non-involute tooth profile feature no base pitches. Therefore, as in non-involute gear pairs, base pitches do not exist, of course, they cannot be equal, and, ultimately, the gear pair is not capable of transmitting a uniform rotation smoothly.

The condition according to which base pitches of a gear and its mating pinion in a geometrically accurate gear pair have to be equal to one another is an important contribution to the scientific theory of gearing. Unfortunately, no name of a gear researcher is known who was the first to derive this significant accomplishment in the theory of gearing. Moreover, even the exact date when this accomplishment was attained also is not known. Hopefully, in the future, both the name and the date of the invention will be identified.

## Tooth Flank Geometry in Geometrically Accurate Intersected-Axes Gearing

For over a century involute parallel-axes gearing was the only kind of gearing for which perfect geometry of the tooth flanks (namely, the involute tooth profile) was known. The desirable geometry of the tooth flanks neither in intersected-axes gearing, nor in crossed-axes gearing, was known for over a century.

It was George Barnard Grant (1849–1917) who proposed (January 14, 1887) a correct method of generation of the tooth flanks in geometrically accurate intersected-axes gearing [24].

Grant's achievement got no extensive application in the industry, as for a long while (and even nowadays) the industry was satisfied with approximate gears that are easier and cheaper in production.

On the author's opinion, G. Grant underestimated his contribution to the scientific theory of gearing. Moreover, there is no evidence that Grant's achievement is properly valued even in nowadays gear community.

### Shishkov Equation of Contact, $\mathbf{n} \cdot \mathbf{V}_{\Sigma} = 0$

The "condition of contact" of the interacting tooth flanks of a gear, $\mathcal{G}$, and a mating pinion, $\mathcal{P}$, is the first scientific result of fundamental importance that can be used in the foundation of the scientific theory of gearing. The "condition of contact" is also known as the "enveloping condition" or "law of contact." The contact condition states that:

Condition of contact: At every point of contact of the tooth flanks of a gear, $\mathcal{G}$, and a mating pinion, $\mathcal{P}$, the projection of the relative velocity vector onto the common perpendicular to the interacting tooth flanks is zero.

The condition of contact of two interacting tooth flanks in a gear pair is known for centuries. Per the author's opinion, this important condition was already known to Camus (1733) [13] or even to Desargues.

As the theory of gearing evolves, new requirements to the theory arose. Since the time when the gear scientists started realizing the importance of the "condition of contact," the forms of its representation were different. In particular, the condition of contact (see Fig. 6.12) required an analytical representation. Numerous attempts were undertaken to derive an appropriate equation that reflects proper condition of contact of a gear, $\mathcal{G}$, and a mating pinion's, $\mathcal{P}$, tooth flanks.

Without going into details of the analysis of this particular problem,[27] it should be stressed here that finally the condition of contact is represented in the form of the dot product of the unit vector of the common perpendicular, $\mathbf{n}$, at point of contact of the tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, and the instant velocity vector, $\mathbf{V}_{\Sigma}$, of the resultant relative motion of the tooth flanks, $\mathcal{G}$ and $\mathcal{P}$. The dot product has to be equal to zero:

$$\mathbf{n} \cdot \mathbf{V}_{\Sigma} = 0 \tag{6.4}$$

[27] For details, the interested reader is referred to the paper by the author: Radzevich, S.P., "Briefly on the Kinematic Method and on the History of the Equation of Contact in the Form of $\mathbf{n} \cdot \mathbf{V} = 0$," In: *Theory of Mechanisms and Machines*, 2010, No. 1. Vol. 15, pp. 42–51. http://tmm.spbstu.ru

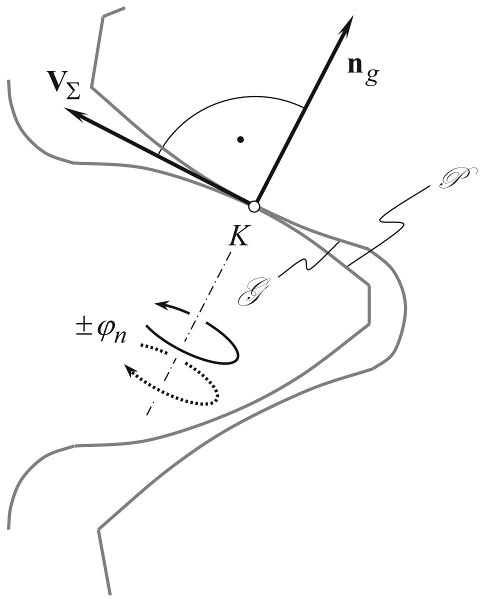



**Fig. 6.12** Permissible instantaneous relative motions in gearing

This was Prof. V.A. Shishkov who proposed (in 1940s–early 1950s, and not later 1948[28]) Eq. (6.4) to describe the condition of contact of two tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, [21, 25].

Here, $\mathbf{n}$ is the unit vector of the common perpendicular, and $\mathbf{V}_\Sigma$ is the linear velocity vector of the instantaneous resultant motion of the gear and the mating pinion.

Equation (6.4) is based on the fact that at common point(s) (points of contact, in other words), the linear velocity vector of the instantaneous resultant motion of the gear and the mating pinion, $\mathbf{V}_\Sigma$, and the unit vector of the common perpendicular, $\mathbf{n}$, have to be perpendicular to one another.

As it follows from the research undertaken by Prof. Radzevich [3], Prof. Shishkov is the first to represent the condition of contact of two smooth regular surfaces in the form of dot product $\mathbf{n} \cdot \mathbf{V}_\Sigma = 0$ of the unit vector of a common perpendicular, $\mathbf{n}$, by the vector of the velocity of the relative motion of the interacting surfaces at a point of their contact. The equation of contact in the form $\mathbf{n} \cdot \mathbf{V}_\Sigma = 0$ is known as "Shishkov equation of contact" [1, 3], and others.

The "Shishkov equation of contact" is a valuable contribution to the scientific theory of gearing. Nowadays, this equation is extensively used by many gear researchers. Unfortunately, this equation is often loosely supposed to be an equation of conjugacy, which is not correct.

The interested reader may wish to go to [3] for more details on "Shishkov equation of contact."

[28]It could happen that the equation of contact, $\mathbf{n} \cdot \mathbf{V} = 0$, can be found out even in earlier (before 1948) publications by Professor *V.A. Shishkov* – in his earlier papers, PhD thesis, and so forth.

## Principal Planes and Reference Systems Associated with Gearing

For a gear pair with a specified set of the design parameters, a corresponding vector diagram for the rotation vectors (as well as for the torques) can be constructed. Several principal directions are associated with a gear pair. These directions are defined by the rotation vectors of a gear, and a mating pinion, the instant rotation vector, and by the center-line. Use of the principal directions allows for construction of a set of principal planes, and principal reference systems associated with a gear pair. By means of the principal planes and principal reference systems, analysis of gearing of all kinds gets significantly simpler.

The set of principal planes is comprised of "pitch-line plane" (or just "$P_{ln}$−plane," for simplicity), "center-line plane" (or just "$C_{ln}$−plane," for simplicity), "normal plane" (or just "$N_{ln}$−plane," for simplicity), and the plane of action, $PA$. All these planes are shown in Fig. 6.13:

"Pitch-line plane" is the plane through the axis of instant rotation, $P_{ln}$, and the center-line, ℄, of the gear pair.

"Center-line plane" is the plane through the center-line, ℄, of the gear pair perpendicular to the pitch line, $P_{ln}$.

"Normal plane" is the plane through the plane-of-action apex, $A_{pa}$, perpendicular to the center-line, ℄, of the gear pair.

"Plane of action" is the plane through the axis of instant rotation (the pitch line), $P_{ln}$, at a transverse pressure angle, $\phi_{t.\ \omega}$, with respect to the center-line, ℄, of the gear pair.

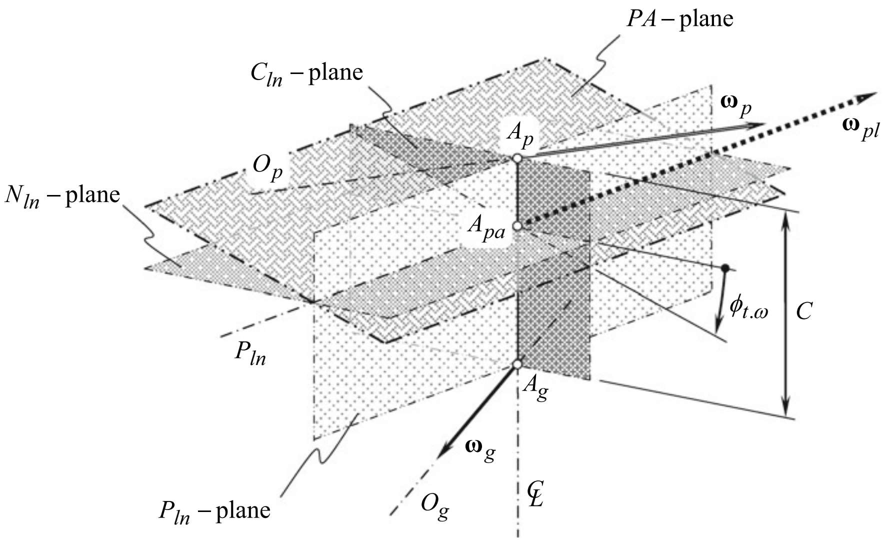



**Fig. 6.13** Principal planes associated with a gear pair: the pitch-line plane (the $P_{ln}$−plane), the center-line plane (the $C_{ln}$−plane), the normal plane (the $N_{ln}$−plane), and the plane of action (the $PA$−plane)

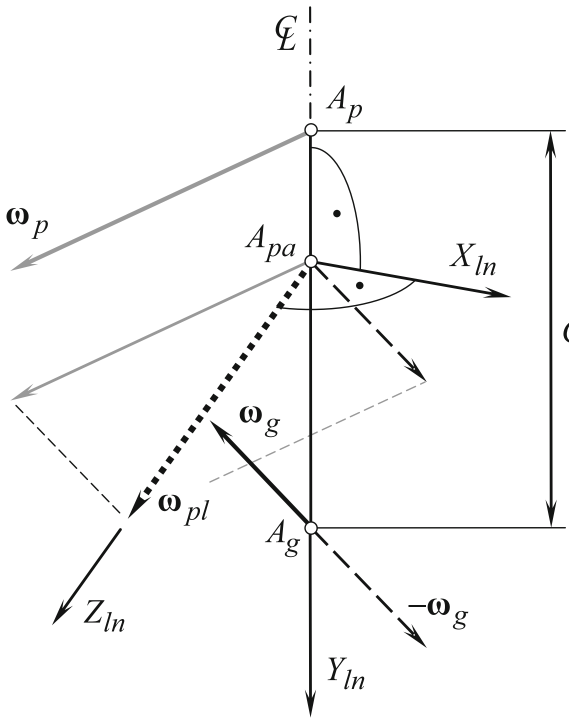



**Fig. 6.14** Principal reference system $X_{ln}Y_{ln}Z_{ln}$ associated with a gear pair

A set of principal reference systems is associated with the vector diagram of a gear pair as shown Fig. 6.14. The rotation vectors, $\boldsymbol{\omega}_g$ and $\boldsymbol{\omega}_p$, of a gear and its mating pinion are at a certain center-distance, $C$, and they cross one another. Points $A_g$ and $A_p$ are the points of intersection of the gear axis of rotation, $O_g$, and the pinion axis of rotation, $O_p$, correspondingly, with the centerline, ℄. The point $A_g$ is referred to as the "gear apex," and the point $A_p$ is referred to as the "pinion apex." The vector of instant rotation, $\boldsymbol{\omega}_{pl}$, of the pinion in relation to the gear is a vector through the point $A_{pa}$. This point is located within the centerline, ℄. The point $A_{pa}$ is referred to as the "plane-of-action apex."

Ultimately, five reference systems are introduced:

- The main reference system, $X_{ln}Y_{ln}Z_{ln}$, associated with the gear pair.
- The stationary gear reference system, $X_{g.\ s}Y_{g.\ s}Z_{g.\ s}$.
- The gear reference system, $X_gY_gZ_g$.
- The stationary pinion reference system, $X_{p.\ s}Y_{p.\ s}Z_{p.\ s}$.
- The pinion reference system, $X_pY_pZ_p$.

These reference systems are referred to as the "principal reference systems" associated with a gear pair.

Contact Geometry: Indicatrix of Conformity $Cnf_R(\mathcal{G}/\mathcal{P})$ at Point of Contact of Tooth Flanks

In order to analytically described the degree of conformity at a point(s) of contact of the tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, a planar characteristic curve was proposed by S. Radzevich [26], late 1970s–at the beginning of 1980s. This characteristic curve is commonly referred to as the "indicatrix of conformity, $Cnf_R(\mathcal{P} \mapsto \mathcal{G})$." The indicatrix of conformity is derived on the premise of "Dupin indicatrices," $Dup(\mathcal{G})$ and $Dup(\mathcal{P})$, of the tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, at a point of their contact.

The equation of the "indicatrix of conformity, $Cnf_R(\mathcal{G}/\mathcal{P})$" at a point of contact of a gear tooth flank, $\mathcal{G}$, and a mating pinion tooth flank, $\mathcal{P}$, is defined of the following structure (see Fig. 6.15):

$$\begin{aligned} Cnf_R(\mathcal{G}/\mathcal{P}) \quad &\Rightarrow \quad r_{cnf}(\varphi,\ \mu) \\ &= r_g(\varphi)\,\mathrm{sgn}\,R_g(\varphi) + r_p(\varphi,\ \mu)\,\mathrm{sgn}\,R_p(\varphi,\ \mu) \end{aligned} \tag{6.5}$$

Here, $R_g$ and $R_p$ are the radii of normal curvature of a gear and a mating pinion's tooth flanks; and $r_p = \sqrt{R_p}$; $\mu$ is the angle of local relative orientation of the tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, at a point $K$ of their contact; and $\varphi$ is the angular parameter of the indicatrix of conformity, $Cnf_R(\mathcal{G}/\mathcal{P})$.

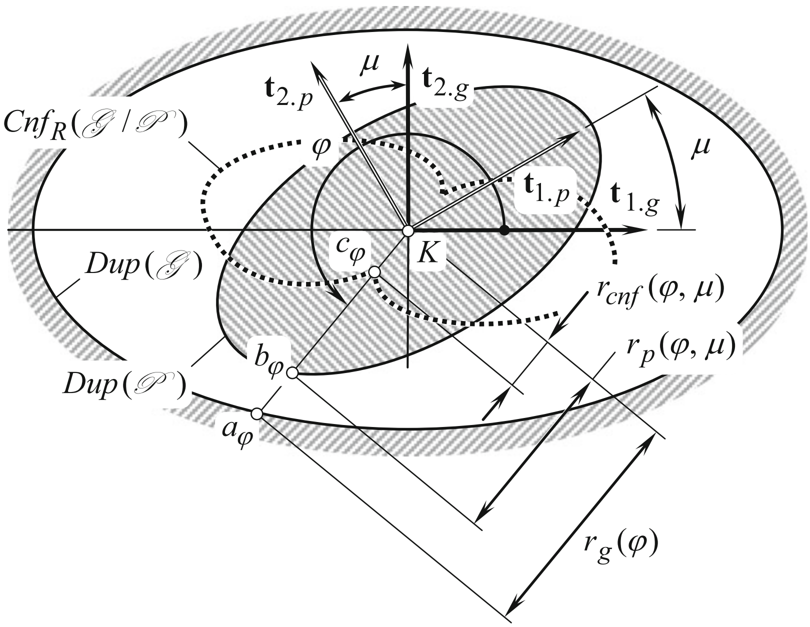



**Fig. 6.15** On the definition of the indicatrix of conformity, $Cnf_R(\mathcal{P} \mapsto \mathcal{G})$, at point of contact of tooth flanks, $\mathcal{G}$ and $\mathcal{P}$. (After Prof. S.P. Radzevich: Radzevich, S.P., *Differential-Geometric Method of Surface Generation*, Dr.Sci. Thesis, Tula, Tula Polytechnic Institute, 1991, 300 pages)

Indicatrix of conformity, $Cnf_R(\mathcal{G}/\mathcal{P})$, at a point of contact of two interacting tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, is vital for designing perfect gear pairs, and, especially, for solving a problem of synthesizing a most favorable gear pair for a particular application.

## Condition of Conjugacy $\mathbf{p}_{ln} \times \mathbf{V}_m \cdot \mathbf{n}_g = 0$ of Interacting Tooth Flanks for Gearing of all Kinds

The condition of conjugacy[29] of two interacting tooth profiles of a gear and a mating pinion is a bit tricky. Informally, the condition of conjugacy can be interpreted in the following manner.

Assume that a profile of one member of a gear pair is given. Then, the tooth profile of the mating member of the gear pair can be generated as an envelope to consecutive positions of the first member in its motion in relation to the second member. Then assume that the tooth profile of the second member of a gear pair is known, and the tooth profile of the first member of the gear pair is generated as an envelope to consecutive positions of the second member in its motion in relation to the first member. Then, compare the obtained tooth profiles of the first member of the gear pair with its original profile. If they are identical to one another, then the interacting tooth flanks are conjugate to one another. Otherwise, the interacting tooth flanks are not conjugate to one another.

The condition of conjugacy of interacting surfaces is more robust than the enveloping condition. All conjugate surfaces are enveloping to one another, but not vice versa – not all enveloping surfaces are conjugate.

In parallel-axes gearing, the problem of conjugacy of the tooth profiles/flanks has been solved in the eighteenth century (~1760) by L. Euler.

In involute gearing (see Fig. 6.16), the line of action, $LA$, and the path of contact, $P_c$, align to one another at every point of contact, $K$, of the tooth flanks $\mathcal{G}$ and $\mathcal{P}$ of the gear and the pinion, correspondingly. This is possible as both the line of action $LA$ and the path of contact $P_c$ are straight lines through the pitch point, $P$, at transverse pressure angle, $\phi_t$, to a perpendicular to the center line. This feature of involute gearing is the root cause of confusion as the line of contact and the path of contact are commonly not distinguished from one another in intersected-axes gearing, as well as in crossed-axes gearing.

At around 2008, condition of conjugacy of the tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, in intersected-axes gearing, and in crossed-axes gearing was formulated by Prof. S. Radzevich. To be conjugate, the tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, have to be designed so as:

(a) To retain the instant line of action, $LA_{\text{inst}}$, within the plane of action.
(b) To ensure that the straight line, $LA_{\text{inst}}$, intersects the axis of instant rotation, $P_{ln}$, at every angular configuration of the gears when they rotate.

[29] Conjugate tooth profiles/surfaces are also known as "*reversibly-enveloping*" profiles/surfaces (or just as $R_e-$profiles/surfaces, for simplicity) [19].

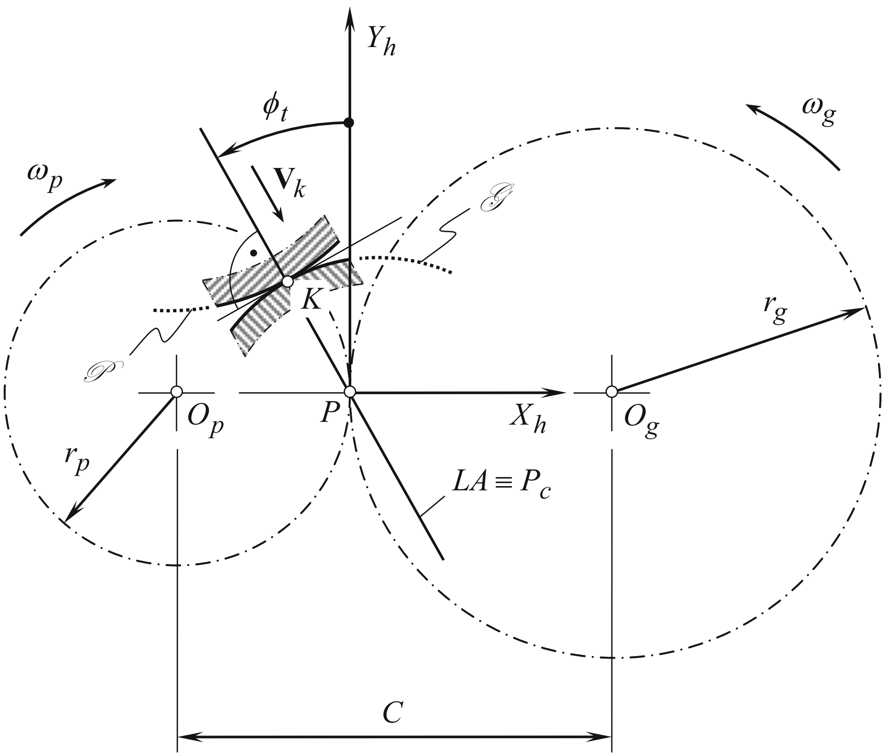



**Fig. 6.16** The line of action, *LA*, and the path of contact, $P_c$, in an involute gearing

Later on, the condition of conjugacy of a gear, $\mathcal{G}$, and a mating pinion's, $\mathcal{P}$, tooth flanks is described analytically by Prof. S. Radzevich (2017) in the form of a triple scalar product $\mathbf{p}_{ln} \times \mathbf{V}_m \cdot \mathbf{n}_g = 0$.

The condition of conjugacy of the tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, of a gear and its mating pinion is of critical importance when designing gears for high-power-density gear pairs, as well as of gear pairs for low-noise/noiseless transmissions.

## Angular Base Pitches: Operating Angular Base Pitch in a Gear Pair

In order to transmit a rotary motion between two shafts, at certain periods of time more than one pair of teeth needs to be engaged in mesh simultaneously. To meet this requirement base pitches of the mating gears must be equal to one another. This fundamental requirement[30] is known only for the cases of perfect parallel-axes gearing with zero axis misalignment.

[30]It is a right point to mention here that the author failed trying to identify the name of a gear researcher who should be credited with this fundamental requirement in the theory of gearing.

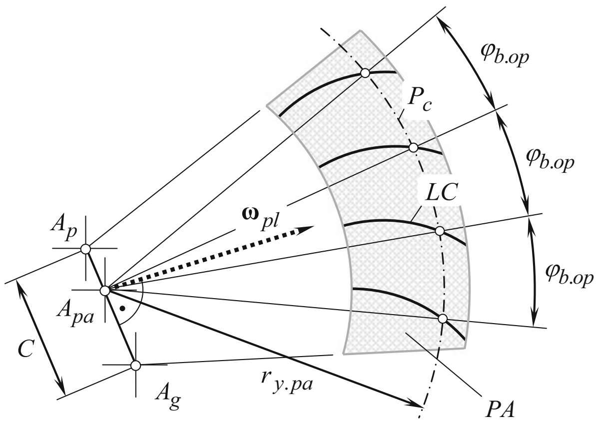



**Fig. 6.17** On the concept of equal angular base pitches of a gear and a mating pinion to operating angular base pitch of a gear pair. (After Prof. *S.P. Radzevich*: Radzevich, S.P., *Theory of Gearing: Kinematics, Geometry, and Synthesis*, 2nd Edition, revised and expanded, *CRC* Press, Boca Raton, FL, 2018, 934 pages)

The earlier discussed in this chapter of book the concept on equal linear base pitches of a gear and a mating pinion in geometrically accurate parallel-axes gearing is evolved (Prof. S. Radzevich, circa 2008 [1]) to the most general case, that is, to the case of crossed-axes gearing (intersected-axes gearing is viewed here as a reduced case of $C_a-$gearing). For this purpose, a "concept of angular operating base pitch of a gear pair, $\varphi_{b.\ op}$" is introduced. A gear angular base pitch, $\varphi_{b.\ g}$, has to be equal to $\varphi_{b.\ op}$, and the pinion angular base pitch, $\varphi_{b.\ p}$, also has to be equal to $\varphi_{b.\ op}$:

$$\begin{cases} \varphi_{b.g} \equiv \varphi_{b.op} \\ \varphi_{b.p} \equiv \varphi_{b.op} \end{cases} \tag{6.6}$$

The concept of equal angular base pitches of a gear and a mating pinion to an operating angular base pitch in the gear pair is illustrated in Fig. 6.17.

The "condition of equality of base pitches" of two mating gears [see a set of Eq. (6.6)] is a valuable contribution to the scientific theory of gearing. This condition is used when designing precision gearing of all kinds.

## Crossed-Axes Gearing with Line Contact between the Tooth Flanks (R-Gearing)

A problem of geometrically accurate parallel-axes gearing with line contact between the tooth flanks has been solved by L. Euler, who proposed (circa ~1760) involute gear tooth profile (or, in other words, "$E_u-$gearing"). A problem of perfect intersected-axis gearing with line contact between the tooth flanks has been solved by G. Grant, who proposed (1887) a method of generation of perfect bevel gear tooth flank. Gearing of this design is referred to as "$G_r-$gearing"). For the first time ever, the problem of geometrically accurate crossed-axes gearing with line contact between the tooth flanks (the so-called $R-$gearing) has been solved (~2008) by Radzevich [1]. Tooth flank $\mathcal{G}$ of a gear (and of a mating pinion $\mathcal{P}$) in crossed-axes gearing of this design is generated by a line of contact, $LC_{\text{des}}$, of a desirable geometry that travels together with the plane of action, $PA$, when the gears rotate. The tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, are viewed as a locus of the desirable line of contact, $LC_{\text{des}}$, considered in a corresponding reference system. The interested reader is referred to [1] for more detail description of the principal features of design of $R-$gearing.

### Scientific Classification of Gearing

An extensive use of vector representation of gear pairs made possible the development of a scientific classification of vector diagrams of gear pairs (Prof. S. Radzevich, circa 2008 [1]). Vector diagrams of gear pairs with constant values of the center-distance, $C$, the crossed-axes angle, $\Sigma$, and the gear ratio, $u$, the so-called $C\Sigma u-$constant gear pairs, as well as the so-called $C\Sigma u-$variable gear pairs, are covered by the classification. The classification of the gear pairs was further evolved to a scientific classification of gear pairs themselves.

### Geometrically Accurate Real Gearing

On the premise of the recent accomplishments in the scientific theory of gearing, a novel gear system is developed by Prof. S. Radzevich at around ~2008 [1]. This gear system is commonly referred to as $S_{pr}-$gearing. If gears in a $S_{pr}-$gear pair are manufactured to the tolerances for the gear accuracy, then the gear pair is insensitive to the axes misalignment that does not exceed the tolerances for the axes' misalignment. This means that the angular base pitch of the gear, and that of the mating pinion, are remained equal to the operating base pitch of the gear pair as long as the axes misalignment is within the tolerance band for the deviations.

### Generalized Form of Equation of Conjugacy of Interacting Tooth Flanks: For Gearing of all Kinds

The considered in this section of the book condition of conjugacy (see Fig. 6.18) of the interacting tooth flanks of a gear and a mating pinion, $\mathcal{G}$ and $\mathcal{P}$, provides a verbal description of the requirements to be meet by conjugate tooth flanks. Recently (2017), an equation of conjugacy of the tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, was derived (Prof. S. Radzevich, circa 2008 [1]):

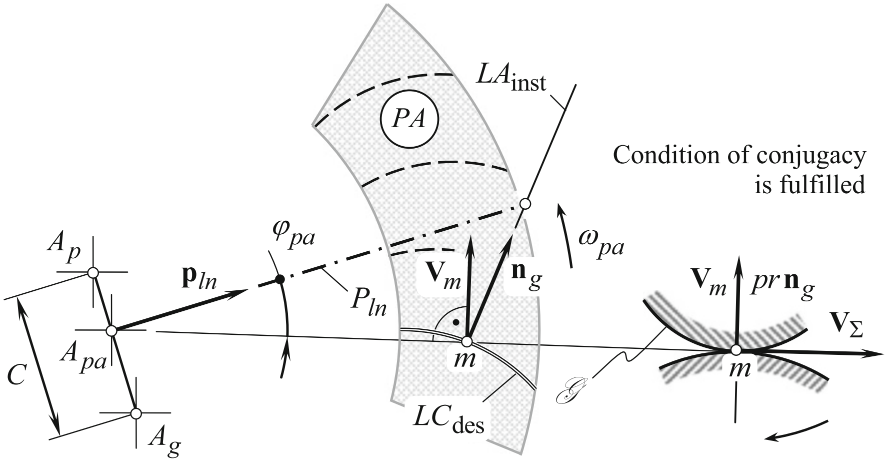



**Fig. 6.18** On derivation of equation of conjugacy, $\mathbf{p}_{ln} \times \mathbf{V}_m \cdot \mathbf{n}_g = 0$, of the tooth flanks, $\mathcal{G}$ and $\mathcal{P}$. (After Prof. S.P. Radzevich: Radzevich, S.P., *Theory of Gearing: Kinematics, Geometry, and Synthesis*, 2nd Edition, revised and expanded, *CRC* Press, Boca Raton, FL, 2018, 934 pages)

$$\mathbf{p}_{ln} \times \mathbf{V}_m \cdot \mathbf{n}_g = 0 \tag{6.7}$$

Here the following are designated (see Fig. 6.18):

$\mathbf{p}_{ln}$ is the unit vector along the axis of instant rotation, $P_{ln}$.
$\mathbf{V}_m$ is the linear velocity vector of a point of a desirable line of contact, $LC_{des}$, between the tooth flanks $\mathcal{G}$ and $\mathcal{P}$.
$\mathbf{n}_g$ is the unit vector of a common perpendicular at point of contact of the tooth flanks $\mathcal{G}$ and $\mathcal{P}$.

If the condition of conjugacy [i.e., specified by Eq. (6.7)] is fulfilled at every point of a desirable line of contact, $LC_{des}$, the gear pair designed this way is capable of transmitting smoothly an input steady rotation to the output shaft.

Accomplishments in the theory of gearing in the post-Eulerian period of evolution of the theory of gearing are briefly summarized immediately below:

- The fundamental theorem of parallel-axes gearing (i.e., the "Camus-Euler-Savary fundamental theorem of gearing") is formulated. Later on, this theorem was published in the book by Robert Willis [14], and sometimes is loosely referred to as "Willis fundamental theorem of gearing," which is incorrect.
- The importance of the "condition of contact" between two interacting tooth flanks (i.e., the "enveloping condition") is realized; various forms of representation of this important condition, both verbal and analytical, are known at that time.
- Investigation into intersected-axes and crossed-axes gearing started at this time.
- A huge mistake in the interpretation of the interaction between the tooth flanks of mating gears has been committed by T. Olivier [7] (1842), and repeated by C. Gochman [8] (1886). All the research in the field of gearing in the years since 1842 through the recent years are significantly affected by this mistake.

- Prof. Shishkov proposed to represent the earlier known condition of contact of the interacting tooth flanks of a gear and a mating pinion in the form of the dot product $\mathbf{n} \cdot \mathbf{V}_{\Sigma} = 0$. This equation of contact is a key equation in the kinematic method of surface generation. Commonly, this equation is referred to as "Shishkov equation of contact," $\mathbf{n} \cdot \mathbf{V}_{\Sigma} = 0$.
- The condition of conjugacy of interacting tooth flanks of a gear and a mating pinion is not understood, and in most cases this important condition is ignored. This is a consequence of the mistake committed by T. Olivier in the nineteenth century.
- The requirement according to which the base pitches of a mating gear and its mating pinion are construed only in part, and only for the case of perfect parallel-axes gearing. The concept of the operating base pitch of a gear pair is not realized at all.

The "fundamental theorem" of parallel-axes gearing and the "contact condition" (i.e., the "enveloping condition") can be considered as the main contribution to the scientific theory of gearing attained at this time.

In the period until the end of the nineteenth century, the development of the tooth flank profile geometry was more or less completed for the case of parallel-axes gearing. Since that time, involute gearing prevailed as the most advantageous shape of the gear teeth flanks.

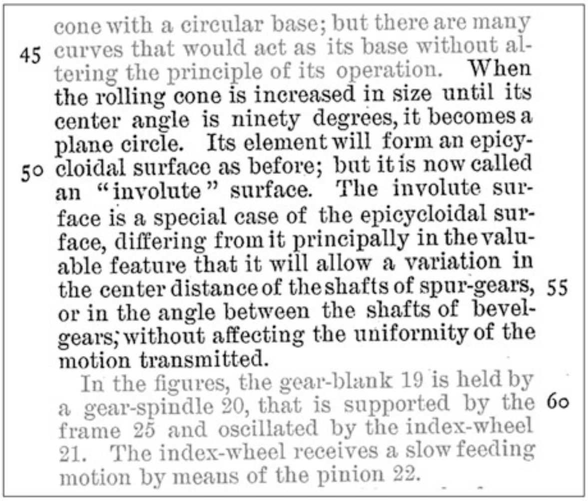
## *6.1.3 Other Contributions to the Field of Geometrically Accurate Gearing*

Regardless of unavailability of the scientific theory of gearing till the beginning of the twenty-first century, gear practitioners on their own have proposed numerous designs of geometrically accurate gearing.

### 6.1.3.1 Grant Bevel Gearing

In this regard, the invention [24] by George Grant[31] (see Fig. 6.19) should be mentioned first of all. The use of the invention [24] allows generating bevel gear tooth flanks for geometrically accurate intersected-axes gearing. This is due to that in one of the possible applications of the invention, ". . . the rolling cone is increased in size until its center angle is ninety degrees, and it becomes a plane circle. Its element

[31] ***George Barnard* Grant** (December 21, 1849–August 16, 1917) is considered one of the founders of gear-cutting industry in the USA (*Grant* established a gear-cutting machine shop in Charlestown, Massachusetts. When this business expanded, he moved the workshop to Boston, expanded it, and named it the *Grant Gear Works*. From this extremely successful establishment evolved the *Philadelphia Gear Works* and the *Cleveland Gear Works*. *George Grant* even wrote several very successful books on the subject, for example, *A Treatise on Gear Wheels*; *A Handbook on the Teeth of Gears, Their Curves, Properties and Practical Construction*, and so forth).


cone with a circular base; but there are many curves that would act as its base without altering the principle of its operation. When the rolling cone is increased in size until its center angle is ninety degrees, it becomes a plane circle. Its element will form an epicycloidal surface as before; but it is now called an "involute" surface. The involute surface is a special case of the epicycloidal surface, differing from it principally in the valuable feature that it will allow a variation in the center distance of the shafts of spur-gears, or in the angle between the shafts of bevel-gears, without affecting the uniformity of the motion transmitted.

In the figures, the gear-blank 19 is held by a gear-spindle 20, that is supported by the frame 25 and oscillated by the index-wheel 21. The index-wheel receives a slow feeding motion by means of the pinion 22.

**Fig. 6.19** The essential of the *G. Grant*'s invention [U.S. Pat. No. 407.437. *Machine for Planing Gear Teeth.*/G.B. Grant (1887)]

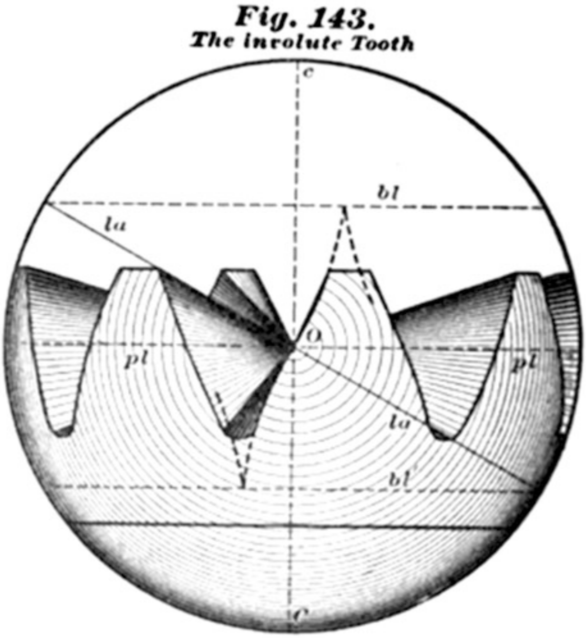



**Fig. 6.20** The involute tooth flank in a bevel gear according to G. Grant [see Fig. 143 in: Grant, G.B., *A Treatise on Gear Wheels*, 6th edition, Philadelphia Gear Works, Inc., Philadelphia, 1893, 105 p.]

will form an epicycloidal surface as before, but it is now called an "involute" surface" (see Fig. 6.19). Therefore, the bevel gear tooth flanks are generated by the describing method adopted to the case of intersected-axes gearing, that is, bevel gearing. This is a significant scientific achievement by G. Grant in the field of scientific theory of gearing. Fig. 6.20 is a good evidence of perfect tooth flank

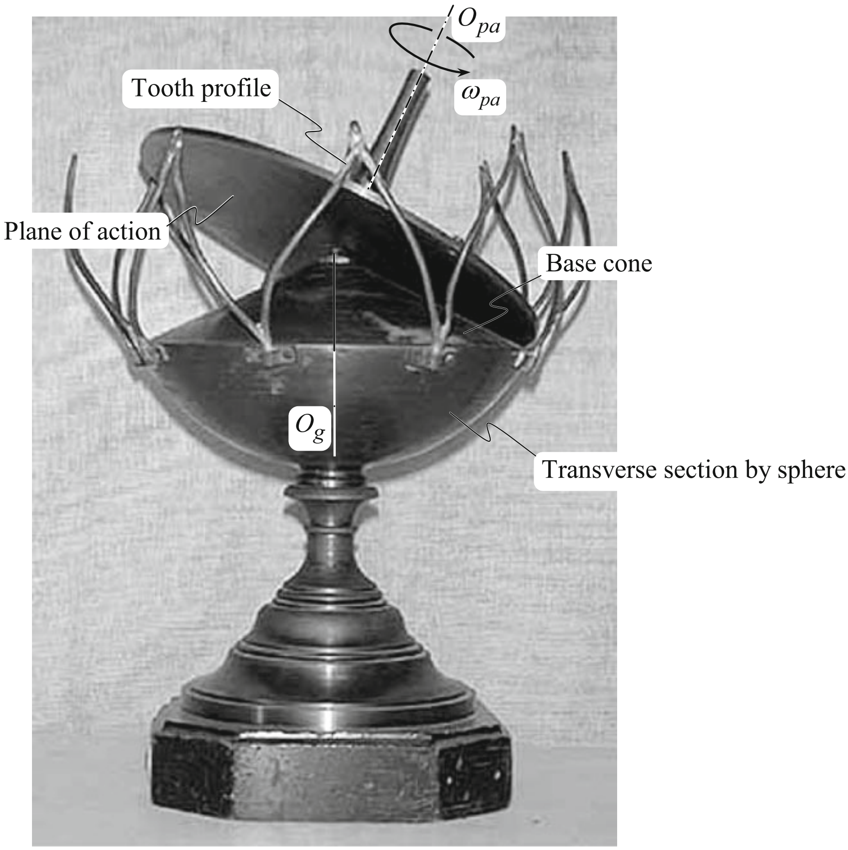



**Fig. 6.21** Demonstration of principal features of meshing in a bevel gear pair

geometry in a bevel gear, correctly realized by G. Grant at the end of the twenty-first century. An elementary device (see Fig. 6.21) was used in the past to demonstrate the principal features of meshing in a bevel gear pair.

The contribution by G. Grant is incomplete, as he proposed only a method of generation of tooth flanks of a gear for intersected-axes gear pairs ($I_a$−gearing). The concept of the "gear/pinion angular base pitch," as well as the concept of the "operating angular base pitch" of a gear pair, was not known to G. Grant.

However, G. Grant was a gear practitioner, and not a researcher, and (per the author's personal opinion) he did not properly value this as his accomplishment, which is of significant importance to the scientific theory of gearing. In addition, in the time of G. Grant, there was no necessity in more accurate bevel gears compared to those produced by the gear generating method. Because of this, the invention by G. Grant was forgotten for over a century.

#### 6.1.3.2 Contribution by Professor N.I. Kolchin

In the mid of the twentieth century, an interesting analytical research in gearing (in bevel gearing in particular) has been undertaken by Professor A.I. Kolchin of the USSR [27]. Professor A.I. Kolchin analytically described the results discovered and known in the public domain before his book was published. However, his contribution to the theory of gearing was important as a profound mathematical analysis of gears has been started from his research [27].

#### 6.1.3.3 Novikov Conformal Gearing

In the late 1940s and at the beginning of 1950s, an extensive research work in the field of gearing has been carried out by Dr. M.L. Novikov[32] in Moscow, at Military Aviation Engineering Academy. Ultimately, a novel design of high-performance gearing was proposed [28, 29]. Later on, the results of the research were summarized in the doctoral thesis [30] and in the monograph [31] by Dr. M.L. Novikov.

The proposed design of gearing features "to-convex-to-concave" contact between the interacting tooth flanks of a gear and a mating pinion. The gear designer is free to design the rest of the gear and the pinion tooth profiles.

When Professor M.L. Novikov carried out his research in the field of conformal gearing, he loosely assumed that in order to transmit a uniform rotary motion, the gear teeth do not need to have special shapes, such as the involute of a circle. He meant that, if a gear is made helical, then the helix itself can ensure uniform angular motion and tooth profiles can then be chosen with a view to minimizing contact stresses. This is a bit confusing: in order to transmit a rotation smoothly, the mating tooth profiles must be either involute or, in a degenerate case, they can feature the "involute tooth point" geometry.

"Novikov gearing" is a type of helical gearing that has a zero length of the field of action, that is, the equality $Z_a = 0$ is valid in "Novikov gearing" (this entails a zero transverse contact ratio, $m_p = 0$, in "Novikov gearing"). The equality of the base pitch of the gear and the pinion, to the operating base pitch of the gear pair, is the principal feature of "Novikov gearing" that distinguishes it from helical non-involute gearing of other types.

It is customary to associate "Novikov gearing"[33] with the patent "Gear Pairs and Cam Mechanisms Having Point System of Meshing" [29]. Evidence can be found out in scientific literature revealing the unfamiliarity of the gear community around the world with this original publication [29] on "Novikov gearing" (see Appendix H

[32]***Mikhail L*. Novikov** (March 25, 1915–August 19, 1957), a famous Soviet gear researcher

[33]The first pair of "*Novikov gearing*" made of aluminum alloy (a pre-prototype) was cut on April 25, 1954, by a disk-type mill cutter. For testing, 15 gear pairs were machined in the summer of 1954 by the disk-type mill cutter. Hobs for cutting gears for "*Novikov gearing*" were proposed later on by Professor *V.N. Kudr'avtsev* (as early as in 1956) – this is a huge mistake committed by Professor *V. N. Kudr'avtsev* to cut gears for "*Novikov gearing*" by hobs.

for details). As early as 1955, before the invention application was filed, a doctoral thesis [30] on the subject had been defended by M.L. Novikov. The author's familiarity with the practice of defending the doctoral thesis adopted in the former Soviet Union allows an assumption that the concept of "Novikov gearing" had been proposed in the late 1940s. After M.L. Novikov was granted with the patent [29], a monograph by him was published [31]. The concept of "Novikov gearing" is discussed in detail in the two aforementioned valuable sources [30, 31]. Unfortunately, none of them are quoted by the gear experts in Western countries and in the USA. This makes it possible a conclusion that gear experts around the world are not familiar with these two valuable sources of information on "Novikov gearing."

Formally, in "Novikov gearing," the tooth flanks have circular arc profile. Actually, as it has been shown later by Professor S.P. Radzevich [32] that "Novikov conformal gearing" is a reduced type of involute gearing in which the involute tooth profile is shrunk to a point, and the rest of the tooth profiles are shaped in the form of a circular arc. Because of this, "Novikov conformal gearing" is a kind of geometrically accurate gearing (a reduced type of involute gearing) that is capable of transmitting a steady rotation smoothly.

#### 6.1.3.4 Contribution by Professor V.a. Gavrilenko

An extensive research in the field of gearing in the 1930s through the 1960s has been carried out by Professor V.A. Gavrilenko[34]. He spent decades on extensive research in the field of gearing, particularly in the geometrical theory of involute gearing. In the author's opinion, the most systematic discussion on involute gearing ever can be found in the monograph by V. Gavrilenko [33]. Unfortunately, the fundamental monographs by V. Gavrilenko are not known for the most of gear experts neither in Europe nor in the USA.

#### 6.1.3.5 Contribution by Jack Phillips

An intensive research into spatial involute gearing was undertaken by Prof. J.R. Phillips[35] [34] who is credited with a new look and in in-depth understanding of the kinematics and the geometry of involute gearing with crossing axes of rotation of driving and of driven gears.

[34]***Vladimir A.*** **Gavrilenko** (June 21, 1899–June 6, 1977), Doctor (Engineering) Sciences and Professor of Mechanical Engineering (Bauman State Technical University, Moscow, Russia)

[35]***Jack Raymond*** **Phillips** (July 18, 1923–January 11, 2009), a famous Australian gear expert (mechanician)

#### 6.1.3.6 Contribution by Walton Musser

In the late 1950s, Walton Musser[36] proposed a novel kind of transmission, the so-called harmonic drive. Although this invention revolutionized the theory of "machines and mechanisms," harmonic drive is not a kind of gear drives in the sense considered in this monograph (gear drives consist of three components, namely, driving gear, driven gear, and the gear housing, while harmonic drive consists of four elements: a wave generator, flex-spline, stationary ring-gear, and the housing). This is the only reason why harmonic drives are not discussed in this monograph; this kind of transmission is out of the scope of the book.

Accomplishments in the field of gearing in that period of time are briefly summarized as follows:

- A breakthrough invention in the field of intersected-axes gearing has been made by G. Grant. He proposed a Machine for Planing Gear Teeth (U.S. Pat. No. 407.437, [24]) that is capable of machining perfect straight bevel gears. The geometry of a straight bevel gear tooth flank (that is equivalent to the involute of a circle in cases of parallel-axes gearing) is proposed by G. Grant for the case of intersected-axes gearing.[37]
- A novel design of conformal gearing was proposed by Dr. M. Novikov [29].

Grant's invention [24] is an important contribution to the theory of gearing. Novikov's invention completely aligns with the well-developed theory of parallel-axes involute gearing, as "Novikov conformal gearing" is a reduced case of involute gearing.

### *6.1.4 Developments in the Field of Approximate Gearing*[38]

To meet the current needs of the industry, practical gear engineers proposed numerous approximate designs of gearing. Initially when the designs were proposed, it loosely assumed that each of them is capable of transmitting a rotation smoothly. Unfortunately, it was shown later on that they do not meet all the requirements perfect gears needs to meet.

[36]***Walton Clarence* Musser** (April 5, 1909–June 8, 1998), a famous American inventor; he is the inventor of the "*harmonic drive*" (1957).

[37]Per the author's opinion, *G. Grant* did not realize the importance of his invention. In the time of *Grant*, the industry was fulfilled with the available on the market approximate gears; no interest to precision (and more costly) bevel gears was indicated by the industry at that time.

[38]For more in detail discussion on manufacture of gears for approximate gearing, the interested reader may wish to go to Chap. 1 "Gears: Brief Notes on the History of Methods of Machining Gears and of Design of Gear Cutting Tools" in the book: Radzevich, S.P., *Gear Cutting Tools*: *Science and Engineering*, CRC press, Boca Raton, Florida, 2017, 606 pages.

#### 6.1.4.1 Samuel Cone Double-Enveloping Worm Gearing

First rudimentary "double-enveloping" worm gear drive was known since the times of Leonardo da Vinci [11]. Nowadays double-enveloping worm gearing was proposed as early as in 1891 by Dr. Friedrich Wilhelm Lorenz[39] of Germany. In his invention Dr. Lorenz proposed methods to generate the worm and the gear of the double-enveloping worm-gear drive, and then he had received two patents for these accomplishments. A bit later (at a round 1920) and independently a similar double-enveloping worm gearing was proposed by Mr. Samuel Cone[40] of the USA. Wilhelm Lorenz and Samuel Cone understood very well the advantages of the drives they had invented, particularly, the increased load capacity due to the higher contact ratio in comparison with that of conventional worm-gear drives. Although the geometry of Lorenz and Cone's drives differs, both types offer this advantage.

Double-enveloping worm gearing is an example of approximate gearing as it does not meet all three fundamental laws of gearing [1].

#### 6.1.4.2 Approximate Bevel Gearing

Early accomplishments in the field of bevel gearing are tightly connected with the name of William Gleason[41]. In 1874, his invention of the straight bevel gear planer for the production of bevel gears with straight teeth substantially advanced the progress of gear making.

The early part of the twentieth century was the beginning of the automotive industry, which required a broader application of bevel gears to transform rotation and power between intersected axes. In the 1920s, automotive industry designers also needed (a) a gear drive to transform motions and power between crossed axes and (b) a lower location for the driving shaft. The Gleason Works engineers met these needs with pioneering developments directed at designing new types of gear drives and the equipment and tools to generate the gears for these drives.

The proposed designs of bevel gears in the nowadays industry are examples of approximate gearing as they are developed and manufactured based on application of the imaginary straight-sided crown gear (basic crown rack). Because of this, nowadays bevel gears of all kinds, that is, straight bevel gears, skew bevel gears, spiral bevel gears, and others, both face-milled and face-hobbed, do not meet all three fundamental laws of gearing [1].

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[39] ***Friedrich Wilhelm* Lorenz** (1842–1924), Doctor of Engineering, inventor, and founder of the *Lorenz Company*

[40] ***Samuel I.* Cone** (1842–1924), a civilian machinist and draftsman, an American inventor of double-enveloping worm gearing

[41] ***William* Gleason** (1836–1922), founder of *The Gleason Works*, Rochester, NY

#### 6.1.4.3 Approximate Crossed-Axes Gearing

The concept of the gearing that operates on crossing shafts can be traced back to the times of Leonardo da Vinci [11].

The need for more accurate and quieter running gears became obvious with the advent of the automobile. Although the hypoid gear was within our manufacturing capabilities by 1916, it was not used practically until 1926, when it was used in the Packard automobile. The hypoid gear made it possible to lower the drive shaft and gain more usable floor space. By 1937 almost all cars used hypoid-geared rear axles.

The success with the design, manufacture, and application of the contemporary crossed-axes gearing is credited in much to two famous gear experts, namely, Nikola Trbojevich (also known as Nicholas Terbo) and Ernest Wildhaber.

Nikola Trbojevich, a world-known research engineer, mathematician, and inventor, was a nephew and friend of Nikola Tesla. Mr. Trbojevich[42] held nearly 200 US and foreign patents, principally in the field of gear design.

Mr. Trbojevich's most notable work that brought him international recognition was the invention of the "Hypoid gear." First published in 1923, it was a new type of spiral bevel gear employing previously unexploited mathematical techniques. The "Hypoid gear" is used on the great majority of all cars, trucks, and military vehicles today. Together with his invention of the tools and machines necessary for its manufacture, the "Hypoid gear" became an integral part of the final drive mechanism of automobiles by 1931. Its effect was immediately apparent in that the overall height of rear-drive passenger automobiles was reduced by at least four inches.

Ernest Wildhaber[43] is one of the most famous inventors in the field of gear manufacture and design. He is granted with 279 patents on gearing, some of which have a broad application in the gear industry because of his work as an engineering consultant for The Gleason Works. The hypoid gear drive is one of the most famous inventions by Dr. Wildhaber. He proposed different pressure angles for the driving and coast tooth sides of a hypoid gear, which allowed him to provide constancy of the tooth top-land.

The proposed designs of crossed-axes gears in the nowadays industry are examples of approximate gearing as they are developed and manufactured based on application of the imaginary crown gear with straight-sided profile (basic crown rack). Because of this, nowadays crossed-axes gears of all types, both face-milled and face-hobbed, do not meet all three fundamental laws of gearing [1].

[42]***Nikola John*** **Trbojevich** (May 21, 1886–December 2, 1973), also known as *Nicholas J. Terbo*, a world-known research engineer, mathematician, and inventor, held the basic patent for the *Hypoid Gear*.

[43]***Ernest*** **Wildhaber** (1893–1979), Doctor of Engineering, h.c., Inventor, and consultant for *The Gleason Works*

#### 6.1.4.4 Face Gearing

Face gearing can be viewed as a reduced case either of intersected-axes gearing, or of crossed-axes gearing when the pitch cone angle increases to the right angle. All known designs of face gearing, both intersected-axes gearing and crossed-axes gearing, are approximate gearing as they do not meet all three fundamental laws of gearing [1]. The face cutting technique used to produce there crossed-axes gears is supplied by these three companies (The Gleason Works, Klingelnberg-Oerlikon, Yutaka Seimitsu Kogyo, LTD) is based upon an empirical and manufacturing technology that predates the World War II.

Accomplishments in the field of gearing in that period of time can be briefly summarized as follows:

- Double-enveloping (approximate) gearing was proposed by Wilhelm Lorenz of Germany (1874), and a bit later (at a round 1920) by Samuel Cone of the USA.
- Design of and methods for machining of approximate hypoid gearing were proposed by Nikola Trbojevich, and later on improved by Ernest Wildhaber, both of the USA.
- Face gearing are widely used in the design of Fellow's gear shaping machines.

The most significant contributions to the field of gearing at that time are made in the field of approximate gearing: to their design and production.

### *6.1.5 Theory of Gearing at the Beginning of the Twenty-First Century: State of the Art*

It should be stated here from the very beginning that no self-consistent (or potentially self-consistent) scientific theory of gearing is developed by the beginning of the twenty-first century (by the year of ~2010).

Among others, a self-consistent scientific theory of gearing must possess two important properties.

First, it must cover all known designs of gears and gearing with no exclusion.

Second, it must cover all (with no exclusion) unknown yet designs of gears and gearing, that is, the theory must possess the property to predict novel designs of gears and gearing.

All the books published so far under the title "Theory of Gearing" [starting from the first (1841) book by Théodore Olivier [7], and ending with the latest publications in the field – by the year of ~2010] consist no scientific theory of gearing. These books cannot be referred to as a "theory of gearing"; rather they are collections of known achievements in the field of gearing, having no ability to predict novel unknown designs of gears and gearing.

No doubt, a scientific theory of gearing is necessary to the gear researchers and practical engineers as it is a powerful tool for the development of novel designs of gears and gearing with a prescribed performance. Such a scientific theory of gearing can be developed now. With that said, it is important to revise the earlier obtained accomplishments in the field of gearing and identify those of them that can be useful in the development of the fundamental scientific theory of gearing.

### 6.1.6 Favorable Approximate Gearing

Easier manufacture is the principal advantage of approximate gearing over perfect gearing. Due to this advantage approximate gears will be used in the industry for a long while.

A theory of favorable approximate gearing can be (and will be) developed on the premises of the scientific theory of gearing [1]. Only in such a scenario approximate gear pairs with favorable design parameters can be designed.

### 6.1.7 Accomplishments in the Field of "Non-circular" Gearing

The most general case of non-circular gears with the crossing axes of rotation is analyzed. For the analysis, a reference system, associated with gear pair in a natural way, is used [the axes of the reference system are along:

(a) The axis of instant rotation, $P_{ln}$.
(b) The center-distance, $\mathbb{C}$.
(c) Perpendicular to these two directions, $P_{ln}$ and $\mathbb{C}$].

In the analysis, the center-distance $C$, the crossed-axes angle $\Sigma$, and the gear ratio $u$, are assumed variable. In particular cases, one or two variable parameters considered of a constant value. Under such the assumption, a classification of perfect $C\Sigma u-$gearing is developed (S. Radzevich, ~2017) [1].

### 6.1.8 Tentative Chronology of the Evolution of the Theory of Gearing

Summarizing the above discussion, the benchmarking achievements in the theory of gearing are schematically outlined in Fig. 6.22.

The proposed chronology begins with an analysis of what was done in pre-Eulerian period of evolution of the gear art. Contributions to the field of gearing by Desargues, de la Hire, and Camus comprise the pre-Eulerian period of evolution of the theory of gearing. In the schematic (see Fig. 6.22), number "0" is assigned to the pre-Eulerian period evolution of the theory of gearing.

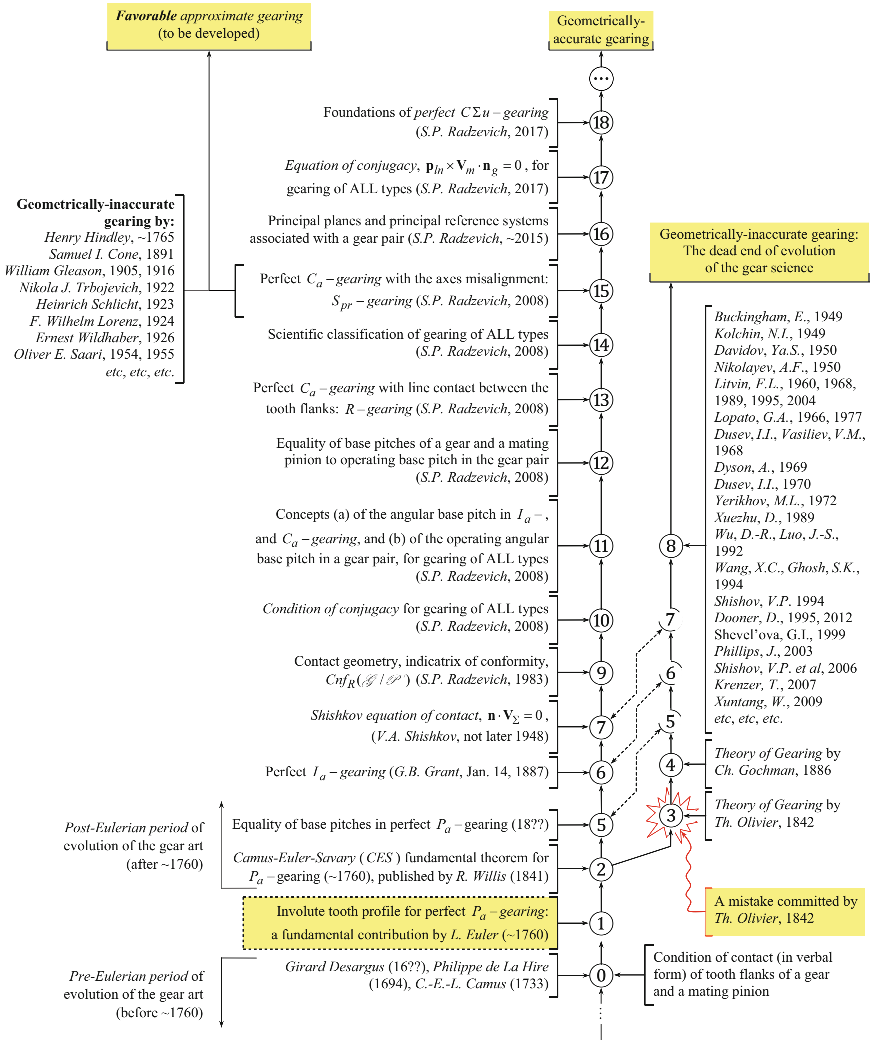


**Fig. 6.22** Tentative chronology of evolution of the scientific theory of gearing

The pre-Eulerian period of evolution of the gear art is followed by the time when the fundamental contribution to the theory of gearing was made by L. Euler. The latter is considered as the origin of the scientific theory of gearing.

Invention of involute gearing[44] by L. Euler (1760) is a benchmarking achievement in the theory of gearing. Per the author's opinion, the origin of the "scientific

[44]It needs to be stressed here that *involute of a circle* itself was known long before the invention of involute gearing by *L. Euler*.

theory of gearing" has to be associated with this accomplishment. In the schematic (see Fig. 6.22), number "1" is assigned to the invention of involute gearing by L. Euler.

The rest of the items in Fig. 6.22 correspond to the principal accomplishments in the theory of gearing that follow the contribution by L. Euler. Where possible, the names of the researchers and dates are associated with the corresponding accomplishments.

The next step in the development of the theory of gearing has been made by L. Euler and F. Savary who are granted with the "fundamental theorem of gearing" (along with Ch. Camus). Later on, in 1841, this theorem has been published in the book by R. Willis [14]. Number "2" is assigned in the schematic (see Fig. 6.22) to the achievement in the scientific theory of gearing. The "Camus-Euler-Savary fundamental theorem of gearing" is valid only for parallel-axes gearing.

In 1842 a huge mistake has been committed by T. Olivier who proposed his version of the theory of gearing based just on the enveloping condition of conjugacy of the interacting tooth flanks of a gear and a mating pinion. The condition of conjugacy of the tooth flanks is not taken into account by T. Olivier. This event is labeled as "3" in the schematic (see Fig. 6.22).

A "parallel" line (the items #3 through #8) corresponds to a wrong way of the evolution of the gear theory. The mistake committed by T. Olivier [7] (1842), and repeated by C. Gochman [8] (1886), significantly impaired further fundamental developments in the theory of gearing ("4," "8," and others in Fig. 6.22). Only approximate gears can be designed following this way. This is the dead end of evolution of the gear science.

The accomplishments in the theory of gearing labeled as "5" trough "7" are applicable in both branches, that is, (a) in the "dead end" of the theory[45] ("4," "8," and others in Fig. 6.22), as well as (b) in the way that leads to the self-consistent scientific theory of gearing [1] ("5" through "14," and others in Fig. 6.22). No perfect intersected-axes and crossed-axes gearing can be designed following this way. No correct tooth flank modification in parallel-axes gearing is possible – only trial and error method can be used to determine the parameters of the tooth flank modification.

The condition that requires equal base pitches of a gear and its mating pinion (only in cases of parallel-axes gearing) is known for a long while (note, the "operating base pitch" of a gear pair is not known yet). Per the author's estimate, this requirement, that is, item "5" in Fig. 6.22, is known since the mid of the nineteenth century. Unfortunately, in the meantime it is not possible to identify the name of the gear scientist who should be credited with this significant accomplishment in the scientific theory of gearing.

Spherical involute in perfect bevel gearing (item "6" in Fig. 6.22) is known since 1887.

[45]It is a right point to stress here that the "*dead end*" in the diagram in Fig. 6.22 means that no "*perfect*" $I_a-$ and $C_a-$gearing are possible; no "*correct*" tooth flank modification in $P_a-$gearing is possible; and trial and error method is dominated.

"Shishkov equation of contact" (item "7" in Fig. 6.22) deserves to be mentioned here, as use of this equation makes possible significant simplifications of the kinematic method of surface generation, especially in cases when both the contact perpendicular, $\mathbf{n}$, and the instant linear velocity vector, $\mathbf{V}_{\Sigma}$, can be determined with no derivatives of equations of the tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, as well as the parameters of the kinematics of a gear pair.

Conditions of contact of the interacting tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, are investigated analytically, and an equation of the indicatrix of conformity, $Cnf_R(\mathcal{G}/\mathcal{P})$, at point of contact of tooth flanks of a gear, $\mathcal{G}$, and a mating pinion, $\mathcal{P}$, (item "9" in Fig. 6.22) is derived [26, 35–37], and others.

Equation of conjugacy $\mathbf{p}_{ln} \times \mathbf{V}_m \cdot \mathbf{n}_g = 0$ (item "17" in Fig. 6.22) of the interacting tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, is derived by Prof. S.P. Radzevich (2017).

Then, the below-listed accomplishments were contributed by Professor S. Radzevich in around 2008:

- Condition of conjugacy of the tooth flanks for gear pairs of all types (item "10" in Fig. 6.22), including intersected-axes gear pairs and crossed-axes gear pairs.
- The concepts of (a) "base pitch" in intersected-axes gear pairs, and crossed-axes gear pairs, and (b) the "operating base pitch" in gear pairs of all types (item "11" in Fig. 6.22).
- The equality of base pitches of a gear and its mating pinion to the "operating base pitch" in gear pairs of all types (item "12" in Fig. 6.22).
- Design of geometrically accurate crossed-axes gearing with line contact between the tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, that is, $R-$gearing (item "13" in Fig. 6.22).
- A scientific classification of vector diagrams of gear pairs of all types (item "14" in Fig. 6.22.
- Design of perfect (crossed-axes) gearing insensitive to the axes misalignment, that is, $S_{pr}-$gearing (item "15" in Fig. 6.22).
- Principal planes and principal reference systems associated with a gear pair are introduced by Professor S. Radzevich in around 2015.
- Equation of conjugacy $\mathbf{p}_{ln} \times \mathbf{V}_m \cdot \mathbf{n}_g = 0$ (item "17" in Fig. 6.22) of the interacting tooth flanks, $\mathcal{G}$ and $\mathcal{P}$, is derived by Prof. S.P. Radzevich (2017).

A theory of favorable approximate gearing will be developed in the future. The discussed scientific theory of gearing is a reliable foundation for the theory of favorable approximate gearing to be developed.

It should be realized that the diagram in Fig. 6.22 is tentative. More accomplishments in the scientific theory of gearing and the corresponding gear researcher's names can be added in Fig. 6.22 if a more in detail investigation into the evolution of the scientific theory of gearing will be undertaken. Only the key (the fundamental) achievements in the scientific theory of gearing are included in the diagram (see Fig. 6.22) in its current stage.

Generally speaking, geometrically accurate gear pairs of any kind can be designed based on the scientific theory of gearing.[46]

[46]Theory of gearing can be viewed as a kind of "*road map*" that helps the user traveling from one point (location) to another point (location) in a most efficient way.

The listed accomplishments form the foundation of the "self-consistent scientific theory of gearing" (Radzevich, S.P., 2012, 2018).

The proposed chronology (see Fig. 6.22) is open for further improvements. Constructive recommendations, comments, and concerns appreciated.

### 6.1.9 *On Other Efforts that Pertain to the Evolution of the Scientific Theory of Gearing*

The author has turned his interest to the evolution of the gear science about a decade ago [3, 38].

Despite gears are extensively used in the industry of many industrially developed countries, not much accomplishments to the theory of gearing are contributed to this end. No accomplishments to the theory of gearing are contributed in the recent years in North America (including the USA and Canada), in Europe (including Germany, Austria, as well as the rest of European countries), and in Asian countries (including, but not limited to China, Taiwan, Japan, and South Korea). In Australia, only the 2003 book by Jack Phillips on General Spatial Involute Gearing [34] deserves to be mentioned in this regard. Production of quality gears in the industry is based in much on the accumulated experience, and not on the means and methods derived from the theory of gearing. Even lead companies in the field of gear design and manufacture indicate poor familiarity with the latest achievements in the theory of gearing. An article [39], as well as many others, is a perfect illustration of poor familiarity of the gear community with the latest achievements in the theory of gearing.

In the recent years, numerous papers on the history of gearing (both in English and in Russian languages) have been authored/co-authored by Babichev, Barmina, Lagutin, Volkov, and others of Russia. All these publications are available in the public domain. A claim on the so-called Russian school of theory of gearing has been aggressively made by the authors. It should be stressed here that all these publications are focused not on the principal accomplishments in the scientific theory of gearing.

The discussion in this section of the book, along with the results of the earlier performed retrospective analysis on the history of evolution of the scientific theory of gearing [1, 2], reveals that this aggressive claim has been made with no sufficient validity. Are there significant accomplishments to the scientific theory of gearing (made by representatives of the so-called Russian school of theory of gearing) that are not taken into account (and not indicated in the chart shown in Fig. 6.22)? Feel free to name them, if any! An appropriate comment will be helpful for the enhancement of our understanding of the evolution of the scientific theory of gearing.

In the published papers and monographs authored even by the leading Soviet/Russian gear researchers, there is no evidence of understanding of the kinematics and geometry of the following:

(a) Novikov gearing (Novikov gearing is missed by Soviets due to their poor professionalism in the theory of gearing; gear experts from Western countries contributed zero to gearing of this particular design) [40].
(b) Spiroid gearing[47] (and geometrically accurate worm gearing in a more general sense).
(c) Perfect intersected-axes and (more generally) crossed-axes gearing.
(d) Perfect gears with the axes misalignment, and so forth.
(e) Still making no difference between enveloping surfaces and conjugate surfaces.
(f) They are not capable of demonstrating that the so-called gearing[48] proposed by a charlatan V.V. Stanovskoi (http://www.ec-gearing.ru/company.php) is a fake.
(g) For decades [for over "50 (!!) years in the theory and practice of gearing"], they carry out a meaningless research on gearing with a "closed line of contact that shrinks" when the gears rotate.

A few more to mention. What can be expected from the less experienced gear researchers of Russia?

Prof. Ya.S. Davidov, one of the Soviet "coryphaeus" in the field of gearing, in his "Memories …" correctly compared all the Russian gear theoreticians with the "swamp" (http://referat.znate.ru/text/index-8600.html). A following dialog took place between Prof. F.L. Litvin and Prof. Ya.S. Davidov, when they were discussing the features of "Novikov gearing": "In one of the conversations with me F.L. Litvin very correctly compared the work of Novikov to the rock thrown into the swamp and caused a stirring of water" (It is likely the comparison of the gear community in the Soviet Union/Russia with a "swamp" makes sense). Can someone ignore this opinion of two well-known Soviet/Russian gear researchers (of Prof. F.L. Litvin, and Prof. Ya.S. Davidov), when they have compared all the Russian gear theoreticians with the "swamp"? This comparison is one more evidence that the claim on the so-called special "Russian school of theory of gearing" is at least doubtful, if not to say more.

The just made conclusion has to be taken into account when the readers meet the meaningless term "Russian school of theory of gearing" (as well as similar terms introduced by Russians in the recent years: "classical school of theory of gearing" and "the gold age of theory of gearing"). In the meantime, experienced readers are skeptical with that and are commonly having a laugh when they read about the so-called Russian school of theory of gearing [9].

[47] After about 40 (!) PhD theses and 5 (!) Dr. Sci theses are defended by these people, how is it permissible to ask a question: "What do we know about spiroid gearing"? What did you do all this time?

[48] Amazingly, but this stupid "gearing" is supported by two doctors of sciences (Dr. *Scherbakov*, *N. R.*, the chairperson of "*Geometry*" department, and Dr. *Bubenchikov*, *A.M.*, the chairperson of "*Theoretical Mechanics"* department, both of Tomsk State University, Russia) *who are granted with scientific degree of Dr.Sci. in mathematics and physics*.

## 6.2 Concluding Remarks

In this chapter of the book a brief overview on the evolution of the scientific theory of gearing is carried out.

All (or, at least, almost all) the principal accomplishments in the scientific theory of gearing are identified and are briefly overviewed in the chapter. The discussion begins with the consideration of the earliest designs of gears. The evolution of the theory of gearing falls into three periods, namely, pre-Eulerian, Eulerian, and post-Eulerian periods of the gear art. The scientific theory of gearing is originated in the Eulerian period of the gear art. Then, the developments in the field of perfect gearing are considered. The contributions by G. Grant, Professor N. Kolchin, Professor M. Novikov, Professor V. Gavrilenko, and others are covered in this discussion.

The developments in the field of approximate gearing is another consideration in this chapter of the book. Here S. Cone double-enveloping worm gearing, approximate bevel gearing, approximate crossed-axes gearing, as well as face gearing are briefly discussed.

A brief summary of the principal accomplishments in the theory of gearing achieved by the beginning of the twenty-first century is provided. The condition of contact of the interacting tooth flanks of a gear and pinion, condition of conjugacy of the interacting tooth flanks of a gear and pinion, and condition of equality of base pitches of the interacting tooth flanks of a gear and pinion are covered in this discussion.

To the best of the author's knowledge, all the principal accomplishments are covered in this text. Where possible, the accomplishments are attributed with corresponding names of the gear researchers, and dates when the contribution has been done.[49] These accomplishments form the foundation of the self-consistent scientific theory of gearing (proposed by Radzevich, S.P. circa 2008 [1]). The scientific theory of gearing is not threatened with destruction, but only superstructure and development are expected (every scientific theory features this property).

Ultimately, a tentative chronology of the evolution of the theory of gearing is proposed.

Among others, the discussion is aimed to initiate an in-depth investigation in the field of the origins of the scientific theory of gearing.

More names of the gear researchers deserve to be mentioned. However, consideration in this section of the book is limited to the evolution only of the theory of gearing. Therefore, the number of names of the researchers is limited only to those who contributed to the kinematics and the geometry of gearing.

[49]Except of the contributions by *L. Euler*, the contributions by other members of the *Hall of Fame* at the *Gear Research Center* (The University of Illinois at Chicago) are out of the scope of the *scientific theory of gearing*, and, thus, are not discussed here.

Brief critical comments on the so-called Russian school of theory of gearing are outlined.

A comprehensive research on the evolution of the theory of gearing is necessary to be undertaken in the nearest future. It is needed that the research be based on in-depth study of the original scientific works of all principal investigators of the topic. The history of engineering is not less important than the engineering itself. The better we know the past, the better we can predict the future.

The discussion in this chapter of the book is helpful for better understanding of the fundamental principles of gearing.

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## *Further Readings*

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